

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

4.4 SHORTEST PATHS

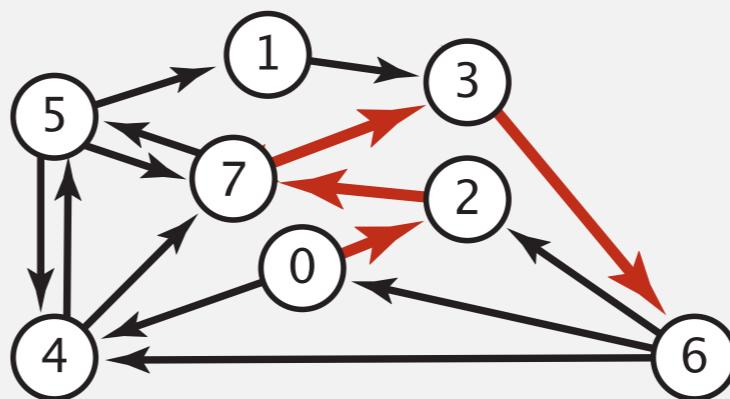
- ▶ *APIs*
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ *negative weights*

Shortest paths in an edge-weighted digraph

Given an edge-weighted digraph, find the shortest path from s to t .

edge-weighted digraph

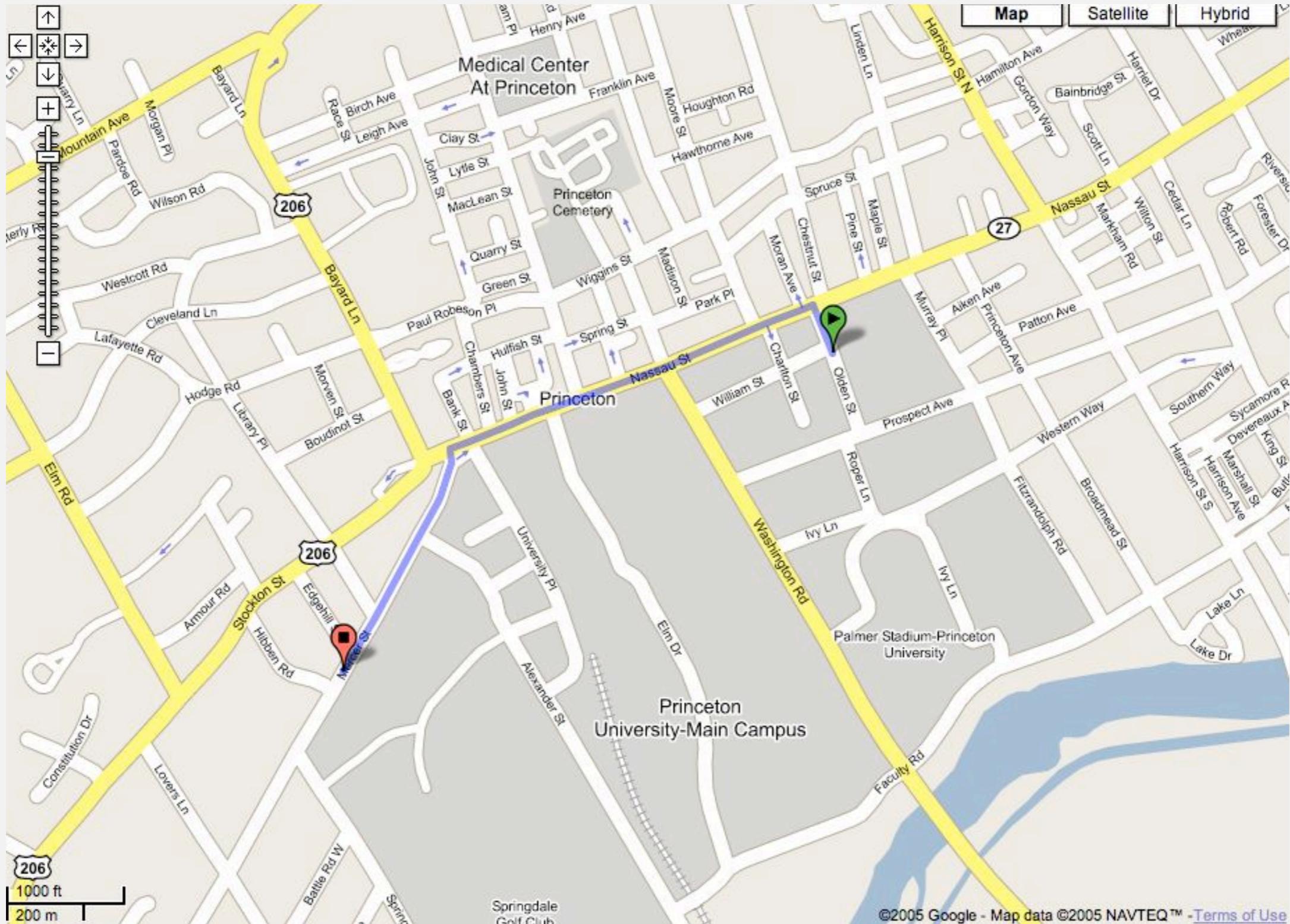
4->5	0.35
5->4	0.35
4->7	0.37
5->7	0.28
7->5	0.28
5->1	0.32
0->4	0.38
0->2	0.26
7->3	0.39
1->3	0.29
2->7	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93



shortest path from 0 to 6

0->2	0.26
2->7	0.34
7->3	0.39
3->6	0.52

Google maps



Car navigation



Shortest path applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.



http://en.wikipedia.org/wiki/Seam_carving



Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

Shortest path variants

Which vertices?

- Single source: from one vertex s to every other vertex.
- Source-sink: from one vertex s to another t .
- All pairs: between all pairs of vertices.

Restrictions on edge weights?

- Nonnegative weights.
- Euclidean weights.
- Arbitrary weights.

Cycles?

- No directed cycles.
- No "negative cycles."

Simplifying assumption. Shortest paths from s to each vertex v exist.

Algorithms

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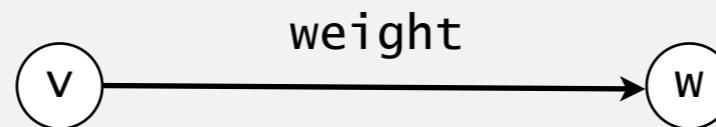
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Weighted directed edge API

```
public class DirectedEdge  
  
    DirectedEdge(int v, int w, double weight)      weighted edge v→w  
  
    int from()                                     vertex v  
  
    int to()                                       vertex w  
  
    double weight()                                weight of this edge  
  
    String toString()                             string representation
```



Idiom for processing an edge e: `int v = e.from(), w = e.to();`

Weighted directed edge: implementation in Java

Similar to Edge for undirected graphs, but a bit simpler.

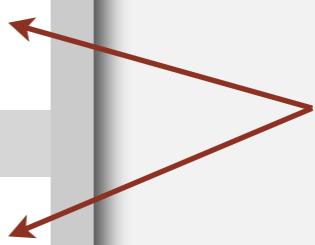
```
public class DirectedEdge
{
    private final int v, w;
    private final double weight;

    public DirectedEdge(int v, int w, double weight)
    {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int from()
    {   return v;   }

    public int to()
    {   return w;   }

    public int weight()
    {   return weight;   }
}
```



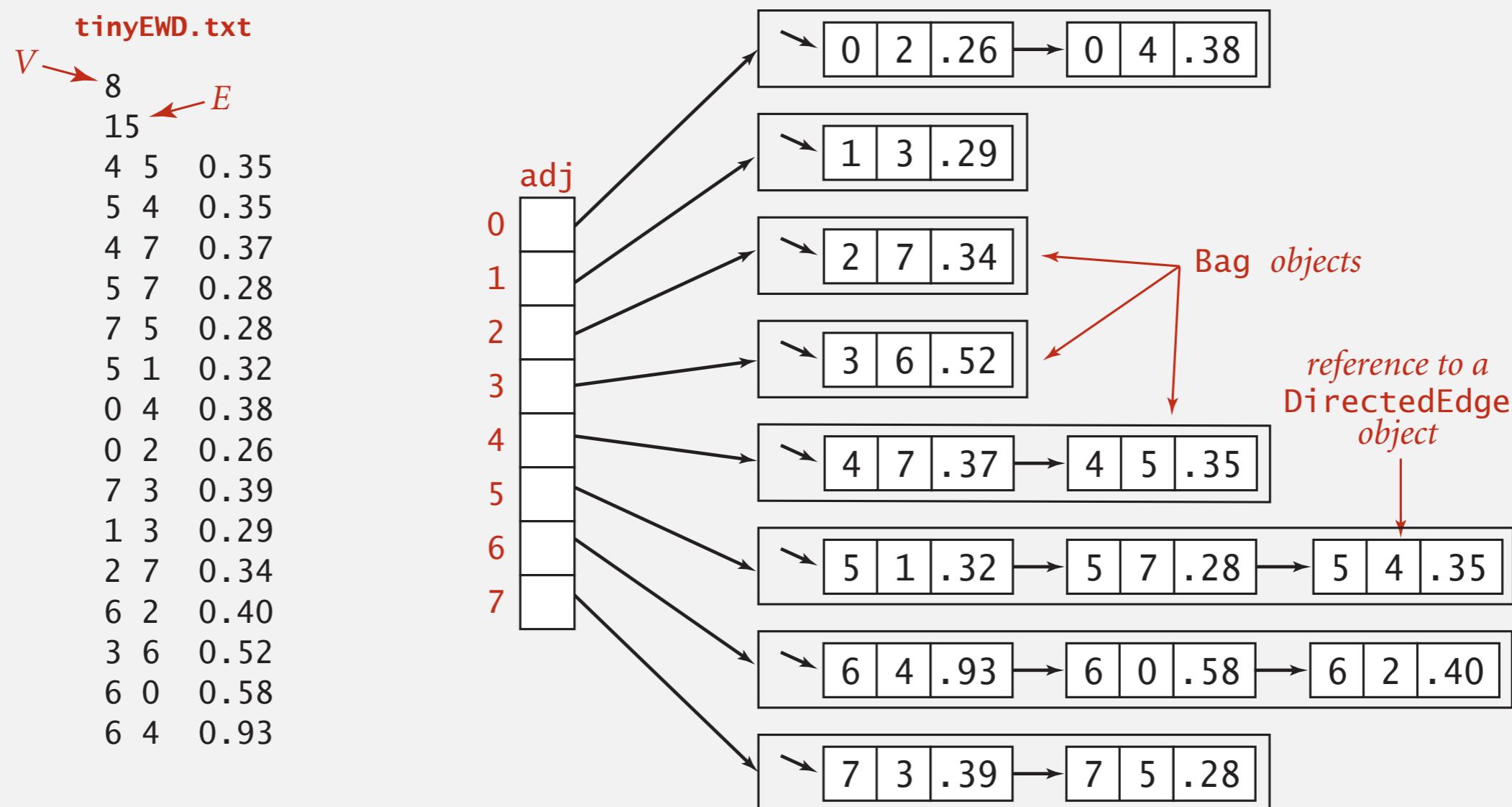
from() and to() replace
either() and other()

Edge-weighted digraph API

public class EdgeWeightedDigraph	
EdgeWeightedDigraph(int V)	<i>edge-weighted digraph with V vertices</i>
EdgeWeightedDigraph(In in)	<i>edge-weighted digraph from input stream</i>
void addEdge(DirectedEdge e)	<i>add weighted directed edge e</i>
Iterable<DirectedEdge> adj(int v)	<i>edges pointing from v</i>
int V()	<i>number of vertices</i>
int E()	<i>number of edges</i>
Iterable<DirectedEdge> edges()	<i>all edges</i>
String toString()	<i>string representation</i>

Conventions. Allow self-loops and parallel edges.

Edge-weighted digraph: adjacency-lists representation



Edge-weighted digraph: adjacency-lists implementation in Java

Same as EdgeWeightedGraph except replace Graph with Digraph.

```
public class EdgeWeightedDigraph
{
    private final int V;
    private final Bag<DirectedEdge>[] adj;

    public EdgeWeightedDigraph(int V)
    {
        this.V = V;
        adj = (Bag<DirectedEdge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<DirectedEdge>();
    }

    public void addEdge(DirectedEdge e)
    {
        int v = e.from();
        adj[v].add(e);
    }

    public Iterable<DirectedEdge> adj(int v)
    { return adj[v]; }
}
```

←
add edge $e = v \rightarrow w$ to
only v 's adjacency list

Single-source shortest paths API

Goal. Find the shortest path from s to every other vertex.

```
public class SP
```

```
    SP(EdgeWeightedDigraph G, int s) shortest paths from s in graph G
```

```
    double distTo(int v) length of shortest path from s to v
```

```
    Iterable <DirectedEdge> pathTo(int v) shortest path from s to v
```

```
    boolean hasPathTo(int v) is there a path from s to v?
```

```
SP sp = new SP(G, s);
for (int v = 0; v < G.V(); v++)
{
    StdOut.printf("%d to %d (%.2f): ", s, v, sp.distTo(v));
    for (DirectedEdge e : sp.pathTo(v))
        StdOut.print(e + " ");
    StdOut.println();
}
```

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```
public class SP
```

```
    SP(EdgeWeightedDigraph G, int s) shortest paths from s in graph G
```

```
    double distTo(int v) length of shortest path from s to v
```

```
    Iterable <DirectedEdge> pathTo(int v) shortest path from s to v
```

```
    boolean hasPathTo(int v) is there a path from s to v?
```

```
% java SP tinyEWG.txt 0
0 to 0 (0.00):
0 to 1 (1.05): 0->4 0.38  4->5 0.35  5->1 0.32
0 to 2 (0.26): 0->2 0.26
0 to 3 (0.99): 0->2 0.26  2->7 0.34  7->3 0.39
0 to 4 (0.38): 0->4 0.38
0 to 5 (0.73): 0->4 0.38  4->5 0.35
0 to 6 (1.51): 0->2 0.26  2->7 0.34  7->3 0.39  3->6 0.52
0 to 7 (0.60): 0->2 0.26  2->7 0.34
```

Algorithms

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4.4 SHORTEST PATHS

- ▶ **APIs**
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ *negative weights*

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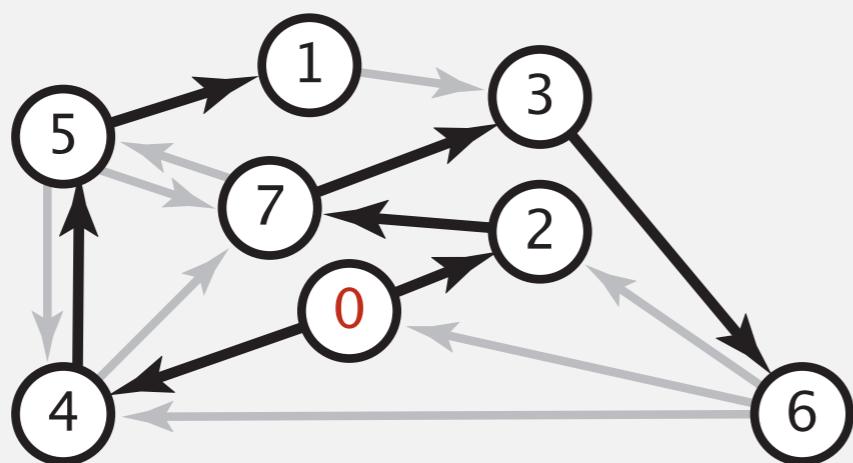
Data structures for single-source shortest paths

Goal. Find the shortest path from s to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- $\text{distTo}[v]$ is length of shortest path from s to v .
- $\text{edgeTo}[v]$ is last edge on shortest path from s to v .



shortest-paths tree from 0

	edgeTo[]	distTo[]
0	null	0
1	5->1	0.32
2	0->2	0.26
3	7->3	0.37
4	0->4	0.38
5	4->5	0.35
6	3->6	0.52
7	2->7	0.34

parent-link representation

Data structures for single-source shortest paths

Goal. Find the shortest path from s to every other vertex.

Observation. A **shortest-paths tree** (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

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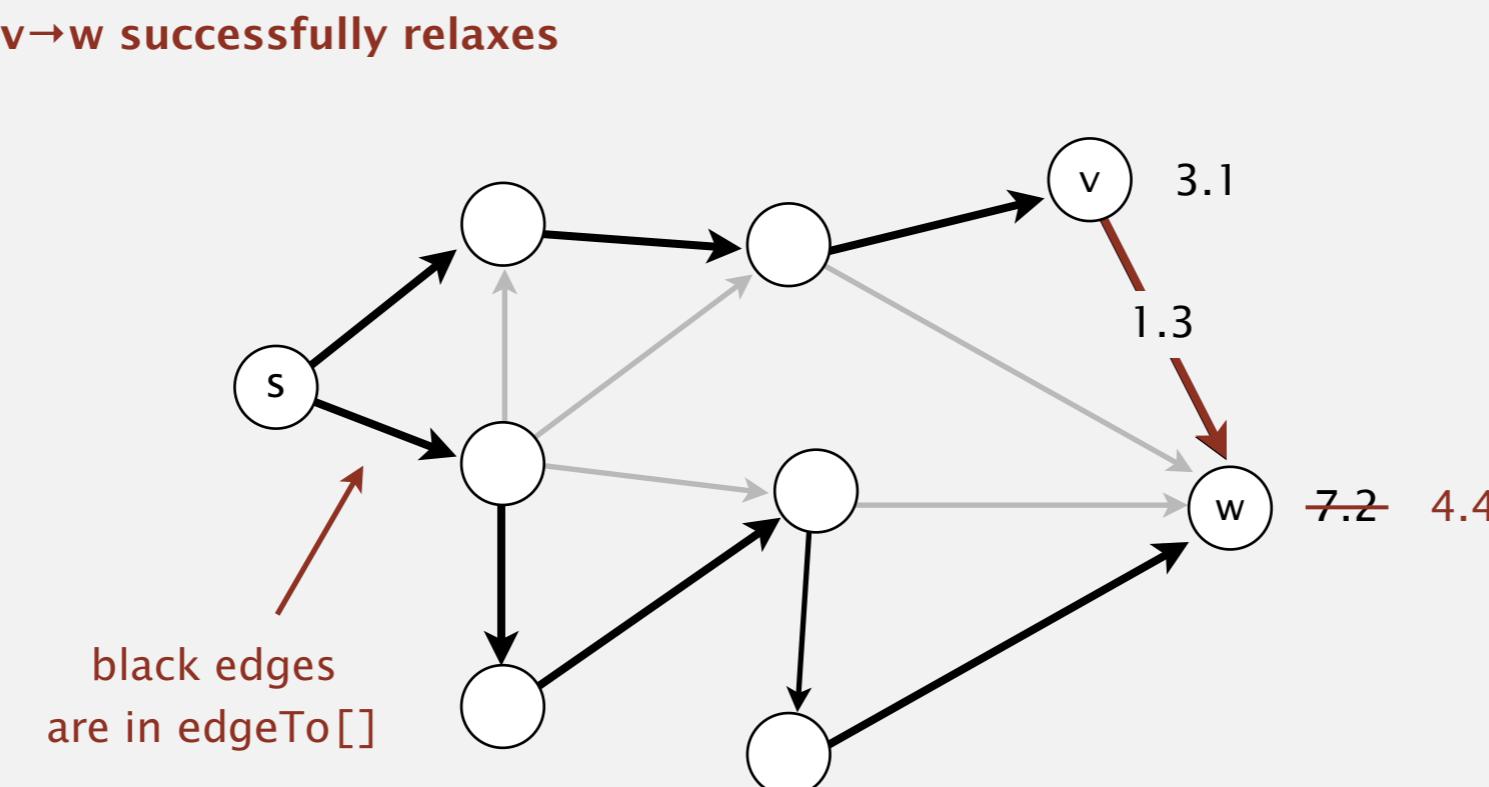
```
public double distTo(int v)
{   return distTo[v]; }

public Iterable<DirectedEdge> pathTo(int v)
{
    Stack<DirectedEdge> path = new Stack<DirectedEdge>();
    for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
        path.push(e);
    return path;
}
```

Edge relaxation

Relax edge $e = v \rightarrow w$.

- $\text{distTo}[v]$ is length of shortest **known** path from s to v .
- $\text{distTo}[w]$ is length of shortest **known** path from s to w .
- $\text{edgeTo}[w]$ is last edge on shortest **known** path from s to w .
- If $e = v \rightarrow w$ gives shorter path to w through v ,
update both $\text{distTo}[w]$ and $\text{edgeTo}[w]$.



Edge relaxation

Relax edge $e = v \rightarrow w$.

- $\text{distTo}[v]$ is length of shortest **known** path from s to v .
- $\text{distTo}[w]$ is length of shortest **known** path from s to w .
- $\text{edgeTo}[w]$ is last edge on shortest **known** path from s to w .
- If $e = v \rightarrow w$ gives shorter path to w through v ,
update both $\text{distTo}[w]$ and $\text{edgeTo}[w]$.

```
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
    }
}
```

Shortest-paths optimality conditions

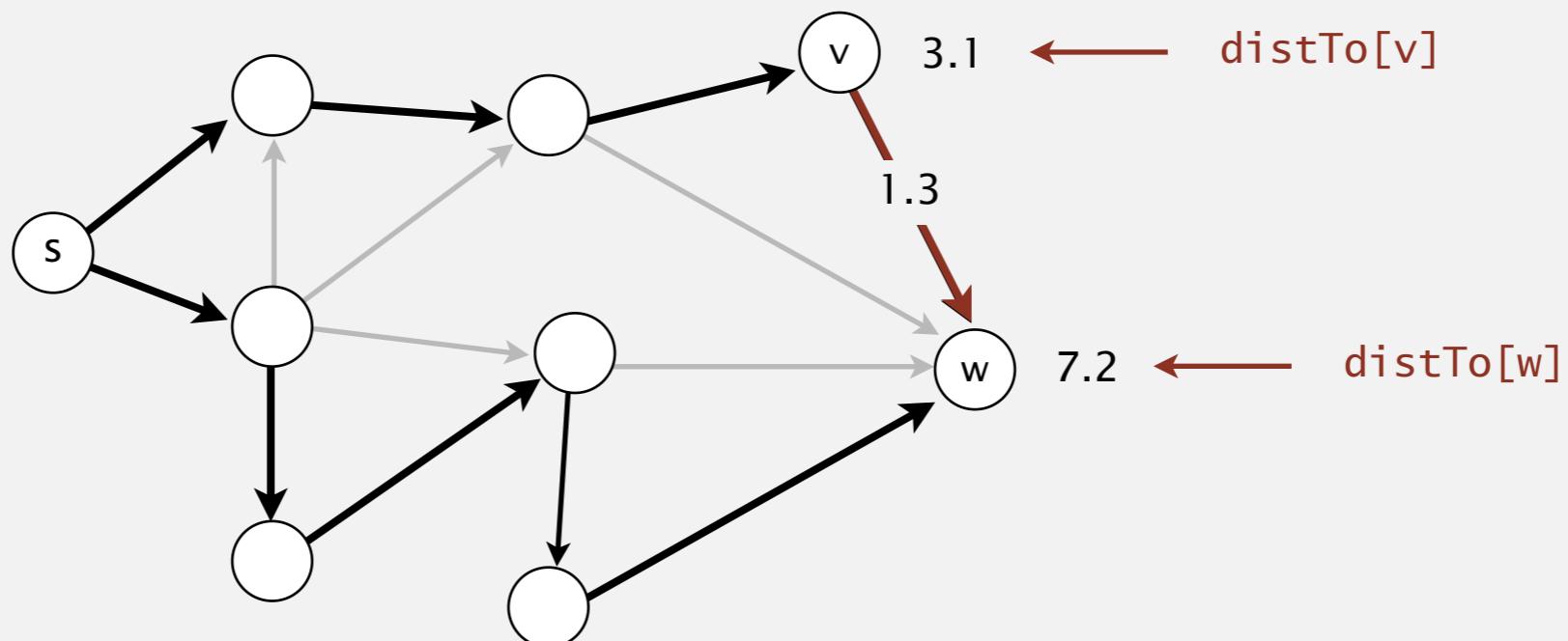
Proposition. Let G be an edge-weighted digraph.

Then $\text{distTo}[]$ are the shortest path distances from s iff:

- $\text{distTo}[s] = 0$.
- For each vertex v , $\text{distTo}[v]$ is the length of some path from s to v .
- For each edge $e = v \rightarrow w$, $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$.

Pf. \Leftarrow [necessary]

- Suppose that $\text{distTo}[w] > \text{distTo}[v] + e.\text{weight}()$ for some edge $e = v \rightarrow w$.
- Then, e gives a path from s to w (through v) of length less than $\text{distTo}[w]$.



Shortest-paths optimality conditions

Proposition. Let G be an edge-weighted digraph.

Then $\text{distTo}[]$ are the shortest path distances from s iff:

- $\text{distTo}[s] = 0$.
- For each vertex v , $\text{distTo}[v]$ is the length of some path from s to v .
- For each edge $e = v \rightarrow w$, $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$.

Pf. \Rightarrow [sufficient]

- Suppose that $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k = w$ is a shortest path from s to w .
- Then, $\text{distTo}[v_1] \leq \text{distTo}[v_0] + e_1.\text{weight}()$
 $\text{distTo}[v_2] \leq \text{distTo}[v_1] + e_2.\text{weight}()$
 \dots
 $\text{distTo}[v_k] \leq \text{distTo}[v_{k-1}] + e_k.\text{weight}()$

$e_i = i^{\text{th}}$ edge on shortest path from s to w

- Add inequalities; simplify; and substitute $\text{distTo}[v_0] = \text{distTo}[s] = 0$:

$$\text{distTo}[w] = \text{distTo}[v_k] \leq e_1.\text{weight}() + e_2.\text{weight}() + \dots + e_k.\text{weight}()$$

weight of shortest path from s to w

- Thus, $\text{distTo}[w]$ is the weight of shortest path to w . ■

weight of some path from s to w

Generic shortest-paths algorithm

Generic algorithm (to compute SPT from s)

Initialize $\text{distTo}[s] = 0$ and $\text{distTo}[v] = \infty$ for all other vertices.

Repeat until optimality conditions are satisfied:

- Relax any edge.
-

Proposition. Generic algorithm computes SPT (if it exists) from s .

Pf sketch.

- Throughout algorithm, $\text{distTo}[v]$ is the length of a simple path from s to v (and $\text{edgeTo}[v]$ is last edge on path).
- Each successful relaxation decreases $\text{distTo}[v]$ for some v .
- The entry $\text{distTo}[v]$ can decrease at most a finite number of times. ■

Generic shortest-paths algorithm

Generic algorithm (to compute SPT from s)

Initialize $\text{distTo}[s] = 0$ and $\text{distTo}[v] = \infty$ for all other vertices.

Repeat until optimality conditions are satisfied:

- Relax any edge.
-

Efficient implementations. How to choose which edge to relax?

Ex 1. Dijkstra's algorithm (nonnegative weights).

Ex 2. Topological sort algorithm (no directed cycles).

Ex 3. Bellman-Ford algorithm (no negative cycles).

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Edsger W. Dijkstra: select quotes

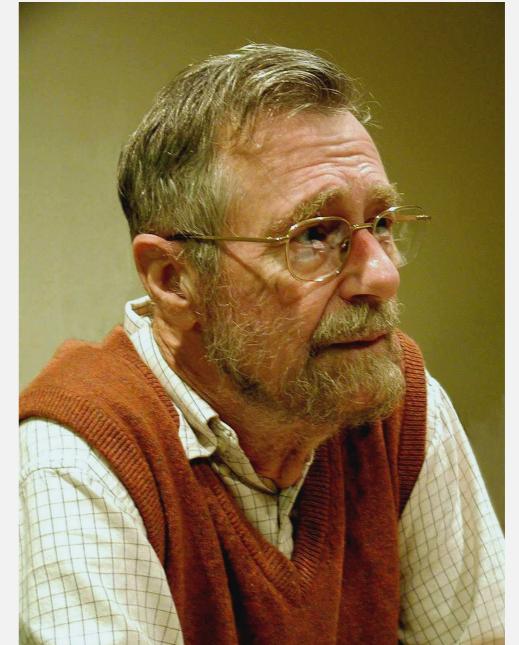
“Do only what only you can do.”

“In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.”

“The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence.”

“It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration.”

“APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.”



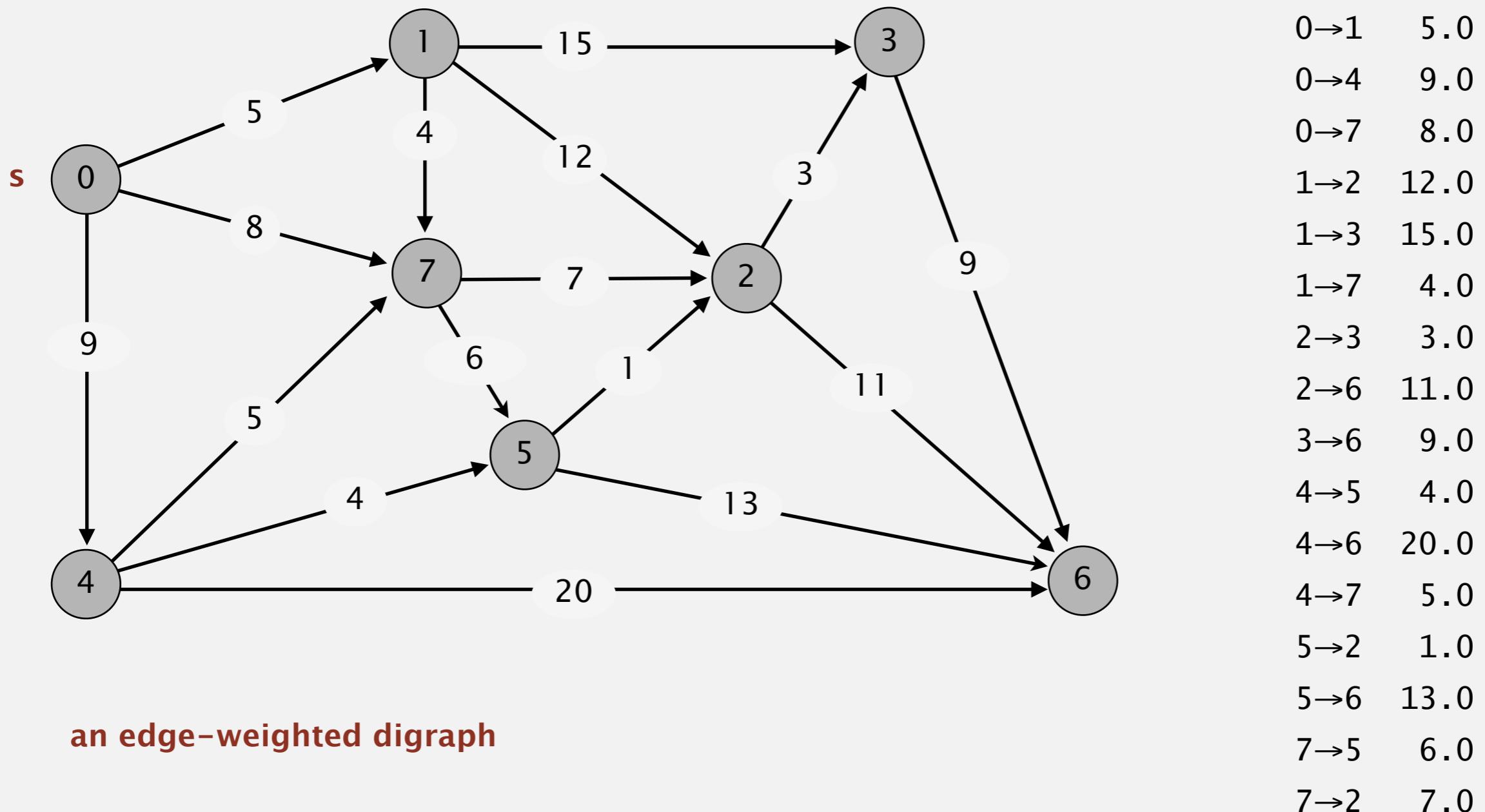
Edsger W. Dijkstra
Turing award 1972

Edsger W. Dijkstra: select quotes



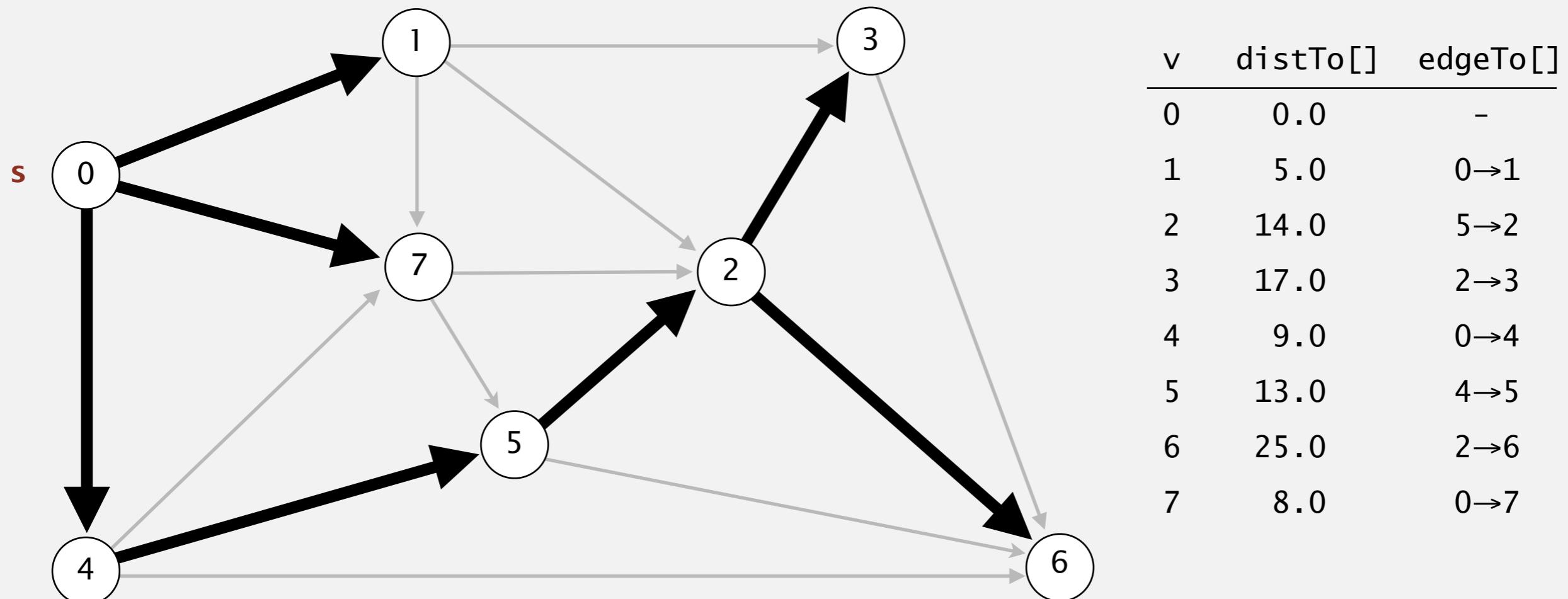
Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest $\text{distTo}[]$ value).
- Add vertex to tree and relax all edges pointing from that vertex.



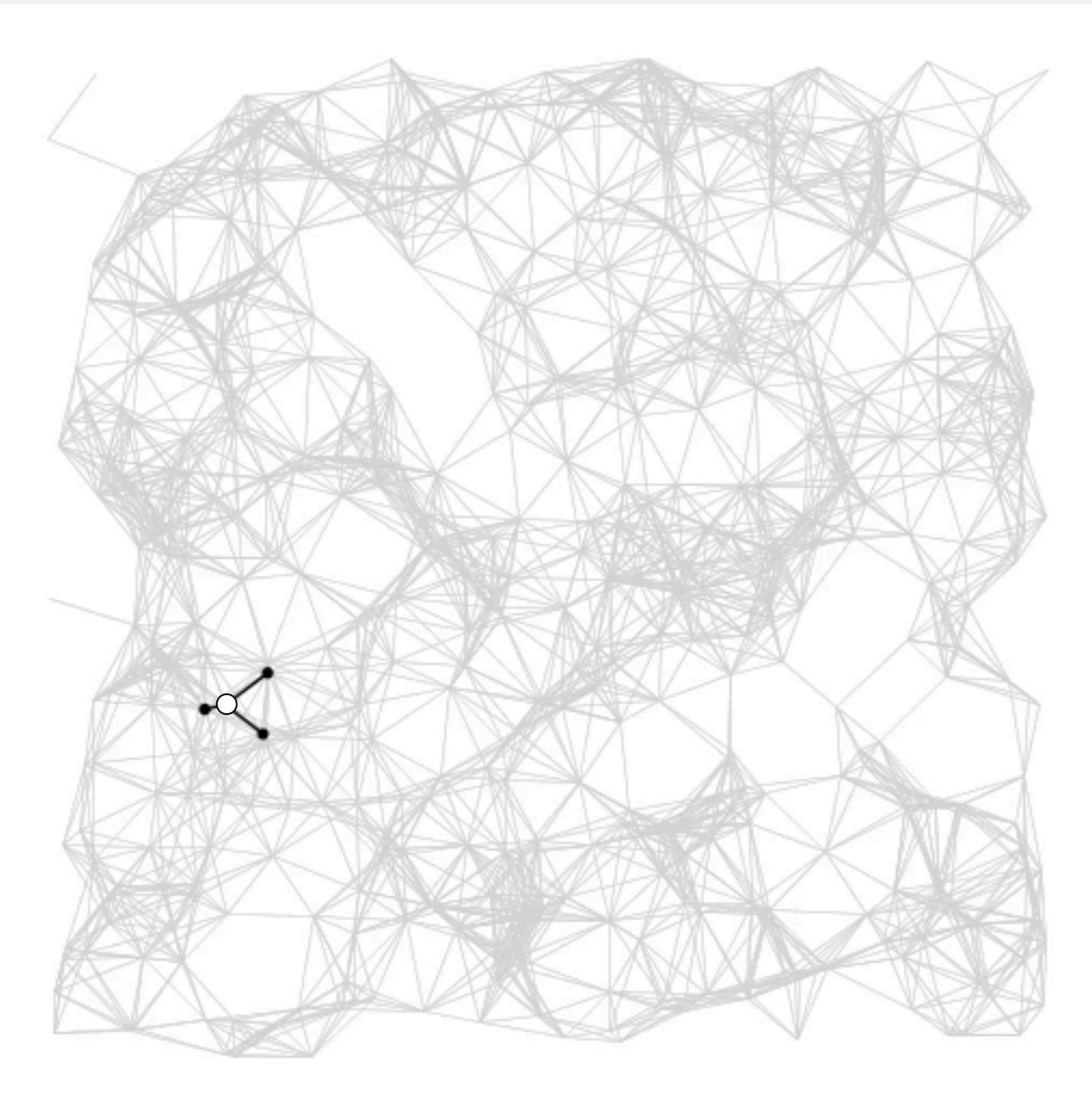
Dijkstra's algorithm demo

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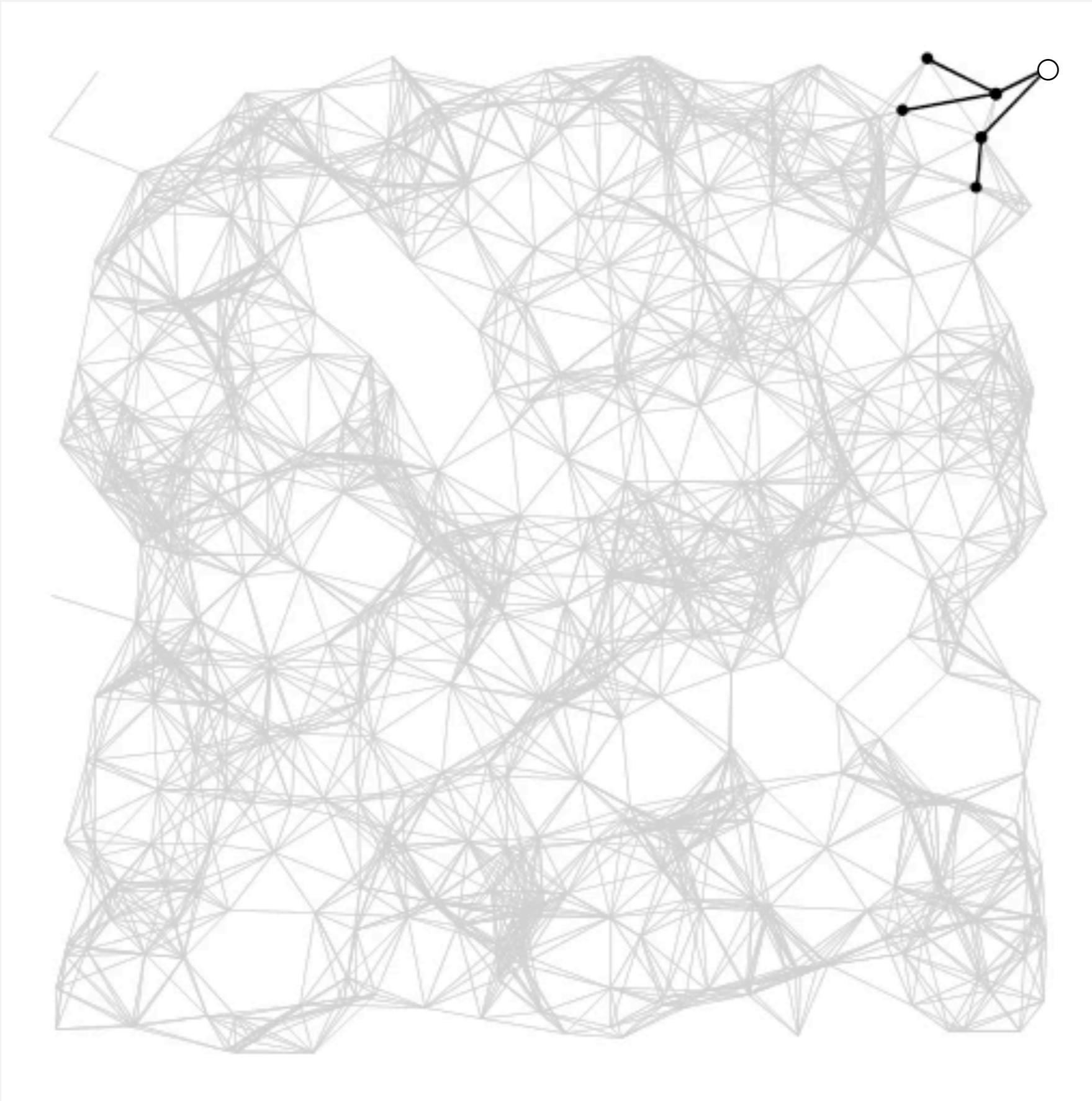


shortest-paths tree from vertex s

Dijkstra's algorithm visualization



Dijkstra's algorithm visualization



Dijkstra's algorithm: correctness proof

Proposition. Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

Pf.

- Each edge $e = v \rightarrow w$ is relaxed exactly once (when v is relaxed), leaving $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$.
- Inequality holds until algorithm terminates because:
 - $\text{distTo}[w]$ cannot increase ← distTo[] values are monotone decreasing
 - $\text{distTo}[v]$ will not change ← we choose lowest distTo[] value at each step
(and edge weights are nonnegative)
- Thus, upon termination, shortest-paths optimality conditions hold. ■

Dijkstra's algorithm: Java implementation

```
public class DijkstraSP
{
    private DirectedEdge[] edgeTo;
    private double[] distTo;
    private IndexMinPQ<Double> pq;

    public DijkstraSP(EdgeWeightedDigraph G, int s)
    {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];
        pq = new IndexMinPQ<Double>(G.V());

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        pq.insert(s, 0.0);
        while (!pq.isEmpty())
        {
            int v = pq.delMin();
            for (DirectedEdge e : G.adj(v))
                relax(e);
        }
    }
}
```

←
relax vertices in order
of distance from s

Dijkstra's algorithm: Java implementation

```
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
        if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
        else                  pq.insert      (w, distTo[w]);
    }
}
```



update PQ

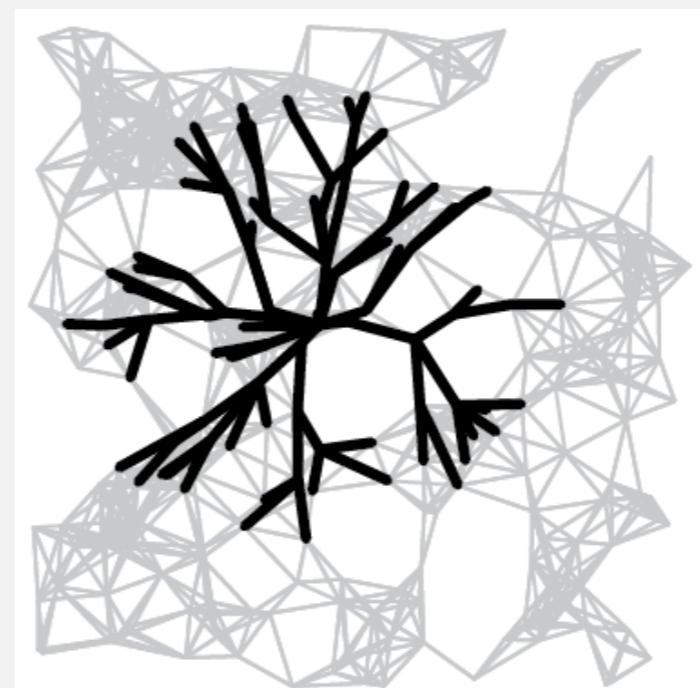
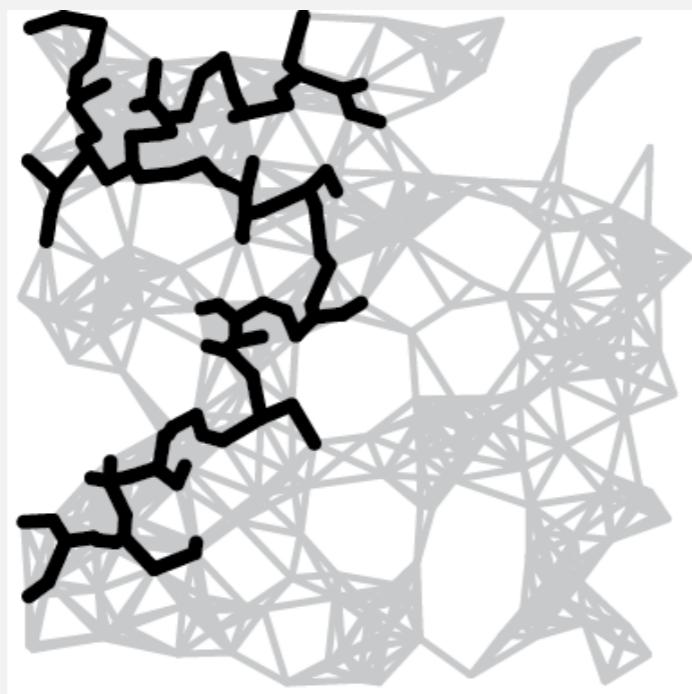
Computing spanning trees in graphs

Dijkstra's algorithm seem familiar?

- Prim's algorithm is essentially the same algorithm.
- Both are in a family of algorithms that compute a graph's spanning tree.

Main distinction: Rule used to choose next vertex for the tree.

- Prim's: Closest vertex to the **tree** (via an undirected edge).
- Dijkstra's: Closest vertex to the **source** (via a directed path).



Note: DFS and BFS are also in this family of algorithms.

Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: V insert, V delete-min, E decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
unordered array	1	V	1	V^2
binary heap	$\log V$	$\log V$	$\log V$	$E \log V$
d-way heap (Johnson 1975)	$\log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap (Fredman-Tarjan 1984)	1^\dagger	$\log V^\dagger$	1^\dagger	$E + V \log V$

\dagger amortized

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

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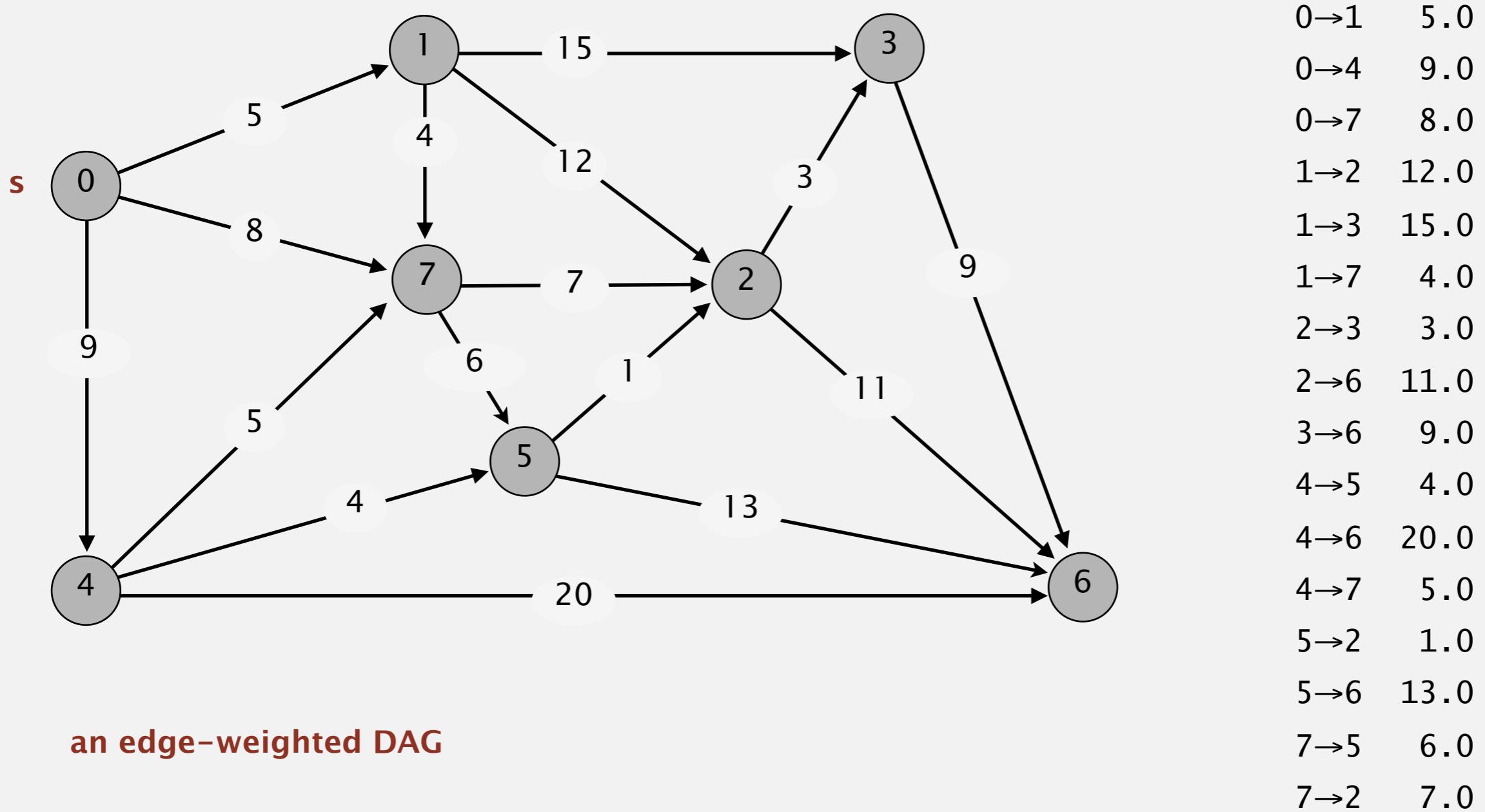
Acyclic edge-weighted digraphs

Q. Suppose that an edge-weighted digraph has no directed cycles.
Is it easier to find shortest paths than in a general digraph?

A. Yes!

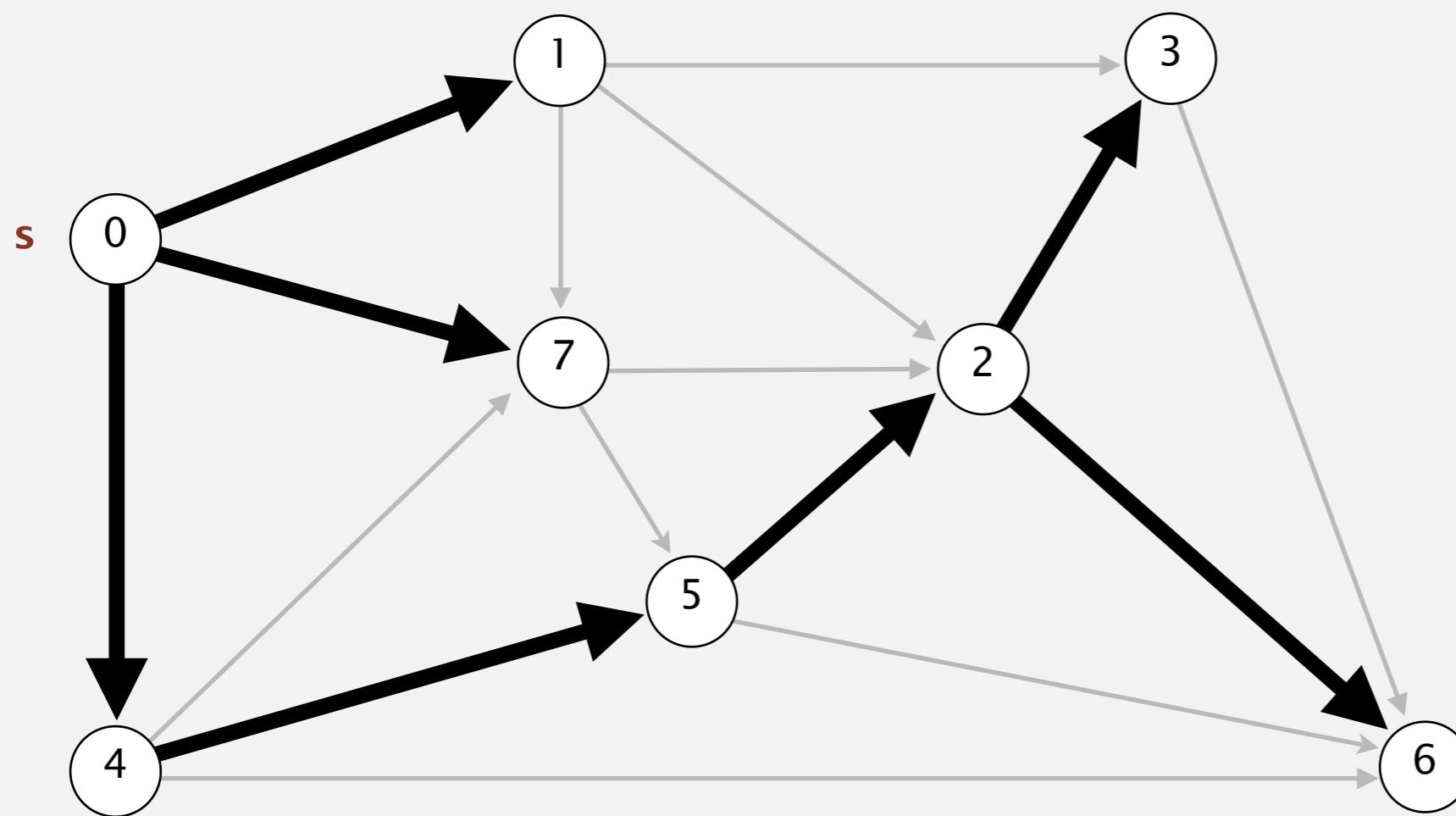
Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges pointing from that vertex.



Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges pointing from that vertex.



shortest-paths tree from vertex s

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

Shortest paths in edge-weighted DAGs

Proposition. Topological sort algorithm computes SPT in any edge-weighted DAG in time proportional to $E + V$.

edge weights
can be negative!

Pf.

- Each edge $e = v \rightarrow w$ is relaxed exactly once (when v is relaxed), leaving $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$.
- Inequality holds until algorithm terminates because:
 - $\text{distTo}[w]$ cannot increase ← $\text{distTo}[]$ values are monotone decreasing
 - $\text{distTo}[v]$ will not change ← because of topological order, no edge pointing to v will be relaxed after v is relaxed
- Thus, upon termination, shortest-paths optimality conditions hold. ■

Shortest paths in edge-weighted DAGs

```
public class AcyclicSP
{
    private DirectedEdge[] edgeTo;
    private double[] distTo;

    public AcyclicSP(EdgeWeightedDigraph G, int s)
    {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        Topological topological = new Topological(G); ← topological order
        for (int v : topological.order())
            for (DirectedEdge e : G.adj(v))
                relax(e);
    }
}
```

Content-aware resizing

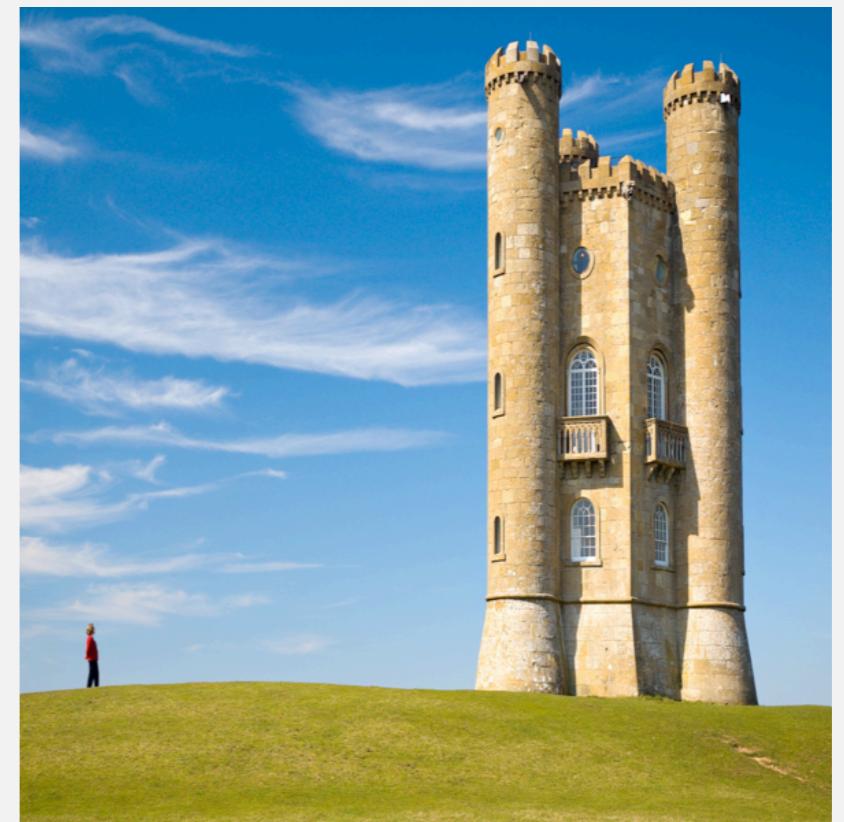
Seam carving. [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.



Shai Avidan
Mitsubishi Electric Research Lab
Ariel Shamir
The interdisciplinary Center & MERL

Content-aware resizing

[Seam carving.](#) [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.



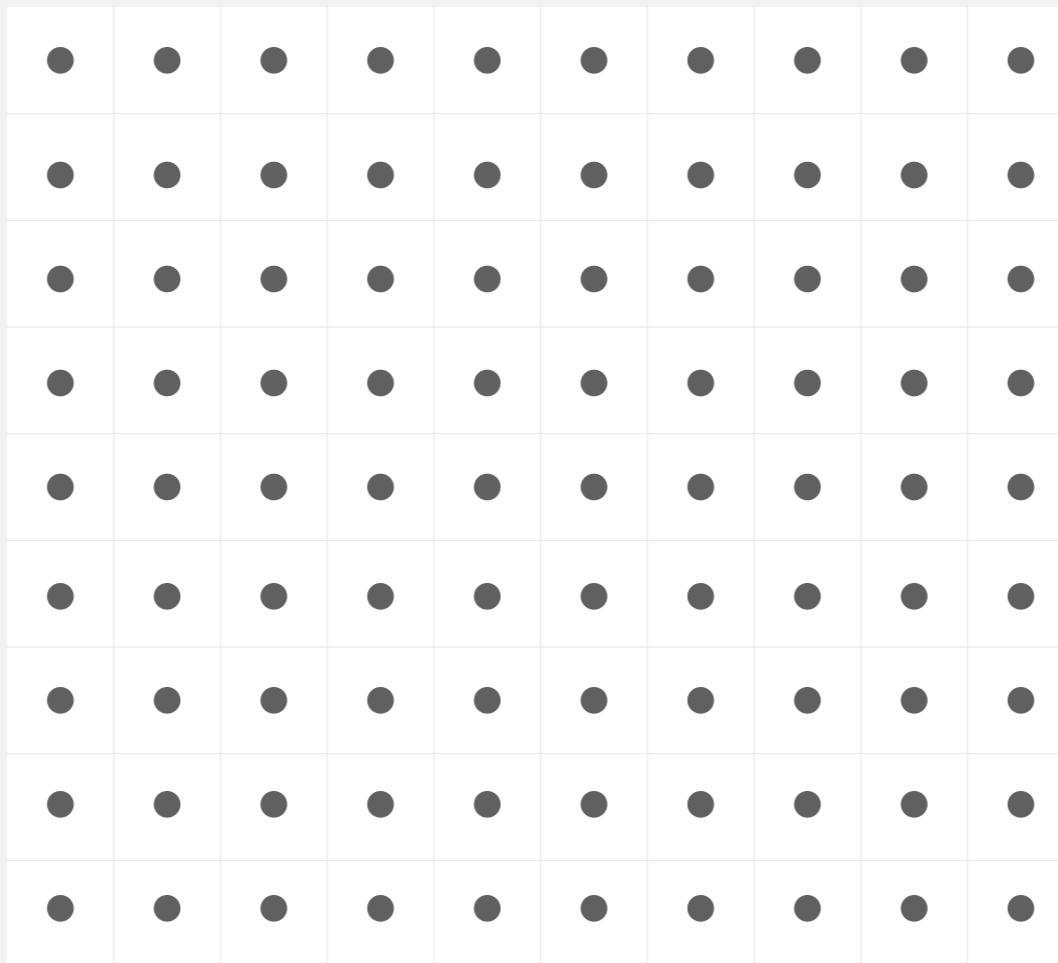
[In the wild.](#) Photoshop CS 5, Imagemagick, GIMP, ...



Content-aware resizing

To find vertical seam:

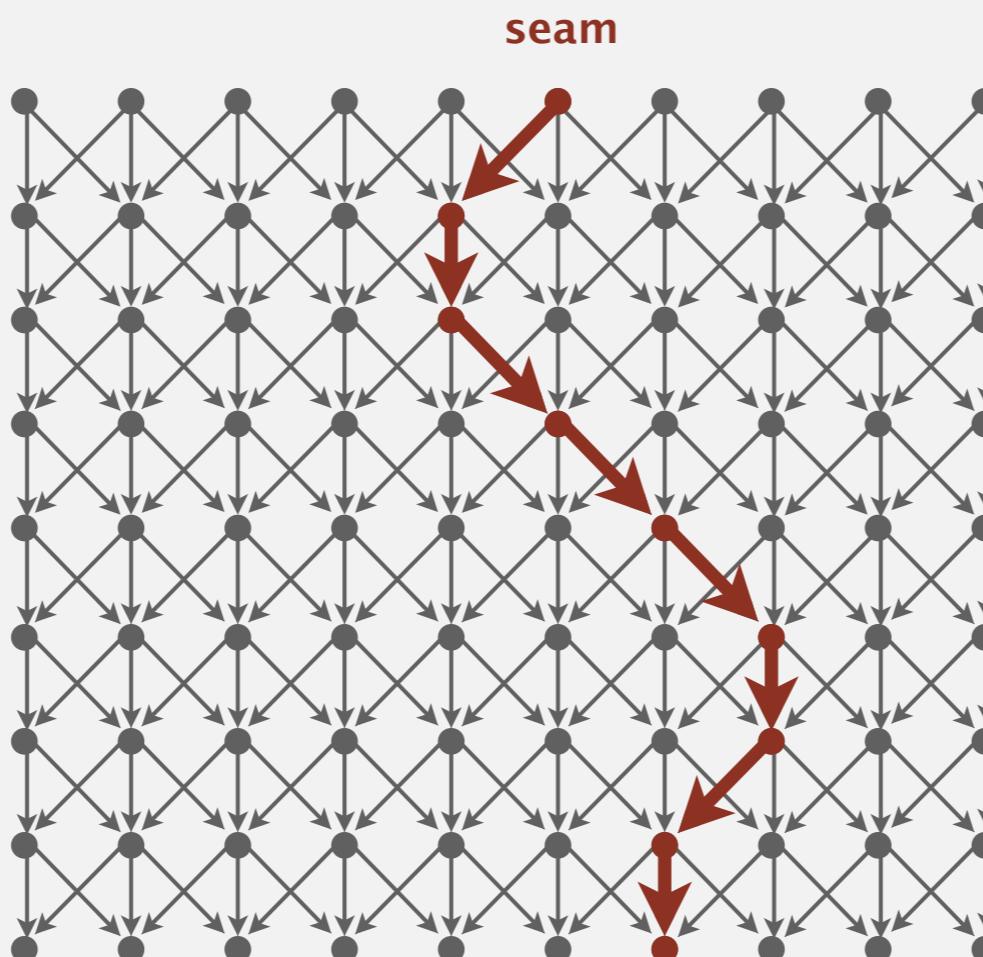
- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.



Content-aware resizing

To find vertical seam:

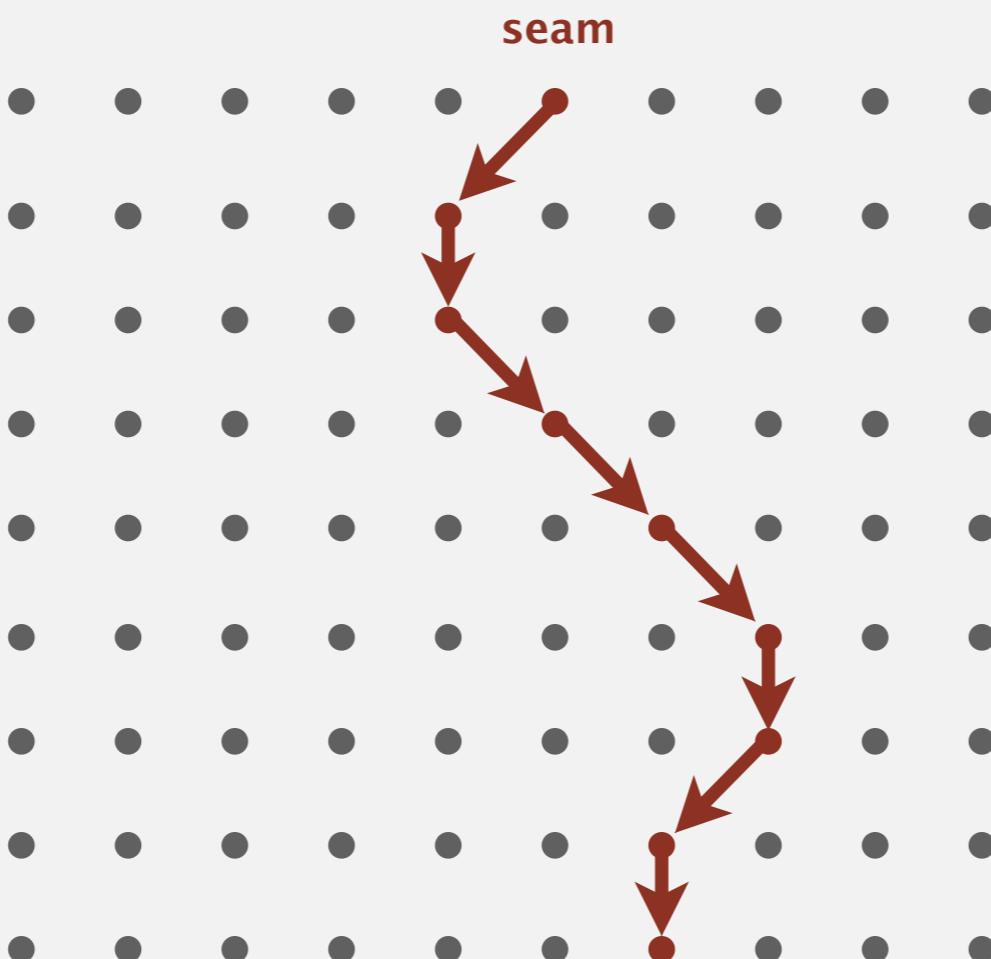
- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.



Content-aware resizing

To remove vertical seam:

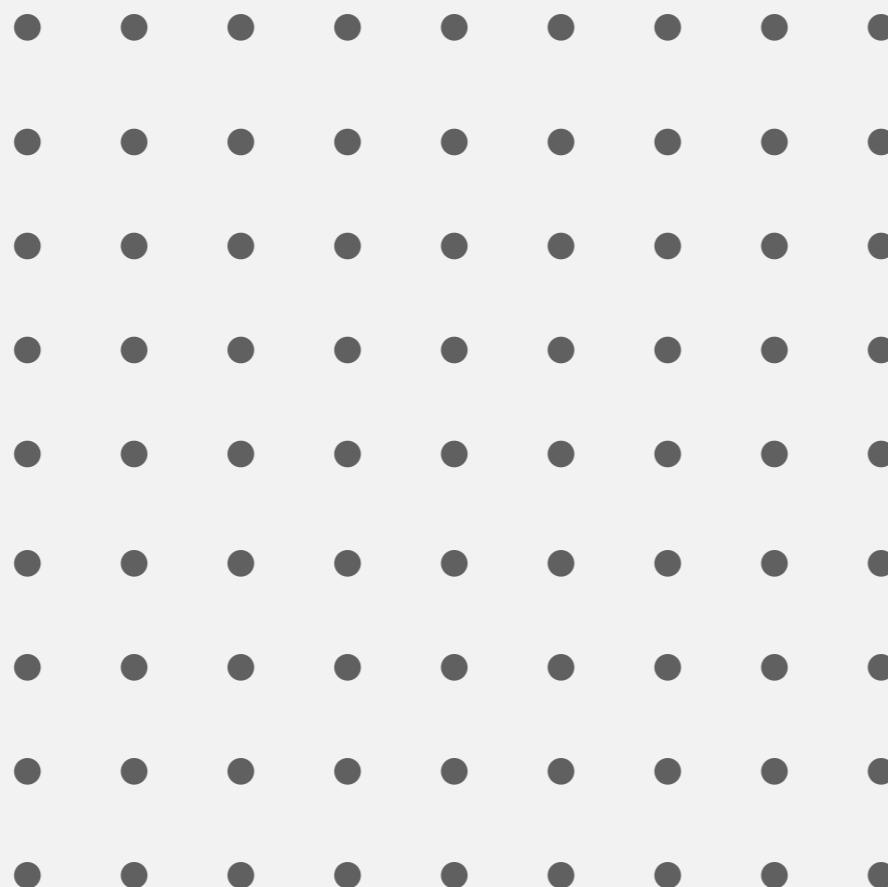
- Delete pixels on seam (one in each row).



Content-aware resizing

To remove vertical seam:

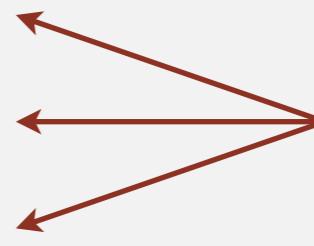
- Delete pixels on seam (one in each row).



Longest paths in edge-weighted DAGs

Formulate as a shortest paths problem in edge-weighted DAGs.

- Negate all weights.
- Find shortest paths.
- Negate weights in result.



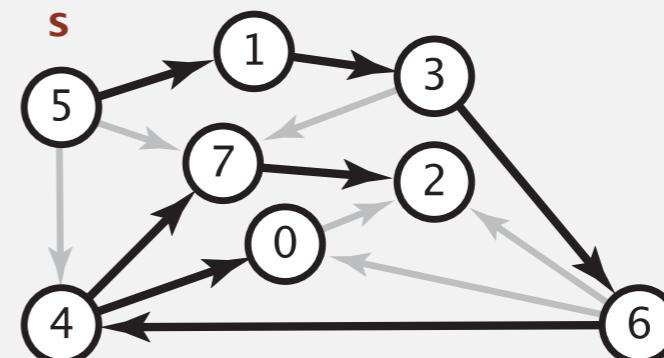
equivalent: reverse sense of equality in relax()

longest paths input

5->4	0.35
4->7	0.37
5->7	0.28
5->1	0.32
4->0	0.38
0->2	0.26
3->7	0.39
1->3	0.29
7->2	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93

shortest paths input

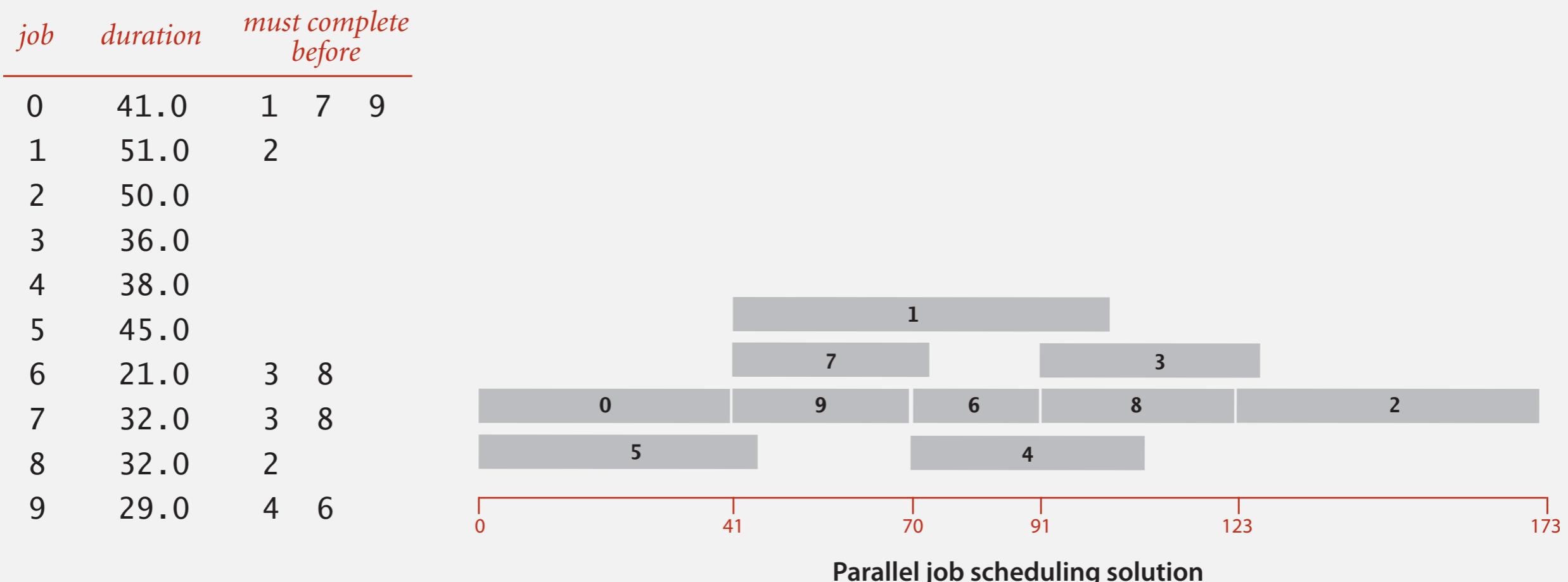
5->4	-0.35
4->7	-0.37
5->7	-0.28
5->1	-0.32
4->0	-0.38
0->2	-0.26
3->7	-0.39
1->3	-0.29
7->2	-0.34
6->2	-0.40
3->6	-0.52
6->0	-0.58
6->4	-0.93



Key point. Topological sort algorithm works even with negative weights.

Longest paths in edge-weighted DAGs: application

Parallel job scheduling. Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

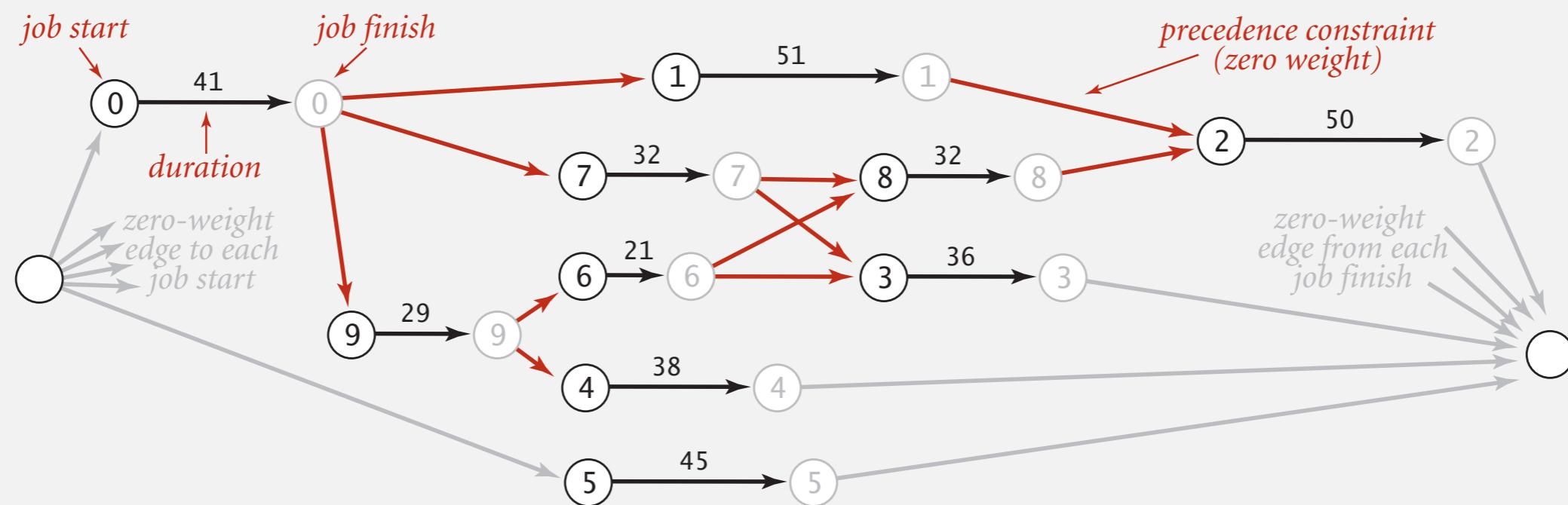


Critical path method

CPM. To solve a parallel job-scheduling problem, create edge-weighted DAG:

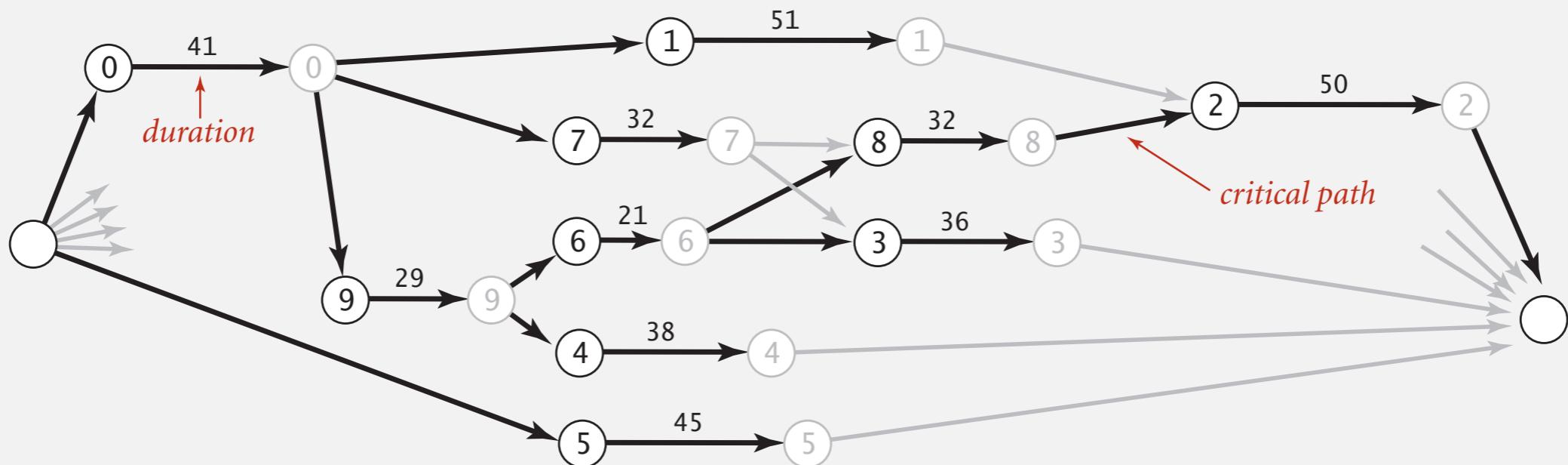
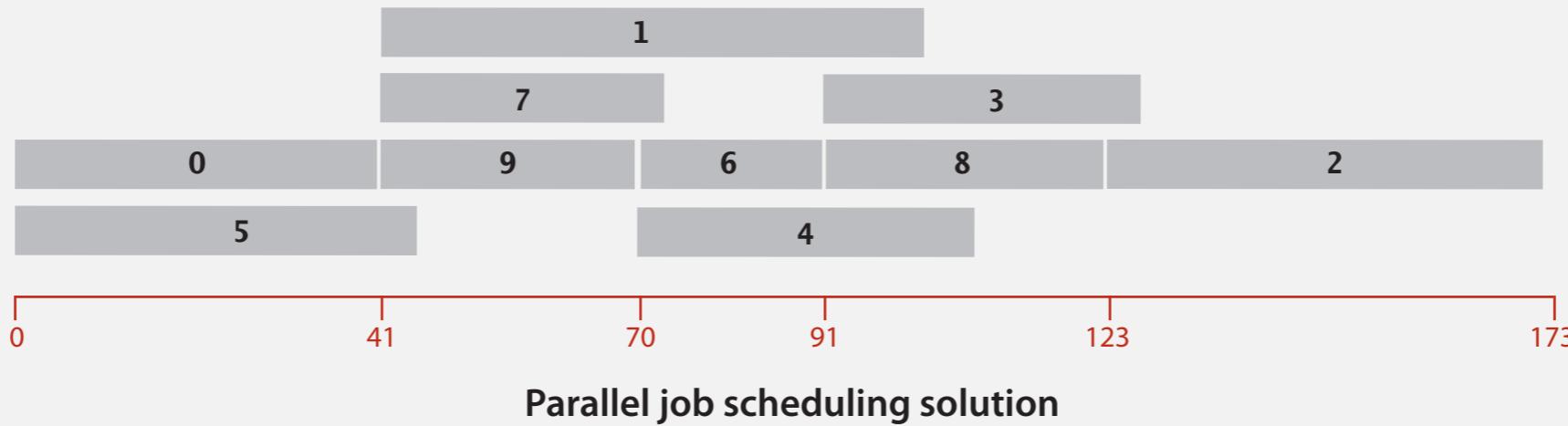
- Source and sink vertices.
- Two vertices (begin and end) for each job.
- Three edges for each job.
 - begin to end (weighted by duration)
 - source to begin (0 weight)
 - end to sink (0 weight)
- One edge for each precedence constraint (0 weight).

<i>job</i>	<i>duration</i>	<i>must complete before</i>		
0	41.0	1	7	9
1	51.0	2		
2	50.0			
3	36.0			
4	38.0			
5	45.0			
6	21.0	3	8	
7	32.0	3	8	
8	32.0	2		
9	29.0	4	6	



Critical path method

CPM. Use **longest path** from the source to schedule each job.



Algorithms

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4.4 SHORTEST PATHS

- ▶ APIs
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ **edge-weighted DAGs**
- ▶ *negative weights*

Algorithms

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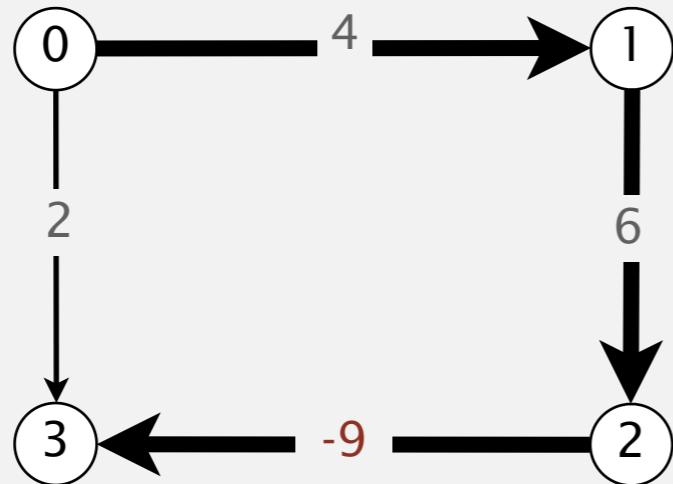
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- ▶ *APIs*
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- ▶ *edge-weighted DAGs*
- ▶ ***negative weights***

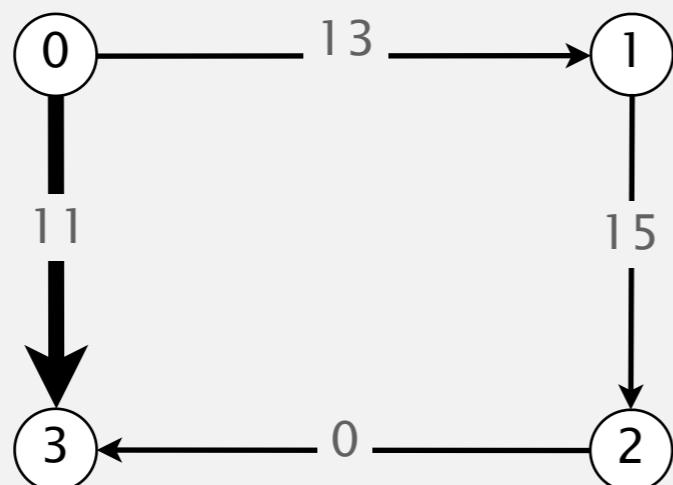
Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.



Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$.

Re-weighting. Add a constant to every edge weight doesn't work.



Adding 9 to each edge weight changes the
shortest path from $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$ to $0 \rightarrow 3$.

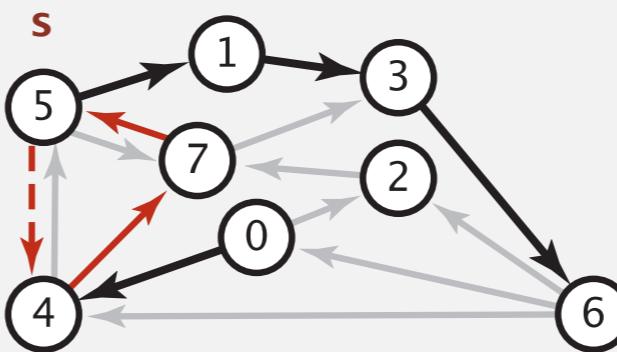
Conclusion. Need a different algorithm.

Negative cycles

Def. A **negative cycle** is a directed cycle whose sum of edge weights is negative.

digraph

4->5	0.35
5->4	-0.66
4->7	0.37
5->7	0.28
7->5	0.28
5->1	0.32
0->4	0.38
0->2	0.26
7->3	0.39
1->3	0.29
2->7	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93



negative cycle $(-0.66 + 0.37 + 0.28)$

$5 \rightarrow 4 \rightarrow 7 \rightarrow 5$

shortest path from 0 to 6

$0 \rightarrow 4 \rightarrow 7 \rightarrow 5 \rightarrow 4 \rightarrow 7 \rightarrow 5 \dots \rightarrow 1 \rightarrow 3 \rightarrow 6$

Proposition. A SPT exists iff no negative cycles.

assuming all vertices reachable from s

Bellman-Ford algorithm

Bellman-Ford algorithm

Initialize $\text{distTo}[s] = 0$ and $\text{distTo}[v] = \infty$ for all other vertices.

Repeat V times:

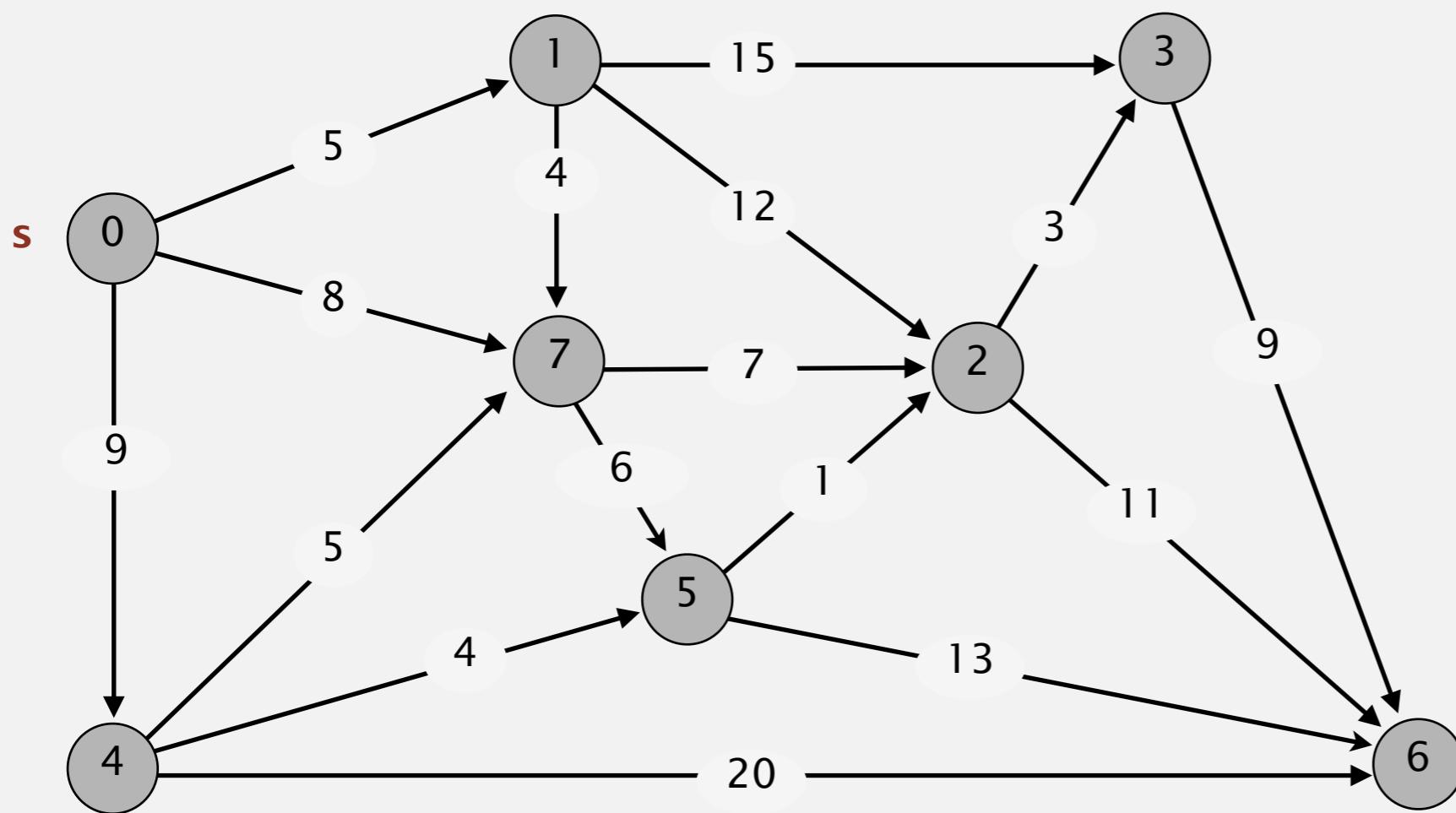
- Relax each edge.

```
for (int i = 0; i < G.V(); i++)
    for (int v = 0; v < G.V(); v++)
        for (DirectedEdge e : G.adj(v))
            relax(e);
```

← pass i (relax each edge)

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.

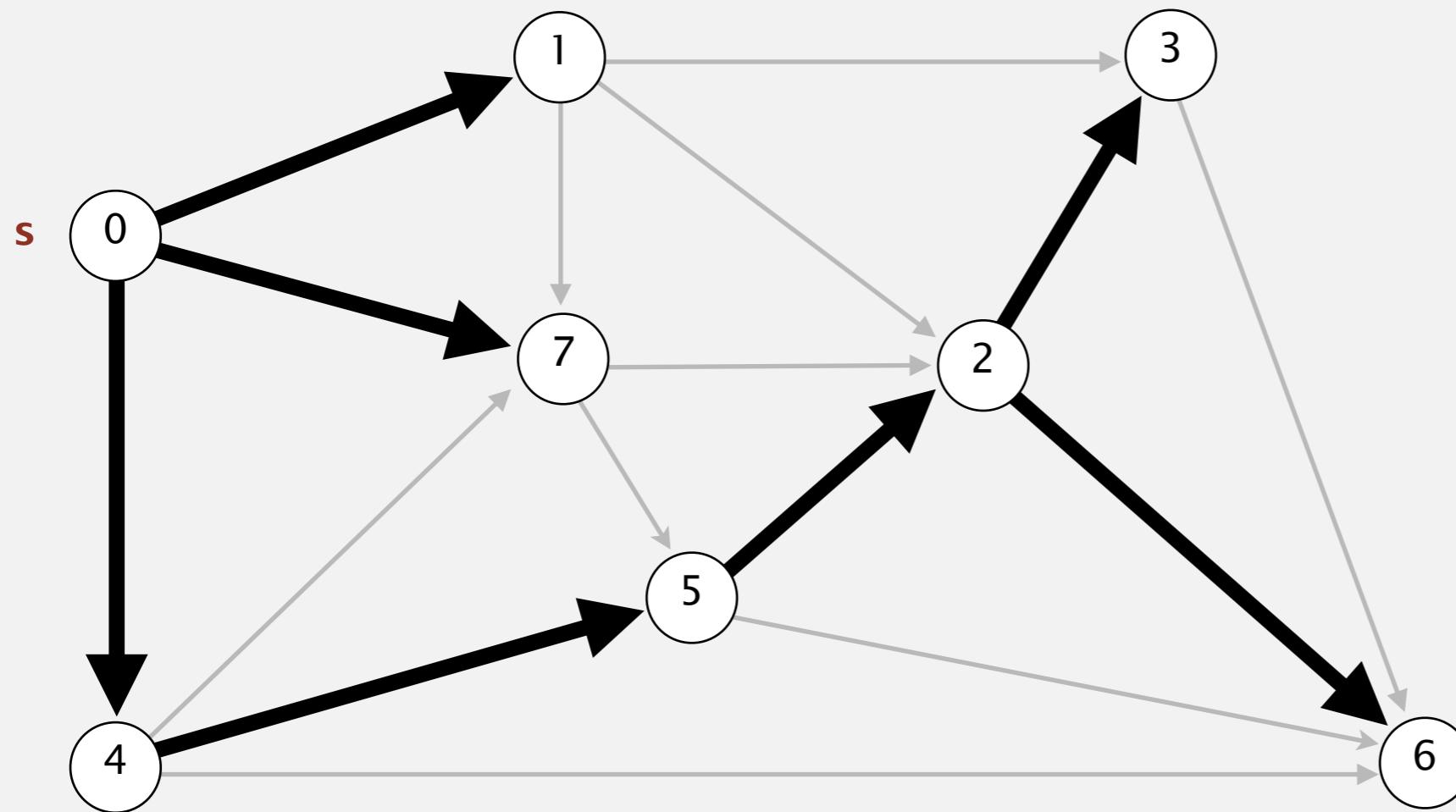


0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

an edge-weighted digraph

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



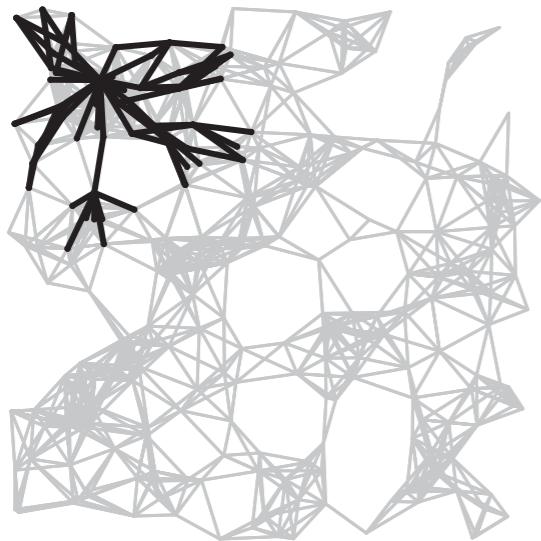
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

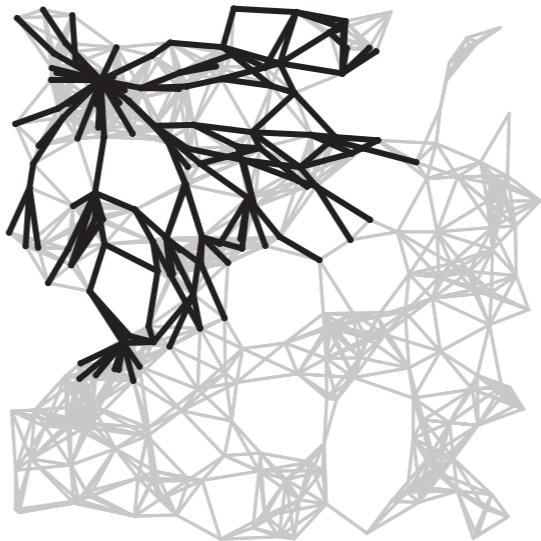
Bellman-Ford algorithm visualization

passes

4



7



10



13



SPT



Bellman-Ford algorithm: analysis

Bellman-Ford algorithm

Initialize $\text{distTo}[s] = 0$ and $\text{distTo}[v] = \infty$ for all other vertices.

Repeat V times:

- Relax each edge.
-

Proposition. Dynamic programming algorithm computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$.

Pf idea. After pass i , found shortest path containing at most i edges.

Bellman-Ford algorithm: practical improvement

Observation. If $\text{distTo}[v]$ does not change during pass i , no need to relax any edge pointing from v in pass $i+1$.

FIFO implementation. Maintain **queue** of vertices whose $\text{distTo}[]$ changed.



be careful to keep at most one copy
of each vertex on queue (why?)

Overall effect.

- The running time is still proportional to $E \times V$ in worst case.
- But much faster than that in practice.

Single source shortest-paths implementation: cost summary

algorithm	restriction	typical case	worst case	extra space
topological sort	no directed cycles	$E + V$	$E + V$	V
Dijkstra (binary heap)	no negative weights	$E \log V$	$E \log V$	V
Bellman-Ford	no negative cycles	$E V$	$E V$	V
Bellman-Ford (queue-based)		$E + V$	$E V$	V

Remark 1. Directed cycles make the problem harder.

Remark 2. Negative weights make the problem harder.

Remark 3. Negative cycles makes the problem intractable.

Finding a negative cycle

Negative cycle. Add two method to the API for SP.

boolean hasNegativeCycle()

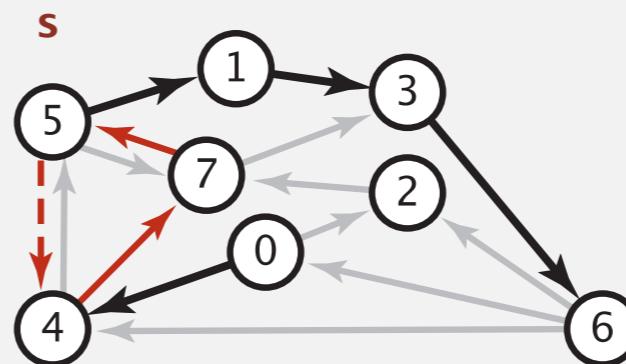
is there a negative cycle?

Iterable <DirectedEdge> negativeCycle()

negative cycle reachable from s

digraph

```
4->5  0.35
5->4 -0.66
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```

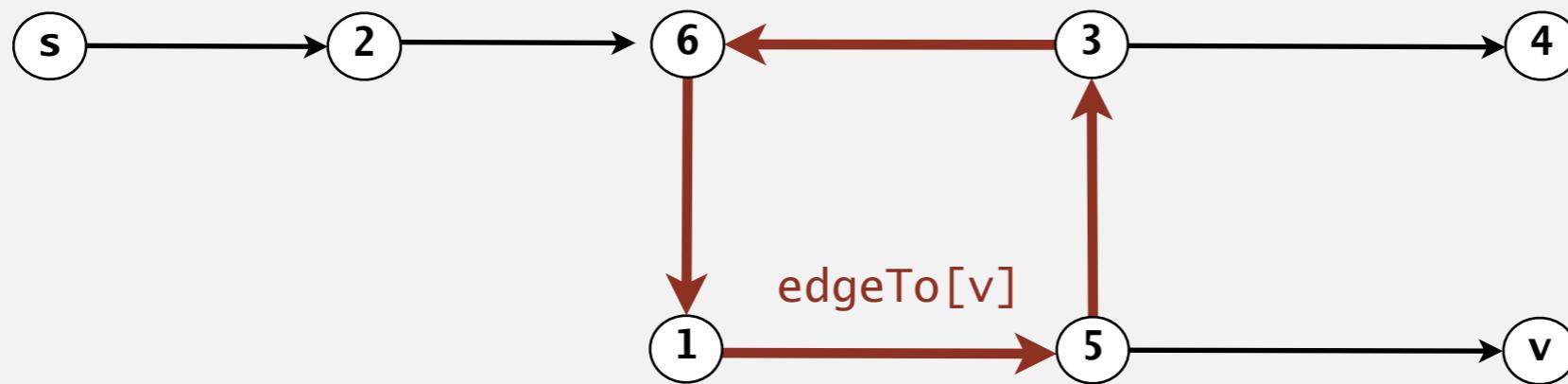


negative cycle (-0.66 + 0.37 + 0.28)

5->4->7->5

Finding a negative cycle

Observation. If there is a negative cycle, Bellman-Ford gets stuck in loop, updating `distTo[]` and `edgeTo[]` entries of vertices in the cycle.



Proposition. If any vertex v is updated in phase v , there exists a negative cycle (and can trace back $\text{edgeTo}[v]$ entries to find it).

In practice. Check for negative cycles more frequently.

Negative cycle application: arbitrage detection

Problem. Given table of exchange rates, is there an arbitrage opportunity?

	USD	EUR	GBP	CHF	CAD
USD	1	0.741	0.657	1.061	1.011
EUR	1.350	1	0.888	1.433	1.366
GBP	1.521	1.126	1	1.614	1.538
CHF	0.943	0.698	0.620	1	0.953
CAD	0.995	0.732	0.650	1.049	1

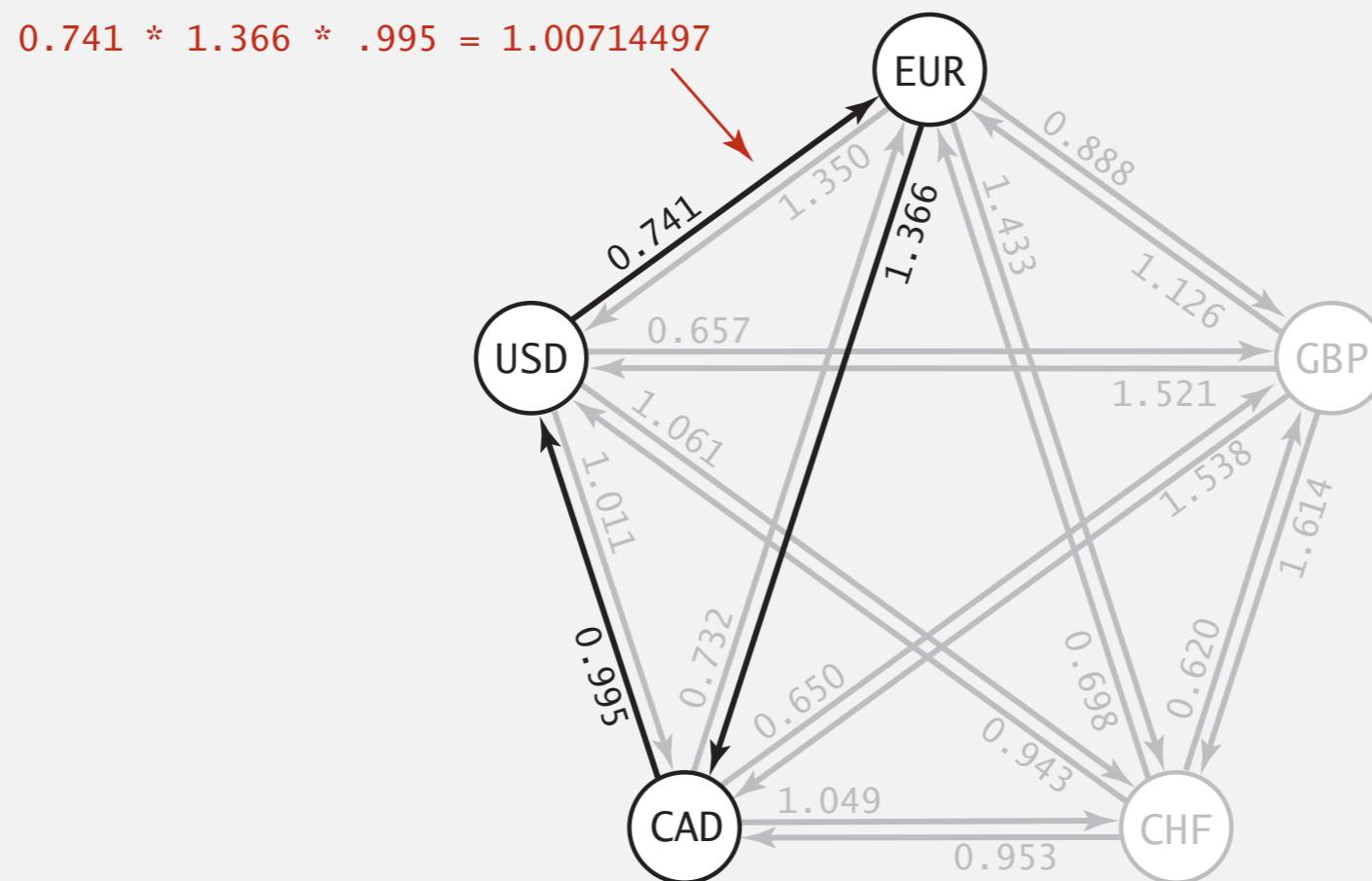
Ex. \$1,000 \Rightarrow 741 Euros \Rightarrow 1,012.206 Canadian dollars \Rightarrow \$1,007.14497.

$$1000 \times 0.741 \times 1.366 \times 0.995 = 1007.14497$$

Negative cycle application: arbitrage detection

Currency exchange graph.

- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find a directed cycle whose product of edge weights is > 1 .

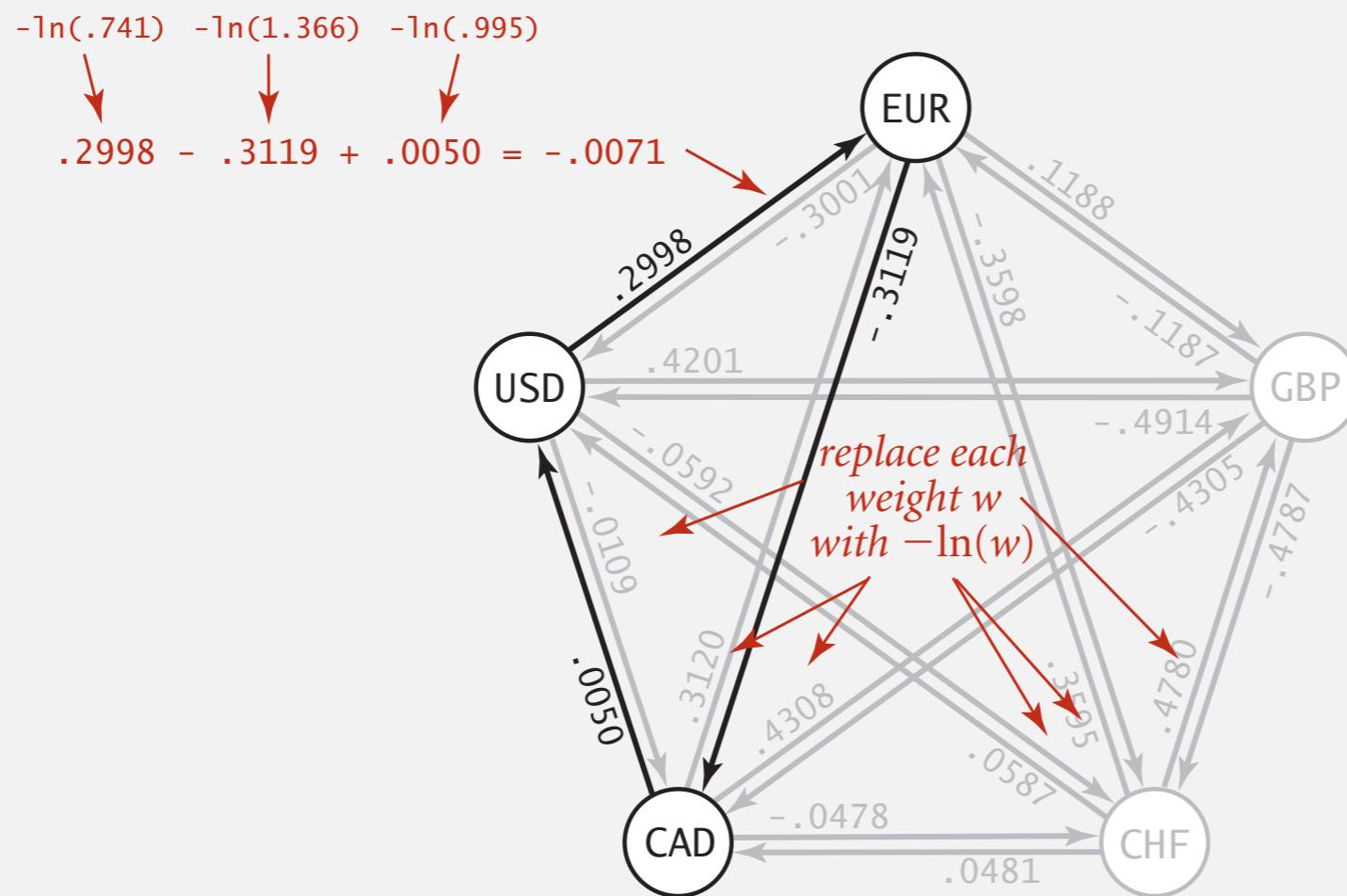


Challenge. Express as a negative cycle detection problem.

Negative cycle application: arbitrage detection

Model as a negative cycle detection problem by taking logs.

- Let weight of edge $v \rightarrow w$ be $-\ln$ (exchange rate from currency v to w).
- Multiplication turns to addition; > 1 turns to < 0 .
- Find a directed cycle whose sum of edge weights is < 0 (negative cycle).



Remark. Fastest algorithm is extraordinarily valuable!

Shortest paths summary

Dijkstra's algorithm.

- Nearly linear-time when weights are nonnegative.
- Generalization encompasses DFS, BFS, and Prim.

Acyclic edge-weighted digraphs.

- Arise in applications.
- Faster than Dijkstra's algorithm.
- Negative weights are no problem.

Negative weights and negative cycles.

- Arise in applications.
- If no negative cycles, can find shortest paths via Bellman-Ford.
- If negative cycles, can find one via Bellman-Ford.

Shortest-paths is a broadly useful problem-solving model.

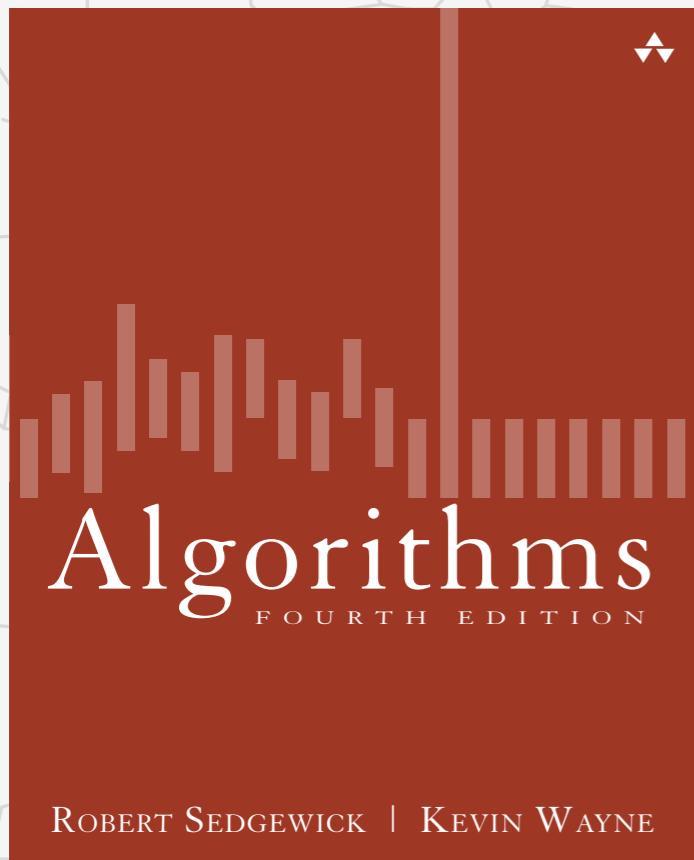
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