IMT SCHOOL FOR ADVANCED STUDIES LUCCA

Data Science & Statistical Learning | II Level Master

Complex Systems Analysis

UNIVERSITÀ DEGLI STUDI FIRENZE

Stability and Bifurcation in Rotors with permanent Shaft Bow on Nonlinear Supports

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- **Bifurcation**: means "**splitting into two branches**"
  - A **point where** a system **shifts behavior qualitatively**:
    - Important to **model dynamics and instability at variation of control parameters**
  - Often seen in **nonlinear systems**:
    - Superposition property, states that, for all **linear systems**, net **response caused by** two or **more stimuli is sum of responses** that would have been caused by **each stimulus** individually
      - In **nonlinear systems no proportional relationship between cause and effect**
    - **Instability** and **unpredictability**: **small changes can lead to big effects**
    - Bifurcation **occurs at some parameters values** so called **bifurcation points**
    - Stable and unstable **fixed points**: **solid line for stable** points and a **broken line for unstable** ones
  - When a **parameter** in model is **changed and trajectory quality varies than** parameter is of **bifurcation**
  - If **new solutions appear direct bifurcation**, in case of **disappearing of solutions bifurcation is inverse**

Some abbreviations: DoF, *Degree of Freedom*; MDoF, *Multi-Degree of Freedom*; PD, *Period Doubling bifurcation*; FE, *Finite Element*; NL, *Nonlinear*; SN, *Saddle Node bifurcation*; LC, *Limit Cycle*; NS, *Neimark-Sacker bifurcation*; LPC, *Limit Point of Cycles*; ODE, *Ordinary Diff. Equation.*

- Stability of fixed points:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$$

- Fixed points (equilibria) ***x★(μ)***:

$$\mathbf{f}(\mathbf{x}^\star, \mu) = 0$$

$$\frac{\partial f_i}{\partial x_j}(\mathbf{x}^\star, \mu) = A_{ij}$$

- let ***x* = *x*★ + *δx***, **A** Jacobian matrix

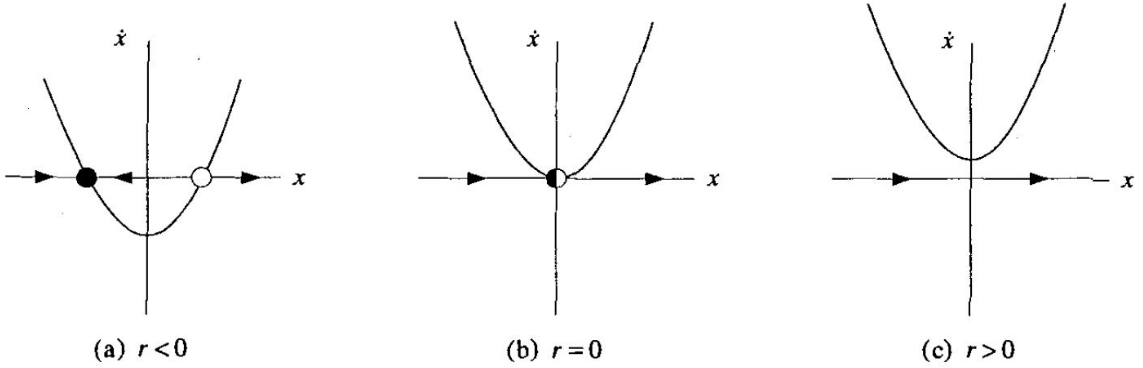
$$\dot{\delta\mathbf{x}} = A\delta\mathbf{x}$$

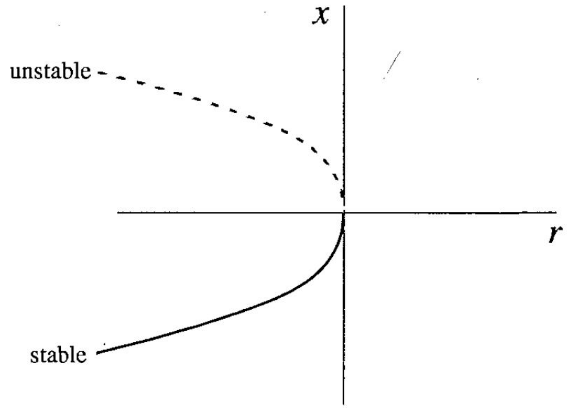
- Eigenvalues $\boldsymbol{a_j}$ of **A** determine stability of equilibria:
  - if **Re(*a*$_j$) > 0** point is unstable
  - It changes **from stable regime to unstable regime when eigenvalue becomes positive**
- **Bifurcation occurs** when a **small smooth change to bifurcation parameter causes a qualitative change in dynamics**
- ***x*★(μ)** depend continuously on **μ** as long as **A** has eigenvalues $\boldsymbol{a_j}$ with non-zero real parts (implicit function theorem)
- If **Re(*a*$_j$) = 0** at **μ = μ$_c$**, then **(*x*★, μ$_c$)** is a bifurcation point
- Branches of solution merge at bifurcation point

- **Saddle-Node Bifurcation**
  - Basic mechanism by which **fixed points are created and destroyed**. As a parameter is varied, two fixed points move toward each other, collide and disappear, **example of a first order system**:

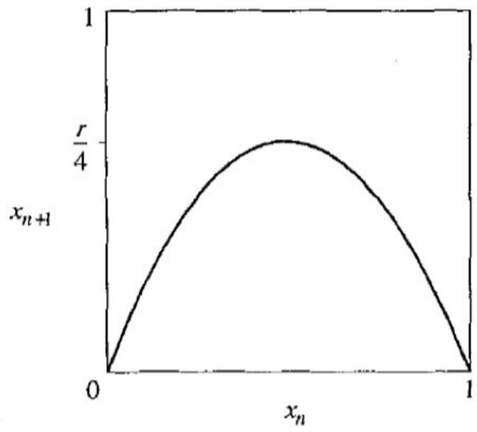
$$\dot{x} = r + x^2$$

  - Where ***r*** is a **parameter**, which may be positive, negative or zero
  - When ***r*** is **negative**, there are **two fixed points**, **one stable and one unstable**
  - As ***r*** **approaches 0** from below, **parabola moves up and two fixed points move toward** each other
  - When ***r* = 0**, fixed points coalesce into a **half-stable fixed point** at ***x** = 0***, which is delicate
  - With **r > 0** when **there are no fixed points**, equilibrium disappears
  - **Bifurcation occurred at r = 0**: vector fields for **r < 0** and **r > 0 are qualitatively different**

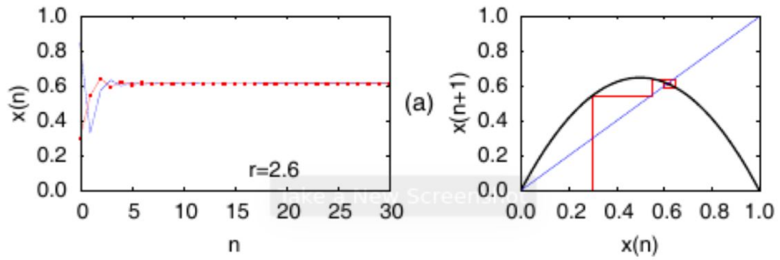




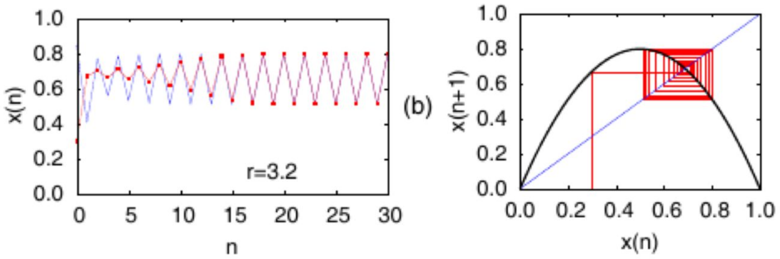
- **Period Doubling Bifurcation**
  - System transits **from stable behavior to oscillatory**: **period of oscillations doubles with each bifurcation**
  - A parameter ***r* is gradually increased**, system might exhibit **stable behavior**, **converging towards a fixed point**
  - At a **critical value of *r***, system **transitions from a single stable fixed point to a stable oscillation with period 2**
    - System now alternates between **two distinct values in a repeating pattern**
  - As **r continues to increase**, system can undergo further **leading to periods 4, 8, 16 and so on**
  - This series of period doubling bifurcations **often leads system towards chaotic behavior**
- Example with **discrete population dynamics, the logistic map**: $x_{n+1} = r x_n (1 - x_n)$



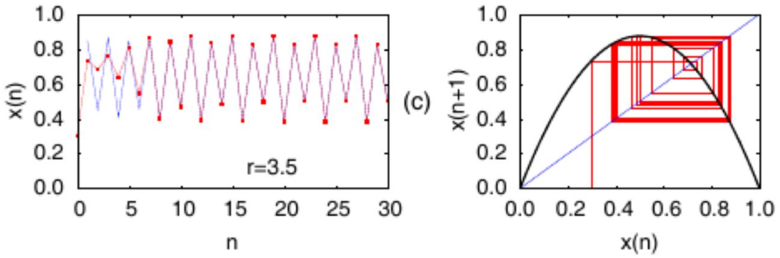
- **Period Doubling Bifurcation**:
  - For each value of ***r***, **control parameter**, **iterate equation** many times, **discarding initial transient**, **reach a steady-state**, system **converge in a single fixed point (stable behavior)**
  - With $\boldsymbol{r < 1} \rightarrow 0$



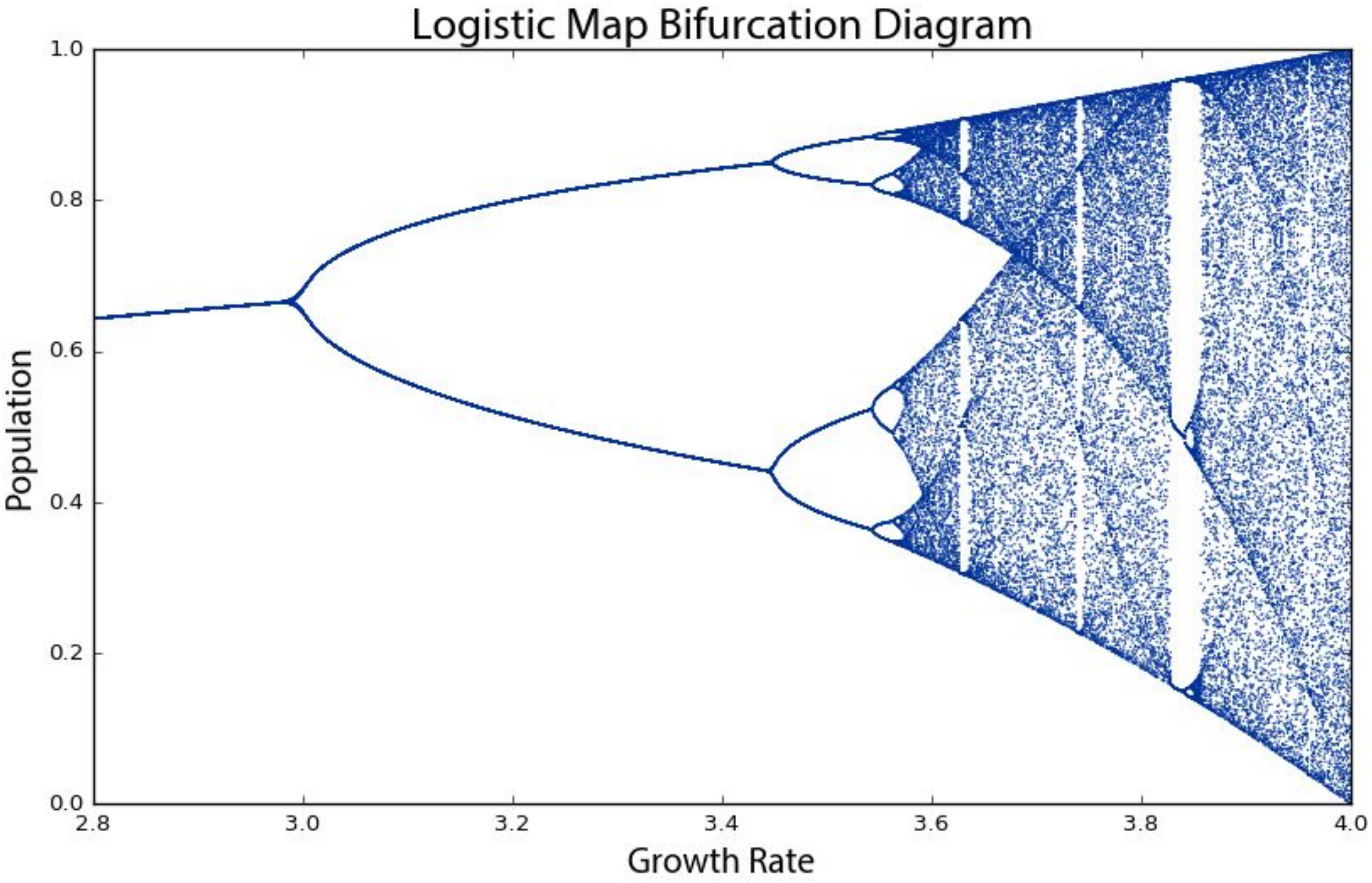
  - ***r* increases at a point where system bifurcates into a stable oscillation with period 2**



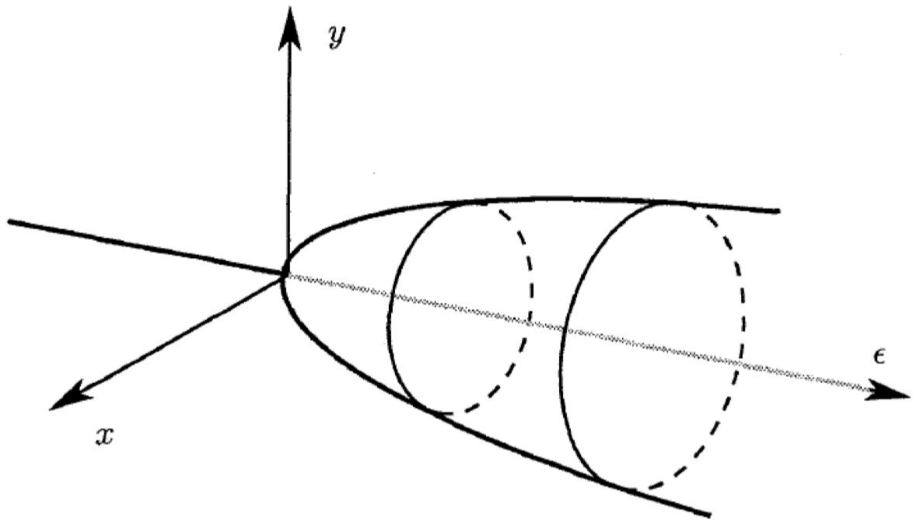
  - Further **increasing r will lead to subsequent bifurcations**, with period of oscillations doubling each time



- **Chaotic behaviour** with ***r* > ~3.57**, with **windows of stable period behaviour** for greater values

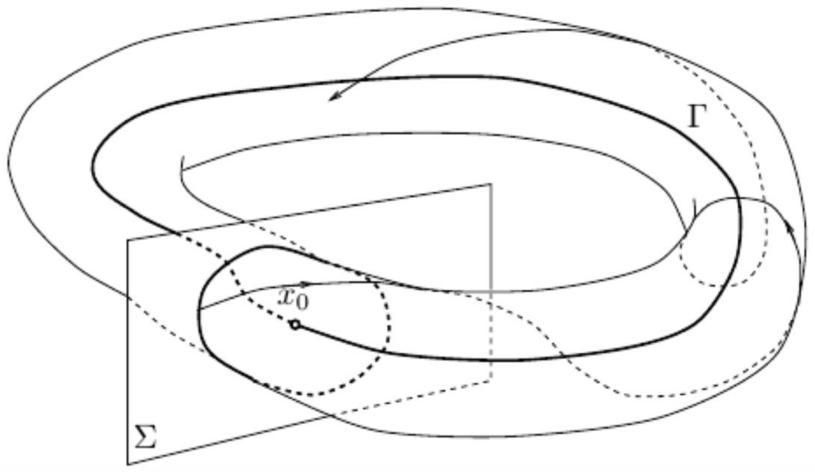


- **Hopf Bifurcation**: oscillatory behavior emerges
  - **Stable equilibrium point of a system transitions into a stable limit cycle** as a control **parameter is varied**
    - Equilibrium point can become unstable at critical value of parameter, it can "loses stability" in a special way:
      - Instead of moving to chaos or remaining unstable, **system settles in a stable periodic orbit, a limit cycle**
    - This limit cycle **represents a regular, repeating pattern of behavior** in system, which **admits stable or unstable periodic orbits close to a circle** with radius $\propto$ $\sqrt{\varepsilon}$ **supercritical** or $\sqrt{-\varepsilon}$ **subcritical case**
  - Hopf bifurcation can describe phenomena like self-sustained oscillations in electronic circuits or behavior of certain chemical reactions



- **Neimark-Sacker Bifurcation:** when a dynamical **system loses stability, leading to a stable torus behavior**
  - System **moves from a stable periodic orbit**, with a predictable pattern, **to an oscillatory state on a stable torus**:
    - Different from doughnut shape, **it's a higher-dimensional structure, like a twisted loop**
    - System orbits around this torus in a complex, yet stable, manner
  - It can describe systems like a forced oscillator with certain resonance conditions

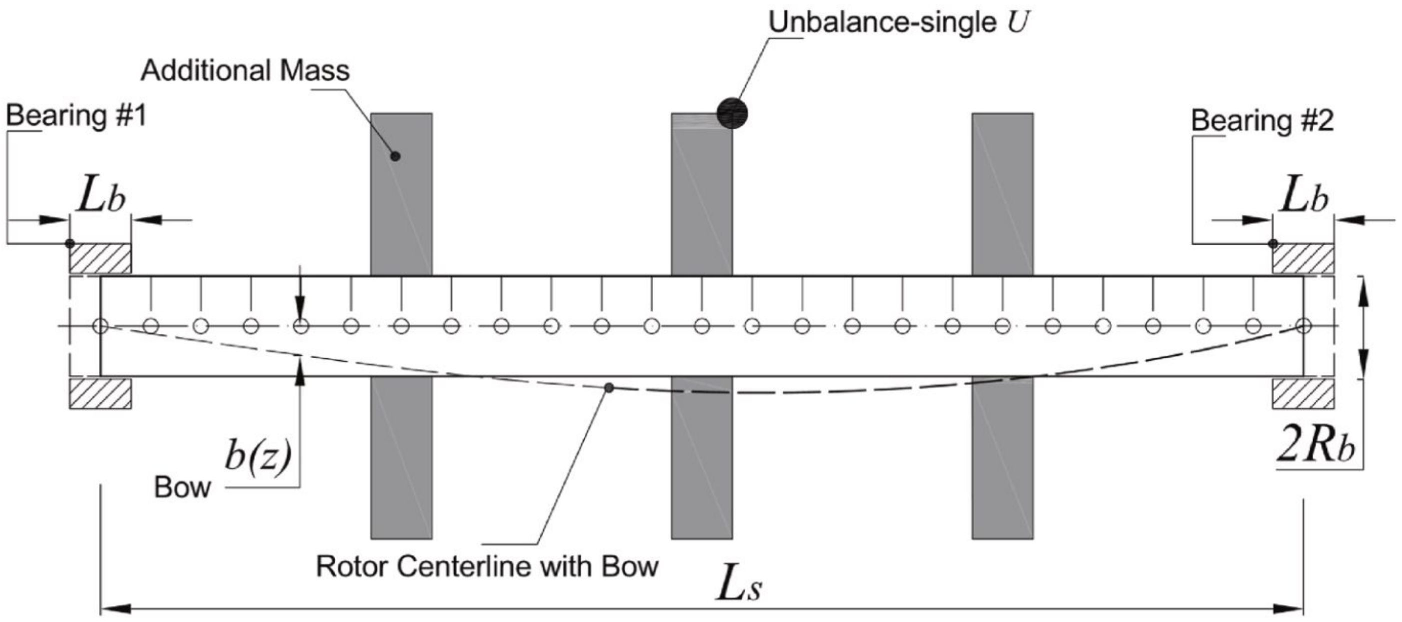
- To **detect Neimark-Sacker bifurcation**, often used **Poincaré sections**:
  - Poincaré section is a method in which **system's trajectory** is **intersected with** a specific **hyperplane** in **phase space**
  - Example of T^2 torus:



- Paper investigates **rotors dynamics stability** through finite element models for **various bow values**:
  - With **permanent bow** and a generic rotor system of a uniform **rotor with three masses**
  - Realistic **rotor of complex geometry** as 2° application
    - **Permanent shaft bow exists in most rotors**: gas turbines, turbopumps and geared shafts
    - Rotors are **mounted on two plain short bearings to include nonlinearity of impedance forces**
- Rotating machines must work within specific **margins of vibration amplitudes for efficiency and reliability**
- **Residual shaft bow** is one of most common rotor faults which may **cause excessive vibration**:
  - E.g. caused by a thermal gradient during start-up-shutdown
- **Aim**: **study** potential of **motion under combined effect of bow and nonlinear journal bearings** in rotor-bearing system
- **Collocation method** is preferred **to evaluate periodic limit cycles** of motion and respective **solution branches of limit are evaluated using numerical continuation methods**:
  - **Numerical continuation** with embedded **collocation method** to study **nonlinear dynamics** of rotor systems
  - **Potential of motion** and **bifurcation** set **evaluated as rotating speed** of system changes, **bifurcation parameter**
  - Even using **pseudo-arc length continuation methods**, it's hard to guarantee identification of full bifurcation sets

- **System 1**: FE model of **generic elastic rotor with bow**
- **System 2**: Dynamic system of a **real turbine rotor mounted on nonlinear journal bearings with bow**
- **Condensed systems** applying model order reduction are **subjected to numerical methods to**:
  - **Evaluate stable or unstable periodic limit cycles** utilizing a collocation type method
    - **Evaluate their branches applying pseudo arc length continuation**
  - **Locate bifurcations of solution branches** and **identify type of bifurcation**:
    - **Period doubling, Neimark-Sacker, Saddle node**
    - Respective quality (supercritical or subcritical) utilizing normal form coefficients
  - Complete bifurcation set and respective solution branches of limit cycles are **evaluated for**:
    - A **specific speed range**: bifurcation parameter
    - Various **combinations of unbalance**: magnitude and distribution
    - **Rotor bow**: magnitude and phase angle

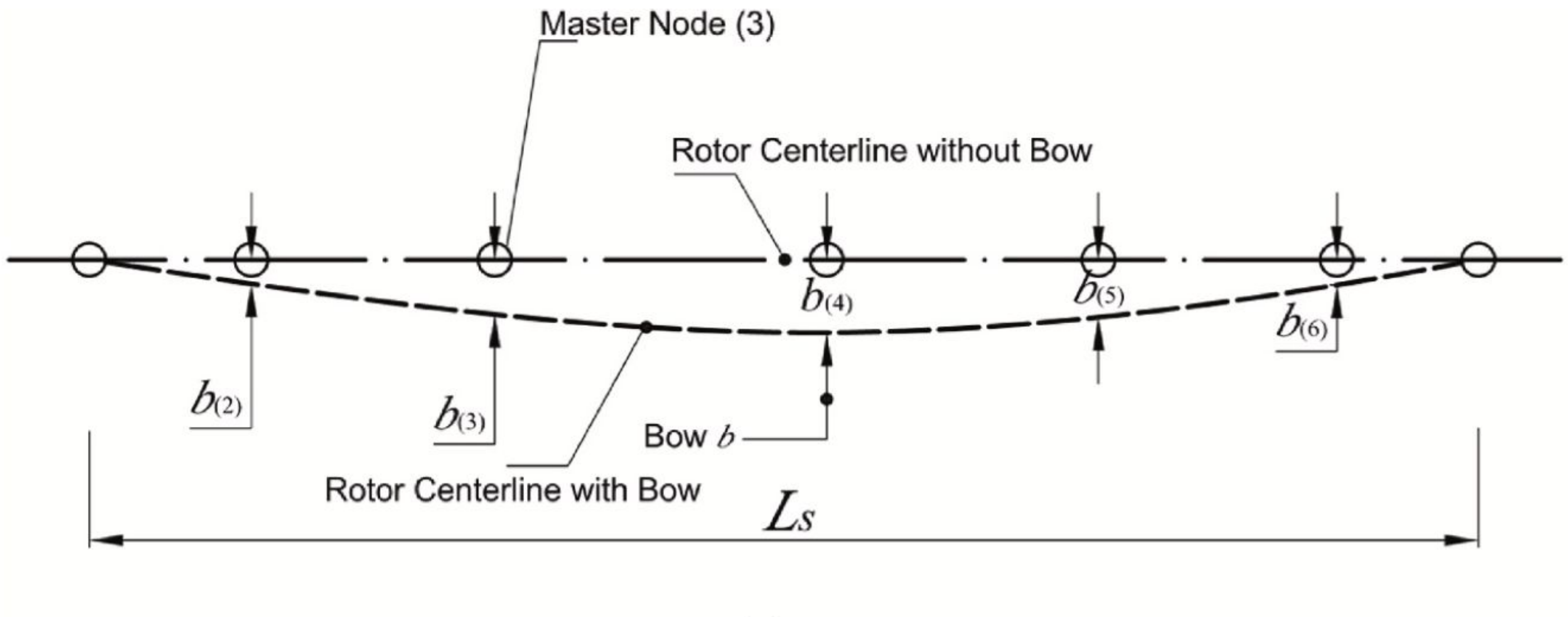
- **Uniform rotor carrying three identical masses placed in symmetry** to model a generic rotor bearing system:
  - Dynamics is modelled with 1D finite beam elements and its mass, gyroscopic and stiffness matrices are composed
  - **Mounted on two identical bearings** which follow short bearing approximation



Bearing span	$L_S = 6$ m	Bearing diameter	$2R_b = 500$ mm
Rotor diameter	$D = 500$ mm	Bearing length	$L_b = 250$ mm
Rotor mass	$M_{tot} = 12000$ kg	Bearing oil dynamic viscosity	$\mu = 0.02$ Pa·s
Operating speed	$\Omega_r = 300$ rad/s	Bearing radial clearance	$c_r = 500$ μm
Young's modulus of shaft	$E = 210$ GPa		
Density of shaft material	$\rho = 7860$ kg/m^3		
Unbalance G2.5	$U \approx 0.1$ kg·m		

- **Physical coordinates** are expressed by $\mathbf{x_i}$ for each degree of freedom (DoF)
- **Bearing forces $\{f_i^B\}$ introduce nonlinearity in system**, as all of them are nonlinear functions of journal kinematics
- Full model matrices are 100 ×100: 24 finite elements, 25 nodes, 4 DoFs per node
- **Most elements** in vectors **$\{f_i^B\}$ bearing forces, $\{f_i^U\}$ unbalance forces, $\{f_i^G\}$ gravity forces in rotor, are zeros**

$$\mathbf{M}\left\{\ddot{x}_i\right\} + (\mathbf{C} + \Omega\mathbf{G})\{\dot{x}_i\} + \mathbf{K}\{x_i\} = \{f_i^B\} + \{f_i^U\} + \{f_i^G\}$$



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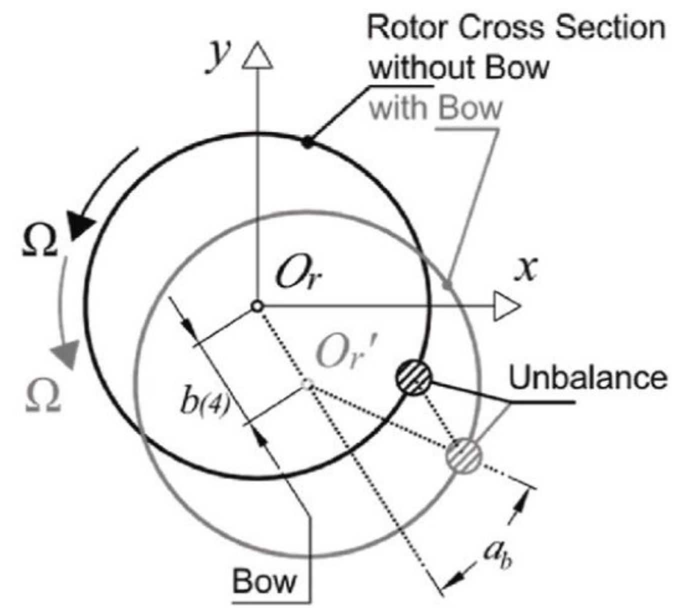


(b)

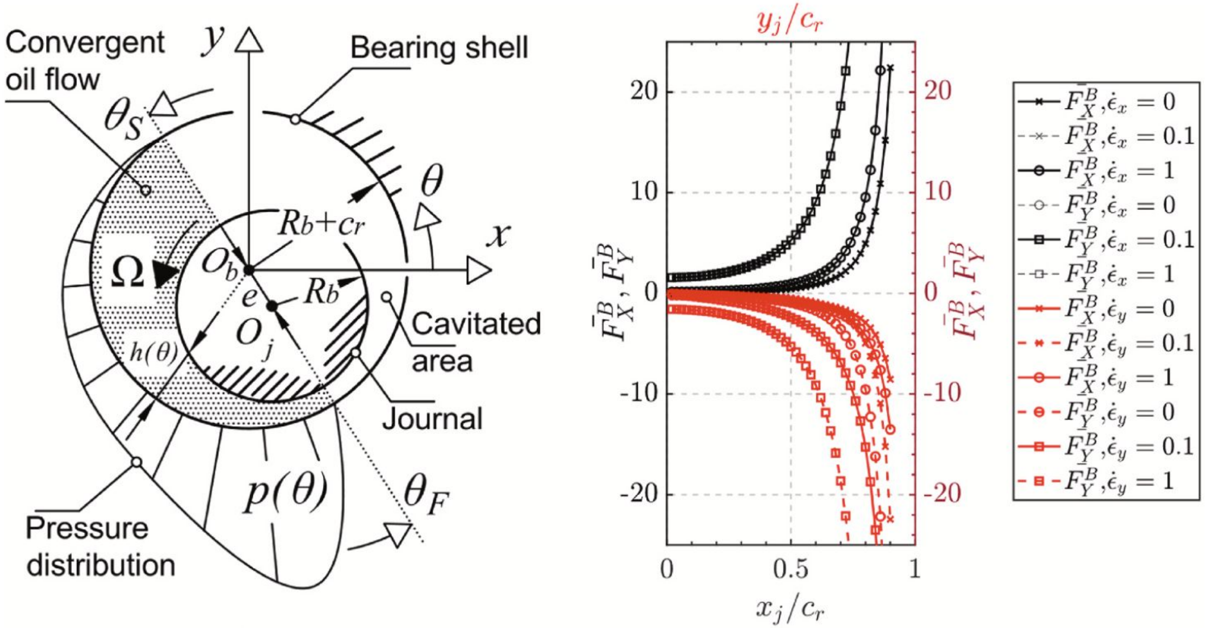
- **Full model is reduced via approximations and matrix manipulation**
- Total number of DoFs included in vector $\{x_{r,i}\}$ is 9, so reduced matrices $\mathbf{M}_r$, $\mathbf{C}_r$, $\mathbf{G}_r$, $\mathbf{K}_r$ are of size 9 × 9

$$[\mathbf{M}_r]\left\{\ddot{x}_{r,i}\right\} + ([\mathbf{C}_r] + \Omega[\mathbf{G}_r])\{\dot{x}_{r,i}\} + [\mathbf{K}_r]\{x_{r,i}\} = \left\{f^B_{r,i}\right\} + \left\{f^U_{r,i}\right\} + \left\{f^G_{r,i}\right\}$$

- **Rotor bow**:
  - Applied a **constant displacement $b_{(i)}$ in respective nodes**, with respect to undeformed rotor centerline
  - At **bearing positions (boundary nodes) is equal to zero**, thus $b_{(1)} = b_{(7)} = 0$
  - *b* is **maximum bow magnitude** and this is assumed to exist **in bearing mid-span**
  - **Bow magnitude *b*** will be **expressed as a percentage of bearing radial clearance c_r**
  - Considering forces generated by bow:

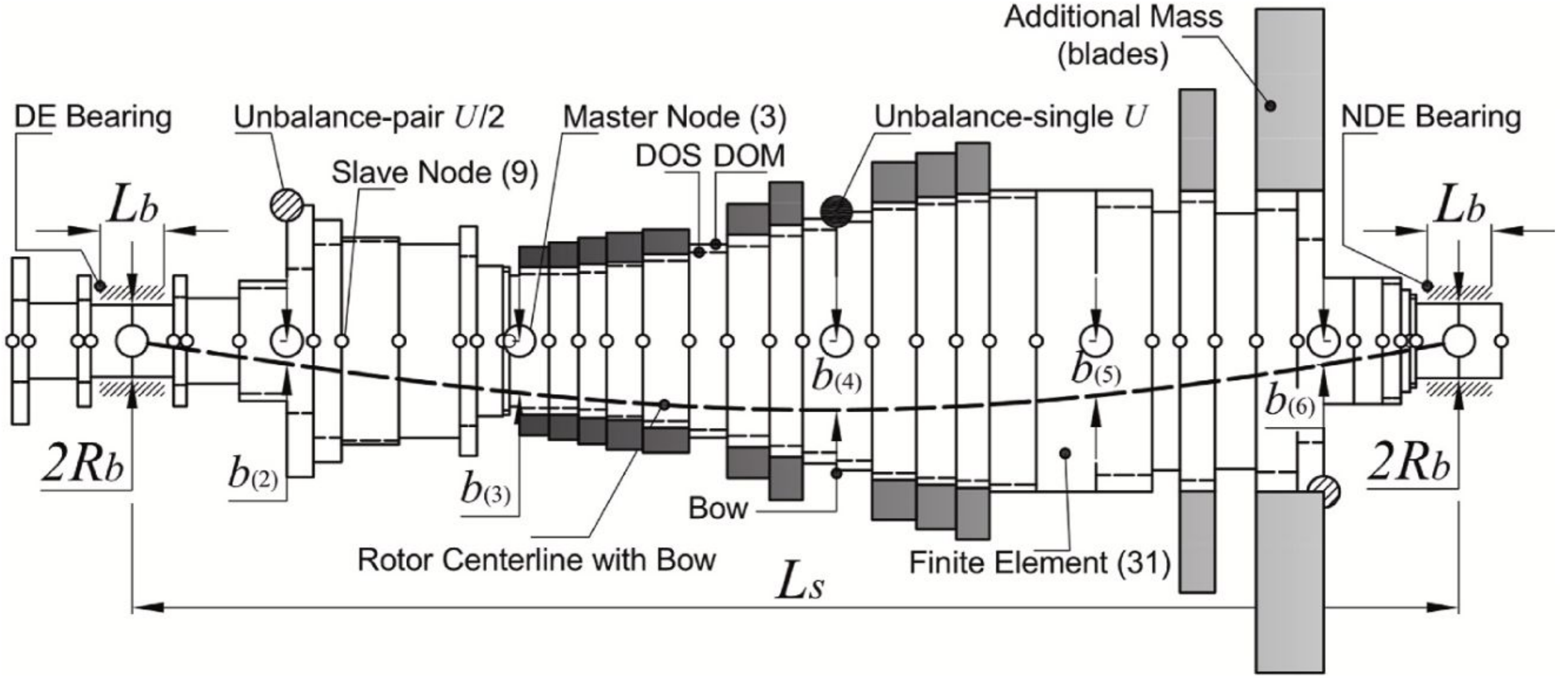
$$\{f_{r,i}\} = \left\{f^B_{r,i}\right\} + \left\{f^U_{r,i}\right\} + \left\{f^G_{r,i}\right\} + \left\{f^{BOW}_{r,i}\right\}$$

- **Journal bearings applied follow short bearing approximation**, incorporating design and operating characteristics of bearings: diameter $2R_b$, bearing length L_b, radial clearance c_r, dynamic viscosity of lubricant μ, operating speed Ω and applied radial load F^B, any combination of these **shows same quality of impedance force nonlinearity**
- **Fluid film impedance forces are nonlinear functions of journal center position** $O_b(x_j, y_j)$ **and of translational velocity**
- Journal kinematics are **expressed in dimensionless forms as** x_j/c_r



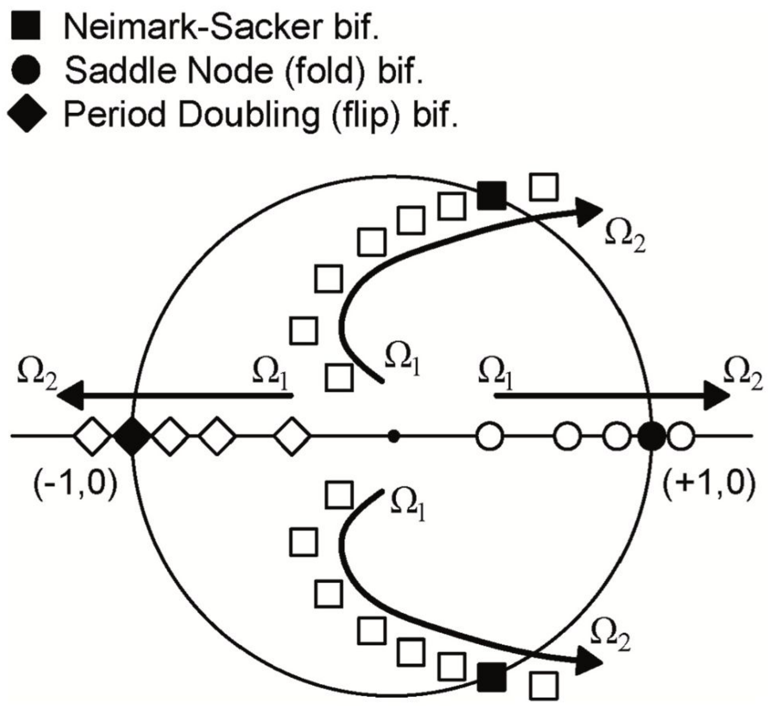
(a) (b)

- **System 2**: **Multi-segment rotor to model a steam turbine**, but any complex geometry can be implemented
- **Unbalance eccentricity e_u follows ISO G-grade definition with service speed of shaft at Ω_r = 420 rad/s** in such systems
- **G-grade 2.5 as reference, therefore $e_u = G/\Omega_r \approx 6.0$ [μm]**
- Motion **equations of System 2** are defined, where reduced matrices **Mr, Cr, Gr, Kr** are of size 11 ×11

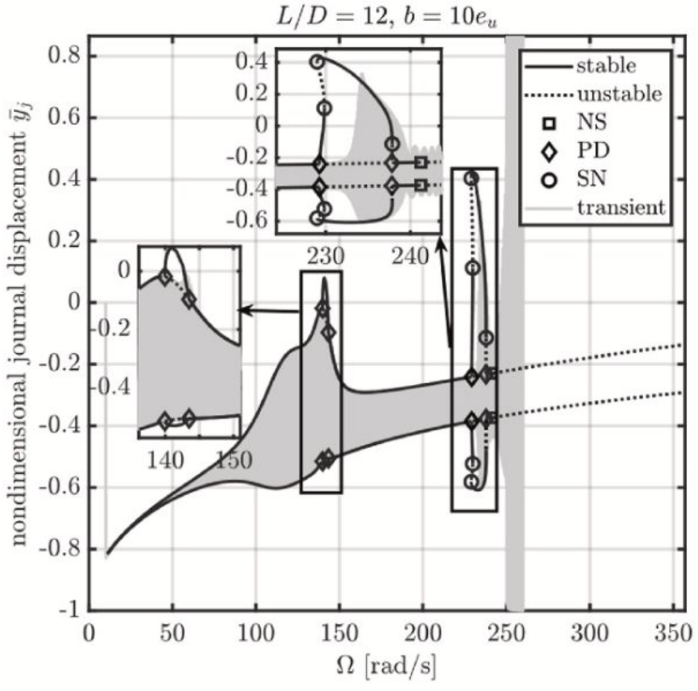


Bearing span	$L_S \approx 4.7$ m	Bearing diameter	$2R_b$ = 300 mm
Rotor equiv. diameter	$D_{eq} \approx 700$ mm	Bearing length	L_b = 150 mm
Rotor mass	$M_{tot} \approx 18000$ kg	Bearing oil dynamic viscosity	μ = 0.03 Pa·s
Operating speed	Ω_r = 420 rad/s	Bearing radial clearance	c_r = 300 μm
Young's modulus of shaft	E = 210 GPa		
Density of shaft material	ρ = 7860 kg/m^3		
Unbalance G2.5	$U \approx 0.1$ kg·m		

- **These systems** can have **asymptotically stable solutions around a fixed point, limit cycle** or also chaotic motions
- A collocation method to compute of periodic limit cycles produced by ODE system, at constant Ω
- **Iterative solution of collocation method return eigenvalues which are directly related to Floquet multipliers of periodic motion**. In this way, **three types of limit cycle bifurcations are found**:
  - **Period doubling bifurcation** (PD), when a **Floquet multiplier crosses unit circle at (− 1, 0)**
  - **Saddle Node bifurcation** (SN) when a **Floquet multiplier crosses unit circle at (+1, 0)**
  - **Secondary Hopf bifurcation**, or **Neimark-Sacker bifurcation** (NS) when **two complex conjugate Floquet multipliers cross unit circle**

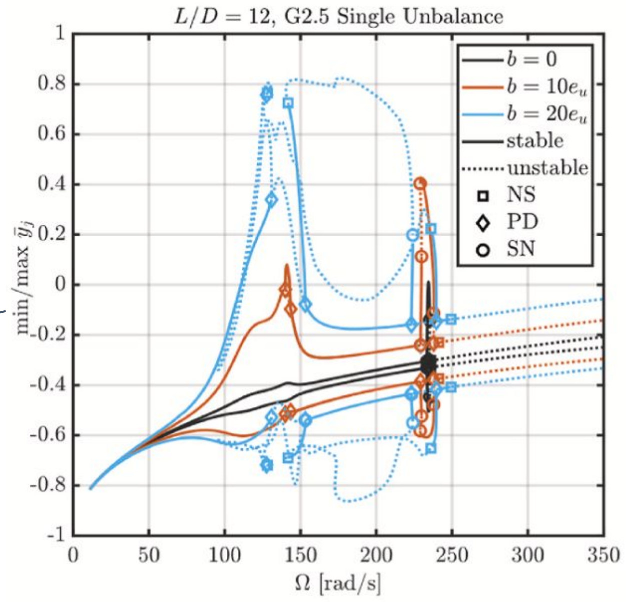


- With ***b*** **= 10** $\boldsymbol{e_u}$
- As **rotor speed approaches** speed of **~140 rad/s**, **system undergoes** a supercritical **period-doubling bifurcation**
- Synchronous **response loses stability and double-period** (sub-synchronous) **stable limit cycle emerges from bifurcation**
  - **Then returns to synchronous limit cycle which regains stability** and **closes double-period solution branch** in a period-doubling bifurcation **at ~148 rad/s**
- Yet **another set of period-doubling bifurcations occur on synchronous limit cycle**
- **As speed increases** further, **response becomes synchronous**, **before undergoing a Neimark-Sacker bifurcation** at which point **it is not possible to evaluate quasi-periodic torus that bifurcates from this point with current formulation**
- Containing **3 saddle-node bifurcations where bifurcating double-period cycle switches stability** every time
- **Generated limit cycles of double period are of much higher amplitude, compared to synchronous response**



(a)

- With ***b* = 20 *e*$_u$**, system can **lose stability before first critical speed** and **not recover**
- **Solution branches change continuously as shaft bow varies**
- System can lose stability even at low speeds with sufficient perturbation



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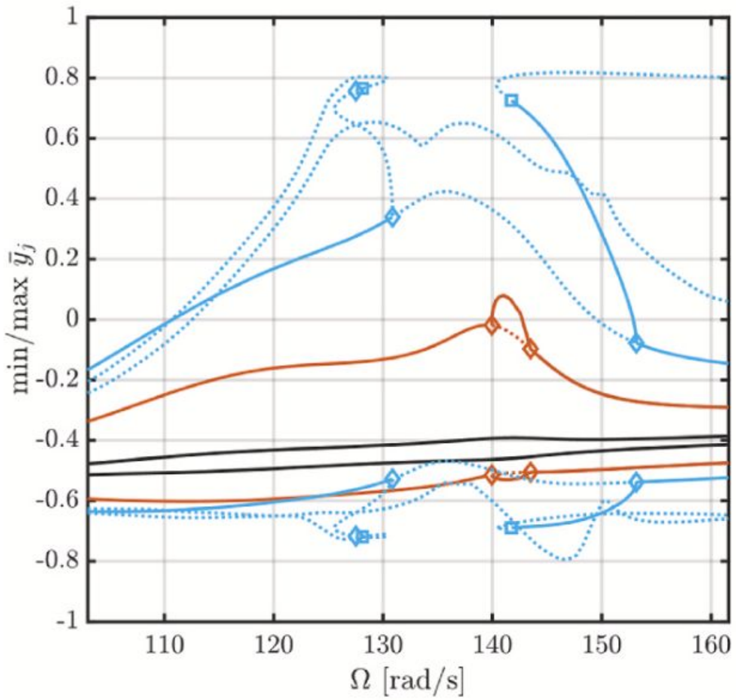

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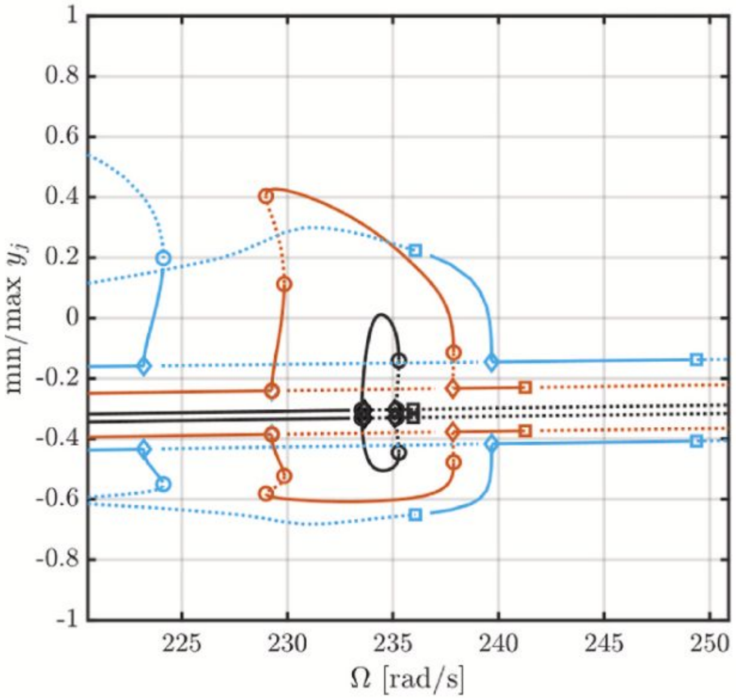
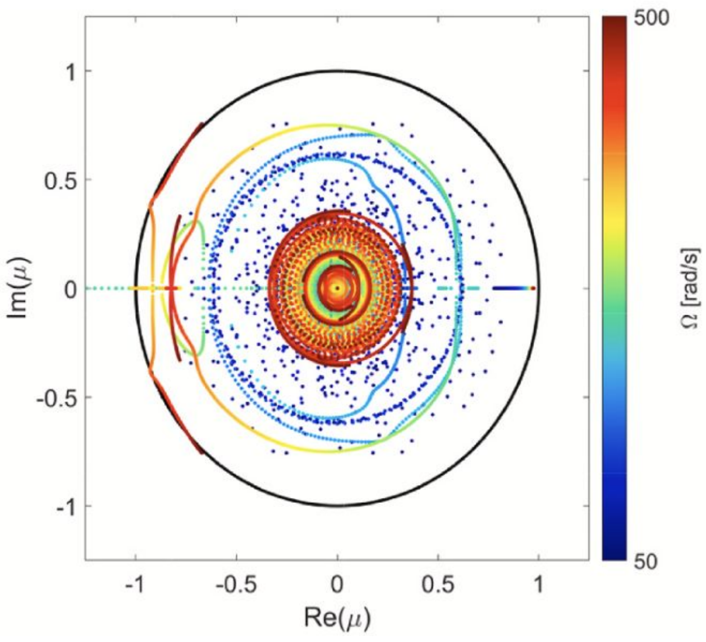
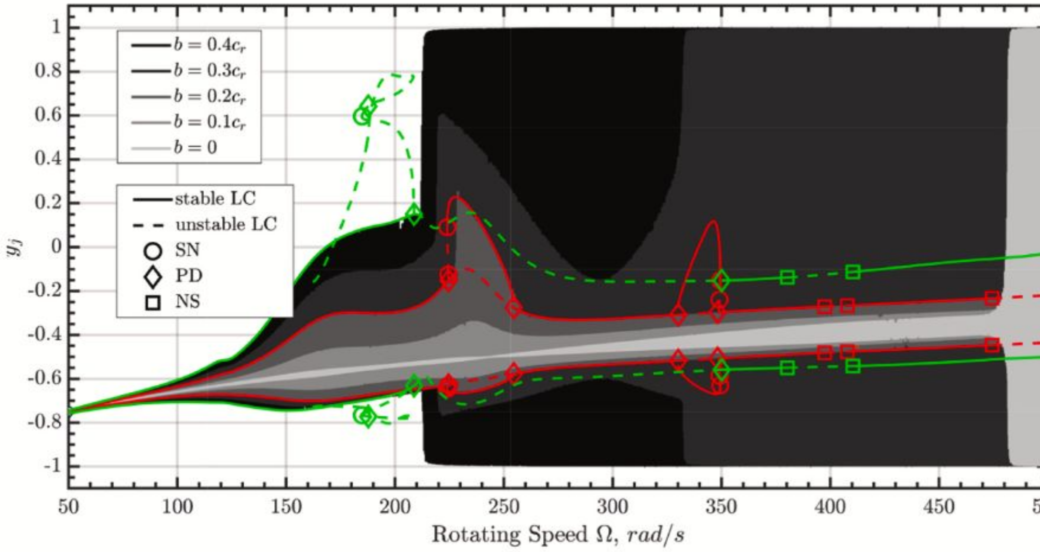


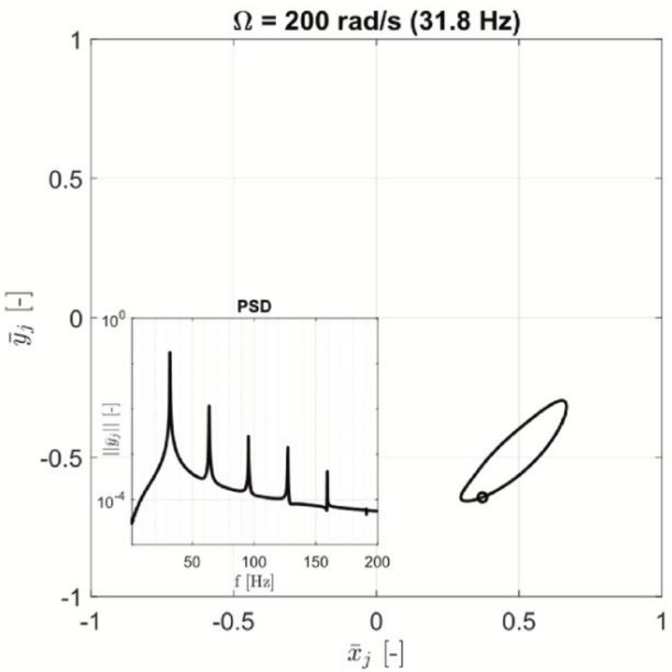
(d)

- **Realistic rotor model**: with rotor bow deformation $b = b/cr = 0, 0.1, 0.2, 0.3$
- **Progress of Floquet multipliers of periodic solutions** for ***b* = 0.2 *cr*** in figure, initially some period doubling bifurcation takes place:
  - One exceeds unit circle in **(− 1, 0)** at **225 *rad/s*** and return at **255 *rad/s***
  - Similarly and at **330 *rad/s*** and **350 *rad/s***
  - **Then** system is governed by **stable elastic response**
  - Until **395 *rad/s*** where a Neimark-Sacker bifurcation takes place, when a conjugate pair of Floquet multipliers cross unit circle
  - At **410 *rad/s*** conjugate pair of Floquet multipliers cross back unit circle and **system recovers global stability through a Neimark-Sacker bifurcation**; **then, elastic response dominates** its motion again
  - Between **395-410 *rad/s*** system executes **stable periodic limit cycle motions close to radial clearance**





- **Trajectories of journal center** at selected speeds with **power spectral density and Poincaré map**
- Trajectories correspond to solutions with bifurcation set for ***b* = 0.2 *cr***. **All trajectories are found to be periodic**
- At **speed < Ω=200 *rad/s***, motion is led by **elastic response of system** and **no bifurcation has been taken place** in system
- At **Ω = 235 *rad/s***, system has **a period doubling bifurcation**:
  - **Period** of motion **is 2T**, **where T is synchronous period** and **Poincaré map includes two discrete points**



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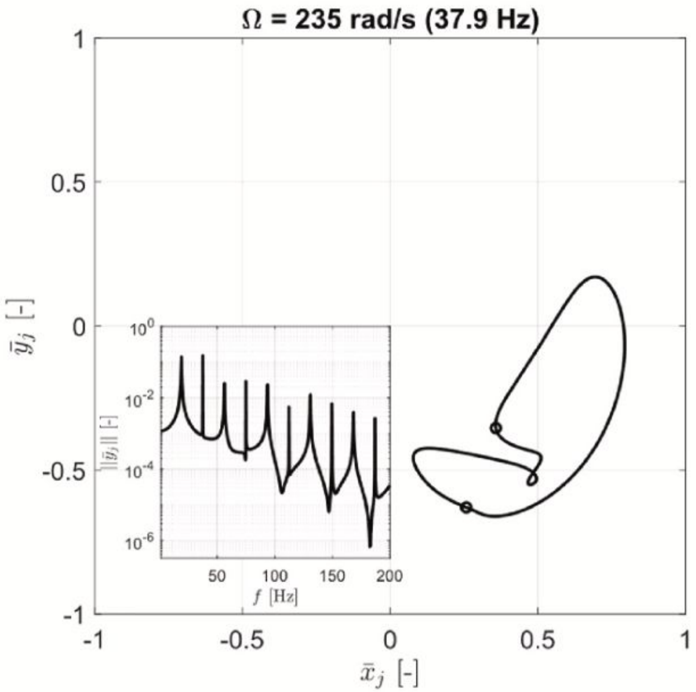

(c)

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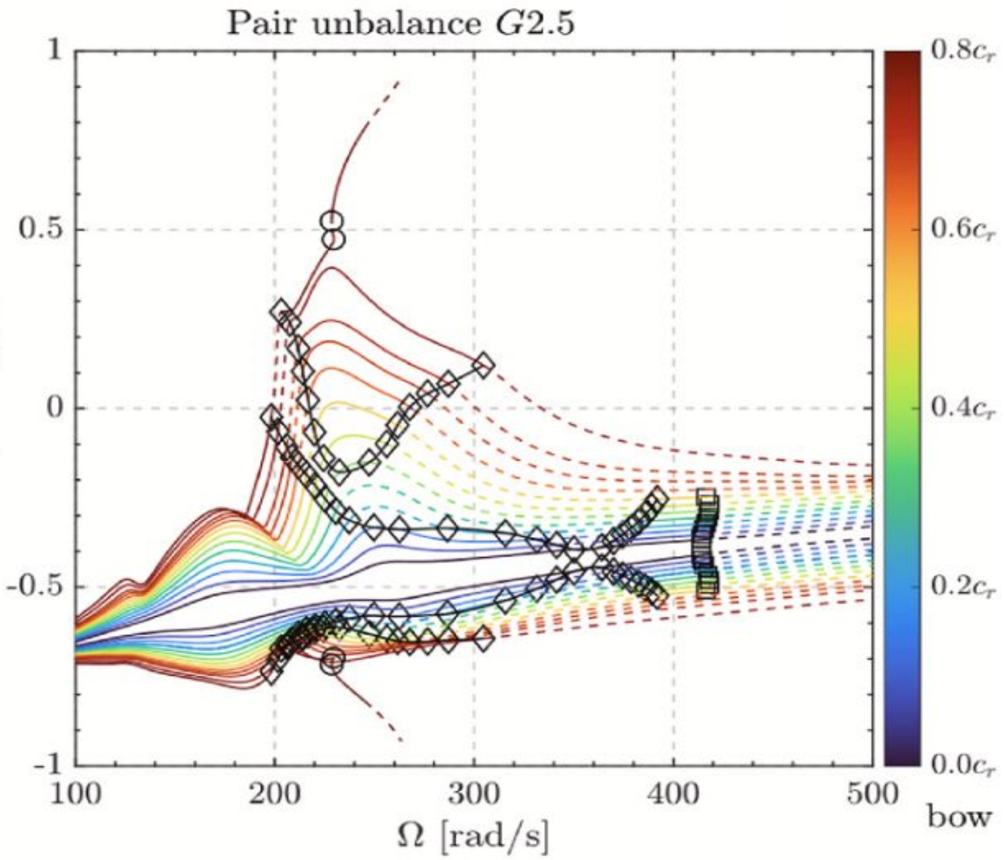
- **Additional solution branches appearing due to existence of bow are initiated at rotating speeds near critical speeds**
- **At most cases** investigated, first (lower speed) **bifurcation occurs as elastic response** of system experiences **resonance**



- **Generic and realistic rotor models** have been studied **on stability and bifurcations** sets when **permanent shaft bow** is introduced
    - Unique source of nonlinearity in systems was nonlinear impedance forces of journal bearings
- **All rotor configurations presented potential to trigger instability when rotor bow exceeded specific values**
    - This was a local instability of synchronous (to rotating speed) branches of limit cycle solutions excited by unbalance
- At most cases, **unstable response** was at **rotating speeds close or higher than critical speed**
- **Periodic or quasi periodic motions were found to evolve after respective bifurcations**: these were attracted by stable solution branches established due to high nonlinear forces of journal bearings
    - When bearing eccentricity is increased and due to physical constraint of bearing radial clearance

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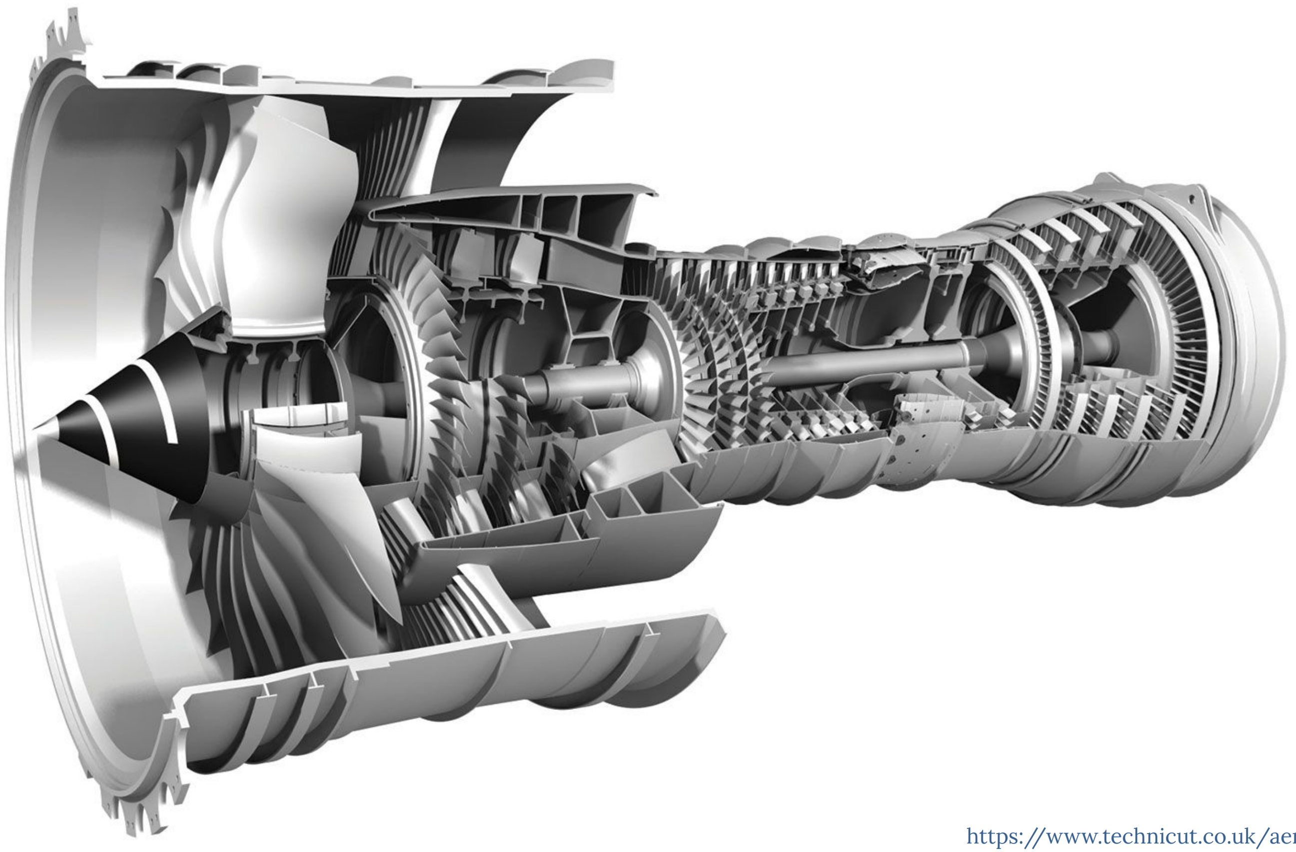

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Thanks for your Attention

Images and text have been gathered from course slides, S. H. Strogatz, *Nonlinear dynamics and chaos* and the paper: *“On the quality of stability and bifurcation sets in rotors with permanent shaft bow on nonlinear supports” | Ioannis Gavalas, Athanasios Chasalevris, Fahimeh Mehralian, R.D. Firouz-Abadi | University of Athens, Greece & Sharif University of Technology, Iran | 29 October 2023”*