Numerical solution for 1D steady heat conduction in a tube subjected to forced and natural convection

Emanuele Kob and Giada Alessi

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Universitat Politècnica de Catalunya - ETSEIB

Abstract

Simulation of a steady, one-dimensional conduction heat transfer through a pipe of constant width subjected to internal forced convection and external natural convection. The objective of this paper is to run few simulations to evaluate the temperature distribution through the pipe, the outlet fluid's conditions, and the total heat transferred across the tube.

The problem solution is obtained through the implementation of the numerical method of finite volumes, while the systems of equations are solved using Gauss-Seidel and TDMA algorithms. This approach requires many attempts to find a compromise between the mesh refinement, computational time, and accuracy of the solution.

Keywords: Steady state heat transfer · Numerical solution · 1D analysis

Contents

Chapter 1

Model definition and resolution methods

1.1 Problem definition

Consider a fluid flow inside a tube of internal diameter D_i, external diameter D_o and length L. The fluid reached steady state conditions and its flow is assumed to be one-dimensional along $\hat{x}$ direction. Moreover, inlet conditions are known (v_{in}, p_{in}, T_{in}). Externally the tube is surrounded by ambient air at a given temperature and pressure, T_{ext}, p_{ext}. Far form the tube the air is at rest. The thermal conductivity of the pipe is high and the thickness is small enough to be considered isothermal in radius direction. Consequently, the temperature distribution of the tube can be considered a function of $\hat{x}$ direction only, $T_t = f(x)$.

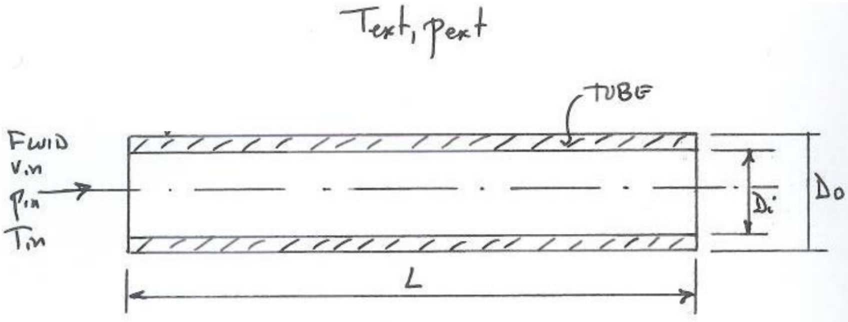


Fig. 1.1: Schematic representation of the model

The objective of the exercise is the evaluation of the average velocity, pressure and temperature of the fluid at the outlet of the tube : $v_{out}, p_{out}, T_{out}$. The total heat flux exchanged $\dot{Q}_w$, together with the inner and outer tube temperatures T_{wi} and T_{wo}, is also being analyzed.

The study of this problem is proposed for three different fluids at different inlet velocities: water, Therminol 66 and air, respectively at $v_{in} = 1$ m/s, 1 m/s and 30 m/s. In the following table, all input data and properties have been summarized.

Parameter	Variable	Value
Thermal conductivity	λ_g	$36[W/(mK)]$
Internal diameter	D_i	$20[mm]$
External diameter	D_o	$24[mm]$
Length	L	$20[m]$
Relative roughness	ε_r	$< 0.0001[m]$
Inlet velocity (case 1,2 and 3)	v_{in}	$1[m/s], 1[m/s], 30[m/s]$
Inlet temperature	T_{in}	$95[°C]$
Inlet pressure	p_{in}	$200000[Pa]$
Ambient temperature	T_{ext}	$20[°C]$
Ambient pressure	p_{ext}	$100000[Pa]$

Table 1.1: Input data of the problem.

1.2 Numerical Solution

The finite volume method is the numerical methodology used to discretize the domain and approximate the differential equations to algebraic equations. The methodology consists in dividing the overall geometry in a network of equally spaced finite volumes, named control volumes. The differential equations are applied to each control volume and approximated by some techniques such as finite differences. Then, it is possible to express the algebraic equations in a system and solve it for the unknowns.

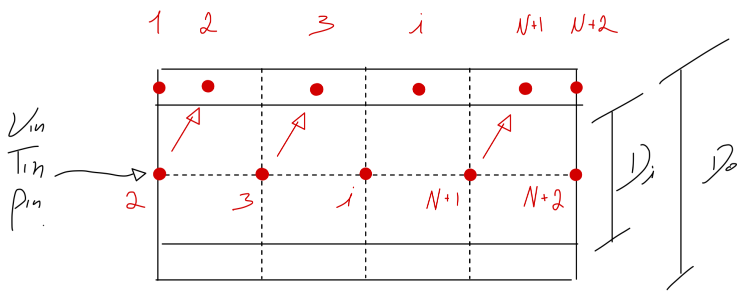


Fig. 1.2: Discretisation with finite volumes

The geometry is discretized in n finite volumes, the $n+2$ nodes for the wall nodes and the $n+1$ surface nodes for the internal fluid model.

1.3 Problem approach

The fluid flows evolution affects the temperature distribution in the pipe which affect the heat transfer to the environment. Since the convection coefficients are unknown, an iterative process is necessary to run multiple computations until convergence of the input variables. It follows a scheme that generally explain the workflow for the resolution of the problem. Later, every box of the block scheme will be breakdown.

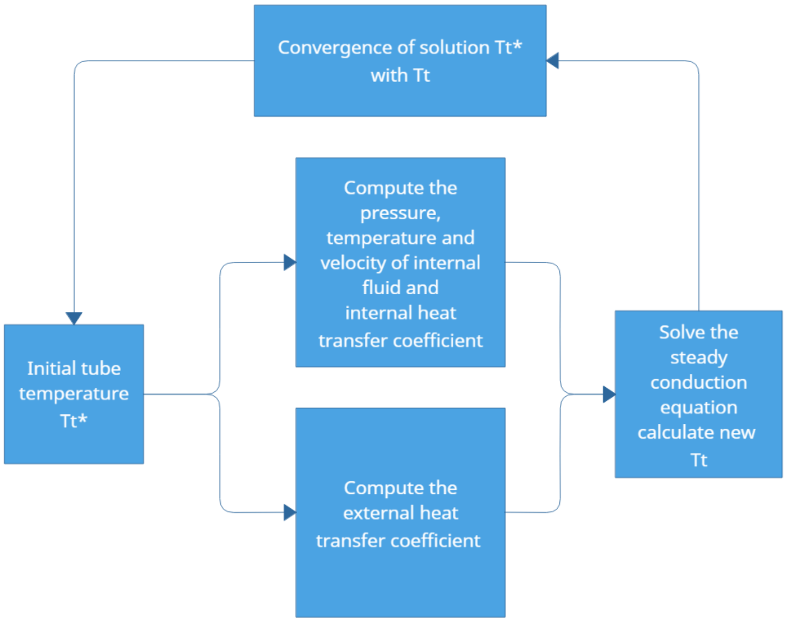


Fig. 1.3: Workflow

1.3.1 Internal fluid

Considering a known value for the wall temperature T_t, the internal fluid problem has three unknowns velocity v, pressure p and temperature T. By applying the Navier-Stockes equations under the steady state and 1D hypothesis to each finite volume it is possible to derive the following system of equations.

$$\begin{cases} \dot{m}(i+1) = \dot{m}_{in} = \rho(i+1)v(i+1)S \\ \\ \dot{m}_{in}(v(i+1) - v(i)) = S(p(i) - p(i+1)) - f_i \dfrac{\rho_i v_i^2}{2} \pi D_i \Delta x_i \\ \\ \dot{m}_{in} C_{p,i}(T(i+1) - T(i)) + \dot{m}_{in} \left(\dfrac{v(i+1)^2 - v(i)^2}{2} \right) = \alpha_i (T_t(i) - T_i) \pi D_i \Delta x_i \\ \\ \rho(i+1) = f(p(i+1) \end{cases} \tag{1.1}$$

The system is applied to the finite volume and the loop runs through every finite volume n. The unknowns are $\rho(i+1)$,$p(i+1)$, $T(i+1)$ and $v(i+1)$. The coefficients f_i, ρ_i, v_i, $C_{p,i}$ and α_i are evaluated at the mean temperature of each volume considering the mean temperature and pressure as

$$\begin{cases} T_i = \frac{1}{2}(T(i+1) + T(i)) \\ \\ p_i = \frac{1}{2}(p(i+1) + p(i)) \end{cases} \tag{1.2}$$

An initial guess vector for the pressure, temperature and velocity needs to be considered to apply the Gauss Seidel method to each volume until convergence.

1.3.1.1 Initial guess

An initial temperature at each surfaces is needed to compute the iteration. The four variables ρ, p, T and v can be evaluated with a linear variation of the temperature according to known boundary values.

$$\begin{cases} T^* = linspace(T_{in}, (T_{ext} + T_{in})/2, N+1) \\ \\ \% = \dfrac{T_{in} - T}{T_{in} - T_t} \\ \\ p^* = \% p_{in} \\ \\ v^* = \% v_{in} \\ \\ \rho^* = f(v) \end{cases} \tag{1.3}$$

The linear variation of the temperature is taken as a reference for the variation of the other variables. The Gauss Seidel method take that value as an input on each control volume step in the computation.

1.3.1.2 Thermophysical properties

The friction factor f_i as well as the internal heat transfer coefficient α_i are computed considering the fluid regime at each control volume. All the necessary relations to evaluate those value are in the following:

*Simplified expressions for water at saturation conditions (range: $T = 273 \div 400\ K$) (**T in K**):*

$$\boldsymbol{\rho}\left(\frac{\boldsymbol{kg}}{\boldsymbol{m^3}}\right) = 847.2 + 1.298T - 2.657 \cdot 10^{-3}T^2; \ \boldsymbol{c_p}\left(\frac{\boldsymbol{J}}{\boldsymbol{kgK}}\right) = 5648.79 - 9.140T + 14.21 \cdot 10^{-3}T^2;$$

$$\boldsymbol{\lambda}\left(\frac{\boldsymbol{W}}{\boldsymbol{mK}}\right) = -0.722 + 7.168 \cdot 10^{-3}T - 9.137 \cdot 10^{-6}T^2; \quad \boldsymbol{\mu}\left(\frac{\boldsymbol{kg}}{\boldsymbol{ms}}\right) = e^{7.867 - 0.077T + 9.04 \cdot 10^{-5}T^2}; \quad \boldsymbol{\beta} = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

Fig. 1.4: Liquid Water thermophysical properties

Therminol 66 thermal oil (range: $T = 273 \div 653\ K$) (**T in K**):

$$\rho\left(\frac{kg}{m^3}\right) = 1164.45 - 0.4389T - 3.21\cdot 10^{-4}T^2; \qquad c_p\left(\frac{J}{kgK}\right) = 658 + 2.82T + 8.97\cdot 10^{-4}T^2$$

$$\lambda\left(\frac{W}{mK}\right) = 0.116 + 4.9\times 10^{-5}T - 1.5\cdot 10^{-7}T^2; \qquad \nu\left(\frac{m^2}{s}\right) = \frac{\mu}{\rho} = e^{-16.096+\frac{586.38}{T-210.65}}; \qquad \beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p$$

Fig. 1.5: Therminol 66 thermal oil thermophysical properties

Simplified expressions for dry air (range: $T = 200 \div 400\ K$) (T in K, p in Pa):

$$\rho = p/(287T); \qquad c_p(J/kgK) = 1031.5 - 0.210T + 4.143\cdot 10^{-4}T^2$$

$$\lambda(W/mK) = 2.728\cdot 10^{-3} + 7.776\cdot 10^{-5}T; \qquad \mu(kg/ms) = \frac{2.5393\cdot 10^{-5}\sqrt{T/273.15}}{1+(122/T)}; \qquad \beta(K^{-1}) = 1/T$$

Fig. 1.6: Dry air thermophysical properties

It i important to remember that the thermophysical properties are evaluated at the mean temperature and pressure of each control volume (T_i,p_i). Those values are useful to evaluate the Reynolds number $R_{e,i} = \rho_i v_i D_i/\mu_i$ and the Prandtl number $P_{r,i} = \mu_i c_{p,i}/\lambda_i$. In the following the typical adimensional number are considered for the current model: forced convection in a cylindrical pipe.

$$\begin{cases} N_{u,i} = \alpha_i D_i/\lambda_i = C R_e^m P_r^n K \\ \\ G_{r,i} = R_{e,i} P_{r,i} D_i/L \end{cases} \tag{1.4}$$

The Nusselt number depends on the constant n ,m, C and K which are empirical constants, which depends on the model geometry and the flow regime defined by the R_e, P_r and Graetz G_z numbers.

Cross-section	D	C	m	n	K	Operating conditions
Circular tube	d	1.86	$\frac{1}{3}$	$\frac{1}{3}$	$\left(\frac{d}{l}\right)^{1/3}\left(\frac{\mu}{\mu_w}\right)^{0.14}$	Laminar flow short tube, Re < 2 000, Gz > 10
	d	3.66	0	0	1	Laminar flow long tube, Re < 2 000, Gz < 10
	d	0.023	0.8	0.4	1	Turbulent flow of gases, Re > 2 000
	d	0.027	0.8	0.33	$\left(\frac{\mu}{\mu_w}\right)^{0.14}$	Turbulent flow of highly viscous liquids, Re > 2000, 0.6 < Pr < 100

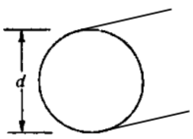


Fig. 1.7: Nusselt number for forced convection inside circular duct at low Mach number

1.3.1.3 Gauss Seidel method for internal fluid flow

For every finite volume a while condition is needed so that the new values ρ, p, T and v are computed and compared with the previous one ρ^*, p^*, T^* and v^* until the max difference between these four values is lower than a convergence ε (typically $10^{-3} \div 10^{-5}$).

Duct	Cross-sectional shape	Friction factor f	Operating conditions
Circular tube		$16\ Re^{-1}$	$Re \leq 2\,000$
		$0.079\ Re^{-0.25}$	$5 \times 10^3 < Re < 3 \times 10^4$ for $\frac{\epsilon}{D} < 0.0001$
Hydraulic Diameter D	$D = a$	$0.046\ Re^{-0.2}$	$3 \times 10^4 < Re < 3 \times 10^6$ for $\frac{\epsilon}{D} < 0.0001$

Fig. 1.8: Friction factors for flow inside a circular duct

1.3.2 External fluid

Given the wall temperature T_t at each control volume, it is possible to evaluate the thermodynamic properties. Also in this case a mean temperature and pressure is considered.

$$T_{m,i} = \frac{1}{2}(T_t(i) + T_{ext}) \tag{1.5}$$

From table 1.6 al thermodynamic properties can be derived. Since the external air is in natural convection, the friction factor is no longer needed and a different Nusselt relation needs to be considered. The flow regime is evaluated by the Rayleigh number $R_{a,i} = P_{r,i}G_{r,i}$, which is a function of the Prandtl number $P_{r,i} = \mu_i c_{p,i}/\lambda_i$ and the Grashof number $Gr = g\beta_i\rho_i^2|T_{t,i} - T_{ext}|X^3/\mu_i^2$.

$$N_{u,i} = \alpha_i D_i/\lambda_i = CR_{a,i}^n K \tag{1.6}$$

Similarly to the forced convection formula, the Nusselt number needs some constant to be calculated. These constants accounts the flow regime evaluated by the Raylight number and the geometry of the external surface.

The flow is considered laminar between $10^3 < R_a < 10^9$ and turbolent above $R_a > 10^9$. It is important to remember that all the thermodynamic properties are evaluated at the film temperature T_m.

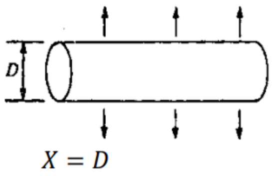
System	Schematic presentation	C	n	K	Operating conditions	References
Exposed surfaces						
Horizontal cylinder		0.47	1/4	1	Laminar flow	3.9
		0.1	1/3	1	Turbulent flow	3.9

Fig. 1.9: Nusselt number for natural convection on a cylindrical surface at low Mach number

1.3.3 Conduction in the pipe thickness

According to the hypothesis each finite volume on the pipe thickness is described by the node variables in the following equation:

$$\sum_i Q_i = 0 \tag{1.7}$$

The steady state energy balance accounts two convective and two conductive heat transfers as can be seen in the following sketch:

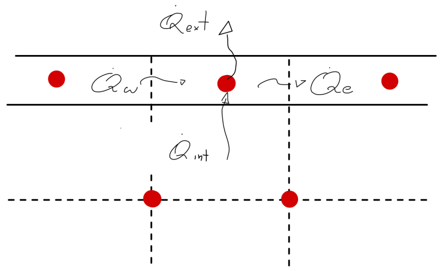


Fig. 1.10: Steady energy balance on the finite volume

$$-\lambda_t S \frac{T_P - T_W}{dX_{PW}} + \lambda_t S \frac{T_E - T_P}{dX_{PE}} + \alpha_i (T_i - T_{tP}) \pi D_i \Delta x_i - \alpha_{ext} (T_{tP} - T_{ext}) \pi D_o \Delta x_i = 0 \tag{1.8}$$

1.3.3.1 Internal Nodes

The general algebraic solution for internal nodes is

$$
\begin{cases}
a_P T_P = a_E T_E + a_W T_W + b_P \\
a_W = S\dfrac{\lambda_t}{\Delta x_i} \\
a_E = S\dfrac{\lambda_t}{\Delta x_i} \\
a_P = a_E + a_W + \alpha_i \pi D_i \Delta x_i + \alpha_{ext} \pi D_o \Delta x_i \\
b_P = \alpha_i T_i \pi D_i \Delta x_i + \alpha_{ext} T_{ext} \pi D_o \Delta x_i
\end{cases}
\tag{1.9}
$$

1.3.3.2 Boundary Nodes

The adiabatic boundary condition simplify the computation since the first and last node are at the same temperature of their neighborhood.

$$
\begin{cases}
T_t(1) = T_t(2) \\
T_t(n+1) = T_t(n+2)
\end{cases}
\tag{1.10}
$$

The algebraic system of equations for the first node can be written as

$$
\begin{cases}
a_P T_P = a_E T_E \\
a_E = S\dfrac{\lambda_t}{\Delta x_i} \\
a_P = a_E
\end{cases}
\tag{1.11}
$$

and for the last node

$$
\begin{cases}
a_P T_P = a_W T_W \\
a_W = S\dfrac{\lambda_t}{\Delta x_i} \\
a_P = a_W
\end{cases}
\tag{1.12}
$$

1.3.3.3 TDMA method for conduction heat transfer

TDMA stays for *tridiagonal matrix algorithm*. It is a **direct method** for solving linear systems of equations. The general algebraic system can be written as a band matrix that has nonzero elements only on the main diagonal and zeros as all other elements. It follows an example of linear equation and its tridiagonal matrix expression.

$$a_i x_{i-1} + b_i x_i + c_i xi + 1 = d_i \tag{1.13}$$

$$\begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix}$$

Fig. 1.11: Tridiagonal Matrix

This numerical solution is a simpler approach then Gauss-Seidel, since it uses just one iteration to get the solution without convergence criteria or initial guess.

The matrix form of the system can be rewritten as

$$M_{ii} \quad T[i] \quad = \quad R[i] \tag{1.14}$$

where M_{ii} is the tridiagonal matrix that contains the parameter $P[i]$

$$\begin{pmatrix} 1 & -P[1] & 0 \\ 0 & 1 & -P[2] \\ 0 & 0 & 1 \end{pmatrix} \tag{1.15}$$

The parameters above can be expressed as

$$\begin{cases} P[i] = \dfrac{a_E[i]}{a_P[i] - a_W[i]P[i-1]} \\ \\ R[i] = \dfrac{b_P[i] + a_W[i]R[i-1]}{a_P[i] - a_W[i]P[i-1]} \end{cases} \tag{1.16}$$

The calculation of the temperature starts from the node $T[n+2]$ and goes back to $T[1]$ with the following algorithm:

$$\begin{cases} T_t[n+2] = R[n+2] \\ \\ T_t[n+1] = P[n+1]T_t[n+2] + R[n+1] \\ \\ ... \\ \\ T_t[1] = P[1]T_t[2] + R[1] \end{cases} \tag{1.17}$$

The vector value that is computed needs to be compared with the initial vector T_t^* given at the beginning of the computation as can be seen in the block scheme 1.3. The whole process is a **while** condition, that continuously updated the $T_t^* = T_t$ and calculate the fluid flow **until the convergence**

$$|T_t[i] - T_t^*[i]| < \epsilon \tag{1.18}$$

Chapter 2

Results and discussion

2.1 Expected output

The numerical and analytical solution are compared in the following table. The results shows different properties evaluated at the outlet of the tube. The analytical solution considers constant thermophysical properties. The computation of the numerical solution is achieved with a discretisation factor equal to $N = 100$ and a convergence criteria $\delta = 10^{-5}$.

Variables	Case 1	Num-sol	Case 2	Num-sol	Case 3	Num-sol
Re_i	61980	61708	4627	4529	54644	55402
Pr_i	1.94	1.941	66.16	67.412	0.70	0.72
K_i	1.00	0.999	0.994	0.995	1.005	1.0004
f_i	0.005	0.0051	0.0097	0.0085	0.005	0.0052
$\alpha_i\,[W/m^2K]$	7742	7743	521	516	185	216
$Gr_o\,[W/m^2K]$	86005	83726	84997	82416	59690	55295
Pr_o	0.70	0.72	0.70	0.72	0.70	0.73
$\alpha_o\,[W/m^2K]$	8.77	8.726	8.73	8.665	7.66	7.487
$U_o\,[W/m^2K]$	8.76	8.622	8.55	8.407	7.29	7.126
$U_i\,[W/m^2K]$	10.51	10.347	10.26	10.088	8.75	8.551
$v_{out}\,[m/s]$	1.00	0.999	1.00	0.999	29.79	29.697
$p_{out}\,[Pa]$	190425	190234	181396	183697	182692	182459
$T_{out}\,[^\circ C]$	94.23	94.231	93.25	93.259	60.75	59.315
$\dot{Q}_w\,[W]$	985	981.43	956	950.44	617	641.39

Table 2.1: Comparison of parameters for Cases 1, 2, and 3.

All the values in the *Num-Sol* column are the value evaluated at the last control volume of the discretized geometry, while the other comparing values are evaluated globally assuming constant thermophysical properties and heat transfer coefficients. Taking into account this fact, it is reasonable to state that the code was successfully implemented. The overall heat transfer coefficient $U_{i,o}$ where evaluated with average values of the internal and external heat transfer coefficients. It follows the results for the temperature, pressure and velocity distribution in the three cases.

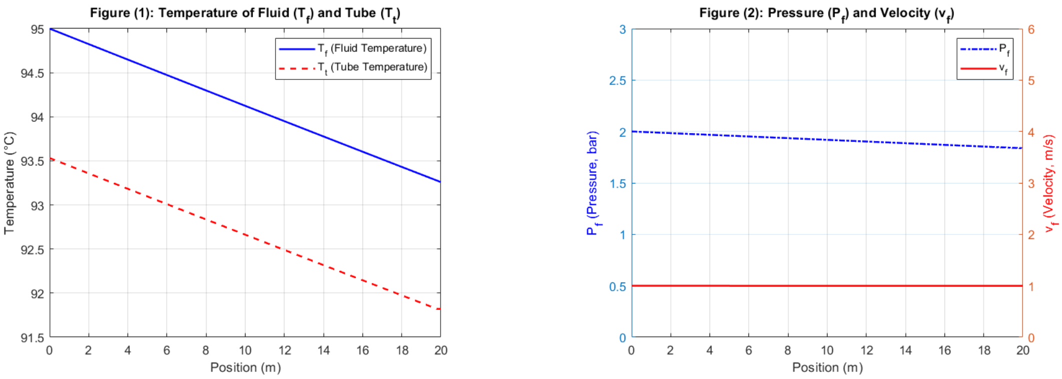


Fig. 2.1: Results for water

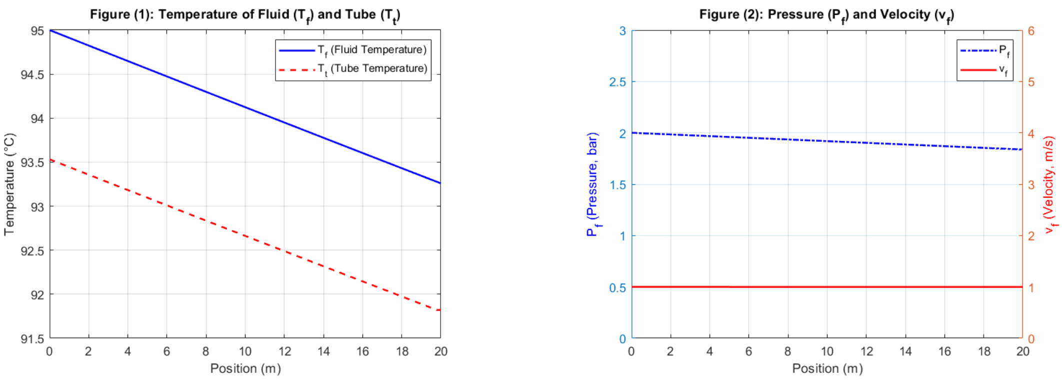


Fig. 2.2: Results for Therminol 66

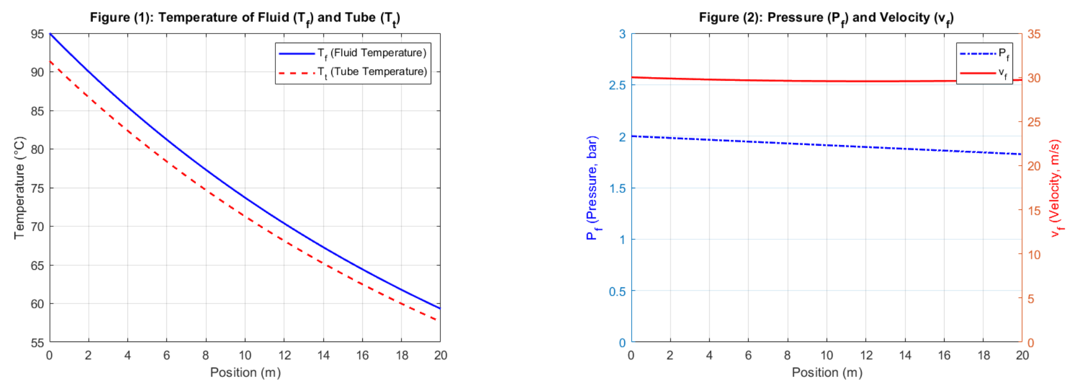


Fig. 2.3: Results for dry air