MINERVA INVESTMENT MANAGEMENT SOCIETY
Quantitative Research Division

Barrier Options

Theoretical and Numerical Approach

Federico Lazzarini	Head of Quantitative Research
Filip Borzecki	Quantitative Research Analyst
Michele Fanelli	Quantitative Research Analyst
Giani Ranzetti	Quantitative Research Analyst

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Abstract

In this primer on derivatives pricing, we explore various types of barrier options, providing a classical Black and Scholes approach and introducing fundamental concepts of stochastic calculus. We also provide an example of a hands-on approach consisting of a numerical approximation with MonteCarlo simulation.

Link to replication code: [LINK]

1 Introduction

A barrier option consists of:

1. a payoff $\phi(S_T)$
2. a barrier B

The payoff $\phi(S_T)$ is paid out at maturity T conditional on the stock price having crossed (or not having crossed) the barrier B at any time $t \in [0, T]$.

- **Knock-in options** are "activated" if the underlying price crosses the barrier: once "knocked-in" the option is valid until maturity whatever happens to the stock price and effectively behaves like a vanilla option. If the barrier is not crossed by maturity the instrument expires worthless.
- **Knock-out options** on the other hand are "deactivated" if the underlying price crosses the barrier. If the latter is hit at any point in time before maturity the instrument expires worthless, otherwise it has the same payoff as a vanilla option.

The barrier is set at inception and can be either above or below the initial stock price; accordingly, barrier options can be classified as either up-and-out, up-and-in, down-and-out, down-and-in (each type can be either a call or a put). Naturally, these instruments are cheaper than vanilla options, which is the main appeal for investors, together with their flexibility; on the other hand they are less traded and thus less liquid.

2 In-Out Parity

An example of the payoff function of a down-and-out (knock-out) call option is the following:

$$\begin{cases} max(S_T - K, 0) & \text{if } S_t > B \text{ for all } t \leq T \\ 0 & \text{otherwise} \end{cases}$$

or in other terms:

$$\begin{cases} (S_T - K)1_F \\ F = \{S_T > K, min_{0 \leq t \leq T}(S_t) > B\} \end{cases}$$

Turns out there exists something akin to the put-call parity equation for barrier options. Take for example a down-and-out call with barrier B and a down-and-in call with the same barrier, underlying, strike and maturity: then a portfolio of the two perfectly replicates the payoff of a vanilla option with the same characteristics. This means that if we denote the price of a down-and-out call option as $C_{do}(S, B, K, T)$ and of a down-and-in as $C_d i(S, B, K, T)$ then:

$$C_{do}(S, B, K, T) + C_{di}(S, B, K, T) = C(S, K, T)$$

This holds true for every type of barrier option, and is especially useful since we can derive an analytic formula for the price of the knock-in option and derive the knock-out price with the in-and-out parity.

3 Pricing a Down-and-In Call Option

By risk neutral valuation:

$$\begin{cases} C_{di}(S, B, K, T) = e^{-rT} E^Q[(S_T - K)1_F] \\ F = \{S_T > K, min_{0\leq t\leq T}(S_t) < B\} \end{cases}$$

By linearity of expectations and recalling the definition of the indicator function $E^P[1_F] = P(F)$ we can write:

$$C_{di}(S, B, K, T) = e^{-rT} E^Q[S_T 1_F] - e^{-rT} K Q(F) \tag{1}$$

Now to go on we need to introduce a few mathematical concepts, namely the **Girsanov theorem**, the **change of numeraire formula**, and the **reflection principle**.

Girsanov Theorem

Let W_t^P be a Wiener process under the measure P. Let k_t be an adapted process and define

$$dL_t = k_t L_t dW_t^P, \quad L_0 = 1.$$

Assume $\mathbb{E}^P[L_T] = 1$ and define Q on $\mathcal{F}_T$ by

$$L_T = \frac{dQ}{dP}.$$

Then

$$dW_t^Q = dW_t^P - k_t dt,$$

defines a Wiener process W_t^Q under Q.

Note that L_t is by construction a martingale (thus $L_t = \mathbb{E}^P[L_T|F_t]$) and is also always positive. It is called a *Radon-Nikodym* derivative and under certain assumptions (including *adaptness* of X) the following properties hold:

- $\mathbb{E}_t^Q[X] = \mathbb{E}_t^P[L_T X]$
- $\mathbb{E}_t^P[X] = \mathbb{E}_t^Q[L_T^{-1} X]$

.

Girsanov theorem heuristically allows us to change the dynamics of an Ito process from a probability measure P to a probability measure Q while keeping the underlying Wiener process coherent to its definition. In fact, this is especially useful to simplify calculations and is the base for the **change of numeraire formula**.

3.1 Change of Numeraire

Define

$$L_T = \frac{dQ^N}{dQ} = \frac{B_0}{B_T}\frac{N_T}{N_0}$$

where Q is the risk-neutral measure, B_t is the money-market account process, N_t is the **numeraire** process, and Q^N is the "new" probability measure based on N.
Now the Girsanov theorem requires $\mathbb{E}^Q[L_T] = 1$. Turns out this is satisfied if we choose

N_t as **any** financial asset; this is easy to check by substitution and recalling the first fundamental theorem of asset pricing:

$$\mathbb{E}^Q\left[\frac{N_T}{B_T}\right] = \frac{N_0}{B_0}$$

which holds for any financial asset N_t.
As an example, choose S_t as the numeraire. Then:

$$L_t = \mathbb{E}^Q\left[\frac{B_0}{N_0}\frac{N_T}{B_T}\middle|F_t\right] = \frac{B_0}{S_0}\frac{S_t}{B_t} = \frac{S_t}{S_0 e^{rt}}$$

We use Ito's Lemma to derive the differential:

$$dL_t = \frac{S_t \sigma}{S_0 e^{rt}} dW_t^Q = \sigma L_t dW_t^Q$$

Thus $k_t = \sigma$ and under the "stock" probability measure Q^S the "new" Wiener process is:

$$dW_t^{Q^S} = dW_t^Q - \sigma dt$$

By now, we can already start simplifying the first part of Equation 1. Recall:

$$C_{di}(S, B, K, T) = e^{-rT} E^Q[S_T 1_F] - e^{-rT} K Q(F)$$

Take the first term and change its probability measure from Q to Q^S:

$$L_T = \frac{B_0}{S_0}\frac{S_T}{B_T} = \frac{S_T}{S_0 e^{rT}} \qquad \text{define the Radon-Nikodym derivative}$$

$$e^{-rT} E^Q[S_T 1_F] = e^{-rT} E^{Q^S}[S_T 1_F L_T^{-1}] \qquad \text{by Equation 3}$$

$$= E^{Q^S}[S_0 1_F] = S_0 Q^S(F)$$

Then, Equation 1 becomes

$$C_{di}(S, B, K, T) = S_0 Q^S(F) - e^{-rT} K Q(F)$$

and we just need to compute $Q(F)$ and $Q^S(F)$ with $F = \{S_T > K, min_{0 \leq t \leq T}(S_t) < B\}$. Note that the two probabilities are different even though they refer to the same event because they are calculated with different probability measures (which we know how to compute using Girsanov theorem). We will now introduce a useful result which allows us to calculate such probabilities.

3.2 Reflection Principle

Reflection principle

Let $(W_t)_{t \geq 0}$ be a standard Wiener process. For a fixed $a > 0$ and $t > 0$, the reflection principle states that:

$$\mathbb{P}(\sup_{0 \leq s \leq t} W_s \geq a) = 2\mathbb{P}(W_t \geq a).$$

In words, the probability that the Wiener process exceeds a at some time $s \leq t$ is twice the probability that it is at least a at time t.

We will provide a short proof:

Let $A = \{\sup_{0\le s\le t} W_s \ge a\}$. Decompose A as

$$A = \{\sup_{0\le s\le t} W_s \ge a, W_t \ge a\} \cup \{\sup_{0\le s\le t} W_s \ge a,\, W_t < a\}.$$

Now note that $\{W_t \ge a\} \subseteq \{\sup_{0\le s\le t} W_s \ge a\}$, implying that:

$$\mathbb{P}(\sup_{0\le s\le t} W_s \ge a, W_t \ge a) = \mathbb{P}(W_t \ge a)$$

Then reflect paths in $\{\sup_{0\le s\le t} W_s \ge a,\, W_t < a\}$ at a, mapping them to $\{W_t \le -a\}$. By symmetry,

$$\mathbb{P}(W_t \le -a) = \mathbb{P}(W_t \ge a).$$

Thus,

$$\mathbb{P}\left(\sup_{0\le s\le t} W_s \ge a\right) = \mathbb{P}(W_t \ge a) + \mathbb{P}(W_t \le -a) = 2\mathbb{P}(W_t \ge a).$$

Which concludes the (somewhat heuristic) proof.

Now the reader may have already guessed how this is useful, but be careful that this basic result only applies to standard Wiener processes, while the event $F = \{S_T \ge K, min_{0\le t\le T}(S_t) < B\}$ includes much more complex stochastic processes. We need to manipulate the terms of F in order to simplify the problem. Apply the logarithm and rewrite F in terms of log prices:

$$F = \left\{\log(S_T) \ge \log(K),\ \min_{0\le t\le T} \log(S_t) \le \log(B)\right\}$$
$$= \left\{\log(S_T) - \log(S_0) \ge \log(K) - \log(S_0),\ \min_{0\le t\le T} \log(S_t) - \log(S_0) \le \log(B) - \log(S_0)\right\}$$

Let:

$$X_t = \log(S_t) - \log(S_0), \quad x = \log(K) - \log(S_0), \quad m = \log(B) - \log(S_0), m_T = \min_{0\le t\le T} X_t$$

Then:

$$F = \{X_T \ge x,\ m_T \le m\}\,.$$

Use Ito's lemma and Girsanov theorem to find the dynamics of X_t under different measures:

$$X_t = \begin{cases} \left(r + \frac{\sigma^2}{2}\right) t + \sigma W_t^S & \text{under } \mathbb{Q}^S \\ \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t^B & \text{under } \mathbb{Q} \end{cases}$$

We consider a lower barrier $B < S_0$, so $m < 0$. Now, the event is much simpler since it involves only (generalized) Brownian motions and minimums of such processes; the trick is now to compute the probability of said event for a standard Brownian motion ($\mu = 0, \sigma = 1$ meaning that in this case $X_t = W_t$) and then change the measure using Girsanov theorem. Using this reasoning define:

$$A := \{W_T \ge x,\ m_T \le m\}, \quad B := \{W_T \le 2m - x\}.$$

We assume $m \leq x$. Then:

$$
\begin{aligned}
P(A) &= P(W_T \geq x,\ m_T \leq m) \\
&= P(W_T \geq x,\ \tau < T) \\
&= P(W_T \leq 2m - x,\ \tau < T) \\
&= P(W_T \leq 2m - x) = P(B) = N\left(\frac{2m - x}{\sqrt{T}}\right),
\end{aligned}
$$

where $N(\cdot)$ denotes the standard normal cumulative distribution function and $\tau \in [0, T]$ is the time at which the process hits the barrier.
It can be shown more generally that for any function g, we have:

$$\mathbb{E}[1_A g(W_T)] = \mathbb{E}[1_B g(2m - W_T)].$$

Now we can apply Girsanov theorem and change the measure to the appropriate Q and Q^S to get:

$$\mathbb{Q}^S(F) = e^{\frac{2\left(r+\frac{\sigma^2}{2}\right)\log\frac{B}{S_0}}{\sigma^2}} N\left(\frac{\log\frac{B^2}{S_0K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$\mathbb{Q}(F) = e^{\frac{2\left(r-\frac{\sigma^2}{2}\right)\log\frac{B}{S_0}}{\sigma^2}} N\left(\frac{\log\frac{B^2}{S_0K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

Now for simplicity define:

$$d_b = \frac{\log\frac{B^2}{S_0K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad \alpha = \frac{2r}{\sigma^2} + 1.$$

Then:

$$c_{di}(S, B, K, T) = S_0\left(\frac{B}{S_0}\right)^{\alpha} N(d_b) - Ke^{-rT}\left(\frac{B}{S_0}\right)^{\alpha-2} N(d_b - \sigma\sqrt{T})$$

Which is the closed form solution for a down-and-in barrier option when the the barrier level is less than the strike price, $B < K$.

Using the In-Out parity, the value of the down-and-out call can be retrieved using:

$$c_{do}(S, B, K, T) = c(S, K, T) - c_{di}(S, B, K, T)$$

On the other hand, as Hull (2021) points out, if $B \geq K$, then:

$$c_{do}(S, B, K, T) = S_0N(x_1) - Ke^{-rT}N(x_1 - \sigma\sqrt{T}) - S_0(\frac{B}{S_0})^{\alpha}N(y_1) + Ke^{-rT}(\frac{B}{S_0})^{\alpha-2}N(y_1 - \sigma\sqrt{T})$$

and:

$$c_{di}(S, B, K, T) = c(S, K, T) - c_{do}(S, B, K, T)$$

where:

$$x_1 = \frac{ln(\frac{S_0}{B})}{\sigma\sqrt{T}} + \frac{\alpha\sigma\sqrt{T}}{2}, \quad y_1 = \frac{ln(\frac{B}{S_0})}{\sigma\sqrt{T}} + \frac{\alpha\sigma\sqrt{T}}{2},$$

3.3 Barrier Correction Framework

An issue with barrier options is the frequency at which the underlying price, S, is observed to identify whether the barrier has been crossed. The analytic formulas provided above assume that S is observed continuously, in real scenarios this may sometimes be the case. However, often the terms of the option contract stipulates that S is observed periodically, for example, everyday at market close. With discrete valuations, there is a possibility that S crossed the barrier between monitoring dates without being observed. This can over- or under- estimate the true option price.

Broadie, Glasserman and Kou (1997) provide a way of adjusting the formulas for a situation where the price of the underlying is observed discretely. They adjust the barrier level, B, by a correction factor based on the conditional probability of the underlying crossing the barrier within an interval, given the start and end prices of that interval. This provides a barrier correction given by:

$$\begin{cases} B_{new} = Be^{\beta\sigma\sqrt{\frac{T}{m}}} \quad if \quad B > S_0 \\ B_{new} = Be^{-\beta\sigma\sqrt{\frac{T}{m}}} \quad if \quad B < S_0 \end{cases}$$

Where $\beta = -\zeta(\frac{1}{2})\sqrt{2\pi} \approx 0.5826$ with ζ being the Reimann zeta function and m the number of monitoring days.

4 Importance Sampling

Importance sampling is a variance reduction technique, useful to simulate rare-events occurring in the lifetime of the option. As a result, this technique is extremely useful to price barrier options with a value close to 0, for instance, a knock-in call with an extreme barrier level. This method of variance reduction has been applied to European options in *Glasserman, Heidelberger and Shahabuddin (1999); Su and Fu (2000)*, to name a few.

The importance sampling framework relies on the **Radon-Nikodyn derivative** to adjust the probability distribution of the underlying asset. This reweights samples from an artificially imposed distribution to more accurately estimate quantities under the target distribution, particularly in regions of interest.

Radon-Nikodyn Derivative

Suppose we are interested in a random variable X under a target distribution $\mathbb{P}$, but we instead sample X from a proposal distribution $\mathbb{Q}$.

The Radon-Nikodym derivative, $\frac{d\mathbb{P}}{d\mathbb{Q}}$, is a function that, for any measurable set A, satisfies:

$$\mathbb{P}(A) = \int_A \frac{d\mathbb{P}}{d\mathbb{Q}} d\mathbb{Q}$$

In practical terms, the Radon-Nikodym derivative corresponds to the likelihood ratio or the weight used to adjust samples from the proposal distribution. For example, if the densities $p(x)$ and $q(x)$ correspond to $\mathbb{P}$ and $\mathbb{Q}$, the derivative is:

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(x) = \frac{p(x)}{q(x)}$$

This ensures that samples drawn from $\mathbb{Q}$ can be properly reweighted to compute expectations under $\mathbb{P}$

Suppose we want to estimate:

$$\alpha = \mathbb{E}[f(X)] \quad \text{where } X \overset{\mathbb{P}}{\sim} f_X$$

The standard Monte Carlo Estimate is then:

$$\hat{\alpha}_n = \frac{1}{n}\sum_{i=0}^{n} f(X_i) \quad \text{where } X_i \sim f_X \text{ i.i.d}$$

Rewriting the first equation:

$$\begin{aligned}
\alpha &= \mathbb{E}[f(X)] \\
&= \int f(x) f_X(x)\, dx \\
&= \int f(x) \underbrace{\frac{f_X(x)}{g(x)}}_{\text{Likelihood Ratio}} g(x)\, dx \\
&= \mathbb{E}^g\left[f(X)\frac{f_X(X)}{g(X)}\right] \text{ where } X \overset{\mathbb{P}^g}{\sim} g
\end{aligned}$$

The Likelihood Ratio or *Radon-Nikodyn* derivative of $\mathbb{P}$ with respect to $\mathbb{P}^g$: $\frac{f_X(x)}{g(x)}$ corresponds to the 'weight' used to adjust samples. It allows to express how probabilities under $\mathbb{P}$ can be computed using samples from $\mathbb{P}^g$.

Under this framework, the Importance Sampling Estimator is:

$$\hat{\alpha_n}^{IS} = \frac{1}{n}\sum_{i=0}^{n} f(X)\frac{f_X(X)}{g(X)} \quad \text{where } X_i \sim g \text{ i.i.d} \tag{2}$$

The Importance Sampling Estimator $\hat{\alpha_n}^{IS}$ is an unbiased and consistent estimate of α because:

$$\begin{aligned}
\lim_{n\to\infty} \hat{\alpha_n}^{\mathrm{IS}} &= \frac{1}{n}\sum_{i=1}^{n} f(X_i)\frac{f_X(X_i)}{g(X_i)} \quad \text{with } X_i \sim g \\
&= \mathbb{E}^g\left[f(X)\frac{f_X(X)}{g(X)}\right] \quad \text{by the law of large numbers} \\
&= \mathbb{E}[f(X)]
\end{aligned}$$

and:

$$\mathbb{E}^g[\hat{\alpha_n}^{\mathrm{IS}}] = \mathbb{E}^g\left[f(X)\frac{f_X(X)}{g(X)}\right] = \mathbb{E}[f(X)]$$

The challenge with importance sampling is finding the distribution g such that the variance is reduced. Because the importance sample estimate is unbiased, a g that satisfies the following inequality must be found:

$$\mathbb{E}^g[(\hat{\alpha_n}^{\mathrm{IS}})^2] < \mathbb{E}[(\hat{\alpha_n})^2]$$

The statement is true if:

$$\begin{aligned}
\mathbb{E}^g\left[\left(f(X)\frac{f_X(X)}{g(X)}\right)^2\right] &< \mathbb{E}\left[(f(X))^2\right] \\
\int\left(f(x)\frac{f_X(x)}{g(x)}\right)^2 g(x)dx &< \int (f(x))^2 f_X(x)dx \\
\int (f(x))^2\frac{(f_X(x))^2}{g(x)}dx &< \int (f(x))^2 f_X(x)dx
\end{aligned}$$

Suppose for a moment that $\alpha = \mathbb{E}[f(X)]$ is known and we set:

$$g(x) = \frac{f(x)f_X(x)}{\alpha}$$

The case of the previous inequality can be re-written as:

$$\begin{aligned}
\int (f(x))^2\frac{(f_X(x))^2}{\frac{f(x)f_X(x)}{\alpha}}dx &= \int \alpha f(x)f_X(x)\,dx = \alpha\int f(x)f_X(x)\,dx \\
&= \alpha^2 < \int (f(x))^2 f_X(x)\,dx
\end{aligned}$$

The inequality is thus satisfied and the Importance Sample Estimator $\hat{\alpha_n}^{IS}$ has *zero* variance. However this is clearly unfeasible in practice because $\alpha = \mathbb{E}[f(X)]$ is unknown. Nevertheless, we can select:

$$g(x) \propto \underbrace{\underbrace{f(x)}_{\text{payoff at } x} \quad \cdot \quad \underbrace{f_X(x)}_{\text{probability of } x}}_{\text{importance of the path } x}$$

Such that the new density $g(x)$ is proportional to the *importance of the path.*

5 Numerical Analysis

In this section, the valuation techniques previously laid out will be applied to price barrier options. The code for these examples can be found here: [LINK]. First as a sanity check for our models, we test the in-out parity, which holds. Naturally, for the parity to hold exactly, the same trajectories should be considered when valuing the knock-in, knock-out and vanilla option. For the remainder of the examples, the starting asset price, $S_0 = 100$, the interest rate, $r = 0.1$, the volatility, $\sigma = 0.3$, the dividend yield, $q = 0$, the maturity, $T = 0.2$ and monitoring frequency, $m = 50$. Assuming 250 trading days a year, 50 simulation steps roughly corresponds to a daily monitoring of the barrier. The strike price, K, barrier level, B and number of simulated trajectories, n, are the only inputs that will vary.

5.1 Closed form Solutions Vs Monte Carlo Simulations

Monte Carlo simulations are run for all barrier option types. Their valuations are then compared with the closed form solutions provided by *Hull* (2002) and partially in Section 4. Then the barrier correction is applied to the closed-form solutions to have a fair comparison with the Monte Carlo valuation methods that have the same discrete monitoring period assumption. The results can be found in Table 1.

Table 1: Comparison of Monte Carlo valuations against closed-form solutions, with and without the barrier correction. Option Parameters: n = 10000, K = 105, H= 110 for up options and B = 90 for down options.

Type	MC	C-F	Deviation	Barrier Corr. C-F	Deviation
Up-and-In Call	4.017900	4.046434	0.71%	4.003110	0.37%
Up-and-Out Call	0.081835	0.043871	86.54%	0.087196	6.15%
Up-and-In Put	0.694430	0.930369	25.36%	0.672580	3.25%
Up-and-Out Put	6.351421	6.080797	4.45%	6.338586	0.2%
Down-and-In Call	0.100304	0.159287	37.03%	0.101733	1.4%
Down-and-Out Call	3.986867	3.931018	1.42%	3.988573	0.04%
Down-and-In Put	5.406075	5.712867	5.37%	5.392596	0.25%
Down-and-Out Put	1.602452	1.298299	23.43%	1.618571	1.0%

The barrier correction mechanism clearly improves valuations. This is due to the fact that without it we are comparing 'apples to oranges' with regards the assumption of continuous or discrete monitoring. Indeed, when the assumptions mismatch, namely with the naive closed-form solutions, a bias is introduced causing some percentage deviations to be unreasonably high. Furthermore, the options with the highest percentage deviations, even across the 2 closed-form valuation models, are those with the smallest valuations. Because percentage deviations are sensitive to the size of the base value, even small absolute differences between the Monte Carlo and closed-form valuations are amplified into larger percentage deviations. This is especially evident for low-valued options, where minor pricing discrepancies become more pronounced in percentage terms.

The next analysis undertaken is the identification of the most efficient number of simulations for the Monte Carlo valuations. With the law of large numbers, as the number of simulated paths increase, the bias and variance of the option price decreases. Nevertheless, more simulations will increase the computational efforts required, motivating the need to find the right balance. In Figure 1 and 2, the 95% confidence interval of Monte Carlo simulations is plotted against the number of simulations for down-and-in and down-and-out call options.

Figure 1 & 2: Comparison of Monte Carlo 95% confidence intervals for down-and-out & down-and-in call options for different numbers of simulations. The benchmark price is the barrier corrected closed-form solution.

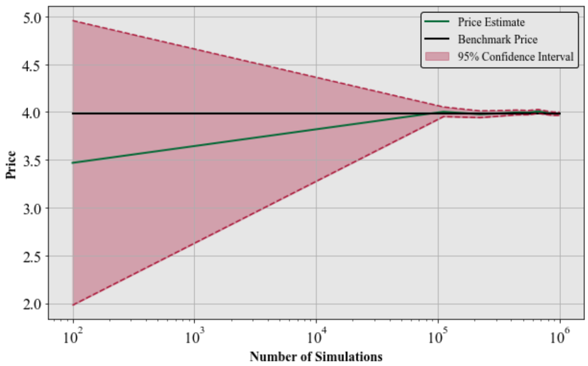

Figure 1: Down-and-In Call

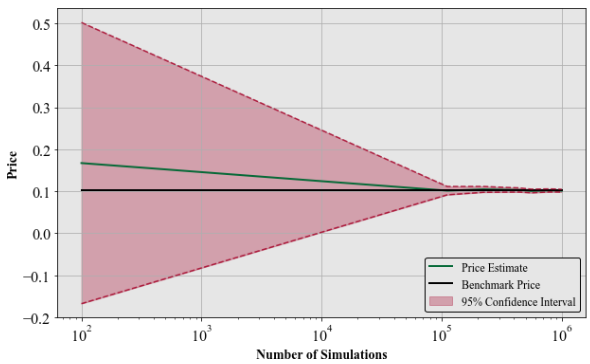

Figure 2: Down-and-Out Call

Figure 1 and 2 demonstrate the convergence of the Monte Carlo valuations to the closed-form valuation as the number of simulation increases. The smallest number of simulations where the Monte Carlo price is both close to the benchmark price and the confidence interval experiences a significant reduction is $n = 100,000$. For that reason, for the remainder of the numerical examples, 100,000 trajectory simulations will be used.

5.2 Importance Sampling

Importance sampling is most effective for pricing options with difficult barriers or strike prices to reach. This is because a pre-determined drift coefficient of the stochastic differential equation can be imposed by introducing a new probability measure using the Girsanov theorem. Figure 3 and 4 plot the trajectories for naive Monte Carlo simulations for a down-and in & down-and-out call with $B = 85$ and $K = 115$. In the down-and in simulations, only 13 paths out of 100,000 both knock into the barrier and finish above the strike price at maturity, This represents 0.01% of the sample. Much computational effort is wasted on paths with negligible contribution to the expected value estimate.

This effort could be reallocated toward successful paths minimizing the variance of the estimate. Down-and-out simulations have 16.99% of trajectories that are successful. This is an improvement on the knock-in counter part but not ideal.

Figure 3 & 4: Monte Carlo path simulations for a down-and-in & down-and-out call with option parameters: B = 85 and K = 115

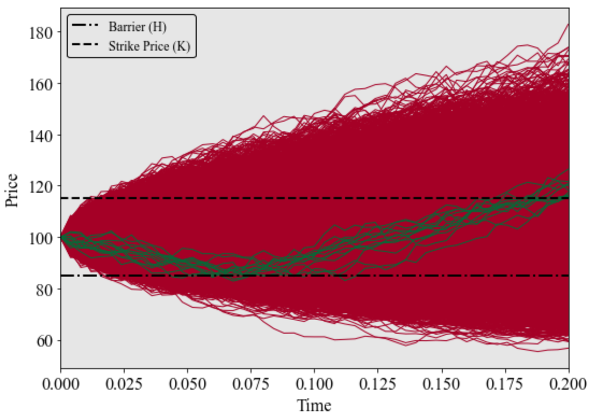

Figure 3: Down-and-In Call

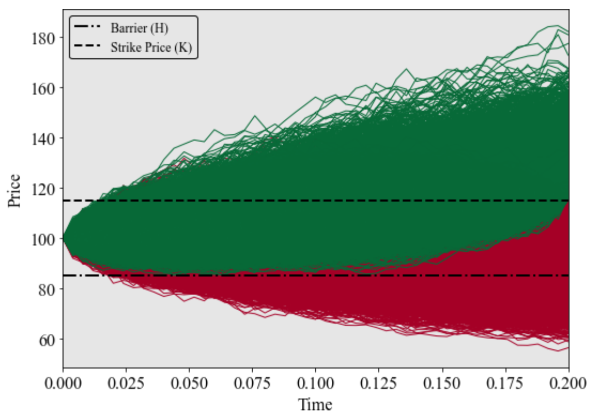

Figure 4: Down-and-Out Call

This motivates the need for importance sampling in which the probability distribution is distorted to make the stochastic process move towards the barrier. Once it knocks in, the process will be nudged towards the strike price. This specific trajectory can be clearly identified in Figure 5, which plots the paths of the importance sampled down-and-in call option. 36.67% of the paths were successful in this sample. Figure 6, which plots the paths of the down-and-out option highlights that only 2 paths were successful in the sample. Hence, importance sampling (as implemented in this paper) is not appropriate to value knock-out options because forcing the trajectories close to the barrier increases the chances of the option knocking out.

Figure 5 & 6: Importance Sampling path simulations for a down-and-in & down-and-out call with option parameters: B = 85 and K = 115

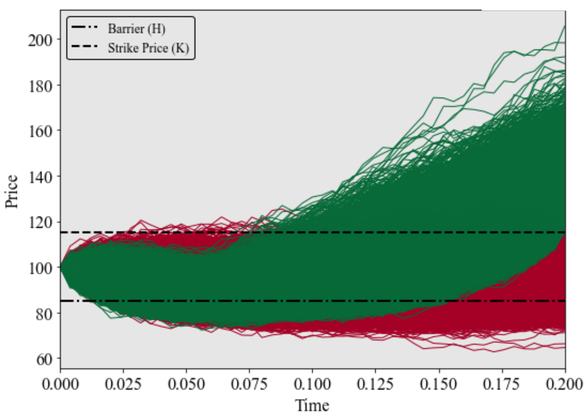

Figure 5: Down-and-In Call

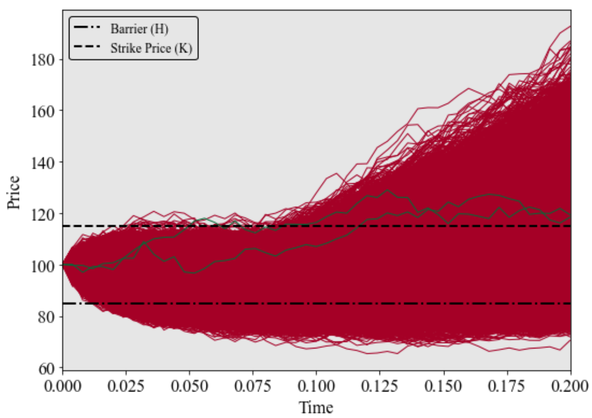

Figure 6: Down-and-Out Call

Importance sampling's poor performance for knock-out valuations is further demonstrated in Table 2. The percentage deviation of the importance sampled price estimate from the benchmark price is 104.79%. The naive Monte Carlo is far superior at 0.35%. The knock-in story is mirrored, the importance sampled estimate is only 1.75% away from the closed form solution. On the other hand, the naive simulation is 31.17%. There is a clear improvement in knock-in valuations using importance sampling.

Table 2: Comparison of naive Monte Carlo and importance sampled valuations against closed-form solutions with the barrier correction. Option Parameters: n = 100000, K = 115, B= 85.

Option Type	C-F	Naive MC	% Deviation	IS MC	% Deviation
Down-and-In Call	0.000612	0.000421	31.17%	0.000601	1.75%
Down-and-Out Call	1.447335	1.452432	0.35%	2.964011	104.79%

The importance sampled price estimate exhibits lower bias as compared to the naive Monte Carlo estimate for down-and-in calls. Fundamentally, importance sampling is a variance reduction technique. As such, the variance of the estimates must also be investigated. A sensitivity analysis is performed on down-and-in calls with different strike prices and barrier levels. Each option is priced using the 2 different frameworks. Figure 7 places these valuations on separate heat maps where the shading is determined by the standard error of the estimate.

Figure 7: Heat maps of the Naive monte Carlo and Importance Sampled variances across down-and-in calls with different strike prices and barrier levels. The price estimate is annotated in each cell, the variance is represented by the shading.

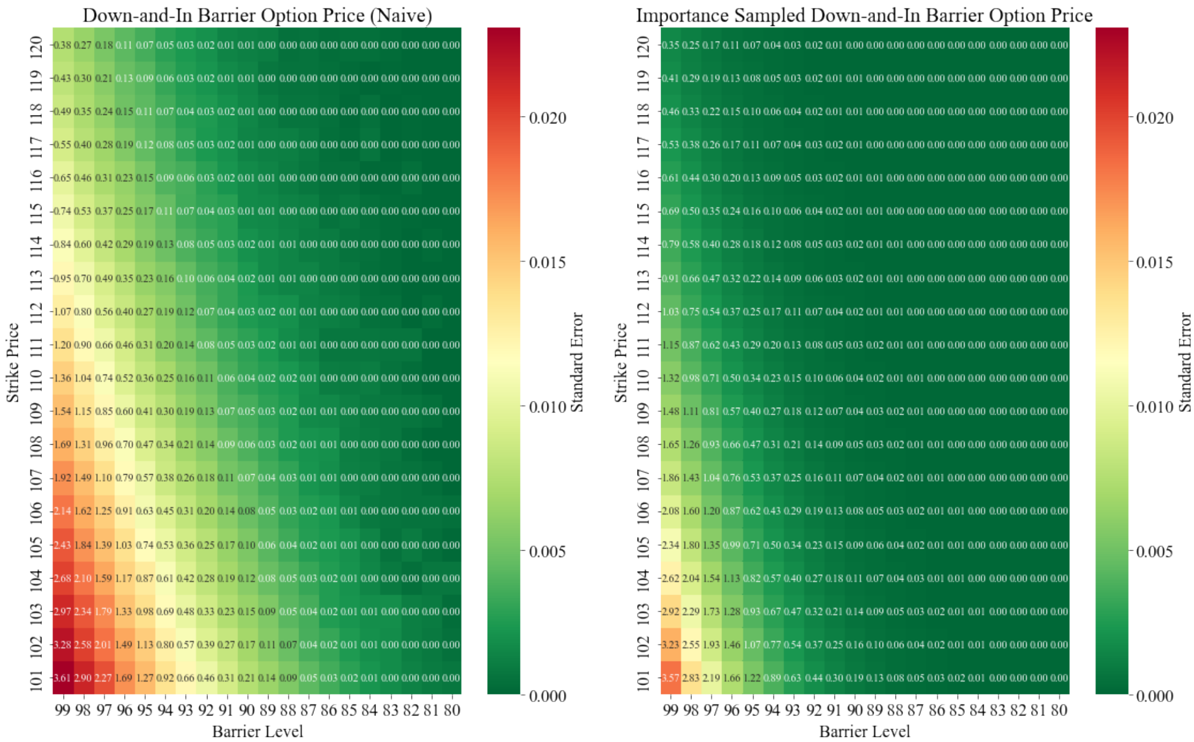

Figure 7 highlights a clear improvement in variance of the price estimate under the importance sampling framework. A much larger portion of the right heat map is shaded in dark green. The price estimates across the heat maps are relatively similar highlighting the low bias of importance sampled valuations. Next, the importance sampling methodology shows significant improvement for pricing options with barriers and strike prices far away from the initial underlying asset price, S_0. Conversely, the variance of estimates for options with closer strike prices and barriers are relatively high, represented by orange and yellow cells in the bottom left corner of the right plot. Finally, in both heat maps, there are higher fluctuations in variances across strike prices (y-axis) than barrier levels (x-axis). This is due to the fact that more trajectories are knocked-in and have a positive payoff when the barrier level is high creating a larger variation in the price estimate, irrespective of valuation framework. This substantiates that importance sampling is more appropriate for pricing options with difficult barrier levels and to a lesser extent strike prices to reach.

6 Limitations

6.1 Limitations of the Black-Scholes approach

The advantages of the closed-form solution provided by the Black-Scholes approaches, both in the context of vanilla and barrier options, come at a cost in the form of required assumptions.

The lognormality of stock prices (which translates to normality of stock returns) required by the model is often rejected in the empirical literature (such as Fama (1965)). In particular, heavy tails of empirical distributions indicate a risk of underestimating the probability of extreme events. This limitation can be overcome with the use of Monte Carlo simulations with distributions better suited to that of stock returns.

Second limitation of the Black-Scholes approach to option pricing is the requirement of constant volatility of the underlying asset. An immediate consequence of this assumption is the underestimation (overestimation) of option prices in high (low) volatility periods. A solution to this issue was proposed by Heston (1993) by introducing stochastic volatility. This approach, apart from relaxing the assumption of constant volatility, also allows to capture the leverage effect – a phenomenon where a drop in a firm' stock price increases its financial leverage (debt-to-equity ratio), amplifying perceived risk and leading to higher volatility in the future.

Since the core of the Black-Scholes model is the Geometric Brownian Motion, the model assumes that no jumps of stock prices will ever occur (i.e. continuity). In practice, even though they are not common, such jumps can occur. To answer this issue and make the model more realistic an extension of the model was introduced in Merton (1976). This solution, however, requires estimation of additional parameters and, thus, increases complexity of the model.

Other limitations of the Black-Scholes approach such as lack of transaction costs, or constant interest rates. All of them can potentially introduce errors and, thus, distort the result.

6.2 Limitations of the Monte-Carlo approach

Since the payoffs of a barrier option depend on the path of the underlying, the Monte Carlo approach seems like a natural solution. Its primary limitation, however, is the distribution assumption. Accuracy of the results is heavily dependent on the degree to which the assumed distribution describes the empirical one. Variance reduction techniques can help adjusted the assumed distribution for the barrier options pricing exercise but, as was shown in previous sections, it is not a one-fit-all solution. Finally, the Monte Carlo approach becomes more computationally expensive when the number of trajectories or, in the context of barrier options, of monitoring periods increases. This means that a trade-off between efficiency and precision needs to be considered for this approach.

7 Conclusion

Barrier options present unique pricing challenges due to their dependence on whether the underlying asset crosses specific price levels during the contract period. This report has demonstrated the application of both analytical and numerical techniques, such as closed-form solutions, Monte Carlo simulations and importance sampling, to address these challenges. While closed-form solutions offer computational efficiency, they rely on idealized assumptions like continuous monitoring, which may not hold in real-world scenarios. Adjustments, such as barrier corrections, significantly improve their accuracy under discrete monitoring conditions, ensuring consistency with the monitoring frequency stipulated in the option contract.

Monte Carlo simulations provide flexibility and adaptability, particularly when pricing barrier options or dealing with complex payoff structures. However, they require substantial computational effort, especially for rare-event scenarios where standard methods fail to converge efficiently. Importance sampling proves to be a powerful variance reduction technique in such cases, effectively focusing computational resources on the most relevant paths. Nonetheless, its limitations, such as poor performance for knock-out options, highlight the need for careful selection of valuation methods based on the specific option characteristics.

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