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Collateral constraints and asset prices[☆]

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ABSTRACT

We study the effects of collateral constraints in an economy populated by investors with nonpledgeable labor incomes and heterogeneous preferences and beliefs. We show that these constraints inflate stock prices and generate spikes and crashes in price-dividend ratios and volatilities, clustering of volatilities, and leverage cycles. They also lead to substantial decreases in interest rates and increases in Sharpe ratios when investors are anxious about hitting constraints due to production crises in the economy. Furthermore, stock prices have large collateral premiums over nonpledgeable incomes. We derive asset prices and stationary distributions of the investors' consumption shares in closed form.

1. Introduction

Financial markets play a key role in facilitating risk sharing and efficient allocation of assets among investors. However, trading in financial assets often entails moral hazard due to investors' incentives to default on their risky positions. The moral hazard can be alleviated by collateralized trades whereby risky positions are backed by financial capital that can be seized in the event of default. The latter arrangement restores the functionality of financial markets at the cost of restricting risk sharing among investors. In this paper, we develop a parsimonious model that sheds new light on the economic effects of such restrictions. In particular, we show how collateralization inflates asset prices, generates repeated booms and busts in the stock market, and leads to spikes, crashes, and clustering of stock return volatilities, persistent periods of loose and binding constraints, and cycles of high and low leverage. Our analysis is facilitated by closed-form solutions and the stationarity of equilibrium processes.

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We consider a pure exchange economy with one consumption good produced by a tree with i.i.d. shocks, similar to Lucas (1978). The economy is populated by two representative investors with heterogeneous constant relative risk aversion (CRRA) preferences over consumption and heterogeneous beliefs about the output growth rate. Each investor receives a fraction of the tree's output as labor income and invests total wealth in financial assets such as bonds and stocks. The investors have limited liability and can re-enter the financial market following defaults on risky positions in financial assets. In the event of default, the financial assets can be seized by counterparties, but labor income cannot be expropriated. The arising moral hazard problem is resolved by requiring risky positions to be backed by collateral in such a way that each investor's total financial wealth stays positive at all times, and hence investors can always pay back to counterparties. We also allow the aggregate consumption to experience rare large negative shocks. These shocks help us explore how mere anxiety about the possibility of a production crisis affects the economy by tightening collateral constraints. Our closed-form solutions allow us to prove some of the results for general model parameters rather than for particular calibrations.

First, we show that collateral constraints increase the prices of all tradable assets with positive cash flows relative to a frictionless economy. Moreover, these increases in prices are more significant when investors are closer to their default boundaries. In particular, the stock price-dividend (P/D) ratio spikes upwards in response to small economic shocks near the default boundaries of investors, giving rise to repeated periods of high and low stock prices.

The intuition for the latter results is as follows. In a frictionless economy, the investors' consumption shares gradually approach zero or one, and hence the economic impact of one of the investors vanishes in the long run (e.g., Blume and Easley, 2006; Yan, 2008; Chabakauri, 2015). The collateral requirements restrict financial losses and protect investors from losing their consumption shares. As a result, the consumption shares are bounded away from zero and one. Moreover, the constraints never bind simultaneously for both investors, and at each moment one of the investors is unconstrained. The unconstrained investor's marginal utility of consumption is proportional to the prices of Arrow-Debreu securities. This marginal utility is expected to be higher in the economy with constraints because the unconstrained investor's consumption is expected to be lower than in the unconstrained economy due to the upper bound on the consumption share, discussed above. Consequently, the prices of Arrow-Debreu securities, and hence the prices of all assets with positive cash flows, are higher in the constrained economy.

The dynamics of the P/D ratio determines the effect of constraints on volatilities. We show that collateral constraints dampen volatilities in bad times, when the aggregate consumption is low, and amplify them in good times, when the aggregate consumption is high. The latter effect makes collateral constraints a useful tool for curbing excessive volatility in bad times. The explanation is that the P/D ratio spikes up both in good and bad times because in good (bad) times the pessimistic (optimistic) investors in our economy lose wealth and may bind on their constraints. As a result, the P/D ratio is procyclical in good times and is countercyclical in bad times. Consequently, the P/D ratio and the dividend move in the same direction in good times and in opposite directions in bad times. Because the stock price is the product of the P/D ratio and the dividend, the stock return volatility increases in good times and decreases in bad times. The volatility experiences spikes and crashes due to the sensitivity of P/D ratios to small shocks when investors are close to hitting their constraints. Moreover, the periods of high and low volatility are persistent because of the persistence of periods when constraints are likely to bind, as discussed below, which gives rise to the clustering of volatilities.

We also derive the distributions of investors' consumption shares in analytic form and show that they are stationary and nondegenerate (i.e., their support is a closed interval rather than a single point). The analysis of these distributions yields three economic insights. First, there is a nontrivial time variation of asset prices in the long run. Second, periods of binding collateral constraints are persistent. That is, the economy stays close to default boundaries for some time because hitting a constraint makes likely hitting it again in the near future due to the slow accumulation of wealth over time. Third, we show that all investors, including those with incorrect beliefs, survive in the long run and can have a large economic impact in equilibrium because the constraints prevent investors from losing their consumption shares, similar to the related literature (e.g., Blume and Easley, 2006; Cao, 2018). We note that the nondegeneracy of consumption share distributions and the persistence of the periods of binding constraints are more difficult to demonstrate than survival, and, to the best of our knowledge, these results are new to the literature.

Next, we show that the mere possibility of a large (albeit unpredictable) drop in the aggregate output next period decreases interest rates and increases Sharpe ratios in the current period when the irrational optimist is close to hitting the collateral constraint. The latter effect only occurs when production crises and collateral constraints are jointly present in the economy. Hence, the collateral constraints amplify the spillover of the production crisis to the financial market. The amplification effect arises because investors "fly to quality" by buying riskless bonds when there is a possibility of hitting the collateral constraint next period. We note that lower interest rates and higher Sharpe ratios can be generated by alternative mechanisms and constraints, discussed in the literature review below. However, the amplification mechanism, to the best of our knowledge, has not been studied before. We also show that investor heterogeneity and the stationarity of equilibrium give rise to cycles of high and low leverage. In particular, the leverage is high when investors are far away from their default boundaries and drops to zero when investors hit their constraints.

Finally, we measure the collateral liquidity premium of the stock versus labor income. This premium arises because dividends and labor incomes are collinear but incomes are nonpledgeable. First, we derive shadow prices of claims to labor incomes such that exchanging marginal units of these claims for the consumption good at shadow prices does not affect investors' welfare. Then, we construct portfolios of stocks that replicate labor incomes. We define the collateral liquidity premium as the percentage difference in the value of the replicating portfolio and the shadow price. The premium from the view of a particular investor widens close to that investor's default boundary and ranges from 0% to 40% in our calibration, which demonstrates the economic significance of collateralization. Moreover, the nontradability of labor income does not contribute to this premium because, in economies with pledgeable labor income, investors circumvent nontradability by taking short positions in the stock.

The paper develops a new methodology for studying the effects of collateralization. This methodology allows us to obtain closed-form equilibrium processes and to prove their properties, which previously could only be studied numerically. For example, we prove that our constraints increase P/D ratios and generate spikes in asset prices and lead to nondegeneracy and stationarity of consumption share distributions. The paper also introduces a tractable discrete-time setting that makes exposition less technical and permits taking continuous-time limits. The tractability and stationarity make our model a convenient benchmark for asset pricing research that can be extended in various directions.

We contribute to the literature by uncovering several new economic effects of collateral constraints. In particular, we show that these constraints give rise to U-shaped P/D ratios that spike when constraints are likely to bind, cause sharp increases and crashes of stock return volatility, magnify volatility in good states and dampen it in bad states, and amplify the effects of crises on interest rates and Sharpe ratios. Our paper is the first to derive the stationary distribution of an investor's consumption share in an economy with collateral constraints in closed form and to use it to demonstrate the persistence of periods of binding constraints and nondegeneracy of the long-run equilibrium. The nondegeneracy of the equilibrium implies time variation of asset prices and cycles of high and low leverage in the long run. The paper also sheds new light on the effects of constraints on interest rates, Sharpe ratios, and collateral premiums that have been reviewed in the related literature. In particular, similar to this literature, the constraints decrease interest rates, increase Sharpe ratios, and give rise to collateral premiums for stocks. In contrast to the previous studies, the effects of constraints on interest rates and Sharpe ratios occur only during periods of anxious economy, when the mere possibility of a production crisis tightens the constraints. We also develop a new approach to capturing collateral premiums and evaluate their economic significance.

Closest to us are papers that study economies in which investors have limited liability and face solvency constraints. Deaton (1991) considers a partial equilibrium model in which investors trade in a riskless asset with an exogenous interest rate and face a nonnegativity constraint on their financial wealth. Detemple and Serrat (2003) also study an economy with a nonnegative wealth in which investors have heterogeneous beliefs and identical risk aversions. They show that this constraint introduces a singularity component into interest rates when the constraint binds, while stock risk premiums have the same structure as in unconstrained economies. In contrast to our work, they do not compute P/D ratios, volatilities, and consumption share distributions and do not study the effects of rare production crises and heterogeneity in preferences.

Chien and Lustig (2010) study a similar constraint in an economy with a continuum of ex-ante identical investors that receive nonpledgeable labor incomes affected by idiosyncratic shocks. They develop similar methods using multipliers in a discrete-time setting, but they do not have closed-form solutions and do not consider differences in beliefs about fundamentals. Lustig and Van Nieuwerburgh (2005) study the role of housing collateral when labor income is nonpledgeable. The main difference of our paper from the latter two papers is that our investors are ex-ante heterogeneous and are not affected by idiosyncratic shocks to labor income. The economic effects of heterogeneity in preferences and beliefs are different from the effects of ex-post heterogeneity in realized idiosyncratic income shocks in the above literature. For example, Krueger and Lustig (2010) show the irrelevance of market incompleteness induced by these income shocks for the risk premiums.

Cao (2018) proves the survival of investors with incorrect beliefs in the long run in economies with general collateral constraints and stationary endowment processes bounded away from zero and shows similar results numerically in an example with nonstationary endowments. Blume et al. (2018) explore potential benefits from imposing trading restrictions, such as natural borrowing constraints, in economies with bounded endowments and investors with heterogeneous beliefs. In contrast to these works, our results do not rely on bounded endowments. Moreover, we derive consumption share distributions in closed form and establish their bimodality, stationarity, and nondegeneracy (i.e., their support is a closed interval rather than a single point). Kubler and Schmedders (2003) prove the existence of stationary equilibria in dynamic economies with general collateral constraints. Rampini and Viswanathan (2018) study household insurance in an economy with collateral constraints with limited enforcement and deep-pocket risk-neutral lenders who provide state-contingent claims to households at zero risk premium. Our model is different from the latter paper in that all investors in our economy are risk averse, and risk premiums are endogenous and time varying. Gromb and Vayanos (2002, 2010, 2018) and Brunnermeier and Pedersen (2009) study economies with CARA investors subject to margin constraints, which have similarities with our constraints. In contrast to their models, in our model all investors have CRRA preferences and interest rates are endogenous.

Geanakoplos (2003, 2009), Fostel and Geanakoplos (2008, 2014), and Geanakoplos and Zame (2014) develop the theory of collateral constraints in two- and three-period economies. Our constraint prevents investors from defaulting in the worst-case scenario as in Geanakoplos (2003, 2009) and leads to higher asset prices as in

Fostel and Geanakoplos (2008). Simsek (2013) studies a two-period economy with a continuum of states and shows that collateral constraints have asymmetric disciplining effects, depending on investor's beliefs, and also shows how defaultable debt endogenously emerges in equilibrium. Biais et al. (2018) study a two-period economy with multiple trees and imperfect collateral pledgeability. In contrast to this literature, we focus on the nonpledgeability of labor income rather than the imperfect pledgeability of assets.

Kehoe and Levine (1993), Kocherlakota (1996), Tsyrennikov (2012), and Osambela (2015) study economies in which investors are weakly better off not defaulting and are permanently excluded from securities markets if they default. Alvarez and Jermann (2000) show that such constraints can be implemented by imposing certain "not too tight" solvency portfolio constraints. Alvarez and Jermann (2001) find that such constraints help explain equity premiums in the US economy. They solve a simple example in closed form and develop a numerical method for the general case. In contrast to this literature, our investors have limited liability and can re-enter the market after a default.

Our paper is related to the literature on the economic effects of borrowing, margin, short-sale, and position limit constraints (e.g., Harrison and Kreps, 1978; Detemple and Murthy, 1997; Basak and Cuoco, 1998; Basak and Croitoru, 2000; 2006; Pavlova and Rigobon, 2008; Gârleanu and Pedersen, 2011; Chabakauri, 2013, 2015; Rytchkov, 2014; Brumm et al., 2015; Buss et al., 2016), portfolio insurance (e.g., Basak, 1995), and value at risk constraints (e.g., Basak and Shapiro, 2001). Our economic results are different from the results in this literature. First, the latter constraints can increase or decrease stock prices depending on whether the investors' risk aversions are greater or less than one (e.g., Chabakauri, 2015), whereas our collateral constraints always increase stock prices irrespective of risk aversions and beliefs. Second, these constraints typically dampen stock return volatility, whereas our collateral constraints amplify them in some states of the economy.

The paper is also related to macro-finance, financial intermediation, and banking literatures that study economies with frictions (Kiyotaki and Moore, 1997; Krusell and Smith, 1998; Brunnermeier and Sannikov, 2014; Klimenko et al., 2016; Kondor and Vayanos, 2019) and to the literature on frictionless economies with heterogeneous investors (e.g., Chan and Kogan, 2002; Basak, 2005; Yan, 2008; Bhamra and Uppal, 2014; Atmaz and Basak, 2018; Massari, 2019; Borovička, 2020, among others).

2. Economic setup

We consider a pure exchange infinite-horizon economy with one consumption good produced by an exogenous Lucas (1978) tree. The economy is populated by two representative heterogeneous investors A and B that hold shares in the tree and receive labor income each period. To facilitate the exposition, we start with a discrete-time economy with dates $t = 0, \Delta t, 2\Delta t, \dots,$ and later take a continuous-time limit.

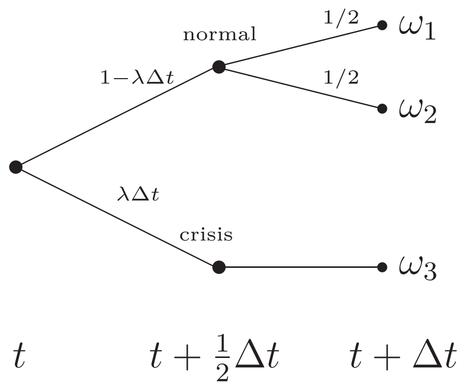

Fig. 1. States of the economy. After time t the economy moves to a normal state with probability $1-\lambda\Delta t$ and to a crisis state with probability $\lambda\Delta t$. Conditional on being in a normal state, the economy moves to either ω_1 or ω_2 with equal probabilities.

At each point of time $t = 0, \Delta t, 2\Delta t, \dots$ the economy is in one of the three states: ω_1, ω_2, and ω_3. With probability $1-\lambda\Delta t$, the economy is either in state ω_1 or state ω_2, which we call normal states, and with probability $\lambda\Delta t$ in state ω_3, which we call the crisis state. Parameter $\lambda > 0$ is the crisis intensity. States ω_1 and ω_2 have probabilities 1/2 conditional on the economy being in a normal state. Fig. 1 depicts the structure of uncertainty.

2.1. Output, financial markets, and investor heterogeneity

At date t, the tree produces $D_t\Delta t$ units of aggregate output, where D_t follows a process

$$\Delta D_t = D_t[\mu_D\Delta t + \sigma_D\Delta w_t + J_D\Delta j_t], \tag{1}$$

where $\mu_D \geq 0$, $\sigma_D > 0$ and $J_D \leq 0$ are output growth mean, volatility, and drop during a crisis, respectively, and $\Delta D_t = D_{t+\Delta t} - D_t$ is the change in output. Processes w_t and j_t are discrete-time analogs of a Brownian motion and Poisson processes, respectively. These processes follow dynamics $w_{t+\Delta t} = w_t + \Delta w_t$ and $j_{t+\Delta t} = j_t + \Delta j_t$, where increments Δw_t and Δj_t are i.i.d. random variables given by

$$\Delta w_t = \begin{cases} +\sqrt{\Delta t}, & \text{in state } \omega_1, \\ -\sqrt{\Delta t}, & \text{in state } \omega_2, \\ \quad 0, & \text{in state } \omega_3, \end{cases}$$

$$\Delta j_t = \begin{cases} 0, & \text{in state } \omega_1, \\ 0, & \text{in state } \omega_2, \\ 1, & \text{in state } \omega_3. \end{cases} \tag{2}$$

It can be easily verified that $\mathbb{E}_t[\Delta w_t|\text{normal}] = 0$ and $\text{var}_t[\Delta w_t|\text{normal}] = \Delta t$, similar to a Brownian motion, where $\mathbb{E}_t[\cdot]$ and $\text{var}_t[\cdot]$ are expectation and variance conditional on time-t information. Parameters μ_D, σ_D, and J_D are such that $D_t > 0$ for all t. Chabakauri (2014) shows that process (1) converges to a continuous-time Lévy process as $\Delta t \to 0$.

The economy is populated by two representative investors A and B. Each investor stands for a continuum of identical investors of unit mass. Fractions l_A and l_B of the aggregate output $D_t\Delta t$ are paid to investors A and B as their labor incomes, respectively, and labor incomes are nontradable. Fractions l_A and l_B can also be interpreted as nontradable shares in the aggregate output such as holdings of illiquid assets. The remaining fraction $1 - l_A - l_B$ is paid as a dividend to the shareholders.

The investors can trade three securities at each date t: (1) a riskless bond in zero net supply, which pays one unit of consumption at date $t + \Delta t$; (2) one stock in net supply of one unit, which is a claim to the stream of dividends $(1 - l_A - l_B)D_t\Delta t$; and (3) a one-period insurance contract in zero net supply, which pays one unit of consumption in the crisis state ω_3 and zero otherwise. Absent any frictions, the market is complete. Market completeness and the absence of idiosyncratic shocks to labor income are required for tractability and allow us to solve the model in closed form. Bond, stock, and insurance prices B_t, S_t, and P_t, respectively, are determined in equilibrium.

2.2. Investor heterogeneity and optimization problems

The investors have heterogeneous CRRA preferences over consumption, given by

$$u_i(c) = \begin{cases} \dfrac{c^{1-\gamma_i}}{1-\gamma_i}, & \text{if } \gamma_i \neq 1, \\ \ln(c), & \text{if } \gamma_i = 1, \end{cases} \tag{3}$$

where $i = A, B$. The investors agree on time-t asset prices and the aggregate output but disagree on the probabilities of states. Both investors have subjective beliefs given by

$$\pi_i(\omega_1) = \frac{1-\lambda_i\Delta t}{2}(1+\delta_i\sqrt{\Delta t}),$$
$$\pi_i(\omega_2) = \frac{1-\lambda_i\Delta t}{2}(1-\delta_i\sqrt{\Delta t}), \quad \pi_i(\omega_3) = \lambda_i\Delta t, \tag{4}$$

where crisis intensities λ_i and disagreement parameters δ_i are such that probabilities (4) are positive. It is immediate to verify that $\pi_i(\omega_1) + \pi_i(\omega_2) + \pi_i(\omega_3) = 1$, and hence $\pi_i(\omega)$ is a probability measure. Throughout the paper, $\mathbb{E}_t^i[\cdot]$ and $\mathrm{var}_t^i[\cdot]$ denote conditional expectations and variances under the probability measure of investor i.

It can be easily verified that time-t conditional expected output growth rate in normal times under the beliefs of investor i is given by

$$\mathbb{E}_t^i\left[\frac{\Delta D_t}{D_t}\middle|\text{normal}\right] = (\mu_D + \delta_i\sigma_D)\Delta t. \tag{5}$$

Therefore, parameter δ_i measures the extent of investor i's pessimism (when $\delta_i < 0$) or optimism (when $\delta_i > 0$) relative to the objective probability measure. For tractability, we assume that investors do not update probabilities over time. We also assume that investor B is weakly less risk averse and is more optimistic than investor A: $\gamma_A \geq \gamma_B$, $\lambda_A \geq \lambda_B$, and $\delta_B \geq \delta_A$. The latter parametric restriction is imposed to simplify the exposition and does not affect the qualitative results in the paper.[1]

At date 0 the investors have certain endowments of financial assets. The total time-t disposable wealth of investor i is given by $W_{it} + l_iD_t\Delta t$, where W_{it} is the financial wealth, defined as the time-t value of all positions in financial assets acquired at the previous date, and $l_iD_t\Delta t$ is the labor income. At date t, investor i allocates wealth to $c_{it}\Delta t$ units of consumption, b_{it} units of bond, and a portfolio of risky assets $n_{it} = (n_{i,St}, n_{i,Pt})$, where $n_{i,St}$ and $n_{i,Pt}$ are units of stock and insurance, respectively.

In a frictionless economy, the financial wealth W_{it} can become negative when investors take risky positions backed by their future labor income. However, in our economy only financial assets are pledgeable, whereas labor incomes are not. Moreover, the investors have limited liability. That is, they can default when their financial wealth becomes negative and then re-enter the market, which gives rise to a moral hazard problem, similar to the related literature (e.g., Chien and Lustig, 2010; Geanakoplos, 2009). This problem is addressed here by requiring the investors to keep their next-period financial wealth $W_{i,t+\Delta t}$ positive at all times so that their pledgeable capital is sufficient to cover all liabilities such as debt and short positions. Intuitively, constraint $W_{i,t+\Delta t} \geq 0$ requires investors to cross-collateralize their pledgeable financial assets in such a way that losses on one position are always offset by gains on the other positions.

Investor $i = A, B$ maximizes expected discounted utility with time discount ρ

$$\max_{c_{it}, b_{it}, n_{it}} \mathbb{E}_t^i\left[\sum_{\tau=t}^{\infty} e^{-\rho\tau} u_i(c_{i\tau})\Delta t\right], \tag{6}$$

subject to the self-financing budget constraints, given by

$$W_{it} + l_iD_t\Delta t = c_{it}\Delta t + b_{it}B_t + n_{it}(S_t, P_t)^\top, \tag{7}$$

$$W_{i,t+\Delta t} = b_{it} + n_{it} \times \left(S_{t+\Delta t} + (1-l_A-l_B)D_{t+\Delta t}\Delta t,\ \mathbf{1}_{\{\omega_{t+\Delta t}=\omega_3\}}\right)^\top \tag{8}$$

and the collateral constraint

$$W_{i,t+\Delta t} \geq 0, \tag{9}$$

where $W_{i,t+\Delta t}$ is the financial wealth at date $t + \Delta t$ given by Eq. (8).

To provide further intuition for the constraint (9), following Gromb and Vayanos (2018), we observe that it is equivalent to the following collateral constraint:

$$W_{it} + (l_iD_t - c_{it})\Delta t \geq \max_{\omega_{t+\Delta t}}\left\{n_{i,st}\left(S_t - \frac{S_{t+\Delta t} + (1-l_A-l_B)D_{t+\Delta t}\Delta t}{1+r_t\Delta t}\right) + n_{i,pt}\left(P_t - \frac{\mathbf{1}_{\{\omega_{t+\Delta t}=\omega_3\}}}{1+r_t\Delta t}\right)\right\}. \tag{10}$$

[1] This assumption makes it easier to see that the consumption share of investor A, $s_t = c^*_{At}/D_t$ (introduced in Section 2.3 below), is countercyclical because in good (bad) times less risk-averse optimists have more (less) wealth and consumption than more risk-averse pessimists. If this assumption is relaxed, all the qualitative results in this paper remain the same. Section IA.2 of the Internet Appendix provides an example of equilibrium processes in an economy where the less risk-averse investor is more pessimistic than the more risk-averse investor and also presents the exact condition for the countercyclicality or procyclicality of the state variable s_t.

The constraint (10) is obtained by substituting bond holding b_{it} from Eq. (7) into Eq. (8) for wealth $W_{i,t+\Delta t}$ and then rearranging terms in the inequality $W_{i,t+\Delta t} \geq 0$. The expression on the right-hand side of the constraint (10) represents the largest possible loss of a risky position evaluated in present-value terms. Therefore, this constraint indicates that the investors are allowed to invest in portfolios of assets using these portfolios as collateral but are required to put up a sufficient amount of their own capital to cover the losses in the worst-case scenario. The coefficients multiplying asset holdings $n_{i,St}$ and $n_{i,Pt}$ in (10) and evaluated at the worst-case state $\omega_{t+\Delta t}$ are endogenous margin requirements that show the investors' own capital invested per unit of asset.

The constraint (10) is similar to collateral constraints in Brunnermeier and Pedersen (2009) and Gromb and Vayanos (2018) with the difference being that we allow investors to "cross-margin" their positions so that one risky asset can be used to cover margins on the other. Brunnermeier and Pedersen (2009) discuss the institutional features of such constraints and point out that it is increasingly possible to "cross-margin."

Remark 1 (Partially pledgeable labor income). Our model can be easily extended to economies where fraction $k_i \in [0, 1]$ of investor i's labor income can be pledged. The requirement to keep the next-period pledgeable wealth nonnegative is then given by

$$W_{i,t+\Delta t} + \underbrace{\frac{k_i l_i}{1 - l_A - l_B}\Big(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t\Big)}_{\text{measure of pledgeable labor income}} \geq 0. \tag{11}$$

The second term in constraint (11) measures the value of the pledgeable income. Let $k_i l_i D_t \Delta t$ be the pledgeable income of investor i. This income is proportional to stock dividends $(1 - l_A - l_B)D_t \Delta t$ and hence can be replicated by a portfolio of $\widehat{n}_i = k_i l_i/(1 - l_A - l_B)$ units of stock with cum dividend value $\widehat{n}_i(S_t + (1 - l_A - l_B)D_t\Delta t)$. The investors can circumvent the nontradability of pledgeable income by shorting stocks against this income. Hence, the claims to pledgeable income are, effectively, tradable and have the same value as the replicating portfolio. The requirement to have positive pledgeable wealth then becomes $W_{i,t+\Delta t} + \widehat{n}_i(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t) \geq 0$, which is equivalent to constraint (11). Lemma A.1 in the Appendix shows that models with $k_i \neq 0$ reduce to models with $k_i = 0$ by a change of variable. Hence, the economic implications of our baseline model with constraint (9) and the model with a more general constraint (11) are the same.

2.3. Equilibrium

Definition. An equilibrium is a set of asset prices $\{B_t, S_t, P_t\}$ and of consumption and portfolio policies $\{c^*_{it}, b^*_{it}, n^*_{it}\}_{i\in\{A,B\}}$ that solve optimization problem (6)for each investor, given processes $\{B_t, S_t, P_t\}$, and consumption and securities markets clear:

$$c^*_{At} + c^*_{Bt} = D_t, \quad b^*_{At} + b^*_{Bt} = 0,$$

$$n^*_{A,St} + n^*_{B,St} = 1, \quad n^*_{A,Pt} + n^*_{B,Pt} = 0. \tag{12}$$

In addition to asset prices, we derive P/D and wealth-consumption ratios $\Psi = S/((1 - l_A - l_B)D)$ and $\Phi_i = W^*_i/c^*_i$, respectively. We also derive the annualized Δt-period riskless interest rate r_t, stock mean return μ_t and volatility σ_t in normal times, and the percentage change of the stock price in the crisis state, denoted by J_t.

We derive the equilibrium in terms of state variable υ_t given by the log-ratio of marginal utilities of investors evaluated at their shares of the aggregate consumption c^*_{it}/D_t:

$$\upsilon_t = \ln\left(\frac{(c^*_{At}/D_t)^{-\gamma_A}}{(c^*_{Bt}/D_t)^{-\gamma_B}}\right). \tag{13}$$

Substituting consumption shares of investors A and B, denoted by $s_t = c^*_{At}/D_t$ and $1 - s_t = c^*_{Bt}/D_t$, into Eq. (13), we express υ_t as a function of s_t:

$$\upsilon_t = \gamma_B \ln(1 - s_t) - \gamma_A \ln(s_t). \tag{14}$$

Variable υ_t is a decreasing function s_t, and hence s_t is an alternative state variable.

We assume that the exogenous model parameters are such that

$$\mathbb{E}^i_t\left[e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_i}\right] < 1, \quad i = A, B. \tag{15}$$

Condition (15) is necessary and sufficient for the existence of equilibrium in homogeneous agent economies populated only by investor A or investor B.

3. Characterization of equilibrium

In this section, we provide the characterization of equilibrium. First, we explain our methodology using a simple example with logarithmic investors and no production crises. Then, we extend the methodology to the general economy described in Section 2. Finally, we derive asset prices and the distributions of investor A's consumption share s_t in closed form in the continuous-time limit of the economy.

3.1. Simple example and discussion of methodology

We sketch our methodology in a simple economy with two logarithmic investors and without production crises ($\lambda_A = \lambda_B = 0$). The investors disagree about the probabilities of states ω_1 and ω_2, given by Eq. (4), and have disagreement parameters $\delta_B = \delta/2$ and $\delta_A = -\delta/2$, where $\delta > 0$. We keep the derivation of equilibrium as close as possible to the derivation in the frictionless economy and clearly outline the sources of tractability of our model.

First, we show that the optimal consumptions satisfy the first order conditions (FOCs)

$$\frac{\xi_{i,t+\Delta t}}{\xi_{it}} = e^{-\rho\Delta t}\frac{(c^*_{i,t+\Delta t})^{-1} + \ell_{i,t+\Delta t}}{(c^*_{it})^{-1}}, \quad i = A, B, \tag{16}$$

where ξ_{it} are the investors' state price densities (SPDs) and $\ell_{i,t+\Delta t} \geq 0$ are the Lagrange multipliers for the collateral constraints $W_{i,t+\Delta t} \geq 0$. Conditions (16) are derived in Lemma 1 below using the standard method of Lagrange multipliers by maximizing the expected discounted utility (6) subject to the collateral constraint $W_{i,t+\Delta t} \geq 0$ and the budget constraint rewritten in the present-value form as follows:

$$c_{it}\Delta t + \mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t}\right] = W_{it} + l_i D_t \Delta t. \tag{17}$$

The budget constraint (17) simply states that the sum of the current consumption and the present value of the next-period financial wealth should be equal to the sum of the current financial wealth and the labor income.

Next, we provide a heuristic derivation of the dynamics of the state variable (13), which in the economy with log-investors is given by $\nu_t = \ln(c^*_{Bt}/c^*_{At})$. Consider first the unconstrained region of the state space where the Lagrange multipliers $\ell_{i,t+\Delta t}$ vanish. Then, the FOCs (16) are the same as in the frictionless economy. Consequently, the dynamics of ν_t in that region is the same as in the frictionless economy and is given by $\nu_{t+\Delta t} = \nu_t + \mu_\nu \Delta t + \sigma_\nu \Delta w_t$, where $\mu_\nu = 0$ and $\sigma_\nu = \delta + o(\Delta t)$.[2]

Let $\bar{\nu}$ and $\underline{\nu}$ denote the values of the state variable when the constraints of investors A and B bind, respectively. We argue below that variable ν_t follows dynamics

$$\nu_{t+\Delta t} = \max\{\underline{\nu}; \min\{\bar{\nu}; \nu_t + \mu_\nu \Delta t + \sigma_\nu \Delta w_t\}\}, \tag{18}$$

where $\mu_\nu = 0$ and $\sigma_\nu = \delta + o(\Delta t)$. The intuition for the dynamics (18) is that when the investors hit their constraints they consume a fraction of labor income and cannot take risky positions that could lead to further financial losses. Hence, the constraints protect the investors' consumption from falling below a certain limit. More formally, suppose investor A may hit the collateral constraint next period. Similar to the derivation of the frictionless dynamics of variable ν_t, we find that $\nu_{t+\Delta t} = \nu_t + \mu_\nu \Delta t + \sigma_\nu \Delta w_t - \ln(1 + \ell_{A,t+\Delta t} c^*_{A,t+\Delta t})$, where $\ell_{A,t+\Delta t} \geq 0$ is the Lagrange multiplier. Hence, $\nu_{t+\Delta t} = \bar{\nu} < \nu_t + \mu_\nu \Delta t + \sigma_\nu \Delta w_t$ if the constraint binds and $\nu_{t+\Delta t} = \nu_t + \mu_\nu \Delta t + \sigma_\nu \Delta w_t$ otherwise. The latter two cases imply that $\nu_{t+\Delta t} = \min\{\bar{\nu}, \nu_t + \mu_\nu \Delta t + \sigma_\nu \Delta w_t\}$. A similar analysis for investor B gives rise to the lower bound $\underline{\nu}$. Combining the results, we obtain the dynamics (18).

The fact that the dynamics (18) of variable ν_t is a capped version of the unconstrained dynamics is the first source of the tractability of our model. The second source of tractability is, as we show next, that the multipliers $\ell_{i,t+\Delta t}$ cancel out in the expressions for wealths W_{it}. In particular, substituting SPDs in Eq. (16) into the budget constraint (17), we notice that the multiplier $\ell_{i,t+\Delta t}$ cancels out due to the complementary slackness condition $\ell_{i,t+\Delta t} W_{i,t+\Delta t} = 0$ for the constraint (9), and the budget constraint becomes

$$W_{it} + l_i D_t \Delta t = c_{it}\Delta t + \mathbb{E}_t^i\left[e^{-\rho\Delta t}\frac{(c^*_{i,t+\Delta t})^{-1}}{(c^*_{it})^{-1}} W_{i,t+\Delta t}\right]. \tag{19}$$

[2] When constraints do not bind and $\ell_{i,t+\Delta t} = 0$, FOCs (16) imply that $\nu_{t+\Delta t} - \nu_t = \ln(c^*_{B,t+\Delta t}/c^*_{Bt}) - \ln(c^*_{A,t+\Delta t}/c^*_{At}) = \ln(\xi_{A,t+\Delta t}/\xi_{At}) - \ln(\xi_{B,t+\Delta t}/\xi_{Bt})$. Moreover, the SPDs are ratios of risk-neutral and physical probabilities so that $\xi_{i,t+\Delta t}/\xi_{it} = \pi^{RN}(\omega_{t+\Delta t})/\pi_i(\omega_{t+\Delta t})$. Consequently, $\nu_{t+\Delta t} - \nu_t = \ln(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t}))$. It can be directly verified that $\ln(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})) = \mu_\nu \Delta t + \sigma_\nu \Delta w_t$, where $\mu_\nu = 0$ and $\sigma_\nu = (\ln(1 + 0.5\delta\sqrt{\Delta t}) - \ln(1 - 0.5\delta\sqrt{\Delta t}))/\sqrt{\Delta t} = \delta + o(\Delta t)$.

From the market clearing condition $c^*_{At} + c^*_{Bt} = D_t$ and the expression $\nu_t = \ln(c^*_{Bt}/c^*_{At})$, we find that $c^*_{At} = D_t/(1 + e^{\nu_t})$ and $c^*_{Bt} = D_t e^{\nu_t}/(1 + e^{\nu_t})$. We also rewrite the financial wealths as $W_{it} = \Phi_i(\nu_t)c^*_{it}$, where $\Phi_i(\nu_t)$ is the wealth-consumption ratio of investor i, and substitute into the budget constraint (19). In particular, for investor A, we find that

$$\Phi_A(\nu_t) = \mathbb{E}_t^A[e^{-\rho\Delta t}\Phi_A(\nu_{t+\Delta t})] + (1 - l_A(1 + e^{\nu_t}))\Delta t. \tag{20}$$

We observe that when $\nu_{t+\Delta t} \in (\underline{\nu}, \bar{\nu})$, the dynamics of variable ν_t and Eq. (20) are the same as in the frictionless economy. Hence, taking limit $\Delta t \to 0$ and proceeding as in the frictionless case, by applying Itô's lemma or Taylor expansions for small Δt, we find that ratio $\Phi_A(\nu_t)$ satisfies the following equation inside the interval $(\underline{\nu}, \bar{\nu})$:

$$\frac{\delta^2}{2}\Phi''_A(\nu) - \frac{\delta^2}{2}\Phi'_A(\nu) - \rho\Phi_A(\nu) + 1 - l_A(1 + e^{\nu}) = 0. \tag{21}$$

The solution of Eq. (21) is given by

$$\Phi_A(\nu) = C_{A-}e^{\varphi_-\nu} + C_{A+}e^{\varphi_+\nu} + \frac{1 - l_A}{\rho} - \frac{l_A}{\rho}e^{\nu}, \tag{22}$$

where $\varphi_\pm = 0.5 \pm \sqrt{1 + 8\rho/\delta^2}$ and constants C_{A+} and C_{A-} can be found from the boundary conditions $\Phi'_A(\underline{\nu}) = \Phi'_A(\bar{\nu}) = 0$. The latter boundary conditions can be derived using Taylor expansions of Eq. (20) near the boundaries. Similarly, for investor B,

$$\Phi_B(\nu) = \left(C_{B-}e^{\varphi_-\nu} + C_{B+}e^{\varphi_+\nu} + \frac{1 - l_B}{\rho}e^{\nu} - \frac{l_B}{\rho}\right)e^{-\nu}. \tag{23}$$

The boundaries are found from the conditions $W_{it} = 0$ or, equivalently, $\Phi_A(\bar{\nu}) = \Phi_B(\underline{\nu}) = 0$.

Finally, we find the P/D ratio Ψ. From the market clearing conditions, $S_t = W_{At} + W_{Bt}$. Hence, ratio Ψ can be expressed as $\Psi_t = (\Phi_{At}s_t + \Phi_{Bt}(1 - s_t))/(1 - l_A - l_B)$. After some algebra, using wealth-consumption ratios (22) and (23), we find that

$$\Psi(\nu) = \frac{C_- e^{\varphi_-\nu} + C_+ e^{\varphi_+\nu}}{1 + e^{\nu}} + \frac{1}{\rho}, \tag{24}$$

where constants $C_\pm$ are found in Corollary 1 below from the boundary conditions.

3.2. Equilibrium in the general case

First, we derive the investors' SPDs ξ_{At} and ξ_{Bt} defined as processes such that asset prices can be expressed as follows (Duffie, 2001, p. 23):

$$B_t = \mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}\right], \tag{25}$$

$$S_t = \mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t)\right], \tag{26}$$

$$P_t = \mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}\mathbf{1}_{\{\omega_{t+\Delta t} = \omega_3\}}\right], \tag{27}$$

where $i = A, B$. The SPD ξ_{it} exists for each investor i due to the absence of arbitrage (Duffie, 2001, p. 4). There is no arbitrage in the economy because zero-investment strategies with nonnegative payoffs are feasible under constraints

(7)–(9). The SPDs ξ_{At} and ξ_{Bt} differ due to differences in beliefs and are linked by the change of measure equation[3]

$$\frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} = \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \frac{\pi_A(\omega_{t+\Delta t})}{\pi_B(\omega_{t+\Delta t})}. \tag{28}$$

We find the SPDs from the investor's first order conditions, reported in Lemma 1.

Lemma 1 (The first order condition). *The SPDs ξ_{it} and optimal consumptions c^*_{it} satisfy the first order conditions*

$$\frac{\xi_{i,t+\Delta t}}{\xi_{it}} = e^{-\rho\Delta t} \frac{(c^*_{i,t+\Delta t})^{-\gamma_i} + \ell_{i,t+\Delta t}}{(c^*_{it})^{-\gamma_i}}, \tag{29}$$

where $\ell_{i,t+\Delta t} \geq 0$ is the Lagrange multiplier for collateral constraint (9) satisfying the complementary slackness condition $\ell_{i,t+\Delta t} W^*_{i,t+\Delta t} = 0$.

We use Lemma 1 to derive the dynamics of state variable ν_t, similar to Section 3.1. When constraints do not bind, the Lagrange multipliers $\ell_{i,t+\Delta t}$ vanish and the conditions (29) are the same as in a frictionless economy. Consequently, the dynamics of the variable ν_t in the unconstrained region of the state space is the same as in the frictionless economy. Next, let $\overline{\nu}$ and $\underline{\nu}$ be the values of the state variable ν_t when constraints (9) of investors A and B bind, respectively. We show that state variable ν_t stays within boundaries $\underline{\nu} \leq \nu_t \leq \overline{\nu}$ because collateral constraints limit the investors' losses of wealth and consumption. The boundaries $\underline{\nu}$ and $\overline{\nu}$ are found from the conditions $W_{it} = 0$, which are equivalent to

$$\Phi_A(\overline{\nu}) = 0, \quad \Phi_B(\underline{\nu}) = 0, \tag{30}$$

where $\Phi_i(\nu_t)$ are wealth-consumption ratios given by Eqs. (A.26) and (A.27) in the Appendix. Proposition 1 below reports the dynamics of ν_t.

Proposition 1 (Closed-form dynamics of the state variable ν_t). *Given the boundaries $\overline{\nu}$ and $\underline{\nu}$, the dynamics of the state variable ν_t is given by*

$$\nu_{t+\Delta t} = \max\{\underline{\nu}; \min\{\overline{\nu}; \nu_t + \mu_\nu \Delta t + \sigma_\nu \Delta w_t + J_\nu \Delta j_t\}\}, \tag{31}$$

where drift μ_ν, volatility σ_ν, and jump J_ν are given in closed form by Eqs. (A.23)–(A.25) in the Appendix. Suppose $\gamma_A \neq \gamma_B$ and/or $\delta_A \neq \delta_B$. Then, for a sufficiently small Δt, the boundaries $\overline{\nu}$ and $\underline{\nu}$ are reflecting (i.e. ν_t does not stay at the boundaries forever).

Dynamics (31) reveals that the constraint does not alter the process for the state variable when the constraint does not bind, and the effects of constraints are captured by the bounds on process ν_t. This property of state variable ν_t plays an important role in establishing the clustering of volatilities and other results in Section 4 below and is difficult to see using numerical methods. Proposition IA.1 in the Internet Appendix proves the existence of finite time-independent bounds $\overline{\nu}$ and $\underline{\nu}$ satisfying Eq. (30). We use the closed-form dynamics (31) to prove the existence and stationarity of equilibrium, derive the SPDs, and study the effects of collateralization on asset prices. Proposition 2 below reports the SPD, asset prices, and their properties.

Proposition 2 (State price density and the effects on asset prices).

(1) *The SPD under the beliefs of investor A is given by*

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t} \left(\frac{s(\nu_{t+\Delta t})}{s(\nu_t)} \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} \times \exp(\max\{0; \nu_t + \mu_\nu \Delta t + \sigma_\nu \Delta w_t + J_\nu \Delta j_t - \overline{\nu}\}), \tag{32}$$

where investor A's time-t consumption share $s(\nu_t)$ solves Eq. (14).

(2) *The P/D ratio $\Psi(\nu_t)$ is uniformly bounded, the stock price S_t is given by*

$$S_t = (1 - l_A - l_B) D_t \mathbb{E}^A_t \left[\sum_{\tau=t+\Delta t}^{+\infty} \frac{\xi_{A\tau}}{\xi_{At}} \frac{D_\tau}{D_t} \right], \tag{33}$$

and the prices of the bond and the insurance contract are given by $B_t = \mathbb{E}^A_t[\xi_{A,t+\Delta t}]/\xi_{At}$ and $P_t = \mathbb{E}^A_t[\xi_{A,t+\Delta t} 1_{\{\omega_{t+\Delta t}=\omega_3\}}]/\xi_{At}$, respectively.

(3) *The prices of bond, stock, and the insurance contract are higher in the economy with collateral constraints than in the frictionless economy, conditional on both economies having the same current output D_t and the state variable ν_t.*

Eq. (32) decomposes the SPD into two terms. The first term is the ratio of marginal utilities of investor A at dates $t + \Delta t$ and t, $e^{-\rho\Delta t}(s(\nu_{t+\Delta t})D_{t+\Delta t})^{-\gamma_A}/(s(\nu_t)D_t)^{-\gamma_A}$, and is the same as in the frictionless economy. The second term captures the effect of the friction on the SPD and is only activated when the constraint of investor A is binding. The SPD of investor B can be obtained along the same lines.

Proposition 2 also demonstrates that collateralization inflates asset prices. This is because the SPD in the constrained economy exceeds its counterpart in the frictionless economy due to the positive Lagrange multiplier $\ell_{i,t+\Delta t}$ in the first order condition (29). We discuss the intuition in Section 4.1. Proposition IA.2 in the Internet Appendix provides the verification theorem for the optimality of investors' optimal strategies.

3.3. Closed-form solution in a continuous-time limit

Next we take continuous-time limit $\Delta t \to 0$ and derive the equilibrium in closed form. Taking the limit allows rewriting the equations for the P/D and wealth-consumption ratios as differential difference equations. For tractability, we derive these ratios in terms of a transformed ratio $\widehat{\Psi}(\nu; \theta)$, which satisfies a simpler equation reported in Lemma 2.

Lemma 2 (Differential difference equation). *In the limit $\Delta t \to 0$, the P/D ratio Ψ and wealth-consumption ratios Φ_i are given by*

$$\Psi(\nu) = \widehat{\Psi}(\nu; -\gamma_A) s(\nu)^{\gamma_A}, \tag{34}$$

[3] Eqs. (25) –(27) can be rewritten as system equations for three unknowns $\pi_i(\omega_k)\xi_{i,t+\Delta t}(\omega_k)/\xi_{it}$, where $k = 1, 2, 3$ for a fixed i. The solution of this system is unique when the matrix of asset payoffs is invertible, and hence $\pi_B(\omega_{t+\Delta t})\xi_{B,t+\Delta t}/\xi_{Bt} = \pi_A(\omega_{t+\Delta t})\xi_{A,t+\Delta t}/\xi_{At}$ for all states.

$$\Phi_i(v) = \frac{(\mathbf{1}_{\{i=A\}} - \mathbf{1}_{\{i=B\}})\widehat{\Psi}(v; 1-\gamma_A) + (\mathbf{1}_{\{i=B\}} - l_i)\widehat{\Psi}(v; -\gamma_A)}{\mathbf{1}_{\{i=A\}}s(v) + \mathbf{1}_{\{i=B\}}(1-s(v))} s(v)^{\gamma_A}, \quad (35)$$

where $s(v)$ solves Eq. (14) and $\widehat{\Psi}(v;\theta)$ satisfies a differential difference equation

$$\frac{\widehat{\sigma}_v^2}{2}\widehat{\Psi}''(v;\theta) + \left(\widehat{\mu}_v + \delta_A\widehat{\sigma}_v + (1-\gamma_A)\sigma_D\widehat{\sigma}_v\right)\widehat{\Psi}'(v;\theta) - \left(\lambda_A + \rho - (1-\gamma_A)(\mu_D + \delta_A\sigma_D) + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)\widehat{\Psi}(v;\theta) + \lambda_A(1+J_D)^{1-\gamma_A}\widehat{\Psi}(\max\{\underline{v}; v+\widehat{J}_v\};\theta) + s(v)^\theta = 0, \quad (36)$$

subject to the reflecting boundary conditions

$$\widehat{\Psi}'(\underline{v};\theta) = 0, \quad \widehat{\Psi}'(\bar{v};\theta) - \widehat{\Psi}(\bar{v};\theta) = 0, \quad (37)$$

where $\widehat{\mu}_v$, $\widehat{\sigma}_v \geq 0$, and $\widehat{J}_v \leq 0$ are constants given by

$$\widehat{\mu}_v = (\gamma_A - \gamma_B)\left(\mu_D - \frac{\sigma_D^2}{2}\right) + \lambda_A - \lambda_B + \frac{\delta_A^2 - \delta_B^2}{2}, \quad (38)$$

$$\widehat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta_B - \delta_A, \quad (39)$$

$$\widehat{J}_v = (\gamma_A - \gamma_B)\ln(1+J_D) + \ln\left(\frac{\lambda_B}{\lambda_A}\right). \quad (40)$$

The boundaries $\bar{v}$ and $\underline{v}$ solve equations

$$\frac{\widehat{\Psi}(\bar{v}; 1-\gamma_A)}{\widehat{\Psi}(\bar{v}; -\gamma_A)} = l_A, \quad \frac{\widehat{\Psi}(\underline{v}; 1-\gamma_A)}{\widehat{\Psi}(\underline{v}; -\gamma_A)} = 1 - l_B. \quad (41)$$

We observe that Eq. (36) is linear, in contrast to economies with constraints directly imposed on trading strategies of investors (e.g., Gârleanu and Pedersen, 2011; Chabakauri, 2013, 2015; Rytchkov, 2014) and is a differential difference equation with a "delayed" argument $v+\widehat{J}_v$ in the fourth term, where $\widehat{J}_v \leq 0$. This term is further complicated by the fact that the argument is restricted to stay above the boundary $\underline{v}$, which gives rise to the term with a peculiar argument $\max\{\underline{v}; v+\widehat{J}_v\}$. This term captures the investors' decisions in anticipation of hitting their constraint.

Before deriving the equilibrium in the general case, we provide analytical P/D ratios for a special case in Corollary 1 below.

Corollary 1 (Analytical price-dividend ratios). *Suppose $\lambda_A = \lambda_B = \lambda > 0$ and $\gamma_A = \gamma_B = \gamma$, where γ is an integer. Then, the P/D ratio $\Psi(v)$ is given by*

$$\Psi(v) = \frac{1}{(1+e^{v/\gamma})^\gamma}\left(C_- e^{\varphi_- v} + C_+ e^{\varphi_+ v} + \sum_{k=0}^{\gamma}\binom{\gamma}{k}\frac{e^{kv/\gamma}}{h(k/\gamma)}\right), \quad (42)$$

where $h(\varphi)$ is a characteristic polynomial of Eq. (36) given by

$$h(\varphi) = \rho - (1-\gamma)(\mu_D + \delta_A\sigma_D) + \frac{(1-\gamma)\gamma}{2}\sigma_D^2 + \lambda(1-(1+J_D)^{1-\gamma}) - (\widehat{\mu}_v + \delta_A\widehat{\sigma}_v + (1-\gamma)\sigma_D\widehat{\sigma}_v)\varphi - \frac{\widehat{\sigma}_v^2\varphi^2}{2}, \quad (43)$$

where φ_- and φ_+ are a negative and positive solutions of equation $h(\varphi) = 0$ and constants $C_\pm$ are given by Eqs. (A.59)–(A.60) in the Appendix, respectively.

In Section 4 below, we argue that the analytical ratio (42) captures some important properties of the P/D ratio, which also holds in the general case with arbitrary risk aversions and crises intensities. Hence, this special case can be used as a tractable benchmark in asset pricing research. Proposition IA.3 in the Internet Appendix presents the closed-form P/D ratio for general CRRA risk aversions and beliefs. Although the general closed-form solution is complex, it provides a constructive proof for the existence of P/D ratios. We also solve Eqs. (36)–(37) for the ratio $\widehat{\Psi}$ using the method of finite differences and double-check that the numerical and closed-form solutions coincide.

We call the interval $v \in [\underline{v}, \underline{v} - \widehat{J}_v]$ in the state space a period of anxious economy, similar to Fostel and Geanakoplos (2008), albeit the investor disagreement does not increase during these periods as in the latter paper. When the economy falls into this state, even a small possibility of a crisis renders the collateral constraint binding and leads to deleveraging. To explore the economic effects of the anxious economy, using the SPD (32), we derive the interest rates r_t and stock risk premium in normal times $\mu_t - r_t$ in Proposition 3 below.

Proposition 3 (The first order condition). *For a sufficiently small interval Δt, the interest rate r_t and the risk premium $\mu_t - r_t$ in normal times are given by*

$$r_t = \begin{cases} \tilde{r}_t + \lambda_A - \lambda_A(1+J_D)^{-\gamma_A}\left(\dfrac{s(\max\{\underline{v}; v_t + \widehat{J}_v\})}{s_t}\right)^{-\gamma_A} + O(\Delta t), & \text{for } \underline{v} < v_t < \bar{v}, \\ -\left(\dfrac{s_t \mathbf{1}_{\{v_t=\bar{v}\}}}{\gamma_A} + \dfrac{(1-s_t)\mathbf{1}_{\{v_t=\underline{v}\}}}{\gamma_B}\right)\dfrac{|\widehat{\sigma}_v|\Gamma_t}{2\sqrt{\Delta t}} + O(1), & \text{for } v = \underline{v} \text{ or } v = \bar{v}, \end{cases} \quad (44)$$

$$\mu_t - r_t = \Gamma_t\left(\sigma_D - \frac{s_t}{\gamma_A}\delta_A - \frac{1-s_t}{\gamma_B}\delta_B + \frac{|\widehat{\sigma}_v|}{2}\left(\frac{(1-s_t)\mathbf{1}_{\{v=\underline{v}\}}}{\gamma_B} - \frac{s_t\mathbf{1}_{\{v=\bar{v}\}}}{\gamma_A}\right)\right)\sigma_t - \lambda_A(1+J_D)^{-\gamma_A}J_t\left(\frac{s(\max\{\underline{v}; v_t + \widehat{J}_v\})}{s_t}\right)^{-\gamma_A} + O(\sqrt{\Delta t}), \quad (45)$$

where $\tilde{r}_t$ is the interest rate in the unconstrained economy without crisis risk, given by

$$\tilde{r}_t = \rho + \gamma_A(\mu_D + \delta_A\sigma_D) - \frac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2 + \left(\frac{\gamma_A\sigma_D\widehat{\sigma}_v - (\widehat{\mu}_v + \delta_A\widehat{\sigma}_v)}{\gamma_B}\right)(1-s_t)\Gamma_t - \widehat{\sigma}_v^2\left(\frac{1}{2\gamma_B^2}(1-s_t)^2\Gamma_t^2 + \frac{1}{2\gamma_A^2\gamma_B^2}s_t(1-s_t)\Gamma_t^3\right), \quad (46)$$

where drift $\widehat{\mu}_v$, volatility $\widehat{\sigma}_v$, and $\widehat{J}_v$ of variable v are given by Eqs. (38)–(40), volatility σ_t in normal times and jump size J_t are given by Eqs. (A.72)–(A.73) in the Appendix, and

$\Gamma_t \equiv \gamma_A\gamma_B/(\gamma_A(1-s_t)+\gamma_B s_t)$ is the risk aversion of a representative investor.

The effects of collateral constraints on interest rates and risk premiums arise due to the investors' concern that a potential crisis can render the constraint binding next period when the economy is close to boundary $\underline{v}$. The third term in the first line of Eq. (44) for the interest rate quantifies the impact of collateralization on precautionary savings due to a downward jump in the aggregate consumption, which we further discuss in Section 4.

Interest rate (44) and risk premium (45) also feature terms with indicator functions $\mathbf{1}_{\{v=\underline{v}\}}$ and $\mathbf{1}_{\{v=\overline{v}\}}$, which are nonzero only at the boundaries $\underline{v}$ and $\overline{v}$. For the interest rate r_t these terms have the order of magnitude proportional to $1/\sqrt{\Delta t}$, and hence the interest rate has singularities at the boundaries $\underline{v}$ and $\overline{v}$ when $\Delta t \to 0$. The intuition is that near the boundaries $\underline{v}$ and $\overline{v}$ even a small shock Δw_t can lead to a default. Thus, when the investor's constraint binds at time t, this investor allocates a larger fraction of income to the riskless asset than in the interior region $\underline{v} < v_t < \overline{v}$, which decreases interest rates.

Similar singularities arise in a continuous-time model of Detemple and Serrat (2003). Our discrete-time analysis sheds new light on these singularities by uncovering their order of magnitude $1/\sqrt{\Delta t}$. Consequently, the per-period rate $r_t\Delta t$ is finite and has an order of magnitude $O(\sqrt{\Delta t})$. Moreover, in contrast to the latter paper, due to production crises, the collateralization in our model affects the interest rates and risk premiums not only at the boundaries but also for the whole period of anxious economy.

3.4. Stationary distribution of consumption share

Absent any frictions, the state variable v follows an arithmetic Brownian motion with a jump. This process is nonstationary and induces nonstationarity in the unconstrained equilibrium where one of the investor's consumption share gradually converges to zero. As a result, with the exception of some knife-edge parameter combinations, only one of the investors has a significant impact on asset prices in the frictionless economy in the long run (e.g., Blume and Easley, 2006; Yan, 2008; Chabakauri, 2015).

Collateral constraints (9) help both investors survive in the long run by protecting them against losing their shares of aggregate consumption beyond certain limits, similar to the previous literature on survival (e.g., Blume and Easley, 2006; Cao, 2018, among others). Our contribution is that we derive the probability density function (PDF) of consumption share s in closed form, show that this PDF is stationary and nondegenerate, and find parameters that determine its shape. The latter result is important because it implies nontrivial time variation of asset prices in the long run. For tractability, we assume that there are no production crises so that $\lambda_A = \lambda_B = 0$. Proposition 4 reports the results.

Proposition 4 (Stationary distribution of consumption share). *Suppose $\lambda_A = \lambda_B = 0$. Then, the PDF $f(s,\tau;\, s_t, t)$ of consumption share s at time τ conditional on observing share s_t at time t is given in closed form by expression (A.84) in the Appendix. Furthermore, the stationary PDF of consumption share s is given by*

$$f(s) = \begin{cases} \dfrac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\left(\dfrac{\gamma_A}{s}+\dfrac{\gamma_B}{1-s}\right) \\ \times \dfrac{((1-s)^{\gamma_B}/s^{\gamma_A})^{2\widehat{\mu}_v/\widehat{\sigma}_v^2}\mathbf{1}_{\{\underline{s}\le s\le \overline{s}\}}}{((1-\underline{s})^{\gamma_B}/\underline{s}^{\gamma_A})^{2\widehat{\mu}_v/\widehat{\sigma}_v^2}-((1-\overline{s})^{\gamma_B}/\overline{s}^{\gamma_A})^{2\widehat{\mu}_v/\widehat{\sigma}_v^2}}, \\ \text{if } \widehat{\mu}_v \neq 0, \\ \left(\dfrac{\gamma_A}{s}+\dfrac{\gamma_B}{1-s}\right)\dfrac{\mathbf{1}_{\{\underline{s}\le s\le \overline{s}\}}}{\gamma_B\ln\left(\frac{1-\underline{s}}{1-\overline{s}}\right)+\gamma_A\ln\left(\frac{\overline{s}}{\underline{s}}\right)}, \quad \text{if } \widehat{\mu}_v = 0, \end{cases} \tag{47}$$

where $\widehat{\mu}_v = (\gamma_A-\gamma_B)(\mu_D-\sigma_D^2/2)+(\delta_A^2-\delta_B^2)/2$, $\widehat{\sigma}_v = (\gamma_A-\gamma_B)\,\sigma_D+\delta_B-\delta_A$, $\mathbf{1}_{\{\underline{s}\le s\le \overline{s}\}}$ is an indicator function, and $\underline{s}$ and $\overline{s}$ are the bounds on the consumption share s, which solve Eq. (14) for $\overline{v}$ and $\underline{v}$, respectively.

Proposition 4 confirms that both investors survive in the long run and consumption share s has a well-defined stationary distribution. The beliefs enter PDF (47) via the ratio of the drift and variance of process v_t, given by $\widehat{\mu}_v/\widehat{\sigma}_v^2$. This ratio determines the relative dominance of investors in the economy. In particular, for bounds $\underline{s}$ and $\overline{s}$ that are symmetric around 0.5, the PDF is concentrated around $\underline{s}$ if $\widehat{\mu}_v > 0$ and around $\overline{s}$ if $\widehat{\mu}_v < 0$. We note that the drift of state variable v can be rewritten as $\widehat{\mu}_v = (\gamma_A-\gamma_B)(\mu_D-\sigma_D^2/2)+(\delta_A-\delta_B)(\delta_A+\delta_B)/2$, and hence the drift is influenced both by the dispersion of beliefs $\delta_A-\delta_B$ and the average bias $(\delta_A+\delta_B)/2$. In the special case $\gamma_A=\gamma_B$ the relative dominance of investors is determined by a simple ratio of the average bias and the dispersion of beliefs:

$$\frac{\widehat{\mu}_v}{\widehat{\sigma}_v^2} = \frac{(\delta_A+\delta_B)/2}{\delta_B-\delta_A}. \tag{48}$$

In another special case where risk aversions are different but beliefs are correct on average, i.e. $(\delta_A+\delta_B)/2=0$, the drift $\widehat{\mu}_v$ is determined by differences in risk aversions.

Fig. 2 a shows the stationary PDF and transition densities $f(s,\, t;\, s_0,\, 0)$ when investors have the same risk aversions $\gamma_A=\gamma_B=2$ and opposite beliefs $\delta_A=-\delta_B=-0.05$. Other model parameters are described in the legend of the figure. From Eq. (5) for the expected dividend growth rate, we observe that the latter disagreement parameters δ_i imply that investor A (B) believes that expected output growth is approximately 10% lower (higher) than under the true probabilities. The stationary PDF is symmetric and bimodal so that both investors occasionally have large consumption shares. Fig. 2b shows two additional distributions when investor A has correct beliefs ($\delta_A=0$) and investor B is optimistic ($\delta_A>0$). In the case $\gamma_A=\gamma_B=2$, $\delta_A=0$, and $\delta_B=0.05$, the stationary distribution peaks at $s=0.9$, which means that the rational investor A has large consumption share in the economy. In the case $\gamma_A=2$, $\gamma_B=1.5$, $\delta_A=0$, and $\delta_B=0.15$, the stationary distribution is bimodal so that both rational and overly optimistic investors occasionally have large consumption shares in the economy.

The economic implication of the bimodality of the stationary PDF under some model parameters is that the periods of binding constraints are likely to be persistent. The

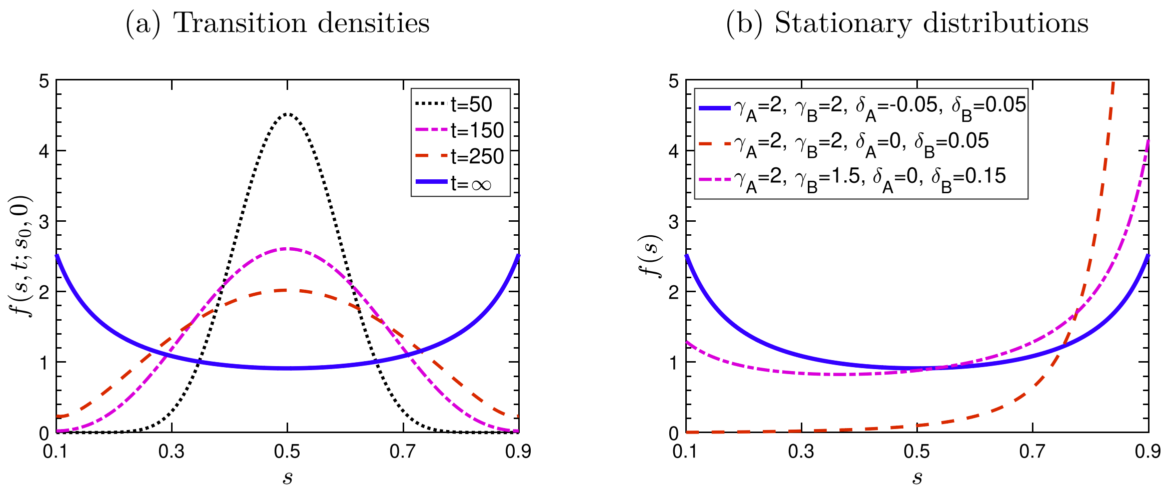

Fig. 2. Convergence to stationary distribution of consumption share $s_t = c^*_{A,t}/D_t$. Panel (a) shows transition densities $f(s,\ t;\ s_0,\ 0)$ for the starting point $s_0 = 0.5$ and the stationary distribution $f(s)$ (i.e., density for $t = \infty$). We set $\gamma_A = 2,\ \gamma_B = 2,\ \mu_D = 0.018,\ \sigma_D = 0.032,\ \lambda_A = \lambda_B = 0,\ \rho = 0.02,\ \delta_A = -0.05$ (i.e., $\mathbb{E}^A[\Delta D_t/D_t|\text{normal}] \approx 0.9\mu_D$), $\delta_B = 0.05$ (i.e., $\mathbb{E}^B[\Delta D_t/D_t|\text{normal}] \approx 1.1\mu_D$), $\underline{s} = 0.1,\ \bar{s} = 0.9,\ l_A = 0.1257$, and $l_B = 0.1178$. Panel (b) shows stationary distributions for different sets of model parameters.

closed-form dynamics (31) for the state variable v helps explain the bimodality of the PDF. From this dynamics, we observe that after hitting a boundary, the process v_t remains in its vicinity for some time. Hence, because variable v follows an arithmetic Brownian motion in the interval $(\underline{v}, \bar{v})$, the probability of hitting the same boundary again is high.

4. Analysis of equilibrium

In this section, we demonstrate the economic implications of our model. In Section 4.1, we show that capital constraints amplify the effect of rare crises on generating lower interest rates and higher Sharpe ratios, lead to spikes and crashes of stock prices and stock return volatilities, amplify volatility in good times and decrease it in bad times, and generate volatility clusters. Section 4.2 measures the economic significance of collateralization by quantifying the collateral premium of the stock.

We study the equilibrium for calibrated parameters. We set the parameters of the aggregate consumption process to $\mu_D = 0.018,\ \sigma_D = 0.032$, and $J_D = -0.25$ and the crisis intensities of investors A and B to $\lambda_A = 0.02$ and $\lambda_B = 0.01$, respectively. The risk aversions are $\gamma_A = \gamma_B = 2$, and the time discount is $\rho = 0.02$. The disagreement parameters are $\delta_A = -\delta_B = -0.05$, and they correspond to the case in which investor A (B) believes that the mean growth rate given by Eq. (5) is approximately equal to $0.9\mu_D$ ($1.1\mu_D$). The shares of labor income $l_A = 0.1257$ and $l_B = 0.1178$ are chosen to generate symmetric bounds on investor A's consumption share: $\underline{s} = 0.1$ and $\bar{s} = 0.9$.[4] We note that all our qualitative results on interest rates, market prices of risk, P/D ratios, and stock return volatilities remain the same when investors have different risk aversions and identical beliefs or when both risk aversions and beliefs are different. Moreover, the results do not depend on whether the more risk-averse investor is more pessimistic or optimistic than the less risk-averse investor.[5]

We plot the equilibrium processes as functions of consumption share $s_t = c^*_{At}/D_t$ because s conveniently lies in the interval (0,1) and is more intuitive than variable v. We observe that consumption share s is countercyclical in the sense that $\text{corr}_t(ds_t, dD_t) < 0$. Intuitively, the aggregate wealth and consumption shift to (away from) investor A following negative (positive) shocks to output because this investor is more pessimistic than investor B. We call a process procyclical (countercyclical) if that process is a decreasing (increasing) function of s. We interpret periods of low (high) s_t as good (bad) times in the economy because during these periods the output D_t is high (low).

4.1. Equilibrium processes

Fig. 3 depicts investor B's leverage/market ratio L_t/S_t and stock holdings n_{Bt} in the constrained (solid line) and unconstrained (dashed line) economies. Fig. 3a demonstrates the cyclicality of leverage. The leverage is low-

[4] Drift μ_D and volatility σ_D are within the ranges considered in the literature (e.g., Basak and Cuoco, 1998; Chan and Kogan, 2002; Rytchkov, 2014). To avoid finding bounds $\underline{s}$ and $\bar{s}$ numerically, we set them exogenously to $\underline{s} = 0.1$ and $\bar{s} = 0.9$ and then recover the shares of labor incomes $l_A = 0.1257$ and $l_B = 0.1178$, which imply these bounds in equilibrium. First, we find $\underline{v}$ and $\bar{v}$ from Eq. (14) for state variable v and then find l_A and l_B from Eq. (41) for the boundaries.

[5] Section IA.2 of the Internet Appendix presents two additional examples of equilibria. The first example shows the equilibrium processes in the economy where investors have different risk aversions but identical beliefs. The second example shows these processes when both risk aversions and beliefs are different, and the less risk-averse investor is also more pessimistic than the more risk-averse investor. The equilibrium processes in these economies have the same features as the processes in our baseline calibration in Section 4.1. In particular, interest rates are lower and Sharpe ratios are higher during anxious times, the price dividend ratios are U-shaped and sensitive to small shocks near the boundaries $\underline{s}$ and $\bar{s}$, and stock return volatilities are higher in good times and lower in bad times.

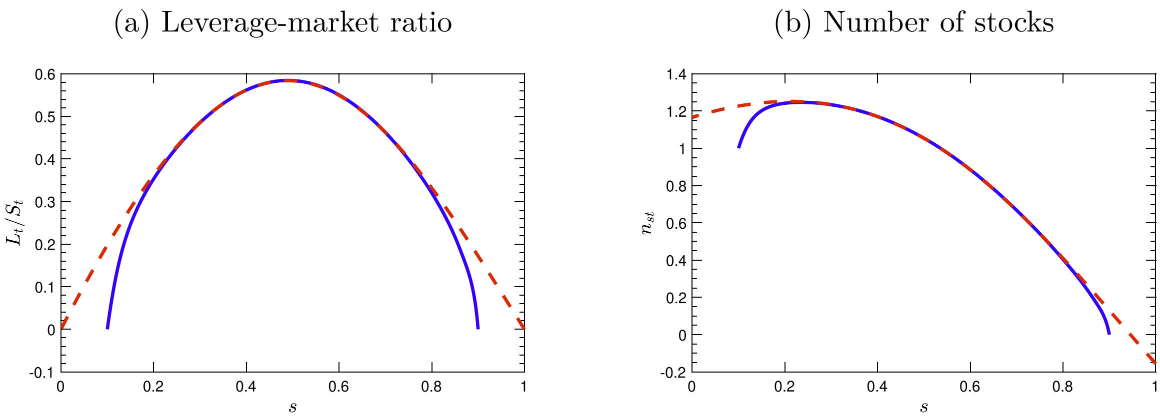

Fig. 3. Leverage and stock holdings of optimistic and less risk-averse investor B. Panels (a) and (b) depict optimistic investor B's leverage/market price ratio L_t/S_t and the number of shares n_{st}, respectively, as functions of consumption share $s_t = c^*_{At}/D_t$. The solid and dashed lines correspond to constrained and unconstrained economies, respectively. The model parameters are $\gamma_A = 2$, $\gamma_B = 2$, $\rho = 0.02$, $\mu_D = 0.018$, $\sigma_D = 0.032$, $J_D = -0.25$, $\lambda_A = 0.02$, $\lambda_B = 0.01$, $\delta_A = -0.05$, $\delta_B = 0.05$, $\underline{s} = 0.1$, $\bar{s} = 0.9$, $l_A = 0.1257$, and $l_B = 0.1178$.

est when either investor A or investor B bind on their constraints. Intuitively, when $s = \bar{s}$, investor B's financial wealth is zero, and hence B lacks collateral and cannot borrow. When $s = \underline{s}$, investor A's financial wealth is zero and the labor income $l_A D_t \Delta t$ is infinitesimally small in the continuous-time limit. The liquidity dries up because investor A cannot supply credit. The leverage cycles are present only in the constrained economy and do not occur in the unconstrained economy where the state variable s is nonstationary and gradually converges to zero or one.

Fig. 3b presents the number of stocks held by investor B. Consider first the unconstrained economy with pledgeable labor income. From Fig. 3b, we observe that the optimistic investor B shorts stocks in the unconstrained economy when consumption share s is close to one. The intuition is that in bad times, following a sequence of negative shocks to output, investor B shorts stocks to finance consumption and backs short positions by the pledgeable labor income. The stream of labor income $l_B D_t \Delta t$ is equivalent to dividends from holding $\widehat{n}_B = l_B/(1 - l_A - l_B)$ units of nontradable shares in the Lucas tree. Short-selling allows the investors to circumvent the nontradability of labor income and to freely adjust the effective share $\hat{n}_B + n_{B,St}$ in the Lucas tree. Overcoming the nontradability of labor incomes makes this economy similar to the nonstationary unconstrained economy. The financial wealth can then become negative. In the constrained economy, the nonnegative wealth constraint precludes investor B from shorting. The trading strategy of investor A equals $1 - n^*_{Bt}$ in equilibrium and can be analyzed similarly. Investor A also has an additional motive to short stocks due to being more pessimistic than investor B.

Fig. 4 depicts the interest rate r_t, Sharpe ratio $(\mu_t - r_t)/\sigma_t$, P/D ratio Ψ, and excess stock return volatility $(\sigma_t - \sigma_D)/\sigma_D$ in the constrained (solid line) and unconstrained (dashed line) economies. Fig. 4a shows the interest rate r_t.[6]

The interest rate declines sharply when the economy enters into an anxious state close to the boundary $\bar{s}$ where even a small possibility of a crisis next period makes the constraint of investor B binding. The intuition is as follows. In the unconstrained economy, a crisis around state $\bar{s}$ generates wealth transfer to the pessimistic investor A and increases her consumption share s above $\bar{s}$. In the constrained economy, consumption share s is capped by $\bar{s}$. Consequently, following a crisis, investor A's marginal utility $(c^*_A)^{-\gamma_A}$ is higher in the constrained than in the unconstrained economy. As a result, investor A is more willing to smooth consumption in the constrained economy, and hence the interest rate declines due to the precautionary savings motive. In particular, the investor buys more bonds, which drives interest rates down. Fig. 4b shows that the Sharpe ratio increases to compensate investor A for buying risky assets from investor B in bad times when consumption share s is large and slightly decreases in good times due to high volatility (as discussed below). We note that the decreases in interest rates and increases in Sharpe ratios during anxious times arise only when both the crises and the constraints are simultaneously present, and hence the crises and constraints reinforce the effects of each other. Eq. (44) for the interest rate and Eq. (45) for the risk premium show that absent any crises ($\lambda_A = \lambda_B = 0$), the constraints affect r_t and $\mu_t - r_t$ only at the boundaries of the state space.

From Fig. 4c, we observe that the collateral constraints increase P/D ratio Ψ relative to the unconstrained economy, in line with Proposition 2. The P/D ratio is also a decreasing (increasing) function of consumption share s near the boundary $\underline{s}$ ($\bar{s}$), which makes it U-shaped and sensitive to small shocks when the constraints are likely to bind. Proposition IA.5 in the Internet Appendix shows that the latter property holds for all model parameters under which the equilibrium exists. We further note that P/D ratio Ψ in Fig. 4c is a convex function of consumption share s, which reinforces its resemblance to a U-shape. Proposition IA.5 also establishes sufficient conditions for the convexity of the P/D ratio in the unconstrained economy Ψ^{unc}

[6] We exclude the singularities in the dynamics of r_t and focus on the dynamics in the unconstrained region because the economy spends an infinitesimal amount of time at the boundaries.

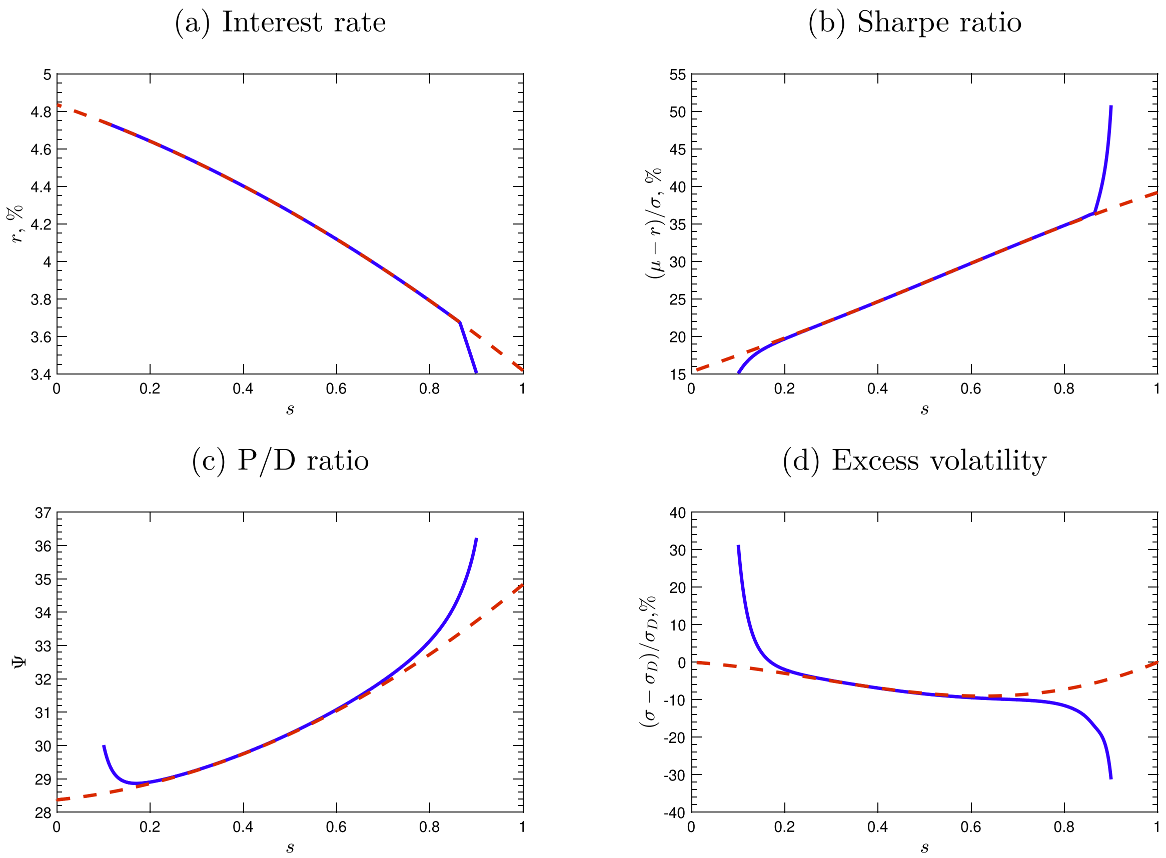

Fig. 4. Equilibrium processes. Panels (a)–(d) show interest rate r_t, Sharpe ratio $(\mu_t - r_t)/\sigma_t$, price-dividend ratio Ψ_t, and excess volatility $(\sigma_t - \sigma_D)/\sigma_D$ as functions of $s_t = c^*_{At}/D_t$ for the constrained (solid lines) and unconstrained (dashed lines) economies. The model parameters are the same as in Fig. 3.

and the difference $\Psi - \Psi^{unc}$. Ratio Ψ is then convex as the sum of the latter two convex components. In particular, Proposition IA.5 shows that ratio Ψ^{unc} is convex when $\gamma_A = \gamma_B \geq 1$, and the difference $\Psi - \Psi^{unc}$ is convex under an additional restriction on crises intensities $\lambda_A = \lambda_B > 0$ (or when $\lambda_A = \lambda_B = 0$ and $\gamma_A \geq \gamma_B \geq 1$).[7] We note that the latter conditions are satisfied for the analytical P/D ratio (42) in Corollary 1.

Next, we discuss the intuition for the fact that P/D ratios in the constrained economy increase more at the boundaries than in the interior of the interval $(\underline{s}, \bar{s})$ and are decreasing (increasing) functions of consumption share s near $\underline{s}$ ($\bar{s}$). Suppose consumption share s is close to the boundary $\bar{s}$, where investor B's constraint is binding but investor A is unconstrained. Because investor A's constraint is loose, the SPD ξ_{At} is proportional to investor A's marginal utility $(c^*_{At})^{-\gamma_A}$. In the constrained economy the consumption share of investor A is capped at $\bar{s} < 1$, whereas in the unconstrained economy it can increase above $\bar{s}$. Therefore, the marginal utility of investor A, and hence the SPD are expected to be higher in the constrained than in the unconstrained economy. Hence, stocks are more valuable in the constrained economy around the boundary $\bar{s}$. The intuition around $\underline{s}$ can be analyzed in a similar way. An additional economic force (explored in Section 4.2 below) contributing to a higher stock price is that the stock can be used as collateral that helps relax the constraint, which gives rise to a premium.

The results in Fig. 4d demonstrate that the constraint makes volatility more procyclical, reducing it in bad times (around $\bar{s}$) and increasing it in good times (around $\underline{s}$). This is because U-shaped P/D ratio in the constrained economy is more procyclical in good times (i.e., around $\underline{s}$) and more countercyclical in bad times (i.e., around $\bar{s}$) than in the unconstrained economy. Stock price $S_t = \Psi_t D_t$ is more volatile in good times (around $\underline{s}$) because both Ψ_t and D_t change in the same direction and is less volatile in bad times (around $\bar{s}$) because Ψ_t and D_t change in opposite directions and partially offset the effects of each other. Lower volatility in bad times is in line with the previous literature on the effects of portfolio constraints on asset prices (e.g., Chabakauri, 2013, 2015; Brunnermeier and Sannikov, 2014, among others). The empirical literature finds that volatility tends to be higher in bad times (e.g., Schwert, 1989). However, high volatility can be explained by high uncertainty about the economic growth and learning effects in bad times (e.g., Veronesi, 1999), which we do not study in

[7] A nonconvex ratio Ψ emerges, for example, when $\gamma_A = \gamma_B < 1$ and $\underline{s} = 0.01$ and $\bar{s} = 0.99$ because the corresponding unconstrained P/D ratio Ψ^{unc} is concave, and ratio Ψ closely follows its contour inside the interval $(\underline{s}, \bar{s})$ and then spikes up at the boundaries, giving rise to nonconvexities.

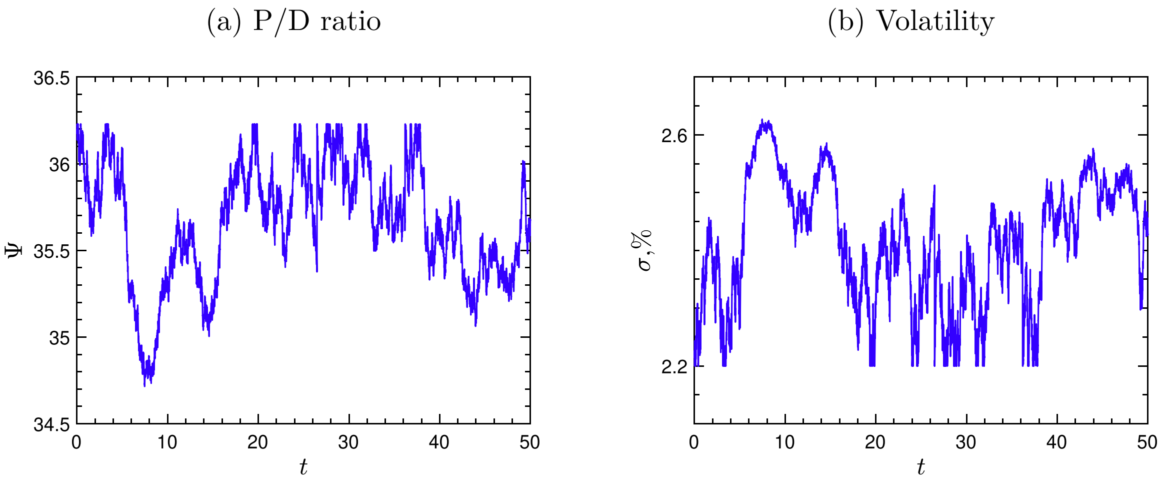

Fig. 5. Simulated P/D ratio Ψ and stock return volatility σ over time. Panels (a) and (b) show the spikes and crashes of simulated P/D ratio and volatility σ, and clustering of volatility σ over the period of 50 years. The model parameters are the same as in Fig. 3.

this paper to focus on the effects of collateral constraints that are not confounded by other effects.

Boundary conditions (37) allow us to explore volatility σ_t near the boundaries $\underline{s}$ and $\bar{s}$ using closed-form expressions in Corollary 2 below.

Corollary 2 (Stock return volatility at the boundaries). *Stock return volatility in normal times σ_t satisfies the following boundary conditions:*

$$\sigma(\underline{s}) = \sigma_D + \frac{\gamma_B \underline{s}\widehat{\sigma}_\nu}{\gamma_A(1-\underline{s}) + \gamma_B \underline{s}} > \sigma_D,$$
$$\sigma(\bar{s}) = \sigma_D - \frac{\gamma_A(1-\bar{s})\widehat{\sigma}_\nu}{\gamma_A(1-\bar{s}) + \gamma_B \bar{s}} < \sigma_D. \tag{49}$$

By continuity, inequalities (49) also hold in the vicinity of the boundaries. Fig. 4d shows that volatility σ_t is steep at the boundaries: it spikes close to $\underline{s}$ and crashes close to $\bar{s}$, consistent with Corollary 2. It also evolves in three regimes of low, medium, and high volatility, which resembles volatility clustering documented in the empirical literature (e.g., Bollerslev, 1987). The distribution of consumption share s in Fig. 2 implies that the economy persists in these clusters for some time.

Fig. 5 plots the simulated dynamics of the P/D ratio and stock return volatility over a period of 50 years. Consistent with our discussion above, the dynamics of the P/D ratio in Fig. 4a exhibits intervals of booms and busts around the times when the collateral constraints become binding. These intervals resemble periods of inflating and deflating bubbles in the economy. The volatility σ in Fig. 4b evolves in clusters of high and low volatility, as explained above.

4.2. Collateral liquidity premium

In this section, we measure the liquidity premium of stocks over labor income arising because stocks can be used as collateral. We consider a marginal representative investor i that does not affect asset prices and characterize this investor's shadow indifference price $\widehat{S}_{it}$ of labor income. We define $\widehat{S}_{it}$ as the price such that exchanging marginal Δl_i claims to the stream of labor income for $\widehat{S}_{it}\Delta l_i$ units of wealth leaves the investor's utility unchanged. Consider the investor i's value function $V_i(W_{it}, \nu_t; l_i)$. Price $\widehat{S}_{it}$ is the solution of equation $V_i(W^*_{it}, \nu_t; l_i) = V_i(W^*_{it} + \widehat{S}_{it}\Delta l_i, \nu_t; l_i - \Delta l_i)$. In the limit $\Delta l_i \to 0$, we find that

$$\widehat{S}_{it} = \frac{\partial V_i(W^*_{it}, \nu_t; l_i)/\partial l_i}{\partial V_i(W^*_{it}, \nu_t; l_i)/\partial W_{it}}. \tag{50}$$

The definition of shadow indifference price $\widehat{S}_{it}$ comes from the literature on the valuation of derivative securities in incomplete markets (e.g., Davis, 1997).

The labor incomes $l_i D_t \Delta t$ are proportional to dividends $(1 - l_A - l_B)D_t\Delta t$. Therefore, if claims on labor incomes were tradable and pledgeable, shadow price $\widehat{S}_{it}$ would have been equal to $S_t/(1 - l_A - l_B)$. However, labor incomes are nontradable and nonpledgeable. Hence, from the view of investor i, the stock enjoys a liquidity premium, defined as

$$\Lambda_{it} = \frac{S_t/(1 - l_A - l_B) - \widehat{S}_{it}}{S_t/(1 - l_A - l_B)}. \tag{51}$$

We find derivatives in Eq. (50) using the envelope theorem. Then, we derive prices $\widehat{S}_{it}$ and show that premiums (51) are positive and large. Proposition 5 reports our results.

Proposition 5 (Shadow prices and the liquidity premium). In the limit $\Delta t \to 0$, investor i's shadow price of a unit of labor income is given by

$$\widehat{S}_{it} = \widehat{\Psi}_i(\nu; -\gamma_A) s(\nu)^{\gamma_A} D_t, \quad i = A, B, \tag{52}$$

where $\widehat{\Psi}_i(\nu;\theta)$ satisfies differential difference Eq. (36) subject to the following boundary conditions for investors A and B

$$\widehat{\Psi}'_A(\underline{\nu};\theta) = 0, \quad \widehat{\Psi}'_A(\bar{\nu};\theta) = 0, \tag{53}$$

$$\widehat{\Psi}'_B(\underline{\nu};\theta) = \widehat{\Psi}_B(\underline{\nu};\theta), \quad \widehat{\Psi}'_B(\bar{\nu};\theta) = \widehat{\Psi}_B(\bar{\nu};\theta). \tag{54}$$

The investors' liquidity premiums for stocks Λ_A and Λ_B are positive, and hence

$$S_t/(1 - l_A - l_B) > \widehat{S}_{At}, \quad S_t/(1 - l_A - l_B) > \widehat{S}_{Bt}. \tag{55}$$

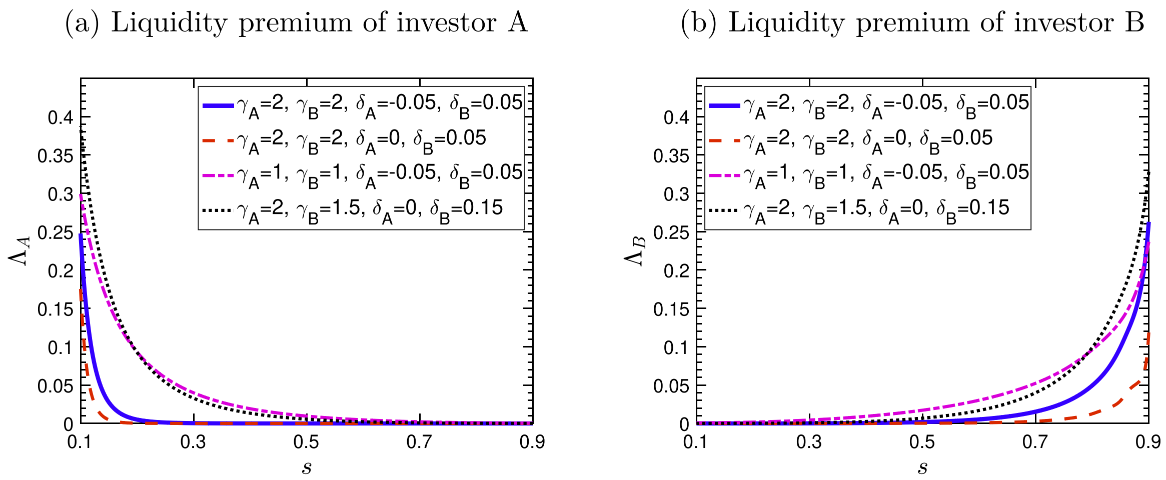

Fig. 6. Collateral liquidity premiums from the view of investors *A* and *B*. The figure shows the collateral liquidity premiums (51) of stocks over nonpledgeable labor incomes from the view of investors *A* and *B* for different sets of risk aversions and beliefs. All other model parameters are the same as in Fig. 3.

The premium $\Lambda_{it} > 0$ arises because the stock can be used as collateral, whereas the labor income cannot. We note that this premium is zero in the frictionless economy, and hence the nontradability of labor income and the possibility of shorting stocks do not contribute to the premium. This is because, as discussed in Section 4.1, in an unconstrained economy with fully pledgeable labor income the investors can circumvent the nontradability of labor income by shorting stocks. We further remark that the shadow prices and liquidity premiums can be found in closed form, similar to stock prices in Section 3, but we do not present them for brevity.

Fig. 6a and b plot the liquidity premiums (51) for investors *A* and *B*, respectively, for different values of risk aversions and beliefs, with other model parameters being the same as in Section 4.1. We observe that investors *A* and *B* have different valuations of their labor incomes due to differences in preferences and beliefs. Their premiums Λ_i are close to zero when the investors are far away from the boundaries where their respective constraints become binding. The premiums increase up to 40%, close to the boundaries where the stock is more valuable for the purposes of relaxing the constraints. Large premiums Λ_{it} imply the economic significance of stock pledgeability. The premiums are smallest for the case $\gamma_A = \gamma_B = 2$, $\delta_A = 0$, and $\delta_B = 0.05$ because the economy in this case is closer to the homogeneous-agent economies than in the other cases depicted in Fig. 6.

5. Conclusion

We develop a parsimonious and tractable theory of asset pricing under collateral constraints. We show that requiring investors to collateralize their trades gives rise to rich dynamics of asset prices and their moments. The constraints lead to booms and busts in stock prices; cause spikes, crashes, and clustering of volatilities; amplify volatilities in good states and dampen them in bad states; decrease interest rates and increase Sharpe ratios when optimistic investors are close to default boundaries; and induce cycles of high and low leverage. The tractability of our model allows us to obtain asset prices and the distributions of consumption shares in closed form.

Appendix A. Proofs

Lemma A.1 (Change of variable). Let $\widehat{n}_i = k_i l_i/(1 - l_A - l_B)$. The maximization of expected discounted utility (6) subject to budget constraints (7) and (8) and constraint (11) is equivalent to maximizing (6) with respect to c_{it}, b_{it}, and $\widetilde{n}_{it}$ subject to the following set of constraints:

$$\widetilde{W}_{it} + l_i D_t \Delta t = c_{it}\Delta t + b_{it} B_t + \widetilde{n}_{it}(S_t, P_t)^\top, \tag{A.1}$$

$$\widetilde{W}_{i,t+\Delta t} = b_{it} + \widetilde{n}_{it} \times (S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t}\Delta t,\ \mathbf{1}_{\{\omega_{t+\Delta t} = \omega_3\}})^\top, \tag{A.2}$$

$$\widetilde{W}_{i,t+\Delta t} \geq 0, \tag{A.3}$$

where $\widetilde{W}_{it} = W_{it} + \widehat{n}_i S_t$ and $\widetilde{W}_{i,t+\Delta t} = W_{i,t+\Delta t} + \widehat{n}_i(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t})$.

Proof of Lemma A.1. Substituting $n_{it} = \widetilde{n}_{it} - (\widehat{n}_i, 0)$ into (7) and (8), we obtain constraints (A.1) and (A.2). Rewriting constraint (11) in terms of variable $\widetilde{W}_{i,t+\Delta t}$, we obtain (A.3). Finally, we note that $\widetilde{W}_{it} = W_{it} + \widehat{n}_i S_t$ is worth $\widetilde{W}_{i,t+\Delta t}$ next period. Hence, (A.1) and (A.2) can be seen as self-financing budget constraints. ■

Lemma A.2 (Concavity of the value function).

(1) *Let $V_i(W_{it}, v_t;\, l_i)$ denote the value function of investor i, where v_t is the state variable. Then, the value function solves the following equation of dynamic programming:*

$$V_i(W_{it}, v_t; l_i) = \max_{c_{it}}\{u_i(c_{it})\Delta t + e^{-\rho\Delta t}\mathbb{E}_t^i[V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)]\}, \tag{A.4}$$

subject to the static budget and collateral constraints:

$$W_{it} + l_i D_t \Delta t = c_{it}\Delta t + \mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t}\right], \quad (A.5)$$

$$W_{i,t+\Delta t} \geq 0. \quad (A.6)$$

(2) *Value function $V_i(W_{it}, \upsilon_t;\, l_i)$ is a concave function of wealth W_{it}.*

Proof of Lemma A.2.

(1) We start by demonstrating the equivalence of the dynamic and static budget constraints (7)-(8) and (A.5), respectively. Multiplying Eq. (8) by $\xi_{i,t+\Delta t}/\xi_{it}$, taking expectation operator $\mathbb{E}_t^i[\cdot]$ on both sides, and using Eq. (25)–(27) for asset prices, we obtain

$$\mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t}\right] = b_{it}B_t + n_{it}(S_t, P_t)^\top. \quad (A.7)$$

From the budget constraint (7), we observe that the right-hand side of (A.7) equals $W_{it} + l_i D_t \Delta t$, and hence we obtain the static budget constraint (A.5). Conversely, if there exists $W_{i,t+\Delta t}$ satisfying constraints (A.5) and (A.6), there exist trading strategies b_{it} and n_{it} that replicate $W_{i,t+\Delta t}$ because the underlying market is effectively complete (i.e., the payoff matrix is invertible). Finally, the dynamic programming Eq. (A.4) is obtained by rewriting the optimization problem (6) in a recursive form.

(2) Consider wealth levels W_{it} and $\widehat{W}_{it}$. Let $\{c^*_{it}, b^*_{it}, n^*_{it}\}$ and $\{\widehat{c}^*_{it}, \widehat{b}^*_{it}, \widehat{n}^*_{it}\}$ be optimal consumptions and portfolios that correspond to W_{it} and $\widehat{W}_{it}$, respectively, and satisfy constraints (7)–(9). For any $\alpha \in [0, 1]$, policies $\{\alpha\widehat{c}^*_{it} + (1-\alpha)c^*_{it}, \alpha\widehat{b}^*_{it} + (1-\alpha)b^*_{it}, \alpha\widehat{n}^*_{it} + (1-\alpha)n^*_{it}\}$ are admissible for wealth $\alpha W_{it} + (1-\alpha)\widehat{W}_{it}$. By concavity of CRRA utilities

$$\begin{aligned} V_i(\alpha W_{it} &+ (1-\alpha)\widehat{W}_{it}, \upsilon_t; l_i) \\ &\geq \sum_{\tau=t}^{\infty} u_i(\alpha\widehat{c}^*_{it} + (1-\alpha)c^*_{it}) \\ &\geq \sum_{\tau=t}^{\infty}\Big(\alpha u_i(\widehat{c}^*_{it}) + (1-\alpha)u_i(c^*_{it})\Big) \\ &= \alpha V_i(W_{it}, \upsilon_t; l_i) + (1-\alpha)V_i(\widehat{W}_{it}, \upsilon_t; l_i). \end{aligned} \quad (A.8)$$

Therefore, $V_i(W_{it}, \upsilon_t;\, l_i)$ is a concave function of wealth. ■

Proof of Lemma 1. Consider the Lagrangian for the optimization problem (A.4) subject to constraints (A.5) and (A.6):

$$\begin{aligned} \mathscr{L} &= u_i(c_{it})\Delta t + e^{-\rho\Delta t}\mathbb{E}_t^i\Bigg[V_i(W_{i,t+\Delta t}, \upsilon_{t+\Delta t}; l_i)\Bigg] \\ &\quad + \eta_{it}\left(W_{it} + l_i D_t\Delta t - c_{it}\Delta t - \mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t}\right]\right) \\ &\quad + \mathbb{E}_t^i\left[e^{-\rho\Delta t}\ell_{i,t+\Delta t}W_{i,t+\Delta t}\right)\Bigg], \end{aligned} \quad (A.9)$$

where the multiplier $\ell_{i,t+\Delta t} \geq 0$ satisfies the complementary slackness condition $\ell_{i,t+\Delta t}W_{i,t+\Delta t} = 0$. Differentiating the Lagrangian (A.9) with respect to c_{it} and $W_{i,t+\Delta t}$, we obtain

$$u_i'(c^*_{it}) = \eta_{it}, \quad (A.10)$$

$$e^{-\rho\Delta t}\left(\frac{\partial V_i(W_{t+\Delta t}, \upsilon_{t+\Delta t}; l_i)}{\partial W} + \ell_{i,t+\Delta t}\right) = \eta_{it}\frac{\xi_{i,t+\Delta t}}{\xi_{it}}. \quad (A.11)$$

By the envelope theorem (e.g, Back, 2010, p. 162),

$$\frac{\partial V_i(W_{i,t+\Delta t}, \upsilon_{t+\Delta t}; l_i)}{\partial W} = u_i'(c^*_{i,t+\Delta t}). \quad (A.12)$$

Substituting the partial derivative of the value function (A.12) and the marginal utility (A.10) into Eq. (A.11), and then dividing both sides of the equation by $u_i'(c^*_{it})$, we obtain the SPD (29). ■

Proof of Proposition 1.

Step 1. Consider the case in which constraints do not bind, and hence $\ell_{i,t+\Delta t} = 0$. Then, using Eq. (13) for the state variable υ_t and the first order conditions (29), we obtain

$$\begin{aligned} \upsilon_{t+\Delta t} - \upsilon_t &= \ln\left(\frac{(c^*_{A,t+\Delta t}/c^*_{At})^{-\gamma_A}}{(c^*_{B,t+\Delta t}/c^*_{Bt})^{-\gamma_B}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) \\ &= \ln\left(\frac{\xi_{A,t+\Delta t}/\xi_{At}}{\xi_{B,t+\Delta t}/\xi_{Bt}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right). \end{aligned} \quad (A.13)$$

From the above equation and the change of measure Eq. (28), which relates SPDs $\xi_{A,t+\Delta t}$ and $\xi_{B,t+\Delta t}$, we obtain the dynamics of υ_t when constraints do not bind:

$$\upsilon_{t+\Delta t} - \upsilon_t = \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right). \quad (A.14)$$

Let $\overline{\upsilon}$ and $\underline{\upsilon}$ be the boundaries satisfying Eq. (30), at which the constraints of investors A and B bind, respectively. Let investor A's constraint be binding so that $\upsilon_{t+\Delta t} = \overline{\upsilon}$, and hence $\ell_{A,t+\Delta t} \geq 0$. Using Eq. (13) for υ_t, first order conditions (29), and $\ell_{A,t+\Delta t} \geq 0$, we obtain

$$\begin{aligned} \overline{\upsilon} - \upsilon_t &\leq \ln\left(\frac{((c^*_{A,t+\Delta t})^{-\gamma_A} + \ell_{A,t+\Delta t})/(c^*_{At})^{-\gamma_A}}{(c^*_{B,t+\Delta t}/c^*_{Bt})^{-\gamma_B}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) \\ &= \ln\left(\frac{\xi_{A,t+\Delta t}/\xi_{At}}{\xi_{B,t+\Delta t}/\xi_{Bt}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) \\ &= \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right). \end{aligned} \quad (A.15)$$

Similarly, for $\upsilon_{t+\Delta t} = \underline{\upsilon}$, we find that $\underline{\upsilon} - \upsilon_t \geq \ln(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})(D_{t+\Delta t}/D_t)^{\gamma_A-\gamma_B})$. The latter two inequalities imply that when the constraint binds, $\upsilon_{t+\Delta t}$ is given by

$$\upsilon_{t+\Delta t} = \max\left\{\underline{\upsilon};\, \min\left\{\overline{\upsilon};\, \upsilon_t + \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right)\right\}\right\}. \quad (A.16)$$

We observe that (A.16) is also satisfied in the unconstrained case in which $\underline{\upsilon} < \upsilon_{t+\Delta t} < \overline{\upsilon}$. It remains to prove that υ_t does not escape $[\underline{\upsilon}, \overline{\upsilon}]$ interval. Consider a marginal investor of type A. We conjecture that υ_t follows dynamics

(A.16) and verify that the consumption choice of investor A indeed implies this dynamics. The analysis for investor B is similar.

We have shown above that υ_t satisfies inequality (A.15) when investor A is constrained. Now, we show the opposite: investor A is constrained when υ_t satisfies (A.15). Hence, $\upsilon_{t+\Delta t}$ cannot exceed $\overline{\upsilon}$. Consider υ_t such that $\upsilon_t + \ln\left(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})(D_{t+\Delta t}/D_t)^{\gamma_A-\gamma_B}\right) > \overline{\upsilon}$ for some $\omega_{t+\Delta t}$ and $\upsilon_t \in (\underline{\upsilon}, \overline{\upsilon})$. Because $\underline{\upsilon} < \upsilon_t < \overline{\upsilon}$, investor A consumes $c^*_{At} = s(\upsilon_t)D_t$, as shown above. We show that the constraint of investor A binds and $c^*_{A,t+\Delta t} = s(\overline{\upsilon})D_{t+\Delta t}$. This consumption level confirms that $\upsilon_{t+\Delta t} = \overline{\upsilon}$ is indeed an equilibrium outcome.

Consider the constraint of investor A at date t in the state $\omega_{t+\Delta t}$ where $\upsilon_{t+\Delta t} = \overline{\upsilon}$:

$$W_{A,t+\Delta t} \geq 0 \equiv \Phi_A(\overline{\upsilon})s(\overline{\upsilon})D_{t+\Delta t}, \tag{A.17}$$

where the last equality holds by the definition of $\overline{\upsilon}$. Using the concavity of the value function, proven in Lemma 1, and condition (A.12) from the envelope theorem, we obtain

$$\begin{aligned} u'_A(c^*_{A,t+\Delta t}) &= \frac{\partial V_A(W_{A,t+\Delta t}, \overline{\upsilon}; l_A)}{\partial W} \\ &\leq \frac{\partial V_A(\Phi_A(\overline{\upsilon})s(\overline{\upsilon})D_{t+\Delta t}, \overline{\upsilon}; l_A)}{\partial W} = u'_A(s(\overline{\upsilon})D_{t+\Delta t}). \end{aligned} \tag{A.18}$$

Because $u'_i(c)$ is a decreasing function, we find that $c^*_{A,t+\Delta t}/D_{t+\Delta t} \geq s(\overline{\upsilon})$.

Investor B is unconstrained when $\upsilon_{t+\Delta t} = \overline{\upsilon}$ and hence has SPD

$$\frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} = e^{-\rho\Delta t}\left(\frac{c^*_{B,t+\Delta t}}{c^*_{Bt}}\right)^{-\gamma_B} = e^{-\rho\Delta t}\left(\frac{(1-s(\overline{\upsilon}))D_{t+\Delta t}}{(1-s(\upsilon_t))D_t}\right)^{-\gamma_B}. \tag{A.19}$$

From the change of measure Eq. (28) and the FOC (29), the SPD of investor A is

$$\begin{aligned} \frac{\xi_{A,t+\Delta t}}{\xi_{At}} &= \frac{\xi_{B,t+\Delta t}}{\xi_{Bt}}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \\ &= e^{-\rho\Delta t}\frac{(c^*_{A,t+\Delta t})^{-\gamma_A} + \ell_{A,t+\Delta t}}{(c^*_{At})^{-\gamma_A}}. \end{aligned} \tag{A.20}$$

From (A.20) and (A.19), we find the Lagrange multiplier:

$$\begin{aligned} \frac{l_{A,t+\Delta t}}{(c^*_{A,t+\Delta t})^{-\gamma_A}} &= \left(\frac{c^*_{A,t+\Delta t}}{c^*_{At}}\right)^{\gamma_A}\left(\frac{(1-s(\overline{\upsilon}))D_{t+\Delta t}}{(1-s(\upsilon_t))D_t}\right)^{-\gamma_B}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} - 1 \\ &\geq \left(\frac{s(\overline{\upsilon})D_{t+\Delta t}}{s(\upsilon_t)D_t}\right)^{\gamma_A}\left(\frac{(1-s(\overline{\upsilon}))D_{t+\Delta t}}{(1-s(\upsilon_t))D_t}\right)^{-\gamma_B}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} - 1 \\ &= \left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right)e^{\upsilon_t-\overline{\upsilon}} - 1 > 0. \end{aligned} \tag{A.21}$$

The first inequality follows from the fact that $c^*_{A,t+\Delta t} \geq s(\overline{\upsilon})D_{t+\Delta t}$, which we proved above. The second equality holds by the definition of state variable (13). The second inequality comes from the assumption that $\upsilon_t + \ln\left(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})(D_{t+\Delta t}/D_t)^{\gamma_A-\gamma_B}\right) > \overline{\upsilon}$. Hence, the Lagrange multiplier $l_{A,t+\Delta t}$ is strictly positive. From the complementary slackness condition, the constraint (A.17) must be binding. Therefore, inequality (A.18) becomes an equality, and hence $c^*_{A,t+\Delta t} = s(\overline{\upsilon})D_{t+\Delta t}$.

Step 2. We now look for coefficients μ_υ, σ_υ and J_υ such that

$$\begin{aligned} \mu_\upsilon\Delta t + \sigma_\upsilon\Delta w_t + J_\upsilon\Delta j_t &= \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) \\ &= \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\right) \\ &\quad + (\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t + \sigma_D\Delta w_t + J_\upsilon\Delta j_t). \end{aligned} \tag{A.22}$$

We write identity (A.22) in each of the states $\omega_{t+\Delta t} \in \{\omega_1, \omega_2, \omega_3\}$ and obtain a system of three linear equations with three unknowns μ_υ, σ_υ and J_υ. Solving this system, we find

$$\mu_\upsilon = \frac{1}{2\Delta t}\left((\gamma_A-\gamma_B)\ln[(1+\mu_D\Delta t)^2 - \sigma_D^2\Delta t] + \ln\left(\frac{1-\lambda_B\Delta t}{1-\lambda_A\Delta t}\right)^2 + \ln\left(\frac{1-\delta_B^2\Delta t}{1-\delta_A^2\Delta t}\right)\right), \tag{A.23}$$

$$\sigma_\upsilon = \frac{1}{2\sqrt{\Delta t}}\left((\gamma_A-\gamma_B)\ln\left(\frac{1+\mu_D\Delta t + \sigma_D\sqrt{\Delta t}}{1+\mu_D\Delta t - \sigma_D\sqrt{\Delta t}}\right) + \ln\left(\frac{(1+\delta_B\sqrt{\Delta t})(1-\delta_A\sqrt{\Delta t})}{(1-\delta_B\sqrt{\Delta t})(1+\delta_A\sqrt{\Delta t})}\right)\right), \tag{A.24}$$

$$J_\upsilon = (\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t + J_D) + \ln\left(\frac{\lambda_B}{\lambda_A}\right) - \mu_\upsilon\Delta t. \tag{A.25}$$

Step 3. Finally, we show that the boundaries are reflecting for a sufficiently small Δt. Assume that $\sigma_\upsilon > 0$; the case $\sigma_\upsilon < 0$ is considered analogously. Then, for a sufficiently small Δt, we observe that $\mu_\upsilon\Delta t - \sigma_\upsilon\sqrt{\Delta t} < 0$ and $\mu_\upsilon\Delta t + \sigma_\upsilon\sqrt{\Delta t} > 0$ because $\sqrt{\Delta t}$-terms dominate Δt-terms, and as $\Delta t \to 0$, $\mu_\upsilon \to \widehat{\mu}_\upsilon$ and $\sigma_\upsilon \to \widehat{\sigma}_\upsilon$, where $\widehat{\mu}_\upsilon$ and $\widehat{\sigma}_\upsilon$ are given by Eqs. (38) and (39), respectively. Moreover, $\widehat{\sigma}_\upsilon > 0$ because $\gamma_A \geq \gamma_B$ and $\delta_B \geq \delta_A$, and at least one of the latter inequalities is strict. Then, the boundaries are reflecting because (1) if $\upsilon_t = \overline{\upsilon}$, then $\upsilon_{t+\Delta t} = \overline{\upsilon} + \mu_\upsilon\Delta t - \sigma_\upsilon\sqrt{\Delta t} < \overline{\upsilon}$ with positive probability, and (2) if $\upsilon_t = \underline{\upsilon}$, then $\upsilon_{t+\Delta t} = \underline{\upsilon} + \mu_\upsilon\Delta t + \sigma_\upsilon\sqrt{\Delta t} > \underline{\upsilon}$ with positive probability. ■

Lemma A.3 (Wealth-consumption ratios). *The investors' wealth-consumption ratios Φ_i are uniformly bounded and given by*

$$\Phi_A(\upsilon_t) = \mathbb{E}^A_t\left[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A} \times \left(\frac{s(\upsilon_\tau)}{s(\upsilon_t)}\right)^{1-\gamma_A}\left(1-\frac{l_A}{s(\upsilon_\tau)}\right)\Delta t\right], \tag{A.26}$$

$$\Phi_B(\upsilon_t) = \mathbb{E}^B_t\left[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_B} \times \left(\frac{1-s(\upsilon_\tau)}{1-s(\upsilon_t)}\right)^{1-\gamma_B}\left(1-\frac{l_B}{1-s(\upsilon_\tau)}\right)\Delta t\right] \tag{A.27}$$

Proof of Lemma A.3. Substituting FOC (29) into the budget constraint (A.5) and using the complementary slackness condition $\ell_{i,t+\Delta t} W^*_{i,t+\Delta t} = 0$, we obtain

$$W^*_{At} = \mathbb{E}^A_t\left[e^{-\rho\Delta t}\left(\frac{c^*_{A,t+\Delta t}}{c^*_{At}}\right)^{-\gamma_A} W^*_{A,t+\Delta t}\right] + (c^*_{At} - l_A D_t)\Delta t. \tag{A.28}$$

Substituting $W^*_{At} = \Phi_{At} c^*_{At}$ and $c^*_{At} = s(\upsilon_t)D_t$ into Eq. (A.28) and iterating, we obtain Eq. (A.26). Let $\bar{s} = s(\underline{\upsilon})$ $\underline{s} = s(\bar{\upsilon})$, where $s(\upsilon)$ is given by Eq. (14). Then, $\bar{s} \geq s \geq \underline{s} > 0$. Using the bounds on s_t, we obtain the following uniform bound on Φ_A:

$$\Phi_A(\upsilon_t) \leq Const \times \mathbb{E}^A_t\left[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A}\Delta t\right]. \tag{A.29}$$

The series on the right-hand side of the latter inequality is convergent due to condition (15) on model parameters. Eq. (A.27) is obtained along the same lines. ■

Proof of Proposition 2.

1) First, we derive the SPD ξ_{At} under the correct beliefs of investor A. When investor A's constraint does not bind, substituting $c^*_{At} = s(\upsilon_t)D_t$ into the first order condition (29), we find that

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t}\left(\frac{s(\upsilon_{t+\Delta t})}{s(\upsilon_t)}\right)^{-\gamma_A}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}. \tag{A.30}$$

Eq. (A.30) is consistent with SPD (32) because when the constraint does not bind, $\upsilon_{t+\Delta t} = \upsilon_t + \mu_\upsilon\Delta t + \sigma_\upsilon\Delta w_t + J_\upsilon\Delta j_t < \bar{\upsilon}$, and hence the exponential term in (32) vanishes.

When the constraint of investor A binds, the constraint of investor B is loose: the constraints cannot bind simultaneously because the stock market would not clear otherwise. Therefore, the ratio $\xi_{B,t+\Delta t}/\xi_{Bt}$ is given by FOC (29) for investor B with $\ell_B = 0$. Using change of measure Eq. (28), we rewrite the latter SPD under the correct beliefs of investor A:

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t}\left(\frac{1-s(\upsilon_{t+\Delta t})}{1-s(\upsilon_t)}\right)^{-\gamma_B}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_B}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}. \tag{A.31}$$

Next, from Eq. (14) for consumption share s we find that $(1-s_t)^{-\gamma_B} = e^{-\upsilon_t}s_t^{-\gamma_A}$. Substituting the latter equality into Eq. (A.31), and also using Eq. (A.22) for the increment $\upsilon_{t+\Delta t} - \upsilon_t$, we obtain

$$\begin{aligned}
\frac{\xi_{A,t+\Delta t}}{\xi_{At}} &= e^{-\rho\Delta t}\left(\frac{s(\upsilon_{t+\Delta t})}{s(\upsilon_t)}\right)^{-\gamma_A}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}\\
&\quad\times e^{\upsilon_t - \upsilon_{t+\Delta t}}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\\
&= e^{-\rho\Delta t}\left(\frac{s(\upsilon_{t+\Delta t})}{s(\upsilon_t)}\right)^{-\gamma_A}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}\\
&\quad\times \exp\{\upsilon_t - \upsilon_{t+\Delta t} + \mu_\upsilon\Delta t + \sigma_\upsilon\Delta w_t + J_\upsilon\Delta j_t\}.
\end{aligned} \tag{A.32}$$

The fact that the constraint of investor A is binding means that $\upsilon_{t+\Delta t} = \bar{\upsilon}$ and $\upsilon_t + \mu_\upsilon\Delta t + \sigma_\upsilon\Delta w_t + J_\upsilon \geq \bar{\upsilon}$ (because otherwise $\upsilon_{t+\Delta t} < \bar{\upsilon}$, and hence the constraint does not bind). Therefore, the exponential term in Eq. (A.32) can be replaced with $\exp(\max\{0, \upsilon_t + \mu_\upsilon\Delta t + \sigma_\upsilon\Delta w_t + J_\upsilon\Delta j_t - \bar{\upsilon}\})$. The latter term vanishes when the constraint of investor A does not bind, and we obtain Eq. (A.30). Therefore, both Eq. (A.30) and Eq. (A.32) are summarized by Eq. (32) for $\xi_{A,t+\Delta t}/\xi_{At}$.

2) From the market clearing condition, $S_t = W_{At} + W_{Bt}$. Dividing by D_t and then rewriting in terms of wealth-consumption ratios, we obtain that $(1 - l_A - l_B)\Psi(\upsilon_t) = \Phi_A(\upsilon_t)s(\upsilon_t) + \Phi_B(\upsilon_t)(1-s(\upsilon_t))$. Hence, $\Psi(\upsilon_t)$ is uniformly bounded because $\Phi_i(\upsilon_t)$ are uniformly bounded by Lemma A.3. The fact that stock price S_t is given by Eq. (33) can be verified by substituting S_t in Eq. (33) into the recursive Eq. (26).

3) In the unconstrained economy, the state variable υ^{unc}_t follows dynamics:

$$\upsilon^{unc}_t = \mu_\upsilon\Delta t + \sigma_\upsilon\Delta w_t + J_\upsilon\Delta j_t. \tag{A.33}$$

Let $U_{t+\Delta t} = U_t + \Delta U_t$ and $V_{t+\Delta t} = V_t + \Delta V_t$, where the increments are given by

$$\Delta U_t = \max\{0; \upsilon_t + \mu_\upsilon\Delta t + \sigma_\upsilon\Delta w_t + J_\upsilon\Delta j_t - \bar{\upsilon}\}, \tag{A.34}$$

$$\Delta V_t = \max\{0; \underline{\upsilon} - \upsilon_t - \mu_\upsilon\Delta t - \sigma_\upsilon\Delta w_t - J_\upsilon\Delta j_t\}. \tag{A.35}$$

The process for the state variable in the constrained economy can be rewritten as

$$\upsilon_{t+\Delta t} = \upsilon_t + \mu_\upsilon\Delta t + \sigma_\upsilon\Delta w_t + J_\upsilon\Delta j_t + \Delta V_t - \Delta U_t. \tag{A.36}$$

If the state variables have the same value at time 0, i.e., $\upsilon_0 = \upsilon^{unc}_0$, we obtain

$$\upsilon_t = \upsilon^{unc}_t + V_t - U_t. \tag{A.37}$$

Next, we prove that the SPD is higher in the constrained economy.

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t}\left(\frac{s(\upsilon_{t+\Delta t})}{s(\upsilon_t)}\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}\exp(\Delta U_t), \tag{A.38}$$

$$\frac{\xi^{unc}_{A,t+\Delta t}}{\xi^{unc}_{At}} = e^{-\rho\Delta t}\left(\frac{s(\upsilon^{unc}_{t+\Delta t})}{s(\upsilon^{unc}_t)}\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}. \tag{A.39}$$

Iterating the above equations, we obtain

$$\frac{\xi_{At}}{\xi_{A0}} = e^{-\rho t}\left(\frac{s(\upsilon_t)}{s(\upsilon_0)}\frac{D_t}{D_0}\right)^{-\gamma_A}\exp(U_t), \tag{A.40}$$

$$\frac{\xi^{unc}_{At}}{\xi^{unc}_{A0}} = e^{-\rho t}\left(\frac{s(\upsilon^{unc}_t)}{s(\upsilon_0)}\frac{D_t}{D_0}\right)^{-\gamma_A}. \tag{A.41}$$

By the definition of $s(\upsilon)$ in Eq. (14), we have $e^\upsilon = (1 - s(\upsilon))^{\gamma_B}\cdot s(\upsilon)^{-\gamma_A}$. Hence,

$$\begin{aligned}
\frac{\xi_{At}/\xi_{A0}}{\xi^{unc}_{At}/\xi^{unc}_{A0}} &= \left(\frac{s(\upsilon_t)}{s(\upsilon^{unc}_t)}\right)^{-\gamma_A}\exp(U_t)\\
&= \left(\frac{s(\upsilon^{unc}_t + V_t - U_t)}{s(\upsilon^{unc}_t)}\right)^{-\gamma_A} e^{\upsilon^{unc}_t}e^{-(\upsilon^{unc}_t - U_t)}\\
&\geq s(\upsilon^{unc}_t - U_t)^{-\gamma_A}e^{-(\upsilon^{unc}_t - U_t)}\cdot s(\upsilon^{unc}_t)^{\gamma_A}e^{\upsilon^{unc}_t}\\
&= (1 - s(\upsilon^{unc}_t - U_t))^{-\gamma_B}\cdot(1 - s(\upsilon^{unc}))^{\gamma_B} \geq 1.
\end{aligned} \tag{A.42}$$

Therefore, we conclude that $\xi_{At}/\xi_{A0} \geq \xi^{unc}_{At}/\xi^{unc}_{A0}$. The latter inequality and the asset prices (25)–(27) imply that prices are higher in the constrained economy. ■

Proof of Lemma 2. The P/D ratio Ψ and wealth-output ratio $\widehat{\Phi}_{it} \equiv W_{it}/D_t$ are functions of the state variable v and satisfy equations:

$$\Psi(v_t) = \mathbb{E}^A_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{D_{t+\Delta t}}{D_t}\left(\Psi(v_{t+\Delta t}) + \Delta t\right)\right], \tag{A.43}$$

$$\widehat{\Phi}_i(v_t) = \mathbb{E}^A_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{D_{t+\Delta t}}{D_t}\widehat{\Phi}_i(v_{t+\Delta t})\right] + \left(\mathbf{1}_{\{i=A\}}s(v_t) + \mathbf{1}_{\{i=B\}}(1-s(v_t)) - l_i\right)\Delta t. \tag{A.44}$$

These equations are obtained by substituting $S_t = (1 - l_A - l_B)D_t\Psi(v_t)$ into Eq. (26) for the stock price and $\widehat{\Phi}_{it} = D_t W_{it}$ into static budget constraints (A.5). Define the following function in discrete time:

$$\widehat{\Psi}(v_t;\theta) = \mathbb{E}^A_t\left[e^{-\rho\Delta t + \Delta U_t}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\right] + s(v_t)^{\theta}\Delta t, \tag{A.45}$$

where ΔU_t is given by Eq. (A.34).

Comparing Eq. (A.45) with Eqs. (A.43) and (A.44) for Ψ and $\widehat{\Phi}_i$, and using the linearity of Eq. (A.45), it is easy to observe that $\Psi(v_t)$ and $\widehat{\Phi}_i(v_t)$ are given by the following equations in terms of $\widehat{\Psi}(v_t;\theta)$:

$$\Psi(v_t) = \widehat{\Psi}(v_t, -\gamma_A)s(v_t)^{\gamma_A} - \Delta t, \tag{A.46}$$

$$\widehat{\Phi}_i(v_t) = ((\mathbf{1}_{\{i=A\}} - \mathbf{1}_{\{i=B\}})\widehat{\Psi}(v; 1-\gamma_A) + (\mathbf{1}_{\{i=B\}} - l_i)\widehat{\Psi}(v; -\gamma_A))s(v)^{\gamma_A}. \tag{A.47}$$

Taking limit $\Delta t \to 0$, and noting that $\Phi_i(v_t) = \widehat{\Phi}_i(v_t)/(\mathbf{1}_{\{i=A\}}s(v_t) + \mathbf{1}_{\{i=B\}}(1-s(v_t)))$, we obtain Eqs. (34) and (35) for $\Psi(v_t)$ and $\Phi_i(v_t)$.

First, we derive the equation for $\widehat{\Psi}(v_t;\theta)$ when v_t belongs to the interior $(\underline{v}, \overline{v})$. For a sufficiently small Δt, we have $\Delta U_t = 0$, where ΔU_t is given by (A.34). Then, we rewrite the expectation of $(D_{t+\Delta t}/D_t)^{1-\gamma_A}\widehat{\Psi}(v_t;\theta)$ as follows:

$$\begin{aligned}\mathbb{E}^A_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\right] &= (1-\lambda_A\Delta t)\mathbb{E}^A_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\middle|\text{normal}\right] \\ &\quad + \lambda_A\Delta t\,\mathbb{E}^A_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\middle|\text{crisis}\right].\end{aligned} \tag{A.48}$$

Noting that in the crisis $D_{t+\Delta t}/D_t = 1 + \mu_v\Delta t + J_D$ and $v_{t+\Delta t} = \max\{\underline{v}; v_t + \mu_v\Delta t + J_v\}$, and in the normal state $D_{t+\Delta t}/D_t = 1 + \mu_D\Delta t + \sigma_D\Delta w_t$ and $v_{t+\Delta t} = v_t + \mu_v\Delta t + \sigma_v\Delta w_t$, using Taylor expansions for $(D_{t+\Delta t}/D_t)^{1-\gamma_A}$ and $\widehat{\Psi}(v_{t+\Delta t};\theta)$, we find

$$\mathbb{E}^A_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)|\text{crisis}\right] = (1+J_D)^{1-\gamma_A}\widehat{\Psi}\left(\max\{\underline{v}; v_t + J_v\};\theta\right). \tag{A.49}$$

$$\begin{aligned}\mathbb{E}^A_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)|\text{normal}\right] &= \left[1 + \left((1-\gamma_A)(\mu_D + \delta_A\sigma_D) - \frac{(1-\gamma_A)\gamma_A\sigma_D^2}{2}\right)\Delta t\right] \\ &\quad \times\widehat{\Psi}(v_t;\theta) + \left(\mu_v + \delta_A\sigma_v + (1-\gamma_A)\sigma_D\sigma_v\right) \\ &\quad \times\widehat{\Psi}'(v_t;\theta)\Delta t + \frac{\sigma_v^2}{2}\widehat{\Psi}''(v_t;\theta)\Delta t + o(\Delta t).\end{aligned} \tag{A.50}$$

Substituting (A.49) and (A.50) into (A.45), we obtain

$$\begin{aligned}\widehat{\Psi}(v_t;\theta) &= \left[1 - \left(\lambda_A + \rho - (1-\gamma_A)(\mu_D + \delta_A\sigma_D) + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)\Delta t\right]\widehat{\Psi}(v_t;\theta) \\ &\quad + (\mu_v + \delta_A\sigma_v + (1-\gamma_A)\sigma_D\sigma_v)\widehat{\Psi}'(v;\theta)\Delta t + \frac{\sigma_v^2}{2}\widehat{\Psi}''(v;\theta)\Delta t \\ &\quad + \lambda_A(1+J_D)^{1-\gamma_A}\widehat{\Psi}(\max\{\underline{v}; v_t + J_v\};\theta)\Delta t + s(v)^{\theta}\Delta t + o(\Delta t).\end{aligned} \tag{A.51}$$

Canceling similar terms, diving by Δt, taking limit $\Delta t \to 0$, and noting that μ_v, σ_v and J_v converge to $\widehat{\mu}_v$, $\widehat{\sigma}_v$, and $\widehat{J}_v$ given by (38)–(40), we obtain Eq. (36) for $\widehat{\Psi}(v_t;\theta)$.

Next, we derive the boundary conditions for $\widehat{\Psi}(v_t;\theta)$. From Eq. (31), the state variable dynamics at lower bound is $v_{t+\Delta t} = \underline{v} + \max\{0, \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t\}$. Let $\Delta v_t = v_{t+\Delta t} - v_t$. Then,

$$\Delta v_t = \max\{0, \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t\}. \tag{A.52}$$

For sufficiently small Δt, the increment Δv_t is positive only in a state in which $\sigma_v\Delta w_t > 0$. In such a state, $\Delta v_t = \mu_v\Delta t + |\sigma_v|\sqrt{\Delta t}$. Therefore, the order of $\mathbb{E}^A_t[\Delta v_t]$ is $\sqrt{\Delta t}$, but second order terms involving Δv_t have lower order:

$$\begin{aligned}&\lim_{\Delta t\to 0}\frac{\mathbb{E}^A_t[\Delta v_t]}{\sqrt{\Delta t}} = \frac{|\widehat{\sigma}_v|}{2}, \\ &\lim_{\Delta t\to 0}\frac{\mathbb{E}^A_t[(\Delta v_t)^2]}{\sqrt{\Delta t}} = \lim_{\Delta t\to 0}\frac{\mathbb{E}^A_t[\Delta v_t\Delta t]}{\sqrt{\Delta t}} = \lim_{\Delta t\to 0}\frac{\mathbb{E}^A_t[\Delta v_t\Delta w_t]}{\sqrt{\Delta t}} \\ &\quad = \lim_{\Delta t\to 0}\frac{\mathbb{E}^A_t[\Delta v_t\Delta j_t]}{\sqrt{\Delta t}} = 0.\end{aligned} \tag{A.53}$$

The Taylor expansion of $\widehat{\Psi}(v_{t+\Delta t};\theta)$ at $v_t = \underline{v}$ is given by

$$\widehat{\Psi}(v_{t+\Delta t};\theta) = \widehat{\Psi}(\underline{v};\theta) + \widehat{\Psi}'(\underline{v};\theta)\Delta v_t + \frac{1}{2}\widehat{\Psi}''(\underline{v};\theta)\Delta v_t^2 + o(\sqrt{\Delta t}). \tag{A.54}$$

In subsequent calculations, we keep terms with order of $\sqrt{\Delta t}$. Using the above results, we obtain the following expansion:

$$\begin{aligned}\mathbb{E}^A_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\right] &= \mathbb{E}^A_t\Big[(1 + \mu_D\Delta t + \sigma_D\Delta w_t + J_v\Delta j_t)^{1-\gamma_A} \\ &\quad \left(\widehat{\Psi}(\underline{v};\theta) + \widehat{\Psi}'(\underline{v};\theta)\Delta v_t + \frac{1}{2}\widehat{\Psi}''(\underline{v};\theta)\Delta v_t^2\right)\Big] \\ &= \widehat{\Psi}(\underline{v};\theta) + \widehat{\Psi}'(\underline{v};\theta)\mathbb{E}^A_t[\Delta v_t] + o(\sqrt{\Delta t}).\end{aligned} \tag{A.55}$$

Substituting (A.55) into (A.45), taking into account that $\Delta U_t = 0$ at $v_t = \underline{v}$, and canceling $\widehat{\Psi}(\underline{v};\theta)$ on both sides, we obtain the first boundary condition $\widehat{\Psi}'(\underline{v};\theta) = 0$.

At the upper bound $v_t = \overline{v}$ investor A is constrained, and hence ΔU_t in (A.34) is positive. From (31) the state variable at the upper bound is

$$\begin{aligned} v_{t+\Delta t} &= \min\{\overline{v}, v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\} \\ &= v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \Delta U_t. \end{aligned} \tag{A.56}$$

The order of $\mathbb{E}_t^A[\Delta U_t]$ is $\sqrt{\Delta t}$, but second order terms involving ΔU_t have order $o(\sqrt{\Delta t})$.

Proceeding in the same way as (A.53)–(A.55), we arrive at

$$\begin{aligned} \widehat{\Psi}(\overline{v};\theta) &= \widehat{\Psi}(\overline{v};\theta) + \left[\widehat{\Psi}(\overline{v};\theta) - \widehat{\Psi}'(\overline{v};\theta)\right]\mathbb{E}_t^A[\Delta U_t] \\ &\quad + o(\sqrt{\Delta t}). \end{aligned} \tag{A.57}$$

Canceling similar terms, taking limit $\Delta t \to 0$, we obtain condition $\widehat{\Psi}(\overline{v};\theta) - \widehat{\Psi}'(\overline{v};\theta) = 0$.

Finally, we derive the equations for $\overline{v}$ and $\underline{v}$. Substituting $\Phi_i(v)$ from Eq. (35) into the equations $\Phi_A(\overline{v}) = 0$ and $\Phi_B(\underline{v}) = 0$, after some algebra, we obtain Eq. (41). ■

Proof of Corollary 1. For $\lambda_A = \lambda_B$ and $\gamma_A = \gamma_B = \gamma$ the differential difference Eq. (36) becomes an ODE because $\widehat{J}_v = 0$. From Eq. (14) for consumption share s we find that $s(v) = 1/(1+e^{v/\gamma})$. Substituting $s(v)$ into (36) and setting $\theta = -\gamma$ we obtain an equation for $\widehat{\Psi}(v;-\gamma)$. It can be directly verified that the solution of Eq. (36) satisfying boundary conditions (37) is given by $\widehat{\Psi}(v;-\gamma) = C_- e^{\varphi_- v} + C_+ e^{\varphi_+ v} + \widehat{\Psi}^{unc}(v)$, where

$$\widehat{\Psi}^{unc}(v) = \sum_{k=0}^{\gamma} \binom{\gamma}{k} \frac{e^{kv/\gamma}}{h(k/\gamma)}, \tag{A.58}$$

and the coefficients are given by

$$C_+ = \frac{(1-\varphi_-)e^{\varphi_-\overline{v}}(\widehat{\Psi}^{unc})'(\underline{v}) - \varphi_- e^{\varphi_-\underline{v}}((\widehat{\Psi}^{unc})(\overline{v}) - (\widehat{\Psi}^{unc})'(\overline{v}))}{\varphi_+(\varphi_- - 1)e^{\varphi_-\overline{v}+\varphi_+\underline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+\overline{v}+\varphi_-\underline{v}}}, \tag{A.59}$$

$$C_- = \frac{(\varphi_+ - 1)e^{\varphi_+\overline{v}}(\widehat{\Psi}^{unc})'(\underline{v}) + \varphi_+ e^{\varphi_+\underline{v}}((\widehat{\Psi}^{unc})(\overline{v}) - (\widehat{\Psi}^{unc})'(\overline{v}))}{\varphi_+(\varphi_- - 1)e^{\varphi_-\overline{v}+\varphi_+\underline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+\overline{v}+\varphi_-\underline{v}}}. \tag{A.60}$$

The P/D ratio is then given by (34), which takes form $\widehat{\Psi}(v;-\gamma)/(1+e^{v/\gamma})^{\gamma}$. ■

Proof of Proposition 3. From Eq. (25) for the bond price and the fact that $1 = B_t(1 + r_t\Delta t)$ we find that the riskless interest rate r_t is given by

$$\begin{aligned} r_t &= \frac{1-\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}]}{\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}]\Delta t} \\ &= \left(\frac{1}{(1-\lambda_A\Delta t)\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}] + \lambda_A\Delta t\,\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}|\text{crisis}]} - 1\right)\frac{1}{\Delta t}, \end{aligned} \tag{A.61}$$

where $\xi_{A,t+\Delta t}/\xi_{At}$ is given by Eq. (32). We separately calculate $\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}]$ and $\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}|\text{crisis}]$, and then take the limit $\Delta t \to 0$.

We start with the derivation of $\mathbb{E}_t^A[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}]$ when $\underline{v} < v_t < \overline{v}$, and hence by continuity, for a sufficiently small Δt the economy is unconstrained next period, so that $\underline{v} < v_{t+\Delta t} < \overline{v}$. In the unconstrained region, we have $\Delta v_t = \mu_v \Delta t + \sigma_v \Delta w_t$ and the SPD is given by (A.30). From the expression for the SPD, using expansions (A.74) and (A.76) from Lemma A.4 below, we obtain

$$\begin{aligned} \mathbb{E}_t^A&\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\middle|\text{normal}\right] \\ &= \mathbb{E}_t^A[((1 + a_t\Delta v_t + b_t(\Delta v_t)^2) \\ &\quad \times (1 - r_A\Delta t - \kappa_A\Delta w_t)|\text{normal}] + o(\Delta t) \\ &= \mathbb{E}_t^A[1 + a_t\Delta v_t + b_t(\Delta v_t)^2 - r_A\Delta t \\ &\quad -\kappa_A\Delta w_t - \kappa_A a_t\Delta v_t\Delta w_t|\text{normal}] + o(\Delta t) \\ &= 1 + (a_t(\widehat{\mu}_v + \delta_A\sigma_D) + b_t\widehat{\sigma}_v^2 - r_A - \kappa_A\delta_A - \kappa_A a_t\widehat{\sigma}_v) \\ &\quad \times \Delta t + o(\Delta t). \end{aligned} \tag{A.62}$$

Conditioning on the crisis state, we have

$$\begin{aligned} \mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\middle|\text{crisis}\right] &= (1-\rho\Delta t)(1+\mu_D\Delta t + J_D)^{-\gamma_A} \\ &\quad \times \left(\frac{s(\max\{\underline{v}, v_t + \mu_v\Delta t + J_v\})}{s(v_t)}\right)^{-\gamma_A} \\ &= (1+J_D)^{-\gamma_A}\left(\frac{s(\max\{\underline{v}, v_t + \widehat{J}_v\})}{s(v_t)}\right)^{-\gamma_A} + o(\Delta t). \end{aligned} \tag{A.63}$$

Substituting a_t and b_t from (A.75) into Eq. (A.62), and then substituting Eq. (A.62) and (A.63) into Eq. (A.61), after simple algebra, we obtain r_t in (44) for the case $\underline{v} < v_t < \overline{v}$.

Now, we derive r_t at the boundaries $\underline{v}$ and $\overline{v}$. The SPD is given by (32). Using expansions (A.74) and (A.76), we obtain the following expansion:

$$\begin{aligned} \mathbb{E}_t^A&\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\middle|\text{normal}\right] \\ &= \mathbb{E}_t^A[((1 + a_t\Delta v_t + b_t(\Delta v_t)^2)(1 - r_A\Delta t - \kappa_A\Delta w_t) \\ &\quad \times(1 + \Delta U_t + 0.5(\Delta U_t)^2)|\text{normal}] + o(\Delta t) \\ &= \mathbb{E}_t^A[1 + a_t\Delta v_t + b_t(\Delta v_t)^2 - r_A\Delta t \\ &\quad -\kappa_A\Delta w_t - \kappa_A a_t\Delta v_t\Delta w_t + \Delta U_t - \kappa_A\Delta w_t\Delta U_t \\ &\quad + a_t\Delta U_t\Delta v_t + 0.5(\Delta U_t)^2|\text{normal}] + O(\Delta t), \end{aligned} \tag{A.64}$$

where ΔU_t is given by Eq. (A.34). Using Eq. (31) for the process v_t and Eq. (A.34) for ΔU_t, for a fixed v_t and sufficiently small Δt, we find that Δv_t and ΔU_t at the boundaries are given by

$$\Delta v_t = \begin{cases} \min(0, \mu_v\Delta t + \sigma_v\Delta w_t), & \text{if } v_t = \overline{v}, \\ \max(0, \mu_v\Delta t + \sigma_v\Delta w_t), & \text{if } v_t = \underline{v}, \end{cases} \tag{A.65}$$

$$\Delta U_t = \begin{cases} 0, & \text{if } v_t < \overline{v}, \\ \max(0, \mu_v\Delta t + \sigma_v\Delta w_t), & \text{if } v_t = \overline{v}, \end{cases} \tag{A.66}$$

We note that for a sufficiently small Δt, the sign of $\mu_v\Delta t + \sigma_v\Delta w_t$ is solely determined by the second term $\sigma_v\Delta w_t$ because it has the order of magnitude $\sqrt{\Delta t}$. Using the latter observation, substituting Eqs. (A.65) and (A.66) into Eq. (A.64) and computing the expectation, we obtain

$$\mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\middle|\text{normal}\right] =$$

$$1+\begin{cases}\dfrac{|\sigma_\nu|(1-a_t)}{2}\sqrt{\Delta t}+O(\Delta t), & \text{if} \quad \nu_t=\overline{\nu},\\ \dfrac{|\sigma_\nu|a_t}{2}\sqrt{\Delta t}+O(\Delta t), & \text{if} \quad \nu_t=\underline{\nu}.\end{cases} \tag{A.67}$$

Substituting (A.67) and (A.63) into Eq. (A.61) for the interest rate r_t, we obtain r_t in Eq. (44) for the case in which ν_t is at the boundary.

To obtain the risk premium, we first decompose stock returns as follows:

$$\frac{\Delta S_t+(1-l_A-l_B)D_{t+\Delta t}\Delta t}{S_t}=\mu_t\Delta t+\sigma_t\Delta w_t+J_t\Delta j_t. \tag{A.68}$$

Multiplying both sides of (A.68) by $\xi_{A,t+\Delta t}/\xi_{At}$ and taking expectations, we obtain

$$\mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{\Delta S_t+(1-l_A-l_B)D_{t+\Delta t}\Delta t}{S_t}\right]=\mu_t\Delta t\mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\right]$$
$$+\sigma_t\mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta w_t\right]+J_t\mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta j_t\right]. \tag{A.69}$$

On the other hand, from Eq. (26) for the stock price, we find that

$$\mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{\Delta S_t+(1-l_A-l_B)D_{t+\Delta t}\Delta t}{S_t}\right]=1-\mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\right]. \tag{A.70}$$

Combining the last two equations and the interest rate (A.61), we obtain

$$\mu_t-r_t=-\left(\sigma_t\mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta w_t\right]+J_t\mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta j_t\right]\right)\times\frac{1+r_t\Delta t}{\Delta t}. \tag{A.71}$$

Then, proceeding in the same way as with the calculation of interest rates and using similar expansions, we obtain Eq. (45) for the risk premium.

Finally, we obtain volatility σ_t and jump size J_t in normal times. In the unconstrained region $\underline{\nu}<\nu_t<\overline{\nu}$, stock price S_t, dividend D_t, and state variable ν_t follow processes $dS_t=S_t[\mu_t d_t+\sigma_t dw_t+J_t dj_t]$, $dD_t=D_t[\mu_D d_t+\sigma_D dw_t+J_D dj_t]$, and $d\nu_t=\widehat{\mu}_\nu dt+\widehat{\sigma}_\nu dw_t+\big(\max\{\underline{\nu};\nu_t+\widehat{J}_\nu\}-\nu_t\big)dj_t$. Applying Itô's lemma to $S_t=(1-l_A-l_B)\widehat{\Psi}(\nu_t;-\gamma_A)s(\nu_t)^{\gamma_A}D_t$, and matching dw_t and dj_t terms, after some algebra, we obtain

$$\sigma_t=\sigma_D+\left(\frac{\widehat{\Psi}'(\nu_t;-\gamma_A)}{\widehat{\Psi}(\nu_t;-\gamma_A)}-\frac{\gamma_A(1-s(\nu_t))}{\gamma_A(1-s(\nu_t))+\gamma_B s(\nu_t)}\right)\widehat{\sigma}_\nu, \tag{A.72}$$

$$J_t=\frac{(1+J_D)\widehat{\Psi}(\max\{\underline{\nu};\nu_t+\widehat{J}_\nu\};-\gamma_A)s(\max\{\underline{\nu};\nu_t+\widehat{J}_\nu\})^{\gamma_A}}{\widehat{\Psi}(\nu_t;-\gamma_A)s(\nu_t)^{\gamma_A}}-1. \tag{A.73}$$

■

Lemma A.4 (Useful expansions).

(1) *For small increment $\Delta\nu_t=\nu_{t+\Delta t}-\nu_t$, the ratio $(s(\nu_{t+\Delta t})/s(\nu_t))^{-\gamma_A}$ has expansion*

$$\left(\frac{s(\nu_{t+\Delta t})}{s(\nu_t)}\right)^{-\gamma_A}=1+a_t\Delta\nu_t+b_t(\Delta\nu_t)^2+o(\Delta t), \tag{A.74}$$

where coefficients a_t and b_t are given by

$$a_t=\frac{(1-s_t)\Gamma_t}{\gamma_B},\quad b_t=\frac{1}{2\gamma_B^2}(1-s_t)^2\Gamma_t^2+\frac{1}{2\gamma_A^2\gamma_B^2}s_t(1-s_t)\Gamma_t^3. \tag{A.75}$$

*$\Gamma_t=\gamma_A\gamma_B/(\gamma_A(1-s)+\gamma_B s)$ is the risk aversion of the representative investor, and s_t is consumption share of investor A that solves*Eq. (14).

(2) *For the case $J_D=0$, the SPD in a one-investor economy can be expanded as follows:*

$$e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}=1-r_A\Delta t-\kappa_A\Delta w_t+o(\Delta t), \tag{A.76}$$

where r_A and κ_A are the riskless rate and the Sharpe ratio in an economy populated only by investor A, given by

$$r_A=\rho+\gamma_A(\mu_D+\delta_A\sigma_D)-\frac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2,\ \kappa_A=\gamma_A\sigma_D. \tag{A.77}$$

Proof of Lemma A.4. (1) We expand the ratio on the left-hand side of (A.74) using Taylor's formula and observe that $a_t=(s(\nu_t)^{-\gamma_A})'/s(\nu_t)^{-\gamma_A}$ and $b_t=0.5(s(\nu_t)^{-\gamma_A})''/s(\nu_t)^{-\gamma_A}$. Differentiating, we obtain the following expressions for a_t and b_t:

$$a_t=-\gamma_A\frac{s'(\nu_t)}{s(\nu_t)},\quad b_t=\frac{\gamma_A(1+\gamma_A)}{2}\left(\frac{s'(\nu_t)}{s(\nu_t)}\right)^2-\frac{\gamma_A}{2}\frac{s''(\nu)}{s(\nu)}. \tag{A.78}$$

To find derivatives $s'(\nu)$ and $s''(\nu)$, we differentiate Eq. (14) twice with respect to ν and obtain two equations for the derivatives:

$$1=-\left(\frac{\gamma_A}{s_t}+\frac{\gamma_B}{1-s_t}\right)s'(\nu_t), \tag{A.79}$$

$$0=\left(\frac{\gamma_A}{s_t^2}-\frac{\gamma_B}{(1-s_t)^2}\right)(s'(\nu_t))^2-\left(\frac{\gamma_A}{s_t}+\frac{\gamma_B}{1-s_t}\right)s''(\nu_t). \tag{A.80}$$

Finding $s'(\nu)$ and $s''(\nu)$ from the system (A.79)–(A.80) and substituting them into expressions (A.78) for coefficients a_t and b_t, after some algebra, we obtain expressions (A.75).

(2) Substituting $D_{t+\Delta t}/D_t$ from (1) into Eq. (A.76), after some algebra, we obtain

$$\begin{aligned}e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}&=e^{-\rho\Delta t}(1+\mu_D\Delta t+\sigma_D\Delta w_t)^{-\gamma_A}\\&=(1-\rho\Delta t)\left(1-\left(\gamma_A\mu_D-\frac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2\right)\Delta t-\gamma_A\sigma_D\right)\\&\quad+o(\Delta t)\\&=1-r_A\Delta t-\kappa_A\Delta w_t+o(\Delta t).\ \blacksquare\end{aligned} \tag{A.81}$$

Proof of Proposition 4. Consider a reflected arithmetic Brownian motion with boundaries $\underline{\nu}$ and $\overline{\nu}$ and dynamics $d\nu_t=\widehat{\mu}_\nu dt+\widehat{\sigma}_\nu dw_t$ when $\underline{\nu}<\nu_t<\overline{\nu}$, where w_t is a Brownian motion. The transition density for this process is given by (see Veestraeten, 2004):

$$\begin{aligned} f_\nu(\nu,\tau;\nu_t,t) &= \frac{1}{\sqrt{2\pi\widehat{\sigma}_\nu^2(\tau-t)}} \\ &\times \sum_{n=-\infty}^{+\infty}\Bigg[\exp\left(-\frac{2\widehat{\mu}_\nu}{\widehat{\sigma}_\nu^2}n(\overline{\nu}-\underline{\nu}) - \frac{(\nu-\nu_t-\widehat{\mu}_\nu(\tau-t)+2n(\overline{\nu}-\underline{\nu}))^2}{2\widehat{\sigma}_\nu^2(\tau-t)}\right) \\ &+\exp\Bigg(-\frac{2\widehat{\mu}_\nu}{\widehat{\sigma}_\nu^2}(\nu_t-\underline{\nu}+n(\overline{\nu}-\underline{\nu})) \\ &- \frac{(\nu-\nu_t-\widehat{\mu}_\nu(\tau-t)+2(\nu_t-\underline{\nu}+n[\overline{\nu}-\underline{\nu}]))^2}{2\widehat{\sigma}_\nu^2(\tau-t)}\Bigg)\Bigg] \\ &+\frac{2\widehat{\mu}_\nu}{\widehat{\sigma}_\nu^2}\sum_{n=0}^{+\infty}\Bigg[\exp\left(-\frac{2\widehat{\mu}_\nu}{\widehat{\sigma}_\nu^2}(\overline{\nu}-\nu+n[\overline{\nu}-\underline{\nu}])\right) \\ &\times\mathcal{N}\left(\frac{\nu_t+\widehat{\mu}_\nu(\tau-t)-\nu-2(\overline{\nu}-\nu+n[\overline{\nu}-\underline{\nu}])}{\widehat{\sigma}_\nu\sqrt{\tau-t}}\right) \\ &-\exp\left(\frac{2\widehat{\mu}_\nu}{\widehat{\sigma}_\nu^2}(\nu-\underline{\nu}+n[\overline{\nu}-\underline{\nu}])\right) \\ &\times\left(1-\mathcal{N}\left(\frac{\nu_t+\widehat{\mu}_\nu(\tau-t)-\nu+2(\nu-\underline{\nu}+n[\overline{\nu}-\underline{\nu}])}{\widehat{\sigma}_\nu\sqrt{\tau-t}}\right)\right)\Bigg], \end{aligned} \quad \text{(A.82)}$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution of a standard normal distribution. By $F_\nu(\nu,\ \tau;\ \nu_t,\ t) = \text{Prob}\{\nu_\tau \le \nu|\nu_t\}$, we denote the corresponding cumulative distribution function of ν conditional on observing ν_t at time t. We observe that $s_t = s(\nu_t)$ is a decreasing function of ν_t implicitly defined by Eq. (14). From the latter equation we also find that $s^{-1}(x) = \gamma_B\ln(1-s) - \gamma_A\ln(s)$. The cumulative distribution function of consumption share s_τ at time τ conditional on observing s_t at time t can then be found as follows:

$$\begin{aligned} F(x,\tau;s_t,t) &= \text{Prob}\{s_\tau \le x|s_t\} \equiv \text{Prob}\{s(\nu_\tau)\le x|s_t\} \\ &= 1-\text{Prob}\{\nu_\tau \le s^{-1}(x)|\nu_t\} \\ &= 1-\text{Prob}\{\nu_\tau \le \gamma_B\ln(1-x)-\gamma_A\ln(x)|\nu_t\} \\ &= 1-F_\nu(\gamma_B\ln(1-x)-\gamma_A\ln(x),\tau;\nu_t,t). \end{aligned} \quad \text{(A.83)}$$

Substituting $\nu_t = \gamma_B\ln(1-s_t) - \gamma_A\ln(s_t)$ into (A.83), differentiating the cumulative distribution function $F(x,\ \tau;\ s_t,\ t)$ with respect to x and setting $x = s$, we find that the transition PDF for s is given by

$$\begin{aligned} f(s,\tau;s_t,t) &= \left(\frac{\gamma_A}{s}+\frac{\gamma_B}{1-s}\right) \\ &\times f_\nu(\gamma_B\ln(1-s)-\gamma_A\ln(s),\tau;\gamma_B\ln(1-s_t)-\gamma_A\ln(s_t),t), \end{aligned} \quad \text{(A.84)}$$

where transition density $f_\nu(\nu,\ \tau;\ \nu_t,\ t)$ is given by Eq. (A.82).

The stationary distribution of variable ν, calculated in Veestraeten (2004), is given by

$$f_\nu(\nu) = \frac{2\widehat{\mu}_\nu}{\widehat{\sigma}_\nu^2}\frac{\exp\left((2\widehat{\mu}_\nu/\widehat{\sigma}_\nu^2)\nu\right)}{\exp\left((2\widehat{\mu}_\nu/\widehat{\sigma}_\nu^2)\overline{\nu}\right)-\exp\left((2\widehat{\mu}_\nu/\widehat{\sigma}_\nu^2)\underline{\nu}\right)}. \quad \text{(A.85)}$$

Proceeding in the same way as for the derivation of transition PDF (A.84), we obtain stationary PDF (47) for consumption share s. The PDF for the case $\widehat{\mu}_\nu = 0$ is obtained from the case $\widehat{\mu}_\nu \ne 0$ by taking the limit $\widehat{\mu}_\nu \to 0$. ■

Proof of Corollary 2. The proof easily follows by substituting boundary conditions (37) into Eq. (A.72) for volatility σ_t at the boundary values $\underline{\nu}$ and $\overline{\nu}$. ■

Proof of Proposition 5. Consider Lagrangian (A.9) for the dynamic optimization of investor i. Differentiating this Lagrangian with respect to l_i and c_{it}, we obtain

$$\begin{aligned} &\frac{\partial V_i(W_{it}^*,\nu_t;l_i)}{\partial l_i} \\ &= \eta_{it}D_t\Delta t + e^{-\rho\Delta t}\mathbb{E}_t^i\left[\frac{\partial V_i(W_{i,t+\Delta t}^*,\nu_{t+\Delta t};l_i)}{\partial l_i}\right], \end{aligned} \quad \text{(A.86)}$$

$$u'(c_{it}^*) = \eta_{it}. \quad \text{(A.87)}$$

By the envelope theorem (e.g., Back, 2010, p. 162),

$$\frac{\partial V_i(W_{it},\nu_t;l_i)}{\partial W} = u_i'(c_{it}^*), \quad \text{(A.88)}$$

$$\frac{\partial V_i(W_{i,t+\Delta t},\nu_{t+\Delta t};l_i)}{\partial W} = u_i'(c_{i,t+\Delta t}^*). \quad \text{(A.89)}$$

Substituting (50), (A.87), (A.88), and (A.89) into Eq. (A.86) and simplifying, we find

$$\widehat{S}_{it} = D_t\Delta t + \mathbb{E}_t^i\left[e^{-\rho\Delta t}\frac{u_i'(c_{i,t+\Delta t}^*)}{u_i'(c_{it}^*)}\widehat{S}_{i,t+\Delta t}\right]. \quad \text{(A.90)}$$

From Eq. (32), we recall that the SPD of investor A is given by

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t+\Delta U_t}\frac{(c_{A,t+\Delta t}^*)^{-\gamma_A}}{(c_{At}^*)^{-\gamma_A}}\frac{D_{t+\Delta t}}{D_t}, \quad \text{(A.91)}$$

where $\Delta U_t = \max\{0;\ \nu_t + \mu_\nu\Delta t + \sigma_\nu\Delta w_t + J_\nu\Delta j_t - \overline{\nu}\}$. Rewriting Eq. (A.90) for investor A in terms of SPD (A.91), we obtain

$$\widehat{S}_{At} = D_t\Delta t + \mathbb{E}_t^A\left[e^{-\Delta U_t}\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\widehat{S}_{A,t+\Delta t}\right]. \quad \text{(A.92)}$$

Following the same steps as in the proof of Lemma 2, we find that $\widehat{S}_{At} = \widehat{\Psi}_i(\nu_t;-\gamma_A)s(\nu_t)^{\gamma_A}D_t$, where $\widehat{\Psi}_i(\nu;\theta)$ satisfies differential difference Eq. (36) with boundary conditions (53).

Iterating Eq. (26) for stock and Eq. (A.92) for shadow prices, we obtain

$$S_t + (1-l_A-l_B)D_t\Delta t = \frac{1}{\xi_t}\mathbb{E}_t^A\left[\sum_{\tau=t}^{\infty}\xi_\tau(1-l_A-l_B)D_\tau\Delta t\right], \quad \text{(A.93)}$$

$$\widehat{S}_{At} = \frac{1}{\xi_t}\mathbb{E}_t^A\left[\sum_{\tau=t}^{\infty}e^{-(U_\tau-U_t)}\xi_\tau D_\tau\Delta t\right]. \quad \text{(A.94)}$$

Inequality $(S_t + (1-l_A-l_B)D_t\Delta t)/(1-l_A-l_B) > \widehat{S}_{At}$ follows from the fact that $U_t = \sum_{\tau=0}^{t}\Delta U_\tau$ is a nondecreasing process. In the continuous-time limit, we obtain that $S_t/(1-l_A-l_B) > \widehat{S}_{At}$. Hence, the liquidity premium Λ_{At} is positive. The derivation of the shadow price of investor B is analogous. ■

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