Calculus 2 (ACSAI)

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Disclaimer

These notes were taken during the "Calculus 2" course given by Professor Alla during the first semester of the Academic Year 2024/2025 at the course "Applied Computer Science and Artificial Intelligence" of Sapienza Università di Roma.
Keep in mind, however, that these notes do **not** replace the course material, so it is suggested to check the professor's resources as well.
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Chapter 1

Multivariable functions

1.1 Defining multivariable functions

Generally speaking, a multivariable function is an application $f : \mathbb{R}^n \to \mathbb{R}$ that, for any input $x \in \mathbb{R}^n$, takes on a value in $\mathbb{R}$.
N.B.: This course will mostly focus on functions in two or three variables.

1.2 The domain and range of a function in two variables

1.2.1 The domain of a function in two variables

Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, the domain of f is the largest set $\Omega \subseteq \mathbb{R}^2$ such that the function is defined for every point in Ω.
Example: Determine the domain of the function $f(x, y) = \sqrt{x^2 + y^2}$.
By the definition of the square root, let $\Omega : x^2 + y^2 \geq 0 \Rightarrow \Omega = \mathbb{R}^2$.

1.2.1.1 Classifying points and regions with respect to a function's domain

Given a function $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, it is possible to classify a generic point $(x_0, y_0) \in \mathbb{R}^2$ with respect to the function's domain as follows:

- A point (x_0, y_0) is said to be an interior point ($(x_0, y_0) \in \Omega$) if there exists a disk of equation $(x - x_0)^2 + (y - y_0)^2 = R^2$ that is fully contained in Ω.

- A point (x_0, y_0) is said to be a boundary point ($(x_0, y_0) \in \delta\Omega$) if every disk of equation $(x - x_0)^2 + (y - y_0)^2 = R^2$ is partially contained in Ω, meaning that some points belong to the domain, whereas other points do not.

- A point (x_0, y_0) is said to be an exterior point ($(x_0, y_0) \notin \Omega$) if there exists a disk of equation $(x - x_0)^2 + (y - y_0)^2 = R^2$ that is not contained in Ω.

In addition, it is possible to state that a given region is open if it contains only its interior points, or closed if it contains its interior and boundary points. Furthermore, a region is said to be bounded if and only if it lies inside of a disk, otherwise the region is said to be unbounded.

1.2.2 The range of a function in two variables

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the range of f is the set $W \subseteq \mathbb{R}$ of all the values that it can take within its domain.
Example: Find the range of the function $f(x, y) = \sqrt{x^2 + y^2}$.
Since the function is irrational, it is possible to conclude that $W = \mathbb{R}^+$.

1.3 Limits of two-variable functions

1.3.1 Formal definition of a limit of a function in two variables

Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, the limit for (x, y) approaching a point (x_0, y_0) is equal to a value $l \in \mathbb{R}$ if, $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that:

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow |f(x, y) - l| < \varepsilon$$

1.3.2 The algebra of limits

Given two functions in two variables f and g whose limits for $(x, y) \to (x_0, y_0)$ are equal to l and m (both finite values) respectively, then the following rules apply:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) \pm g(x, y) = l \pm m$$

$$\lim_{(x,y)\to(x_0,y_0)} k \cdot f(x, y) = k \cdot l \ (k \in \mathbb{R})$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) \cdot g(x, y) = l \cdot m$$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y)}{g(x, y)} = \frac{l}{m} \text{ (assume, for simplicity, } m \neq 0\text{)}$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l^m \text{ (assume, for simplicity, } m \in \mathbb{R}^+\text{)}$$

1.3.3 Determining whether a limit exists or not

In order to prove that a limit exists, the aim is to prove that the value it takes is the same for any path passing through the point (x_0, y_0).
However, since there are infinite paths passing trough a point, this test is actually used to find a counterexample to show that a limit does **not** exist as it is sufficient to find two different paths $y = p_1(x)$ and $y = p_2(x)$, both passing through (x_0, y_0) such that the limit yields different results when considered with respect to these two paths.
Example: Show that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{x^3 + xy^2}$$

Let $y = mx$ and compute the limit in terms of x:

$$l_1 = \lim_{x\to 0} \frac{x^4 + (mx)^4}{x^3 + x(mx)^2} = \lim_{x\to 0} \frac{x^4(1 + m^4)}{x^3(1 + m^2)} = \lim_{x\to 0} \frac{x(1 + m^4)}{1 + m^2} = 0 \;\forall\; m \in \mathbb{R}$$

Now, pick $y = \sqrt{x}$ and compute again the limit in terms of x:

$$l_2 = \lim_{x\to 0} \frac{x^4 + (\sqrt{x^4})}{x^3 + x(\sqrt{x})^2} = \lim_{x\to 0} \frac{x^2(x^2 + 1)}{x^2(x + 1)} = \lim_{x\to 0} \frac{x^2 + 1}{x + 1} = 1$$

Since it was possible to find two paths such that $l_1 \neq l_2$, the limit does not exist.

1.3.4 Computing limits using polar coordinates

In order to show that a limit exists, it is simpler to compute it using polar coordinates:

$$\begin{cases} x - x_0 = \rho\cos\theta \\ y - y_0 = \rho\sin\theta \end{cases} \quad \text{for } \rho > 0, \theta \in [0, 2\pi] \Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{\rho\to 0} f(\rho,\theta)$$

In this case, the limit exists if and only if its result does not depend on the value of θ.
Example: Use polar coordinates to evaluate the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + yx^2}{x^2 + y^2}$$

Switch to polar coordinates and rewrite the limit:

$$\lim_{\rho\to 0} \frac{\rho^3\cos^3\theta + \rho\sin\theta\rho^2\cos^2\theta}{\rho^2\cos^2\theta + \rho^2\sin^2\theta} = \lim_{\rho\to 0} \rho(\cos^3\theta + \sin\theta\cos^2\theta) = 0$$

Observe that, since the result of the limit is the same $\forall\; \theta \in [0, 2\pi]$, it is possible to conclude that the limit exists and is equal to 0.

1.4 Continuity in $\mathbb{R}^2$

A function $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be continuous in (x_0, y_0) if it is defined in (x_0, y_0), meaning that $(x_0, y_0) \in \Omega$, and $\exists \lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.
In particular, a function is said to be continuous if it is continuous in all points of its domain.
Example: Show that the following function is continuous in $\mathbb{R}^2$:

$$f(x, y) = \begin{cases} \frac{1-\cos(xy)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Observe that it is possible to state that the function is certainly continuous in $\mathbb{R}^2 \setminus \{(0, 0)\}$, so the aim is to study the function's behaviour as (x, y) approaches $(0, 0)$:

$$\lim_{(x,y)\to(0,0)} \frac{1 - \cos(xy)}{x^2 + y^2}$$

Recall that $\lim\limits_{t\to 0} \frac{1-\cos t}{t^2} = \frac{1}{2}$, meaning that, by asymptotic comparison, it is possible to let $1 - \cos t \approx \frac{t^2}{2}$, meaning that:

$$\lim_{(x,y)\to(0,0)} \frac{1 - \cos(xy)}{x^2 + y^2} \approx \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2(x^2 + y^2)} = \frac{0}{0}$$

Switch to polar coordinates to solve the limit:

$$\lim_{\rho\to 0} \frac{\rho^2 \cos^2\theta \rho^2 \sin^2\theta}{2(\rho^2 \cos^2\theta + \rho^2 \sin^2\theta)} = \lim_{\rho\to 0} \frac{\rho^2 \cos^2\theta \sin^2\theta}{2} = 0$$

Therefore, since $\exists \lim\limits_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$, the function is continuous in $(0, 0)$ as well, thus being continuous over all $\mathbb{R}^2$.

Chapter 2

Partial derivatives

2.1 Defining partial derivatives by means of the incremental ratio

A function $f : \mathbb{R}^2 \to \mathbb{R}$ can actually be derived with respect to either of its variables, meaning that it is possible to define its partial derivatives by means of the following incremental ratios:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$
$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Example: Using the incremental ratio, find the partial derivatives of the function $f(x, y) = x^2 y$.

- With respect to x:

$$f_x(x, y) = \lim_{h \to 0} \frac{(x+h)^2 y - x^2 y}{h} = \lim_{h \to 0} \frac{h(2xy + hy)}{h} = 2xy$$

- With respect to y:

$$f_y(x, y) = \lim_{h \to 0} \frac{x^2(y+h) - x^2 y}{h} = \lim_{h \to 0} \frac{x^2 h}{h} = x^2$$

2.1.1 Generalizing partial derivatives for functions in n variables

Most particularly, this procedure can be generalized for functions of n variables as well: in fact, given a function $f : \mathbb{R}^n \to \mathbb{R}$, its partial derivative with respect to the variable x_i can be computed as:

$$f_{x_i}(x) = \lim_{h \to 0} \frac{f(x + e_i h) - f(x)}{h}, \text{ where } e_i \text{ is the } i^{th} \text{ standard unit of } \mathbb{R}^n.$$

2.2 Derivability and the gradient of a function

Generally speaking, a function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be derivable at a point (x_0, y_0) if it is possible to compute all its partial derivatives and said derivatives are all continuous in (x_0, y_0).
When this happens, it is possible to define the gradient of the function as a vector whose entries are the function's partial derivatives, meaning that:

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y))$$

Example: Find the gradient vector of the function $f(x, y) = x^2 y$.
Knowing that $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2$, it is possible to conclude that:

$$\nabla f(x, y) = (2xy, x^2)$$

2.3 Tangent planes

While the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ was used to represent the slope of its tangent line, partial derivatives in $\mathbb{R}^2$ are now used to represent tangent planes.
Generally speaking, a tangent plane has standard equation $ax + by + cz + d = 0$, with the slopes of the plane being given by the function's partial derivatives, meaning that, if a function $f : \mathbb{R}^2 \to \mathbb{R}$ is derivable at a point (x_0, y_0), it is possible to write the equation of a tangent plane as:

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

Example: Find the tangent plane to the function $f(x, y) = x^2 + y^2$ in the point $P(1, 1)$.
Since the function's partial derivatives are $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, it is possible to find the equation of the tangent plane as:

$$z = 2(x - 1) + 2(y - 1) + 2 = 2x + 2y - 2$$

2.4 Differentiability in $\mathbb{R}^2$

2.4.1 Defining differentiability

While derivability and differentiability were considered to be equivalent concepts for functions of a single variable, this is not the case in multivariable calculus.
In fact, a function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be differentiable at a point (x_0, y_0) if and only if it can be approximated with its tangent plane in a neighbourhood of that point, resulting in:

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - f_x(x_0, y_0)h - f_y(x_0, y_0)k}{\sqrt{h^2 + k^2}} = 0$$

Example: Check if the function $f(x,y) = x^2 + 3xy + y - 2$ is differentiable at $(0,0)$.
Start by applying the differentiability criterion:

$$\lim_{(h,k)\to(0,0)} \frac{h^2 + 3hk + k - 2 + 2 - k}{\sqrt{h^2 + k^2}} = \frac{0}{0}$$

Switch to polar coordinates to solve the limit:

$$\lim_{\rho\to 0} \frac{\rho^2\cos^2\theta + 3\rho^2\cos\theta\sin\theta}{\sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta}} = \lim_{\rho\to 0} \rho(\cos^2\theta + 3\cos\theta\sin\theta) = 0 \ \forall\ \theta \in [0, 2\pi]$$

Since the differentiability criterion is satisfied, the function is indeed differentiable at $(0,0)$.

2.4.2 Linking continuity, derivability and differentiability in $\mathbb{R}^2$

If a function $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point (x_0, y_0), then it is possible to state that the function will also be continuous and derivable at (x_0, y_0) and that there exists a tangent plane to the function with equation $z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$.
However, unlike in single-variable calculus, the converse is also true because, if a function $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ is continuous and derivable at a point (x_0, y_0), with both its partial derivatives also being continuous at (x_0, y_0), then it will be differentiable in (x_0, y_0) as well.
Example: An alternative way to show that $f(x,y) = x^2 + 3xy + y - 2$ is differentiable at $(0,0)$ is by observing that the function is continuous at $(0,0)$ and that, since $f_x(x,y) = 2x + 3y$ and $f(x,y) = 3x + 1$, $\nabla f(0,0) = (0,1)$, which means that the function is derivable in $(0,0)$ as well, allowing to conclude that it must indeed be differentiable in the point.

2.5 Defining directional derivatives: the rate of change

Keep in mind that, for a function $f : \mathbb{R}^2 \to \mathbb{R}$, its partial derivatives represent an incremental ratio with respect to a standard unit's direction.
For this reason, directional derivatives act as a generalization of partial derivatives with respect to a generic direction represented by a unit vector $\vec{u} = (u_1, u_2)$, meaning that they represent the function's "rate of change" with respect to the given direction, which, at a point (x_0, y_0), is defined as the following:

$$\frac{\partial f}{\partial u}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + u_1 h, y_0 + u_2 h) - f(x_0, y_0)}{h}$$

N.B.: Observe that, sometimes, the direction vector can also be expressed as a generic vector $\vec{v}$: in this case, if $\vec{v}$ is not a unit vector, it must first be normalized

into one by setting $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$.
Example: Compute the directional derivative of the function $f(x, y) = x^2 + y^2$ with respect to the vector $\vec{v} = (1, 1)$ in the point $P(1, 2)$.
Start by observing that $\|\vec{v}\| = \sqrt{2} \neq 1$, so $\vec{v}$ must first be normalised into a unit vector $\vec{u} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

$$\frac{\partial f}{\partial u}(1, 2) = \lim_{h \to 0} = \frac{(1 + \frac{\sqrt{2}}{2}h)^2 + (2 + \frac{\sqrt{2}}{2}h)^2 - 5}{h} = \lim_{h \to 0} \frac{h(3\sqrt{2} + h)}{h} = 3\sqrt{2}$$

2.5.1 Computing directional derivatives via gradient

If a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point (x_0, y_0), then it will also be derivable with respect to any direction, allowing to calculate its directional derivative by means of the dot product between the gradient vector and the direction vector:

$$\frac{\partial f}{\partial u}(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u} = f_x(x_0, y_0) \cdot u_1 + f_y(x_0, y_0) \cdot u_2$$

Example: Compute the directional derivative of the function $f(x, y) = x^2 + y^2$ with respect to the unit vector $\vec{u} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ in the point $P(1, 2)$.
Observe that the function is continuous, derivable and differentiable in $\mathbb{R}^2$, so, knowing that $\nabla f(x, y) = (2x, 2y)$, it is possible to find the directional derivative in $(1, 2)$ by simply evaluating:

$$\frac{\partial f}{\partial u}(1, 2) = (2 \cdot \frac{\sqrt{2}}{2}) + (4 \cdot \frac{\sqrt{2}}{2}) = 3\sqrt{2}$$

2.6 Higher-order partial derivatives

Similarly to single-variable calculus, it is possible to evaluate the second-order derivatives of a function in two variables as well.
However, since it is possible to derive a function in two variables with respect to either x or y, a function $f : \mathbb{R}^2 \to \mathbb{R}$ actually allows to define the following four second-order partial derivatives:

$$f_{xx} = \lim_{h \to 0} \frac{f_x(x + h, y) - f_x(x, y)}{h}$$
$$f_{xy} = \lim_{h \to 0} \frac{f_x(x, y + h) - f_x(x, y)}{h}$$
$$f_{yx} = \lim_{h \to 0} \frac{f_y(x + h, y) - f_y(x, y)}{h}$$
$$f_{yy} = \lim_{h \to 0} \frac{f_y(x, y + h) - f_y(x, y)}{h}$$

2.6.1 Schwarz's theorem

Given a function $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, if the function's partial derivatives f_x and f_y are continuous at a point $(x_0, y_0) \in \Omega$, then its mixed second-order derivatives are symmetrical, meaning that $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.
Example: Compute the second-order derivatives of $f(x, y) = x \cos y + ye^x$.
Start by computing the function's first-order partial derivatives:

$$f_x(x, y) = \cos y + ye^x$$
$$f_y(x, y) = -x \sin y + e^x$$

At this point, it is possible to compute the second-order partial derivatives in the following way:

$$f_{xx}(x, y) = ye^x$$
$$f_{xy}(x, y) = -\sin y + e^x$$
$$f_{yx}(x, y) = -\sin y + e^x$$
$$f_{yy}(x, y) = -x \cos y$$

Most particularly, observe that, since both partial derivatives are continuous in $\mathbb{R}^2$, then it is possible to state that the mixed second-order derivatives will be symmetrical in $\mathbb{R}^2$, allowing to conclude that $f_{xy}(x, y) = f_{yx}(x, y)\ \forall\ (x, y) \in \mathbb{R}^2$.

Chapter 3

Extreme values

3.1 Stationary points in $\mathbb{R}^2$

Assuming a function $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ is derivable, any point $(x_0, y_0) \in \Omega$ such that $\nabla f(x_0, y_0) = 0$ is said to be a stationary point, and it can be classified as follows:

- (x_0, y_0) is an absolute maximum if $f(x_0, y_0) \geq f(x, y) \ \forall \ (x, y) \in \Omega$.
- Given a disk of equation $\beta_R(x_0, y_0) = (x - x_0)^2 + (y - y_0)^2 = R^2$, (x_0, y_0) is a relative maximum if $f(x_0, y_0) \geq f(x, y) \ \forall (x, y) \in \beta_R$.
- (x_0, y_0) is an absolute minimum if $f(x_0, y_0) \leq f(x, y) \ \forall \ (x, y) \in \Omega$.
- Given a disk of equation $\beta_R(x_0, y_0) = (x - x_0)^2 + (y - y_0)^2 = R^2$, (x_0, y_0) is a relative minimum if $f(x_0, y_0) \leq f(x, y) \ \forall \ (x, y) \in \beta_R$.
- (x_0, y_0) is said to be a saddle point if, for every disk β_R, some points satisfy $f(x, y) \leq f(x_0, y_0)$, whereas other points satisfy $f(x, y) \geq f(x_0, y_0)$.

In particular, the stationary points of a function can be found by computing its partial derivatives f_x and f_y and letting $\nabla f(x, y) = 0$, which is equivalent to solving the following system:

$$\begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases}$$

Example: Find the stationary points of the function $f(x, y) = 3x^2 + y^2 - x^3 y$. Start by finding the function's stationary points by solving $\nabla f(x, y) = 0$:

$$\begin{cases} 6x - 3x^2 y = 0 \\ 2y - x^3 = 0 \end{cases}$$

Factorise $6x - 3x^2 y = 3x(2 - xy) = 0$ and consider each solution separately:

- First factor: $3x = 0$

$$\begin{cases} 3x = 0 \\ 2y - x^3 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

- Second factor: $2 - xy = 0$

$$\begin{cases} 2 - xy = 0 \\ 2y - x^3 = 0 \end{cases}$$

Substitute $2y - x^3 = 0 \Leftrightarrow y = \frac{x^3}{2}$ and solve the system:

$$\begin{cases} 2 - xy = 0 \\ y = \frac{x^3}{2} \end{cases} \Rightarrow \begin{cases} 2 - \frac{x^4}{2} = 0 \\ y = \frac{x^3}{2} \end{cases} \Rightarrow \begin{cases} x = -\sqrt{2} \\ y = -\sqrt{2} \end{cases} \vee \begin{cases} x = \sqrt{2} \\ y = \sqrt{2} \end{cases}$$

Therefore, the function's stationary points are $P_1(0,0)$, $P_2(-\sqrt{2},-\sqrt{2})$ and $P_3(\sqrt{2},\sqrt{2})$.

3.2 Defining the nature of stationary points through the Hessian test

Generally speaking, the nature of a stationary point (x_0, y_0) can be determined by computing the Hessian matrix, which, for a function $f : \mathbb{R}^2 \to \mathbb{R}$, will be of the type:

$$Hf(x_0, y_0) = \begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}$$

In fact, it is possible to state that:

- If $det(Hf(x_0, y_0)) > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a minimum point.
- If $det(Hf(x_0, y_0)) > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a maximum point.
- If $det(Hf(x_0, y_0)) < 0$, then (x_0, y_0) is a saddle point.
- If $det(Hf(x_0, y_0)) = 0$, then the test is inconclusive for the point (x_0, y_0).

Example: Study the stationary points of the function $f(x,y) = 3x^2 + y^2 - x^3y$. Knowing that the function's stationary points are $P_1(0,0)$, $P_2(-\sqrt{2},-\sqrt{2})$ and $P_3(\sqrt{2},\sqrt{2})$, use the second-order derivatives to compute the generic Hessian matrix and study each point separately:

$$Hf(x,y) = \begin{bmatrix} 6 - 6xy & -3x^2 \\ -3x^2 & 2 \end{bmatrix}$$

- For $P_1(0,0)$, the Hessian matrix becomes:

$$Hf(0,0) = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow det(Hf(0,0)) = 12 > 0$$

  Since $f_{xx}(0,0) = 6 > 0$, the point $P_1(0,0)$ is a minimum point.

- For $P_2(-\sqrt{2}, -\sqrt{2})$, the Hessian matrix becomes:

$$Hf(-\sqrt{2}, -\sqrt{2}) = \begin{bmatrix} -6 & -6 \\ -6 & 2 \end{bmatrix} \Rightarrow det(Hf(-\sqrt{2}, -\sqrt{2})) = -48 < 0$$

  The point $P_2(-\sqrt{2}, -\sqrt{2})$ is a saddle point.

- For $P_3(\sqrt{2}, \sqrt{2})$, the Hessian matrix becomes:

$$Hf(0,0) = \begin{bmatrix} -6 & -6 \\ -6 & 2 \end{bmatrix} \Rightarrow det(Hf(\sqrt{2}, \sqrt{2})) = -48 < 0$$

  The point $P_3(\sqrt{2}, \sqrt{2})$ is a saddle point.

3.2.1 Generalizing the Hessian test in $\mathbb{R}^n$

Assuming $x_0 \in \mathbb{R}^n$ is a stationary point for a function in n variables $f : \mathbb{R}^n \to \mathbb{R}$, it is possible to study its nature by generalizing the Hessian test and studying the eigenvalues of the Hessian matrix.
In fact, if $x_0 \in \mathbb{R}^n$ is a stationary point for a function $f : \mathbb{R}^n \to \mathbb{R}$, it is possible to state that:

- If $Hf(x_0)$ is positive-definite, meaning that all its eigenvalues are strictly positive, then x_0 is a minimum point.
- If $Hf(x_0)$ is negative-definite, meaning that all its eigenvalues are strictly negative, then x_0 is a maximum point.
- If $Hf(x_0)$ is indefinite, meaning that some eigenvalues are strictly positive while others are strictly negative, then x_0 is a saddle point.
- If $Hf(x_0)$ is semi-definite, meaning that at least one of its eigenvalues is equal to zero, then the test is inconclusive for x_0.

Example: Study the stationary points of $f(x,y,z) = x^2+y^2+z^2-3x-4y-5z$.
Start by finding the function's stationary points by solving $\nabla f(x,y,z) = 0$:

$$\begin{cases} 2x-3=0 \\ 2y-4=0 \\ 2z-5=0 \end{cases} \Rightarrow \begin{cases} x=\frac{3}{2} \\ y=2 \\ z=\frac{5}{2} \end{cases}$$

The point $P(\frac{3}{2}, 2, \frac{5}{2})$ is therefore the function's sole stationary point.
At this point, use the second-order derivatives and compute the generic Hessian

matrix:

$$Hf(x, y, z) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Therefore, consider the Hessian for the stationary point:

$$Hf(\frac{3}{2}, 2, \frac{5}{2}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Since the eigenvalues are $\lambda_{1,2,3} = 2$ (one eigenvalue of multiplicity 3), the matrix $Hf(x_0)$ is positive-definite, meaning that $P(\frac{3}{2}, 2, \frac{5}{2})$ is a minimum point.

Chapter 4

Curves

4.1 Defining curves

A curve is an application $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^n$ (usually, $n = 2$ or 3) whose behaviour is expressed by using the following parametric equations:

$$\gamma(t) = \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad , t \in I$$

Example: A circumference of radius r can be expressed using the following equation:

$$\gamma(t) = \begin{cases} r \cos t \\ r \sin t \end{cases} \quad , t \in [0, 2\pi]$$

In addition, it is also possible to derive curves from Cartesian equations.
Example: Given the Cartesian equation $y = 2x+1$, it is possible to parametrize $x(t) = t$ and write y in terms of x in order to derive the following parametric equation:

$$\gamma(t) = \begin{cases} t \\ 2t + 1 \end{cases}$$

4.2 Closed curves

A curve $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ is said to be closed in the interval $[a, b]$ if $\gamma(a) = \gamma(b)$.
Example: Show that the unit circle $\gamma(t) = (\cos t, \sin t)$ is closed in the interval $[0, 2\pi]$.
Let $\gamma(0) = (1, 0)$ and $\gamma(2\pi) = (1, 0)$: since $\gamma(0) = \gamma(2\pi)$, it is possible to conclude that the curve is indeed closed in the interval $[0, 2\pi]$.

4.3 Simple curves

A curve $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ is said to be simple in the interval (a, b) if it is one-to-one in the interval, meaning that, given two points $t_1, t_2 \in (a, b)$, it must hold that $\gamma(t_1) = \gamma(t_2) \Leftrightarrow t_1 = t_2$.
Observe that, for a curve to be simple, it is sufficient to check that at least one of its components is an injective function in the interval.
Example: Show that the spiral curve $\gamma(t) = (t \cos t, t \sin t)$ is simple in the interval $[0, 4\pi]$.
Pick two values $t_1, t_2 \in (0, 4\pi)$ and consider the following system:

$$\begin{cases} t_1 \cos t_1 = t_2 \cos t_2 \\ t_1 \sin t_1 = t_2 \sin t_2 \end{cases}$$

Actually, notice that it is possible to recover that $\gamma(t_1) = \gamma(t_2) \Leftrightarrow t_1 = t_2$ by construction of the spiral, meaning that the curve is indeed simple in $(0, 4\pi)$.

4.4 Regular curves

A curve $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ is said to be regular in the interval (a, b) if the tangent vector $\gamma'(t) = (x'(t), y'(t), z'(t))$ is component-wise continuous in the interval, and $\gamma'(t) \neq 0 \ \forall \ t \in (a, b)$.
Example: Show that $\gamma(t) = (t^3 - t, t^2 - 1)$ is regular in the interval $[-2, 2]$.
Let $\gamma'(t) = (3t^2 - 1, 2t)$ and consider the system $\gamma'(t) = 0$:

$$\begin{cases} 3t^2 - 1 = 0 \\ 2t = 0 \end{cases} \Rightarrow \begin{cases} t = 0 \\ -1 = 0 \end{cases}$$

Since the system is incompatible, it holds that $\gamma'(t) \neq 0 \ \forall \ t \in (-2, 2)$, meaning that the curve is indeed regular in the interval.

4.4.1 Computing tangent unit vectors

Assuming a curve $\gamma : I \to \mathbb{R}^n$ is regular in I, it is possible to normalize the tangent vector at a given point $P \in \mathbb{R}^n$ into a tangent unit vector by writing:

$$\gamma_u'(t_0) = \frac{\gamma'(t_0)}{\|\gamma'(t_0)\|} \text{ with } t_0 \in I \text{ such that } \gamma(t_0) = P$$

4.5 The length of a curve

Generally speaking, if a curve $\gamma : [a, b] \to \mathbb{R}^n$ is regular in (a, b), then it is possible to evaluate its length in the interval as:

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \ dt$$

Example: Find the length of the curve $\gamma(t) = (r\cos t, r\sin t)$ in the interval $[0, 2\pi]$.
Start by computing $\gamma'(t) = (-r\sin t, r\cos t)$: since $\gamma'(t) \neq 0 \ \forall \ t \in (0, 2\pi)$, the curve is regular in the interval.
In particular, given $\|\gamma'(t)\| = r$, it is possible to find the length of the curve in the interval by computing the following integral:

$$L(\gamma) = \int_0^{2\pi} r \ dt = r[t]_0^{2\pi} = r(2\pi - 0) = 2\pi r$$

4.6 Line integrals

Given a function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ and a curve $\gamma : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$, the line integral of f with respect to γ basically represents the area of the function along the curve.
Most particularly, by applying a "change of variable", this integral can be evaluated as follows:

$$\int_\gamma f(x, y) \ dt = \int_a^b f(x(t), y(t)) \cdot \|\gamma'(t)\| \ dt$$

N.B.: Keep in mind that, sometimes, a curve may be parametrized in different ways depending on the direction of traversal: when this happens, the result of the line integral may vary according to the parametrization, although it is often the case that the results will only differ by their sign.
Example: Compute the line integral of the function $f(x, y) = \frac{x}{1+y^2}$ with respect to the curve $\gamma(t) = (\cos t, \sin t)$, with $t \in [0, \frac{\pi}{2}]$.
Let $\gamma'(t) = (-\sin t, \cos t)$, meaning that $\|\gamma'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$, so compute the following integral:

$$\int_\gamma \frac{x}{1+y^2} \ dt = \int_0^{\frac{\pi}{2}} \frac{\cos t}{1+\sin^2 t} \ dt$$

Apply a change of variable: let $u = \sin t$, for $u \in [0, 1]$, and notice that $\frac{du}{dt} = \cos t$.
Therefore, it is possible to solve the integral in terms of u:

$$\int_0^1 \frac{1}{1+u^2} \ du = [\arctan u]_0^1 = \frac{\pi}{4}$$

4.6.1 Line integrals with respect to a piece-wise curve

Assuming the curve is piece-wise, meaning that $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$, then it is possible to evaluate the line integral of the function with respect to γ by applying additivity, meaning that:

$$\int_{\gamma} f(x,y)\ dt = \int_{\gamma_1} f(x,y)\ dt + \cdots + \int_{\gamma_n} f(x,y)\ dt$$

Example: Compute the line integral of the function $f(x,y) = \sin(xy)$ with respect to the square with vertices $(0,0),(1,0),(1,1),(0,1)$.
Start by finding an appropriate equation to represent the square: by additivity, let $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where:

- $\gamma_1(t) = (t,0)$ for $t \in [0,1]$ $(\|\gamma_1'(t)\| = 1)$.
- $\gamma_2(t) = (1,t)$ for $t \in [0,1]$ $(\|\gamma_2'(t)\| = 1)$.
- $\gamma_3(t) = (t,1)$ for $t \in [1,0]$ $(\|\gamma_3'(t)\| = 1)$.
- $\gamma_4(t) = (0,t)$ for $t \in [1,0]$ $(\|\gamma_4'(t)\| = 1)$.

For this reason, the line integral can be evaluated in the following ways:

$$\int_{\gamma} \sin(xy)\ dt = \int_0^1 \sin(0t)\ dt + \int_0^1 \sin(t)\ dt + \int_1^0 \sin(t)\ dt + \int_1^0 \sin(0t)\ dt$$

Most particularly, by simplifying the equation and by applying the properties of integrals, it is possible to recover that:

$$\int_{\gamma} \sin(xy)\ dt = \int_0^1 \sin(t)\ dt - \int_0^1 \sin(t)\ dt = 0$$

4.7 Using curves to set constraints over maxima and minima

In single-variable calculus, if a function $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ satisfied Weierstrass' theorem in the interval, then it was guaranteed to have a maximum and a minimum in it, also allowing to limit the study of extreme points to $[a,b]$ by only studying the function in the stationary points belonging to the interval and then comparing these values with $f(a)$ and $f(b)$ as well.
This procedure can be generalized for multivariable functions as well: in fact, given a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, setting a constraint over maxima and minima to D consists in finding the function's stationary points by setting $\nabla f(x,y) = 0$ (provided that the solutions belong to the region) and then checking its behaviour at the boundaries of D by parametrizing it as a (piece-wise) curve and checking eventual "corners" as well.
At this point, it is enough to check the values that the function takes on at these points to determine its maxima and minima within the region D.
Example: Find the maxima and minima of $f(x,y) = x^2 + y^2 - xy + x + y$ within the region $D = \{x \leq 0, y \leq 0, x + y \geq -3\}$.

Start by finding the function's stationary points by solving $\nabla f(x,y) = 0$:

$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \Rightarrow \begin{cases} 2x - y + 1 = 0 \\ 2y - x + 1 = 0 \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = -1 \end{cases}$$

Therefore, $P_1(-1,-1) \in D$ is a first candidate to be a maximum/minimum point.
Now, study the boundary region by parametrizing it as a curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where:

- $\gamma_1(t) = (0,t)$, for $t \in [0,-3]$ (in this case, the direction of traversal is inverted).
  Restrict the function to the curve by letting $f_{\gamma_1}(t) = t^2 + t$ and study its stationary points by considering $f'_{\gamma_1}(t) = 2t + 1 = 0$, allowing to find a stationary point $P_2(0, -\frac{1}{2}) \in D$.
  Furthermore, by studying the extremes of the interval, it is possible to recover the corners $P_3(0,0) \in D$ and $P_4(0,-3) \in D$.

- $\gamma_2(t) = (t, -t-3)$, for $t \in [0,-3]$ (in this case, the direction of traversal is inverted).
  Restrict the function to the curve by letting $f_{\gamma_2}(t) = 3t^2 + 9t + 6$ and study its stationary points by considering $f'_{\gamma_2}(t) = 6t + 9 = 0$, allowing to find a stationary point $P_5(-\frac{3}{2}, -\frac{3}{2}) \in D$.
  Furthermore, by studying the extremes of the intervals, it is possible to recover the corners $P_4(0,-3)$, which is in common with the piece γ_1, and $P_6(-3,0) \in D$.

- $\gamma_3(t) = (t,0)$, for $t \in [-3,0]$.
  Restrict the function to the curve by letting $f_{\gamma_3}(t) = t^2 + t$ and find its stationary points by considering $f_{\gamma_3}(t) = 2t + 1 = 0$, allowing to find a stationary point $P_7(-\frac{1}{2}, 0) \in D$.
  Furthermore, by studying the extremes of the interval, it is possible to recover again the corners $P_6(-3,0)$, which is in common with the piece γ_2, and $P_3(0,0)$, which is in common with the piece γ_1.

Lastly, by comparing the values that the function takes at these points, it is possible to conclude that $P_4(0,-3)$ and $P_6(-3,0)$ are points of maximum, whereas $P_5(-\frac{3}{2}, -\frac{3}{2})$ is a minimum point.

Chapter 5

Multiple integrals

5.1 Double integrals

5.1.1 Generic construction of a double integral

Remember that, in single-variable calculus, integrating a function $f : [a, b] \to \mathbb{R}$ consisted in finding its area over the interval $[a, b]$.
Similarly, given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, integration over the region D consists in finding the volume of the function within the surface defined by the region.
For this reason, it is possible to partition D as a sequence of functions $p_j(x)$ in order to rewrite it as:

$$D = \{((x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ \alpha(x) \leq y \leq \beta(x))\},\ \alpha(x) = p_0(x) < \cdots < p_n(x) = \beta(x)$$

Most particularly, notice that each of these partitions defines a volume such that, if v represents a lower bound for the total volume and V represents an upper bound for it, it is possible to carry out the following approximation:

$$v \leq \sum_{j=0}^{n} p_j(x) \leq V$$

However, as $n \to \infty$, the approximation gets better and better, allowing to state that:

$$\sum_{j=0}^{n} p_j(x) = \int\int_D f(x, y)\ dxdy$$

Generally speaking, if a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, it is also integrable in D, meaning that:

$$\left| \int\int_D f(x, y)\ dxdy \right| < +\infty$$

5.1.2 Properties of double integrals

Generally speaking, integrable functions have the following properties:

- By linearity, it is possible to state that:

$$\int\int_D (\alpha f(x,y)+\beta g(x,y))\ dxdy = \alpha \int\int_D f(x,y)\ dxdy+\beta \int\int_D g(x,y)\ dxdy$$

- Assuming $D = D_1 \cup \cdots \cup D_n$, it is possible to apply additivity to state that:

$$\int\int_D f(x,y)\ dxdy = \sum_{i=1}^{n} \int\int_{D_i} f(x,y)\ dxdy$$

- If $f(x,y) = 1$, then the integral of the function over D represents the area of the region.

5.1.3 Double integrals in a rectangular region

Given a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$, if R is a rectangular region, meaning that it is defined as $R = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$, then it is possible to integrate f over R in the following two ways:

$$\int\int_R f(x,y)\ dxdy = \int_c^d \int_a^b f(x,y)\ dxdy = \int_a^b \int_c^d f(x,y)\ dydx$$

Example: Evaluate the following integral using both versions of the double integral.

$$\int\int_R \frac{1}{(x+y)^2}\ dxdy \text{ , where } R = [1,2] \times [3,4]$$

- Start by integrating with respect to x:

$$\int_3^4 \left(\int_1^2 \frac{1}{(x+y)^2}\ dx \right) dy = \int_3^4 \left[\frac{-1}{x+y} \right]_1^2 dy = \int_3^4 \frac{1}{1+y} - \frac{1}{2+y}\ dy$$

$$\int_3^4 \frac{1}{1+y} - \frac{1}{2+y}\ dy = [\ln|1+y| - \ln|2+y|]_3^4 = \ln(\frac{25}{24})$$

- Start by integrating with respect to y:

$$\int_1^2 \left(\int_3^4 \frac{1}{(x+y)^2}\ dy \right) dx = \int_1^2 \left[\frac{-1}{x+y} \right]_3^4 dx = \int_1^2 \frac{1}{x+3} - \frac{1}{x+4}\ dx$$

$$\int_1^2 \frac{1}{x+3} - \frac{1}{x+4}\ dx = [\ln|x+3| - \ln|x+4|]_1^2 = \ln(\frac{25}{24})$$

Remember that, regardless of the integration order, the result of the integral should always be the same.

5.1.3.1 Integral decoupling

Sometimes, given a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$, with $R = [a, b] \times [c, d]$, if the function can be written as $f(x, y) = \alpha(x)\beta(y)$, it is possible to integrate f over R by decoupling, meaning that:

$$\int\int_R f(x,y)\ dxdy = \int_a^b \alpha(x)\ dx \cdot \int_c^d \beta(y)\ dy$$

5.1.4 Double integrals in a non-rectangular region

It is often the case, however, that, given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, D is not a rectangular region.
For this reason, it is very useful to rewrite D either as a normal domain with respect to either x or y.
In this case, if a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, it is possible to state that:

- If $D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ \alpha(x) \leq y \leq \beta(x)\}$ is a normal domain with respect to x, then:

  $$\int\int_D f(x,y)\ dxdy = \int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\ dydx$$

  Example: Evaluate the integral of the function $f(x, y) = xy$ over the region $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 0 \leq y \leq x\}$.
  Since D is written as a normal domain with respect to x, compute the integral as follows:

  $$\int\int_D xy\ dxdy = \int_0^1 \left(\int_0^x xy\ dy\right) dx = \int_0^1 x\left[\frac{y^2}{2}\right]_0^x dx = \int_0^1 \frac{x^3}{8}\ dx = \left[\frac{x^4}{2}\right]_0^1 = \frac{1}{8}$$

- If $D = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d,\ \gamma(y) \leq x \leq \delta(y)\}$ is a normal domain with respect to y, then:

  $$\int\int_D f(x,y)\ dxdy = \int_c^d \int_{\gamma(y)}^{\delta(y)} f(x,y)\ dxdy$$

  Example: Evaluate the integral of the function $f(x, y) = xy$ over the region $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1,\ y \leq x \leq 1\}$.
  Since D is written as a normal domain with respect to y, compute the integral as follows:

  $$\int\int_D xy\ dxdy = \int_0^1 \left(\int_y^1 xy\ dx\right) dy = \int_0^1 y\left[\frac{x^2}{2}\right]_y^1 dy = \int_0^1 \frac{y}{2} - \frac{y^3}{2}\ dy = \left[\frac{y^2}{4} - \frac{y^4}{8}\right]_0^1 = \frac{1}{8}$$

5.1.4.1 Switching between normal domains

Generally speaking, given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, if D is defined as a normal domain with respect to x, it is possible to rewrite it as a normal domain with respect to y in the following way:

$$D = \{(x, y) \in \mathbb{R}^2 \mid \alpha(a) \le y \le \alpha(b),\ \beta^{-1}(y) \le x \le \alpha^{-1}(y)\}$$

Without loss of generality, it is possible to apply the same logic when D is defined as a normal domain with respect to y, meaning that it is possible to rewrite it as a normal domain with respect to x in the following way:

$$D = \{(x, y) \in \mathbb{R}^2 \mid \gamma(c) \le x \le \gamma(d),\ \delta^{-1}(x) \le y \le \gamma^{-1}(x)\}$$

Example: Rewrite the region $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 4,\ \frac{x^2}{2} \le y \le 2x\}$ as a normal domain with respect to y.
Pick $\alpha(x) = \frac{x^2}{2}$ and $\beta(x) = 2x$ and recover $\alpha(0) = 0$ and $\alpha(4) = 8$.
Both functions are invertible in $[0, 4]$, so define $\alpha^{-1}(y) = \sqrt{2y}$ and $\beta^{-1}(y) = \frac{y}{2}$, and rewrite the region as $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 8,\ \frac{y}{2} \le y \le \sqrt{2y}\}$.

5.2 Triple integrals

Remember that, from a geometrical point of view, integrating the function $f(x, y) = 1$ over a region $D \subset \mathbb{R}^2$ consisted in finding the area of D.
Similarly, integrating the function $f(x, y, z) = 1$ over a region $\Omega \subset \mathbb{R}^3$ allows to find the volume of Ω.
Therefore, given a region $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid a \le z \le b,\ (x, y) \in D \subset \mathbb{R}^2\}$, if a function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in Ω, it will also be integrable in the region and it is possible to evaluate the integral in the following way:

$$\int\int\int_\Omega f(x, y, z)\ dxdydz = \int_a^b \left(\int\int_D f(x, y, z)\ dxdy \right) dz$$

On the other hand, if $\alpha(x, y) \le z \le \beta(x, y)$, it is simpler to solve the integral by exploiting double integrals in the following way:

$$\int\int\int_\Omega f(x, y, z)\ dxdydz = \int\int_D \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x, y, z)\ dz \right) dxdy$$

N.B.: Without loss of generality, these procedures can applied for any ordering of the three variables.
Example: Using triple integrals, find the volume of the following region Ω:

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le z \le 4,\ x^2 + y^2 \le z^2\}$$

Notice that it is possible to find a region $D_z \subset \mathbb{R}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le z^2\}$, meaning that, assuming $f(x, y, z) = 1$, it is possible to integrate f over Ω in the following way:

$$\int\int\int_{\Omega} dxdydz = \int_0^4 \left(\int\int_{D_z} dxdy \right) dz$$

By definition of D_z, it is possible to solve the double integral by switching polar coordinates:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad \text{for } \rho \in [0, z] \text{ and } \theta \in [0, 2\pi]\text{, and let } |det(J)| = \rho$$

Therefore, it is possible to obtain:

$$\int\int_{D_z} dxdy = \int_0^z \int_0^{2\pi} \rho \; d\theta d\rho = \int_0^z \rho \; d\rho \int_0^{2\pi} \; d\rho = \left[\frac{\rho^2}{2}\right]_0^z [\theta]_0^{2\pi} = \pi z^2$$

It is therefore possible to conclude that:

$$\int\int\int_{\Omega} dxdydz = \int_0^4 \pi z^2 \; dz = \pi\left[\frac{z^3}{3}\right]_0^4 = \frac{64\pi}{3}$$

5.3 Integration by change of variables

Sometimes, given a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, integrating the function over D may be a tricky task.
For this, reason, it is possible to apply a change of variable and find some functions $f_1, \ldots, f_n$ such that:

$$\begin{cases} x_1 = f_1(u_1, \ldots, u_n) \\ x_2 = f_2(u_1, \ldots, u_n) \\ \ldots \\ x_n = f_n(u_1, \ldots, u_n) \end{cases} \text{, with each variable } u_1, \ldots, u_n \text{ being in a fixed range.}$$

Most particularly, if J is the Jacobian matrix obtained by computing the partial derivatives of the various functions with respect to the new variables, it is possible to state that, if a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is continuous in D, it is possible to compute the integral in the following way:

$$\int \cdots \int_D f(x_1, \ldots, x_n) \; dx_1 \ldots dx_n = \int \cdots \int_D f(u_1, \ldots, u_n)|det(J)| \; du_1 \ldots du_n$$

5.3.1 Integration by polar coordinates

Sometimes, given a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, it might be more convenient to integrate f over D by using polar coordinates by applying the following change of variable:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad \text{for } \rho \in [0,R],\ \theta \in [0,2\pi]$$

For this reason, define the Jacobian matrix as:

$$J(\rho,\theta) = \begin{bmatrix} x_\rho & x_\theta \\ y_\rho & y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix} \Rightarrow |det(J)| = \rho$$

In this, case, if a function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous in D, then it is also integrable in the region and it is possible to evaluate the integral in the following way:

$$\int\int_D f(x,y)\ dxdy = \int\int_D f(\rho,\theta)\cdot|det(J)|\ d\rho d\theta$$

Example: Use polar coordinates to evaluate the following integral:

$$\int\int_D x^2 - y^2\ dxdy, \text{ where } D = \{x \le 0,\ y \ge 0,\ x^2 + y^2 \le 1\}$$

Notice that D describes the interior and boundary points of the unit circle that lie on the second quadrant of the $xy-$plane, so apply a change of variable:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad \text{for } \rho \in [0,1],\ \theta \in \left[\frac{\pi}{2},\pi\right]$$

For this reason, observe that D can be rewritten as a region $D' = [0,1] \times [\frac{\pi}{2},\pi]$, and, by switching to polar coordinates, it is possible to calculate the integral in the following way:

$$\int\int_D x^2 - y^2\ dxdy = \int_0^1 \int_{\frac{\pi}{2}}^{\pi} (\rho^2\cos^2\theta - \rho^2\sin^2\theta)\cdot\rho\ d\theta d\rho$$

Knowing that $\rho^2 cos^2\theta - \rho^2 sin^2\theta = \rho^2\cos 2\theta$, it is possible to solve the integral by applying decoupling and conclude that:

$$\int_0^1 \int_{\frac{\pi}{2}}^{\pi} (\rho^2\cos^2\theta - \rho^2\sin^2\theta)\cdot\rho\ d\theta d\rho = \int_0^1 \int_{\frac{\pi}{2}}^{\pi} \rho^3\cos 2\theta\ d\theta d\rho = \left[\frac{\rho^4}{4}\right]_0^1 \left[\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{\pi} = 0$$

5.3.2 Integration by cylindrical coordinates

Sometimes, a simpler way to compute the integral of a function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ over Ω is by switching to cylindrical coordinates, meaning that:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \quad \text{for } \rho \in [0,R],\ \theta \in [0,2\pi],\ z \in [a,b]$$

For this reason, define the Jacobian matrix as:

$$J(\rho, \theta, z) = \begin{bmatrix} x_\rho & x_\theta & x_z \\ y_\rho & y_\theta & y_z \\ z_\rho & z_\theta & z_z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |det(J)| = \rho$$

In this case, if a function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in Ω, then it is also integrable in the region and it is possible to evaluate the integral in the following way:

$$\int\int\int_\Omega f(x, y, z)\ dxdydz = \int\int\int_\Omega f(\rho, \theta, z) \cdot |det(J)|\ d\rho d\theta dz$$

Example: Use cylindrical coordinates to compute the following integral:

$$\int\int\int_\Omega x\ dxdydz \text{ over } \Omega = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq z \leq 2 - x^2 - y^2\}$$

Start by noticing that, by definition of the region Ω, it is possible to recover that $x^2+y^2 \leq 1$, meaning that it is possible to apply the following change of variables:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta & \text{for } \rho \in [0, 1],\ \theta \in [0, 2\pi],\ z \in [x^2 + y^2, 2 - x^2 - y^2] \\ z = z \end{cases}$$

However, since $x^2 + y^2 = \rho^2$, it is possible to rewrite $z \in [\rho^2, 2 - \rho^2]$, meaning that it is possible to compute the integral in the following way:

$$\int_0^1 \int_0^{2\pi} \int_{\rho^2}^{2-\rho^2} \rho\cos\theta \cdot \rho\ dz d\theta d\rho = \int_0^1 \int_0^{2\pi} \rho^2 \cos\theta(2 - 2\rho^2)\ d\theta d\rho$$

At this point, it is possible to solve the integral by applying decoupling, allowing to conclude that:

$$\int_0^1 2\rho^2 - 2\rho^4\ d\rho \int_0^{2\pi} \cos\theta\ d\theta = \left[\frac{2\rho^3}{3} - \frac{2\rho^5}{5}\right]_0^1 \cdot [\sin\theta]_0^{2\pi} = 0$$

5.3.3 Integration by spherical coordinates

Another alternative to compute the integral of a function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ over Ω is by switching to spherical coordinates, meaning that:

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi & \text{for } \rho \in [0, R],\ \theta \in [0, 2\pi],\ \varphi \in [0, \pi] \\ z = \rho\cos\varphi \end{cases}$$

For this reason, define the Jacobian matrix as:

$$J(\rho, \theta, \varphi) = \begin{bmatrix} x_\rho & x_\theta & x_\varphi \\ y_\rho & y_\theta & y_\varphi \\ z_\rho & z_\theta & z_\varphi \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\sin\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix} \Rightarrow |det(J)| = \rho^2\sin\varphi$$

In this case, if a function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in Ω, then it is also integrable in the region and it is possible to evaluate the integral in the following way:

$$\int\int\int_{\Omega} f(x,y,z)\ dxdydz = \int\int\int_{\Omega} f(\rho,\theta,\varphi)\cdot|det(J)|\ d\rho d\theta d\varphi$$

Example: Use spherical coordinates to compute the following integral:

$$\int\int\int_{\Omega} \frac{1}{x^2+y^2+z^2}+x+y+z\ dxdydz \text{ over } \Omega = \{1 \le x^2+y^2+z^2 \le 4,\ z \ge 0\}$$

Notice that Ω describes a dome-like region given by a hemisphere of radius 2 but without the part belonging to the hemisphere of radius 1, meaning that it is possible to apply the following change of variable:

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases} \quad \text{for } \rho \in [1,2],\ \theta \in [0,2\pi],\ \varphi \in \left[0, \frac{\pi}{2}\right]$$

It is therefore possible to rewrite the integral in the following way:

$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_1^2 \left(\frac{1}{\rho^2}+\rho\cos\theta\sin\varphi+\rho\sin\theta\sin\varphi+\rho\cos\varphi\right)|-\rho^2\sin\varphi|\ d\rho d\theta d\varphi$$

In order to simplify computations, start by integrating with respect to θ because, by periodicity of sine and cosine, some components will cancel out, whereas the other components do not depend on θ, resulting in:

$$\int_0^{2\pi} \sin\varphi+\rho^3\cos\theta\sin^2\varphi+\rho^3\sin\theta\sin^2\varphi+\rho^3\cos\varphi\sin\varphi\ d\theta = 2\pi(\sin\varphi+\rho^3\cos\varphi\sin\varphi)$$

At this point, solve the integral as a double integral by integrating first with respect to φ and then with respect to ρ, meaning that:

$$2\pi\int_1^2\int_0^{\frac{\pi}{2}} \sin\varphi+\rho^3\cos\varphi\sin\varphi\ d\varphi d\rho = 2\pi\int_1^2 \left[-\cos\varphi+\frac{\rho^3\sin^2\varphi}{2}\right]_0^{\frac{\pi}{2}} d\rho = 2\pi\int_1^2 1+\frac{\rho^3}{2}\ d\rho$$

It is therefore possible to conclude that:

$$2\pi\int_1^2 1+\frac{\rho^3}{2}\ d\rho = 2\pi\left[\rho+\frac{\rho^4}{8}\right]_1^2 = \frac{23\pi}{4}$$