

(1) BLACK-SCHOLES - European Options PLAIN VANILLA with continuous dividends

Call Option Price : $C(0) = S(0) e^{-qT} \cdot N(d_1) - K e^{-rT} \cdot N(d_2)$

$$d_1 = \frac{\ln(S(0)/K) + (r - q + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

Put Option Price : $P(0) = K e^{-rT} N(-d_2) - S(0) e^{-qT} N(-d_1)$

$$d_1 = \frac{\ln(S(0)/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

→ We show some computations for the Greeks (CALL OPTION). We consider :

$$\text{DELTA} = \Delta = \frac{\partial C}{\partial S} \quad , \quad \text{VEGA} = \gamma = \frac{\partial C}{\partial \sigma} \quad , \quad \text{THETA} = \Theta = \frac{\partial C}{\partial T}$$

$$\text{RHO} = \rho = \frac{\partial C}{\partial r} \quad , \quad \text{EPSILON} = \varepsilon = \frac{\partial C}{\partial q} \quad , \quad \text{GAMMA} = \Gamma = \frac{\partial \Delta}{\partial S}$$

Derivation of Δ : $\frac{\partial C}{\partial S} = \frac{\partial}{\partial S} \left[\underbrace{S e^{-qT} N(d_1)}_{\text{DERIVATIVE OF A PRODUCT}} - K e^{-rT} N(d_2) \right]$

$$\text{DERIVATIVE OF A PRODUCT : } \frac{\partial}{\partial x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{\partial C}{\partial S} = e^{-qT} N(d_1) + S e^{-qT} \frac{\partial N(d_1)}{\partial S} - K e^{-rT} \frac{\partial N(d_2)}{\partial S} \quad (1)$$

NOTE : $\frac{\partial N(d_i)}{\partial S} = \frac{\partial N(d_i)}{\partial d_i} \cdot \frac{\partial d_i}{\partial S}$

↓
CHAIN RULE

↳ $\frac{\partial}{\partial d_i} \int_{-\infty}^{d_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_i^2} = N'(d_i)$

From (1) : $\frac{\partial C}{\partial S} = \boxed{e^{-qT} N(d_1)} + \underbrace{S e^{-qT} \cdot N'(d_1) \cdot \frac{\partial d_1}{\partial S} - K e^{-rT} N'(d_2) \frac{\partial d_2}{\partial S}}_{\text{We show that this is zero}} \quad (2)$

We show that this is zero (3)

NOTE : $\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S}$ since $d_2 = d_1 - \frac{1}{2}\sigma^2 T$

From (3) : $\underbrace{\left(S e^{-qT} e^{-\frac{d_1^2}{2}} - K e^{-rT} e^{-\frac{d_2^2}{2}} \right)}_{\text{Focus on this and show that it is zero. (4)}} \cdot \frac{\partial d_1}{\partial S} \cdot \frac{1}{\sqrt{2\pi}}$

Focus on this and show that it is zero. (4)

Consider (4) : $S \cdot \exp \left\{ -qT - \frac{\left[\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)T \right]^2}{2\sigma^2 T} \right\} = K \exp \left\{ -rT - \frac{\left[\ln\left(\frac{S}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)T \right]^2}{2\sigma^2 T} \right\}$

$S \exp \left\{ \left[-2\sigma^2 T q - \cancel{\ln^2\left(\frac{S}{K}\right)} - 2\ln\left(\frac{S}{K}\right)\left(r - q + \frac{1}{2}\sigma^2\right)T - \left(r - q + \frac{1}{2}\sigma^2\right)^2 T^2 \right] \cdot \frac{1}{2\sigma^2 T} \right\}$

$$= K \exp \left\{ \left[-2b^2 T^2 x - \cancel{\ln^2 \left(\frac{s}{K} \right)} - 2 \ln \left(\frac{s}{K} \right) \left(x - q - \frac{1}{2} b^2 \right) T - \left(x - q - \frac{1}{2} b^2 \right)^2 T^2 \right] \cdot \frac{1}{2b^2 T} \right\}$$

$$S \exp \left\{ \left[-\cancel{2b^2 T^2 q} - 2 \ln \left(\frac{s}{K} \right) \left(x - q + \frac{1}{2} b^2 \right) T - \cancel{\left(x - q \right)^2 T^2} - \cancel{\left(x - q \right) b^2 T^2} - \cancel{\frac{b^4 T^2}{4}} \right] \cdot \frac{1}{2b^2 T} \right\}$$

$$- K \exp \left\{ \left[-\cancel{2b^2 T^2 x} - 2 \ln \left(\frac{s}{K} \right) \left(x - q - \frac{1}{2} b^2 \right) T - \cancel{\left(x - q \right)^2 T^2} + \cancel{\left(x - q \right) b^2 T^2} - \cancel{\frac{b^4 T^2}{4}} \right] \cdot \frac{1}{2b^2 T} \right\} = 0$$

$$S \exp \left\{ - \ln \left(\frac{s}{K} \right) b^2 T \cdot \frac{1}{2b^2 T} \right\} = K \exp \left\{ \ln \left(\frac{s}{K} \right) b^2 T \cdot \frac{1}{2b^2 T} \right\}$$

$$\exp \left\{ \cancel{\ln \left(\frac{s}{K} \right) - \frac{1}{2} \ln \left(\frac{s}{K} \right)} \right\} = \exp \left\{ \frac{1}{2} \ln \left(\frac{s}{K} \right) \right\} \quad \checkmark$$

$$\gamma = \frac{\partial C}{\partial b} = \frac{\partial}{\partial b} \left[S e^{-qT} N(d_1) \right] - \frac{\partial}{\partial b} \left[K e^{-rT} N(d_2) \right]$$

$$= S e^{-qT} N'(d_1) \frac{\partial d_1}{\partial b} - K e^{-rT} N'(d_2) \frac{\partial d_1}{\partial b} + K e^{-rT} N'(d_2) \sqrt{T}$$

$$= K e^{-rT} N'(d_2) \sqrt{T}$$

$$\textcircled{4} = \frac{\partial C}{\partial T} = \frac{\partial}{\partial T} \left[S e^{-qT} N(d_1) \right] - \frac{\partial}{\partial T} \left[K e^{-rT} N(d_2) \right]$$

$$= -q S e^{-qT} N(d_1) + S e^{-qT} N'(d_1) \frac{\partial d_1}{\partial T} + r K e^{-rT} N(d_2) - K e^{-rT} N'(d_2) \frac{\partial d_2}{\partial T}$$

$$\begin{aligned}
&= -q S e^{-qT} N(d_1) + S e^{-qT} \cancel{N'(d_1)} \frac{\partial d_1}{\partial T} + \pi K e^{-\pi T} N(d_2) - K e^{-\pi T} \cancel{N'(d_2)} \frac{\partial d_2}{\partial T} + \\
&\quad - \frac{1}{2\sqrt{T}} \partial K e^{-\pi T} N'(d_2) = -q S e^{-qT} N(d_1) + \pi K e^{-\pi T} N(d_2) - \frac{\partial}{2\sqrt{T}} K e^{-\pi T} N'(d_2)
\end{aligned}$$

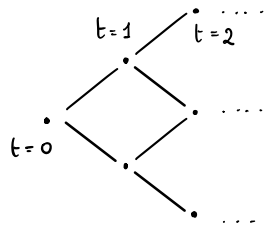
$$\begin{aligned}
\rho &= \frac{\partial C}{\partial \pi} = \frac{\partial}{\partial \pi} \left[S e^{-qT} N(d_1) \right] - \frac{\partial}{\partial \pi} \left[K e^{-\pi T} N(d_2) \right] \\
&= S e^{-qT} \cancel{N'(d_1)} \frac{\partial d_1}{\partial \pi} + T K e^{-\pi T} N(d_2) - K e^{-\pi T} \cancel{N'(d_2)} \frac{\partial d_2}{\partial \pi} = T K e^{-\pi T} N(d_2)
\end{aligned}$$

$$\begin{aligned}
\varepsilon &= \frac{\partial C}{\partial q} = \frac{\partial}{\partial q} \left[S e^{-qT} N(d_1) \right] - \frac{\partial}{\partial q} \left[K e^{-\pi T} N(d_2) \right] \\
&= -T S e^{-qT} N(d_1) + S e^{-qT} \cancel{N'(d_1)} \frac{\partial d_1}{\partial q} - K e^{-\pi T} \cancel{N'(d_2)} \frac{\partial d_2}{\partial q} \\
&= -T S e^{-qT} N(d_1)
\end{aligned}$$

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial}{\partial S} e^{-qT} N(d_1) = e^{-qT} N'(d_1) \cdot \frac{\partial d_1}{\partial S} = \frac{1}{S 2\sqrt{T}} e^{-qT} N'(d_1)$$

$$\text{NOTE: } \frac{\partial d_1}{\partial S} = \frac{\partial}{\partial S} \frac{\ln\left(\frac{S}{K}\right) + \left(\pi - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{1}{S \sigma\sqrt{T}}$$

Computation of Greeks in the BINOMIAL MODEL



$$\Delta = \frac{C_1(u) - C_1(d)}{S_1(u) - S_1(d)}$$

where

$$S_1(u) = S \cdot u, \quad S_1(d) = S \cdot d$$

$C_1(u)$ = price of the option at $t=1$ when the asset is going up

$$\Delta_1(u) = \frac{C_2(uu) - C_2(ud)}{S_2(uu) - S_2(ud)}$$

$$\Delta_1(d) = \frac{C_2(ud) - C_2(dd)}{S_2(ud) - S_2(dd)}$$

$$\Gamma = \frac{\Delta_1(u) - \Delta_1(d)}{S_1(u) - S_1(d)}$$

$$\theta = \frac{C_2(ud) - C_0}{2 \Delta t \cdot 365}$$

$$\rho = \frac{C_0(r+1\%) - C_0(r-1\%)}{100 \cdot 2\%}$$

$$\gamma = \frac{C_0(2+1\%) - C_0(2-1\%)}{100 \cdot 2\%}$$

• Graphic

OPTION CALCULATOR	
Frame 1 : WELCOME MESSAGE	
Frame 2 : INPUTS	<div>CALCULATE</div> <div>RESET</div> <div>Frame 3</div>
BINOMIAL TREE (CRR) Frame 6	BLACK SCHOLES Frame 7
Frame 4 : OUTPUTS	Frame 5 : OUTPUTS