Formulae used for the OPTION CALCULATOR APP

(1) BLACK - SCHOLES - European Options PLAIN VANILLA with continuous dividends

Call Option Price :

$$C(0) = S(0)e^{-qT} \cdot N(d_1) - Ke^{-rT} \cdot N(d_2)$$

$$d_1 = \frac{\ln\left(S(0)/K\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Put Option Price :

$$P(0) = Ke^{-rT}N(-d_2) - S(0)e^{-qT}N(-d_1)$$

$$d_1 = \frac{\ln\left(S(0)/K\right) + \left(r - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

→ We show some computations for the Greeks (CALL OPTION). We consider:

$$\text{DELTA} = \Delta = \frac{\partial C}{\partial S}, \quad \text{VEGA} = \nu = \frac{\partial C}{\partial \sigma}, \quad \text{THETA} = \Theta = \frac{\partial C}{\partial T}$$

$$\text{RHO} = \rho = \frac{\partial C}{\partial r}, \quad \text{EPSILON} = \varepsilon = \frac{\partial C}{\partial q}, \quad \text{GAMMA} = \Gamma = \frac{\partial \Delta}{\partial S}$$

Derivation of Δ :

$$\frac{\partial C}{\partial S} = \underbrace{\frac{\partial}{\partial S}\left[Se^{-qT}N(d_1)\right.}_{\text{DERIVATIVE OF A PRODUCT: } \frac{\partial}{\partial x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)} \left. - Ke^{-rT}N(d_2)\right]$$

$$\frac{\partial C}{\partial S} = e^{-qT}N(d_1) + S e^{-qT}\frac{\partial N(d_1)}{\partial S} - K e^{-rT}\cdot\frac{\partial N(d_2)}{\partial S} \qquad (1)$$

NOTE : $$\frac{\partial N(d_i)}{\partial S} \underset{\text{CHAIN RULE}}{=} \underbrace{\frac{\partial N(d_i)}{\partial d_i}} \cdot \frac{\partial d_i}{\partial S}$$

$$\hookrightarrow \frac{\partial}{\partial d_i}\int_{-\infty}^{d_i}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_i^2} = N'(d_1)$$

From (1) : $$\frac{\partial C}{\partial S} = \boxed{e^{-qT}N(d_1)} + \underbrace{S e^{-qT}\cdot N'(d_1)\cdot\frac{\partial d_1}{\partial S} - K e^{-rT}N'(d_2)\frac{\partial d_2}{\partial S}}_{\text{We show that this is zero} \quad (3)} \qquad (2)$$

NOTE : $\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S}$ since $d_2 = d_1 - \sigma\sqrt{T}$

From (3) : $$\underbrace{\left(S e^{-qT}\cdot e^{-\frac{d_1^2}{2}} - K e^{-rT} e^{-\frac{d_2^2}{2}}\right)}_{\text{Focus on this and show that it is zero. (4)}}\cdot\frac{\partial d_1}{\partial S}\,\frac{1}{\sqrt{2\pi}}$$

Consider (4) : $$S\cdot\exp\left\{-qT - \frac{\left[\ln\left(\frac{S}{K}\right)+\left(r-q+\frac{1}{2}\sigma^2\right)T\right]^2}{2\sigma^2 T}\right\} = K\exp\left\{-rT - \frac{\left[\ln\left(\frac{S}{K}\right)+\left(r-q-\frac{1}{2}\sigma^2\right)T\right]^2}{2\sigma^2 T}\right\}$$

$$S\exp\left\{\left[-2\sigma^2T^2q - \cancel{\ln^2\left(\frac{S}{K}\right)} - 2\ln\left(\frac{S}{K}\right)\left(r-q+\frac{1}{2}\sigma^2\right)T - \left(r-q+\frac{1}{2}\sigma^2\right)^2T^2\right]\cdot\frac{1}{2\sigma^2T}\right\}$$

$$= K\exp\left\{\left[-2\sigma^2T^2 r - \cancel{\ln^2\left(\frac{S}{K}\right)} - 2\ln\left(\frac{S}{K}\right)\left(r-q-\frac{1}{2}\sigma^2\right)T - \left(r-q-\frac{1}{2}\sigma^2\right)^2T^2\right]\cdot\frac{1}{2\sigma^2T}\right\}$$

$$S\exp\left\{\left[-\cancel{2\sigma^2T^2q} - 2\ln\left(\frac{S}{K}\right)\left(r-q+\frac{1}{2}\sigma^2\right)T - \cancel{(r-q)^2T^2} - \cancel{(r-q)\sigma^2T^2} - \cancel{\frac{\sigma^4T^2}{4}}\right]\cdot\frac{1}{2\sigma^2T}\right\}$$

$$- K\exp\left\{\left[-\cancel{2\sigma^2T^2r} - 2\ln\left(\frac{S}{K}\right)\left(r-q-\frac{1}{2}\sigma^2\right)T - \cancel{(r-q)^2T^2} + \cancel{(r-q)\sigma^2T^2} - \cancel{\frac{\sigma^4T^2}{4}}\right]\cdot\frac{1}{2\sigma^2T}\right\} = 0$$

$$S\exp\left\{-\ln\left(\frac{S}{K}\right)\cancel{\sigma^2T}\cdot\frac{1}{2\cancel{\sigma^2T}}\right\} = K\exp\left\{\ln\left(\frac{S}{K}\right)\cancel{\sigma^2T}\cdot\frac{1}{2\cancel{\sigma^2T}}\right\}$$

$$\exp\left\{\ln\left(\frac{S}{K}\right) - \cancel{\frac{1}{2}\ln\left(\frac{S}{K}\right)}\right\} = \exp\left\{\cancel{\frac{1}{2}\ln\left(\frac{S}{K}\right)}\right\}$$

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$$\nu = \frac{\partial C}{\partial\sigma} = \frac{\partial}{\partial\sigma}\left[Se^{-qT}N(d_1)\right] - \frac{\partial}{\partial\sigma}\left[Ke^{-rT}N(d_2)\right]$$

$$= Se^{-qT}N'(d_1)\frac{\partial d_1}{\partial\sigma} - Ke^{-rT}N'(d_2)\frac{\partial d_1}{\partial\sigma} + Ke^{-rT}N'(d_2)\sqrt{T}$$

$$\boxed{= Ke^{-rT}N'(d_2)\sqrt{T}}$$

$$\Theta = \frac{\partial C}{\partial T} = \frac{\partial}{\partial T}\left[Se^{-qT}N(d_1)\right] - \frac{\partial}{\partial T}\left[Ke^{-rT}N(d_2)\right]$$

$$= -qSe^{-qT}N(d_1) + Se^{-qT}N'(d_1)\frac{\partial d_1}{\partial T} + rKe^{-rT}N(d_2) - Ke^{-rT}N'(d_2)\frac{\partial d_2}{\partial T}$$

$$= -qSe^{-qT}N(d_1) + Se^{-qT}\cancel{N'(d_1)}\frac{\partial d_1}{\partial T} + rKe^{-rT}N(d_2) - Ke^{-rT}\cancel{N'(d_2)}\frac{\partial d_1}{\partial T} +$$

$$-\frac{1}{2\sqrt{T}}\sigma K e^{-rT}N'(d_2) = -qSe^{-qT}N(d_1) + rKe^{-rT}N(d_2) - \frac{\sigma}{2\sqrt{T}}Ke^{-rT}N'(d_2)$$

$$\rho = \frac{\partial C}{\partial r} = \frac{\partial}{\partial r}\left[Se^{-qT}N(d_1)\right] - \frac{\partial}{\partial r}\left[Ke^{-rT}N(d_2)\right]$$

$$= Se^{-qT}\cancel{N'(d_1)}\frac{\partial d_1}{\partial r} + T\cdot Ke^{-rT}N(d_2) - Ke^{-rT}\cancel{N'(d_2)}\frac{\partial d_2}{\partial r} = TKe^{-rT}N(d_2)$$

$$\varepsilon = \frac{\partial C}{\partial q} = \frac{\partial}{\partial q}\left[Se^{-qT}N(d_1)\right] - \frac{\partial}{\partial q}\left[Ke^{-rT}N(d_2)\right]$$

$$= -T\cdot S e^{-qT}N(d_1) + Se^{-qT}\cancel{N'(d_1)}\frac{\partial d_1}{\partial q} - Ke^{-rT}\cancel{N'(d_2)}\frac{\partial d_2}{\partial q}$$

$$= -T\cdot S\cdot e^{-qT}N(d_1)$$

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial}{\partial S}e^{-qT}N(d_1) = e^{-qT}\cdot N'(d_1)\cdot\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}}e^{-qT}N'(d_1)$$

NOTE: $\frac{\partial d_1}{\partial S} = \frac{\partial}{\partial S}\frac{\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{1}{S\sigma\sqrt{T}}$