On the Study of a Probability Distribution for Precipitation Totals[1]

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ABSTRACT

Some of the probability distribution models for precipitation totals and their applications are briefly reviewed. The general properties of a probability distribution model which is a mixture of gamma and Poisson distributions are discussed. A new approximation is given for the solution of likelihood equations and the efficiency of the estimators is obtained. An application also is made using the monthly precipitation totals.

1. Introduction

Various functional expressions have long been proposed as theoretical probability distribution models for rainfall totals by many meteorologists and statisticians. Some satisfactory models have been found to describe historical rainfall data. Among these probability models, the gamma distribution has been extensively used for some of its desirable properties.

The gamma distribution with two parameters is the special case of the Pearson Type III distributions when the location parameter is zero. It's probability density function is given by the equation

$$f(x) = \frac{1}{\beta^\gamma \Gamma(\gamma)} x^{\gamma-1} \exp(-x/\beta), \quad \gamma, \beta > 0, x > 0,$$

where γ and β are the shape and scale parameters, respectively. Γ is the usual gamma function.

For a given data set, moment estimates of γ and β may be obtained as

$$\tilde{\gamma} = \bar{x}^2/s^2 \quad \text{and} \quad \tilde{\beta} = s^2/\bar{x}.$$

Here $\bar{x}$ and s^2 are the sample means and variance, respectively. In earlier times these estimates were used but later maximum likelihood (ML) estimates of the parameters replaced them. ML estimates can be obtained by solving the following equations:

$$\hat{\gamma}\hat{\beta} = \bar{x},$$

$$\log\hat{\gamma} - \psi(\hat{\gamma}) = A,$$

where $\psi(\hat{\gamma})$ is the digamma function and $A = \log\bar{x} - (\sum_{i=1}^{n} \log x_i)/n$. Thom (1958) obtained the approximate solution for $\hat{\gamma}$ by using the asymptotic expansion of the left-hand side of the above equation as

$$\hat{\gamma} = \frac{1 + (1 + 4A/3)^{1/2}}{4A}.$$

A satisfactory approximation is obtained for large values of γ. But, for small values of γ some corrections or modifications are necessary in order to obtain a more reliable estimate. Greenwood and Durand (1960) derived some numerical computation formulas to find more precise estimates. On the other hand, Bowman and Shenton (1970) proposed a simple correction factor for the computed $\hat{\gamma}$. Moreover, in 1973 they derived nearly unbiased estimators (Shenton and Bowman, 1973).

The gamma distribution has been used or verified as a model by Barger and Thom (1949), Mooley and Crutcher (1968), Neyman and Scott (1967), Schickedanz (1967), Schickedanz and Decker (1969), Simpson (1972), Thom and Vestal (1968) and others. Some known techniques relative to gamma distribution were reviewed by Gupta and Panchapakesan (1980).

Apart from the gamma distribution some other distribution functions also have been used especially for particular purposes. Suzuki (1980) summarized some of these distributions. The log-normal distribution is often fitted to the amount of precipitation for short time intervals caused by some factors such as cumulus clouds or weather modification experiments (Mielke and Johnson, 1973). In order to explain the long-tailed property of the distribution of rainfall amounts Mielke (1973) introduced a positively-skewed two-parameter distribution model as

$$f(x) = (\alpha/\beta)[\alpha + (x/\beta)^\alpha]^{-(\alpha+1)/\alpha} \quad x \geqslant 0.$$

Suzuki (1964) proposed a three-parameter model as

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a generalized form of the gamma distribution

$$f(x) = [\alpha\beta^{\nu/\alpha}/\Gamma(\nu/\alpha)] \exp(-\beta x^{\alpha})x^{\nu-1}, \quad x \geq 0,$$

where α, β and ν are the parameters of the distributions.

A remarkable distribution model was suggested by Fisher and Cornish (1960) for rainfall over arid regions. Todorovic and Yevjevich (1969) and Bernier and Fandeaux (1970) applied this model to various sets of rainfall data. Buishand (1977) used the same model for monthly totals of some stations in the Netherlands, Germany and Belgium. He obtained a satisfactory fitting and concluded that, especially for the data sets with a large proportion of zero values, the underlying distribution fits the data well and could be preferred over the gamma distribution.

The objectives of the present study are to introduce and explain some properties of the distribution suggested by Fisher and Cornish (1960) and obtain an approximate solution for the ML estimators of the parameters. The derivation of the probability density function of the distribution is given in Section 2. In Section 3, the two different approaches, namely, numerical solution and approximate solution for the ML estimators are explained. The asymptotic efficiency of the moment estimators relative to the ML estimators is discussed in Section 4. The last section is devoted to discussing the results of an application of the model to a long-period record of monthly rainfall totals.

2. Derivation of the model

The derivation of the probability model which is considered here was described by Buishand (1977) as follows: Let us assume that rainfall occurs as instantaneous showers according to a Poisson process with mean intensity, or rate, $1/\mu$, so that the number of showers in a time interval with length t is Poisson-distributed with mean t/μ. Rainfall amounts of single showers are assumed to be (i) independent of the process of their occurrence, (ii) mutually independent, and (iii) exponentially distributed with mean $1/\rho$.

Assuming the constant rainfall amounts for each occurrence of rainfall instead of exponentially distributed rainfall amounts, DeBoer (1958) used a similar process to describe the distribution of rainfall totals for a period of length at least 30 days.

If the stationarity assumption of the Poisson process holds, the probability of getting at least x amount of rainfall in the time interval of length t can be obtained as

$$P(X \leq x) = P(N = 0) + \sum_{k=1}^{\infty} P(X \leq x|N = k)\cdot P(N = k), \quad (1)$$

where the random variable N indicates the number of showers in the same interval. For a given value of $N = k$, X is the sum of k independently and identically distributed exponential variables. Therefore, the distribution of X is gamma with shape parameter k and scale parameter $1/\rho$. On the other hand, N is Poisson-distributed with mean $\theta = t/\mu$. Thus (1) becomes

$$P(X \leq x) = e^{-\theta} + \sum_{k=1}^{\infty} \frac{\theta^k \exp(-\theta)}{k!} \int_0^x \frac{\rho^k y^{k-1}}{\Gamma(k)} e^{-\rho y} dy.$$

From the equation above, it follows that the random variable x has a positive probability at $x = 0$ with

$$P(x = 0) = \exp(-\theta)$$

and has a density for $x > 0$ given by

$$f(x) = \exp\left(-\theta - \frac{\lambda x}{\theta}\right) \Sigma \frac{\lambda^k x^{k-1}}{k!(k-1)!}, \quad x > 0,$$

where $\lambda = \rho\theta$. For $x > 0$ the probability density function also can be expressed as

$$f(x) = \exp\left(-\theta - \frac{\lambda x}{\theta}\right)\left(\frac{\lambda}{x}\right)^{1/2} I_1[2(\lambda x)^{1/2}].$$

Here I_1 stands for a modified Bessel function of order 1.

Cumulants of the distribution are given by the relation (Fisher and Cornish, 1960) $\kappa_r = r!\theta^{r+1}/\lambda^r$. From this relation, central moments can easily be obtained:

$$\mu = \mu_1' = \theta^2/\lambda, \quad (2)$$

$$\mu_2 = 2\theta^3/\lambda^2, \quad (3)$$

$$\mu_3 = 6\theta^4/\lambda^3,$$

$$\mu_4 = (24\theta + 12\theta^2)\theta^4/\lambda^4.$$

The coefficient of skewness is $\gamma = \mu_3/\mu_2^{3/2} = 3/(2\theta)^{1/2}$. This shows that the distribution is positively skewed and that the amount of skewness depends inversely on the square root of the parameter θ.

3. Estimation of the parameters

The easiest way of estimating the parameters is by the method of moments. Equating the first and second central moments to sample moments, the moment estimates of the parameters can be found as

$$\tilde{\theta} = 2\bar{x}^2/s^2, \quad (4)$$

$$\tilde{\lambda} = 4\bar{x}^3/s^4. \quad (5)$$

Although the method of moments is easy to apply in fitting frequency distributions, it is the exceptional case when this method proves to be fully efficient in estimating the parameters (Thom, 1958). However, asymptotically efficient estimators may be obtained as follows by the method of maximum likelihood.

Suppose that there are n independent observations of which m are nonzero and $n - m$ are zero. Let also the observations be denoted by $x_1, x_2, \ldots, x_n$. The likelihood function may be written as

$$L(\lambda, \theta) = \prod_{i=1}^{n} \exp[-\theta - (\lambda x_i/\theta)]h(x_i, \lambda), \quad (6)$$

where

$$h(x_i, \lambda) = \begin{cases} \sum_{k=1}^{\infty} \dfrac{\gamma^y x_i^{k-1}}{k!(k-1)!}, & x > 0 \\ 1, & x = 0. \end{cases}$$

The logarithm of the likelihood function becomes

$$\log L(\lambda, \theta) = -n\theta - \frac{\lambda}{\theta} \sum x_i + \sum_{i=1}^{n} \log h(x_i, \lambda). \quad (7)$$

ML estimates of the parameters are obtained by maximizing Eq. (6) or equivalently (7). The ML equations are

$$\frac{\partial \log L}{\partial \theta} = -n + \frac{\lambda}{\theta^2} \sum_{i=1}^{n} x_i = 0,$$

$$\frac{\partial \log L}{\partial \lambda} = -\frac{\sum x_i}{\theta} + \sum_{i=1}^{n} \frac{h'(x_i, \lambda)}{h(x_i, \lambda)},$$

where $h'(x_i, \lambda)$ is the derivative of $h(x_i, \lambda)$. Solving the first equation above for θ and substituting it in the second equation gives

$$G(\lambda) = -(n \sum x_i/\lambda)^{1/2} + \sum_{i=1}^{n} \frac{h'(x_i, \lambda)}{h(x_i, \lambda)} = 0. \quad (8)$$

The solution for λ can be found by the iteration formula of Newton-Raphson as

$$\lambda_l = \lambda_{l-1} - G(\lambda_{l-1}) \Big/ \left[\frac{\partial G(\lambda)}{\partial \lambda}\right]_{\lambda = \lambda_{l-1}}.$$

After solving for $\hat{\lambda}$ by iteration, the ML estimate of θ is obtained by the formula

$$\hat{\theta} = (\hat{\lambda} \sum x_i/n)^{1/2}. \quad (9)$$

As may be seen from the equations above, computational procedure for estimating the parameters is rather tedious. It is also noted that the success in obtaining convergence depends on selecting good initial values of λ.

By considering the difficulties in the computational procedures and some other disadvantages, we developed approximate solutions of the ML equations for the distribution. Writing $h(x_i, \lambda)$ and $h'(x_i, \lambda)$ in (8) in terms of the modified Bessel function of order 1 as

$$h(x_i, \lambda) = \left(\frac{\lambda}{x}\right)^{1/2} I_1[2(\lambda x)^{1/2}],$$

$$h'(x_i, \lambda) = \frac{1}{2(\lambda x)^{1/2}} I_1[2(\lambda x)^{1/2}] + I'[2(\lambda x)^{1/2}],$$

their ratio is

$$\frac{h'(x_i, \lambda)}{h(x_i, \lambda)} = \frac{1}{2\lambda} + \left(\frac{x}{\lambda}\right)^{1/2} \frac{I_1'[2(\lambda x)^{1/2}]}{I_1[2(\lambda x)^{1/2}]}, \quad (10)$$

where I_1' stands for the derivative of I_1. It can be shown that for large values of λx the asymptotic expansion of the ratio above is given by

$$\frac{I_1'[2(\lambda x)^{1/2}]}{I_1[2(\lambda x)^{1/2}]} = 1 - \frac{1}{4(\lambda x)^{1/2}} + \frac{3}{32\lambda x} + \frac{3}{64(\lambda^3 x^3)^{1/2}} + \cdots. \quad (11)$$

Derivation of this expansion is given in the appendix. The approximation increases in accuracy with larger λ and x. However, we are interested in approximating λ, not the ratio. Taking the first four terms in (11) it follows from (10) that

$$\frac{h'(x_i, \lambda)}{h(x_i, \lambda)} \approx \left(\frac{x}{\lambda}\right)^{1/2} + \frac{1}{4\lambda} + \frac{3}{32(x\lambda^3)^{1/2}}. \quad (12)$$

Substituting this in (8) we find after simplification

$$[\sum (x_i)^{1/2} - (n \sum x_i)^{1/2}]\left[\lambda^{-1/2} + \frac{m}{4\lambda} + \left(\frac{3}{32} \sum x_i^{-1/2}\right)\right]\lambda^{-3/2} \approx 0.$$

This equation also can be rewritten as

$$\frac{3B}{32}\delta^3 + \frac{m}{4}\delta^2 + A\delta = 0,$$

where

$$\left.\begin{aligned} A &= [\sum x_i^{1/2} - (n \sum x_i)^{1/2}] \\ B &= \sum x_i^{-1/2} \\ \delta &= (\lambda)^{-1/2} \end{aligned}\right\}. \quad (13)$$

Besides $\delta_1 = 0$, the other two roots of (13) are given by

$$\delta_{2,3} = \frac{-4m \pm 4(m^2 - 6AB)^{1/2}}{3B}.$$

Using the pertinent root we finally obtain an approximate solution for λ as

$$\hat{\lambda} = \left\{\frac{m + (m^2 - 6AB)^{1/2}}{8A}\right\}^2. \quad (14)$$

Together with Eq. (9), (14) gives the ML estimates for the parameters of the distribution considered here.

Discrepancies between the exact values and the approximated values may occur due to using only the first four terms in (11). A satisfactory approximation can be obtained for $2(\lambda x)^{1/2} > 1$, or equivalently for $x > 1/(4\lambda)$. As it turns out for the precipitation totals of a given period in our case this inequality almost always holds. Therefore, under general conditions, it may be expected that equation (14) provides a good approximation to the exact value of the ML estimates.

4. Variances and covariances of estimators

Approximate variances and covariances of the moment estimators may be obtained by expanding Eqs. (4) and (5) in Taylor series about the population mean and variance as

$$\tilde{\theta} \approx \left(\frac{2\mu^2}{\mu_2}\right) + (\bar{x} - \mu)\left(\frac{4\mu}{\mu_2}\right) + (s^2 - \mu_2)\left(-\frac{2\mu^2}{\mu_2{}^2}\right),$$

$$\tilde{\lambda} \approx \left(\frac{4\mu^3}{\mu_2{}^2}\right) + (\bar{x} - \mu)\left(\frac{12\mu^2}{\mu_2{}^2}\right) + (s^2 - \mu_2)\left(-\frac{8\mu^3}{\mu_2{}^3}\right).$$

Using the results in (2) and (3), we obtain

$$\mathrm{Var}(\tilde{\theta}) \approx \frac{4\lambda^2}{\theta^2}\mathrm{Var}(\bar{x}) + \frac{\lambda^4}{4\theta^4}\mathrm{Var}(s^2) - \frac{2\lambda^3}{\theta^3}\mathrm{Cov}(\bar{x}, s^2), \quad (15)$$

$$\mathrm{Var}(\tilde{\lambda}) \approx \frac{9\lambda^4}{\theta^4}\mathrm{Var}(\bar{x}) + \frac{\lambda^6}{\theta^6}\mathrm{Var}(s^2) - \frac{6\lambda^5}{\theta^5}\mathrm{Cov}(\bar{x}, s^2), \quad (16)$$

$$\mathrm{Cov}(\tilde{\theta}, \tilde{\lambda}) \approx \frac{48\mu^3}{\mu_2{}^3}\mathrm{Var}(\bar{x}) + \frac{16\mu^5}{\mu_2{}^5}\mathrm{Var}(s^2) - \frac{56\mu^4}{\mu_2{}^4}\mathrm{Cov}(\bar{x}, s^2). \quad (17)$$

For sufficiently large n, it can be shown that

$$\mathrm{Var}(\bar{x}) = \frac{\mu_2}{n} = \frac{2\theta^3}{n\lambda^2},$$

$$\mathrm{Var}(s^2) \approx \frac{\mu_4 - \mu_2{}^2}{n} = \frac{8\theta^5(\theta + 3)}{n\lambda^4},$$

$$\mathrm{Cov}(\bar{x}, s^2) \approx \frac{\mu_3}{n} = \frac{6\theta^4}{n\lambda^3}.$$

Substituting these in (15), (16) and (17), we finally get

$$\mathrm{Var}(\tilde{\theta}) \approx \frac{2\theta(\theta + 1)}{n},$$

$$\mathrm{Var}(\tilde{\lambda}) \approx \frac{8\lambda^2}{n}\left(1 + \frac{3}{4\theta}\right),$$

$$\mathrm{Cov}(\tilde{\theta}, \tilde{\lambda}) \approx \frac{\lambda(4\theta + 3)}{n}.$$

The asymptotic variances and covariances of the ML estimators can be obtained from the expectation of the second derivatives. The inverse of the variance–covariance matrix is given by

$$V^{-1} = \{\sigma^{ij}\}, \quad (18)$$

where σ^{ij} is defined by

$$\sigma^{ij} = -E\left\{\frac{\partial^2 \log L}{\partial\alpha_i\partial\alpha_j}\right\} \quad (19)$$

and α's are the parameters of the distribution (Kendall and Stuart, 1973).

Considering the number of nonzero observations as a random variable, from (2) and (7) it can easily be shown that

$$-E\left\{\frac{\partial^2 \log L}{\partial\theta^2}\right\} = \frac{2n}{\theta}, \quad (20)$$

$$-E\left\{\frac{\partial^2 \log L}{\partial\theta\partial\lambda}\right\} = -\frac{n}{\lambda}, \quad (21)$$

$$-E\left\{\frac{\partial^2 \log L}{\partial\lambda^2}\right\} = -E\left[\sum_{i=1}^{n}\left\{\frac{x_i}{\lambda} - \left[\frac{h'(x_i, \lambda)}{h(x_i, \lambda)}\right]^2\right\}\right] = n\left[\varphi(\lambda, \theta) - \frac{\theta^2}{\lambda^2}\right], \quad (22)$$

where $\varphi(\lambda, \theta) = E[h'(x, \lambda)/h(x, \lambda)]^2$. From (18) and (19) variances and the covariance may be obtained as

$$\mathrm{Var}(\hat{\theta}) = \frac{[\lambda^2\varphi(\lambda, \theta) - \theta^2]\theta}{n[2\lambda^2\varphi(\lambda, \theta) - 2\theta^2 - \theta]},$$

$$\mathrm{Var}(\hat{\lambda}) = \frac{2\lambda^2}{n[2\lambda^2\varphi(\lambda, \theta) - 2\theta^2 - \theta]},$$

$$\mathrm{Cov}(\hat{\lambda}, \theta) = \frac{\lambda\theta}{n[2\lambda^2\varphi(\lambda, \theta) - 2\theta^2 - \theta]}.$$

For Eq. (22) using the result of (12), the approximate expression for the expectation becomes

$$-E\left\{\frac{\partial^2 \log L}{\partial\lambda^2}\right\} \approx \frac{n}{2\lambda^2\theta}\left(\theta^2 + \frac{\theta}{2} + \frac{3}{32}\right).$$

This together with the Eqs. (20) and (21) gives the following results for the approximate variances and covariances of the ML estimators of the parameters:

$$\mathrm{Var}(\hat{\theta}) \approx \frac{\theta}{2n}\left(\frac{32\theta^2}{16\theta + 3} + 1\right),$$

TABLE 1. Estimates of the parameters.

	Exact ML procedure		Approx. ML procedure		Method of moments	
Month	λ	θ	λ	θ	λ	θ
1	9.3853	5.1552	9.1162	5.0808	5.5758	3.9735
2	22.6896	6.8996	22.4118	6.8572	21.7178	6.7502
3	15.5620	7.6969	15.3675	7.6487	12.7395	6.9640
4	22.3245	9.1267	22.1952	9.1003	17.3463	8.0450
5	13.1039	7.3752	12.9647	7.3360	10.0259	6.4511
6	12.1472	7.0333	11.9710	6.9821	9.5798	6.2459
7	8.6973	5.6250	8.4290	5.5375	6.9034	5.0114
8	23.4771	8.7617	23.3223	8.7328	19.6982	8.0256
9	7.7397	4.9370	7.4677	4.8494	5.7828	4.2674
10	15.4991	6.6659	15.2529	6.6127	17.2310	7.0284
11	21.9159	7.9184	21.6219	7.8651	25.4992	8.5412
12	14.9519	6.3624	14.7238	6.3137	13.0193	5.9370

$$\mathrm{Var}(\hat{\lambda}) \approx \frac{64\lambda^2\theta}{n(16\theta + 3)},$$

$$\mathrm{Cov}(\hat{\theta}, \hat{\lambda}) \approx \frac{2\lambda\theta^2}{n(\theta + 3/16)}.$$

The asymptotic efficiencies of the moment estimators compared with maximum likelihood estimators may be obtained as

$$\mathrm{Eff}(\tilde{\theta}) \approx \frac{8\theta^2 + 4\theta + (3/4)}{16\theta^2 + 19\theta + 3},$$

$$\mathrm{Eff}(\tilde{\lambda}) \approx \frac{8\theta^2}{16\theta^2 + 15\theta + (9/4)}.$$

It is interesting to note that the efficiency of the moment estimators depends on the parameter θ only. The asymptotic efficiencies for both of the estimators tends to 50% approximately when θ is large. For small θ, efficiencies of $\tilde{\theta}$ and $\tilde{\lambda}$ approach 1/4 and 0, respectively.

5. An application of the probability model using the monthly rainfall totals

In order to make an empirical comparison between the different methods of estimating the parameters, only one meteorological station was selected. $n = 83$ years of records for monthly precipitation totals were available for Whitestown meteorological station, Indiana. Parameters of the underlying probability distribution were estimated for each month of the year by applying the three estimation procedures, namely, exact ML procedure, approximate ML procedure and method of moments. Results are shown in Table 1. As may be seen from the table, parameter estimates obtained by the approximate method are very close to the exact values obtained by the numerical method. The discrepancies between the exact values and the approximate values for $\hat{\lambda}$ and $\hat{\theta}$ are almost within the 3.5 and 1.8% level, respectively. It will be noted that the approximate method always underestimates the parameters of the distribution.

The method of moments seems to give poor estimates for the parameters. The estimates have large differences from the exact values obtained by the numerical method.

6. Summary and conclusion

A mixture of Poisson and gamma distributions have been presented. Approximate solutions for the likelihood equations and the efficiency of the estimators were obtained. Because of the simplicity of the formulas given in (9) and (14), it is useful in fitting data when the exact solutions for the parameters are not required. The definition of the distribution leads to a more efficient estimate for the probability of zero amount of precipitation.

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APPENDIX

The Asymptotic Expansion of $h'(x, \lambda)|h(x, \lambda)$

The equation

$$\frac{d^2y}{d^2x} + \frac{1}{x}\frac{dy}{dx} - y = 0 \tag{A1}$$

is called a modified Bessel equation of zero order. A solution of this equation except for a constant factor that remains finite when $x = 0$ is denoted by $I_0(x)$ and is given by

$$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \tag{A2}$$

which is called the modified Bessel function of the first kind of zero order. For large x it can be shown that the asymptotic expansion of $I_0(x)$ is (Bowman, 1958)

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}}\left\{1 + \frac{1}{8x} + \frac{1^2 \cdot 3^2}{2!}\frac{1}{(8x)^2} + \cdots\right\}. \quad \text{(A3)}$$

Taking the derivative with respect to x we obtain

$$I'_0(x) = \frac{e^x}{(2\pi x)^{1/2}}\left\{1 - \frac{3}{8x} - \frac{15}{128x^2} - \frac{105}{1024x^3} - \cdots\right\}. \quad \text{(A4)}$$

Assuming that the ratio $I_0(x)/I'_0(x)$ can be expanded in a series of the form

$$\frac{I_0(x)}{I'_0(x)} = 1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \cdots, \quad \text{(A5)}$$

we substitute (3) and (4) for $I_0(x)$ and $I'_0(x)$, respectively, in the above and obtain, after collecting the like terms,

$$1 + \frac{1}{8x} + \frac{9}{128x^2} + \frac{75}{1024x^3} = 1 + \left(c_1 - \frac{3}{8}\right)\frac{1}{x} + \left(c_2 - \frac{3c_1}{8} - \frac{15}{128}\right)\frac{1}{x^2} + \left(c_3 - \frac{3c_2}{8} - \frac{15c_1}{128} - \frac{105}{1024}\right)\frac{1}{x^3}. \quad \text{(A6)}$$

By equating the coefficients, it follows that the equation is formally satisfied by the series, provided the coefficients are given by

$$c_1 = 1/2, \quad c_2 = 3/8, \quad c_3 = 3/8; \quad \text{(A7)}$$

and hence,

$$\frac{I_0(x)}{I'_0(x)} = 1 + \frac{1}{2x} + \frac{3}{8x^2} + \frac{3}{8x^3} + \cdots \quad \text{(A8)}$$

(Bowman, 1958). On the other hand, the modified Bessel function of the first kind of first order is defined as

$$I_1(x) = \frac{x}{2} + \frac{x^3}{2^2 \cdot 4^2} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \cdots. \quad \text{(A9)}$$

The relation between $I_1(x)$ and $I_0(x)$ can be expressed as

$$\frac{I'_1(x)}{I_1(x)} = \frac{I_0(x)}{I'_0(x)} - \frac{1}{x} \quad \text{(A10)}$$

and, therefore, the asymptotic expansion for $I'_1(x)/I_1(x)$ can be written as

$$\frac{I'_1(x)}{I_1(x)} = 1 - \frac{1}{2x} + \frac{3}{8x^2} + \frac{3}{8x^3} + \cdots. \quad \text{(A11)}$$

Writing

$$h(x, \lambda) = \left(\frac{\lambda}{x}\right)^{1/2} I_1[2(\lambda x)^{1/2}], \quad \text{(A12)}$$

$$h'(x, \lambda) = \frac{1}{2(\lambda x)^{1/2}} I_1[2(\lambda x)^{1/2}] + I'_1[2(\lambda x)^{1/2}], \quad \text{(A13)}$$

we have

$$\frac{h'(x, \lambda)}{h(x, \lambda)} = \frac{1}{2\lambda} + \left(\frac{x}{\lambda}\right)^{1/2} \frac{I'_1[2(\lambda x)^{1/2}]}{I_1[2(\lambda x)^{1/2}]}. \quad \text{(A14)}$$

From (11), we finally obtain

$$\frac{h'(x, \lambda)}{h(x, \lambda)} = \left(\frac{x}{\lambda}\right)^{1/2} + \frac{1}{4\lambda} + \frac{3}{32(x\lambda^3)^{1/2}} + \cdots. \quad \text{(A15)}$$

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