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Asset prices in an ambiguous economy

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Abstract Models with ambiguity averse preferences have the potential to explain some pricing anomalies on financial markets. However, the models used in applications make additional assumptions, beyond ambiguity aversion, on the structure of the investor's preferences. Therefore, it is not clear how to disentangle the effect of ambiguity aversion from other features of preferences on equilibrium prices. This paper offers a general theory of asset pricing assuming only ambiguity aversion. Price indeterminacy may result in equilibrium when preferences are not smooth. A set of priors, which is identifiable in all the models used in applications, contains the relevant information to price assets. Ambiguity enriches the standard pricing formula by an additional stochastic discount factor and we calculate its explicit form for various models.

Keywords Asset pricing · Ambiguity aversion · Indeterminacy

JEL Classification C62 · G11

1 Introduction and overview of the results

Classical models of intertemporal asset pricing, *à la* Lucas [27], assume that agents evaluate streams of consumption through (recursive) expected utility. According to the *rational expectation hypothesis*, each investor has a *unique* (subjective) probability coinciding with the law governing the "true" data generating process. However, investors typically have little

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information concerning the process governing returns and they are be unable to single out a unique probability (a model) to evaluate them. Using the words of Pastor and Stambaugh [33]: “Even after observing 206 year of data (1802–2007), investors do not know the parameters of the return-generating process, especially the parameters related to the conditional expected return.”

Intertemporal asset pricing taking into account model uncertainty represents a fruitful generalization of the rational expectation hypothesis (for example, Epstein and Wang [11]; Hansen and Sargent [21]). Model uncertainty offers a possible explanation to some observed anomalies of the financial markets, such as the equity premium puzzle. This conclusion follows from a plethora of theoretical results appeared in the literature during the last decades. Despite its popularity, asset pricing under ambiguity lacks a unified theory since each of the work in the literature uses a different model of intertemporal decision under ambiguity. The aim of this work is to enhance our understanding of the role of ambiguity aversion in asset pricing, *disentangling the model-specific effect from the effect of ambiguity aversion alone*. In other words, we want to isolate the effect of ambiguity aversion, meant as preference for hedging, on asset prices and to understand how the results change when adding more structure to the models, for example, constant absolute ambiguity aversion (translation invariance). We set up our *unified* approach in a Lucas tree economy where assets (trees) have ambiguous returns. The paper makes three main contributions: first, price indeterminacy may result in equilibrium if the utility of the representative investor is not smooth. Second, the main information to price assets and to characterize equilibria is contained in a set of (subjective) probabilities that can be identified in all models used in applications (see Sect. 3.2). Moreover, the latter is related to the (Clarke) subdifferential of the certainty equivalent operator at the optimum. Lastly, an “ambiguity” stochastic discount factor may enrich the standard pricing formula. These general results hold regardless of how the investor evaluates ambiguous continuation plans, hence they are not affected by model-specific factors.

1.1 Overview of the results

In our model, we use a recursive utility that includes many standard recursive models of decision under ambiguity [8,26,29,38] and it is given by

$$V_t(c) = u(c_t) + \beta I_t(V_{t+1}(c))$$

where I_t is an ambiguity averse (i.e. a quasiconcave) certainty equivalent operator.

We show that equilibrium prices q_t satisfy the following general Euler (in)equalities

$$\min_{p\in C_t} \beta E_p\left[\frac{u'(c_{t+1})}{u'(c_t)}(q_{t+1}+d_{t+1})\right] \le q_t \le \max_{p\in C_t} \beta E_p\left[\frac{u'(c_{t+1})}{u'(c_t)}(q_{t+1}+d_{t+1})\right]$$

Despite their similarity with Epstein and Wang [11], the inequalities above are not generated by a MaxMin intertemporal utility, but they hold for a much larger class of ambiguity averse preferences. The main element, the set C_t of “time-t relevant probabilities”, contains those probabilities which are relevant to *identifying candidate solutions* conditional on the observed realizations of past returns. A first consequence of the previous Euler inequality is the possibility of price indeterminacy, namely, a multiplicity of prices that supports equilibrium (see Lemma 2). This extends the results of Epstein and Wang [11] to a much wider class of preferences. Price indeterminacy crucially depends on the non-smoothness of the certainty equivalent operators, since the set of time-t relevant probabilities is related to the (Clarke) subdifferential of the value function at the optimum. Therefore, for smooth preferences, C_t is a singleton and prices are determinate.

Secondly, ambiguity aversion generates a family of ambiguity stochastic discount factors (SDF), M^a_{t+1}, that enrich the standard pricing formula to

$$1 = E_p\left[M^a_{t+1} M_{t+1} R_{t+1} \middle| \mathscr{F}_t\right] \tag{SDF}$$

The structure of M^a_{t+1} is completely determined by the optimizing behavior of the investor, namely, the probabilities contained in C_t. Different models produce different probabilities, hence different stochastic discount factors M^a_{t+1}. The first part of the paper is devoted to studying the existence and the properties of equilibrium prices. In the second part, we calculate equilibrium prices for specific recursive models of choice under ambiguity and we give an explicit form to the ambiguity discount factors M^a_{t+1}.

The literature on asset pricing under ambiguity is extensive, see Guidolin and Rinaldi [17] for a survey. The current paper is a generalization of Epstein and Wang [11][1], where it is assumed that the agent is MaxMin expected utility maximizer. Epstein and Schneider [10], Ozsoylev and Werner [32] and Illeditsch [23] also studied models of asset pricing with MaxMin preferences and learning. Asset pricing with robust preferences (also called Multiplier preferences) is studied in Hansen and Sargen [21], Hansen et al. [22] and Maenhout [30]. Skiadas [39] studied asset pricing with scale-invariant (positive-homogeneous) utility. The smooth ambiguity model has been recently applied to asset pricing in Collard et al. [6], where they address the historical equity premium puzzle assuming the economy evolves according to a hidden-state variable process. Recently, Ju and Miao [24] developed an asset pricing model under ambiguity using the smooth ambiguity averse model and a non-additive aggregation of current utility and continuation value, allowing a three-way separation of ambiguity aversion, risk aversion and inter-temporal substitution. The result is a three-way decomposition of the stochastic discount factor that takes into account, ambiguity, risk and inter-temporal substitution. The aim of the present work is to unify the many approaches to asset pricing under ambiguity listed above and to provide results that are consequences of ambiguity aversion alone. The present work is also related to the work of Rigotti et al. [35]. Though they analyze risk sharing in a static economy with multiple agents, they identify a set of probabilities that correspond to our time-t relevant probabilities distorted by marginal utilities. A possible line for future research is to merge the two approaches and study a dynamic extension of the work of Rigotti et al. [35] along the lines of the present work.

2 Intertemporal utility

2.1 Setup

In this section we set the framework in which inter-temporal consumption-investment problems are defined and we give conditions for the existence and uniqueness of the inter-temporal utility we use to price assets.

Let the set of states Ω be a compact metric space endowed with its Borel σ-algebra $\mathscr{B}(\Omega)$. Denote by $\mathscr{M}^+_1(\Omega)$ the space of probability measures on Ω, it is a compact Polish space[2]. Time is discrete and the decision maker observes at time $t \in \mathbb{N}$ a realization $\omega_t \in \Omega$. The measurable space given by $(\Omega^\infty, \mathscr{B}(\Omega^\infty))$, where $\mathscr{B}(\Omega^\infty)$ is the product Borel σ-algebra, represents possible paths (or histories). Let $\omega^t \triangleq (\omega_1, \ldots, \omega_t)$ denote a path up to

[1] The same authors extended these results to more general state space and consumption processes in Epstein and Wang [12]. A continuous time version is studied in Chen and Epstein [4].

[2] In the weak* topology $\sigma(\mathscr{M}^+_1(\Omega), C(\Omega))$.

time t and, for all t, Ω^t the set of all such paths. It is canonically embedded in Ω^∞, so it becomes a measurable space when endowed with the (cylinders) product σ-algebra $\mathscr{B}(\Omega^t)$. A consumption process $c = (c_t)$ is *adapted* if $c_t : \Omega^\infty \to \mathbb{R}^n$ is $\mathscr{B}(\Omega^t)$-measurable for all t. Moreover, we say it is *continuous* if c_t is continuous for all t and *real-valued* if $c_t : \Omega^\infty \to \mathbb{R}$. The space of adapted continuous real-valued processes is denoted by $\mathrm{AC}(\Omega^\infty, \mathscr{B}(\Omega^\infty))$, it becomes a Banach space when endowed with the weighted sup-norm

$$\|c\| \triangleq \sup_t \sup_{\omega^t} \frac{\left|c_t(\omega^t)\right|}{b^t} \quad \text{for some } b \geq 1$$

The space $\mathscr{D}$ of consumption processes is given by

$$\mathscr{D} \triangleq \left\{c \in \mathrm{AC}(\Omega^\infty, \mathscr{B}(\Omega^\infty)) : \|c\| < \infty \;\text{ and }\; c_t(\omega^t) \geq 0, \;\text{ for all } t \geq 1 \text{ and} \omega^t \in \Omega^t\right\}$$

We assume the decision maker evaluates infinite consumption streams according to the recursive utility

$$V_t(c, \omega^t) = u\left(c_t(\omega^t)\right) + \beta I\left(V_{t+1}(c, \omega^{t+1}); \omega^t\right) \tag{1}$$

where I is a locally Lipschitz, quasiconcave, normalized[3] and monotone (w.r.t. pointwise order) certainty equivalent operator. Ambiguity aversion of the investor is represented by quasiconcavity or preference for *hedging* (see Cerreia-Vioglio et al. [2] for a discussion). The dependence of $I(\cdot, \omega^t)$ on $\omega^t \in \Omega^t$ allows the investor to change the certainty equivalent operator across time. A weak assumption we impose, is a form of continuity of the certainty equivalent operator with respect to histories of observed states. Namely, continuity of the map

$$\omega^t \longmapsto I(\cdot\,; \omega^t) \tag{Cont.}$$

It is always satisfied when Ω is a finite space. As a technical assumption, it is necessary to prove the existence of a continuous recursive utility. In the case of a functional form that does not vary with time, i.e. $I(\cdot\,; \omega^t) = I(\cdot)$, continuity is automatically satisfied (for example in Strzalecki [40]). Allowing for a time-varying utility we are not assuming IID ambiguity (in the sense of Epstein and Schneider [9]), that is, the investor may learn about the underlying ambiguity.

2.2 Special cases

In this section, we illustrate special cases of the recursive utility in Eq. (1), that have been proposed in the literature. In Epstein and Wang [11] the intertemporal utility is given by

$$V_t(c, \omega^t) = u\left(c_t(\omega^t)\right) + \beta \min_{p \in \mathscr{P}(\omega^t)} \int V_{t+1}(c, \omega^{t+1}) dp$$

a dynamic extension of MaxMin expected utility of Gilboa and Schmeidler [15]. Maccheroni et al. [29] proposed an axiomatization of recursive variational preferences, which generalizes the recursive multiple priors of Epstein and Schneider [8]. In the recursive case, the intertemporal utility is given by

$$V_t(c, \omega^t) = u\left(c_t(\omega^t)\right) + \beta \min_{p \in \mathscr{M}_1^+(\Omega)} \left(\int V_{t+1}(c, \omega^{t+1}) dp + \eta_t(\omega^t, p)\right)$$

[3] A functional $I : C(\Omega) \to \mathbb{R}$ is quasiconcave if upper contour sets, $\{f \in C(\Omega) : I(f) \geq r\}$, are convex, for all $r \in \mathbb{R}$. We say that I is normalized if $I(1_\Omega r) = r$, for all $r \in \mathbb{R}$.

where $\eta_t(\omega^t, p)$ is a penalization function. When $\eta_t(\omega^t, p) = D(p\|p^*)$ is the Kullback–Leibler divergence, we recover the robust preferences of Hansen and Sargent [21]. Similarly, a recursive version of the smooth ambiguity model of Klibanoff et al. [25] has been axiomatized by Klibanoff et al. [26]. The intertemporal utility is of the form

$$V_t(c, \omega^t) = u\left(c_t(\omega^t)\right) + \beta\phi^{-1}\left[\int_{\Theta} \phi\left(\int V_{t+1}(c, \omega^{t+1})dp_\theta(\omega_{t+1}|\omega^t)\right) d\mu(\theta|\omega^t)\right]$$

where $\phi(\cdot)$ characterizes ambiguity attitude and Θ is a (finite) set of parameters that determines the transition probability $p_\theta(\omega_{t+1}|\omega^t)$. In general, Klibanoff et al. [25] preferences are not locally Lipschitz, so we are focusing on a subclass that satisfies locally Lipschitz continuity (a behavioral condition to guarantee local Lipschitz can be found in Ghirardato and Siniscalchi [14, Online Appendix]). It is worth noticing that in applications, assumptions on ϕ often imply global Lipschitzianity (see for example, Collard et al. [6]). An additional model is the recursive version, axiomatized by Siniscalchi [38], of the Vector expected utility of Siniscalchi [36] . The form of intertemporal utility is

$$V_t(c, \omega^t) = u(c_t(\omega^t)) + \beta E_{p_t}\left[V_{t+1}(c, \omega^{t+1})\right] + A_{t,\omega^t}\left(\beta E_p\left[\xi^{t,\omega^t} \cdot V_{t+1}(c, \omega^{t+1})\right]\right)$$

where p_t is a baseline probability, ξ^{t,ω^t} are the adjustment factors and A_{t,ω^t} is an adjustment function.

The present approach can be generalized to include other interesting cases of recursive utilities that separates risk attitude and intertemporal substitution. Since we focus on the role of ambiguity only, asset pricing with general intertemporal utility with nonlinear aggregators as in Marinacci and Montrucchio [31], may represent an insightful extension of this paper.

2.3 Time and preferences

Dynamic models of decision under ambiguity can be dynamically inconsistent, meaning the investor's ex ante contingent preferences may not correspond to those ex post. The recursive structure we imposed in (1) implies a form of dynamic consistency. Since we allow for a time-dependent certainty equivalent operator, we need a more precise definition. Dynamic consistency will be useful in characterizing equilibria in our economy. We say that the recursive utility of Eq. (1) is dynamically consistent if, for all $t \in \mathbb{N}$ and $\omega^t \in \Omega^t$, $c', c \in \mathscr{D}$ and $T \geq t$, $V_t(c', \omega^t) \geq V_t(c, \omega^t)$ if:

DC1. $c'_\tau = c_\tau$ for $\tau = t, \dots T-1$
DC2. $V_T(c', \omega^T) \geq V_T(c, \omega^T)$ for all $\omega^T \in \Omega^T$

and $V_t(c', \omega^t) > V_t(c, \omega^t)$ if DC2 is strict on a non-null event. We introduced the definition of dynamic consistency here, since it is important to characterize equilibrium prices. Dynamic consistency is a normatively appealing but restrictive requirement. For example, in a finite state-space setting, Gumen and Savochkin [18] proved that many classes of dynamically consistent preferences are dynamically unstable: after conditioning they collapse to expected utility.

Dynamic consistency is always satisfied under recursive expected utility, however, for non-expected utility models it imposes additional structure to the functional representing preferences. For example, in the case of a multiple priors, dynamic consistency makes priors *rectangular*,[4] as proved in Epstein and Schneider [8]. Alternative restrictions induced by

[4] Equivalent properties have been proposed in the literature on dynamic risk measures see Delbaen [7] and Riedel [34].

dynamic consistency are found in Maccheroni et al. [29] for variational preferences, in Siniscalchi [38] for vector expected utility and in Klibanoff et al. [26] for the smooth ambiguity model.[5]

2.4 Existence and uniqueness

This section is devoted to establishing sufficient conditions for the existence and uniqueness of a utility function that solves Eq. (1). Existence of a recursive utility satisfying (1) is guaranteed by monotonicity (w.r.t. pointwise order) and an application of Tarski's fixed point theorem, once the set of consumption processes $\mathscr{D}$ is endowed with the pointwise order. Therefore, the recursive utility in Eq. (1) *always* exists. Considering uniqueness, in the case of a globally Lipschitz (for example in Epstein and Wang [11], Maccheroni et al. [29] and Siniscalchi [38]) certainty equivalent I_t, uniqueness follows from standard contraction techniques. Our assumption of a locally Lipschitz certainty equivalent precludes the contraction argument, nonetheless, the next theorem shows that under an appropriate relation between the discount rate β and the rate of growth of consumption processes, a unique recursive utility exists (the same condition is used in Epstein and Wang [11]).

Theorem 1 *Suppose I is locally Lipschitz continuous, if $\beta b < 1$ then, for each $c \in \mathscr{D}$ there exists a unique $V(c) \in \mathscr{D}$ satisfying Eq.* (1).

$b \geq 1$ is a parameter regulating the speed of growth of consumption processes defined in Sect. 2. Locally Lipschitz continuity of I is a weak requirement, it is always satisfied when I is translation invariant or concave. However, for more general classes of preferences as recursive uncertainty averse or Monotone Bernoullian and Continuous of Cerreia-Vioglio et al. [1], global or local Lipschitzianity does not necessarily hold. A behavioral condition that ensures a locally Lipschitz representation of preferences is provided in Ghirardato and Siniscalchi [14, Online Appendix]. As hinted earlier, we assume that all the models in the next sections will satisfy locally Lipschitz continuity.

3 The economy

3.1 A Lucas tree economy

We consider the same extension of Lucas pure exchange economy as Epstein and Wang [11]. Assume $u' > 0, u'(0) = \infty$. There is a single good with total supply available at any time and state modeled by an endowment process $e = (e_t) \in \mathscr{D}$. We assume for simplicity, that the endowment process has a time-homogeneous Markov structure, so there is a function e^* such that

$$e_t(\omega^t) = e^*(\omega_t), \quad t \geq 1, \quad \omega^t \in \Omega^t$$

with $e^*(\omega) > 0$ for all $\omega \in \Omega$. In each period, N securities are traded at prices $q_i = \left(q_{i,t}\right) \in \mathscr{D}, i \in 1, \ldots, N$. Each security pays a dividend $d_1 = \left(d_{i,t}\right) \in \mathscr{D}$ and it is available in zero net supply at all times and states of the world. The consumer optimizes her intertemporal utility by choosing consumption and portfolio allocations in the current and future periods. A pair

[5] Ambiguity sensitive preferences display a trade-off between dynamic consistency and consequentialism. Beyond the Epstein and Schneider [8] approach, other methods of updating ambiguous beliefs have been proposed: the dynamic consistent but not consequentialist approach of Hanany and Klibanoff [19] and Hanany et al. [20] and the sophisticated but not dynamically consistent approach of Siniscalchi [37].

(c, θ), with $c \in \mathscr{D}$ and $\theta = (\theta_t)$ a continuous process, represents a plan of consumption and portfolio allocations at each period t. We say that (c, θ) is (t, ω^t)-*feasible* if for all $\tau \geq t$,

$$q_\tau \theta_\tau + c_\tau = \theta_{\tau-1}[q_\tau + d_\tau] + e_\tau, \quad \theta_{t-1}(\omega^{t-1}) \triangleq 0$$
$$\inf_{i,\tau,\omega^\tau} \theta_{i,\tau}(\omega^\tau) > -\infty$$

The first condition is an intertemporal budget constraint, whereas the second is a weak restriction on short sales. Let ${}^t c \triangleq (c_t, c_{t+1}, \ldots)$ denote the continuation consumption process and denote ${}^t c|\omega^t \triangleq (c_\tau)_{\tau=t}^\infty$ the continuation of c conditional to history ω^t. A (t, ω^t)-feasible plan (c, θ) is (t, ω^t)-*optimal*, if

$$V_t({}^t c|\omega^t, \omega^t) \geq V_t({}^t c'|\omega^t, \omega^t), \quad \text{for all } (t, \omega^t)\text{-feasible plans } (c', \theta')$$

An *equilibrium* is a price process $(q_t) \in \mathscr{D}$ such that $(e, 0)$ is (t, ω^t)-optimal for all $t \geq 1$ and $\omega^t \in \Omega^t$.

3.2 Ambiguous consumption capital asset pricing model (ACCAPM)

This section is devoted to derive the Euler equations. As in Epstein and Wang [11], we use a two-period perturbation of an equilibrium process. The intuition is that the investor is always at her optimum along the equilibrium path implying that any perturbation must be sub-optimal. The next result is a differential characterization of optimality with respect to a today-tomorrow perturbation. To avoid technical complications, we assume that $I(\cdot, \omega_t)$ is nice[6] at e^* for all t and ω_t.

Lemma 2 *Let $e \in \mathscr{D}$ be a positive, Markovian and time-homogeneous consumption process with $e_t(\omega_t) = e^*(\omega_t)$. Let $h = (h_t)_1^\infty$, with $h_1 \in \mathbb{R}$, $h_2 \in C(\Omega)$ and $h_t = 0$ for all $t \neq 1, 2$. Then, a necessary condition for optimality i.e.*

$$0 \in \operatorname*{argmax}_{\xi} V_t(e^* + \xi(h_1, h_2, 0, \ldots), \omega^t)$$

is

$$0 = u'(e^*(\omega_t))h_1 + \beta \int u'(e^*)h_2 dp \quad \text{for some } p \in \partial I\left(V_{t+1}(e^*); \omega_t\right) \tag{2}$$

where $\partial I(V_{t+1}(e^*); \omega_t)$ is the Clarke subdifferential (see "Appendix 1" for the definition) of $I(\cdot; \omega_t)$ at $V_{t+1}(e^*)$. The condition in Lemma 2 is sufficient when $\xi \longmapsto (V_t(e^* + \xi(h_1, h_2, 0, \ldots), \omega^t)$ is a concave function. For example, when u is concave and I_t is concave (e.g. MaxMin, Variational, Confidence preferences).

Similarly to Epstein and Wang [11] and Maccheroni et al. [29], we can identify an element of $\partial V_t(e^*)$ as an element of $\mathbb{R} \times \mathscr{M}_1^+(\Omega)$ given by

$$\partial V_t(e^*) = \left\{\left(u'(e^*(\omega_t)), u'(e^*)dp\right) : \exists p \in \partial I\left(V_{t+1}(e^*); \omega_t\right)\right\}$$

where $u'(e^*)dp$ is a measure with density $u'(e^*)$ with respect to p.

Assume (q_t) is a equilibrium. Given (t, ω_t), let a perturbation (c, θ) defined as $c_\tau = e_\tau$ and $\theta_\tau = 0$ for all $\tau \neq t, t+1$, for $\Delta \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, $c_t = e_t - \xi(\Delta \cdot q_t)$, $\theta_t = \xi\Delta$, $\theta_{t+1} = 0$ and $c_{t+1} = e_{t+1} + \xi\Delta \cdot (q_{t+1} + d_{t+1})$. ξ and Δ represent, respectively, the size and direction of the perturbation. Optimality of the initial policy means that any perturbation must leave

[6] A functional $I(\cdot, \omega_t)$ is nice at c if $0 \notin \partial I(c, \omega_t)$. Niceness excludes that an investor considers possible the degenerate measure that assigns zero measure to *all* events.

the decision maker worse off. Let $h_t \triangleq -\Delta \cdot q_t$ and $h_{t+1} \triangleq \Delta \cdot (q_{t+1} + d_{t+1})$. By Lemma 2, Eq. (2) can be rewritten as

$$0 = -u'(e^*(\omega_t))\Delta \cdot q_t + \beta \int u'(e^*)[\Delta \cdot (q_{t+1} + d_{t+1})]dp \quad \text{for some } p \in \partial I(V_{t+1}(e^*); \omega_t)) \tag{3}$$

If we normalize the subdifferential, we can define a correspondence $C(e^*; \omega_t) : \Omega \rightrightarrows \mathscr{M}_1^+(\Omega)$, as

$$C(e^*; \omega_t) \triangleq \left\{ \frac{p}{p(\Omega)} : p \in \partial I(V_{t+1}(e^*); \omega_t) \right\} \tag{4}$$

by the assumption of local niceness and the properties of Clarke subdifferential (see Proposition 5), $C(e^*; \omega_t) \subseteq \mathscr{M}_1^+(\Omega)$ is a weak* compact and convex set. The set $C(e^*; \omega_t)$ contains the *time-t relevant probabilities*. They are definable regardless of the functional form of intertemporal utility and they are the main information needed to price assets. In the next section we will impose a condition on this set to prove existence of equilibria.

Equation (3) allows to define an interval of prices given by

$$\min_{p \in C(e^*; \omega_t)} \beta \int u'(e^*)\Delta \cdot (q_{t+1} + d_{t+1})dp \leq u'(e^*(\omega_t))\Delta \cdot q_t \leq \max_{p \in C(e^*; \omega_t)} \beta \int u'(e^*)\Delta \cdot (q_{t+1} + d_{t+1})dp$$

Or equivalently

$$\min_{p \in C(e^*; \omega_t)} \left\{ \beta E_p \left[\frac{u'(e_{t+1})}{u'(e^*(\omega_t)} \Delta \cdot (q_{t+1} + d_{t+1}) \right] - \Delta \cdot q_t \right\} \leq 0, \quad \forall \Delta \in \mathbb{R}^N \tag{5}$$

The previous equation is more general than the corresponding equation in Epstein and Wang [11] since we do not assume any specific functional form for I, beyond (locally Lipschitz) continuity and quasiconcavity. However, the interpretation is similar, *optimality may be consistent with a non-degenerate interval of prices*. It is also clear that the possibility of price indeterminacy crucially depends on the lack of smoothness of the functional representing investor's preferences. If the set $C(e^*; \omega_t)$ is a singleton, equilibrium prices are determined. We can conclude that market price indeterminacy depends on ambiguity aversion combined with non-smoothness of preferences. This is consistent with the result of Epstein and Wang [11], but alternative models to the MaxMin can yield different predictions.

Proceeding as in Epstein and Wang [11], we can rewrite Eq. (5) in a more tractable form such as

$$\min_{p \in C(e^*; \omega_t)} \max_i \left\{ \left| \beta E_p \left[\frac{u'(e_{t+1})}{u'(e^*(\omega_t))} (q_{i,t+1} + d_{i,t+1}) \right] - q_{i,t} \right| \right\} = 0 \tag{6}$$

Equation (6) is more transparent in the case $N = 1$, i.e. a unique asset is traded, then it becomes

$$\min_{p \in C(e^*; \omega_t)} \beta E_p \left[\frac{u'(e_{t+1})}{u'(e^*(\omega_t))} (q_{t+1} + d_{t+1}) \right] \leq q_t \leq \max_{p \in C(e^*; \omega_t)} \beta E_p \left[\frac{u'(e_{t+1})}{u'(e^*(\omega_t))} (q_{t+1} + d_{t+1}) \right] \tag{7}$$

Inequalities (7) are the Euler inequalities that an equilibrium price has to satisfy.

4 Existence and characterization of equilibria

As mentioned in the introduction, existence and characterization of equilibria require additional assumptions on the behavior of time-t relevant probabilities with respect to information arrival. In particular, the set $C(e^*, \omega_t)$ has to vary smoothly with information's arrival. The assumption of time-homogeneity and Markov allows us to impose a continuity condition only with respect to the information observed at that node. We call this assumption *Strict Feller* as in Epstein and Wang [11].

Assumption 1 (*Strict Feller*) The map $\omega_t \longmapsto \operatorname{argmin}_{p\in C(e^*;\omega)} \int f dp$ is continuous for all $f \in C(\Omega)$.

Strict Feller is trivially satisfied with a finite state-space or under recursive expected utility or, more importantly, when the certainty equivalent operator $I(\cdot, \omega^t)$ does not depend on ω^t, namely with IID ambiguity. It is also satisfied when the certainty equivalent operator is smooth, since $C(e^*; \omega)$ is a singleton.

We are ready to state the main theorem of this section:

Theorem 3 (Existence and characterization of equilibria)

(i) $(q_t) \in \mathscr{D}$ *is an equilibrium, if and only if, it satisfies Eq.* (6).
(ii) *Under Strict Feller (Assumption* 1*), there exists an equilibrium.*

Theorem 3 fully characterizes equilibria as the price processes satisfying Eq. (6). There are three main results coming from Theorem 3. Equilibrium prices may be indeterminate: possible different selections solving the sequence of Euler inequalities will generate different equilibrium prices. However, this is possible only if the functional representing preferences is not smooth. For smooth representations, the equilibrium price is unique. Second, the set of probabilities determining Euler (in)equalities is completely characterized by the Clarke subdifferential. Lastly, equilibria exist under a fairly general condition.

It is worth noticing that, in the particular case of IID ambiguity, time-homogeneous and Markov return of the asset and a non-degenerate interval of equilibrium prices, any fluctuation of prices can only be caused by indeterminacy, hence there is room for volatility of the sunspot type. Again, this is not a phenomenon due to the MaxMin decision rule, but it holds under general models of ambiguity averse choice that are not smooth.

By Theorem 3, an equilibrium price is given by the recursive relation

$$q_t^* = \beta E_{p_t}\left[\frac{u'(e_{t+1})}{u'(e^*(\omega_t))}(q_{t+1}^* + d_{t+1})\right]$$

The existence of the ambiguity stochastic discount factor M_{t+1}^a we announced in the introduction is hidden in the probability p_t. In Sect. 5, we give explicit formulae for the ambiguity stochastic discount factors.

5 Calculation for specific models

In this section we calculate the equilibrium price of assets, assuming specific certainty equivalents in Eq. (1) and we explicitly describe the corresponding ambiguity stochastic discount factor M_t^a. For simplicity, we consider an economy with a single asset.

As anticipated in the introduction, to each model we can attach an ambiguity stochastic discount factor M^a_{t+1} that satisfies the formula

$$1 = E_p\left[M^a_{t+1}M_{t+1}R_{t+1}\middle|\mathscr{F}_t\right]$$

M^a_{t+1} is fully characterized by the priors contained in $C_t(e^*;\omega_t)$, since each model corresponds to a different formulation for the time-t relevant probabilities.

5.1 Uncertainty averse preferences

We first focus on the most general class of Uncertainty Averse preference (UA) of Cerreia-Vioglio et al. [2]. They impose a minimal restriction to the ambiguity attitude of a DM, such as the uncertainty aversion axiom of Gilboa and Schmeidler. It means that if a continuation plan $V_t(c_t,\omega^t)$ is indifferent to another $V_t(c'_t,\omega^t)$, then any mixture of them is weakly preferred to both. Many models of decision under ambiguity are special cases[7] of UA preferences. Consider first the static model - each prospect is evaluated through

$$I(f) = \inf_{p\in\mathscr{M}_1^+(\Omega)} G\left(\int f dp, p\right)$$

where u is a Bernoulli utility, $G:\mathbb{R}\times\mathscr{M}_1^+(\Omega)\to(-\infty,\infty]$ is a quasi-convex, monotone in the first argument, normalized and lower-semicontinuous function. UA preferences are not locally Lipschitz continuous in general, however, we will *assume* that they satisfy the locally Lipschitz continuity axiom of [14, Online Appendix], hence we can apply the results developed in the previous sections. Define

$$\text{dom}\, G \triangleq \bigcup\left\{p\in\mathscr{M}_1^+(\Omega) : G(x,p)<\infty\right\}$$

and let $\mathscr{C}^* = \text{cl }(\text{dom}\, G)$. Then, a dynamic formulation is given by

$$V_t(c,\omega^t) = u(c_t(\omega^t)) + \beta \min_{p\in\mathscr{C}_t^*} G_t\left(\int V_{t+1}(c,\omega^{t+1})dp, p\right) \tag{8}$$

By Ghirardato and Siniscalchi [13], the Clarke subdifferential of a certainty equivalent representing uncertainty averse preferences, can be written as:[8]

$$C(e^*;\omega_t) = \operatorname*{argmin}_{p\in\mathscr{M}_1^+(\Omega)} G_t\left(\int V_{t+1}(e^*)dp, p\right) \tag{9}$$

Therefore Eq. (7) can be rewritten as:

$$\min_{p\in C(e^*;\omega_t)} \beta E_p\left[\left(\frac{u'(e_{t+1})}{u'(e_t)}\right)(q_{t+1}+d_{t+1})\right] \le q_t \le \max_{p\in C(e^*;\omega_t)} \beta E_p\left[\left(\frac{u'(e_{t+1})}{u'(e_t)}\right)(q_{t+1}+d_{t+1})\right]$$

where the set $C(e^*;\omega_t)$ is define din Eq. (9). With UA preferences, the relevant probabilities are the ones minimizing the cost function G, generalizing the result of Epstein and Wang

[7] Variational preferences [28], MaxMin [15], Confidence preferences [3], uncertainty averse Smooth Ambiguity [25].

[8] Assuming niceness of I at $V_{t+1}(e^*)$.

[11] for MaxMin preferences. An equilibrium price satisfies

$$q_t = \beta E_{p^*}\left[\left(\frac{u'(e_{t+1})}{u'(e_t)}\right)(q_{t+1} + d_{t+1})\right]$$

The difference from a price coming from rational expectation with recursive expected utility is in the prior p^*.

5.2 Special cases: homothetic and variational

A further specialization of uncertainty averse preferences sheds new lights on the nature of the ambiguous stochastic discount factor M^a_{t+1}. If we assume that the agent has homothetic recursive preference in the sense of Chateauneuf and Faro [3], then the function G of the uncertain averse preferences is equal to $G_t(m, p) = \frac{m}{\eta_t(p)}$ and

$$V_t(c, \omega^t) = u(c_t(\omega_t)) + \beta \min_{p \in \mathscr{C}_t^*} \frac{\int V_{t+1}(c, \omega^{t+1}) dp}{\eta_t(\omega^t, p)}$$

for some concave and upper hemicontinuous function $\eta_t : \mathscr{C}_t^* \to [0, 1]$. The latter represents the investor's "confidence" on each priors. Higher values of η correspond to more confident priors. The set of time-t relevant probabilities is given by:

$$C(e^*; \omega_t) = \operatorname*{argmin}_{p \in \mathscr{M}_1^+(\Omega)} \frac{\int V_{t+1}(e^*) dp}{\eta_t(\omega^t, p)}$$

Therefore, Eq. (7) becomes

$$\min_{p \in C(e^*;\omega_t)} \beta E_p\left[\left(\frac{u'(e_{t+1})}{u'(e_t)}\right)(q_{t+1} + d_{t+1})\right] \leq q_t \leq \max_{p \in C(e^*;\omega_t)} \beta E_p\left[\left(\frac{u'(e_{t+1})}{u'(e_t)}\right)(q_{t+1} + d_{t+1})\right]$$

With homothetic preferences, the priors generating equilibrium prices depends on the investor's "confidence" about the likelihood of the subjective probability.

Another relevant particular case of uncertainty averse preferences is given by the recursive Variational preferences, Maccheroni et al. [29], where $G_t(m, p) = m + \eta_t(p)$ and

$$V_t(c, \omega^t) = u(c_t(\omega_t)) + \beta \min_{p \in \mathscr{M}_1^+(\Omega)} \left(\int V_{t+1}(c, \omega^{t+1}) + \eta_t(\omega^t, p)\right)$$

where η_t is a cost function associated to each prior p. We have

$$C(e^*; \omega_t) = \operatorname*{argmin}_{p \in \mathscr{M}_1^+(\Omega)} \int V_{t+1}(e^*) + \eta_t(\omega^t, p)$$

An interesting subcase is represented by recursive multiplier preferences (sometime referred to as robust preferences), where the cost function is given by $\eta_t(\omega_t, p) \triangleq \kappa D(p \| \hat{p}_t)$, for some $\kappa \in [0, \infty]$ where

$$D(p \| \hat{p}_t) = \int p \log \frac{p}{\hat{p}_t}, \quad \text{if } p \ll \hat{p}_t$$

and $D(p\|\hat{p}_t) = \infty$ otherwise, is the Kullback–Leibler distance of p with respect to $\hat{p}_t$ and $\hat{p}_t$ is a reference measure. It is well known that

$$V_t(c,\omega^t) = u(c_t(\omega_t)) + \beta \min_{p\in\mathscr{M}_1^+(\Omega)} \left(\int V_{t+1}(c,\omega^{t+1})dp + \kappa D(p\|\hat{p}_t)\right)$$

is equivalent to

$$V_t(c,\omega^t) = u(c_t(\omega_t)) - \kappa\beta \log\left(\int \exp\left(-\frac{V_{t+1}(c,\omega^{t+1})}{\kappa}\right) d\hat{p}_t\right)$$

Since $D(\cdot\|\hat{p}_t) : \mathscr{M}_1^+(\Omega) \to [0,\infty]$ is strictly convex, $C(e^*,\omega)$ is a singleton and there is a *unique* equilibrium price (q_t). It is determined by (see Maccheroni et al. [29])

$$q_t = \beta \frac{E_{\hat{p}_t}\left[\frac{u'(e_{t+1})}{u'(e_t)}(q_{t+1}+d_{t+1})\exp\left(-\frac{V_{t+1}(e^*)}{\kappa}\right)\right]}{E_{\hat{p}_t}\left[\exp\left(-\frac{V_{t+1}(e^*)}{\kappa}\right)\right]}$$

the usual ambiguity stochastic discount factor related to robust control preferences. The multiplier preferences are smooth, therefore equilibrium prices are determined uniquely and there is no room for price indeterminacy.

5.3 Second order expected utility

The robust preference of Hansen and Sargent [21] are related to a class of preferences axiomatized by Grant et al. [16]. The Second Order Subjective expected utility generalizes expected utility introducing a second order function that determines the ambiguity attitude of the investors, in the spirit of Klibanoff et al. [25]:

$$V_t(c,\omega^t) = u(c_t(\omega^t)) + \beta\phi^{-1}\left(\int \phi(V_{t+1}(c,\omega^{t+1}))dp_t\right)$$

Ambiguity aversion corresponds to a concave, and we assume smooth, ϕ. Using the derivation of the inverse rule, we can write each element $\mu \in \partial I(V_{t+1}(c,\omega^t))$ as

$$\int f d\mu = \int \frac{\phi'(V_{t+1}(c))}{\phi'\left(\phi^{-1}\left(E_p\left[\phi(V_{t+1}(c))\right]\right)\right)} f dp$$

for all $f \in C(\Omega)$. We can rewrite Eq. (7) as

$$q_t = \beta E_{p_t}\left[\tilde{\xi}_{t,p}\left(\frac{u'(e_{t+1})}{u'(e_t)}\right)(q_{t+1}+d_{t+1})\right]$$

where $\tilde{\xi}_t(\omega)$ is given by a normalized version of:

$$\xi_{t,p}(\omega) = \frac{\phi'(V_{t+1}(e^*))(\omega)}{\phi'\left(\phi^{-1}\left(E_p\left[\phi(V_{t+1}(e^*))\right]\right)\right)}$$

Since ϕ is concave, the investor is ambiguity averse and $\phi'(V_{t+1}(e^*))(\omega)$ is high when the expected value of the continuation is low, so that the ambiguity stochastic discount factor increases the probability of observing a bad outcome, reflecting pessimism. Since the second order expected utility is a smooth model the equilibrium price is unique and price indeterminacy cannot occur.

In the particular case of $\phi(r) = -\exp\left(-\frac{r}{\kappa}\right)$, the second order expected utility is equivalent to the robust preferences of Hansen and Sargent [21] (see the previous section), hence:

$$V_t(c, \omega^t) = u(c_t(\omega^t)) - \beta\kappa \ln\left(\int \exp\left(-\frac{V_{t+1}(c, \omega^{t+1})}{\kappa}\right) dp_t\right)$$

and the expression of equilibrium prices is well-known (see Maccheroni et al. [29])

$$q_t = \beta \frac{E_{p_t}\left[\left(\frac{u'(e_{t+1})}{u'(e_t)}\right)(q_{t+1} + d_{t+1}) \exp\left(-\frac{V_{t+1}(e^*)}{\kappa}\right)\right]}{E_{p_t}\left[\exp\left(-\frac{V_{t+1}(e^*)}{\kappa}\right)\right]}$$

5.4 Smooth ambiguity model

The smooth ambiguity model of Klibanoff et al. [25] is among the most popular models used in applications. According to the recursive version of the model (see Klibanoff et al. [26]), an ambiguous consumption stream is evaluated by (we assume a locally Lipschitz continuous certainty equivalent operator):

$$V_t(c, \omega^t) = u(c_t(\omega^t)) + \beta\phi^{-1}\left[\int_\Theta \phi\left(\int V_{t+1}(c, \omega^{t+1}) dp_{t,\theta}\right) d\mu(\theta|\omega^t)\right]$$

where Θ is a space of (unobserved) parameters $\theta \in \Theta$. $p_{t,\theta} \triangleq p_\theta(\omega_{t+1}|\omega^t)$ is the probability under p_θ that the next observed state of the world is ω_{t+1} given ω^t and μ represents the investor's prior on Θ. Assuming an everywhere differentiable ϕ, the expression for equilibrium prices is similar to the case of second order expected utility and Eq. (7) becomes

$$q_t = \beta E_{\mu(\theta|\omega^t)}\left[\tilde{\xi}_t(\theta) E_{p_{t,\theta}}\left[\left(\frac{u'(e_{t+1})}{u'(e_t)}\right)(q_{t+1} + d_{t+1})\right]\right]$$

where $\tilde{\xi}_t(\theta)$ is a distortion affecting $\mu(\theta|\omega^t)$ given by a monotone function of:

$$\xi_t(\theta) \triangleq \frac{\phi'\left(E_{p_{t,\theta}}\left[V_{t+1}(e^*)\right]\right)}{\phi'\left(\phi^{-1}\left(E_{\mu(\theta|\omega^t)}\left[\phi\left(E_{p_{t,\theta}}\left[V_{t+1}(e^*)\right]\right)\right]\right)\right)}$$

Again, concavity of ϕ, hence ambiguity aversion, decreases the subjective likelihood of observing good outcomes, reflecting pessimism.

5.5 Vector expected utility

Recursive vector expected utility has been axiomatized in Siniscalchi [38], where an application to a Lucas economy similar to the present work is provided. In the case of recursive vector expected utility, the form of intertemporal utility is given by

$$\begin{aligned} V_t(c, \omega^t) = {} & u(c_t(\omega_t)) + \beta E_{p_t(\omega_t, \cdot)}\left[V_{t+1}(c, \omega^{t+1})\right] \\ & + A_{t,\omega^t}\left(\beta E_{p_t(\omega_t, \cdot)}\left[\xi^{t,\omega^t} \cdot V_{t+1}(c, \omega^{t+1})\right]\right) \end{aligned}$$

By Ghirardato and Siniscalchi [13] and Siniscalchi [38], Eq. (7) becomes

$$q_t = \beta E_{p_t(\omega_t,\cdot)} \left[\left\{ 1 + \sum_{0 \le i \le M} \frac{\partial A_{t,\omega_t} \left(E_{p_t(\omega_t,\cdot)}[\beta V_{t+1}(e^*) \cdot \xi^{t,\omega_t}] \right)}{\partial \phi_i} \xi_i^{t,\omega_t} \right\} \right.$$
$$\left. \frac{u'(e_{t+1})}{u'(e_t)} (q_{t+1} + d_{t+1}) \right]$$

where $\phi_i \in \mathbb{R}^n$ and $\partial A/\phi_i$ is a partial derivative. The functional form of vector expected is particularly suitable for a direct comparison with the EU case. Here ambiguity attaches an additional piece to the standard stochastic discount factor who exacerbates the effects of high or low dividend. The analysis follows that of Siniscalchi [38]. To gain intuition, assume a Clarke regular A and let $n = 1$, so that only an adjustment factor exists. Under ambiguity aversion, A is concave, together with the fact that $A(0) = 0$ is a maximum, $A'_{t,\omega_t} \left(E_{p_t(\omega_t,\cdot)}[\beta V_{t+1}(e^*) \cdot \xi^{t,\omega_t}] \right) \le 0$. To see the effect on discounting, if the investor expects a high endowment tomorrow, u' is low, and this effect is reinforced by $-A'(\cdot)$. Therefore, ambiguity aversion increases the discount of good outcomes, with respect to the benchmark EU case.

6 Conclusion

We price assets with ambiguous returns in a Lucas tree economy with a representative ambiguity averse investor. We provide a unified approach that abstracts from the specific models used in the literature. A multiplicity of prices may support equilibria when the functional representing preference is not smooth. Equilibrium prices are determined by a set of subjective probabilities related to the Clarke subdifferential of the functional representing investor's preferences. Equilibria exist under fairly general conditions.

Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

Appendix A: Non-smooth optimization

For completeness we report the main results that are used in the proofs.

Definition 4 Let $U \subset X$ an open subset of a Banach space X. Let f a locally Lipschitz functional $f : U \to \mathbb{R}$. For every $x \in U$ and $y \in X$, the Clarke (upper) derivative of f in x in the direction y is

$$f^\circ(x; y) = \limsup_{\substack{z \to x \\ t \downarrow 0}} \frac{f(z + ty) - f(y)}{t}$$

The Clarke (sub)differential of f at x is the set

$$\partial f(x) = \left\{ x^* \in X^* : \langle x^*, x \rangle \le f^\circ(x; y), \forall y \in X \right\}$$

Next proposition collects some useful properties of Clarke differential (see Clarke [5]).

Proposition 5 *Let $f : U \to \mathbb{R}$ be locally Lipschitz. Then:*

1. *$\partial f(x)$ is a nonempty, convex, weak* compact subset of X^*.*
2. *For every $v \in X$, $f^\circ(x; v) = \max_{x^* \in \partial f(x)} \langle x^*, v\rangle$*
3. *If f is convex, $\partial f(x)$ is the usual subdifferential of convex analysis.*
4. *(Lebourg Mean Value Theorem) For any $x, y \in X$, there exists $\gamma \in (0, 1)$, such that*

$$f(x) - f(y) \in \langle \partial f(\gamma x + (1-\gamma)y), x - y\rangle$$

A function $f : X \to \mathbb{R}$ on a Banach space X is Clarke regular at x if for all $v \in X$,

$$f^\circ(x; v) = f'(x; v)$$

where $f'(x; v) = \lim_{t\downarrow 0} \frac{f(x+tv)-f(x)}{t}$ is the usual directional derivative.

Theorem 6 (Clarke [5], Chain Rule) *Let F be a map from a Banach space X to another Banach space Y and g a real-valued function on Y. Suppose F is strictly differentiable at x and g is locally Lipschitz at $F(x)$. Then $f = g \circ F$ is Lipschitz at x and*

$$\partial f(x) \subseteq \partial g(F(x)) \circ D_s F(x)$$

Equality holds if g or $-g$ is regular at $F(x)$ or, if F maps every neighborhood of x to a set which is dense in a neighborhood of $F(x)$.

The interpretation (see [5, Remark 2.3.11], is that every element $z^* \in \partial f(x)$ can be represented as a composition of a map $y^* \in \partial g(F(x))$ and $D_s F(x)$ such that $\langle z^*, v\rangle = \langle y^*, D_s F(x)(v)\rangle$.

Appendix B: Proofs

We need the following preliminary result:

Lemma 7 *Fix $c \in \mathscr{D}$ and let V_t given as in Eq. (1). Define for each $T \in \mathbb{N}$, $\left\{V_t^T(c)\right\}_{t=1}^{\infty} \in \mathscr{D}$, $V_t^T \triangleq 0$ if $t > T$ and*

$$V_t^T(c, \omega_t) \triangleq u(c_t(\omega_t)) + \beta I\left(V_{t+1}^T(c, \omega_t)\right)$$

for $0 \leq t \leq T$. If $\beta b < 1$, then $\lim_{T\to\infty} V_t^T(c, \omega_t) = V_t(c, \omega_t)$ for all t and ω_t.

where b is defined in Sect. 2 and it is a parameter limiting the speed of growth of consumption.

Proof of Lemma 7 For any $t \leq T$ and ω_t, notice that $V_t^T(c, \omega_t) \leq V_t(c, \omega_t) \leq V_t^T(c, \omega_t) + \|V(c)\| \beta^{T-t+1} b^{T+1}$.

Proof of Theorem 1 Take $V(c)$, $V'(c)$ that solve Eq. (1), (by Tarksi's fixed point theorem at least one exists). Fix an arbitrary t then,

$$
\begin{aligned}
V_t(c,\omega^t) - \bar{V}_t^T(c,\omega^t) &= \beta E_{p_t}\left[V_{t+1}(c,\omega^t) - \bar{V}_{t+1}^T(c,\omega^t)\right] \\
&\qquad \text{by Lebourg Mean Value Theorem (4. in Prop. 5).} \\
&= \beta E_{p_t}\left[\beta I(V_{t+1}(c,\omega^t)) - \beta I(\bar{V}_{t+1}^T(c,\omega^t))\right] \text{ by definition} \\
&= \beta E_{p_t}\left[\beta E_{p_{t+1}}\left[V_{t+1}(c,\omega^t) - \bar{V}_{t+1}^T(c,\omega^t)\right]\right] \text{ by Lebourg MVT.} \\
&\le \beta E_{p_t}\left[\cdots\left[\beta E_{p_{t+T+1}}\left[V_{t+T+1}(c,\omega^{t+T})\right]\right]\right] \\
&\le (\beta b)^{T-t+1} b^t \, \|V(c)\| \overset{T\to\infty}{\longrightarrow} 0 \quad \text{by } \beta b < 1
\end{aligned}
$$

Applying the same argument to $\bar{V}_t(c,\omega^t) - V_t^T(c,\omega^t)$ concludes the proof.

Proof of Lemma 2 By definition

$$V_t(e^* + \xi(h_1, h_2, 0, \ldots), \omega^t) = u(e^*(\omega_t) + \xi h_1) + \beta I\left(V_{t+1}(e^* + \xi h_2, \omega^t); \omega_t\right)$$

which is equal to

$$V_t(e^* + \xi(h_1, h_2, 0, \ldots), \omega^t) = u(e^*(\omega_t) + \xi h_1) + \beta I\left(u(e^* + \xi h_2 + \phi(\omega_{t+1}); \omega_t\right)$$

where $\phi(\omega_{t+1}) = I\left(V_{t+2}(e^*, \omega_t); \omega^{t+1}\right)$. Optimality of $\xi = 0$ implies by Theorem 6 and the consequent discussion that

$$0 = u'(e^*(\omega_t))h_1 + \beta \int u'(e^*)h_2 dp, \quad \text{for some } p \in \partial I(u(e^*) + \phi(\omega_{t+1}); \omega_t)$$

The proof is concluded noticing that $u(e^*) + \phi(\omega_{t+1}) = V_{t+1}(e^*)$.

We first introduce a concept of equilibrium that is weaker then global equilibrium. First, given (t, ω^t), let (c', θ') a plan with $c' = \left\{c'_t\right\}_1^\infty$, $c'_\tau = 0$ for all $\tau \neq t, t+1$, $c'_t \in \mathbb{R}$, $c'_{t+1} \in C(\Omega)$, $\theta'_\tau = 0$ for all $\tau \neq t$, $\theta'_t = \Delta$ for $\Delta \in \mathbb{R}^n$. We define a *two-period perturbation* of a (t, ω^t)-feasible plan (c, θ) as $(c + \xi c', \theta + \xi\theta')$ for sufficiently small $\xi \in \mathbb{R}$ such that $(c + \xi c', \theta + \xi\theta')$ is (t, ω^t)-feasible. We say a (t, ω^t)-feasible plan is *myopically optimal* if

$$V_t({}^t c|\omega^t, \omega^t) \ge V_t({}^t c'|\omega^t, \omega^t), \quad \text{for all two-period perturbation } (c', \theta')$$

Then we call $\{q_t\} \in \mathscr{D}$ a *myopic equilibrium* if $(e, 0)$ is myopically optimal for all $t \ge 1$, $\omega^t \in \Omega^t$.

Proof of Theorem 3 (i). We need the following fact.

Fact 1 *If q satisfies Eq. (6) then it is a myopic equilibrium.*

Proof of Fact 1 First note that, if q is a myopic equilibrium, then $V_t(e) \ge V_t(c, \omega_t)$ for all two-period perturbations c and it is equivalent to e being a local maximum for $\xi \mapsto V(e + \xi c, \omega_t)$. By [5, Proposition 2.3.2], we have $0 \in \partial V(e + \xi c, \omega_t)$ at $\xi = 0$. To prove Fact 1, assume that q is a myopic equilibrium, and Eq. (6) is not satisfied. Let

$$\Gamma^*(e^*, \omega_t) \triangleq \operatorname*{argmin}_{p \in C(e^*;\omega_t)} \int \frac{u'(e_{t+1})}{u'(e_t)}(q_{t+1} + d_{t+1})dp$$

Then, for some $t \in \mathbb{N}$ and $\omega^t \in \Omega^t$ with $\theta_t = \Delta$,

$$\beta E_{\pi_t(\omega^t;\cdot)}\left\{u'(e_{t+1})\theta_t \cdot (q_{t+1} + d_{t+1})\right\} > \theta_t \cdot q_t u'(e_t) \tag{10}$$

where $\pi_t : \Omega_t \to \mathscr{M}^+(\Omega_t)$, with $\pi_t(\omega^t;\cdot) \in \Gamma^*(e^*, \omega_t)$ for each t and ω_t. Now let $c_\tau = e_\tau$ for all $\tau \neq t, t+1$ $c_t \triangleq e_t - \theta_t \cdot q_t$, $c_{t+1} = e_{t+1} + \theta_t \cdot (q_{t+1} + d_{t+1})$ and $\theta_\tau = 0$ for all $\tau \neq t$ and , it is (t, ω_t)-feasible, but inequality Eq. 10 is equivalent to

$$\frac{d}{d\xi} V_t(e + \xi c, \omega_t)\Big|_{\xi=0} > 0$$

for a two-period perturbation c, contradicting the assumption that q is a myopic equilibrium.

To conclude the proof of (i), we will show that myopic equilibria are global. Assume by contradiction that q is a myopic equilibrium that is not global. Then, there is a (t, ω^t)-feasible path (for all t, ω^t) $c \in \mathscr{D}$ (assume for simplicity that c is time-homogeneous and Markov) such that, for some t and some ω_t,

$$V_t(c, \omega_t) \geq V_t(e^*) + 2\epsilon$$

for some $\epsilon > 0$. Choose an integer N, such that, if a consumption plan $c'_\tau = c_\tau$ for $\tau = t, \dots, t+N$, then $V_t(c', \omega_t) \geq V(c, \omega_t) + \epsilon$. It follows that for such paths

$$V_t(c', \omega_t) \geq V_t(e^*) + \epsilon$$

Define a family of N consumption processes $\hat{c}^{t+i}$, $i = 0, \dots, N-1$, such that

$$\hat{c}^{t+i} \triangleq \begin{cases} c_{t+j+1} & \text{for all } j \in [0, i] \\ e_t & \text{otherwise} \end{cases}$$

Then at t,

$$V_t(\hat{c}^{t+N-1}, \omega_t) > V_t(e^*) \tag{11}$$

Moreover notice that, $\hat{c}^{t+N-2}$ coincides with e from $t-N-1$ onward, whereas $\hat{c}^{t+N-1}$ differs from e only at $t+N-1$. Since q is a myopic equilibrium, it is robust for a two-period perturbation, hence, a posteriori for a one-period perturbation, then

$$V_{t+N-1}(\hat{c}^{t+N-2}) = V_{t+N-1}(e^*) \geq V_{t+N-1}(\hat{c}^{t+N-1})$$

Dynamic consistency DC1-DC2 implies

$$V_t(\hat{c}^{t+N-2}) \geq V_t(\hat{c}^{t+N-1}) \tag{12}$$

Together Eqs. (11) and (12) imply

$$V_t(\hat{c}^{t+N-2}, \omega_t) > V_t(e^*)$$

Repeating the argument until time t, we get $V_t(\hat{c}^t, \omega_t) > V_t(e^*)$ but $\hat{c}^t$ is a one-period perturbation of e, and this contradict the fact that q is a myopic equilibrium.

(ii). Given Assumption 1, Micheal's Selection theorems implies the existence of a continuous selection, $\hat{\pi}_t(\omega^t;\cdot) \in \Gamma^*(e^*, \omega_t)$ and $\omega_t \in \Omega^t$, such that

$$q_{i,t} = \beta E_{\hat{\pi}(\omega^t;\cdot)} \left[\frac{u'(e_{t+1})}{u'(e^*(\omega_t))} (q_{t+1} + d_{t+1}) \right]$$

Then define an operator $T : \mathscr{D} \to \mathscr{D}$ as

$$(T_i f)_t(\omega^t) = \beta E_{\hat{\pi}(\omega^t;\cdot)} \left[f_{t+1} + u'(e_{t+1}) d_{i,t+1} \right]$$

It clearly has a unique fixed point $T_i \hat{f}_i = \hat{f}_i$, now let $\hat{f}_{i,t} = u'(e^*(\omega_t)) q_{i,t}$. Then it is a global equilibrium.

References

1. Cerreia-Vioglio, S., Ghirardato, P., Maccheroni, F., Marinacci, M., Siniscalchi, M.: Rational preferences under ambiguity. Econ. Theory **48**(2), 341–375 (2011)
2. Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., Montrucchio, L.: Uncertainty averse preferences. J. Econ. Theory **146**(4), 1275–1330 (2011)
3. Chateauneuf, A., Faro, J.H.: Ambiguity through confidence functions. J. Math. Econ. **45**(9–10), 535–558 (2009)
4. Chen, Z., Epstein, L.G.: Ambiguity, risk, and asset returns in continuous time. Econometrica **70**(4), 1403–1443 (2002)
5. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
6. Collard, F., Mukerji, S., Sheppard, K., Tallon, J.: Ambiguity and the historical equity premium. Technical Report (2011)
7. Delbaen, F.: The structure of m-Stable sets and in particular of the set of risk neutral measures. In: Émery, M., Yor, M. (eds.) Memoriam Paul-André Meyer. Lecture Notes in Mathematics, vol. 1874, pp. 215–258. Springer, Berlin (2006)
8. Epstein, L., Schneider, M.: Recursive multiple-priors. J. Econ. Theory **113**(1), 1–31 (2003a)
9. Epstein, L.G., Schneider, M.: IID: independently and indistinguishably distributed. J. Econ. Theory **113**(1), 32–50 (2003b)
10. Epstein, L.G., Schneider, M.: Ambiguity, information quality, and asset pricing. J. Financ. **63**(1), 197–228 (2008)
11. Epstein, L.G., Wang, T.: Intertemporal asset pricing under knightian uncertainty. Econometrica **62**(2), 283–322 (1994)
12. Epstein, L.G., Wang, T.: Uncertainty, risk-neutral measures and security price booms and crashes. J. Econ. Theory **67**(1), 40–82 (1995)
13. Ghirardato, P., Siniscalchi, M.: A more robust definition of multiple priors. Carlo Alberto Notebook 144, Collegio Carlo Alberto (2010)
14. Ghirardato, P., Siniscalchi, M.: Ambiguity in the small and in the large. Econometrica **80**, 2827–2847 (2012)
15. Gilboa, I., Schmeidler, D.: Maxmin expected utility with non-unique prior. J. Math. Econ. **18**(2), 141–153 (1989)
16. Grant, S., Polak, B., Strzalecki, T.: Second-Order Expected Utility. Mimeo, New York (2009)
17. Guidolin, M., Rinaldi, F.: Ambiguity in asset pricing and portfolio choice: a review of the literature. Theory Decis. **74**(2), 183–217 (2013)
18. Gumen, A., Savochkin, A.: Dynamically stable preferences. J. Econ. Theory **148**(4), 1487–1508 (2013)
19. Hanany, E., Klibanoff, P.: Updating preferences with multiple priors. Theor. Econ. **2**(3), 261–298 (2007)
20. Hanany, E., Klibanoff, P., et al.: Updating ambiguity averse preferences. BE J. Theor. Econ. **9**(1), 407 (2009)
21. Hansen, L.P., Sargent, T.J.: Robust control and model uncertainty. Am. Econ. Rev. **91**(2), 60–66 (2001)
22. Hansen, L.P., Sargent, T.J., Tallarini, T.D.J.: Robust permanent income and pricing. Rev. Econ. Stud. **66**(4), 873–907 (1999)
23. Illeditsch, P.K.: Ambiguous information, portfolio inertia, and excess volatility. J. Financ. **66**(6), 2213–2247 (2011)
24. Ju, N., Miao, J.: Ambiguity, learning, and asset returns. Econometrica **80**(2), 559–591 (2012)
25. Klibanoff, P., Marinacci, M., Mukerji, S.: A smooth model of decision making under ambiguity. Econometrica **73**(6), 1849–1892 (2005)
26. Klibanoff, P., Marinacci, M., Mukerji, S.: Recursive smooth ambiguity preferences. J. Econ. Theory **144**(3), 930–976 (2009)
27. Lucas, R.E.J.: Asset prices in an exchange economy. Econometrica **46**(6), 1429–45 (1978)
28. Maccheroni, F., Marinacci, M., Rustichini, A.: Ambiguity aversion, robustness, and the variational representation of preferences. Econometrica **74**(6), 1447–1498 (2006a)
29. Maccheroni, F., Marinacci, M., Rustichini, A.: Dynamic variational preferences. J. Econ. Theory **128**(1), 4–44 (2006b)
30. Maenhout, P.: Robust portfolio rules and asset pricing. Rev. Financ. Stud. **17**(4), 951–983 (2004)
31. Marinacci, M., Montrucchio, L.: Unique solutions for stochastic recursive utilities. J. Econ. Theory **145**(5), 1776–1804 (2010)
32. Ozsoylev, H., Werner, J.: Liquidity and asset prices in rational expectations equilibrium with ambiguous information. Econ. Theory **48**(2), 469–491 (2011)
33. Pastor, L., Stambaugh, R.F.: Are stocks really less volatile in the long run? J. Financ. **67**(2), 431–478 (2012)

34. Riedel, F.: Optimal stopping with multiple priors. Econometrica **77**(3), 857–908 (2009)
35. Rigotti, L., Shannon, C., Strzalecki, T.: Subjective beliefs and ex ante trade. Econometrica **76**(5), 1167–1190 (2008)
36. Siniscalchi, M.: Vector expected utility and attitudes toward variation. Econometrica **77**(3), 801–855 (2009)
37. Siniscalchi, M.: Dynamic choice under ambiguity. Theor. Econ. **6**(3), 379–421 (2011a)
38. Siniscalchi, M.: Recursive vector expected utility. Technical Report (2011b)
39. Skiadas, C.: Scale-invariant asset pricing and consumption/portfolio choice with general attitudes toward risk and uncertainty. Math. Financ. Econ. **7**(4), 431–456 (2013)
40. Strzalecki, T.: Temporal resolution of uncertainty and recursive models of ambiguity aversion. Econometrica **81**(3), 1039–1074 (2013)