Numerical Simulation of Voter Models

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Abstract. The voter models tries to address the problem of describing the evolution of opinions in a population of individuals. This is an of course an abstraction. Nevertheless one can think about building an increasingly complex model, taking into account how real decision are made. In this project, I try to simulate different voter models, trying to estimate an important parameter as the consensus time and the dynamics.

Keywords: Voter Model · Simulation · Consensus Time.

1 Introduction

The voter is an idealized description of how opinions can spread in a network on individuals. Each voter can be thought as a node of a generic network taking different values depending on the number of opinions(e.g. 0/1 or more states). In the original model there is notion of right and wrong opinion, even if a mechanism of "competing propaganda" can be added, describing the news media effects or others. The scope of the simulation is reaching consensus(all nodes with the same state/opinion) and measuring in how many steps this can be achieved. In this model the consensus is theoretically, inevitable: we will see that with the simulation this is not always the case.

1.1 The project

The model is as simple as important in probability and statistical physics and it is only soluble when voters are places on regular lattices or complete graphs; it can be further generalized to include realistic processes in decision making.

Here we are going to treat first the classic voter model in different lattices(1-D, Square, Fully-connected) and then reproduce the results obtained with the confident voting[1] both symmetric and asymmetric case. The average consensus time for some lattice has been computed as the time evolution and the distribution probability of the consensus times.
All the simulations have been performed using Python language and optimized using Numba library. When possible, for example to compute the average quantities, the code has been parallelized with the same library.

2 Classic Voter Model

In the classic model, as previously said, the voters are placed in at the nodes of a generic graph. The possible states are $+1$ or -1.
The algorithm is extremely simple:

1. Select a node (i) in the graph, all nodes being uniformly distributed
2. Select randomly and uniformly one neighbour (j) and adopt the state of that node
3. Repeat until consensus is reached

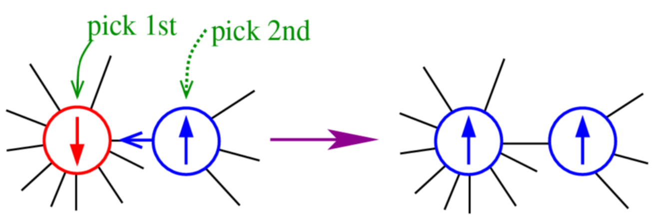

Fig. 1. Schematization of the algorithm

In this model, the agent has no self-confidence and adopts the state of neighbour without resistance. This feature can be modified adding a sort of stubborness. We call the initial 'magnetization' $m_0 = \frac{N_\uparrow - N_\downarrow}{N}$, being N the number of nodes: it is global parameter for the system.
The average consensus time $T(m_0)$ depends strongly on the initial conditions and such consensus is reached when the global parameter $M = \frac{\sum_{i=1}^{N} \sigma_i}{N}$ reaches 1 or -1. The average magnetization can be expressed as $m =< M >$.
Denoting $\sigma_i = \pm 1$ as the state of the voter at the site i , we can express the transition probability w_i as,

$$w_i = \frac{1}{2}(1 - \frac{\sigma_i}{z} \sum_{j \in \partial i} \sigma_j)$$

This probability is linear in the number of disagreeing neighbours since all the neighbours are uniformly picked at random.
One could write the master equation for the site (i):

$$\frac{d\sigma_i}{dt} = -2 w_i \sigma_i$$

Inserting the rate w_i into the equation and averaging of many realizations one has:

$$< \frac{d\sigma_i}{dt} >= - < \sigma_i > + \frac{1}{z} \sum_{j \in \partial i} < \sigma_j > \quad = \quad 0$$

Again, summing over all sites and supposing the initial magnetization is 0. The in-coming 'flux' of probability from all the neighbours site at (i) balances the out-coming one and the average magnetization is conserved.

2.1 Consensus Time

One can evaluate theoretically the consensus time for a generic graph. Let's change first the basis in which the "order parameter" is expressed, from ± 1 to $0, 1$.

$$\rho = \frac{m+1}{2}$$

ρ is the density of $\uparrow$ voters. Since the initialization is conserved, so is m, and, therefore ρ. One can use then the formalism of the backward Kolmogorov equation. After a time δt the density can switch to $\rho \pm \delta\rho$ according to the following equation:

$$T(\rho) = w_{\rho \to \rho+\delta\rho}[T(\rho+\delta\rho)+\delta t] + w_{\rho\to\rho-\delta\rho}[T(\rho-\delta\rho)+\delta t] + w_{\rho\to\rho}[T(\rho)+\delta t]$$

An opinion can be changed, in principle, after N steps; therefore, a single update takes $\delta t = \frac{1}{N}$. Going to the continuum limit we can write the differential equation

$$N\rho(1-\rho)\frac{d^2T}{d\rho^2} = -1$$

The solution is:

$$T(\rho) = -N[(1-\rho)ln(1-\rho) + \rho ln(\rho)]$$

This N dependence is characteristic of all eucledian Voter models of dimension $d \geq 3$. For $d = 2$ one has $T \propto Nln(N)$, and $d = 1$, $T \propto N^2$.

2.2 Simulation

The simulation has been done over three kinds of lattices: fully-connected, 1-D and Square lattice. In the following I will show the plots of the different cases. In particular, the average consensus time $\mathbb{E}[\tau]$ rescaled of N.

Fully Connected Graph: In a complete or fully-connected graph, each node is connected to everyone else but itself. The simulation follows the rules reported before, with initial condition $m_0 = 0$. The consensus time can be expressed as $T(\rho) = T(m_0)$, just makind the substitution and one has $T(m_0) = c(m_0)N$.

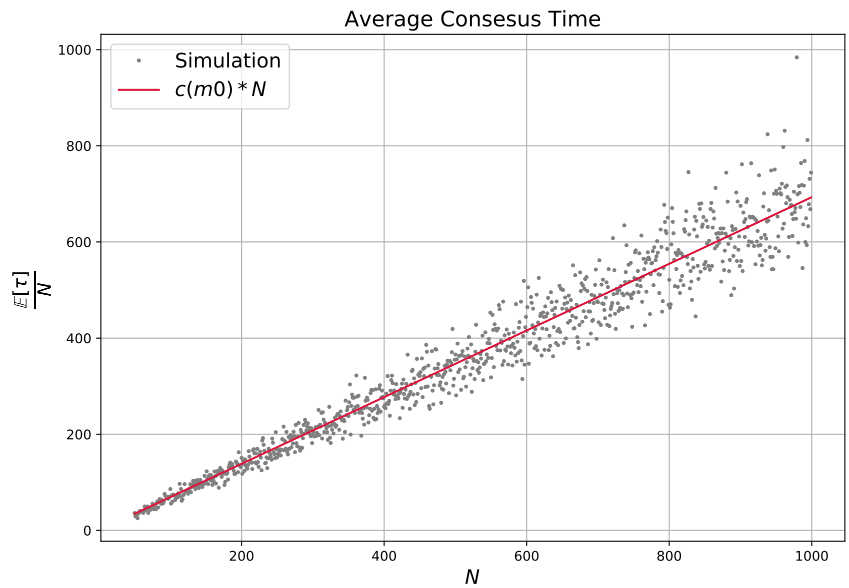

Fig. 2. $m_0 = 0$. Each point is obtained with 50 realizations, from N = 50 to N = 1000.

The simulations reproduces quite well the linear behavior of the theoretical function, with the slope depending on the initial density and therefore fixed, once the magnetization is fixed. The fluctuations seem to increase with the number of nodes.

Square lattice: In $d = 2$, I implemented a square lattice with periodic boundary conditions. The neighbours to choose from are just four, with equal probability.

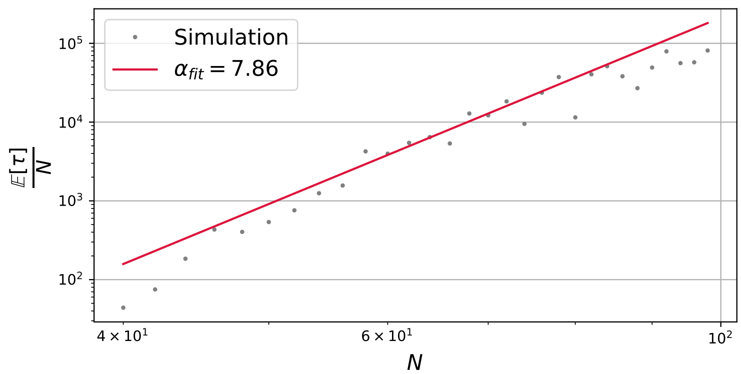

Fig. 3. $m_0 = 0$. Each point 5 realizations, in log-log scale. The result is bad, which I think is due to the low computing power available. N is restricted to 100.

1D lattice: At long times, the agents form two large domain with different states and the evolution is determined by the evolution of the interface, between this two domains. The average consensus time is shown below and reproduces the theoretical behavior.

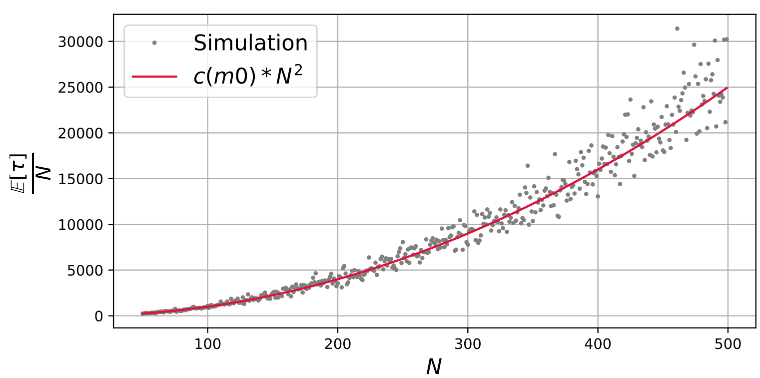

Fig. 4. $m_0 = 0$. Each point is obtained with 50 realizations, from N = 50 to N = 500.

3 Confident Voting

An evident problem of realism in the classic voter model is the lack of confidence of the agents. Each individual assigns a certain weight to a specific belief or opinion and the mechanism by which this happens may vary substantially: an important one is the mechanism of reinforcement. A person requires multiple contacts with persons with different opinions to change its. In this framework, there two ways in which an agent can behave: we can suppose this is a two opinion system, with states A and B. The transitions are schematized below:

$$A_c \rightleftarrows A_u \rightleftarrows B_u \rightleftarrows B_c$$

$$\begin{tikzcd} A_c \arrow[r, shift left] & A_u \arrow[l, shift left] \arrow[r, shift left] \arrow[rr, bend right, red] & B_u \arrow[l, shift left] \arrow[r, shift left] \arrow[ll, bend left, red] & B_c \arrow[l, shift left] \end{tikzcd}$$

The label 'c' and 'u' means respectively, confident and unsure state. The variables are therefore A_c, A_u, B_u and B_c and represent the densities of agents in that state. All transition have rate 1 ideally, even in the simulation, when two voter interact and two outcomes can happen, they are equally probable.

Marginal: The first transition line is the marginal behavior. An unsure agent that changes opinion is also unsure and can switch back or become confident in one update. The rate equations for A_c, A_u (symmetrical for B state) are:

$$\dot{A_c} = -(B_c + B_u)A_c + A_cA_u$$
$$\dot{A_u} = (B_c + B_u)A_c - A_cA_u + B_uA_c - B_cA_u$$

The possible pair-interactions are listed below:

$$\begin{aligned} A_uB_u &\to A_uA_u \quad \mathit{or} \quad B_uB_u; & A_cB_c &\to A_cB_u \quad \mathit{or} \quad A_uB_c \\ A_cA_u &\to A_cA_u \quad \mathit{or} \quad A_cA_c; & B_cB_u &\to B_cB_u \quad \mathit{or} \quad B_cB_c \\ B_uA_c &\to B_uA_u \quad \mathit{or} \quad A_cA_u; & B_cA_u &\to B_uA_u \quad \mathit{or} \quad B_cB_u \end{aligned}$$

Extremal: In the extremal behavior, an unsure agent becomes confident of the opposite after one update. It is compared often to St. Paul which became extremely confident about his faith after persecuting Christians. it requires two interactions with an opposite opinion voter to switch back to the original state.

$$\dot{A_c} = -B_cA_c + B_uA_u + A_cA_u$$
$$\dot{A_u} = B_cA_c - B_uA_u - A_cA_u + B_uA_c - B_cA_u$$

The possible pair-interactions are listed below:

$$\begin{aligned} A_uB_u &\to A_uA_c \quad \mathit{or} \quad B_uB_c; & A_cB_c &\to A_cB_u \quad \mathit{or} \quad B_uA_c \\ A_cA_u &\to A_cA_u \quad \mathit{or} \quad A_cA_c; & B_cB_u &\to B_cB_u \quad \mathit{or} \quad B_cB_c \\ B_uA_c &\to B_uA_u \quad \mathit{or} \quad A_cA_c; & B_cA_u &\to B_uA_u \quad \mathit{or} \quad B_cB_c \end{aligned}$$

3.1 Time Evolution

It is interesting to do an analysis of the fixed point.

Symmetric System: The simplest case is the symmetric one. *Marginal.* We can fix the densities as following: $A_c = B_c$ and $A_u = B_u$. We obtain the following equations, $\dot{A}_c = -\dot{A}_u = -A_0^2$ with solution:

$$A_u = \frac{1}{2} - A_c$$

$$A_c(t) = \frac{A_c(0)}{1 + A_c(0)t}$$

Evidently, as time goes, the density of confident voters vanishes and the one of unsure voters tends to oscillate around $\frac{1}{2}$. The plot in **Fig. 5.**, below, shows a qualitative correspondence with the rate equations, although, the solution(above) does not seem to follow the simulation: a far greater number of nodes would only leave the mismatch intact but a discretization problem could be accountable.

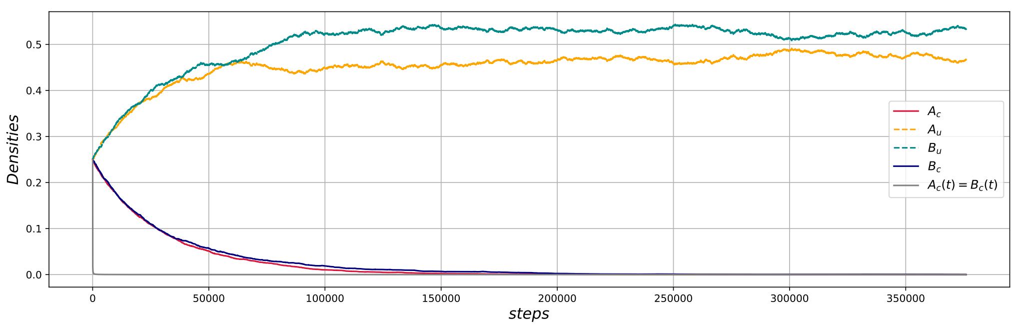

Fig. 5. Marginal time evolution on fully-connected graph (N=10000) with symmetric initial conditions. The exponential decay of the confident agents in blue and red, the theoretical solution in grey(it does not match the simulation!).

Extremal. The rate equation yeld

$$\dot{A}_c = -\dot{A}_u = A_c^2 + \frac{1}{2}A_c - \frac{1}{4} = -(A_c - \lambda_+)(A_c - \lambda_-)$$

$$\lambda_\pm = \frac{1}{4}(-1 \pm \sqrt{5}) \approx 0.31, -0.81$$

The fixed point λ_+ is a stable fixed point and the solution approaches this value at long times. Since we have that $A_c = \frac{1}{2} - A_c = \lambda_+$, all four densities are present at long times. Also in this case, the simulation does not reproduce precisely the theoretical predictions. The plot below, in **Fig. 6.**, shows an initial exponential drive of the confident densities towards the eigenvalue λ_+, but then they all deviate from that behavior reaching consensus every time. Again here, a problem of discretization could be accountable or the random number generator.

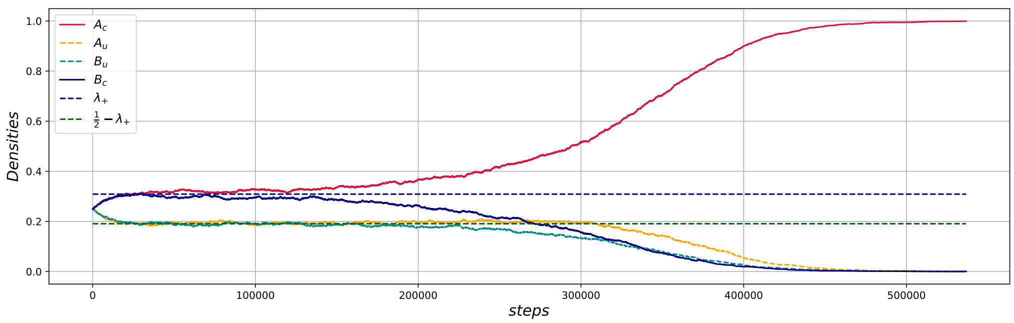

Fig. 6. Extremal time evolution on fully-connected graph (N=8000) with symmetric initial conditions. The eigenvalue λ_+ in blue. The theoretical unsure density in green.

Non-symmetric System: In case there is slight asymmetry in the densities of voters, the systems is driven towards a stable fixed point which is consensus. The dynamics is controlled by two time scales: one that is $O(1)$ and the other $O(ln(N))$. In a complete graph, we can introduce the N dependence by the deviation from the symmetric density as,

$$A_c(0) = \frac{1}{2} + \epsilon, \qquad \epsilon = \frac{1}{N}$$

$$B_c(0) = \frac{1}{2} - \epsilon, \qquad \epsilon = \frac{1}{N}$$

In this sense the asymmetry is given only by changing an state's node. In the following simulation we will set $A_u = B_u = 0$. In the marginal model in **Fig. 7.** [2] the unsure densities approach the unstable point $\frac{1}{2}$ and stay there for a time of order $ln(N)$, but after the start both start vanishing reaching consensus. For the extremal model,in **Fig. 7.**, a similar behavior occurs, but in this case the system reaches the stable fix point $A_c = \lambda_+$ and $A_u = \frac{1}{2} - \lambda_+$, before consensus. Follows, the time evolution for both model present in various papers.

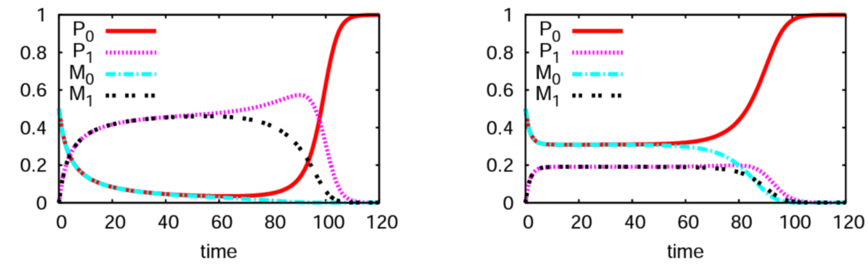

Fig. 7. left: Marginal model time evolution. right: extremal time evolution. Asymmetric initial conditions with $A_c(0) = 0.50001$ and $B_c(0) = 0.49999$. [2]

For comparison, I show the same plots made with my simulation (next page). From the results of the simulation we can deduce that there is a really qualitative

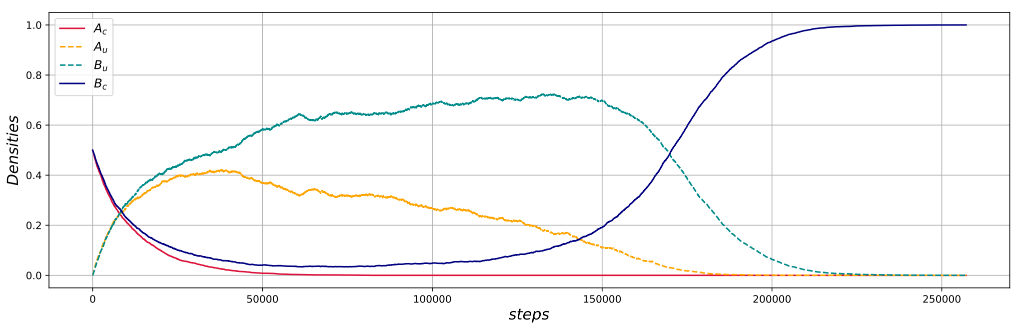

Fig. 8. Marginal Time evolution from the simulation. N = 5000. $A_c(0) = 0.50002$ and $B_c(0) = 0.49998$. The confident in blue reaches consensus

agreement, even if the curves are not perfectly shaped. But that is only a problem of computational power and can be further improved. To conclude, in both model we observe a period in which the system remains close to an unstable fixed point before reaching consensus, which is a stable fixed point.
The state space with the fixed points, if represented, it shows a tetrahedron with the condition that, at each time, $A_c + A_u + B_c + B_u = 1$.

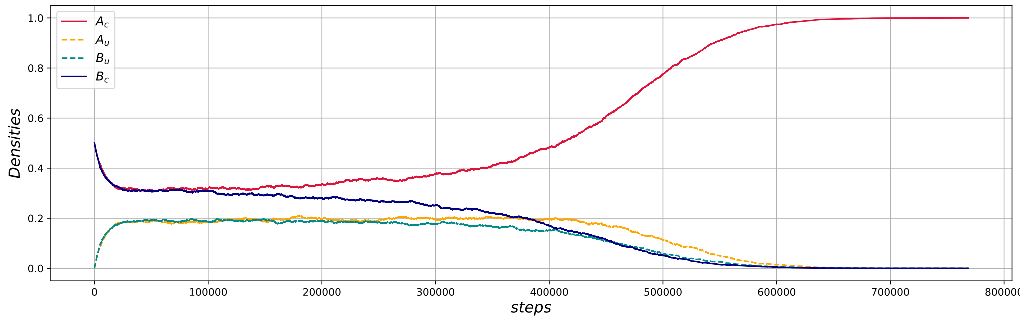

Fig. 9. Extremal Time Evolution from the simulation. N = 10000. $A_c(0) = 0.50001$ and $B_c(0) = 0.49999$. The confident voter in red reaches consensus.

Time evolution on other lattices: What follows is the same model implemented with the 1D and Square lattice, with periodic boundary conditions. It differs a bit from the classic voter model, because here, the magnetization of the transitions is not conserved, so this leads to formation of domains.

1. One dimensional.
In the one dimensional case, the evolution is determined by the evolution of the interfaces that form. The behavior of the interface would be similar to a Random Walk in 1-D, but in this case the hopping probability but with probability $\frac{1}{4}$. Below is shown the result of the simulation for both marginal and extremal. In both models, the system seems to be close to an unstable fixed point, as

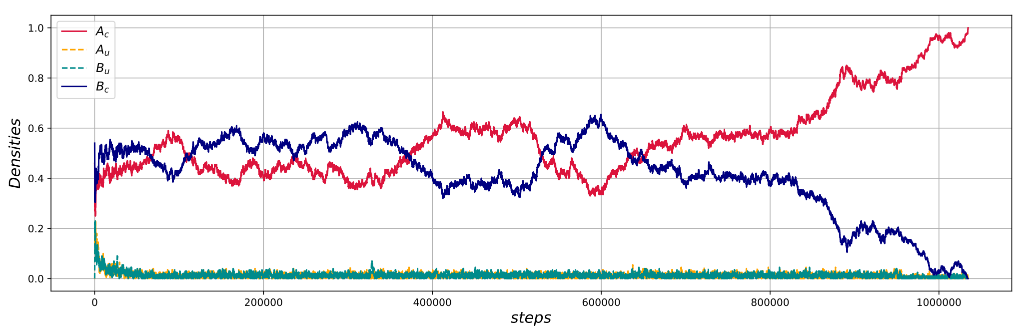

Fig. 10. Marginal Time evolution on 1D graph. N = 200, with near-symmetric initial conditions drawn from binomial distributions.

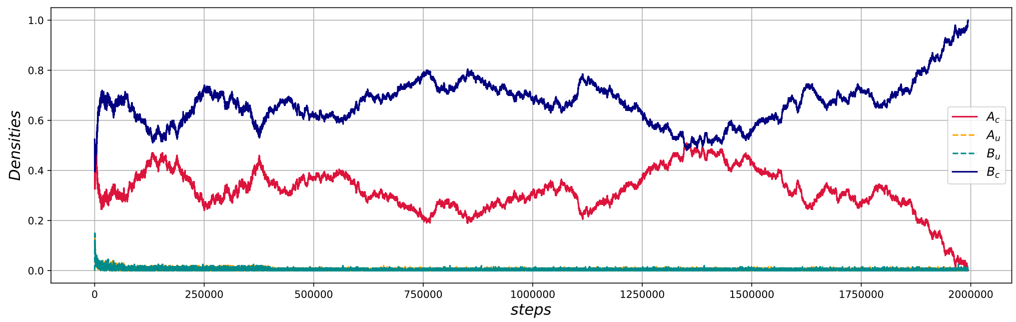

Fig. 11. Extremal Time evolution on 1D graph. N = 200, with near-symmetric initial conditions drawn from binomial distributions.

the fully connected case: the confident voters' densities oscillate around $\frac{1}{2}$, while the unsure voters density increases rapidly before suddenly decreasing towards what seems a constant value until consensus is reached. Globally, the time evolution does not differ that much from one model to another. Of course, larger the asymmetry of initial conditions, the smaller the probability that the confident densities 'cross' each other and fluctuate.
2. Square lattice.
The behavior in the square lattice is more rich, as there are different ways in which the system can reach consensus as we will see later in the consensus time distribution. What follows are original plots of time evolutions.

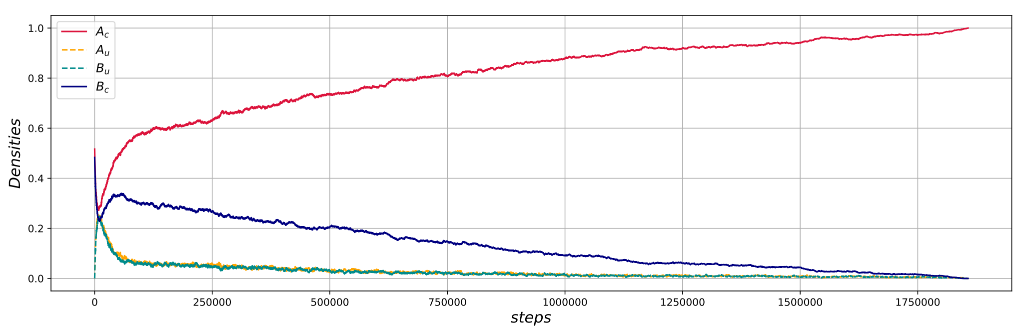

Fig. 12. Marginal Time evolution on square lattice 50x50, with near-symmetric initial conditions drawn from binomial distributions.

Both marginal and extremal model are characterized by a rapid inversion of densities, confident and unsure, after which the system tends to consensus. The density of unsure voters decays particularly slowly in both cases reaching zero at consensus. Also in the case of square lattice, there are no main differences to highlight, even if the inversion peak and the behavior after that look a bit different.

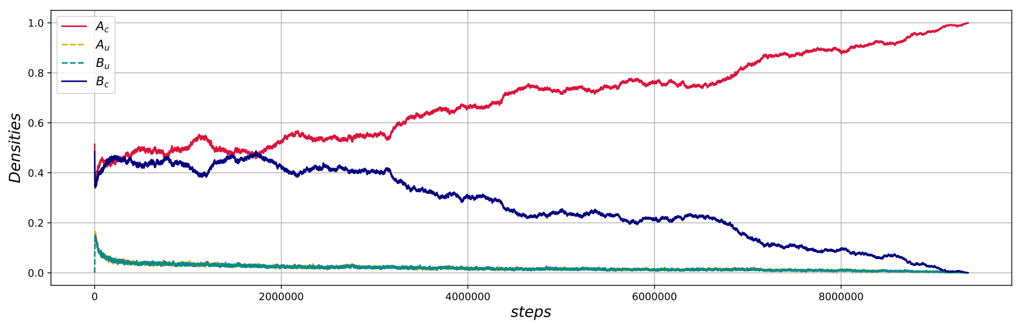

Fig. 13. Extremal Time evolution on square lattice 50x50, with near-symmetric initial conditions drawn from binomial distributions.

3.2 Time distribution in the Square Lattice

As we previously said, there are two ways in which the square lattice confident voting reaches consensus: for both marginal and extremal, in fact, the consensus is reached through domain coarsening, a single opinion droplet, and in a stripe-like way. In both cases, confident voters diffuse diffuse in a 'sea' of alternating unsure agents of both opinions. The last behavior occurs in many non-equilibrium systems(*Ising Model with zero-temperature Glauber Dynamics, majority voter model, naming game, AB model, etc.*... it is an open subject trying to understand why this symmetry braking occurs in many non-equilibrium systems, in which the dynamics is symmetric in both dimensions, x and y. It is

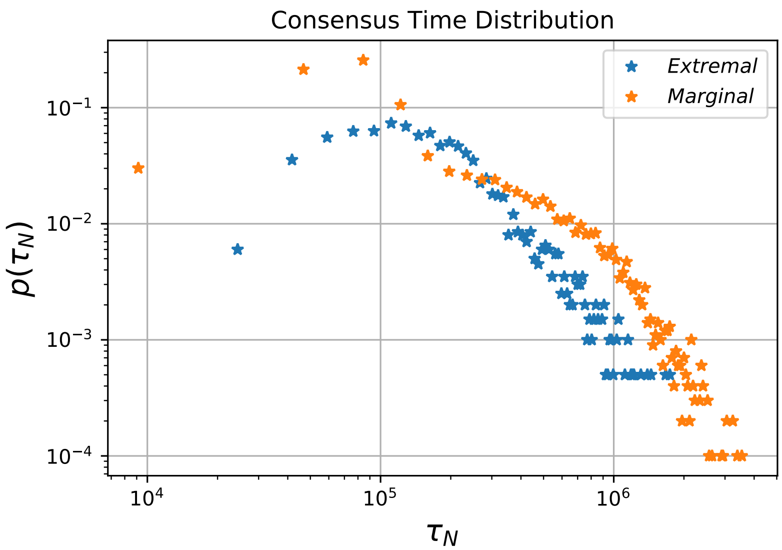

Fig. 14. Consensus time distribution for both marginal and extremal models. Marginal distribution has been realized with 10000 points, extremal with 2000 points. 200 bins for both models.

evident from this plot, more for the marginal model, that, there are two time scales in the Time distribution. The first time scale, hence the first peak should scale with N, while the other part of the distributon, even if not clearly visible, should scale as $N^{\frac{3}{2}}$. The linear scaling corresponds to consensus by coarsening: the length of the droplet l grows like $\sqrt{t}$. When it reaches L, consensus is reached. Therefore, at the end one has $L = \sqrt{t}$: since in the square lattice $N \propto L^2$, we have that average consensus time $T_N \propto N$.

3.3 Statistical Analysis

In order to understand better the behavior of the simulation, we now perform a statistical comparative analysis of both marginal and extremal model, with a further modification towards a stochastic model in which the transition happen with a certain rate. The previous section was devoted, actually, to a simulation of mean-field signals expressed by the mean field rate equation. There was not, in this sense, any diffusion or drift term, no matter how complicated.
It is better to analyze statistical quantities and see what they can tell us more about the model; Each signal of the simulation has been averaged to obtain the mean curves. It is a square lattice of dimensions 20x20. For this purpose, one can start by looking at the mean signals of the densities of both marginal and extremal model, in **Fig. 15.** Even if they are supposed to be symmetric ini-

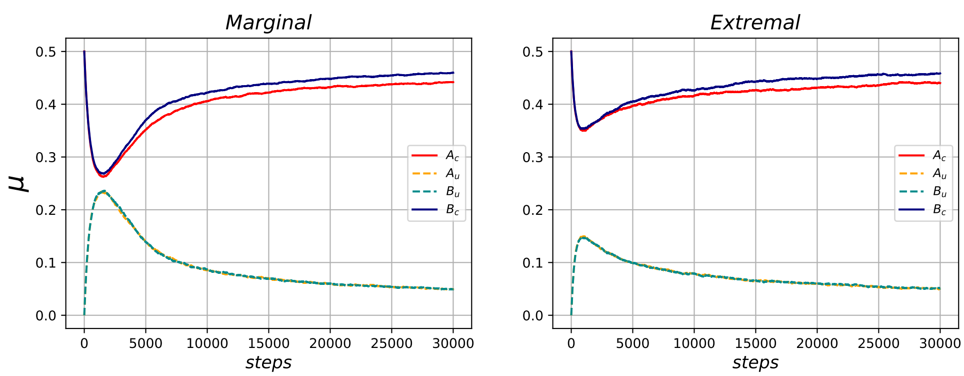

Fig. 15. Mean Time Evolution on square lattice $20x20$, with on-average symmetric initial conditions drawn from binomial distributions. 200 simulations and 30000 steps.

tial conditions, the correspondent behavior is really difficult to reach, because in every simulation, perfect symmetric initial densities are impossible to achieve, especially in a such a "small" lattice. Therefore, one should consider only the mean time evolutions as symmetric. Regarding to that, one should notice that both plots of the densities of confident voters A_c, B_c tend to $\frac{1}{2}$, as consensus is reached on average half of the times for both opinions, since $< A_c(0) + B_c(0) >= 1$. Furthermore, the difference in the models resides in the maximum density peaks, which is reached by the marginal model. The explanation for this, intuitively, is that in the marginal model, in order to be confident of the opposite opinion, the agent needs to update his state twice, being therefore slower in the transition. This fact facilitates the dynamics of the unsure agents from one state to the other; they are able to diffuse in a larger portion of the lattice in less steps than the marginal model, in which *"extremal"* transitions from unsure to opposite confident are possible.

Now let's consider the coefficient of variation to compare the two time series of the models: it is defined as $CV = \frac{\sigma}{\mu}$. It is clear that the dispersion of the

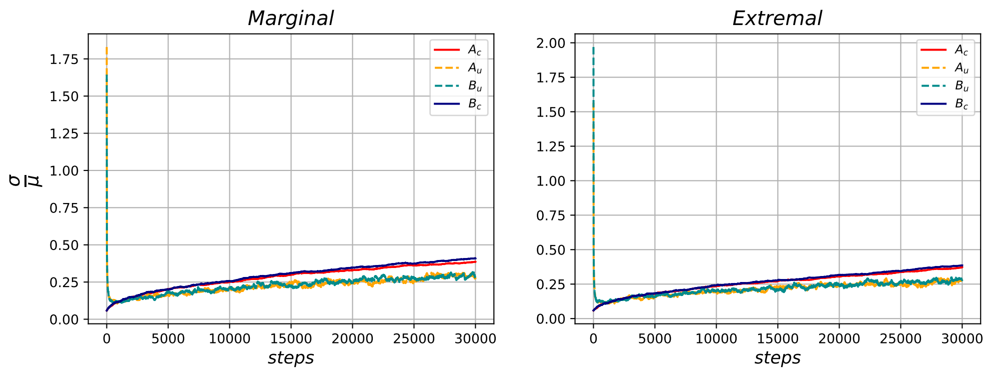

Fig. 16. Coefficient of Variation on square lattice 20x20, with on-average symmetric initial conditions drawn from binomial distributions. 200 simulations and 30000 steps.

signals of the two model behave the same with respect to the mean; it increases with time which means that either the variance is somehow time dependent or the mean value decreases or both. Since we have seen in the previous page, that the mean has an opposite behavior for confident and unsure voters, we deduce what is shown in the variance plot in **Fig. 17.**, below. There looks to be an

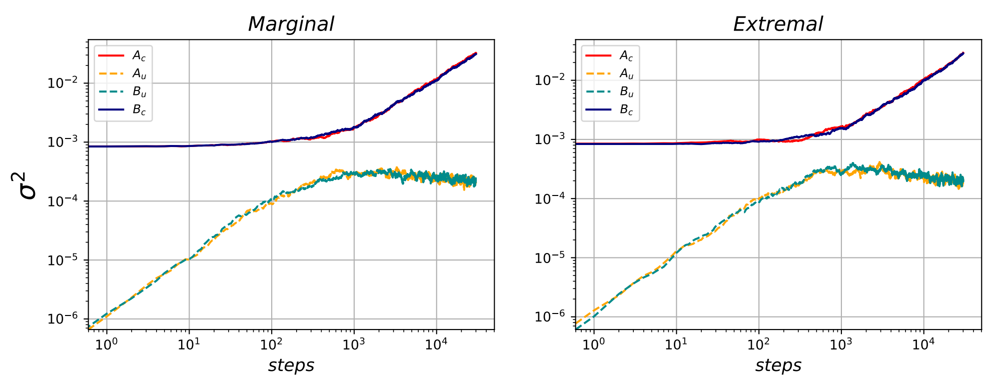

Fig. 17. Time dependent variance in log-log scale on square lattice 20x20, with on-average symmetric initial conditions drawn from binomial distributions. 200 simulations and 30000 steps.

exchange of behavior between confident and unsure voters, near the peak point, maximum for unsure and minimum for confident voters. At the beginning, there is diffusion(Random Walk paths) by the unsure agents while the confident voter show a sub-diffusive behavior. The transition then lasts for more than a decade(*in log-scale*), after which the confident voters change and assume a diffusive motion in the lattice while the unsure voters become heavily sub-diffuse; and this is true for both models.

3.4 Stochastic Model

One can think of modifying the mean-field in order to inject external noise into system. To do this, instead of implementing rate 1 transition, there can be different values, as for example the one depicted below:

$$\begin{tikzcd} A_c \arrow[loop left, "1-p"] \arrow[r, shift left, "p"] & A_u \arrow[l, shift left, "0.33"] \arrow[loop above, "0.33"] \arrow[r, shift left, "0.33"] & B_u \arrow[l, shift left, "0.33"] \arrow[loop above, "0.33"] \arrow[r, shift left, "0.33"] & B_c \arrow[l, shift left, "p"] \arrow[loop right, "1-p"] \end{tikzcd}$$

$$\begin{tikzcd} A_c \arrow[loop left, "1-p"] \arrow[r, shift left, "p"] & A_u \arrow[l, shift left, "0.33"] \arrow[loop above, "0.33"] \arrow[r, shift left, "0.33"] \arrow[rr, bend right, red] & B_u \arrow[l, shift left, "0.33"] \arrow[loop above, "0.33"] \arrow[r, shift left, "0.33"] \arrow[ll, bend left, red] & B_c \arrow[l, shift left, "p"] \arrow[loop right, "1-p"] \end{tikzcd}$$

In this diagram representing stochastic rate equations, almost all the parameters are fixed such that from unsure states was equally-likely to move in another unsure opposite state, a more confident state or staying put(sorry for the bad diagram): this is an arbitrary decision, as one can model this parameters to better reproduce real decision processes. There is only one degree of freedom which is the rate p, determining the probability(per unit time) at which a confident agent can change his mind and become unsure. Now, the degree of complexity here can be important, as those rates can be homogeneous or heterogeneous, following for example a particular power-law distribution: in this case I choose the same rate per each node in the lattice.

In the simulation, at each step, after the update following the marginal or extremal model, follows a further stochastic update replicating the diagram below. As it turns out, the choice of p is important and changes qualitatively the time evolution of the system, as well as the variance diagram.

Simulation: In the following are showed the plots of mean time evolution and variance for different values of $p < \frac{1}{2}$.

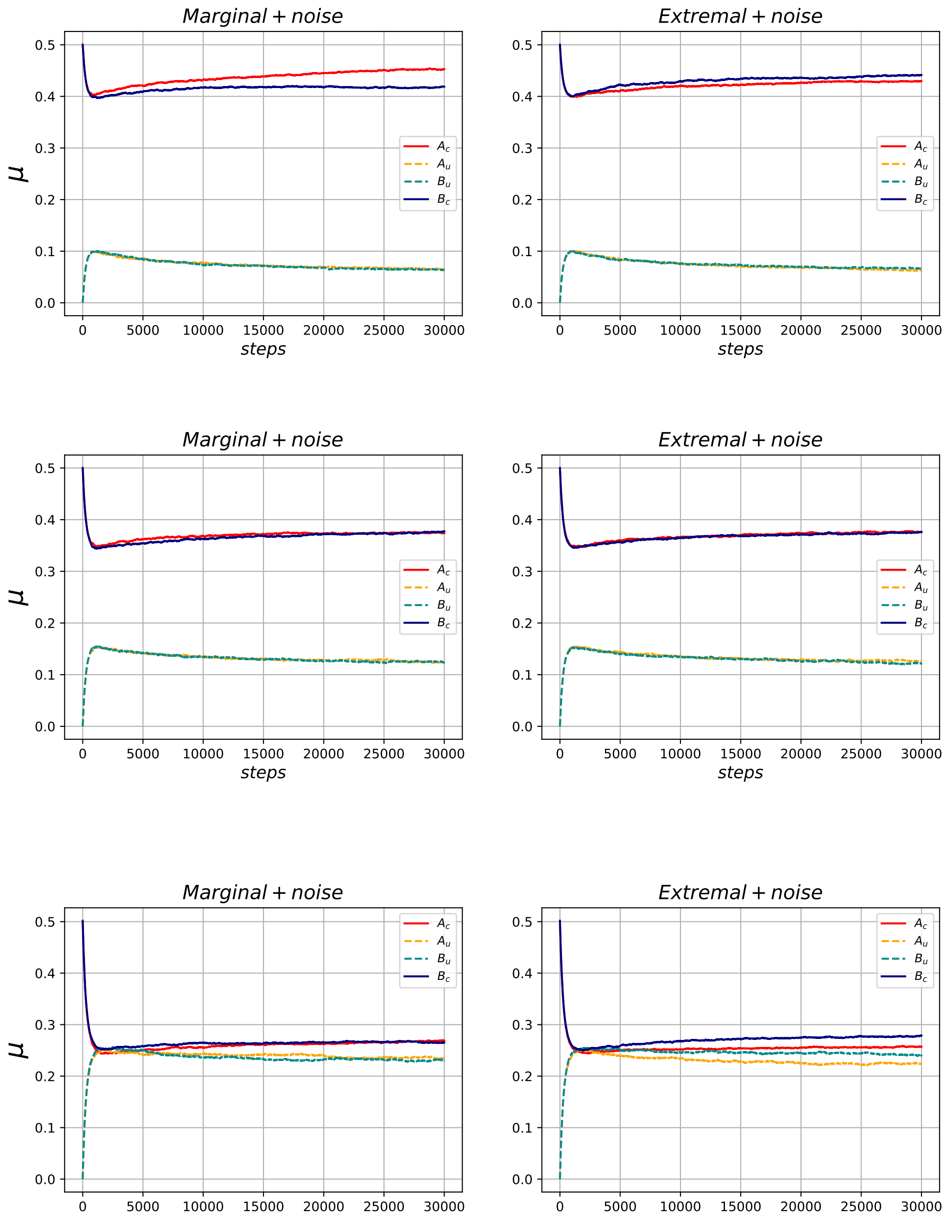

Fig. 18. Mean Time Evolution on square lattice $20x20$, with on-average symmetric initial conditions drawn from binomial distributions. 200 simulations and 30000 steps. From Top to Down: $p = \frac{1}{10}, p = \frac{1}{4}, p = \frac{1}{2}$. Increasing the rate of *"indecision"* of confident voters, changes the value of the peak. There is an inversion of densities of confident and unsure agents at $p = \frac{1}{2}$.

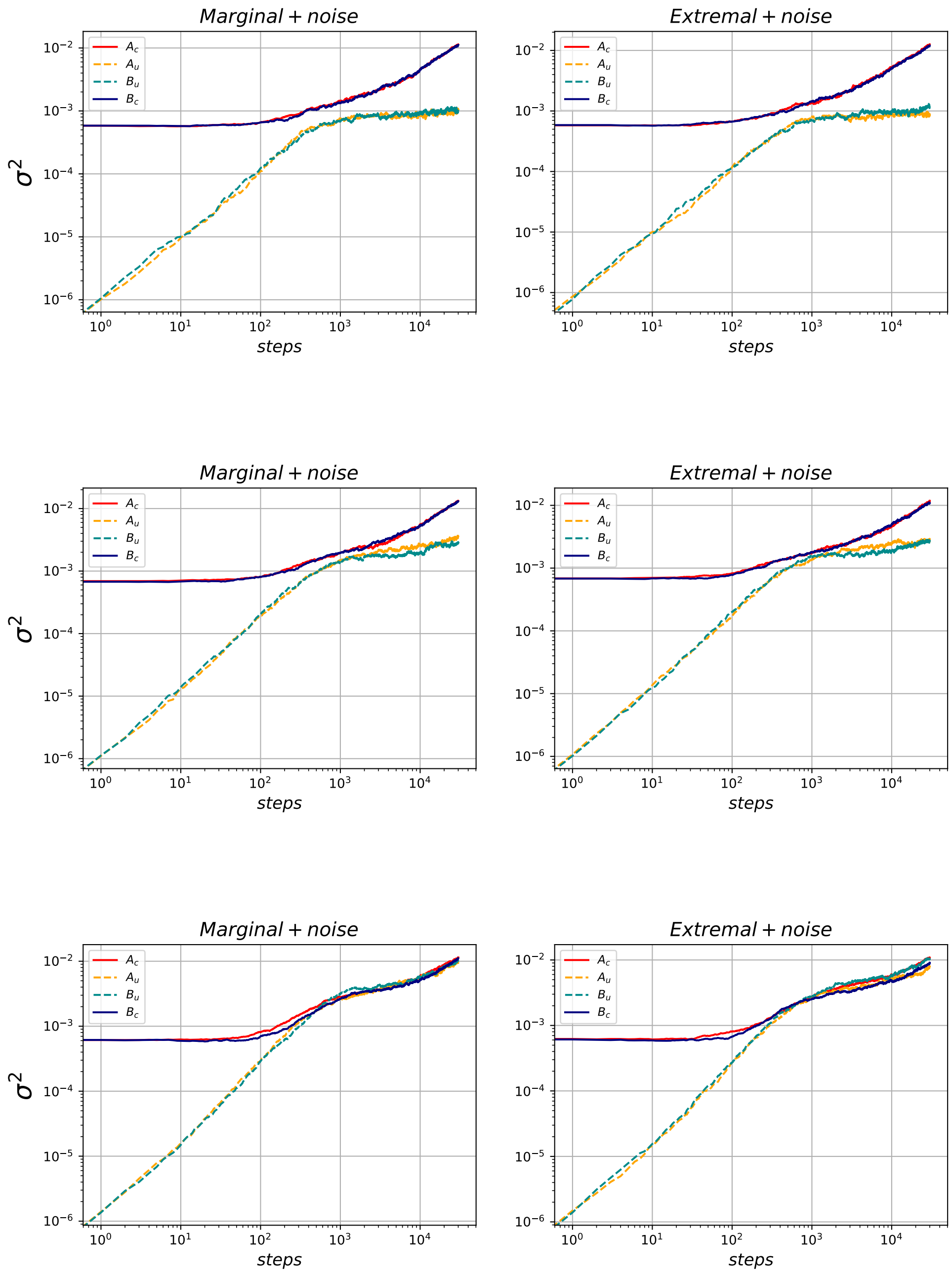

Fig. 19. Variance in log-log scale on square lattice 20x20, with on-average symmetric initial conditions drawn from binomial distributions. 200 simulations and 30000 steps. From Top to Down: $p = \frac{1}{10}, p = \frac{1}{4}, p = \frac{1}{2}$.

The rate parameter p regulates the dynamics of the unsure voters such that, if the value is small, the confident voters are less likely to change their minds and therefore there is wider gap in the mean densities plot; the consensus time is also smaller. Increasing the value of p, the gap closes definitely at $p = \frac{1}{2}$, increasing a lot the average consensus time. Over this value there is superposition of the curves up to $p = 1$; at this point the density of confident voters can reach its minimum and the average consensus time its maximum.

The plot of the variance instead, shows that the presence of the noise changes qualitatively the diffusion properties of the agents. In this sense, while the peak in the densities does not change much the linear and constant behavior with respect to the noiseless case, in the relaxation part the confident voters become more sub-diffusive in the lattice: as one can see, in fact, the curve in the variance plot is less than linear. For what concerns the unsure agents instead, the behavior is less sub-diffusive with respect to the noiseless case.
Increasing the value of p, the unsure agents' variance plot shows a progressive decreasing of the sub-diffusive behavior(the slop increases), until the confident and unsure behavior match at $p = \frac{1}{2}$. One can suppose that increasing even more p, the relaxation of the unsure voters can become super-diffusive.

A further study can be conducted to understand better the stochastic equations, introducing more degrees of freedom in the rates, for example.