On the Mathematical Modeling and Simulation of Crowd Motion

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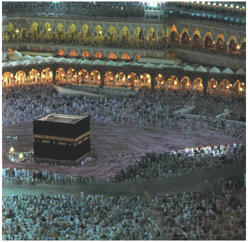
Outline

Pedestrian motion

- Empirical studied of human crowds started about 50 years ago.
- Nowadays there is a large literature on different micro- and macroscopic approaches available.
- Challenges: microscopic interactions not clearly defined, multiscale effects, finite size effects,.....



Mathematical modeling - microscopic level

1. *Force based models:* position of a particle is determined by forces acting on it e.g. Newton dynamics, stochastic differential equations (SDE), ...
   Newton equations of motion

$$\begin{aligned} dX_i &= V_i dt \\ dV_i &= F_i(X_1, \dots, X_N, V_1, \dots, V_N)dt + \sigma_i dB_i^t. \end{aligned}$$

   $X_i = X_i(t)$ is the location of the i-th particle, $V_i = V_i(t)$ its velocity, F_i the forces acting on it, and dB_i some additive noise.

2. *Stochastic optimal control* Each agent wants to minimize a cost functional

$$J_i(v_1, v_2, \dots, v_N) = \mathbb{E}(\int_0^T L_i(X_i, V_i) + F(X_1, \dots, X_N)dt)$$

   under the constraint that

$$dX_i = V_i dt + \sigma_i dB_i^t.$$

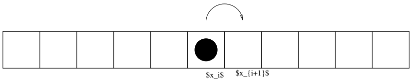
   where L and F denote the running cost.

Mathematical modeling - microscopic level

1. *Lattice based models:* Consider a domain divided into a cells and each cell center represents a possible position of a particle. Particle may jump from one cell to another with a certain transition probability, e.g. cellular automata, ..

   Time continuous, discrete in space random walk for the probability p_i to find a particle at a discrete lattice point x_i:

$$\frac{\partial p_i}{\partial t} = \mathcal{T}^{-}_{i+1} p_{i+1} + \mathcal{T}^{+}_{i-1} p_{i-1} - (\mathcal{T}^{+}_{i} + \mathcal{T}^{-}_{i}) p_i$$



Example: Social force model [1]

- Each pedestrian changes his/her velocity V_i according to:

$$\frac{dV_i}{dt} = \frac{v_i^0 \mathbf{e}_i^0 - V_i}{\tau_i} + \frac{1}{m_i} \sum_{i \neq j} \underbrace{f_{ij}}_{\text{interaction forces}} ,$$

  where v_i^0 is a desired velocity in direction $\mathbf{e}_i^0$, τ_i is the characteristic time and m_i the mass of the i-th particle.

- Interaction forces:

$$f_{ij} = \underbrace{A_i \exp\left(\frac{R_{ij} - d_{ij}}{B_i}\right) \cdot \mathbf{n}_{ij}}_{\text{repulsion}} + \underbrace{k(R_{ij} - d_{ij}) \cdot \mathbf{n}_{ij}}_{\text{body force}} - \underbrace{c_{ij}\mathbf{n}_{ij}}_{\text{attraction}} + \dots$$

  where R_{ij} is the sum of the radii, d_{ij} the distance between two pedestrians and the normalized vector pointing from pedestrian j to i.

[1] D. Helbing and P. Molnar, *Social force model for pedestrian dynamics*, Phys. Rev. E. 51, 1995

Nonlinear diffusion transportation models

In the macroscopic limit $N \to \infty$ one usually obtains a nonlinear transport-diffusion equations of the form

$$\rho_t = \operatorname{div}(D(\rho)\underbrace{\nabla(E'(\rho) - V + W * \rho)}_{:=v}).$$

- $V = V(x)$ is an external potential energy (e.g. confinement,....),
- $D = D(\rho)$ denotes the nonlinear diffusion/mobility
- $E = E(\rho)$ an entropy/internal energy.
- $W = W(x)$ is an interaction energy.

Classic fluid dynamic based models: Based on the conservation of mass

$$\rho_t = \operatorname{div}(\rho v)$$

with a particularly chosen velocity v (which depends on the specific modeling assumptions).
Traffic flow: Lighthill-Whitman and Richards model,....
Pedestrian motion: Colombo, Rossini, Venuti, ...
Review on literature: Bellomo & Dogbe 2011.

Hughes model for pedestrian flow [2]

1. Speed of pedestrians depends on the density of the surrounding pedestrian flow

$$v = f(\rho)u, \quad |u| = 1.$$

2. Pedestrians have a common sense of the task (called potential ϕ)

$$u = -\frac{\nabla \phi}{|\nabla \phi|}.$$

3. Pedestrians try to minimize their travel time, but want to avoid high densities

$$|\nabla \phi| = \frac{1}{g(\rho)f(\rho)}.$$

Saturation effects are included via the function $f(\rho) = (\rho_{\max} - \rho)$, where $\rho_{\max}$ denotes the maximum density.

[2] Hughes, R. *A continuum theory for the flow of pedestrians*, Transportation Research Part B, 36, 507-535, 2002

Hughes' model for pedestrian flow (with $g(\rho) = 1$):

$$\partial_t \rho - \mathrm{div}(\rho f^2(\rho) \nabla \phi) = 0$$
$$|\nabla \phi| = \frac{1}{f(\rho)}$$

Analytic issues:

- fully coupled system; nonlinear hyperbolic conservation law.
- density dependent stationary Hamilton Jacobi equation $\Rightarrow \phi \in C^{0,1}$ only.

Let us consider the regularized system:

$$\partial_t \rho^\varepsilon - \mathrm{div}((\rho^\varepsilon f^2(\rho^\varepsilon) \nabla \phi^\varepsilon) = \varepsilon \Delta \rho^\varepsilon$$
$$-\delta_1 \Delta \phi^\varepsilon + |\nabla \phi^\varepsilon| = \frac{1}{f(\rho^\varepsilon) + \delta_2}.$$

1D : solution ρ^ε converges to an entropy solution for $\varepsilon \to 0$, but $\delta_1 > 0, \delta_2 > 0$![3]

[3] M. Di Francesco, P.A. Markowich, J.-F. Pietschmann and MTW, *On Hughes model for pedestrian flow: the one-dimensional case*, JDE, 2011

Mean field games [4]

Microscopic model

N-player stochastic differential game

$$\inf_{V_i \in \mathcal{A}} \mathbb{E}\left[\int_0^T f(t, X_i, V_i, \rho)dt + g(\rho, X_i, t = T)\right]$$
$$dX_i = V_i dt + \sigma dB_i \;,\; X_i(t = 0) = x.$$

Transient macroscopic model

Calculate Nash equilibrium, limiting equations as $N \to \infty$ gives time dependent mean field game: Find (ϕ, ρ) such that

$$\frac{\partial \phi}{\partial t} + \nu \Delta \phi - H(x, \nabla \phi) = 0$$
$$\frac{\partial \rho}{\partial t} - \nu \Delta \rho - \operatorname{div}\left(\frac{\partial H}{\partial p}(x, \nabla \phi)\rho\right) = 0,$$

with the initial and end conditions

$$\phi(x, T) = g[\rho(x, T)], \;\; \rho(x, 0) = \rho_0(x),$$

where H is the Legendre transform of the running cost f.

[4] P.-L. Lions, J.-M. Lasry, *Mean field games*, Japan. J. Math., 2, 229-260, 2007

Link to optimal control problems

If the running cost f has the form

$$f(x,t,v,\rho) = L(x,t,v)\rho,$$

then the MFG can be written as an optimal control problem. For example let us consider the kinetic energy $f(x,t,v) = \frac{1}{2}\rho|v|^2$, then

$$\inf_v \left[\frac{1}{2} \int_0^T \int_\Omega \rho|v|^2 \, dxdt + g(\rho(T)) \right]$$

under the constraint that

$$\frac{\partial \rho}{\partial t} = \nu \Delta \rho - \operatorname{div}(\rho v), \quad \rho(x,0) = \rho_0(x).$$

The formal optimality condition is $v = \nabla \phi$ and therefore the adjoint equation reads as

$$\frac{\partial \phi}{\partial t} + \nu \Delta \phi - \frac{1}{2}|\phi|^2 = 0$$

with the terminal condition $\phi(x,T) = g'(\rho(T))$.

An optimal control approach for fast exit scenarios [5]

- Let us consider an evacuation or fast exit scenario, i.e. a room with one or several exits from which a groups wants to leave as fast as possible.
- Each individual tries to find the optimal trajectory to the exit, taking into account the distance to the exit, the density of people and other costs.

Figure: Fast-exit experiment conducted at the TU Delft

[5] M. Burger, M. Di Francesco, P.A. Markowich, MTW, *Mean field games with nonlinear mobilities in pedestrian dynamics*, accepted at DCDS, 2014.

Fast exit of particles

- Let $X(t)$ denote the trajectory of a particle, which wants to leave the domain Ω as fast as possible. The exit time is defined as:

$$T_{exit}(X) = \sup\{t > 0 \mid X(t) \in \Omega\}.$$

- Fastest path is chosen such that

$$\frac{1}{2}\int_0^{T_{exit}} |V(t)|^2 \, dt + \frac{\alpha}{2} T_{exit}(X) \to \min_{(X,V)} .$$

subject to $\dot{X}(t) = V(t)$, $X(0) = X_0$.

- For the Dirac measure $\mu = \delta_{X(t)}$ and a final time T sufficiently large we can write:

$$T_{exit} = \int_0^T \int_\Omega d\delta_{X(t)} dt.$$

- This yields the equivalence between the continuum formulation and the particle formulation, i.e.

$$\int_0^T \int_\Omega |v(x,t)|^2 d\mu dt = \int_0^T \int_\Omega |v(x,t)|^2 d\delta_{X(t)} dt = \int_0^{T_{exit}} |v(X(t),t)|^2 dt,$$

by mapping the Eulerian to the Lagrangian coordinates.

- Hence the minimization for the particle problem can be written as a continuum problem

$$I_T(\mu, v) = \frac{1}{2}\int_0^T \int_\Omega |v(x,t)|^2 d\mu dt + \frac{\alpha}{2}\int_0^T \int_\Omega d\mu dt,$$

subject to $\partial_t \mu + \nabla \cdot (\mu v) = 0,\ \mu\mid_{t=0} = \delta_{X_0}$.

- For a stochastic particle $d\mu = \rho\ dx$ and a final time T sufficiently large, the minimization reads as

$$I_T(\rho, v) = \frac{1}{2}\int_0^T \int_\Omega \rho(x,t)\ |v(x,t)|^2\ dxdt + \frac{1}{2}\int_0^T \int_\Omega \rho(x,t)\ dxdt,$$

subject to $\partial_t \rho + \nabla \cdot (\rho v) = \frac{\sigma^2}{2}\Delta\rho,\ \rho(x,0) = \rho_0(x)$.

Optimality conditions

- Formal optimality conditions; calculated via the Lagrangian with dual variable ϕ:

$$L_T(\rho, v, \phi) = I_T(\rho, v) + \int_0^T \int_\Omega (\partial_t \rho + \nabla \cdot (v\rho) - \frac{\sigma^2}{2}\Delta\rho)\phi \, dx \, dt.$$

- For the optimal solution we have

$$\begin{aligned} 0 &= \partial_v L_T(\rho, v, \phi) = \rho v - \rho \nabla \phi \\ 0 &= \partial_\rho L_T(\rho, v, \phi) = \frac{1}{2}|v|^2 + \frac{\alpha}{2} - \partial_t \phi - v \cdot \nabla \phi - \frac{\sigma^2}{2}\Delta\phi, \end{aligned}$$

  with the additional terminal condition $\phi(x, T) = 0$.

- Inserting $v = \nabla\phi$ we obtain the following system (with MFG structure):

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \nabla \phi) - \frac{\sigma^2}{2}\Delta\rho &= 0 \\ \partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + \frac{\sigma^2}{2}\Delta\phi &= \frac{\alpha}{2}. \end{aligned}$$

Mean field games and crowding

We consider the following generalization of the optimal control problem:

$$I_T(\rho, v) = \frac{1}{2}\int_0^T \int_\Omega F(\rho)|v(x,t)|^2 dx\ dt + \frac{1}{2}\int_0^T \int_\Omega E(\rho)\ dx\ dt,$$

subject to

$$\partial_t \rho + \nabla \cdot (G(\rho)v) = \frac{\sigma^2}{2}\Delta\rho, \text{ with initial condition } \rho(x, t=0) = \rho_0(x).$$

Motivation:

- $G = G(\rho)$ corresponds to a nonlinear mobility, e.g. $G(\rho) = \rho(\rho_{\max} - \rho)$.
- $F = F(\rho)$ corresponds to transport costs created by large densities. For example:

$$F(\rho) \to \infty \text{ as } \rho \to \rho_{\max}.$$

- $E = E(\rho)$ may model active avoidance of jams, in particular by penalizing large density regions.

Relation to the classical model by Hughes

- Let $H(\rho) = \frac{G^2}{F} = \rho f(\rho)^2$, $E(\rho) = \alpha\rho$ and $\sigma = 0$. Then we obtain the following optimality conditions:

$$\partial_t \rho + \nabla \cdot (\rho\ f(\rho)^2 \nabla\phi) = 0$$

$$\partial_t \phi + \frac{f(\rho)}{2}(f(\rho) + 2\rho f'(\rho))|\nabla\phi|^2 = \frac{\alpha}{2}$$

- For large T we expect equilibration of ϕ backward in time, hence we consider

$$\partial_t \rho + \nabla \cdot (\rho\ f(\rho)^2 \nabla\phi) = 0$$

$$(f(\rho) + 2\rho f'(\rho))|\nabla\phi|^2 = \frac{\alpha}{f(\rho)},$$

  which has a similar structure as the Hughes model

$$\partial_t \rho + \nabla \cdot (\rho\ f(\rho)^2 \nabla\phi) = 0$$

$$|\nabla\phi| = \frac{1}{f(\rho)}.$$

- Setting $f(\rho) = \rho_{\max} - \rho$ and $\alpha = 1$ yields $f(\rho) + 2\rho f'(\rho) = \rho_{\max} - 3\rho$, thus for small densities the behavior is similar, but the singular point is $\rho = \frac{\rho_{\max}}{3}$.

Boundary conditions

On the Neumann boundary Γ_N we clearly have no outflux, hence naturally

$$(-\frac{\sigma^2}{2}\nabla\rho + j)\cdot n = 0.$$

At an exit Γ_E: the outflux depends on how fast people can leave the room.

$$(-\frac{\sigma^2}{2}\nabla\rho + j)\cdot n = \underbrace{\beta\rho}_{\text{outflow}}.$$

Boundary conditions for the adjoint variable ϕ can be obtained via Lagrangian and result in

$$\frac{\sigma^2}{2}\nabla\phi\cdot n + \beta\phi = 0 \text{ on } \Gamma_E \quad \text{and} \quad -\frac{\sigma^2}{2}\nabla\phi\cdot n = 0 \text{ on } \Gamma_N.$$

Analysis of the optimal control model

Let $\rho_{\max} > 0$ denote the maximum density and $\Upsilon = [0, \rho_{\max}]$. Let $F = G = H$ which satisfy the following assumptions:

(A1) $F = F(\rho) \in C^1(\mathbb{R})$, F bounded, $E = E(\rho) \in C^1(\mathbb{R})$ and $F(\rho) \geq 0$, $E(\rho) \geq 0$ for $\rho \in \Upsilon$.

Existence of minimizes is guaranteed if

(A2) $E = E(\rho)$ is convex.

To ensure that the minimizes satisfy $\rho \in \Upsilon$, we need the following assumption on F:

(A3) $F(0) > 0$ if $\rho \in \Upsilon$ and $F = 0$ otherwise.

Uniqueness holds for:

(A4) $F = F(\rho)$ is concave.

We consider the optimization problem on the set $V \times Q$, i.e. $I_T(\rho, v) : V \times Q \to \mathbb{R}$, where V and Q are defined as follows

$$V = L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \text{ and } Q = L^2(\Omega \times (0, T)).$$

Abstract problem formulation in different variables

- Then the optimization problem reads as

$$\min_{(\rho,v)\in V\times Q} I_T(\rho,v) \text{ such that } \partial_t\rho = \frac{\sigma^2}{2}\Delta\rho - \operatorname{div}(F(\rho)v),$$

respectively in momentum formulation

$$\min_{(\rho,j)\in V\times Q} I_T(\rho,j) \text{ such that } \partial_t\rho = \frac{\sigma^2}{2}\Delta\rho - \operatorname{div}(j).$$

- In the case of non-concave H we use the variable $w = \sqrt{F(\rho)}v$. Then the minimization problem becomes:

$$\min_{(\rho,w)\in V\times Q} \frac{1}{2}\int_0^T\int_\Omega (|w|^2 + E(\rho))\,dxdt \ \text{ such that } \ \partial_t\rho = \frac{\sigma^2}{2}\Delta\rho - \operatorname{div}(\sqrt{F(\rho)}w).$$

Theorem (Existence in the general case)

Let $\rho_0 \in L^2(\Omega)$. Let (A1) and (A2) be satisfied, $\sigma > 0$ and $w = \sqrt{F(\rho)}v$. Then the variational problem has at least a weak solution $(\rho, w) \in V \times Q$ with initial condition ρ_0. If in addition (A3) is satisfied, then $\rho \in \Upsilon$.

Idea of the proof: Let $(\rho_k, w_k) \in V \times Q$ be a minimizing sequence. Both sequences ρ_k and w_k are bounded, hence we can extract a weakly convergent sub-sequence with limit $(\hat{\rho}, \hat{w})$. Using the Lemma of Aubin and Lions we can show that $(\hat{\rho}, \hat{w})$ is admissible and the lower semicontinuity of the objective functional implies the it is indeed a minimizer.

Theorem (Existence for concave mobility)

Let (A1), (A2), (A3), and (A4) be satisfied, $\sigma > 0$ and $j = F(\rho)v$. Then the variational problem has at least one minimize $(\rho, j) \in L^\infty(\Omega \times (0, T)) \times Q$ such that $\rho(x) \in \Upsilon$ for almost every $x \in \Omega$. If E is strictly convex the minimize is unique.

We solve the optimal control problem using a steepest descent method given by

1. Solve the forward equation for $\rho = \rho(x,t)$

$$\partial_t \rho = \frac{\sigma^2}{2}\Delta\rho - \nabla\cdot(F(\rho)v)$$

$$(-\frac{\sigma^2}{2}\Delta\rho + \nabla\cdot F(\rho)v)\cdot n = \beta\rho \text{ on } \Gamma_E \text{ and } (-\frac{\sigma^2}{2}\Delta\rho + \nabla\cdot F(\rho)v)\cdot n = 0 \text{ on } \Gamma_N,$$

   using Newton's method for the implicit Euler discretization and a mixed hybrid DG method for the discretization in space.

2. Calculate the backward evolution of the adjoint variable ϕ using the density ρ

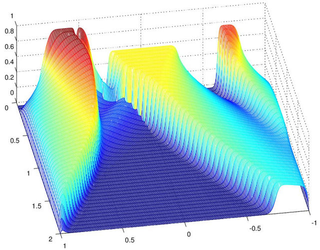
$$-\partial_t\phi - \frac{\sigma^2}{2}\Delta\phi - G'(\rho)v\cdot\nabla\phi = -\frac{1}{2}F'(\rho)|v|^2 - \frac{1}{2}E'(\rho)$$

$$-\frac{\sigma^2}{2}\nabla\phi\cdot n - \beta\phi = 0 \text{ on } \Gamma_E \text{ and } -\frac{\sigma^2}{2}\nabla\phi\cdot n = 0 \text{ on } \Gamma_N,$$

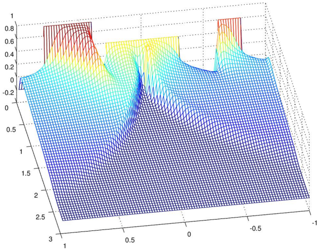
   using an implicit in time discretization and a mixed hybrid DG method.

3. Update the velocity $v = v - \tau(F(\rho)v - G(\rho)\nabla\phi)$.
4. Go to (1) until convergence of the functional.

Fast exit for three groups



(a) Solution of the classical Hughes model



(b) Solution of the mean field optimal control approach

Understanding the Hughes model

- Let us consider N particles with position $X_k = X_k(t)$ and the empirical density

$$\rho^N(t) = \frac{1}{N} \sum_{k=1}^{N} \delta(x - X_k(t)).$$

- To define the cost functional in a proper way we introduce the smoothed approximation ρ_g^N by

$$\rho_g^N(t) = (\rho^N * g)(x,t) = \frac{1}{N} \sum_{k=1}^{N} g(x - X_k(t)),$$

  where g is a sufficiently smooth positive kernel.

We freeze the empirical density ρ^N and look for the optimal trajectory of each particle, i.e.

$$C(X;\rho(t)) = \min_{(X,V)} \frac{1}{2} \int_t^{T+t} \frac{|V(s)|^2}{G(\rho_g^N(\xi(s;t))} ds + \frac{1}{2} T_{exit}(X,V),$$

subject to $\frac{d\xi}{ds} = V(s)$ and $\xi(0) = X(t)$.

Understanding the Hughes model

- For sufficiently large $T > T_{exit}$:

$$T_{exit}(X, V) - t = \int_t^{T+t} \int_\Omega \delta_{\xi(s)} \, dx \, ds.$$

Since the macroscopic version of $\rho^N(t)$ converges to the mean field $\rho(t)$ (at least in the weak-* sense of measures as $N \to \infty$), we replace $\rho_g^N(t)$ by $\rho(t)$ and obtain:

$$C(X; \rho(t)) = \min_{(\mu, w)} J(\mu, w) = \frac{1}{2} \int_t^{T+t} \int_\Omega \left(\frac{w^2(x, s)}{G(\rho(\xi(s; t)))} + 1 \right) d\mu ds,$$

subject to $\partial_s \mu + \nabla \cdot (\mu w) = 0$ with $\mu(t = 0) = \delta_X$.

- The formal optimality conditions can be calculated via the Lagrange functional

$$0 = \partial_w L_X = w - G(\rho)\nabla\psi,$$
$$0 = \partial_\mu L_X = \partial_s \psi + \frac{1}{2} G(\rho(t))|\nabla\psi|^2 - \frac{1}{2}.$$

  with $\psi(T + t) = 0$, $\psi(t) = -\phi$

- For $T \to 0$ the behavior at $s = t$ represents the long-time behavior of the HJE. Hence we expect $\phi = -\psi(t)$ to solve

$$G(\rho(t))|\nabla\psi|^2 = 1.$$

Then we recover the Hughes model by choosing $G(\rho) = f(\rho)^2$ i.e.

$$\partial_t \rho + \nabla \cdot (\rho f(\rho)^2 \nabla\phi) = 0,$$
$$|\nabla\phi| = \frac{1}{f(\rho)}.$$

Numerical simulations of the particle model

Consider N particles located at positions $X_i = X_i(t)$, $i = 1, \dots N$. In every time step we update their position by:

1. Determine the empirical density

$$\rho_g^N(x,t) = \frac{1}{N} \sum_{i=1}^{N} g(x - X_i(t)),$$

   where g denotes a Gaussian to approximate the Dirac deltas.

2. Solve the Eikonal equation[6]

$$|\nabla \phi| = \frac{1}{f(\rho_g^N)}$$

   with boundary conditions $\phi = 0$ at the exit and $\phi = 1$ at obstacles.

3. Update each position by $X_i(t + \Delta t) = X_i(t) + \Delta t(\rho_g^N f(\rho_g^N)^2 \nabla \phi)$.

[6] J. Qian, Y.-T. Zhang, H.-K. Zhao, Fast sweeping methods for Eikonal equations on triangular mesh, SIAM J. Numer. Anal. 45(1), 2007

Future work

- Analysis of the problem for more general functions E, F and $G \Rightarrow$ non convex optimization problems
- Simulations in 2D, numerical solvers for non-convex optimization problems.
- Other generalization of the Hughes model for pedestrian flow, e.g. include local vision.
- Multiscale problems in crowd dynamics: crowd-leader interactions or the motion of small social groups in large crowds.
- Inverse problems: How many leaders are necessary to guide a crowd efficiently ?

Thank you very much for your attention !

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