Formulae used for the OPTION CALCULATOR APP

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(1) BLACK - SCHOLES - European Options PLAIN VANILLA with continuous dividends

Call Option Price :

$$C(0) = S(0)e^{-qT} \cdot N(d_1) - Ke^{-rT} \cdot N(d_2)$$

$$d_1 = \frac{\ln\left(S(0)/K\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Put Option Price :

$$P(0) = Ke^{-rT}N(-d_2) - S(0)e^{-qT}N(-d_1)$$

$$d_1 = \frac{\ln\left(S(0)/K\right) + \left(r - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

→ We show some computations for the Greeks (CALL OPTION). We consider:

$$\text{DELTA} = \Delta = \frac{\partial C}{\partial S}, \quad \text{VEGA} = \nu = \frac{\partial C}{\partial \sigma}, \quad \text{THETA} = \Theta = \frac{\partial C}{\partial T}$$

$$\text{RHO} = \rho = \frac{\partial C}{\partial r}, \quad \text{EPSILON} = \varepsilon = \frac{\partial C}{\partial q}, \quad \text{GAMMA} = \Gamma = \frac{\partial \Delta}{\partial S}$$

Derivation of Δ :

$$\frac{\partial C}{\partial S} = \frac{\partial}{\partial S}\left[\underbrace{Se^{-qT}N(d_1)}_{} - Ke^{-rT}N(d_2)\right]$$

DERIVATIVE OF A PRODUCT : $\frac{\partial}{\partial x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

$$\frac{\partial C}{\partial S} = e^{-qT} N(d_1) + S e^{-qT} \frac{\partial N(d_1)}{\partial S} - K e^{-rT} \cdot \frac{\partial N(d_2)}{\partial S} \qquad (1)$$

NOTE : $$\underbrace{\frac{\partial N(d_i)}{\partial S}}_{} \underset{\text{CHAIN RULE}}{=} \underbrace{\frac{\partial N(d_i)}{\partial d_i}} \cdot \frac{\partial d_i}{\partial S}$$

$$\hookrightarrow \frac{\partial}{\partial d_i} \int_{-\infty}^{d_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_i^2} = N'(d_1)$$

From (1) : $$\frac{\partial C}{\partial S} = \boxed{e^{-qT} N(d_1)} + \underbrace{S e^{-qT} \cdot N'(d_1) \cdot \frac{\partial d_1}{\partial S} - K e^{-rT} N'(d_2) \frac{\partial d_2}{\partial S}}_{\text{We show that this is zero} \quad (3)} \qquad (2)$$

NOTE : $\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S}$ since $d_2 = d_1 - \sigma\sqrt{T}$

From (3) : $$\underbrace{\left(S e^{-qT} \cdot e^{-\frac{d_1^2}{2}} - K e^{-rT} e^{-\frac{d_2^2}{2}} \right)}_{\text{Focus on this and show that it is zero. } (4)} \cdot \frac{\partial d_1}{\partial S} \frac{1}{\sqrt{2\pi}}$$

Consider (4) : $$S \cdot \exp\left\{ -qT - \frac{\left[\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)T \right]^2}{2\sigma^2 T} \right\} = K \exp\left\{ -rT - \frac{\left[\ln\left(\frac{S}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)T \right]^2}{2\sigma^2 T} \right\}$$

$$S \exp\left\{ \left[-2\sigma^2 T^2 q - \cancel{\ln^2\left(\frac{S}{K}\right)} - 2\ln\left(\frac{S}{K}\right)\left(r - q + \frac{1}{2}\sigma^2\right)T - \left(r - q + \frac{1}{2}\sigma^2\right)^2 T^2 \right] \cdot \frac{1}{2\sigma^2 T} \right\}$$

$$= K \exp\left\{\left[-2\sigma^2T^2r - \cancel{\ln^2\left(\frac{S}{K}\right)} - 2\ln\left(\frac{S}{K}\right)\left(r-q-\frac{1}{2}\sigma^2\right)T - \left(r-q-\frac{1}{2}\sigma^2\right)^2T^2\right]\cdot\frac{1}{2\sigma^2T}\right\}$$

$$S\exp\left\{\left[-\cancel{2\sigma^2T^2q} - 2\ln\left(\frac{S}{K}\right)\left(r-q+\frac{1}{2}\sigma^2\right)T - \cancel{(r-q)^2T^2} - \cancel{(r-q)\sigma^2T^2} - \cancel{\frac{\sigma^4T^2}{4}}\right]\cdot\frac{1}{2\sigma^2T}\right\}$$

$$- K\exp\left\{\left[-\cancel{2\sigma^2T^2r} - 2\ln\left(\frac{S}{K}\right)\left(r-q-\frac{1}{2}\sigma^2\right)T - \cancel{(r-q)^2T^2} + \cancel{(r-q)\sigma^2T^2} - \cancel{\frac{\sigma^4T^2}{4}}\right]\cdot\frac{1}{2\sigma^2T}\right\} = 0$$

$$S\exp\left\{-\ln\left(\frac{S}{K}\right)\cancel{\sigma^2T}\cdot\frac{1}{2\cancel{\sigma^2T}}\right\} = K\exp\left\{\ln\left(\frac{S}{K}\right)\cancel{\sigma^2T}\cdot\frac{1}{2\cancel{\sigma^2T}}\right\}$$

$$\exp\left\{\ln\left(\frac{S}{K}\right) - \cancel{\frac{1}{2}\ln\left(\frac{S}{K}\right)}\right\} = \exp\left\{\cancel{\frac{1}{2}\ln\left(\frac{S}{K}\right)}\right\}$$

$$\nu = \frac{\partial C}{\partial \sigma} = \frac{\partial}{\partial\sigma}\left[Se^{-qT}N(d_1)\right] - \frac{\partial}{\partial\sigma}\left[Ke^{-rT}N(d_2)\right]$$

$$= Se^{-qT}N'(d_1)\frac{\partial d_1}{\partial\sigma} - Ke^{-rT}N'(d_2)\frac{\partial d_1}{\partial\sigma} + Ke^{-rT}N'(d_2)\sqrt{T}$$

$$\boxed{= Ke^{-rT}N'(d_2)\sqrt{T}}$$

$$\Theta = \frac{\partial C}{\partial T} = \frac{\partial}{\partial T}\left[Se^{-qT}N(d_1)\right] - \frac{\partial}{\partial T}\left[Ke^{-rT}N(d_2)\right]$$

$$= -qSe^{-qT}N(d_1) + Se^{-qT}N'(d_1)\frac{\partial d_1}{\partial T} + rKe^{-rT}N(d_2) - Ke^{-rT}N'(d_2)\frac{\partial d_2}{\partial T}$$

$$= -qSe^{-qT}N(d_1) + Se^{-qT}\cancel{N'(d_1)}\frac{\partial d_1}{\partial T} + rKe^{-rT}N(d_2) - Ke^{-rT}\cancel{N'(d_2)}\frac{\partial d_1}{\partial T} +$$

$$- \frac{1}{2\sqrt{T}}\sigma K e^{-rT}N'(d_2) = -qSe^{-qT}N(d_1) + rKe^{-rT}N(d_2) - \frac{\sigma}{2\sqrt{T}}Ke^{-rT}N'(d_2)$$

$$\rho = \frac{\partial C}{\partial r} = \frac{\partial}{\partial r}\left[Se^{-qT}N(d_1)\right] - \frac{\partial}{\partial r}\left[Ke^{-rT}N(d_2)\right]$$

$$= Se^{-qT}\cancel{N'(d_1)}\frac{\partial d_1}{\partial r} + T\cdot Ke^{-rT}N(d_2) - Ke^{-rT}\cancel{N'(d_2)}\frac{\partial d_2}{\partial r} = TKe^{-rT}N(d_2)$$

$$\varepsilon = \frac{\partial C}{\partial q} = \frac{\partial}{\partial q}\left[Se^{-qT}N(d_1)\right] - \frac{\partial}{\partial q}\left[Ke^{-rT}N(d_2)\right]$$

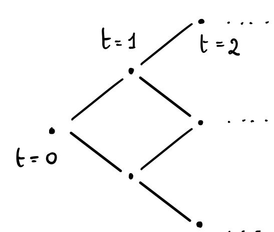
$$= -T\cdot S e^{-qT}N(d_1) + Se^{-qT}\cancel{N'(d_1)}\frac{\partial d_1}{\partial q} - Ke^{-rT}\cancel{N'(d_2)}\frac{\partial d_2}{\partial q}$$

$$= -T\cdot S\cdot e^{-qT}N(d_1)$$

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial}{\partial S}e^{-qT}N(d_1) = e^{-qT}N'(d_1)\cdot\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}}e^{-qT}N'(d_1)$$

NOTE : $\frac{\partial d_1}{\partial S} = \frac{\partial}{\partial S}\frac{\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{1}{S\sigma\sqrt{T}}$

Computation of Greeks in the BINOMIAL MODEL



$$\Delta = \frac{C_1(u) - C_1(d)}{S_1(u) - S_1(d)}$$

where $S_1(u) = S \cdot u$, $S_1(d) = S \cdot d$

$C_1(u)$ = price of the option at $t=1$ when the asset is going up

$$\Delta_1(u) = \frac{C_2(uu) - C_2(ud)}{S_2(uu) - S_2(ud)} \qquad \Delta_1(d) = \frac{C_2(ud) - C_2(dd)}{S_2(ud) - S_2(dd)}$$

$$\Gamma = \frac{\Delta_1(u) - \Delta_1(d)}{S_1(u) - S_1(d)} \qquad \theta = \frac{C_2(ud) - C_0}{2\,\Delta t}$$

$$\rho = \frac{C_0(r + 1\%)}{C_0} - 1 \qquad \nu = \frac{C_0(\sigma + 1\%)}{C_0} - 1$$

- Graphic