Calculus 2 Exercises

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Disclaimer

This document contains the solutions to the exercise sheets provided by Professor Alla during the ACSAI Calculus 2 course that was given during the Academic Year 2024/2025.
Keep in mind, however, that these solutions are **not** official and may therefore contain mistakes, so it is suggested to double check eventual computations.
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Contents

Chapter 1

Exercise Sheet 1: Domain and limits of functions in two variables

1.1 Exercise 1

Define the domain of the following functions:

- $f(x, y) = \ln(4 - 4x^2 - y^2)$

$$\Omega : 4 - 4x^2 - y^2 > 0 \Rightarrow \Omega : 4x^2 - y^2 < 4$$

N.B.: Observe that the domain is an ellipse with semi-axes 1 and 2.

- $f(x, y) = \sqrt{x^3 - y} + \frac{1}{y - \ln x}$

$$\Omega : \begin{cases} x^3 - y \geq 0 \\ y - \ln x \neq 0 \\ x > 0 \end{cases} \Rightarrow \Omega : \begin{cases} y \leq x^3 \\ y \neq \ln x \\ x > 0 \end{cases}$$

- $f(x, y) = \ln(\frac{y - x^2}{x - y})$

$$\Omega : \begin{cases} \frac{y - x^2}{x - y} > 0 \\ x - y \neq 0 \end{cases} \Rightarrow \Omega : \begin{cases} y - x^2 > 0 \\ x - y > 0 \\ x - y \neq 0 \end{cases} \vee \begin{cases} y - x^2 < 0 \\ x - y < 0 \\ x - y \neq 0 \end{cases}$$

$$\Omega : \begin{cases} y > x^2 \\ y < x \\ y \neq x \end{cases} \vee \begin{cases} y < x^2 \\ y > x \\ y \neq x \end{cases}$$

- $f(x,y) = \sqrt{x^2 - y^2} + \ln(x^2 + y^2)$

$$\Omega : \begin{cases} x^2 - y^2 \geq 0 \\ x^2 + y^2 > 0 \end{cases} \quad \Rightarrow \Omega : \begin{cases} -|x| \leq y \leq |x| \\ (x,y) \neq (0,0) \end{cases}$$

1.2 Exercise 2

Show that the following limits do not exist and verify that, using polar coordinates, the result depends on the value of θ:

- $\lim\limits_{(x,y)\to(0,0)} \frac{2y^2}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} \frac{2y^2}{x^2+y^2} = \frac{0}{0}$$

Consider the path $y = mx$, and recompute the limit in terms of $x \to 0$:

$$\lim_{x\to 0} \frac{2(mx)^2}{x^2+(mx)^2} = \lim_{x\to 0} \frac{2m^2x^2}{x^2(1+m^2)} = \frac{2m^2}{1+m^2}$$

Notice that the limit does not exist because depends on the value of m. Indeed, if the limit is computed in polar coordinates, it is possible to show that:

$$\lim_{\rho\to 0} \frac{2\rho^2 \sin^2\theta}{\rho^2\cos^2\theta + \rho^2\sin^2\theta} = \lim_{\rho\to 0} \frac{2\rho^2\sin^2\theta}{\rho^2} = 2\sin^2\theta$$

Therefore, it is possible to conclude that this limit does not exist as it depends on the value of θ.

- $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-2xy+y^3}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-2xy+y^3}{x^2+y^2} = \frac{0}{0}$$

Consider the path $y = mx$, and recompute the limit in terms of $x \to 0$:

$$\lim_{x\to 0} \frac{x^2 - 2x(mx) + (mx)^3}{x^2+(mx)^2} = \lim_{x\to 0} \frac{x^2(1-2m+m^3x)}{x^2(1+m^2)} = \frac{1-2m}{1+m^2}$$

Notice that the limit does not exist because it depends on the value of m. Indeed, if the limit is computed in polar coordinates, it is possible to show that:

$$\lim_{\rho\to 0} \frac{\rho^2\cos^2\theta - 2\rho\cos\theta\rho\sin\theta + \rho^3\sin^3\theta}{\rho^2\cos\theta + \rho^2\sin^2\theta} = \lim_{\rho\to 0} \frac{\rho^2(\cos^2\theta - 2\cos\theta\sin\theta + \rho\sin^3\theta)}{\rho^2} = \cos^2\theta - \sin 2\theta$$

Therefore, it is possible to conclude that this limit does not exist as it depends on the value of θ.

1.3 Exercise 3

Compute the following limits:

- $\lim\limits_{(x,y)\to(0,0)} \frac{x(e^{xy}-1)}{x^2+y^2}$
  Start by applying special limits for asymptotic approximation:
  $$\lim_{t\to 0} \frac{e^t-1}{t} = 1 \Rightarrow e^t - 1 \approx t, \text{ meaning that } e^{xy}-1 \approx xy$$
  Therefore, it is possible to simplify computations:
  $$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \frac{0}{0}$$
  Solve the limit by switching to polar coordinates:
  $$\lim_{\rho\to 0} \frac{\rho^2\cos^2\theta\rho\sin\theta}{\rho^2\cos^2\theta + \rho^2\sin^2\theta} = \lim_{\rho\to 0} \frac{\rho^3\cos^2\theta\sin\theta}{\rho^2} = \lim_{\rho\to 0} \rho\cos^2\theta\sin\theta = 0 \ \forall\ \theta \in [0, 2\pi]$$

- $\lim\limits_{(x,y)\to(0,0)} \frac{x(e^{x+y}-1)}{x^2+y^2}$
  Start by applying special limits for asymptotic approximation:
  $$\lim_{t\to 0} \frac{e^t-1}{t} = 1 \Rightarrow e^t - 1 \approx t, \text{ meaning that } e^{x+y}-1 \approx x+y$$
  Therefore, it is possible to simplify computations:
  $$\lim_{(x,y)\to(0,0)} \frac{x(x+y)}{x^2+y^2} = \frac{0}{0}$$
  Solve the limit by switching to polar coordinates:
  $$\lim_{\rho\to 0} \frac{\rho\cos\theta(\rho\cos\theta+\rho\sin\theta)}{\rho^2\sin^2\theta + \rho^2\sin^2\theta} = \lim_{\rho\to 0} \frac{\rho^2\cos\theta(\cos\theta + sin\theta)}{\rho^2} = \cos^2\theta + \cos\theta\sin\theta$$
  Notice that, in this case, the limit does not exist as its result depends on the value of θ.

- $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2y)}{\sqrt{x^2+y^2}}$
  Start by applying special limits for asymptotic approximation:
  $$\lim_{t\to 0} \frac{\sin t}{t} = 1 \Rightarrow \sin t \approx t \Rightarrow \sin(x^2y) \approx x^2y$$
  Therefore, it is possible to simplify computations:
  $$\lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{x^2+y^2}} = \frac{0}{0}$$

Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0}\frac{\rho^2\cos^2\theta\rho\sin\theta}{\sqrt{\rho^2\cos^2\theta+\rho^2\sin^2\theta}}=\lim_{\rho\to 0}\frac{\rho^3\cos^2\theta\sin\theta}{\sqrt{\rho^2}}=\lim_{\rho\to 0}\rho^2\cos^2\theta\sin\theta=0\ \forall\ \theta\in[0,2\pi]$$

- $\lim\limits_{(x,y)\to(1,0)}\frac{y^5}{((x-1)^2+y^2)^2}+\frac{\ln(y+1)}{y}+\frac{\sin^2(x-1)}{x-1}$ Start by applying special limits for asymptotic approximation:

$$\lim_{t\to 0}\frac{\ln(t+1)}{t}=1\Rightarrow\ln(t+1)\approx t$$

$$\lim_{t\to 0}\frac{\sin t}{t}=1\Rightarrow\sin t\approx t\Rightarrow\sin(x-1)\approx x-1$$

Therefore, it is possible to simplify computations:

$$\lim_{(x,y)\to(0,0)}\frac{y^5}{((x-1)^2+y^2)}+\frac{y}{y}+\frac{(x-1)^2}{x-1}=\lim_{(x,y)\to(0,0)}\frac{y^5}{((x-1)^2+y^2)}+1+(x-1)=\frac{0}{0}$$

Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0}\frac{\rho^5\sin^5\theta}{(\rho^2\cos^2\theta+\rho^2\sin^2\theta)^2}+1+\rho\cos\theta=\lim_{\rho\to 0}\frac{\rho^5\sin^5\theta}{(\rho^2)^2}+1=\lim_{\rho\to 0}\rho\sin^5\theta+1=1\ \forall\ \theta\in[0,2\pi]$$

Chapter 2

Exercise Sheet 2: Continuity and partial derivatives

2.1 Exercise 1

Determine the continuity of the following functions in $\mathbb{R}^2$:

- First function:

$$f(x,y) = \begin{cases} \frac{e^{xy^2}-1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

The function is defined to be surely continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$, meaning that the goal is to check the limit of the function as $(x,y) \to (0,0)$:

By asymptotic comparison, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy^2}-1}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy^2}{\sqrt{x^2+y^2}} = \frac{0}{0}$

Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0} \frac{\rho\cos\theta\rho^2\sin^2\theta}{\sqrt{\rho^2\cos^2\theta+\rho^2\sin^2\theta}} = \lim_{\rho\to 0} \frac{\rho^3\cos\theta\sin^2\theta}{\rho} = \lim_{\rho\to 0} \rho^2\cos\theta\sin^2\theta = 0 \ \forall\ \theta \in [0, 2\pi]$$

Since $\exists \displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$, the function is continuous in $(0,0)$ as well, and therefore it is continuous all over $\mathbb{R}^2$.

- Second function:

$$f(x,y) = \begin{cases} \frac{x^2y^3}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

The function is defined to be surely continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$, meaning that the goal is to check the limit of the function as $(x, y) \to (0,0)$:

$$\lim_{(x,y=\to(0,0)} \frac{x^2y^3}{(x^2+y^2)^2} = \frac{0}{0}$$

Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0} \frac{\rho^2\cos^2\theta\rho^3\sin^3\theta}{(\rho^2\cos^2\theta+\rho^2\sin^2\theta)^2} = \lim_{\rho\to 0} \frac{\rho^5\cos^2\theta\sin^3\theta}{(\rho^2)^2} = \lim_{\rho\to 0} \rho\cos^2\theta\sin^3\theta = 0 \ \forall\ \theta \in [0, 2\pi]$$

Since $\exists \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$, the function is continuous in $(0,0)$ as well, and therefore it is continuous all over $\mathbb{R}^2$.

2.2 Exercise 2

Compute the domain and partial derivatives of the following functions:

- $f(x,y) = (x+y)(x-y)$.
  $\Omega = \mathbb{R}^2$.

  $$f_x(x,y) = x - y + x + y = 2x$$
  $$f_y(x,y) = x - y - (x-y) = -2y$$

  N.B.: In this case, it is possible to simplify computations by rewriting $f(x,y) = x^2 - y^2$.

- $f(x,y) = \frac{x^2}{x^2+y}$.
  $\Omega : x^2 + y \neq 0 \Rightarrow \Omega : y \neq -x^2$.

  $$f_x(x,y) = \frac{2x(x^2+y) - x^2 \cdot 2x}{(x^2+y)^2} = \frac{2xy}{(x^2+y)^2}$$
  $$f_y(x,y) = \frac{-x^2}{(x^2+y)} \text{ (in this case, use } \frac{d(\frac{1}{f(x)})}{dx} = -\frac{f'(x)}{f^2(x)})$$

- $f(x,y) = \sqrt{x+2y}$.
  $\Omega : x + 2y \geq 0 \Rightarrow \Omega : y \geq -\frac{x}{2}$.

  $$f_x(x,y) = \frac{1}{2\sqrt{x+2y}}$$
  $$f_y(x,y) = \frac{2}{2\sqrt{x+2y}} = \frac{1}{\sqrt{x+2y}}$$

- $f(x,y) = \ln(x^2+y^2)$.
  $\Omega : x^2+y^2 > 0 \Rightarrow \Omega : (x,y) \neq (0,0) \Rightarrow \Omega = \{\mathbb{R}^2 \setminus (0,0)\}$.

$$f_x(x,y) = \frac{2x}{x^2+y^2}$$
$$f_y(x,y) = \frac{2y}{x^2+y^2}$$

- $f(x,y) = e^{x^2y}$.
  $\Omega = \mathbb{R}^2$.

$$f_x(x,y) = 2xye^{x^2y}$$
$$f_y(x,y) = x^2e^{x^2y}$$

- $f(x,y) = \frac{e^{x^2+y^2}}{y}$.
  $\Omega : y \neq 0$.

$$f_x(x,y) = \frac{2xe^{x^2+y^2}}{y} \text{ (in this case, treat } \frac{1}{y} \text{ as a constant)}$$
$$f_y(x,y) = \frac{2ye^{x^2+y^2}y - e^{x^2+y^2}}{y^2} = \frac{e^{x^2+y^2}(2y^2-1)}{y^2}$$

- $f(x,y) = \ln(x+y)x^2 + y\sin x$.
  $\Omega : x+y > 0 \Rightarrow \Omega : y > -x$.

$$f_x(x,y) = \frac{x^2}{x+y} + 2x\ln(x+y) + y\cos x$$
$$f_y(x,y) = \frac{x^2}{x+y} + \sin x \text{ (in this case, treat } x^2 \text{ as a constant)}$$

- $f(x,y) = \ln(\frac{x^2+y^2-1}{e^x})$.

$$\Omega : \frac{x^2+y^2-1}{e^x} > 0 \Rightarrow \Omega : \begin{cases} x^2+y^2-1 > 0 \\ e^x > 0 \end{cases} \vee \begin{cases} x^2+y^2-1 < 0 \\ e^x < 0 \end{cases}$$

  Notice, however, that the second system of inequalities is incompatible as $\nexists\ x \in \mathbb{R}$ such that $e^x < 0$, meaning that, knowing that $e^x > 0\ \forall\ x \in \mathbb{R}$, it is possible to conclude that $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 1\}$.

$$f_x(x,y) = \frac{e^x}{x^2+y^2-1} \cdot \frac{2xe^x - (x^2+y^2-1)e^x}{(e^2)^2} = \frac{2x-(x^2+y^2-1)}{x^2+y^2-1}$$

$$f_y(x,y) = \frac{e^x}{x^2+y^2-1} \cdot \frac{2y}{e^x} = \frac{2y}{x^2+y^2-1} \text{ (in this case, treat } \frac{1}{e^x} \text{ as a constant)}$$

- $f(x,y) = \ln(\frac{e^{x^2}}{y+1}) + \sqrt{\cos y}$.

$$\Omega : \begin{cases} \frac{e^{x^2}}{y+1} > 0 \\ \cos y \geq 0 \end{cases} \Rightarrow \Omega : \begin{cases} e^{x^2} > 0 \\ y+1 > 0 \\ \cos y \geq 0 \end{cases} \vee \begin{cases} e^{x^2} < 0 \\ y+1 < 0 \\ \cos y \geq 0 \end{cases}$$

However, since the second system is incompatible, it is possible to conclude that:

$$\Omega : \begin{cases} e^{x^2} > 0 \\ y+1 > 0 \\ \cos y \geq 0 \end{cases} \Rightarrow \Omega : y \in (-1, \frac{\pi}{2}]$$

$$f_x(x,y) = \frac{y+1}{e^{x^2}} \cdot 2xe^{x^2} = 2x(y+1) \text{ (treat } \frac{1}{y+1} \text{ and } \sqrt{\cos y} \text{ as constants)}$$

$$f_y(x,y) = \frac{y+1}{e^{x^2}} \cdot \frac{-e^{x^2}}{(y+1)^2} - \frac{-\sin y}{2\sqrt{\cos y}} = -(\frac{1}{y+1} + \frac{\sin y}{2\sqrt{\cos y}}) \text{ (treat } e^{x^2} \text{ as a constant)}$$

- $f(x,y) = \ln(x^2+y^2) - \frac{1}{\sqrt{2x+3y}} + e^{\sin(x^2+y)}$.

$$\Omega : \begin{cases} x^2+y^2 > 0 \\ 2x+3y > 0 \end{cases} \Rightarrow \Omega : \begin{cases} (x,y) \neq (0,0) \\ y > -\frac{2x}{3} \end{cases}$$

$$f_x(x,y) = \frac{2x}{x^2+y^2} - (-\frac{1}{(\sqrt{2x+3y})^2} \frac{2}{2\sqrt{2x+3y}}) + 2xe^{\sin(x^2+y)} \cos(x^2+y)$$

$$f_y(x,y) = \frac{2y}{x^2+y^2} - (-\frac{1}{(\sqrt{2x+3y})^2} \frac{3}{2\sqrt{2x+3y}}) + e^{\sin(x^2+y)} \cos(x^2+y)$$

2.3 Exercise 3

Compute the gradient of the following functions at the given point using the incremental ratio and, afterwards, verify that the result is the same by computing the partial derivatives:

- $f(x,y) = xye^{x+y}$ in $(-1,1)$.
  Find the partial derivatives by applying the definition of incremental ratio:

  1. With respect to x:

  $$f_x(-1,1) = \lim_{h \to 0} \frac{f(-1+h,1) - f(-1,1)}{h} = \lim_{h \to 0} \frac{(h-1)e^{h-1+1} - (-e^{-1+1})}{h}$$

  Apply special limits to conclude that:

  $$f_x(-1,1) = \lim_{h \to 0} \frac{he^h - (e^h - 1)}{h} = \lim_{h \to 0} \frac{he^h}{h} - \frac{e^h - 1}{h} = \lim_{h \to 0} e^h - 1 = 0$$

2. With respect to y:

$$f_y(-1,1) = \lim_{h\to 0} \frac{f(-1,1+h) - f(-1,1)}{h} = \lim_{h\to 1} \frac{-(1+h)e^{-1+h+1} - (-e^{-1+1})}{h}$$

Apply special limits to conclude that:

$$f_y(-1,1) = \lim_{h\to 0} \frac{-he^h - (e^h - 1)}{h} = \lim_{h\to 0} \frac{-he^h}{h} - \frac{e^h - 1}{h} = \lim_{h\to 0} -e^h - 1 = -2$$

Therefore, $\nabla f(-1,1) = (0,-2)$.
Indeed, it is possible to recover the same result by directly computing the partial derivatives:

$$f_x(x,y) = ye^{x+y} + xye^{x+y} \Rightarrow f_x(-1,1) = 0$$
$$f_y(x,y) = xe^{x+y} + xye^{x+y} \Rightarrow f_y(-1,1) = -2$$

Therefore, $\nabla f(-1,1) = (0,-2)$.

- $f(x,y) = (x-y)\sin(x^2+y)$ in $(0,0)$.
Find the partial derivatives by applying the definition of incremental ratio:

1. With respect to x:

$$f_x(x,y) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h\sin(h^2)}{h} = \lim_{h\to 0} \sin(h^2) = 0$$

2. With respect to y:

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{-h\sin h}{h} = \lim_{h\to 0} -\sin h = 0$$

Therefore $\nabla f(0,0) = (0,0)$.
Indeed, it is possible to recover the same result by directly computing the partial derivatives:

$$f_x(x,y) = \sin(x^2+y) + (x-y)2x\cos(x^2+y) \Rightarrow f(0,0) = 0$$
$$f_y(x,y) = -\sin(x^2+y) + (x-y)\cos(x^2+y) \Rightarrow f(0,0) = 0$$

Therefore, $\nabla f(0,0) = (0,0)$.

- $f(x,y) = \ln(xy+1)e^{-x}$ in $(1,0)$.
Find the partial derivatives by applying the definition of incremental ratio:

1. With respect to x:

$$f_x(1,0) = \lim_{h\to 0} \frac{f(1+h,0) - f(1,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

2. With respect to y:

$$f_y(1,0) = \lim_{h\to 0} \frac{f(1,h) - f(1,0)}{h} = \lim_{h\to 0} \frac{\ln(h+1)e^{-1}}{h} = e^{-1} \text{ as } \lim_{h\to 0} \frac{\ln(h+1)}{h} = 1$$

Therefore, $\nabla f(1,0) = (0, e^{-1})$.
Indeed, it is possible to recover the same result by directly computing the partial derivatives:

$$f_x(x,y) = \frac{ye^{-x}}{xy+1} + \ln(xy+1)(-e^{-x}) \Rightarrow f_x(1,0) = 0$$

$$f_y(x,y) = \frac{xe^{-x}}{xy+1} \text{ (in this case, treat } e^{-x} \text{ as a constant) } \Rightarrow f_y(1,0) = e^{-1}$$

Therefore, $\nabla f(1,0) = (0, e^{-1})$.

2.4 Exercise 4

Compute the gradient and the tangent plane in the given point:

- $f(x,y) = x^3 + y^3 - 3xy$ in $(2,2)$.
  Start by computing the partial derivatives:

$$f_x(x,y) = 3x^2 - 3y \Rightarrow f_x(2,2) = 6$$
$$f_y(x,y) = 3y^2 - 3x \Rightarrow f_y(2,2) = 6$$

  Therefore, $\nabla f(2,2) = (6,6)$, while the tangent plane will be given by:

$$z = \nabla f(2,2)\cdot(x-2, y-2) + f(2,2) = 6(x-2) + 6(x-2) + 4 = 6x - 6y - 20$$

- $f(x,y) = \frac{x-y}{x+y}$ in $(1,1)$.
  Start by computing the partial derivatives:

$$f_x(x,y) = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2} \Rightarrow f_x(1,1) = \frac{1}{2}$$
$$f_y(x,y) = \frac{-(x+y)-(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2} \Rightarrow f_y(1,1) = -\frac{1}{2}$$

  Therefore, $\nabla f(1,1) = (\frac{1}{2}, -\frac{1}{2})$, while the tangent plane will be given by:

$$z = \nabla f(1,1)\cdot(x-1, y-1) + f(1,1) = \frac{1}{2}(x-1) - \frac{1}{2}(y-1) = \frac{1}{2}x - \frac{1}{2}y$$

- $f(x,y) = e^{xy}$ in $(1,5)$.
  Start by computing the partial derivatives:

$$f_x(x,y) = ye^{xy} \Rightarrow f_x(1,5) = 5e^5$$
$$f_y(x,y) = xe^{xy} \Rightarrow f_y(1,5) = e^5$$

  Therefore, $\nabla f(1,5) = (5e^5, e^5)$, while the tangent plane will be given by:

$$z = \nabla f(1,5)\cdot(x-1, y-5) + f(1,5) = 5e^5(x-1) + e^5(y-5) + e^5 = 5e^5x + e^5y - 9e^5$$

Chapter 3

Exercise Sheet 3: Continuity and differentiability

3.1 Exercise 1

Discuss continuity and differentiability in $\mathbb{R}^2$ for the following functions:

- First function:

$$f(x,y) = \begin{cases} \frac{e^{xy^2}-1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

The function is defined to be surely continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$, meaning that the goal is to check the limit of the function as $(x,y) \to (0,0)$:

By asymptotic comparison, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy^2}-1}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy^2}{\sqrt{x^2+y^2}} = \frac{0}{0}$

Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0} \frac{\rho\cos\theta\rho^2\sin^2\theta}{\sqrt{\rho^2\cos^2\theta+\rho^2\sin^2\theta}} = \lim_{\rho\to 0} \frac{\rho^3\cos\theta\sin^2\theta}{\rho} = \lim_{\rho\to 0} \rho^2\cos\theta\sin^2\theta = 0 \;\forall\; \theta \in [0,2\pi]$$

Since $\exists \displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$, the function is continuous in $(0,0)$ as well, and therefore it is continuous all over $\mathbb{R}^2$.
Observe that the function is also differentiable in $\mathbb{R}^2 \setminus \{(0,0)\}$: in order to check differentiability at $(0,0)$ it is possible to state that the function is differentiable in the point if and only if:

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - f_x(0,0)h - f_y(0,0)k}{\sqrt{h^2+k^2}} = 0$$

Most particularly, apply the definition of incremental ratio to find the function's partial derivatives:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h}\frac{e^{h\cdot 0^2} - 1}{\sqrt{h^2}} = \lim_{h\to 0} \frac{h\cdot 0^2}{h\sqrt{h^2}} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h}\frac{e^{0\cdot h^2} - 1}{\sqrt{h^2}} = \lim_{h\to 0} \frac{0\cdot h^2}{h\sqrt{h^2}} = 0$$

At this point, start by applying special limits for asymptotic approximation:

$$\lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\frac{e^{hk^2}-1}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\frac{hk^2}{\sqrt{h^2+k^2}} = \frac{0}{0}$$

Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0} \frac{\rho\cos\theta\rho^2\sin^2\theta}{\rho^2\cos^2\theta + \rho^2\sin^2\theta} = \lim_{\rho\to 0} \frac{\rho^3\cos\theta\sin^2\theta}{\rho^2} = \lim_{\rho\to 0} \rho\cos\theta\sin^2\theta = 0 \ \forall\ \theta\in[0,2\pi]$$

Therefore, since the condition is satisfied, the function is indeed differentiable in $(0,0)$ as well, and therefore it is differentiable all over $\mathbb{R}^2$.

- Second function:

$$f(x,y) = \begin{cases} \frac{x^2y^3}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

The function is defined to be surely continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$, meaning that the goal is to check the limit of the function as $(x,y)\to(0,0)$:

$$\lim_{(x,y=\to(0,0)} \frac{x^2y^3}{(x^2+y^2)^2} = \frac{0}{0}$$

Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0} \frac{\rho^2\cos^2\theta\rho^3\sin^3\theta}{(\rho^2\cos^2\theta + \rho^2\sin^2\theta)^2} = \lim_{\rho\to 0} \frac{\rho^5\cos^2\theta\sin^3\theta}{(\rho^2)^2} = \lim_{\rho\to 0} \rho\cos^2\theta\sin^3\theta = 0 \ \forall\ \theta\in[0,2\pi]$$

Since $\exists \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$, the function is continuous in $(0,0)$ as well, and therefore it is continuous all over $\mathbb{R}^2$.

Observe that the function is also differentiable in $\mathbb{R}^2 \setminus \{(0,0)\}$: in order to check differentiability at $(0,0)$ it is possible to state that the function is differentiable in the point if and only if:

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - f_x(0,0)h - f_y(0,0)k}{\sqrt{h^2+k^2}} = 0$$

Most particularly, apply the definition of incremental ratio to find the function's partial derivatives:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h} \frac{h^2 \cdot 0^3}{(h^2)^2} = \lim_{h\to 0} \frac{0}{h^3} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h} \frac{0^2 h^3}{(h^2)^2} = \lim_{h\to 0} \frac{0}{h^2} = 0$$

Therefore, it is possible to conclude that:

$$\lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}} \frac{h^2k^3}{(h^2+k^2)^2} = \frac{0}{0}$$

Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0} \frac{1}{\sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta}} \frac{\rho^2\cos^2\theta\rho^3\sin^3\theta}{(\rho^2\cos^2\theta + \rho^2\sin^2\theta)^2} = \lim_{\rho\to 0} \frac{\rho^5\cos^2\theta\sin^3\theta}{\rho\cdot(\rho^2)^2} = \cos^2\theta\sin^3\theta$$

In this case, the limit does not exist as the result depends on the value of θ.

Therefore, since the condition is not satisfied, the function is not differentiable at $(0,0)$.

3.2 Exercise 2

Given $a \in \mathbb{R}$, consider the following function:

$$f(x,y) = \begin{cases} \frac{x^3y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ a & \text{if } (x,y) = (0,0) \end{cases}$$

- Discuss continuity of f in $(0,0)$ with respect to the parameter a.
  By definition, the function will be continuous at $(0,0)$ if and only if $\exists \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.
  Therefore, start by computing the limit of the function as $(x,y)\to(0,0)$:

  $$\lim_{(x,y)\to 0} \frac{x^3y}{x^2+y^2} = \frac{0}{0}$$

  Solve the limit by switching to polar coordinates:

  $$\lim_{\rho\to 0} \frac{\rho^3\cos^3\theta\rho\sin\theta}{\rho^2\cos^2\theta + \rho^2\sin^2\theta} = \lim_{\rho\to 0} \frac{\rho^4\cos^3\theta\sin\theta}{\rho^2} = \lim_{\rho\to 0} \rho^2\cos^3\theta\sin\theta = 0 \ \forall\ \theta \in [0, 2\pi]$$

  Therefore, the function will be continuous at $(0,0)$ for $f(0,0) = a = 0$.

- Is the function differentiable in $(0,0)$?
  By definition, the function is differentiable at $(0,0)$ if and only if:

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-f_x(0,0)h-f_y(0,0)k}{\sqrt{h^2+k^2}} = 0$$

  Most particularly, apply the definition of incremental ratio to find the function's partial derivatives:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \frac{1}{h}\frac{h^3\cdot 0}{h^2} = \lim_{h\to 0} 0 = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h}\frac{0^3\cdot h}{h^2} = \lim_{h\to 0} \frac{0}{h^2} = 0$$

  Therefore, it is possible to conclude that:

$$\lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}} \frac{h^3 k}{h^2+k^2} = \frac{0}{0}$$

  Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0} \frac{1}{\sqrt{\rho^2\cos^2\theta+\rho^2\sin^2\theta}} \frac{\rho^3\cos^3\theta\rho\sin\theta}{\rho^2\cos^2\theta+\rho^2\sin^2\theta} = \lim_{\rho\to 0} \frac{\rho^4\cos^3\theta}{\rho\cdot\rho^2} = \lim_{\rho\to 0} \rho\cos^3\theta\sin\theta = 0 \ \forall\ \theta\in[0,2\pi]$$

  Therefore, since the condition is satisfied, the function is differentiable at $(0,0)$, provided that it is assumed to be continuous as well.

3.3 Exercise 3

Consider the following function:

$$f(x,y) = \frac{\sin^\alpha(xy)}{\sqrt{x^2+y^2}},\ \alpha > 0$$

- Compute the domain of f.

$$\Omega : \begin{cases} x^2+y^2 \geq 0 \\ x^2+y^2 \neq 0 \end{cases} \Rightarrow \Omega : x^2+y^2 > 0 \Rightarrow \Omega : (x,y) \neq (0,0)$$

  Therefore, it is possible to conclude that $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$.

- Discuss for which value of α it is possible to extend the function for continuity outside its domain.
  By definition, the function will be continuous at a point (x_0,y_0) if and only if $\exists \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.
  Most particularly, the function is surely continuous in $\mathbb{R}^2 \setminus (0,0)$, so the goal is to check for continuity at the point $(0,0)$ by checking the limit of

the function as $(x, y) \to (0, 0)$:

By asymptotic approximation, $\lim_{(x,y)\to(0,0)} \frac{\sin^\alpha(xy)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{(xy)^\alpha}{\sqrt{x^2+y^2}} = \frac{0}{0}$

Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0} \frac{(\rho\cos\theta\rho\sin\theta)^\alpha}{\sqrt{\rho^2\cos^2\theta+\rho^2\sin^2\theta}} = \lim_{\rho\to 0} \frac{\rho^{2\alpha}\cos^\alpha\theta\sin^\alpha\theta}{\rho} = \begin{cases} 0 & \text{if } 2\alpha-1>0 \\ \nexists & \text{if } 2\alpha-1\leq 0 \end{cases}$$

Therefore, it is possible to extend the function for continuity by writing it as:

$$f(x,y) = \begin{cases} \frac{\sin^\alpha(xy)}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}, \text{ provided that } \alpha > \frac{1}{2}$$

- After having extended the function f in $\mathbb{R}^2$, compute $\nabla f(0,0)$.
  Assuming $\alpha > \frac{1}{2}$ to guarantee continuity, find the function's partial derivatives by applying the definition of incremental ratio:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h}\frac{\sin^\alpha(h\cdot 0)}{\sqrt{h^2}} = \lim_{h\to 0} \frac{0}{h^2} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h}\frac{\sin^\alpha(0\cdot h)}{\sqrt{h^2}} = \lim_{h\to 0} \frac{0}{h^2} = 0$$

  Therefore, it is possible to conclude that $\nabla f(0,0) = (0,0)$.

- Discuss for which value of α f is differentiable in $(0,0)$.
  Assuming $\alpha > \frac{1}{2}$ to guarantee continuity and differentiability, the function is said to be differentiable at $(0,0)$ if and only if:

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-f_x(0,0)h-f_y(0,0)k}{\sqrt{h^2+k^2}} = 0$$

  Start by applying special limits for asymptotic approximation:

$$\lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}}\frac{\sin^\alpha(hk)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{(hk)^\alpha}{h^2+k^2} = \frac{0}{0}$$

  Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0} \frac{(\rho\cos\theta\rho\sin\theta)^\alpha}{\rho^2\cos^2\theta+\rho^2\sin^2\theta} = \lim_{\rho\to 0} \frac{\rho^{2\alpha}\cos^\alpha\theta\sin^\alpha\theta}{\rho^2} = \begin{cases} 0 & \text{if } 2\alpha-2>0 \\ \nexists & \text{if } 2\alpha-2\leq 0 \end{cases}$$

  Therefore, since the condition is satisfied for $\alpha > 1$, the function will also be differentiable at $(0,0)$ if and only if $\alpha > 1$.

3.4 Exercise 4

Determine if the function is differentiable at the given point:

- $f(x,y) = \sqrt{1-xy^2}$ in $P(1,0)$.
  The function is said to be differentiable at $P(1,0)$ if and only if:

$$\lim_{(h,k)\to(0,0)} \frac{f(1+h,k)-f(1,0)-f_x(1,0)h-f_y(1,0)k}{\sqrt{h^2+k^2}} = 0$$

  Most particularly, start by finding the function's partial derivatives by applying the definition of incremental ratio:

$$f_x(1,0) = \lim_{h\to 0} \frac{f(1+h,0)-f(1,0)}{h} = \lim_{h\to 0} \frac{\sqrt{1-(1+h)0^2}-\sqrt{1}}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$f_y(1,0) = \lim_{h\to 0} \frac{f(1,h)-f(1,0)}{h} = \lim_{h\to 0} \frac{\sqrt{1-h^2}-\sqrt{1}}{h} = \lim_{h\to 0} \frac{-2h}{2\sqrt{1-h^2}} = 0 \text{ by De L'Hopital's rule.}$$

  At this point, it is possible to conclude that:

$$\lim_{(h,k)\to(0,0)} \frac{\sqrt{1-(1+h)k^2}-\sqrt{1}}{\sqrt{h^2+k^2}} = \frac{0}{0}$$

  Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0} \frac{\sqrt{1-(1+\rho\cos\theta)\rho^2\sin^2\theta}-\sqrt{1}}{\sqrt{\rho^2\cos^2\theta+\rho^2\sin^2\theta}} = \lim_{\rho\to 0} \frac{\sqrt{1-\rho^2\sin^2\theta-\rho^3\cos\theta\sin^2\theta}-\sqrt{1}}{\rho} = \frac{0}{0}$$

  Solve the limit by applying De L'Hopital's rule:

$$\lim_{\rho\to 0} = \frac{-2\rho\sin^2\theta-3\rho^2\cos\theta\sin^2\theta}{2\sqrt{1-\rho^2\sin^2\theta-\rho^3\cos\theta\sin^2\theta}} = 0 \ \forall\ \theta \in [0,2\pi]$$

  Therefore, since the condition is satisfied, the function is differentiable at $(1,0)$.
  N.B.: It is also possible to recover differentiability by knowing that the function is continuous and differentiable at $(1,0)$.

- $f(x,y) = x^2y + e^{2xy}$ in $P(1,1)$.
  The function is said to be differentiable at $P(1,1)$ if and only if:

$$\lim_{(h,k)\to(0,0)} \frac{f(1+h,1+k)-f(1,1)-f_x(1,1)h-f_y(1,1)k}{\sqrt{h^2+k^2}} = 0$$

  Most particularly, start by finding the function's partial derivatives by applying the definition of incremental ratio:

$$f_x(1,1) = \lim_{h\to 0} \frac{f(1+h,1)-f(1,1)}{h} = \lim_{h\to 0} \frac{(h+1)^2+e^{2(h+1)}-(1+e^2)}{h} = 2+2e^2$$

$$f_y(1,1) = \lim_{h\to 0} \frac{f(1,1+h) - f(1,1)}{h} = \lim_{h\to 0} \frac{(h+1) + e^{2(h+1)} - (1+e^2)}{h} = 1{+}2e^2$$

At this point, it is possible to conclude that:

$$\lim_{(h,k)\to(0,0)} \frac{(h+1)^2(k+1) + e^{2(h+1)(k+1)} - (1+e^2) - (2+2e^2)h - (1+2e^2)k}{\sqrt{h^2+k^2}} = \frac{0}{0}$$

Start by doing some algebra and expand the following values:

$$(h+1)^2(k+1) = (h^2+2h+1)(k+1) = h^2k + h^2 + 2hk + 2h + k + 1$$

$$2(h+1)(k+1) = 2(hk+h+k+1)$$

For this reason, it is possible to rewrite the limit in the following way:

$$\lim_{(h,k)\to(0,0)} \frac{h^2k + h^2 + 2hk + e^{2(hk+h+k+1)} - e^2 - 2e^2h - 2e^2k}{\sqrt{h^2+k^2}} = \frac{0}{0}$$

Solve the limit by switching to polar coordinates:

$$\lim_{\rho\to 0} \frac{\rho^2\cos^2\theta\rho\sin\theta + \rho^2\cos^2\theta + 2\rho\cos\theta\rho\sin\theta + e^{2(\rho\cos\theta\rho\sin\theta+\rho\cos\theta+\rho\sin\theta+1)} - e^2 - 2e^2\rho\cos\theta - 2e^2\rho\sin}{\sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta}}$$

Focus on the exponential terms and apply special limits for asymptotic approximation:

$$\lim_{\rho\to 0} e^{2(\rho^2\cos\theta\sin\theta+\rho\cos\theta+\rho\sin\theta+1)} - e^2 = \lim_{\rho\to 0} e^2(e^{2(\rho^2\cos\theta\sin\theta+\rho\cos\theta+\rho\sin\theta)} - 1) \text{ and exploit } \lim_{t\to 0}\frac{e^t-1}{t} = 1$$

This approximation allows to conclude that:

$$\lim_{\rho\to 0} \frac{\rho^3\cos^2\theta\sin\theta + \rho^2\cos^2\theta + 2\rho^2\cos\theta\sin\theta + 2e^2(\rho^2\cos\theta\sin\theta + \rho\cos\theta + \rho\sin\theta) - 2e^2\rho\cos\theta - 2e^2\rho\sin}{\rho}$$

$$\lim_{\rho\to 0} \frac{\rho(\rho^2\cos^2\theta\sin\theta + \rho\cos^2\theta + 2\rho\sin\theta\cos\theta + 2e^2\rho\cos\theta\sin\theta + 2e^2\cos\theta + 2e^2\sin\theta - 2e^2\cos\theta - 2e^2\sin\theta}{\rho}$$

$$\lim_{\rho\to 0} \rho^2\cos^2\theta\sin\theta + \rho\cos^2\theta + 2\rho\sin\theta\cos\theta + 2e^2\rho\cos\theta\sin\theta = 0 \ \forall\ \theta \in [0, 2\pi]$$

Therefore, since the condition is satisfied, the function is differentiable at $(1,1)$.
N.B.: It is also possible to recover differentiability by knowing that the function is continuous and differentiable at $(1,0)$.

3.5 Exercise 5

Consider the following function:

$$f(x,y) = y^2 e^{-\frac{x^2}{y^2}}$$

- Compute the domain of f.
  $\Omega : y^2 \neq 0 \Rightarrow \Omega = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$

- Compute $f(0,0)$ using limits.
  Since computing the actual limit of f as $(x,y) \to (0,0)$ can be tricky, it is possible to exploit the squeeze theorem by bounding the function.
  Start by noticing that, since the function is defined as the product between a quadratic term and an exponential function, then it is for sure non-negative in $\mathbb{R}^2$, meaning that $f(x,y) \geq 0$.
  In addition, since $-\frac{x^2}{y^2} \leq 0 \; \forall \; (x,y) \in \mathbb{R}^2$, it is possible to state that $e^{-\frac{x^2}{y^2}} \leq 1$ in $\mathbb{R}^2$, meaning that $f(x,y) \leq y^2$.
  Therefore, by applying the squeeze theorem, it is possible to conclude that:

  $$0 \leq \lim_{(x,y)\to(0,0)} y^2 e^{-\frac{x^2}{y^2}} \leq \lim_{(x,y=\to(0,0)} y^2 \Rightarrow 0 \leq \lim_{(x,y)\to(0,0)} y^2 e^{-\frac{x^2}{y^2}} \leq 0 \lim_{(x,y)\to(0,0)} y^2 e^{-\frac{x^2}{y^2}} = 0$$

- Compute $f_x(0,0)$ and $f_y(0,0)$ using the definition.
  Find the partial derivatives by applying the definition of incremental ratio:

  1. With respect to x:

  $$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

  2. With respect to y:

  $$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^2 e^{-\frac{0}{h^2}}}{h} = \lim_{h\to 0} h = 0$$

- Is the function differentiable at $(0,0)$?
  By definition, the function will be differentiable at $(0,0)$ if and only if the following holds true:

  $$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - f_x(0,0)h - f_y(0,0)k}{\sqrt{h^2+k^2}} = 0$$

  $$\lim_{(h,k)\to(0,0)} \frac{k^2 e^{-\frac{h^2}{k^2}}}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{k^2}{\sqrt{h^2+k^2}} = \frac{0}{0} \text{ by squeeze theorem on } k^2 e^{-\frac{h^2}{k^2}} \leq k^2.$$

  At this point, solve the limit by switching to polar coordinates:

  $$\lim_{\rho\to 0} \frac{\rho^2 \sin^2\theta}{\sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta}} = \lim_{\rho\to 0} \frac{\rho^2\sin^2\theta}{\rho} = \lim_{\rho\to 0} \rho \sin^2\theta = 0 \; \forall \; \theta \in [0, 2\pi]$$

  Therefore, since the condition is satisfied, the function is indeed differentiable at $(0,0)$.

3.6 Exercise 6

Using the definition with the incremental ratio, compute the following directional derivatives and, afterwards, verify that the result is the same by using the formula with the gradient:

- $f(x, y) = e^x y$ along the direction $v = (3, 4)$ at the point $(2, 0)$.
  Notice that v is not a unit vector, so start by normalising it:

  $$\|v\| = \sqrt{3^2 + 4^2} = 5 \Rightarrow u = \frac{v}{\|v\|} = (\frac{3}{5}, \frac{4}{5})$$

  Therefore, by applying the definition of rate of change, it is possible to conclude that:

  $$\frac{\partial f}{\partial u}(2, 0) = \lim_{h \to 0} \frac{f(2 + \frac{3}{5}h, \frac{4}{5}h) - f(2, 0)}{h} = \lim_{h \to 0} \frac{e^{2+\frac{3}{5}h} \frac{4}{5}h}{h} = \frac{4e^2}{5}$$

  Indeed, it is possible to recover the same result by directly computing the directional derivative:

  $$f_x(x, y) = e^x y \Rightarrow f_x(2, 0) = 0$$
  $$f_y(x, y) = e^x \Rightarrow f_y(2, 0) = e^2$$

  Therefore, it is possible to conclude that:

  $$\frac{\partial f}{\partial u}(2, 0) = \nabla f(2, 0) \cdot u = \frac{4e^2}{5}$$

- $f(x, y) = x^2 + y^2 - xy$ along the direction $v = (1, 1)$ at the point $(0, 0)$.
  Notice that v is not a unit vector, so start by normalising it:

  $$\|v\| = \sqrt{1^2 + 1^2} = \sqrt{2} \Rightarrow u = \frac{v}{\|v\|} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

  Therefore, by applying the definition of rate of change, it is possible to conclude that:

  $$\frac{\partial f}{\partial u}(0, 0) = \lim_{h \to 0} \frac{f(\frac{\sqrt{2}}{2}h, \frac{\sqrt{2}}{2}h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{1}{2}h^2 + \frac{1}{2}h^2 - \frac{1}{2}h^2}{h} = \lim_{h \to 0} \frac{h^2}{2h} = 0$$

  Indeed, it is possible to recover the same result by directly computing the directional derivative:

  $$f_x(x, y) = 2x - y \Rightarrow f_x(0, 0) = 0$$
  $$f_y(x, y) = 2y - x \Rightarrow f_y(0, 0) = 0$$

  Therefore, it is possible to conclude that:

  $$\frac{\partial f}{\partial u}(0, 0) = \nabla f(0, 0) \cdot u = 0$$

3.7 Exercise 7

Compute the directional derivative along the direction v at the point P for the following functions:

- $f(x,y) = \ln(\frac{1}{x^2+y^2}) + 2y$ along $v = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ at the point $P(1,1)$.
  Start by recovering the function's gradient vector:

$$f_x(x,y) = \frac{-2x}{(x^2+y^2)^2}(x^2+y^2) = \frac{-2x}{x^2+y^2} \Rightarrow f_x(1,1) = -1$$

$$f_y(x,y) = \frac{-2y}{(x^2+y^2)^2}(x^2+y^2) + 2 = \frac{-2y}{x^2+y^2} + 2 \Rightarrow f_y(1,1) = 1$$

  Therefore, given $\nabla f(1,1) = (-1,1)$, it is possible to conclude that:

$$\frac{\partial f}{\partial v}(1,1) = \nabla f(1,1) \cdot u = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$$

- $f(x,y) = \ln(\frac{x^2+y^2+1}{e^x})$ along $v = (0,1)$ at the point $P(0,1)$.
  Start by recovering the function's gradient vector:

$$f_x(x,y) = \frac{e^x}{x^2+y^2+1} \cdot \frac{2xe^x - (x^2+y^2+1)e^x}{(e^x)^2} = \frac{2x - (x^2+y^2+1)}{x^2+y^2+1} \Rightarrow f_x(0,1) = -1$$

$$f_y(x,y) = \frac{e^x}{x^2+y^2+1} \cdot \frac{2y}{e^x} = \frac{2y}{x^2+y^2+1} \Rightarrow f_y(0,1) = 1$$

  Therefore, given $\nabla f(0,1) = (1,1)$, it is possible to conclude that:

$$\frac{\partial f}{\partial v}(0,1) = \nabla f(0,1) \cdot u = 0 + 1 = 1$$

Chapter 4

Exercise Sheet 4: Second derivatives and stationary points

4.1 Exercise 1

Compute the first and second partial derivatives of the following functions:

- $f(x, y) = \sin(2x + 3y)$.
  Start by computing the first-order partial derivatives:

$$f_x(x, y) = 2\cos(2x + 3y)$$
$$f_y(x, y) = 3\cos(2x + 3y)$$

  Therefore, it is possible to compute the second-order partial derivatives:

$$f_{xx}(x, y) = -4\sin(2x + 3y)$$
$$f_{xy}(x, y) = -6\sin(2x + 3y)$$
$$f_{yx}(x, y) = -6\sin(2x + 3y)$$
$$f_{yy}(x, y) = -9\sin(2x + 3y)$$

- $f(x, y) = \ln(x^2 + y)$.
  Start by computing the first-order partial derivatives:

$$f_x(x, y) = \frac{2x}{x^2 + y}$$
$$f_y(x, y) = \frac{1}{x^2 + y}$$

Therefore, it is possible to compute the second-order partial derivatives:

$$f_{xx}(x,y) = \frac{2(x^2+y) - 2x \cdot 2x}{(x^2+y)^2} = \frac{2y-2x^2}{(x^2+y)^2}$$
$$f_{xy}(x,y) = \frac{-2x}{(x^2+y)^2}$$
$$f_{yx}(x,y) = \frac{-2x}{(x^2+y)^2}$$
$$f_{yy}(x,y) = \frac{-1}{(x^2+y)^2}$$

- $f(x,y) = e^{\frac{2x}{x+3y}}$.
  Start by computing the first-order partial derivatives:

$$f_x(x,y) = \frac{2(x+3y)-2x}{(x+3y)^2} e^{\frac{2x}{x+3y}} = \frac{6y}{(x+3y)^2} e^{\frac{2x}{x+3y}}$$
$$f_y(x,y) = \frac{-6x}{(x+3y)^2} e^{\frac{2x}{x+3y}}$$

  Therefore, it is possible to compute the second-order partial derivatives:

$$f_{xx}(x,y) = \frac{-12y(x+3y)}{((x+3y)^2)^2} e^{\frac{2x}{x+3y}} + \frac{6y}{(x+3y)^2} \frac{6y}{(x+3y)^2} e^{\frac{2x}{x+3y}} = \frac{-12xy}{(x+3y)^4} e^{\frac{2x}{x+3y}}$$
$$f_{xy}(x,y) = \frac{6(x+3y)^2 - 36y(x+3y)}{((x+3y)^2)^2} e^{\frac{2x}{x+3y}} + \frac{6y}{(x+3y)^2}(\frac{-6x}{(x+3y)^2}) e^{\frac{2x}{x+3y}} = \frac{(6x^2-90y^2)e^{\frac{2x}{x+3y}}}{(x+3y)^4}$$
$$f_{yx}(x,y) = \frac{-6(x+3y)^2 - 12x(x+3y)}{((x+3y)^2)^2} e^{\frac{2x}{x+3y}} + (\frac{-6x}{(x+3y)^2}) \frac{6y}{(x+3y)^2} e^{\frac{2x}{x+3y}} = \frac{(6x^2-90y^2)e^{\frac{2x}{x+3y}}}{(x+3y)^4}$$
$$f_{yy}(x,y) = \frac{-36x(x+3y)}{((x+3y)^2)^2} e^{\frac{2x}{x+3y}} + (\frac{-6x}{(x+3y)^2})(\frac{-6x}{((x+3y)^2)^2}) e^{\frac{2x}{x+3y}} = \frac{-108xy}{(x+3y)^4} e^{\frac{2x}{x+3y}}$$

- $f(x,y) = \cos(x^2+xy+y^2)$.
  Start by computing the first-order partial derivatives:

$$f_x(x,y) = -(2x+y)\sin(x^2+xy+y^2)$$
$$f_y(x,y) = -(x+2y)\sin(x^2+xy+y^2)$$

  Therefore, it is possible to compute the second-order partial derivatives:

$$f_{xx}(x,y) = -2\sin(x^2+xy+y^2) - (2x+y)^2\cos(x^2+xy+y^2)$$
$$f_{xy}(x,y) = -\sin(x^2+xy+y^2) - (2x+y)(x+2y)\cos(x^2+xy+y^2)$$
$$f_{yx}(x,y) = -\sin(x^2+xy+y^2) - (x+2y)(2x+y)\cos(x^2+xy+y^2)$$
$$f_{yy}(x,y) = -2\sin(x^2+xy+y^2) - (x+2y)^2\cos(x^2+xy+y^2)$$

N.B.: In all cases, whenever the function is continuous, Schwarz's theorem guarantees $f_{xy}(x,y) = f_{yx}(x,y)$.

4.2 Exercise 2

Discuss which kind of stationary points you obtain from the following functions:

- $f(x, y) = x^3 + y^3 + xy$.
  Start by finding the function's stationary points by solving $\nabla f(x, y) = 0$:

  $$\begin{cases} 3x^2 + y = 0 \\ 3y^2 + x = 0 \end{cases} \quad \text{from the first equation, let } y = -3x^2.$$

  $$\begin{cases} y = -3x^2 \\ 27x^4 + x = 0 \end{cases} \Rightarrow \begin{cases} y = -3x^2 \\ x(27x^3 + 1) = 0 \end{cases} \Rightarrow \begin{cases} y = -3x^2 \\ x = 0 \lor x = -\frac{1}{3} \end{cases}$$

  Therefore, it is possible to conclude that $P_1(0, 0)$ and $P_2(-\frac{1}{3} - \frac{1}{3})$ are the function's stationary points.
  At this point, compute the Hessian matrix:

  $$Hf(x, y) = \begin{bmatrix} 6x & 1 \\ 1 & 6y \end{bmatrix}$$

  Consider each point separately:

  1. $P_1(0, 0)$

     $$Hf(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow det(Hf(0, 0)) = -1$$

     Therefore, since $det(Hf(0, 0)) < 0$, $P_1(0, 0)$ is a saddle point.
  2. $P_2(-\frac{1}{3}, -\frac{1}{3})$

     $$Hf(-\frac{1}{3}, -\frac{1}{3}) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow det(Hf(-\frac{1}{3}, -\frac{1}{3})) = 3$$

     Therefore, since $det(Hf(-\frac{1}{3}, -\frac{1}{3})) > 0$ and $f_{xx}(-\frac{1}{3}, -\frac{1}{3}) = -2 < 0$, $P_2(-\frac{1}{3}, -\frac{1}{3})$ is a maximum point.

- $f(x, y) = x^3 - y^3 + xy$.
  Start by finding the function's stationary points by solving $\nabla f(x, y) = 0$:

  $$\begin{cases} 3x^2 + y = 0 \\ -3y^2 + x = 0 \end{cases} \quad \text{from the second equation, let } x = 3y^2.$$

  $$\begin{cases} 27y^4 + y = 0 \\ x = 3y^2 \end{cases} \Rightarrow \begin{cases} y(27y^3 + 1) = 0 \\ x = 3y^2 \end{cases} \Rightarrow \begin{cases} y = 0 \lor y = -\frac{1}{3} \\ x = 3y^2 \end{cases}$$

  Therefore, it is possible to conclude that $P_1(0, 0)$ and $P_2(\frac{1}{3}, -\frac{1}{3})$ are the function's stationary points.

At this point, compute the Hessian matrix:

$$Hf(x,y) = \begin{bmatrix} 6x & 1 \\ 1 & -6y \end{bmatrix}$$

Consider each point separately:

1. $P_1(0,0)$

$$Hf(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow det(Hf(0,0)) = -1$$

   Therefore, since $det(Hf(0,0)) < 0$, $P_1(0,0)$ is a saddle point.

2. $P_2(\frac{1}{3}, -\frac{1}{3})$

$$Hf(\frac{1}{3}, -\frac{1}{3}) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow det(Hf(\frac{1}{3}, -\frac{1}{3})) = 3$$

   Therefore, since $det(Hf(\frac{1}{3}, -\frac{1}{3})) > 0$ and $f_{xx}(\frac{1}{3}, -\frac{1}{3}) = 2 > 0$, $P_2(\frac{1}{3}, -\frac{1}{3})$ is a minimum point.

- $f(x,y) = x^3 + 3xy^2 - 15x - 12y$.
  Start by finding the function's stationary points by solving $\nabla f(x,y) = 0$:

$$\begin{cases} 3x^2 + 3y^2 - 15 = 0 \\ 6xy - 12 = 0 \end{cases} \quad \text{simplify } x = \pm\sqrt{5-y^2} \text{ and } xy = 2.$$

  Consider the solutions to the first equation separately to find the stationary points:

  1. $x = -\sqrt{5-y^2}$

$$\begin{cases} x = -\sqrt{5-y^2} \\ -y\sqrt{5-y^2} = 2 \end{cases}$$

     Assuming $-y\sqrt{5-y^2} \geq 0$, square both sides of the second equation and solve the system via quartic equations:

$$\begin{cases} x = -\sqrt{5-y^2} \\ y^2(5-y^2) = 4 \end{cases} \quad \text{let } t = y^2$$

$$\begin{cases} x = -\sqrt{5-t} \\ 5t - t^2 - 4 = 0 \end{cases} \quad \Rightarrow t = 1 \vee t = 4$$

     However, since $t = y^2$, the equation actually has four solutions, meaning that it is possible to find the stationary points $P_1(-2,-1)$, $P_2(-2,1)$, $P_3(-1,-2)$ and $P_4(-1,2)$.

2. $x = \sqrt{5 - y^2}$

$$\begin{cases} x = \sqrt{5-y^2} \\ y\sqrt{5-y^2} = 2 \end{cases}$$

Assuming $y\sqrt{5-y^2} \geq 0$, square both sides of the equation and solve the system via quadric equations:

$$\begin{cases} x = \sqrt{5-y^2} \\ y^2(5-y^2) = 4 \end{cases} \qquad \text{let } t = y^2$$

$$\begin{cases} x = \sqrt{5-t} \\ 5t - t^2 - 4 = 0 \end{cases} \qquad \Rightarrow t = 1 \vee t = 4$$

However, since $t = y^2$, the equation actually has four solutions, meaning that it is possible to find the stationary points $P_5(2,-1)$, $P_6(2,1)$, $P_7(1,-2)$ and $P_8(1,2)$.

At this point, compute the Hessian matrix:

$$Hf(x,y) = \begin{bmatrix} 6x & 6y \\ 6y & 6x \end{bmatrix}$$

Consider each point separately:

1. $P_1(-2,-1)$

$$Hf(-2,-1) = \begin{bmatrix} -12 & -6 \\ -6 & -12 \end{bmatrix} \Rightarrow det(Hf(-2,-1)) = 108$$

Therefore, since $det(Hf(-2,-1)) > 0$ and $f_{xx}(-2,-1) = -12 < 0$, $P_1(-2,-1)$ is a maximum point.

2. $P_2(-2,1)$

$$Hf(-2,1) = \begin{bmatrix} -12 & 6 \\ 6 & -12 \end{bmatrix} \Rightarrow det(Hf(-2,1)) = 108$$

Therefore, since $det(Hf(-2,1)) > 0$ and $f_{xx}(-2,1) = -12 < 0$, $P_2(-2,1)$ is a maximum point.

3. $P_3(-1,-2)$

$$Hf(-1,-2) = \begin{bmatrix} -6 & -12 \\ -12 & -6 \end{bmatrix} \Rightarrow det(Hf(-1,-2)) = -108$$

Therefore, since $det(Hf(-1,-2)) < 0$, $P_3(-1,-2)$ is a saddle point.

4. $P_4(-1,2)$

$$Hf(-1,2) = \begin{bmatrix} -6 & 12 \\ 12 & -6 \end{bmatrix} \Rightarrow det(Hf(-1,2)) = -108$$

Therefore, since $det(Hf(-1,2)) < 0$, $P_4(-1,2)$ is a saddle point.

5. $P_5(2,-1)$

$$Hf(2,-1) = \begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix} \Rightarrow det(Hf(2,-1)) = 108$$

Therefore, since $det(Hf(2,-1)) > 0$ and $f_{xx}(2,-1) = 12 > 0$, $P_5(2,-1)$ is a minimum point.

6. $P_6(2,1)$

$$Hf(2,1) = \begin{bmatrix} 12 & 6 \\ 6 & 12 \end{bmatrix} \Rightarrow det(Hf(2,-1)) = 108$$

Therefore, since $det(Hf(2,1)) > 0$ and $f_{xx}(2,1) = 12 > 0$, $P_6(2,1)$ is a minimum point.

7. $P_7(1,-2)$

$$Hf(1,-2) = \begin{bmatrix} 6 & -12 \\ -12 & 6 \end{bmatrix} \Rightarrow det(Hf(1,-2)) = -108$$

Therefore, since $det(Hf(1,-2)) < 0$, $P_7(1,-2)$ is a saddle point.

8. $P_8(1,2)$

$$Hf(1,2) = \begin{bmatrix} 6 & 12 \\ 12 & 6 \end{bmatrix} \Rightarrow det(Hf(1,-2)) = -108$$

Therefore, since $det(Hf(1,2)) < 0$, $P_8(1,2)$ is a saddle point.

- $f(x,y) = 2(x^2 + y^2 + 1) - (x^4 + y^4)$.
Start by finding the function's stationary points by solving $\nabla f(x,y) = 0$:

$$\begin{cases} 4x - 4x^3 = 0 \\ 4y - 4y^3 = 0 \end{cases} \quad \text{from the first equation, let } 4x - 4x^3 = 0 \Leftrightarrow x = 0 \lor x = -1 \lor x = 1.$$

Consider each solution separately to find the stationary points:

1. $x = 0$

$$\begin{cases} x = 0 \\ 4y(1 - y^2) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \lor y = \pm 1 \end{cases}$$

Therefore, it is possible to find the stationary points $P_1(0,0)$, $P_2(0,-1)$ and $P_3(0,1)$.

2. $x = -1$

$$\begin{cases} x = -1 \\ 4y(1 - y^2) = 0 \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = 0 \lor y = \pm 1 \end{cases}$$

Therefore, it is possible to find the stationary points $P_4(-1,0)$, $P_5(-1,-1)$ and $P_6(-1,1)$.

3. $x = 1$

$$\begin{cases} x = 1 \\ 4y(1-y^2) = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \vee y = \pm 1 \end{cases}$$

   Therefore, it is possible to find the stationary points $P_7(1,0)$, $P_8(1,-1)$ and $P_9(1,1)$.

At this point, compute the Hessian matrix:

$$Hf(x,y) = \begin{bmatrix} 4-12x^2 & 0 \\ 0 & 4-12y^2 \end{bmatrix}$$

Consider each point separately:

1. $P_1(0,0)$

$$Hf(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow det(Hf(0,0)) = 16$$

   Therefore, since $det(Hf(0,0)) > 0$ and $f_{xx}(0,0) = 4 > 0$, $P_1(0,0)$ is a minimum point.

2. $P_2(0,-1)$

$$Hf(0,-1) = \begin{bmatrix} 4 & 0 \\ 0 & -8 \end{bmatrix} \Rightarrow det(Hf(0,-1)) = -32$$

   Therefore, since $det(Hf(0,-1)) < 0$, $P_2(0,-1)$ is a saddle point.

3. $P_3(0,1)$

$$Hf(0,1) = \begin{bmatrix} 4 & 0 \\ 0 & -8 \end{bmatrix} \Rightarrow det(Hf(0,1)) = -32$$

   Therefore, since $det(Hf(0,1)) < 0$, $P_3(0,1)$ is a saddle point.

4. $P_4(-1,0)$

$$Hf(-1,0) = \begin{bmatrix} -8 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow det(Hf(-1,0)) = -32$$

   Therefore, since $det(Hf(-1,0)) < 0$, $P_4(-1,0)$ is a saddle point.

5. $P_5(-1,-1)$

$$Hf(-1,-1) = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix} \Rightarrow det(Hf(-1,-1)) = 64$$

   Therefore, since $det(Hf(-1,-1)) > 0$ and $f_{xx}(-1,-1) = -8 < 0$, $P_5(-1,-1)$ is a maximum point.

6. $P_6(-1,1)$

$$Hf(-1,1) = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix} \Rightarrow det(Hf(-1,1)) = 64$$

Therefore, since $det(Hf(-1,1)) > 0$ and $f_{xx}(-1,1) = -8 < 0$, $P_6(-1,1)$ is a maximum point.

7. $P_7(1,0)$

$$Hf(1,0) = \begin{bmatrix} -8 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow det(Hf(1,0)) = -32$$

Therefore, since $det(Hf(1,0)) < 0$, $P_7(1,0)$ is a saddle point.

8. $P_8(1,-1)$

$$Hf(1,-1) = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix} \Rightarrow det(Hf(1,-1)) = 64$$

Therefore, since $det(Hf(1,-1)) > 0$ and $f_{xx}(1,-1) = -8 < 0$, $P_8(1,-1)$ is a maximum point.

9. $P_9(1,1)$

$$Hf(1,1) = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix} \Rightarrow det(Hf(1,1)) = 64$$

Therefore, since $det(Hf(1,1)) > 0$ and $f_{xx}(1,1) = -8 < 0$, $P_9(1,1)$ is a maximum point.

- $f(x,y) = 2x^4 - 16x^2y^2 + x$.
  Start by finding the function's stationary points by solving $\nabla f(x,y) = 0$:

  $$\begin{cases} 8x^3 - 32xy^2 + 1 = 0 \\ -32x^2y = 0 \end{cases} \qquad \text{from the second equation, let } 32x^2y = 0 \Leftrightarrow x = 0 \vee y = 0$$

  Consider two separate systems:

  $$\begin{cases} x = 0 \\ 1 = 0 \end{cases} \vee \begin{cases} y = 0 \\ 8x^3 + 1 = 0 \end{cases}$$

  Therefore, since the first system has no solutions, it is possible to conclude that $P_1(-\frac{1}{2},0)$ is the function's sole stationary point.
  At this point, compute the Hessian matrix:

  $$Hf(x,y) = \begin{bmatrix} 24x^2 - 32y^2 & -64xy \\ -64xy & -32x^2 \end{bmatrix}$$

  Consider the matrix for $P_1(-\frac{1}{2},0)$:

  $$Hf(-\frac{1}{2},0) = \begin{bmatrix} 6 & 0 \\ 0 & -8 \end{bmatrix} \Rightarrow det(Hf(-\frac{1}{2},0)) = -48$$

  Therefore, since $det(Hf(-\frac{1}{2},0)) < 0$, $P_1(-\frac{1}{2},0)$ is a saddle point.

- $f(x,y,z) = (x^3 - x)(y^2 + z^2 - 1)$.
  Start by finding the function's stationary points by solving $\nabla f(x,y,z) = 0$:

  $$\begin{cases} (3x^2-1)(y^2+z^2-1) = 0 \\ 2y(x^3-x) = 0 \\ 2z(x^3-x) = 0 \end{cases} \qquad \text{choose } 2y(x^3-x) = 0 \Leftrightarrow y = 0 \vee x = 0 \vee x = \pm 1$$

  Consider each solution separately to find the stationary points:

  1. $y = 0$

     $$\begin{cases} y = 0 \\ (3x^2-1)(z^2-1) = 0 \\ 2z(x^3-x) = 0 \end{cases} \qquad \text{choose } 2z(x^3-x) = 0 \Leftrightarrow z = 0 \vee x = 0 \vee x = \pm 1$$

     Again, consider each solution separately:

     (a) $z = 0$

     $$\begin{cases} y = 0 \\ z = 0 \\ -(3x^2-1) = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ z = 0 \\ x = \pm\frac{\sqrt{3}}{3} \end{cases}$$

     Therefore, it is possible to find the stationary points $P_1(-\frac{\sqrt{3}}{3}, 0, 0)$ and $P_2(\frac{\sqrt{3}}{3}, 0, 0)$.

     (b) $x = 0$

     $$\begin{cases} y = 0 \\ x = 0 \\ -(z^2-1) = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 0 \\ z = \pm 1 \end{cases}$$

     Therefore, it is possible to find the stationary points $P_3(0,0,-1)$ and $P_4(0,0,1)$.

     (c) $x = -1$

     $$\begin{cases} y = 0 \\ x = -1 \\ 2(z^2-1) = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = -1 \\ z = \pm 1 \end{cases}$$

     Therefore, it is possible to find the stationary points $P_5(-1,0,-1)$ and $P_6(-1,0,1)$.

     (d) $x = 1$

     $$\begin{cases} y = 0 \\ x = 1 \\ 2(z^2-1) = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 1 \\ z = \pm 1 \end{cases}$$

Therefore, it is possible to find the stationary points $P_7(1, 0, -1)$ and $P_8(1, 0, 1)$.

2. $x = 0$

$$\begin{cases} x = 0 \\ -(y^2 + z^2 - 1) = 0 \\ 0 = 0 \end{cases}$$

Since this system has infinite solutions, parametrize $y^2 + z^2 = 1$ as a unit circle and consider $P_9(0, \cos\theta, \sin\theta)$, with $\theta \in [0, 2\pi]$ to be a stationary point for the function.

3. $x = -1$

$$\begin{cases} x = -1 \\ 2(y^2 + z^2 - 1) = 0 \\ 0 = 0 \end{cases}$$

Since this system has infinite solutions, parametrize $y^2 + z^2 = 1$ as a unit circle and consider $P_{10}(-1, \cos\theta, \sin\theta)$, with $\theta \in [0, 2\pi]$ to be a stationary point for the function.

4. $x = 1$

$$\begin{cases} x = 1 \\ 2(y^2 + z^2 - 1) = 0 \\ 0 = 0 \end{cases}$$

Since this system has infinite solutions, parametrize $y^2 + z^2 = 1$ as a unit circle and consider $P_{11}(1, \cos\theta, \sin\theta)$, with $\theta \in [0, 2\pi]$ to be a stationary point for the function.

At this point, compute the Hessian matrix:

$$Hf(x, y, z) = \begin{bmatrix} 6x(y^2 + z^2 - 1) & 2y(3x^2 - 1) & 2z(3x^2 - 1) \\ 2y(3x^2 - 1) & 2(x^3 - x) & 0 \\ 2z(3x^2 - 1) & 0 & 2(x^3 - x) \end{bmatrix}$$

Consider each point separately:

1. $P_1(-\frac{\sqrt{3}}{3}, 0, 0)$

$$Hf(-\frac{\sqrt{3}}{3}, 0, 0) = \begin{bmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & \frac{4\sqrt{3}}{9} & 0 \\ 0 & 0 & \frac{4\sqrt{3}}{9} \end{bmatrix} \Rightarrow \lambda_1 = 2\sqrt{3}, \lambda_{2,3} = \frac{4\sqrt{3}}{9}$$

Therefore, since the Hessian matrix is positive-definite, $P_1(-\frac{\sqrt{3}}{3}, 0, 0)$ is a minimum point.

2. $P_2(\frac{\sqrt{3}}{3}, 0, 0)$

$$Hf(\frac{\sqrt{3}}{3}, 0, 0) = \begin{bmatrix} -2\sqrt{3} & 0 & 0 \\ 0 & -\frac{4\sqrt{3}}{9} & 0 \\ 0 & 0 & -\frac{4\sqrt{3}}{9} \end{bmatrix} \Rightarrow \lambda_1 = -2\sqrt{3}, \lambda_{2,3} = -\frac{4\sqrt{3}}{9}$$

Therefore, since the Hessian matrix is negative-definite, $P_2(\frac{\sqrt{3}}{3}, 0, 0)$ is a maximum point.

3. $P_3(0, 0, -1)$

$$Hf(0, 0, -1) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

In this case, the eigenvalues must be found by solving the characteristic polynomial:

$$\lambda I - Hf = \begin{bmatrix} \lambda & 0 & -2 \\ 0 & \lambda & 0 \\ -2 & 0 & \lambda \end{bmatrix} \Rightarrow det(\lambda I - Hf) = \lambda^3 - 4\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -2, \lambda_3 = 2$$

Therefore, since the matrix is semi-definite, the test is inconclusive for $P_3(0, 0, -1)$.

4. $P_4(0, 0, 1)$

$$Hf(0, 0, 1) = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

In this case, the eigenvalues must be found by solving the characteristic polynomial:

$$\lambda I - Hf = \begin{bmatrix} \lambda & 0 & 2 \\ 0 & \lambda & 0 \\ 2 & 0 & \lambda \end{bmatrix} \Rightarrow det(\lambda I - Hf) = \lambda^3 - 4\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -2, \lambda_3 = 2$$

Therefore, since the matrix is semi-definite, the test is inconclusive for $P_4(0, 0, 1)$.

5. $P_5(-1, 0, -1)$

$$Hf(-1, 0, 1) = \begin{bmatrix} 0 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}$$

In this case, the eigenvalues must be found by solving the characteristic polynomial:

$$\lambda I - Hf = \begin{bmatrix} \lambda & 0 & 4 \\ 0 & \lambda & 0 \\ 4 & 0 & \lambda \end{bmatrix} \Rightarrow det(\lambda I - Hf) = \lambda^3 - 16\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -4, \lambda_3 = 4$$

Therefore, since the matrix is semi-definite, the test is inconclusive for $P_5(-1,0,-1)$.

6. $P_6(-1,0,1)$

$$Hf(-1,0,1) = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

In this case, the eigenvalues must be found by solving the characteristic polynomial:

$$\lambda I - Hf = \begin{bmatrix} \lambda & 0 & -4 \\ 0 & \lambda & 0 \\ -4 & 0 & \lambda \end{bmatrix} \Rightarrow det(\lambda I - Hf) = \lambda^3 - 16\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -4, \lambda_3 = 4$$

Therefore, since the matrix is semi-definite, the test is inconclusive for $P_6(-1,0,1)$.

7. $P_7(1,0,-1)$

$$Hf(1,0,-1) = \begin{bmatrix} 0 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}$$

In this case, the eigenvalues must be found by solving the characteristic polynomial:

$$\lambda I - Hf = \begin{bmatrix} \lambda & 0 & 4 \\ 0 & \lambda & 0 \\ 4 & 0 & \lambda \end{bmatrix} \Rightarrow det(\lambda I - Hf) = \lambda^3 - 16\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -4, \lambda_3 = 4$$

Therefore, since the matrix is semi-definite, the test is inconclusive for $P_7(1,0,-1)$.

8. $P_8(1,0,1)$

$$Hf(1,0,1) = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

In this case, the eigenvalues must be found by solving the characteristic polynomial:

$$\lambda I - Hf = \begin{bmatrix} \lambda & 0 & -4 \\ 0 & \lambda & 0 \\ -4 & 0 & \lambda \end{bmatrix} \Rightarrow det(\lambda I - Hf) = \lambda^3 - 16\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -4, \lambda_3 = 4$$

Therefore, since the matrix is semi-definite, the test is inconclusive for $P_8(1,0,1)$.

9. $P_9(0, \cos\theta, \sin\theta)$

$$Hf(0, \cos\theta, \sin\theta) = \begin{bmatrix} 0 & -2\cos\theta & -2\sin\theta \\ -2\cos\theta & 0 & 0 \\ -2\sin\theta & 0 & 0 \end{bmatrix}$$

In this case, the eigenvalues must be found by solving the characteristic polynomial:

$$\lambda I - Hf = \begin{bmatrix} \lambda & 2\cos\theta & 2\sin\theta \\ 2\cos\theta & \lambda & 0 \\ 2\sin\theta & 0 & \lambda \end{bmatrix} \Rightarrow det(\lambda I - Hf) = \lambda^3 - 4\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_{2,3} = \pm 2$$

Therefore, since the matrix is semi-definite, the test is inconclusive for $P_9(0, \cos\theta, \sin\theta)$.

10. $P_{10}(-1, \cos\theta, \sin\theta)$

$$Hf(-1, \cos\theta, \sin\theta) = \begin{bmatrix} 0 & 4\cos\theta & 4\sin\theta \\ 4\cos\theta & 0 & 0 \\ 4\sin\theta & 0 & 0 \end{bmatrix}$$

In this case, eigenvalues must be found by solving the characteristic polynomial:

$$\lambda I - Hf = \begin{bmatrix} \lambda & -4\cos\theta & -4\sin\theta \\ -4\cos\theta & \lambda & 0 \\ -4\sin\theta & 0 & \lambda \end{bmatrix} \Rightarrow det(\lambda I - Hf) = \lambda^3 - 16\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_{2,3} = \pm 4$$

Therefore, since the matrix is semi-definite, the test is inconclusive for $P_{10}(-1, \cos\theta, \sin\theta)$.

11. $P_{11}(1, \cos\theta, \sin\theta)$

$$Hf(1, \cos\theta, \sin\theta) = \begin{bmatrix} 0 & 4\cos\theta & 4\sin\theta \\ 4\cos\theta & 0 & 0 \\ 4\sin\theta & 0 & 0 \end{bmatrix}$$

In this case, eigenvalues must be found by solving the characteristic polynomial:

$$\lambda I - Hf = \begin{bmatrix} \lambda & -4\cos\theta & -4\sin\theta \\ -4\cos\theta & \lambda & 0 \\ -4\sin\theta & 0 & \lambda \end{bmatrix} \Rightarrow det(\lambda I - Hf) = \lambda^3 - 16\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_{2,3} = \pm 4$$

Therefore, since the matrix is semi-definite, the test is inconclusive for $P_{11}(1, \cos\theta, \sin\theta)$.

- $f(x, y, z) = x^2 + y^2 + (z^2 - 1)^2 + 12$.
  Start by finding the function's stationary points by solving $\nabla f(x, y, z) = 0$:

$$\begin{cases} 2x = 0 \\ 2y = 0 \\ 4z(z^2-1) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \vee z = \pm 1 \end{cases}$$

Therefore, it is possible to conclude that the function's stationary points are $P_1(0,0,0)$, $P_2(0,0,-1)$ and $P_3(0,0,1)$.
At this point, compute the Hessian matrix:

$$Hf(x,y,z) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 12z^2-4 \end{bmatrix}$$

Consider each point separately:

1. $P_1(0,0,0)$

$$Hf(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2, \lambda_3 = -4$$

Therefore, since the Hessian matrix is indefinite, $P_1(0,0,0)$ is a saddle point.

2. $P_2(0,0,-1)$

$$Hf(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2, \lambda_3 = 8$$

Therefore, since the Hessian matrix is positive-definite, $P_2(0,0,-1)$ is a minimum point.

3. $P_3(0,0,1)$

$$Hf(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2, \lambda_3 = 8$$

Therefore, since the Hessian matrix is positive-definite, $P_3(0,0,1)$ is a minimum point.

Chapter 5

Exercise Sheet 5: Curves

5.1 Exercise 1

Consider the following parametric equations:

$$\gamma_1 : \begin{cases} x(t) = 2 - t \\ y(t) = 4t - t^2 \end{cases} \quad , \, t \in [0, 4]$$

$$\gamma_2 : \begin{cases} x(t) = t - 6 \\ y(t) = 4 - t \end{cases} \quad , \, t \in [4, 6]$$

$$\gamma_3 : \begin{cases} x(t) = 3 \cos t \\ y(t) = 5 \sin t \end{cases} \quad , \, t \in [\pi, 2\pi]$$

- Write the Cartesian equations of the curves.

  1. First curve:
     Start by writing t in terms of one of the two parameters:

     $$x(t) = 2 - t \Rightarrow t = 2 - x$$

     At this point, substitute the value of t into the parametric equation of y to recover the corresponding Cartesian equation:

     $$y(t) = 4(2-x)-(2-x)^2 = 8-4x-4+4x-x^2 = 4-x^2 \Rightarrow y = 4-x^2$$

  2. Second curve:
     Start by writing t in terms of one of the two parameters:

     $$x(t) = t - 6 \Rightarrow t = x + 6$$

     At this point, substitute the value of t into the parametric equation of y in order to recover the corresponding Cartesian equation:

     $$y(t) = 4 - (x + 6) = 4 - x - 6 = -x - 2 \Rightarrow y = -x - 2$$

3. Third curve:
   Observe that this is actually the parametrization of an ellipse, meaning that the corresponding Cartesian equation will be given by:

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

- Determine if the curves are closed and/or simple.

1. First curve:
   By definition, the curve is said to be closed in $[0, 4]$ if and only if $\gamma_1(0) = \gamma_1(4)$:

$$\gamma_1(0) = (2, 0) \neq (-2, 0) = \gamma_1(4)$$

   Therefore, the curve is not closed in the interval $[0, 4]$.
   In addition, the curve is said to be simple in $(0, 4)$ if and only if, by picking $a, b \in (0, 4)$, it holds that $\gamma_1(a) = \gamma_1(b)$:

$$\begin{cases} 2 - a = 2 - b \\ 4a - a^2 = 4b - b^2 \end{cases} \Rightarrow \begin{cases} a = b \\ 4b - b^2 = 4b - b^2 \end{cases}$$

   Therefore, since the condition is satisfied, the curve is indeed simple in $(0, 4)$.
   N.B.: Alternatively, it is possible to directly confirm that the curve is simple by noticing that the first component is an injective function.

2. Second curve:
   By definition, the curve is said to be closed in $[4, 6]$ if and only if $\gamma_2(4) = \gamma_2(6)$:

$$\gamma_2(4) = (-2, 0) \neq (0, -2) = \gamma_2(6)$$

   Therefore, the curve is not closed in the interval $[4, 6]$.
   In addition, the curve is said to be simple in $(4, 6)$ if and only if, by picking $a, b \in (4, 6)$, it holds that $\gamma_2(a) = \gamma_2(b) \Leftrightarrow a = b$:

$$\begin{cases} a - 6 = b - 6 \\ 4 - a = 4 - b \end{cases} \Rightarrow \begin{cases} a = b \\ a = b \end{cases}$$

   Therefore, since the condition is satisfied, the curve is indeed simple in $(4, 6)$.
   N.B.: Alternatively, it is possible to directly confirm that the curve is simple by noticing that both its components are injective functions.

3. Third curve:
   By definition, the curve is said to be closed in $[\pi, 2\pi]$ if and only if $\gamma_3(\pi) = \gamma_3(2\pi)$:

$$\gamma_3(\pi) = (-3, 0) \neq \gamma_3(2\pi) = (3, 0)$$

Therefore, the curve is not closed in the interval $[\pi, 2\pi]$.
In addition, the curve is said to be simple in $(\pi, 2\pi)$ if and only if, by picking $a, b \in (\pi, 2\pi)$, it holds that $\gamma_3(a) = \gamma_3(b) \Leftrightarrow a = b$:

$$\begin{cases} 3\cos a = 3\cos b \\ 5\sin a = 5\sin b \end{cases} \Rightarrow \begin{cases} a = -b \\ -\sin b = \sin b \end{cases} \vee \begin{cases} a = b \\ \sin b = \sin b \end{cases}$$

Therefore, since the condition is satisfied, the curve is indeed simple in $(\pi, 2\pi)$.

- Find the tangent vector and the unit tangent vector to γ_1 at the point $P(1, 3)$.
Start by finding the point $t_0 \in [0, 4]$ such that $\gamma_1(t_0) = P$:

$$\begin{cases} 2 - t_0 = 1 \\ 4t_0 - t_0^2 = 3 \end{cases} \Rightarrow \begin{cases} t_0 = 1 \\ 4 - 1 = 3 \end{cases} \Rightarrow t_0 = 1 \in [0, 4]$$

At this point, find the generic parametric equation of the tangent vector of γ_1:

$$\gamma_1' : \begin{cases} x'(t) = -1 \\ y'(t) = 4 - 2t \end{cases}, \ t \in [0, 4]$$

Therefore, by plugging into the equation $t_0 = x$, it is possible to find the tangent vector at the point P as:

$$\gamma_1'(1) = (-1, 2)$$

Lastly, it is possible to recover the unit tangent vector at the point P by normalizing the tangent vector:

$$\|\gamma_1'(1)\| = \sqrt{1 + 4} = \sqrt{5} \Rightarrow \gamma_{1_u}'(1) = (-\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5})$$

- Find the tangent vector and the unit tangent vector to γ_3 at the point $P(-\frac{3\sqrt{3}}{2}, -\frac{5}{2})$.
Start by finding the point $t_0 \in [\pi, 2\pi]$ such that $\gamma_3(t_0) = P$:

$$\begin{cases} 3\cos t_0 = -\frac{3\sqrt{3}}{2} \\ 5\sin t_0 = -\frac{5}{2} \end{cases} \Rightarrow \begin{cases} \cos t_0 = -\frac{\sqrt{3}}{2} \\ \sin t_0 = -\frac{1}{2} \end{cases} \Rightarrow t_0 = \frac{7\pi}{6} \in [\pi, 2\pi]$$

At this point, find the generic parametric equation of the tangent vector of γ_3:

$$\gamma_3' : \begin{cases} x'(t) = -3\sin t \\ y'(t) = 5\cos t \end{cases}$$

Therefore, by plugging into the equation $t_0 = \frac{7\pi}{6}$, it is possible to find the tangent vector at the point P as:

$$\gamma_3'(\frac{7\pi}{6}) = (\frac{3}{2}, -\frac{5\sqrt{3}}{2})$$

Lastly, it is possible to recover the unit tangent vector at the point P by normalizing the tangent vector:

$$\|\gamma_3'(\frac{7\pi}{6})\| = \sqrt{\frac{9}{4} + \frac{75}{4}} = \sqrt{21} \Rightarrow \gamma_{3_u}'(\frac{7\pi}{6}) = (\frac{\sqrt{21}}{14}, -\frac{5\sqrt{7}}{14})$$

5.2 Exercise 2

Consider the curve $\gamma(t) = (e^t, \sqrt{2}t)$, for $t \in [-1, 1]$.

- Determine if the curve is closed.
  By definition, the curve is said to be closed in $[-1, 1]$ if and only if $\gamma(-1) = \gamma(1)$:

  $$\gamma(-1) = (e^{-1}, -\sqrt{2}) \neq (e, \sqrt{2}) = \gamma(1)$$

  Therefore, the curve is not closed in $[-1, 1]$.

- Determine if the curve is simple.
  By definition, the curve is said to be simple in $(-1, 1)$ if and only if, by picking $a, b \in (-1, 1)$, it holds that $\gamma(a) = \gamma(b) \Leftrightarrow a = b$:

  $$\begin{cases} e^a = e^b \\ \sqrt{2}a = \sqrt{2}b \end{cases} \Rightarrow \begin{cases} a = b \\ a = b \end{cases}$$

  Therefore, since the condition is satisfied, the curve is indeed simple in $(-1, 1)$.
  N.B.: Alternatively, it is possible to directly confirm that the curve is simple by noticing that both components are injective functions.

- Determine if the curve is regular.
  By definition, the curve is said to be regular in $(-1, 0)$ if and only if the tangent vector $\gamma'(t)$ is component-wise continuous and $\gamma'(t) \neq 0 \; \forall \; t \in (-1, 1)$.
  Start by computing $\gamma'(t) = (e^t, \sqrt{2})$ and study $\gamma'(t) = 0$:

  $$\begin{cases} e^t = 0 \\ \sqrt{2} = 0 \end{cases} \text{, which is an incompatible system.}$$

  Therefore, since the condition is satisfied, the curve is indeed regular in $(-1, 1)$.

- Determine the tangent vector at the point $P(1,0)$.
  Start by finding the point $t_0 \in [-1,1]$ such that $\gamma(t_0) = P$:

$$\begin{cases} e^{t_0} = 1 \\ \sqrt{2}t_0 = 0 \end{cases} \Rightarrow t_0 = 0 \in [-1,1]$$

  Therefore, given the generic equation of the tangent vector $\gamma'(t) = (e^t, \sqrt{2})$, it is possible to conclude that $\gamma'(0) = (1, \sqrt{2})$.

- Determine the Cartesian equation of the curve.
  Start by writing t in terms of one of the two parameters:

$$x(t) = e^t \Rightarrow t = \ln x$$

  At this point, substitute the value of t into the parametric equation of y to recover the corresponding Cartesian equation:

$$y(t) = \sqrt{2}\ln x \Rightarrow y = \sqrt{2}\ln x$$

5.3 Exercise 3

Compute the length of the following curves:

- First curve:

$$\gamma : \begin{cases} x(t) = t^2 \\ y(t) = \frac{t^3}{3} - t \end{cases} , t \in [0,1]$$

  Start by computing the curve's tangent vector and show that it is regular in $(0,1)$:

$$\gamma'(t) : \begin{cases} x'(t) = 2t \\ y'(t) = t^2 - 1 \end{cases} \Rightarrow \begin{cases} 2t = 0 \\ t^2 - 1 = 0 \end{cases} \text{, which is an incompatible system.}$$

  At this point, find the norm of the tangent vector:

$$\|\gamma'(t)\| = \sqrt{4t^2 + t^4 - 2t^2 + 1} = \sqrt{t^4 + 2t^2 + 1} = \sqrt{(t^2+1)^2} = t^2 + 1$$

  Therefore, it is possible to conclude that:

$$L(\gamma) = \int_0^1 t^2 + 1 \ dt = [\frac{t^3}{3} + t]_0^1 = \frac{4}{3}$$

- Second curve:

$$\gamma : \begin{cases} x(t) = 4t^3 - 3t \\ y(t) = 6t^2 \end{cases} , t \in [0,1]$$

Start by computing the curve's tangent vector and show that it is regular in $(0,1)$:

$$\gamma' : \begin{cases} x'(t) = 12t^2 - 3 \\ y'(t) = 12t \end{cases} \Rightarrow \begin{cases} 12t^2 - 3 = 0 \\ 12t = 0 \end{cases} \text{, which is an incompatible system.}$$

At this point, find the norm of the tangent vector:

$$\|\gamma'(t)\| = \sqrt{144t^4 - 72t^2 + 9 + 144t^2} = \sqrt{144t^4 + 72t^2 + 9} = \sqrt{(12t^2+3)^2} = 12t^2+3$$

Therefore, it is possible to conclude that:

$$L(\gamma) = \int_0^1 12t^2 + 3 \; dt = [4t^3 + 3t]_0^1 = 7$$

5.4 Exercise 4

Consider the curve $r : \mathbb{R} \to \mathbb{R}^2$ with parametric equations:

$$r(t) = (t^2, t^3)$$

- Determine if the curve is closed.
  Observe that the curve cannot be closed in $\mathbb{R}$ because, by taking the limits for $t \to -\infty$ and $t \to +\infty$, it is possible to notice that:

$$\lim_{t \to -\infty} r(t) = (+\infty, -\infty) \neq (+\infty, +\infty) = \lim_{t \to +\infty} r(t)$$

- Determine if the curve is simple.
  By definition, the curve is said to be simple in $\mathbb{R}$ if and only if, by picking $a, b \in \mathbb{R}$, it holds that $r(a) = r(b) \Leftrightarrow a = b$:

$$\begin{cases} a^2 = b^2 \\ a^3 = b^3 \end{cases} \Rightarrow \begin{cases} a = -b \\ -b^3 = b^3 \end{cases} \vee \begin{cases} a = b \\ b^3 = b^3 \end{cases}$$

  Therefore, since the condition is satisfied, the curve is indeed simple in $\mathbb{R}$.
  N.B.: Alternatively, it is possible to directly confirm that the curve is simple by noticing that the second component is an injective function.

- Determine if the curve is regular.
  By definition, the curve is said to be regular in $\mathbb{R}$ if and only if the tangent vector $r'(t)$ is component-wise continuous and $r'(t) \neq 0 \; \forall \; t \in \mathbb{R}$.
  Start by computing $r'(t) = (2t, 3t^2)$ and study $r'(t) = 0$:

$$\begin{cases} 2t = 0 \\ 3t^2 = 0 \end{cases} \Leftrightarrow t = 0 \in \mathbb{R}$$

  Therefore, since the condition is not satisfied, the curve cannot be regular in $\mathbb{R}$.

- Compute the length of the curve starting from the point $P_1(0,0)$ to the point $P_2(1,1)$.
  Start by finding the points $a, b \in \mathbb{R}^3$ such that $r(a) = P_1$ and $r(b) = P_2$:

$$\begin{cases} a^2 = 0 \\ a^3 = 0 \end{cases} \Leftrightarrow a = 0$$

$$\begin{cases} b^2 = 1 \\ b^3 = 1 \end{cases} \Leftrightarrow b = 1$$

  Show that the curve is regular at least in $(0,1)$:

$$r'(t) = (2t, 3t^2) \Rightarrow \begin{cases} 2t = 0 \\ 3t^2 = 0 \end{cases} \Leftrightarrow t = 0 \notin (0,1)$$

  At this point, compute the norm of the tangent vector:

$$\|r'(t)\| = \sqrt{4t^2 + 9t^4} = \sqrt{t^2(4 + 9t^2)} = t\sqrt{4 + 9t^2}$$

  Therefore, it is possible to conclude that:

$$L(r) = \int_0^1 t\sqrt{4 + 9t^2}\ dt$$

  Apply a change of variable: let $u = 4 + 9t^2$, for $u \in [4, 13]$, and consider $\frac{du}{dt} = 18t$, resulting in:

$$L(r) = \frac{1}{18}\int_4^{13} \sqrt{u}\ du = \frac{1}{18}\frac{2}{3}[u^{\frac{3}{2}}]_4^{13} = \frac{1}{27}(\sqrt{13^3} - \sqrt{4^3}) = \frac{(\sqrt{13^3} - 8)}{27}$$

5.5 Exercise 5

Compute the length of the curve $y = \frac{2}{3}\sqrt{(x-1)^3}$, for $x \in [1, 4]$.
Start by parametrizing the Cartesian equation of the curve by setting $x(t) = t$ and plug in this value into the equation of $y(t)$, resulting in:

$$\gamma : \begin{cases} x(t) = t \\ y(t) = \frac{2}{3}\sqrt{(t-1)^3} \end{cases}, \ t \in [1, 4]$$

Compute the equation of the tangent vector and show that it is regular in $(1, 4)$:

$$\gamma' : \begin{cases} x'(t) = 1 \\ y'(t) = \sqrt{t-1} \end{cases} \Rightarrow \begin{cases} 1 = 0 \\ \sqrt{t-1} = 0 \end{cases}, \text{ which is an incompatible system.}$$

At this point, compute the norm of the tangent vector:

$$\|\gamma'(t)\| = \sqrt{1 + t - 1} = \sqrt{t}$$

Therefore, it is possible to conclude that:

$$L(\gamma) = \int_1^4 \sqrt{t}\ dt = [\frac{t^{\frac{3}{2}}}{\frac{3}{2}}]_1^4 = \frac{2}{3}(\sqrt{64} - 1) = \frac{2}{3}(8-1) = \frac{14}{3}$$

5.6 Exercise 6

Let $r : [-1, 0] \to \mathbb{R}^3$ be the curve in space with parametric equation:

$$r(t) = (3t^2, 1 + 3t, 2t^3)$$

- Determine if the curve is closed.
  By definition, the curve is said to be closed in $[-1, 0]$ if and only if $r(-1) = r(0)$:

  $$r(-1) = (3, -2, -2) \neq (0, 1, 0) = r(0)$$

  Therefore, the curve is not closed in the interval $[-1, 0]$.

- Determine if the curve is simple.
  By definition, the curve is said to be simple in $(-1, 0)$ if and only if, by picking $a, b \in (-1, 0)$, it holds that $r(a) = r(b) \Leftrightarrow a = b$:

  $$\begin{cases} 3a^2 = 3b^2 \\ 1 + 3a = 1 + 3b \\ 2a^3 = 2b^3 \end{cases} \Rightarrow \begin{cases} b^2 = b^2 \\ a = b \\ b^3 = b^3 \end{cases}$$

  Therefore, since the condition is satisfied, the curve is indeed simple in $(-1, 0)$.
  N.B.: Alternatively, it is possible to directly confirm that the curve is simple by noticing that the second and third components are injective functions and that the first component, while not injective in $\mathbb{R}$, is injective at least in $(-1, 0)$.

- Determine if the curve is regular.
  By definition, the curve is said to be regular in $(-1, 0)$ if and only if the tangent vector $r'(t)$ is component-wise continuous and $r'(t) \neq 0\ \forall\ t \in (-1, 0)$.
  Start by computing $r'(t) = (6t, 3, 6t^2)$ and study $r'(t) = 0$:

  $$\begin{cases} 6t = 0 \\ 3 = 0 \\ 6t^2 = 0 \end{cases} \text{, which is an incompatible system.}$$

  Therefore, since the condition is satisfied, the curve is indeed regular in $(-1, 0)$.

- Compute the length of the curve.
  Knowing that the curve is regular in $(-1, 0)$, start by computing the norm of the tangent vector:

$$\|r'(t)\| = \sqrt{36t^2 + 9 + 36t^4} = \sqrt{(6t^2 + 3)^2} = 6t^2 + 3$$

  Therefore, it is possible to conclude that:

$$L(r) = \int_{-1}^{0} 6t^2 + 3 \; dt = [2t^3 + 3t]_{-1}^{0} = 5$$

- Compute the tangent vector to $r(t)$ at the point $P(0, 1, 0)$.
  Start by finding the point $t_0 \in [-1, 0]$ such that $r(t_0) = P$:

$$\begin{cases} 3t_0^2 = 0 \\ 1 + 3t_0 = 1 \\ 2t_0^3 = 0 \end{cases} \Rightarrow t_0 = 0 \in [-1, 0]$$

  Therefore, given the tangent vector $r'(t) = (6t, 3, 6t^2)$, it is possible to conclude that $r'(0) = (0, 3, 0)$.

5.7 Exercise 7

Compute the length of the segment of $\mathbb{R}^3$ that links $P_1(1, 2, 3)$ to $P_2(0, 4, 1)$.
Start by considering the generic parametric equation of a segment in $\mathbb{R}^3$:

$$\gamma : \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} \text{, with } a, b, c \in \mathbb{R}.$$

Assume that the segment is defined in $[0, 1]$, and, by picking $(x_0, y_0, z_0) = P_1$, rewrite this equation in the following way:

$$\gamma : \begin{cases} x = 1 + at \\ y = 2 + bt \\ z = 3 + ct \end{cases} \quad \text{with } t = 0$$

However, the segment must also pass through P_2, meaning that it is possible to find a, b, c by solving the following system:

$$\begin{cases} 0 = 1 + at \\ 4 = 2 + bt \\ 1 = 3 + ct \end{cases} \quad \text{with } t = 1 \Rightarrow \begin{cases} a = -1 \\ b = 2 \\ c = -2 \end{cases}$$

Therefore, the parametric equation of the segment will be given by:

$$\gamma : \begin{cases} x = 1 - t \\ y = 2 + 2t \\ z = 3 - 2t \end{cases}$$

At this point, compute the tangent vector and find its norm:

$$\gamma' : \begin{cases} x' = -1 \\ y' = 2 \\ z' = -2 \end{cases} \quad \Rightarrow \|\gamma'(t)\| = \sqrt{1+4+4} = \sqrt{9} = 3$$

Therefore, it is possible to conclude that:

$$L(\gamma) = \int_0^1 3 \; dt = [3t]_0^1 = 3$$

Chapter 6

Exercise Sheet 6: Curves and line integrals

6.1 Exercise 1

Let γ be the curve with Cartesian equation $y = x^3$, with $x \in [0, 1]$:

- Write the parametric equation for γ.
  Start by parametrizing $x(t) = t$, for $t \in [0, 1]$, allowing to conclude that:

$$\gamma : \begin{cases} x(t) = t \\ y(t) = t^3 \end{cases} \quad , t \in [0, 1]$$

- Determine if the curve is simple or regular.
  By definition, the curve is said to be simple in $(0, 1)$ if and only if, by picking $a, b \in (0, 1)$, it holds that $\gamma(a) = \gamma(b) \Leftrightarrow a = b$:

$$\begin{cases} a = b \\ a^3 = b^3 \end{cases} \quad \Leftrightarrow a = b$$

  Therefore, since the condition is satisfied, the curve is indeed simple in $(0, 1)$.
  N.B.: Alternatively, it is possible to directly confirm that the curve is simple by noticing that both components are injective functions in the interval $(0, 1)$.
  On the other hand, the curve us defined to be regular in the interval $(0, 1)$ if and only if the tangent vector $\gamma'(t)$ is component-wise continuous and $\gamma'(t) \neq 0 \ \forall \ t \in (0, 1)$.
  Start by computing $\gamma'(t) = (1, 3t^2)$ and study $\gamma'(t) = 0$:

$$\begin{cases} 1 = 0 \\ 3t^2 = 0 \end{cases} \quad \text{, which is an incompatible system.}$$

Therefore, since the condition is satisfied, the curve is indeed regular in $(0,1)$.

- Compute the line integral of $f(x,y) = x^3 + y$ with respect to γ.
Knowing that $\gamma'(t) = (1, 3t^2)$, let $\|\gamma'(t)\| = \sqrt{1+9t^4}$ and compute the following integral:

$$\int_\gamma x^3 + y \ dt = \int_0^1 (t^3 + t^3)\sqrt{1+9t^4} \ dt = \int_0^1 2t^3\sqrt{1+9t^4} \ dt$$

Apply a change of variable and let $u = 1+9t^4$, for $u = [1,,10]$, and recover $\frac{du}{dt} = 36t^3$, allowing to conclude that:

$$\frac{2}{36}\int_1^{10} \sqrt{u} \ du = \frac{1}{18}[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}]_1^{10} = \frac{1}{27}((10)^{\frac{3}{2}} - 1) = \frac{\sqrt{1000}-1}{27}$$

6.2 Exercise 2

Let γ be the arc of a hyperbola $y = \frac{1}{x}$, with $x \in [\frac{1}{2}, 2]$:

- Write the parametric equation for γ.
Start by parametrizing $x(t) = t$, for $t \in [\frac{1}{2}, 2]$, allowing to conclude that:

$$\gamma : \begin{cases} x(t) = t \\ y(t) = \frac{1}{t} \end{cases}, t \in [\frac{1}{2}, 2]$$

- Determine if the curve is regular.
By definition, the curve is said to be regular in the interval $(\frac{1}{2}, 2)$ if and only if the tangent vector $\gamma'(t)$ is component-wise continuous and $\gamma'(t) \neq 0 \ \forall \ t \in [\frac{1}{2}, 2]$.
Start by computing $\gamma'(t) = (1, -\frac{1}{t^2})$ and study $\gamma'(t) = 0$:

$$\begin{cases} 1 = 0 \\ -\frac{1}{t^2} = 0 \end{cases}, \text{which is an incompatible system.}$$

Therefore, since the condition is satisfied, the curve is indeed regular in $(\frac{1}{2}, 2)$.

- Compute the line integral of $f(x,y) = x^5$ with respect to γ.
Knowing that $\gamma'(t) = (1, -\frac{1}{t^2})$, let $\|\gamma'(t)\| = \frac{\sqrt{t^4+1}}{t^2}$ and compute the following integral:

$$\int_\gamma x^5 \ dt = \int_{\frac{1}{2}}^2 t^5 \frac{\sqrt{t^4+1}}{t^2} \ dt = \int_{\frac{1}{2}}^2 t^3\sqrt{t^4+1} \ dt$$

Apply a change of variable and let $u = t^4 + 1$, for $t \in [\frac{17}{16}, 17]$, and recover $\frac{du}{dt} = 4t^3$, allowing to conclude that:

$$\frac{1}{4}\int_{\frac{17}{16}}^{17} \sqrt{u}\ du = \frac{1}{4}[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}]_{\frac{17}{16}}^{17} = \frac{1}{6}((17)^{\frac{3}{2}} - (\frac{17}{16})^{\frac{3}{2}}) = \frac{\sqrt{17^3} - \sqrt{(\frac{17}{16})^3}}{6}$$

6.3 Exercise 3

Compute the following line integrals:

- $f(x, y) = x^5$, with respect to the curve γ given by $y = \frac{1}{x}$, for $x \in [\frac{1}{2}, 2]$.
  Start by parametrizing γ from the Cartesian equation:

  $$\gamma : \begin{cases} x(t) = t \\ y(t) = \frac{1}{t} \end{cases}, \ t \in [\frac{1}{2}, 2]$$

  Compute the norm of the tangent vector:

  $$\gamma'(t) = (1, -\frac{1}{t^2}) \Rightarrow \|\gamma'(t)\| = \sqrt{1 + \frac{1}{t^4}} = \sqrt{\frac{t^4+1}{t^4}} = \frac{\sqrt{t^4+1}}{t^2}$$

  Therefore, it is possible to compute the line integral in the following way:

  $$\int_\gamma x^5\ dt = \int_{\frac{1}{2}}^{2} t^5 \frac{\sqrt{t^4+1}}{t^2}\ dt = \int_{\frac{1}{2}}^{2} t^3\sqrt{t^4+1}\ dt$$

  Apply a change of variable: let $u = t^4 + 1$, for $u \in [\frac{17}{16}, 17]$, and consider $\frac{du}{dt} = 4t^3$, allowing to conclude that:

  $$\frac{1}{4}\int_{\frac{17}{16}}^{17} \sqrt{u}\ du = \frac{1}{4}\frac{2}{3}[u^{\frac{3}{2}}]_{\frac{17}{16}}^{17} = \frac{\sqrt{17^3} - \sqrt{(\frac{17}{16})^3}}{6}$$

- $f(x, y) = \sin(xy)$, with respect to the square with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$.
  By plotting the square, let $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ and consider each curve separately:

  1. Let $\gamma_1(t) = (t, 0)$, for $t \in [0, 1]$.
     Compute the norm of the tangent vector:

     $$\gamma_1'(t) = (1, 0) \Rightarrow \|\gamma_1'(t)\| = \sqrt{1+0} = 1$$

     Therefore, it is possible to compute the line integral in the following way:

     $$\int_{\gamma_1} \sin(xy)\ dt = \int_0^1 \sin(0t)\ dt = \int_0^1 0\ dt = 0$$

2. Let $\gamma_2(t) = (1, t)$, for $t \in [0, 1]$.
   Compute the norm of the tangent vector:

   $$\gamma_2'(t) = (0, 1) \Rightarrow \|\gamma_2'(t)\| = \sqrt{0+1} = 1$$

   Therefore, it is possible to compute the line integral in the following way:

   $$\int_{\gamma_2} \sin(xy)\ dt = \int_0^1 \sin t\ dt = [-\cos t]_0^1 = 1 - \cos 1$$

3. Let $\gamma_3(t) = (t, 1)$, for $t \in [1, 0]$ (in this case, the direction of traversal is inverted).
   Compute the norm of the tangent vector:

   $$\gamma_3'(t) = (1, 0) \Rightarrow \|\gamma_3'(t)\| = \sqrt{1+0} = 1$$

   Therefore, it is possible to compute the line integral in the following way:

   $$\int_{\gamma_3} \sin(xy)\ dt = \int_1^0 \sin t\ dt = -\int_0^1 \sin t\ dt = -[-\cos t]_0^1 = \cos 1 - 1$$

4. Let $\gamma_4(t) = (0, t)$, for $t \in [1, 0]$ (in this case, the direction of traversal is inverted).
   Compute the norm of the tangent vector:

   $$\gamma_4'(t) = (0, 1) \Rightarrow \|\gamma_4'(t)\| = \sqrt{0+1} = 1$$

   Therefore, it is possible to compute the line integral in the following way:

   $$\int_{\gamma_4} \sin(xy)\ dt = \int_1^0 \sin(0t)\ dt = -\int_0^1 0\ dt = 0$$

Therefore, by applying integral additivity, it is possible to conclude that:

$$\int_{\gamma} \sin(xy)\ dt = 0 + (1 - \cos 1) + (\cos 1 - 1) + 0 = 0$$

- $f(x, y) = \sqrt{x^2 + y^2}$, with respect to $\gamma = (2(\cos t + t \sin t), 2(\sin t - t \cos t))$ for $t \in [0, 2\pi]$.
  Start by computing the norm of the tangent vector:

  $$\gamma'(t) = (2t \cos t, 2t \sin t) \Rightarrow \|\gamma'(t)\| = \sqrt{4t^2 \cos^2 t + 4t^2 \sin^2 t} = 2t$$

  Therefore, it is possible to compute the line integral in the following way:

  $$\int_{\gamma} \sqrt{x^2 + y^2}\ dt = \int_0^{2\pi} \sqrt{(2(\cos t + t \sin t))^2 + (2(\sin t - t \cos t))^2} \cdot 2t\ dt = \int_0^{2\pi} 2\sqrt{t^2 + 1} \cdot 2t\ dt$$

Apply a change of variable: let $u = t^2+1$, for $u \in [1, 4\pi^2+1]$, and consider $\frac{du}{dt} = 2t$, allowing to conclude that:

$$2\int_1^{4\pi^2+1} \sqrt{u}\ du = 2\frac{2}{3}[u^{\frac{3}{2}}]_1^{4\pi^2+1} = \frac{4(\sqrt{(4\pi^2+1)^3}-1)}{3}$$

6.4 Exercise 4

Given the curve γ with parametric equation:

$$\gamma : \begin{cases} x = t^2 \\ y = \frac{t^3}{3} - 1 \end{cases}, \ t \in [0,1]$$

- Determine if the curve is closed, simple or regular.
  By definition, the curve will be closed in $[0,1]$ if and only if $\gamma(0) = \gamma(1)$:

  $$\gamma(0) = (0,-1) \neq (1, -\frac{2}{3}) = \gamma(1)$$

  Therefore, the curve is not closed in the interval $[0,1]$.
  In addition, the curve is said to be simple in $(0,1)$ if and only if, by picking $a, b \in (0,1)$ it holds that $\gamma(a) = \gamma(b) \Leftrightarrow a = b$:

  $$\begin{cases} a^2 = b^2 \\ \frac{a^3}{3} - 1 = \frac{b^3}{3} - 1 \end{cases} \Rightarrow \begin{cases} a = -b \\ -\frac{b^3}{3} = \frac{b^3}{3} \end{cases} \vee \begin{cases} a = b \\ \frac{b^3}{3} = \frac{b^3}{3} \end{cases}$$

  Therefore, since the condition is satisfied, the curve is indeed simple in $(0,1)$.
  N.B.: Alternatively, it is possible to directly confirm that the curve is simple by noticing that the second component is an injective function and, similarly, the first component, while not injective in $\mathbb{R}$, is injective at least in $(0,1)$.
  Lastly, the curve is said to be regular in $(0,1)$ if and only if the tangent vector $\gamma'(t)$ is component-wise continuous and $\gamma'(t) \neq 0 \ \forall \ t \in (0,1)$.
  Start by computing $\gamma'(t) = (2t, t^2)$ and study $\gamma'(t) = 0$:

  $$\begin{cases} 2t = 0 \\ t^2 = 0 \end{cases} \Leftrightarrow t = 0 \notin (0,1)$$

  Therefore, since the condition is satisfied, the curve is indeed regular in $(0,1)$.

- Compute $L(\gamma)$.
  Knowing that the curve is regular in its domain, it is possible to compute its length through its norm:

  $$\|\gamma'(t)\| = \sqrt{(2t)^2 + (t^2)^2} = t\sqrt{4+t^2}$$

Therefore, it is possible to conclude that:

$$L(\gamma) = \int_0^1 \|\gamma'(t)\| \, dt = \int_0^1 t\sqrt{4+t^2} \, dt$$

Apply a change of variable and let $u = 4 + t^2$, for $u \in [4, 5]$, and recover $\frac{du}{dt} = 2t$, resulting in:

$$L(\gamma) = \frac{1}{2}\int_4^5 \sqrt{u} \, du = \frac{1}{2}[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}]_4^5 = \frac{1}{3}(5^{\frac{3}{2}} - 4^{\frac{3}{2}}) = \frac{\sqrt{125} - 8}{3}$$

- Compute the following line integral:

$$\int_\gamma \frac{e^{x^{\frac{3}{2}} - 3y}}{x+4} \, dt$$

By using the information recovered about the tangent vector, it is possible to compute the integral in the following way:

$$\int_\gamma \frac{e^{x^{\frac{3}{2}} - 3y}}{x+4} \, dt = \int_0^1 \frac{e^{t^{2 \cdot \frac{3}{2}} - 3(\frac{t^3}{3} - 1)}}{t^2+4} \cdot t\sqrt{t^2+4} \, dt = \int_0^1 \frac{e^3 t}{\sqrt{t^2+4}} \, dt$$

Apply a change of variable and let $u = t^2 + 4$, for $u \in [0, 5]$, and recover $\frac{du}{dt} = 2t$, allowing to conclude that:

$$\frac{e^3}{2}\int_4^5 \frac{1}{\sqrt{u}} \, du = \frac{e^3}{2}[2\sqrt{u}]_4^5 = e^3(\sqrt{5} - \sqrt{4})$$

6.5 Exercise 5

Compute the following line integrals:

- $f(x, y) = x$ with respect to $\gamma = (t, t^2)$, for $t \in [0, 2]$.
  Start by finding the norm of the tangent vector:

$$\gamma'(t) = (1, 2t) \Rightarrow \|\gamma'(t)\| = \sqrt{1 + 4t^2}$$

Therefore, it is possible to compute the line integral in the following way:

$$\int_\gamma x \, dt = \int_0^2 t\sqrt{1+4t^2} \, dt$$

Apply a change of variable: let $u = 1 + 4t^2$, for $u \in [1, 17]$, and consider $\frac{du}{dt} = 8t$, resulting in:

$$\frac{1}{8}\int_1^{17} \sqrt{u} \, du = \frac{1}{8}\frac{2}{3}[u^{\frac{3}{2}}]_1^{17} = \frac{\sqrt{17^3} - 1}{12}$$

- $f(x, y) = \sqrt{1 - y^2}$ with respect to $\gamma = (\sin t, \cos t)$, for $t \in [0, \pi]$.
  Start by finding the norm of the tangent vector:

$$\gamma'(t) = (\cos t, -\sin t) \Rightarrow \|\gamma'(t)\| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

  Therefore, it is possible to compute the line integral in the following way:

$$\int_\gamma \sqrt{1 - y^2}\ dt = \int_0^\pi \sqrt{1 - \cos^2 t}\ dt = \int_0^\pi \sin t\ dt = [-\cos t]_0^\pi = 2$$

- $f(x, y) = \frac{x}{1+y^2}$ with respect to $\gamma = (\cos t, \sin t)$, for $t \in [0, \frac{\pi}{2}]$.
  Start by finding the norm of the tangent vector:

$$\gamma'(t) = (-\sin t, \cos t) \Rightarrow \|\gamma'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

  Therefore, it is possible to compute the line integral in the following way:

$$\int_\gamma \frac{x}{1 + y^2}\ dt = \int_0^{\frac{\pi}{2}} \frac{\cos t}{1 + \sin^2 t}\ dt$$

  Apply a change of variable: let $u = \sin t$, for $u \in [0, 1]$, and consider $\frac{du}{dt} = \cos t$, resulting in:

$$\int_0^1 \frac{1}{1 + u^2}\ du = [\arctan u]_0^1 = \frac{\pi}{4}$$

- $f(x, y) = y^2$ with respect to $\gamma = (t, e^t)$, for $t \in [0, \ln 2]$.
  Start by finding the norm of the tangent vector:

$$\gamma'(t) = (1, e^t) \Rightarrow \|\gamma'(t)\| = \sqrt{1 + (e^t)^2} = \sqrt{1 + e^{2t}}$$

  Therefore, it is possible to compute the line integral in the following way:

$$\int_\gamma y^2\ dt = \int_0^{\ln 2} e^{2t}\sqrt{1 + e^{2t}}\ dt$$

  Apply a change of variable: let $u = 1 + e^{2t}$, for $u \in [2, 5]$, and consider $\frac{du}{dt} = 2e^{2t}$, resulting in:

$$\frac{1}{2}\int_2^5 \sqrt{u}\ du = \frac{1}{2}\frac{2}{3}[u^{\frac{3}{2}}]_2^5 = \frac{\sqrt{125} - \sqrt{8}}{3}$$

- $f(x, y, z) = x + z$ with respect to $\gamma = (t, \frac{3\sqrt{2}}{2}t^2, t^3)$, for $t \in [0, 1]$.
  Start by finding the norm of the tangent vector:

$$\gamma'(t) = (1, 3\sqrt{2}t, 3t^2) \Rightarrow \|\gamma'(t)\| = \sqrt{1 + 18t^2 + 9t^4}$$

  Therefore, it is possible to compute the line integral in the following way:

$$\int_\gamma x + z\ dt = \int_0^1 (t + t^3)\sqrt{1 + 18t^2 + 9t^4}\ dt$$

Apply a change of variable: let $u = 1 + 18t^2 + 9t^4$, for $u \in [1, 28]$, and consider $\frac{du}{dt} = 36t + 36t^3 = 36(t + t^3)$, resulting in:

$$\frac{1}{36}\int_1^{28} \sqrt{u}\ du = \frac{1}{36}\frac{2}{3}[u^{\frac{3}{2}}]_1^{28} = \frac{\sqrt{28^3} - 1}{54}$$

- $f(x, y, z) = \sqrt{z}$ with respect to $\gamma = (\cos t, \sin t, t^2)$, for $t \in [0, \pi]$.
  Start by finding the norm of the tangent vector:

  $$\gamma'(t) = (-\sin t, \cos t, 2t) \Rightarrow \|\gamma'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4t^2} = \sqrt{1 + 4t^2}$$

  Therefore, it is possible to compute the line integral in the following way:

  $$\int_\gamma \sqrt{z}\ dt = \int_0^\pi \sqrt{t^2}\sqrt{1 + 4t^2}\ dt = \int_0^\pi t\sqrt{1 + 4t^2}\ dt$$

  Apply a change of variable: let $u = 1 + 4t^2$, for $u \in [1, 1 + 4\pi^2]$, and consider $\frac{du}{dt} = 8t$, resulting in:

  $$\frac{1}{8}\int_1^{1+4\pi^2} \sqrt{u}\ du = \frac{1}{8}\frac{2}{3}[u^{\frac{3}{2}}]_1^{1+4\pi^2} = \frac{\sqrt{(1 + 4\pi^2)^3} - 1}{12}$$

6.6 Exercise 6

Compute the following line integrals:

- $f(x, y) = x + y$ with respect to the triangle with vertices $(0, 0), (0, 1), (1, 0)$. By plotting the triangle, let $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ and consider each curve separately:

  1. Let $\gamma_1(t) = (0, t)$, for $t \in [0, 1]$.
     Compute the norm of the tangent vector:

     $$\gamma_1'(t) = (0, 1) \Rightarrow \|\gamma_1'(t)\| = \sqrt{0 + 1} = 1$$

     Therefore, it is possible to compute the line integral in the following way:

     $$\int_{\gamma_1} x + y\ dt = \int_0^1 t\ dt = [\frac{t^2}{2}]_0^1 = \frac{1}{2}$$

  2. Let $\gamma_2(t) = (t, 1 - t)$, for $t \in [0, 1]$.
     Compute the norm of the tangent vector:

     $$\gamma_2'(t) = (1, -1) \Rightarrow \|\gamma_2'(t)\| = \sqrt{1 + 1} = \sqrt{2}$$

     Therefore, it is possible to compute the line integral in the following way:

     $$\int_{\gamma_2} x + y\ dt = \int_0^1 (t + (1 - t))\sqrt{2}\ dt = \sqrt{2}\int_0^1\ dt = \sqrt{2}[t]_0^1 = \sqrt{2}$$

3. Let $\gamma_3(t) = (t, 0)$, for $t \in [1, 0]$ (in this case, the direction of traversal is inverted).
Compute the norm of the tangent vector:

$$\gamma_3'(t) = (1, 0) \Rightarrow \|\gamma_3'(t)\| = \sqrt{1+0} = 0$$

Therefore, it is possible to compute the line integral in the following way:

$$\int_{\gamma_3} x + y \; dt = \int_1^0 t \; dt = -\int_0^1 t \; dt = -[\frac{t^2}{2}]_0^1 = -\frac{1}{2}$$

Therefore, by applying integral additivity, it is possible to conclude that:

$$\int_{\gamma} x + y \; dt = \frac{1}{2} + \sqrt{2} - \frac{1}{2} = \sqrt{2}$$

N.B.: The result of this integral may change according to the direction of traversal considered for the curve.

- $f(x, y) = xy$ with respect to the quarter of ellipse of equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that is contained in the first quadrant.
Start by parametrizing the Cartesian equation of the curve:

$$\gamma : \begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}, \; t \in [0, \frac{\pi}{2}]$$

At this point, compute the norm of the tangent vector:

$$\gamma'(t) = (-2\sin t, 3\cos t) \Rightarrow \|\gamma'(t)\| = \sqrt{4\sin^2 t + 9\cos^2 t} = \sqrt{4 + 5\cos^2 t}$$

Therefore, it is possible to compute the line integral in the following way:

$$\int_{\gamma} xy \; dt = \int_0^{\frac{\pi}{2}} 6\cos t \sin t \sqrt{4 + 5\cos^2 t} \; dt$$

Apply a change of variable: let $u = 4 + 5\cos^2 t$, for $u \in [9, 4]$, and consider $\frac{du}{dt} = -10\cos t \sin t$, resulting in:

$$-\frac{6}{10}\int_9^4 \sqrt{u} \; du = \frac{3}{5}\int_4^9 \sqrt{u} \; du = \frac{3}{5}\frac{2}{3}[u^{\frac{3}{2}}]_4^9 = \frac{2}{5}(27 - 8) = \frac{38}{5}$$

6.7 Exercise 7

The cycloid is the curve described by a point on the circumference of a circle of radius R as the circle rolls along the x-axis without slipping. This curve has parametric equation given by:

$$\gamma(t) = (R(t - \sin t), R(1 - \cos t)), \; t \in [0, 2\pi]$$

- Determine if the curve is closed and regular.
  By definition, the curve will be closed in $[0, 2\pi]$ if and only if $\gamma(0) = \gamma(2\pi)$:

$$\gamma(0) = (0, 0) \neq (2\pi R, 0) = \gamma(2\pi)$$

  Therefore, the curve is not closed in the interval $[0, 2\pi]$ unless $R = 0$.
  On the other hand, the curve is defined to be regular in the interval $(0, 2\pi)$ if and only if the tangent vector $\gamma'(t)$ is component-wise continuous and $\gamma'(t) \neq 0 \ \forall \ t \in (0, 2\pi)$.
  Start by computing $\gamma'(t) = (R(1 - \cos t), R \sin t)$ and study $\gamma'(t) = 0$:

$$\begin{cases} R(1 - \cos t) = 0 \\ R \sin t = 0 \end{cases} \quad \Leftrightarrow t = 0 \notin (0, 2\pi) \vee t = 2\pi \notin (0, 2\pi)$$

  Therefore, since the condition is satisfied, the curve is indeed regular in $(0, 2\pi)$.

- Compute the length of the curve.
  Knowing that the curve is regular in its domain, it is possible to compute its length through its norm:

$$\|\gamma'(t)\| = \sqrt{(R(1 - \cos t))^2 + (R \sin t)^2} = \sqrt{2} R \sqrt{1 - \cos t}$$

  Most particularly, by exploiting the identity $1 - \cos t = 2 \sin^2(\frac{t}{2})$, it is possible to simplify the previous result to:

$$\|\gamma'(t)\| = \sqrt{2} R \sqrt{2 \sin^2(\frac{t}{2})} = 2R \sin(\frac{t}{2})$$

  Therefore, it is possible to conclude that:

$$L(\gamma) = \int_0^{2\pi} \|\gamma'(t)\| \ dt = \int_0^{2\pi} 2R \sin(\frac{t}{2}) \ dt$$

  Apply a change of variable and let $u = \frac{t}{2}$, for $u \in [0, \pi]$, and let $\frac{du}{dt} = \frac{1}{2}$, resulting in:

$$L(\gamma) = \int_0^{\pi} 2R \sin u \cdot 2 \ du = 4R[-\cos u]_0^{\pi} = 4R[1 - (-1)] = 8R$$

Chapter 7

Exercise Sheet 7: Local extrema

7.1 Exercise 1

Compute the absolute maximum and minimum of the following functions constrained to the indicated region.

- $f(x,y) = x^2y + xy^2 - xy$, $D = \{(x,y) \in \mathbb{R}^2 : y \geq x-1,\ y \leq 1,\ x \geq 0\}$.
  Start by finding the stationary points (x,y) such that $\nabla f(x,y) = 0$:

  $$\begin{cases} 2xy + y^2 - y = 0 \\ x^2 + 2xy - x = 0 \end{cases} \Rightarrow \begin{cases} y(2x+y-1) = 0 \\ x(x+2y-1) = 0 \end{cases}$$

  Consider each solution to the first equation separately:

  1. $y = 0$:

     $$\begin{cases} y = 0 \\ x(x-1) = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 0 \vee x = 1 \end{cases}$$

     In this case, it is possible to find the stationary points $P_1(0,0) \in D$ and $P_2(1,0) \in D$.

  2. $2x + y - 1 = 0$:

     $$\begin{cases} y = 1-2x \\ x(x+2-4x-1) = 0 \end{cases} \Rightarrow \begin{cases} y = 1-2x \\ x(1-3x) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \end{cases} \vee \begin{cases} x = \frac{1}{3} \\ y = \frac{1}{3} \end{cases}$$

     In this case, it is possible to find the stationary points $P_3(0,1) \in D$, which also happens to be one of the corners of the region, and $P_4(\frac{1}{3}, \frac{1}{3}) \in D$.

At this point, study the boundaries (and eventual corners) of the region by parametrizing it as a curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$:

1. Let $\gamma_1(t) = (t, t-1)$, for $t \in [0, 2]$.
   Restrict the function to the curve by letting $f_{\gamma_1}(t) = 2t^3 - 4t^2 + 2t$ and find the corresponding stationary points:
   $$f'_{\gamma_1}(t) = 6t^2 - 8t + 2 = 0 \Rightarrow t = \frac{1}{3} \vee t = 1$$
   Therefore, is possible to find the stationary points $P_5(\frac{1}{3}, -\frac{2}{3}) \in D$ and $P_2(1, 0)$, which was already found via $\nabla f(x, y) = 0$.
   Furthermore, by studying the extremes of the interval, it is possible to recover the corners $P_6(0, -1) \in D$ and $P_7(2, 1) \in D$.
2. Let $\gamma_2(t) = (t, 1)$, for $t \in [2, 0]$ (in this case, the direction of traversal is inverted).
   Restrict the function to the curve by letting $f_{\gamma_2}(t) = t^2$ and find the corresponding stationary points:
   $$f'_{\gamma_2}(t) = 2t = 0 \Rightarrow t = 0$$
   Therefore, it is possible to find again the point $P_3(0, 1)$, which was already found via $\nabla f(x, y) = 0$.
   Furthermore, by studying the extremes of the interval, it is possible to recover again the corners $P_7(2, 1)$, which was already found via γ_1 and $P_3(0, 1)$.
3. Let $\gamma_3(t) = (0, t)$, for $t \in [1, -1]$ (in this case, the direction of traversal is inverted).
   Restrict the function to the curve and let $f_{\gamma_3}(t) = 0$, which means that there is no stationary point to study as the function is 0 in this component.
   On the other hand, by studying the extremes of the interval, it is possible to find again the corners $P_3(0, 1)$ and $P_6(0, -1) \in D$, which were already found via $\nabla f(x, y) = 0$ and γ_1 respectively.

Lastly, compare the values the function takes at the found points:

1. $P_1(0, 0) \Rightarrow f(0, 0) = 0$.
2. $P_2(1, 0) \Rightarrow f(1, 0) = 0$.
3. $P_3(0, 1) \Rightarrow f(0, 1) = 0$.
4. $P_4(\frac{1}{3}, \frac{1}{3}) \Rightarrow f(\frac{1}{3}, \frac{1}{3}) = -\frac{1}{27}$.
5. $P_5(\frac{1}{3}, -\frac{2}{3}) \Rightarrow f(\frac{1}{3}, -\frac{2}{3}) = \frac{8}{27}$.
6. $P_6(0, -1) = 0$.
7. $P_7(2, 1) = 4$.

Therefore, it is possible to conclude that $P_7(2, 1)$ is a local maximum, whereas $P_4(\frac{1}{3}, \frac{1}{3})$ is a local minimum.

- $f(x,y) = x^3 - xy + y^3$ within the triangle with vertices $(0,0), (0,1), (1,0)$.
  Start by finding the stationary points (x,y) such that $\nabla f(x,y) = 0$:

  $$\begin{cases} 3x^2 - y = 0 \\ -x + 3y^2 = 0 \end{cases} \Rightarrow \begin{cases} y = 3x^2 \\ 27x^4 - x = 0 \end{cases} \Rightarrow \begin{cases} y = 3x^2 \\ x(27x^3 - 1) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = \frac{1}{3} \\ y = \frac{1}{3} \end{cases}$$

  Therefore, it is possible to find the stationary points $P_1(0,0) \in D$, which also happens to be one of the corners of the region, and $P_2(\frac{1}{3}, \frac{1}{3}) \in D$.
  N.B.: Without loss of generality, the same conclusion is reached by setting $x = 3y^2$ instead.
  At this point, study the boundaries (and eventual corners) of the region by parametrizing it as a curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$:

  1. Let $\gamma_1(t) = (0,t)$, for $t \in [0,1]$.
     Restrict the function to the curve by letting $f_{\gamma_1}(t) = t^3$ and find the corresponding stationary points:

     $$f'_{\gamma_1}(t) = 3t^2 = 0 \Rightarrow t = 0$$

     Therefore, it is possible to find again the corner $P_1(0,0)$, which was already found via $\nabla f(x,y) = 0$.
     Furthermore, by studying the extremes of the interval, it is possible to recover the corners $P_1(0,0)$ and $P_3(0,1) \in D$.
  2. Let $\gamma_2(t) = (t, 1-t)$, for $t \in [0,1]$.
     Restrict the function to the curve by letting $f_{\gamma_2}(t) = 4t^2 - 4t + 1$ and find the corresponding stationary points:

     $$f'_{\gamma_2}(t) = 8t - 4 = 0 \Rightarrow t = \frac{1}{2}$$

     Therefore, it is possible to find the stationary point $P_4(\frac{1}{2}, \frac{1}{2}) \in D$.
     Furthermore, by studying the extremes of the interval, it is possible to recover the corners $P_3(0,1)$, which was already found via γ_1, and $P_5(1,0) \in D$.
  3. Let $\gamma_3(t) = (t, 0)$, for $t \in [1,0]$ (in this case, the direction of traversal is inverted).
     Restrict the function to the curve and let $f_{\gamma_3}(t) = t^3$ and find the corresponding stationary points:

     $$f'_{\gamma_3}(t) = 3t^2 = 0 \Rightarrow t = 0$$

     Therefore, it is possible to find again the corner $P_1(0,0)$, which was already found via $\nabla f(x,y) = 0$ and then through γ_1.
     Furthermore, by studying the extremes of the interval, it is possible to recover again the corners $P_5(1,0)$ and $P_1(0,0)$, which were found via γ_2 and $\nabla f(x,y) = 0$ and γ_1 respectively.

  Lastly, compare the values the function takes at the found points:

1. $P_1(0,0) \Rightarrow f(0,0) = 0$.
2. $P_2(\frac{1}{3}, \frac{1}{3}) \Rightarrow f(\frac{1}{3}, \frac{1}{3}) = -\frac{1}{27}$.
3. $P_3(0,1) \Rightarrow f(0,1) = 1$.
4. $P_4(\frac{1}{2}, \frac{1}{2}) \Rightarrow f(\frac{1}{2}, \frac{1}{2}) = 0$.
5. $P_5(1,0) \Rightarrow f(1,0) = 1$.

Therefore, it is possible to conclude that $P_3(0,1)$ and $P_5(1,0)$ are local maxima, whereas $P_2(\frac{1}{3}, \frac{1}{3})$ is a local minimum.

- $f(x,y) = x^3 - xy^2$, $D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.
Start by finding the stationary points (x,y) such that $\nabla f(x,y) = 0$:

$$\begin{cases} 3x^2 - y^2 = 0 \\ -2xy = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Therefore, it is possible to find a first candidate $P_1(0,0) \in D$, which also happens to be one of the corners of the region.
At this point, study the boundaries (and eventual corners) of the region by parametrizing it as a curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:

1. Let $\gamma_1(t) = (t, 0)$, for $t \in [0,1]$.
Restrict the function to the curve by letting $f_{\gamma_1}(t) = t^3$ and find the corresponding stationary points:

$$f'_{\gamma_1}(t) = 3t^2 = 0 \Rightarrow t = 0$$

Therefore, it is possible to recover again the corner $P_1(0,0)$, which was already found via $\nabla f(x,y) = 0$.
Furthermore, by studying the extremes of the interval, it is possible to find the corners $P_1(0,0)$ and $P_2(1,0) \in D$.
2. Let $\gamma_2(t) = (1, t)$, for $t \in [0,1]$.
Restrict the function to the curve by letting $f_{\gamma_2}(t) = 1 - t^2$ and find the corresponding stationary points:

$$f'_{\gamma_2}(t) = -2t = 0 \Rightarrow t = 0$$

Therefore, it is possible to recover again the corner $P_2(1,0)$, which was already found as a corner via γ_1.
Furthermore, by studying the extremes of the interval, it is possible to find the corners $P_2(1,0)$ and $P_3(1,1) \in D$.
3. Let $\gamma_3(t) = (t, 1)$, for $t \in [1,0]$ (in this case, the direction of traversal is inverted).
Restrict the function to the curve by letting $f_{\gamma_3}(t) = t^3 - t$ and find the corresponding stationary points:

$$f'_{\gamma_3}(t) = 3t^2 - 1 = 0 \Rightarrow t = \pm\frac{\sqrt{3}}{3}$$

Therefore, discard $P(-\frac{\sqrt{3}}{3}, 1) \notin D$ and focus on the stationary point $P_4(\frac{\sqrt{3}}{3}, 1) \in D$.
Furthermore, by studying the extremes of the interval, it is possible to recover again the corners $P_3(1,1)$, which was already found via γ_2, and $P_5(0,1) \in D$.

4. Let $\gamma_4(t) = (0, t)$, for $t \in [1, 0]$ (in this case, the direction of traversal is inverted).
   Restrict the function to the curve and let $f_{\gamma_4}(t) = 0$, which means that there is no stationary point to study as the function is 0 in this component.
   On the other hand, by studying the extremes of the interval, it is possible to find again the corners $P_5(0,1)$ and $P_1(0,0)$, which were already found via γ_3 and γ_1 respectively.

Lastly, compare the values the function takes at the found points:

1. $P_1(0,0) \Rightarrow f(0,0) = 0$.
2. $P_2(1,0) \Rightarrow f(1,0) = 1$.
3. $P_3(1,1) \Rightarrow f(1,1) = 0$.
4. $P_4(\frac{\sqrt{3}}{3}, 1) \Rightarrow f(\frac{\sqrt{3}}{3}, 1) = -\frac{2\sqrt{3}}{9}$.
5. $P_5(0,1) \Rightarrow f(0,1) = 0$.

Therefore, it is possible to conclude that $P_2(1,0)$ is a local maximum, whereas $P_4(\frac{\sqrt{3}}{3}, 1)$ is a local minimum.

- $f(x,y) = x^2 + 3y^2 - x$, $D = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \leq -x+1,\ y \geq x-1\}$.
  Start by finding the stationary points (x,y) such that $\nabla f(x,y) = 0$:

$$\begin{cases} 2x - 1 = 0 \\ 6y = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ y = 0 \end{cases}$$

  Therefore, it is possible to find a first candidate $P_1(\frac{1}{2}, 0) \in D$.
  At this point, study the boundaries (and eventual corners) of the region by parametrizing it as a curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$:

  1. Let $\gamma(t) = (t, t-1)$, for $t \in [0,1]$.
     Restrict the function to the curve by letting $f_{\gamma_1}(t) = 4t^2 - 7t + 3$ and find the corresponding stationary points:

$$f'_{\gamma_1}(t) = 8t - 7 = 0 \Rightarrow t = \frac{7}{8}$$

     Therefore, it is possible to find the stationary point $P_2(\frac{7}{8}, -\frac{1}{8}) \in D$.
     Furthermore, by studying the extremes of the interval, it is possible to recover the corners $P_3(0,-1) \in D$ and $P_4(1,0) \in D$.

2. Let $\gamma_2(t) = (t, 1-t)$, for $t \in [1, 0]$ (in this case, the direction of traversal is inverted).
   Restrict the function to the curve by letting $f_{\gamma_2}(t) = 4t^2 - 7t + 3$ and find the corresponding stationary points:
$$f'_{\gamma_2}(t) = 8t - 7 = 0 \Rightarrow t = \frac{7}{8}$$
   Therefore, it is possible to find the stationary point $P_5(\frac{7}{8}, \frac{1}{8}) \in D$.
   Furthermore, by studying the extremes of the interval, it is possible to recover the corners $P_4(1, 0)$, which was already found via γ_1, and $P_6(0, 1) \in D$.
3. Let $\gamma_3(t) = (0, t)$, for $t \in [1, -1]$ (in this case, the direction of traversal is inverted).
   Restrict the function to the curve by letting $f_{\gamma_3}(t) = 3t^2$ and find the corresponding stationary points:
$$f'_{\gamma_3}(t) = 6t = 0 \Rightarrow t = 0$$
   Therefore, it is possible to find the stationary point $P_7(0, 0) \in D$.
   Furthermore, by studying the extremes of the interval, it is possible to recover again the corners $P_6(0, 1)$ and $P_3(0, -1)$, which were already found via γ_1 and γ_2 respectively.

Lastly, compare the values the function takes at the found points:

1. $P_1(\frac{1}{2}, 0) \Rightarrow f(\frac{1}{2}, 0) = -\frac{1}{4}$.
2. $P_2(\frac{7}{8}, -\frac{1}{8}) \Rightarrow f(\frac{7}{8}, -\frac{1}{8}) = -\frac{1}{16}$.
3. $P_3(0, -1) \Rightarrow f(0, -1) = 3$.
4. $P_4(1, 0) \Rightarrow f(1, 0) = 0$.
5. $P_5(\frac{7}{8}, \frac{1}{8}) \Rightarrow f(\frac{7}{8}, \frac{1}{8}) = -\frac{1}{16}$.
6. $P_6(0, 1) \Rightarrow f(0, 1) = 3$.
7. $P_7(0, 0) \Rightarrow f(0, 0) = 0$.

Therefore, it is possible to conclude that $P_3(0, -1)$ and $P_6(0, 1)$ are local maxima, whereas $P_1(\frac{1}{2}, 0)$ is a local minimum.

- $f(x, y) = 2x^2 + y^2 - x$, $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.
  Start by finding the stationary points (x, y) such that $\nabla f(x, y) = 0$:
$$\begin{cases} 4x - 1 = 0 \\ 2y = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{4} \\ y = 0 \end{cases}$$
  Therefore, it is possible to recover a first candidate as $P_1(\frac{1}{4}, 0) \in D$.
  At this point, study the boundaries (and eventual corners) of the region by parametrizing it as a curve $\gamma(t) = (\cos t, \sin t)$, for $t \in [0, 2\pi]$.
  Restrict the function to the curve by letting $f_\gamma(t) = \cos^2 t - \cos t + 1$ and

find the corresponding stationary points:

$$f'_\gamma(t) = -2\sin t\cos t + \sin t = \sin t(1-2\cos t) = 0 \Rightarrow t = 0 \vee t = \pi \vee t = 2\pi \vee t = \frac{\pi}{3} \vee t = \frac{5\pi}{3}$$

Therefore, it is possible to find stationary points $P_2(1,0) \in D$, $P_3(-1,0) \in D$, $P_4(\frac{1}{2}, \frac{\sqrt{3}}{2}) \in D$ and $P_5(\frac{1}{2}, -\frac{\sqrt{3}}{2}) \in D$.
Most particularly, observe that there are no corners to study.
Lastly, compare the values the function takes at the found points:

1. $P_1(\frac{1}{4}, 0) \Rightarrow f(\frac{1}{4}, 0) = -\frac{1}{8}$.
2. $P_2(1,0) \Rightarrow f(1,0) = 1$.
3. $P_3(-1,0) \Rightarrow f(-1,0) = 3$.
4. $P_4(\frac{1}{2}, \frac{\sqrt{3}}{2}) \Rightarrow f(\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{3}{4}$.
5. $P_5(\frac{1}{2}, -\frac{\sqrt{3}}{2}) \Rightarrow f(\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{3}{4}$.

Therefore, it is possible to conclude that $P_3(-1,0)$ is a local maximum, whereas $P_1(\frac{1}{4}, 0)$ is a local minimum.

- $f(x,y) = x^2 - y^2 + x + 1$, $D = \{(x,y) \in \mathbb{R}^2 : 4x^2 + y^2 \leq 1\}$.
Start by finding the stationary points (x,y) such that $\nabla f(x,y) = 0$:

$$\begin{cases} 2x + 1 = 0 \\ -2y = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2} \\ y = 0 \end{cases}$$

Therefore, it is possible to recover a first candidate as $P_1(-\frac{1}{2}, 0) \in D$.
At this point, study the boundaries (and eventual corners) of the region by parametrizing it as a curve $\gamma(t) = (\frac{\cos t}{2}, \sin t)$, for $t \in [0, 2\pi]$.
Restrict the function to the curve by letting $f_\gamma(t) = \frac{5\cos^2 t}{4} + \frac{\cos t}{2}$ and find the corresponding stationary points:

$$f'_\gamma(t) = -\frac{5\sin t\cos t}{2} - \frac{\sin t}{2} = -\frac{\sin t}{2}(5\cos t + 1) = 0 \Rightarrow t = 0 \vee t = \pi \vee t = 2\pi \vee \arccos(-\frac{1}{5}) \vee 2\pi - \arccos(-\frac{1}{5})$$

Therefore, it is possible to find the stationary points $P_2(\frac{1}{2}, 0) \in D$, $P_1(-\frac{1}{2}, 0)$, which was already found via $\nabla f(x,y) = 0$, $P_3(-\frac{1}{10}, \sqrt{\frac{24}{25}}) \in D$ and $P_4(-\frac{1}{10}, -\sqrt{\frac{24}{25}}) \in D$.
Most particularly, observe that there are not corners to study.
Lastly, compare the values the function takes at the found points:

1. $P_1(-\frac{1}{2}, 0) \Rightarrow f(-\frac{1}{2}, 0) = \frac{3}{4}$.
2. $P_2(\frac{1}{2}, 0) \Rightarrow f(\frac{1}{2}, 0) = \frac{7}{4}$.
3. $P_3(-\frac{1}{10}, \sqrt{\frac{24}{25}}) \Rightarrow f(-\frac{1}{10}, \sqrt{\frac{24}{25}}) = -\frac{1}{20}$.
4. $P_4(-\frac{1}{10}, -\sqrt{\frac{24}{25}}) \Rightarrow f(-\frac{1}{10}, -\sqrt{\frac{24}{25}}) = -\frac{1}{20}$.

Therefore, it is possible to conclude that $P_2(\frac{1}{2}, 0)$ is a local maximum, whereas $P_3(-\frac{1}{10}, \sqrt{\frac{24}{25}})$ and $P_4(-\frac{1}{10}, -\sqrt{\frac{24}{25}})$ are local minima.

- $f(x, y) = \sin(x+y)\cos(x-y)$, $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \pi,\ 0 \leq y \leq \pi\}$.
Start by finding the stationary points (x, y) such that $\nabla f(x, y) = 0$:

$$\begin{cases} \cos(x+y)\cos(x-y) - \sin(x+y)\cos(x-y) = 0 \\ \cos(x+y)\cos(x-y) + \sin(x+y)\sin(x-y) = 0 \end{cases} \Rightarrow \begin{cases} \cos((x+y)+(x-y)) = 0 \\ \cos((x+y)-(x-y)) = 0 \end{cases}$$

$$\begin{cases} \cos 2x = 0 \\ \cos 2y = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{\pi}{4} + \frac{k\pi}{2} \\ y = \frac{\pi}{4} + \frac{k\pi}{2} \end{cases} \quad \text{for } k \in \{0, 1\}.$$

By combining the various values, it is possible to find the stationary points $P_1(\frac{\pi}{4}, \frac{\pi}{4}) \in D$, $P_2(\frac{\pi}{4}, \frac{3\pi}{4}) \in D$, $P_3(\frac{3\pi}{4}, \frac{\pi}{4}) \in D$ and $P_4(\frac{3\pi}{4}, \frac{3\pi}{4}) \in D$.
At this point, study the boundaries (and eventual corners) of the region by parametrizing it as a curve $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$:

1. Let $\gamma_1(t) = (t, 0)$, for $t \in [0, \pi]$.
Restrict the function to the curve by letting $f_{\gamma_1}(t) = \sin t \cos t$ and find the corresponding stationary points:

$$f'_{\gamma_1}(t) = \cos^2 t - \sin^2 t = \cos 2t = 0 \Rightarrow t = \frac{\pi}{4} \vee t = \frac{3\pi}{4}$$

Therefore, it is possible to find the stationary points $P_5(\frac{\pi}{4}, 0) \in D$ and $P_6(\frac{3\pi}{4}, 0) \in D$.
Furthermore, by studying the extremes of the interval, it is possible to find the corners $P_7(0, 0) \in D$ and $P_8(\pi, 0) \in D$.

2. Let $\gamma_2(t) = (\pi, t)$, for $t \in [0, \pi]$.
Restrict the function to the curve by letting $f_{\gamma_2}(t) = \sin t \cos t$ and find the corresponding stationary points:

$$f'_{\gamma_2}(t) = \cos^2 t - \sin^2 t = \cos 2t = 0 \Rightarrow t = \frac{\pi}{4} \vee t = \frac{3\pi}{4}$$

Therefore, it is possible to find the stationary points $P_9(\pi, \frac{\pi}{4}) \in D$ and $P_{10}(\pi, \frac{3\pi}{4}) \in D$.
Furthermore, by studying the extremes of the interval, it is possible to find the corners $P_8(\pi, 0)$, which was already found as a corner via γ_1, and $P_{11}(\pi, \pi) \in D$.

3. Let $\gamma_3(t) = (t, \pi)$, for $t \in [\pi, 0]$ (in this case, the direction of traversal is inverted).
Restrict the function to the curve by letting $f_{\gamma_3}(t) = \sin t \cos t$ and find the corresponding stationary points:

$$f'_{\gamma_3}(t) = \cos^2 t - \sin^2 t = \cos 2t = 0 \Rightarrow t = \frac{\pi}{4} \vee t = \frac{3\pi}{4}$$

Therefore, it is possible to find the stationary points $P_{12}(\frac{\pi}{4}, \pi) \in D$ and $P_{13}(\frac{3\pi}{4}, \pi) \in D$.
Furthermore, by studying the extremes of the interval, it is possible to find the corners $P_{11}(\pi, \pi)$, which was already found as a corner via γ_2, and $P_{14}(0, \pi) \in D$.

4. Let $\gamma_4(t) = (0, t)$, for $t \in [\pi, 0]$ (in this case, the direction of traversal is inverted).
Restrict the function to the curve by letting $f_{\gamma_4}(t) = \sin t \cos t$ and find the corresponding stationary points:

$$f'_{\gamma_4}(t) = \cos^2 t - \sin^2 t = \cos 2t = 0 \Rightarrow t = \frac{\pi}{4} \vee t = \frac{3\pi}{4}$$

Therefore, it is possible to find the stationary points $P_{15}(0, \frac{\pi}{4}) \in D$ and $P_{16}(0, \frac{3\pi}{4}) \in D$.
Furthermore, by studying the extremes of the interval it is possible to find the corners $P_{17}(0, \pi) \in D$ and $P_7(0, 0)$, which was already found as a corner via γ_1.

Lastly, compare the values the function takes at the found points:

1. $P_1(\frac{\pi}{4}, \frac{\pi}{4}) \Rightarrow f(\frac{\pi}{4}, \frac{\pi}{4}) = 1$.
2. $P_2(\frac{\pi}{4}, \frac{3\pi}{4}) \Rightarrow f(\frac{\pi}{4}, \frac{3\pi}{4}) = 0$.
3. $P_3(\frac{3\pi}{4}, \frac{\pi}{4}) \Rightarrow f(\frac{3\pi}{4}, \frac{\pi}{4}) = 0$.
4. $P_4(\frac{3\pi}{4}, \frac{3\pi}{4}) \Rightarrow f(\frac{3\pi}{4}, \frac{3\pi}{4}) = -1$.
5. $P_5(\frac{\pi}{4}, 0) \Rightarrow f(\frac{\pi}{4}, 0) = \frac{1}{2}$.
6. $P_6(\frac{3\pi}{4}, 0) \Rightarrow f(\frac{3\pi}{4}, 0) = -\frac{1}{2}$.
7. $P_7(0, 0) \Rightarrow f(0, 0) = 0$.
8. $P_8(\pi, 0) \Rightarrow f(\pi, 0) = 0$.
9. $P_9(\pi, \frac{\pi}{4}) \Rightarrow f(\pi, \frac{\pi}{4}) = \frac{1}{2}$.
10. $P_{10}(\pi, \frac{3\pi}{4}) \Rightarrow f(\pi, \frac{3\pi}{4}) = -\frac{1}{2}$.
11. $P_{11}(\pi, \pi) \Rightarrow f(\pi, \pi) = 0$.
12. $P_{12}(\frac{\pi}{4}, \pi) \Rightarrow f(\frac{\pi}{4}, \pi) = \frac{1}{2}$.
13. $P_{13}(\frac{3\pi}{4}, \pi) \Rightarrow f(\frac{3\pi}{4}, \pi) = -\frac{1}{2}$.
14. $P_{14}(0, \pi) \Rightarrow f(0, \pi) = 0$.
15. $P_{15}(0, \frac{\pi}{4}) \Rightarrow f(0, \frac{\pi}{4}) = \frac{1}{2}$.
16. $P_{16}(0, \frac{3\pi}{4}) \Rightarrow f(0, \frac{3\pi}{4}) = -\frac{1}{2}$.
17. $P_{17}(0, \pi) \Rightarrow f(0, \pi) = 0$.

Therefore, it is possible to conclude that $P_1(\frac{\pi}{4}, \frac{\pi}{4})$ is a local maximum, whereas $P_4(\frac{3\pi}{4}, \frac{3\pi}{4})$ is a local minimum.

Chapter 8

Exercise Sheet 8: Double integrals

8.1 Exercise 1

Compute the following double integrals:

- $f(x, y) = x + y$ over $Q = [0, 1] \times [0, 2]$.
  Apply Fubini's theorem for double integrals:

$$\int_0^2 \int_0^1 x + y \; dxdy = \int_0^2 [\frac{x^2}{2} + xy]_0^1 \; dy = \int_0^2 \frac{1}{2} + y \; dy = [\frac{y}{2} + \frac{y^2}{2}]_0^2 = 3$$

- $f(x, y) = x(1 - y)$ over the square of vertices $(1, 0), (2, 0), (1, 1), (2, 1)$.
  By plotting the square on a graph, observe that the region can be written as $Q = [1, 2] \times [0, 1]$.
  Therefore, it is possible to apply Fubini's theorem for double integrals:

$$\int_0^1 \int_1^2 x(1-y) \; dxdy = \int_0^1 (1-y)[\frac{x^2}{2}]_1^2 \; dy = \int_0^1 \frac{3(1-y)}{2} \; dy = \frac{3}{2}[y - \frac{y^2}{2}]_0^1 = \frac{3}{4}$$

- $f(x, y) = \frac{1}{x+y}$ over $Q = [0, 1] \times [0, 1]$.
  Observe that the function is not continuous at $(0, 0) \in Q$, meaning that, technically, the function cannot be integrated over Q.
  However, if Fubini's theorem is applied, it is possible to discover the following:

$$\int_0^1 \int_0^1 \frac{1}{x+y} \; dxdy = \int_0^1 [\ln|x+y|]_0^1 \; dy = \int_0^1 \ln(1+y) - \ln y \; dy$$

  At this point, apply integral linearity and solve each integral separately:

1. First integral:

$$\int_0^1 \ln(1+y)\ dy = [(1+y)\ln(1+y) - (1+y)]_0^1 = 2\ln 2 - 1$$

2. Second integral:

$$\int_0^1 \ln y\, dy = \lim_{a\to 0^+} \int_a^1 \ln y\, dy = \lim_{a\to 0^+} [a\ln a - a]_a^1 = \lim_{a\to 0^+} -1 - a\ln a - a = 0\cdot\infty$$

The problem lies in $a\ln a$, so, let $a = \frac{1}{\frac{1}{a}}$ and apply De L'Hopital's rule:

$$\lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}} = \lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}} = \lim_{a\to 0^+} \frac{1}{a}\cdot(-a^2) = \lim_{a\to 0^+} -a = 0$$

For this reason, it is possible to conclude that the function is integrable and:

$$\int_0^1 \ln y\ dy = -1$$

It is thus possible to conclude that:

$$\int_0^1 \int_0^1 \frac{1}{x+y}\ dxdy = 2\ln 2 - 1 - (-1) = 2\ln 2$$

Therefore, the function is actually integrable in an improper way in Q.

- $f(x,y) = \frac{1}{(x+2y)^2}$ over $Q = [3,4]\times[1,2]$.
Apply Fubini's theorem for double integrals:

$$\int_1^2 \int_3^4 \frac{1}{(x+2y)^2}\ dxdy = \int_1^2 [-\frac{1}{x+2y}]_3^4\ dy = \int_1^2 \frac{1}{3+2y} - \frac{1}{4+2y}\ dy = [\frac{\ln|\frac{3+2y}{4+2y}|}{2}]_1^2 = \frac{\ln(\frac{21}{20})}{2}$$

- $f(x,y) = \frac{x}{1+y}$ over $Q = [0,1]\times[2,3]$.
Apply Fubini's theorem for double integrals:

$$\int_0^1 \int_2^3 \frac{x}{1+y}\ dydx = \int_0^1 x[\ln|1+y|]_2^3\ dx = (\ln 4 - \ln 3)\int_0^1 x\ dx = \ln(\frac{4}{3})[\frac{x^2}{2}]_0^1 = \frac{\ln(\frac{4}{3})}{2}$$

- $f(x,y) = \frac{x}{1+xy}$ over $Q = [0,1]\times[0,2]$.
Apply Fubini's theorem for double integrals:

$$\int_0^1 \int_0^2 \frac{x}{1+xy}\ dydx$$

Apply a change of variable: let $u = 1+xy$, for $u \in [1, 1+2x]$, and consider $\frac{du}{dy} = x$, resulting in:

$$\int_0^1 \int_1^{1+2x} \frac{1}{u}\ dudx = \int_0^1 [\ln|u|]_1^{1+2x}\ dx = \int_0^1 \ln(1+2x)\ dx$$

Apply another change of variable: let $t = 1+2x$, for $t \in [1,3]$, and consider $\frac{dt}{dx} = 2$, resulting in:

$$\frac{1}{2}\int_1^3 \ln t \; dt = \frac{1}{2}[t\ln t - t]_1^3 = \frac{3\ln 3 - 2}{2}$$

- $f(x,y) = xye^{x^2+y^2}$ over the rectangle of vertices $(0,0),(2,0),(0,3),(2,3)$. By plotting the rectangle on a graph, observe that the region can be written as $Q = [0,2]\times[0,3]$.
  Most particularly, by the properties of exponentials, let $xye^{x^2+y^2} = xe^{x^2}ye^{y^2}$ and solve the integral by applying decoupling:

$$\int_0^3\int_0^2 xye^{x^2+y^2} \; dxdy = \int_0^2 xe^{x^2} \; dx \int_0^3 ye^{y^2} \; dy$$

  Consider each integral separately:

  1. Integral with respect to x:

  $$\int_0^2 xe^{x^2} \; dx$$

     Apply a change of variable: let $u = x^2$, for $u \in [0,4]$, and consider $\frac{du}{dx} = 2x$, resulting in:

  $$\frac{1}{2}\int_0^4 e^u \; du = \frac{1}{2}[e^u]_0^4 = \frac{e^4-1}{2}$$

  2. Integral with respect to y:

  $$\int_0^3 ye^{y^2} \; dy$$

     Apply a change of variable: let $t = y^2$, for $t \in [0,9]$, and consider $\frac{dt}{dy} = 2y$, resulting in:

  $$\frac{1}{2}\int_0^9 e^t \; dt = \frac{1}{2}[e^t]_0^9 = \frac{e^9-1}{2}$$

  Therefore, it is possible to conclude that:

$$\int_0^3\int_0^2 xye^{x^2+y^2} \; dxdy = \frac{e^4-1}{2}\cdot\frac{e^9-1}{2} = \frac{e^{13}-e^9-e^4+1}{4}$$

- $f(x,y) = x\sin(xy)$ over $Q = [1,2]\times[2,3]$.
  Apply Fubini's theorem for double integrals:

$$\int_1^2\int_2^3 x\sin(xy) \; dydx$$

Apply a change of variable: let $u = xy$, for $u \in [2x, 3x]$, and consider $\frac{du}{dy} = x$, resulting in:

$$\int_1^2 \int_{2x}^{3x} \sin u \, dudx = \int_1^2 [-\cos u]_{2x}^{3x} \, dx = \int_1^2 \cos 2x - \cos 3x \, dx = [\frac{\sin 2x}{2} - \frac{\sin 3x}{3}]_1^2$$

8.2 Exercise 2

Write the following domains as a normal domain with respect to x and as a normal domain with respect to y:

- The triangle with vertices $(1, 0), (2, 0), (2, 1)$.
  1. With respect to x:
  $$D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 2, \ 0 \leq y \leq x - 1\}$$
  2. With respect to y:
  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1, \ y + 1 \leq x \leq 2\}$$
- The triangle with vertices $(0, 0), (1, 0), (\frac{1}{2}, 1)$.
  1. With respect to x:
  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \frac{1}{2}, \ 0 \leq y \leq 2x\} \cup \{(x, y) \in \mathbb{R}^2 : \frac{1}{2} \leq x \leq 1, \ 0 \leq y \leq 2 - 2x\}$$
  2. With respect to y:
  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1, \ \frac{y}{2} \leq x \leq 1 - \frac{y}{2}\}$$
- The region bounded by the lines $y = 2x$, $y = x$ and $y = -x + 1$.
  1. With respect to x:
  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \frac{1}{3}, \ x \leq y \leq 2x\} \cup \{(x, y) \in \mathbb{R}^2 : \frac{1}{3} \leq x \leq \frac{1}{2}, \ x \leq y \leq 1 - x\}$$
  2. With respect to y:
  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \frac{1}{2}, \ \frac{y}{2} \leq x \leq y\} \cup \{(x, y) \in \mathbb{R}^2 : \frac{1}{2} \leq y \leq \frac{2}{3}, \ \frac{y}{2} \leq x \leq 1 - y\}$$
- $D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \ y \leq 0, \ y \geq x - 3\}$.
  1. With respect to x:
  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 3, \ x - 3 \leq y \leq 0\}$$

2. With respect to y:

$$D = \{(x, y) \in \mathbb{R}^2 : -3 \le y \le 0,\ 0 \le x \le 3 - y\}$$

- The region bounded by $y = x^2$ and $y = 1$.

  1. With respect to x:

  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1,\ x^2 \le y \le 1\}$$

  2. With respect to y:

  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 1,\ 0 \le x \le \sqrt{y}\}$$

- The region bounded by $y = \sqrt{x}$, $y = 1 - x$ and the x-axis.

  1. With respect to x:

  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le \frac{3 - \sqrt{5}}{2},\ 0 \le y \le \sqrt{x}\} \cup \{(x, y) \in \mathbb{R}^2 : \frac{3 - \sqrt{5}}{2} \le x \le 1,\ 0 \le y \le 1 - x\}$$

  2. With respect to y:

  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le \frac{-1 + \sqrt{5}}{2},\ y^2 \le x \le 1 - y\}$$

- The quarter of the unit circle contained in the second quadrant.

  1. With respect to x:

  $$D = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 0,\ 0 \le y \le \sqrt{1 - x^2}\}$$

  2. With respect to y:

  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 1,\ -\sqrt{1 - y^2} \le x \le 0\}$$

8.3 Exercise 3

Compute the area of the following regions:

- The triangle with vertices $(0, 0), (2, 0), (1, 1)$.
  By plotting the triangle on a graph, write the corresponding region as a normal domain with respect to y:

  $$D = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 1,\ y \le x \le 2 - y\}$$

  Therefore, apply Fubini's theorem for double integrals and rewrite the integral in the following way:

  $$A(D) = \int\int_D dxdy = \int_0^1 \int_y^{2-y} dxdy = \int_0^1 2 - 2y\ dy = [2y - y^2]_0^1 = 1$$

- $D = \{(x, y) \in \mathbb{R}^2 : x \leq 0,\ y \geq 0,\ y \leq x + 2\}$.
  By plotting the corresponding region on a graph, rewrite D as a normal domain with respect to x:

$$D = \{(x, y) \in \mathbb{R}^2 :\ -2 \leq x \leq 0,\ 0 \leq y \leq x + 2\}$$

  Therefore, apply Fubini's theorem for double integrals and rewrite the integral in the following way:

$$A(D) = \int\int_D dxdy = \int_{-2}^{0}\int_{0}^{x+2} dydx = \int_{-2}^{0} x + 2\ dx = [\frac{x^2}{2} + 2x]_{-2}^{0} = 2$$

- The region bounded between $y = -x^2 + 1$ and $y = 0$.
  By plotting the corresponding region on a graph, rewrite D as a normal domain with respect to x:

$$D = \{(x, y) \in \mathbb{R}^2 :\ -1 \leq x \leq 1,\ 0 \leq y \leq 1 - x^2\}$$

  Therefore, apply Fubini's theorem for double integrals and rewrite the integral in the following way:

$$A(D) = \int\int_D dxdy = \int_{-1}^{1}\int_{0}^{1-x^2} dydx = \int_{-1}^{1} 1-x^2\ dx = [x-\frac{x^3}{3}]_{-1}^{1} = \frac{4}{3}$$

- The region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 2$.
  By plotting the region on a graph, rewrite it as a normal domain with respect to x:

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2,\ 0 \leq y \leq \sqrt{x}\}$$

  Therefore, apply Fubini's theorem for double integrals and rewrite the integral in the following way:

$$A(D) = \int\int_D dxdy = \int_{0}^{2}\int_{0}^{\sqrt{x}} dydx = \int_{0}^{2} \sqrt{x}\ dx = [\frac{x^{\frac{3}{2}}}{\frac{3}{2}}]_{0}^{2} = \frac{4\sqrt{2}}{3}$$

- $D = \{(x, y) \in \mathbb{R}^2 : y \geq x^2 - 1,\ y \leq -x^2 + 1\}$.
  By plotting the corresponding region on a graph, rewrite D as a normal domain with respect to x:

$$D = \{(x, y) \in \mathbb{R}^2 :\ -1 \leq x \leq 1,\ x^2 - 1 \leq y \leq 1 - x^2\}$$

  Therefore, apply Fubini's theorem for double integrals and rewrite the integral in the following way:

$$A(D) = \int\int_D dxdy = \int_{-1}^{1}\int_{x^2-1}^{1-x^2} dydx = \int_{-1}^{1} 2-2x^2\ dx = [2x-\frac{2x^3}{3}]_{-1}^{1} = \frac{8}{3}$$

8.4 Exercise 4

Compute the integral of $f(x, y) = xe^y$ over the triangle with vertices $(0, 0), (-1, 0), (0, -1)$. By plotting the triangle on a graph, rewrite D as a normal domain with respect to x:

$$D = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 0, \ -1 - x \leq y \leq 0\}$$

Therefore, apply Fubini's theorem for double integrals and rewrite the integral in the following way:

$$\int_{-1}^{0} \int_{-1-x}^{0} xe^y \, dydx = \int_{-1}^{0} x[e^y]_{-1-x}^{0} \, dx = \int_{-1}^{0} x(1-e^{-1-x}) \, dx = \int_{-1}^{0} x-xe^{-1-x} \, dx$$

At this point, apply integral linearity and solve each integral separately:

1. First integral:

$$\int_{-1}^{0} x \, dx = [\frac{x^2}{2}]_{-1}^{0} = -\frac{1}{2}$$

2. Second integral:

$$\int_{-1}^{0} xe^{-1-x} \, dx = \int_{-1}^{0} \frac{xe^{-x}}{e} \, dx$$

Apply integration by parts and let $f'(x) = e^{-x}$ and $g(x) = x$:

$$\int_{-1}^{0} xe^{-1-x} \, dx = \frac{1}{e}([-xe^{-x}]_{-1}^{0} - \int_{-1}^{0} -e^{-x} \, dx) = \frac{1}{e}[-xe^{-x} - e^{-x}]_{-1}^{0} = -\frac{1}{e}$$

Therefore, it is possible to conclude that:

$$\int\int_D xe^y \, dxdy = -\frac{1}{2} - (-\frac{1}{e}) = \frac{2-e}{2e}$$

8.5 Exercise 5

Compute the following integrals:

- $f(x, y) = x$ over $D = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, \ 0 \leq y \leq 1 - x^2\}$.
  Since D is expressed as a normal domain with respect to x, apply Fubini's theorem for double integrals and solve the integral in the following way:

$$\int_{-1}^{1} \int_{0}^{1-x^2} x \, dydx = \int_{-1}^{1} x[y]_0^{1-x^2} \, dx = \int_{-1}^{1} x(1-x^2) \, dx = [\frac{x^2}{2} - \frac{x^4}{4}]_{-1}^{1} = 0$$

  N.B.: It is possible to directly conclude that the integral is equal to 0 by observing the symmetry of $x(1 - x^2)$, which is an odd function in the interval $[-1, 1]$.

- $f(x,y) = x^2 + y$ over $D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2,\ 0 \leq y \leq \frac{x}{2}\}$.
  Since D is expressed as a normal domain with respect to x, apply Fubini's theorem for double integrals and solve the integral in the following way:

$$\int_0^2 \int_0^{\frac{x}{2}} x^2 + y \; dydx = \int_0^2 [x^2 y + \frac{y^2}{2}]_0^{\frac{x}{2}} \; dx = \int_0^2 \frac{x^3}{2} + \frac{x^2}{8} \; dx = [\frac{x^4}{8} + \frac{x^3}{24}]_0^2 = \frac{7}{3}$$

- $f(x,y) = xy$ over the triangle of vertices $(0,0), (0,1), (1,1)$.
  By plotting the triangle on a graph, write the corresponding region as a normal domain with respect to x:

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq x\}$$

  Therefore, apply Fubini's theorem for double integrals and solve the integral in the following way:

$$\int_0^1 \int_0^x xy \; dydx = \int_0^1 x[\frac{y^2}{2}]_0^x \; dx = \int_0^1 \frac{x^3}{2} \; dx = [\frac{x^4}{8}]_0^1 = \frac{1}{8}$$

- $f(x,y) = y^3$ over the triangle of vertices $(0,2), (1,1), (3,2)$.
  By plotting the triangle on a graph, write the corresponding region as a normal domain with respect to y:

$$D = \{(x,y) \in \mathbb{R}^2 : 1 \leq y \leq 2,\ 2 - y \leq x \leq 2y - 1\}$$

  Therefore, apply Fubini's theorem for double integrals and solve the integral in the following way:

$$\int_1^2 \int_{2-y}^{2y-1} y^3 \; dxdy = \int_1^2 y^3 [x]_{2-y}^{2y-1} \; dy = \int_1^2 3y^4 - 3y^3 \; dy = [\frac{3y^5}{5} - \frac{3y^4}{4}]_1^2 = \frac{93}{5} - \frac{45}{4}$$

- $f(x,y) = ye^{-x^2}$ over $D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{x}\}$.
  Since D is expressed as a normal domain with respect to x, apply Fubini's theorem for double integrals and solve the integral in the following way:

$$\int_0^1 \int_0^{\sqrt{x}} ye^{-x^2} \; dydx = \int_0^1 e^{-x^2} [\frac{y^2}{2}]_0^{\sqrt{x}} \; dx = \int_0^1 \frac{x}{2} e^{-x^2} \; dx$$

  Apply a change of variable: let $u = -x^2$, for $u \in [0, -1]$, and consider $\frac{du}{dx} = -2x$, resulting in:

$$-\frac{1}{4} \int_0^{-1} e^u \; du = \frac{1}{4} \int_{-1}^0 e^u \; du = \frac{1}{4} [e^u]_{-1}^0 = \frac{1 - e^{-1}}{4}$$

Chapter 9

Exercise Sheet 9: Double and triple integrals

9.1 Exercise 1

Compute the following double integrals:

- $f(x,y) = x^2$ over $D = \{(x,y) \in \mathbb{R}^2 : 4 \le x^2 + y^2 \le 9\}$. Solve the integral by switching to polar coordinates:

  $$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \text{, for } \rho \in [2,3] \text{ and } \theta \in [0, 2\pi] \ \Rightarrow |det(J)| = \rho$$

  Therefore, it is possible to rewrite the integral in the following way:

  $$\int\int_D x^2 \ dxdy = \int_2^3 \int_0^{2\pi} \rho^2 \cos^2\theta \cdot \rho \ d\rho d\theta = \int_2^3 \int_0^{2\pi} \rho^3 \cos^2\theta \ d\rho d\theta$$

  Solve the integral by applying decoupling:

  $$\int_2^3 \rho^3 \ d\rho \int_0^{2\pi} \cos^2\theta \ d\theta = [\frac{\rho^4}{4}]_2^3[\frac{\theta}{2} + \frac{\sin 2\theta}{4}]_0^{2\pi} = \frac{65\pi}{4}$$

- $f(x,y) = x^2 + y^2$ over $D = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4, \ x \ge 0, \ y \ge 0\}$. Solve the integral by switching to polar coordinates:

  $$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \text{, for } \rho \in [1,2] \text{ and } \theta \in [0, \frac{\pi}{2}] \ \Rightarrow |det(J)| = \rho$$

  Therefore, it is possible to rewrite the integral in the following way:

  $$\int\int_D x^2 + y^2 \ dxdy = \int_1^2 \int_0^{\frac{\pi}{2}} (\rho^2\cos^2\theta + \rho^2\sin^2\theta) \cdot \rho \ d\rho d\theta = \int_1^2 \int_0^{\frac{\pi}{2}} \rho^3 \ d\rho d\theta$$

Solve the integral by applying decoupling:

$$\int_1^2 \rho^3 \ d\rho \int_0^{\frac{\pi}{2}} \ d\theta = [\frac{\rho^4}{4}]_1^2 [\theta]_0^{\frac{\pi}{2}} = \frac{15}{4}\frac{\pi}{2} = \frac{15\pi}{8}$$

- $f(x, y) = y^3$ over $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, \ x \geq |y|\}$.
  Solve the integral by switching to polar coordinates:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}, \text{ for } \rho \in [0, 1] \text{ and } \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}] \ \Rightarrow |det(J)| = \rho$$

Therefore, it is possible to rewrite the integral in the following way:

$$\int\int_D y^3 \ dxdy = \int_0^1 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \rho^3 \sin^3\theta \cdot \rho \ d\rho d\theta = \int_0^1 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \rho^4 \sin^3\theta \ d\rho d\theta$$

Notice that this integral is actually equal to 0 by symmetry of $\sin^3\theta$, which is an odd function in the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

- $f(x, y) = xy^2$ over $D = \{x^2 + y^2 \leq 1, \ y \geq 0, \ x \geq 0, \ y \geq -x + 1\}$.
  Apply integral additivity and let $D = D_1 \setminus (D_2 \cup D_3)$ and consider each region separately:

1. Let $D_1 = \{x^2 + y^2 \leq 1, \ y \geq 0, \ x \geq 0\}$ and solve the integral by switching to polar coordinates:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \text{ for } \rho \in [0, 1] \text{ and } \theta \in [0, \frac{\pi}{2}] \ \Rightarrow |det(J)| = \rho$$

Therefore, it is possible to rewrite the integral in the following way:

$$\int_0^1 \int_0^{\frac{\pi}{2}} \rho\cos\theta\rho^2\sin^2\theta \cdot \rho \ d\rho d\theta = \int_0^1 \int_0^{\frac{\pi}{2}} \rho^4 \cos\theta\sin^2\theta \ d\rho d\theta$$

Solve the integral by applying decoupling:

$$\int_0^1 \rho^4 \ d\rho \int_0^{\frac{\pi}{2}} \cos\theta\sin^2\theta \ d\theta = [\frac{\rho^5}{5}]_0^1 [\frac{\sin^3\theta}{3}]_0^{\frac{\pi}{2}} = \frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$$

2. By plotting, let $D_2 = \{0 \leq x \leq \frac{1}{2}, \ x \leq y \leq 1 - x\}$ and apply Fubini's theorem for double integrals:

$$\int_0^{\frac{1}{2}} \int_x^{1-x} xy^2 \ dydx = \int_0^{\frac{1}{2}} x[\frac{y^3}{3}]_x^{x-1} \ dx = \int_0^{\frac{1}{2}} \frac{x(1-x)^3}{3} - \frac{x \cdot x^3}{3} \ dx$$

Rewrite the integral in the following way:

$$\int_0^{\frac{1}{2}} \frac{x - 3x^2 + 3x^3 - 2x^4}{3} \ dx = [\frac{x^2}{6} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{2x^5}{15}]_0^{\frac{1}{2}} = \frac{11}{960}$$

3. Let $D_3 = \{x^2 + y^2 \leq 1,\ x \geq 0,\ 0 \leq y \leq x\}$ and solve the integral by switching to polar coordinates:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad \text{for } \rho \in [0,1] \text{ and } \theta \in [0, \frac{\pi}{4}] \ \Rightarrow |det(J)| = \rho$$

Therefore, it is possible to rewrite the integral in the following way:

$$\int_0^1 \int_0^{\frac{\pi}{4}} \rho\cos\theta\rho^2\sin^2\theta \cdot \rho \ d\rho d\theta = \int_0^1 \int_0^{\frac{\pi}{4}} \rho^4 \cos\theta \sin^2\theta \ d\rho d\theta$$

Solve the integral by applying decoupling:

$$\int_0^1 \rho^4 \ d\rho \int_0^{\frac{\pi}{4}} \cos\theta\sin^2\theta \ d\theta = [\frac{\rho^5}{5}]_0^1[\frac{\sin^3\theta}{3}]_0^{\frac{\pi}{4}} = \frac{1}{5} \cdot \frac{2\sqrt{2}}{24} = \frac{\sqrt{2}}{60}$$

Therefore, it is possible to conclude that:

$$\int\int_D xy^2 \ dxdy = \frac{1}{15} - (\frac{11}{960} + \frac{\sqrt{2}}{60}) = \frac{53 - 16\sqrt{2}}{960}$$

- $f(x,y) = x + y$ over $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2,\ y \leq x^2,\ x \geq 0\}$.
Apply integral additivity and let $D = D_1 \cup D_2$ and consider each region separately:

1. Let $D_1 = \{0 \leq x \leq 1,\ -\sqrt{2 - x^2} \leq y \leq x^2\}$ and apply Fubini's theorem for double integrals:

$$\int_0^1 \int_{-\sqrt{2-x^2}}^{x^2} x{+}y \, dydx = \int_0^1 [xy{+}\frac{y^2}{2}]_{-\sqrt{2-x^2}}^{x^2} \, dx = \int_0^1 x^3{+}\frac{x^4}{2}{+}x\sqrt{2-x^2}{-}\frac{2-x^2}{2} \ dx$$

For simplicity, apply integral linearity and solve each integral separately:

(a) First integral:

$$\int_0^1 x^3 \ dx = [\frac{x^4}{4}]_0^1 = \frac{1}{4}$$

(b) Second integral:

$$\int_0^1 \frac{x^4}{2} \ dx = [\frac{x^5}{10}]_0^1 = \frac{1}{10}$$

(c) Third integral:

$$\int_0^1 x\sqrt{2 - x^2} \ dx$$

Apply a change of variable: let $u = 2 - x^2$, for $u \in [2, 1]$, and consider $\frac{du}{dx} = -2x$, resulting in:

$$-\frac{1}{2}\int_2^1 \sqrt{u}\ du = \frac{1}{2}\int_1^2 \sqrt{u}\ du = \frac{1}{2}[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}]_1^2 = \frac{2\sqrt{2}-1}{3}$$

(d) Fourth integral:

$$\int_0^1 \frac{2-x^2}{2}\ dx = \int_0^1 1 - \frac{x^2}{2}\ dx = [x - \frac{x^3}{6}]_0^1 = \frac{5}{6}$$

Therefore, it is possible to conclude that:

$$\int\int_{D_1} x + y\ dxdy = \frac{1}{4} + \frac{1}{10} + \frac{2\sqrt{2}-1}{3} - \frac{5}{6} = \frac{40\sqrt{2}-49}{60}$$

2. Let $D_2 = \{1 \leq x \leq \sqrt{2},\ -\sqrt{2-x^2} \leq y \leq \sqrt{2-x^2}\}$ and apply Fubini's theorem for double integrals:

$$\int_1^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} x+y\ dydx = \int_1^{\sqrt{2}} [xy+\frac{y^2}{2}]_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\ dx = \int_1^{\sqrt{2}} 2x\sqrt{2-x^2}\ dx$$

Apply a change of variable: let $u = 2-x^2$, for $u \in [1, 0]$, and consider $\frac{du}{dx} = -2x$, resulting in:

$$-\int_1^0 \sqrt{u}\ du = \int_0^1 \sqrt{u}\ du = [\frac{u^{\frac{3}{2}}}{\frac{3}{2}}]_0^1 = \frac{2}{3}$$

Therefore, it is possible to conclude that:

$$\int\int_D x + y\ dxdy = \frac{40\sqrt{2}-49}{60} + \frac{2}{3} = \frac{40\sqrt{2}-9}{60}$$

- $f(x, y) = y\cos x$ over $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ y \geq 1 - x\}$.
  By plotting the corresponding region on a graph, rewrite D as a normal domain with respect to x:

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 1 - x \leq y \leq \sqrt{1-x^2}\}$$

Therefore, apply Fubini's theorem for double integrals and rewrite the integral in the following way:

$$\int_0^1\int_{1-x}^{\sqrt{1-x^2}} y\cos x\ dydx = \int_0^1 \cos x[\frac{y^2}{2}]_{1-x}^{\sqrt{1-x^2}}\ dx = \int_0^1 (x - x^2)\cos x\ dx$$

At this point, apply integral linearity and solve each integral separately:

1. First integral:
$$\int_0^1 x\cos x \; dx$$
Apply integration by parts and let $f'(x) = \cos x$ and $g(x) = x$:
$$\int_0^1 x\cos x \; dx = [x\sin x]_0^1 - \int_0^1 \sin x \; dx = [x\sin x + \cos x]_0^1 = \sin 1 + \cos 1 - 1$$
2. Second integral:
$$\int_0^1 x^2 \cos x \; dx$$
Apply integration by parts and let $f'(x) = \cos x$ and $g(x) = x^2$:
$$\int_0^1 x^2 \cos x = [x^2 \sin x]_0^1 - \int_0^1 2x \sin x \; dx$$
Again, apply integration by parts and let $r'(x) = \sin x$ and $s(t) = 2x$:
$$\int_0^1 2x\sin x \; dx = [-2x\cos x]_0^1 - \int_0^1 -2\cos x \; dx = [-2x\cos x + 2\sin x]_0^1$$
This allows to conclude that:
$$\int_0^1 x^2 \cos x \; dx = [x^2 \sin x + 2x\cos x - 2\sin x]_0^1 = 2\cos 1 - \sin 1$$

Therefore, it is possible to conclude that:
$$\int\int_D y\cos x \; dxdy = \sin 1 + \cos 1 - 1 - (2\cos 1 - \sin 1) = 2\sin 1 - \cos 1 - 1$$

- $f(x,y) = e^{x-y}$, where D is the region bounded by $x+y=4$, $3x+y=4$ and $x+3y=4$.
Apply integral additivity and let $D = D_1 \cup D_2$ and consider each region separately:

1. Let $D_1 = \{0 \le x \le 1,\ 4-3x \le y \le 4-x\}$ and apply Fubini's theorem for double integrals:
$$\int_0^1 \int_{4-3x}^{4-x} e^x e^{-y} \; dydx = \int_0^1 e^x [-e^{-y}]_{4-3x}^{4-x} \; dx = \int_0^1 \frac{e^{4x} - e^{2x}}{e^4} \; dx = \frac{1}{e^4}[\frac{e^{4x}}{4} - \frac{e^{2x}}{2}]_0^1 = \frac{e^4 - 2e^2 + 1}{4e^4}$$
2. Let $D_2 = \{1 \le x \le 4,\ \frac{4-x}{3x} \le y \le 4-x\}$ and apply Fubini's theorem for double integrals:
$$\int_1^4 \int_{\frac{4-x}{3}}^{4-x} e^x e^{-y} \; dydx = \int_1^4 e^x [-e^{-y}]_{\frac{4-x}{3}}^{4-x} \; dx = \int_1^4 \frac{e^{\frac{4x}{3}}}{e^{\frac{4}{3}}} - \frac{e^{2x}}{e^4} \; dx = [\frac{3e^{\frac{4x}{3}}}{4e^{\frac{4}{3}}} - \frac{e^{2x}}{2e^4}]_1^4 = \frac{e^6 - 3e^2 + 2}{4e^4}$$

Therefore, it is possible to conclude that:

$$\int\int_D e^{x-y}\ dxdy = \frac{e^4-2e^2+1}{4e^4} + \frac{e^6-3e^2+2}{4e^4} = \frac{e^6+e^4-5e^2+3}{4e^4}$$

9.2 Exercise 2

Compute the volume of the following regions:

- $\Omega = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 1 \leq y \leq 2,\ z \geq 0,\ z \leq 4-x-y\}$.
  Start by extracting $Q = [0,1] \times [1,2]$ and $z \in [0, 4-x-y]$ from the definition of Ω, resulting in:

  $$V(\Omega) \int\int\int_\Omega\ dxdydz = \int\int_Q \int_0^{4-x-y}\ dzdxdy = \int\int_Q 4-x-y\ dxdy$$

  At this point, apply Fubini's theorem for double integrals:

  $$V(\Omega) = \int_1^2 \int_0^1 4-x-y\ dxdy = \int_1^2 [4x-\frac{x^2}{2}-yx]_0^1\ dy = \int_1^2 \frac{7}{2}-y\ dy = [\frac{7y}{2}-\frac{y^2}{2}]_1^2 = 2$$

- $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z \leq 2\}$.
  Due to the definition of Ω, it is useful to switch to cylindrical coordinates:

  $$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \quad \text{for } \rho \in [0,\sqrt{2}],\ \theta \in [0,2\pi] \text{ and } z \in [\rho^2, 2].$$

  Knowing that, in cylindrical coordinates, $|det(J)| = \rho$, it is possible to conclude that:

  $$V(\Omega) = \int\int\int_\Omega\ dxdydz = \int_0^{\sqrt{2}} \int_0^{2\pi} \int_{\rho^2}^2 \rho\ d\rho d\theta dz$$

  $$V(\Omega) = 2\pi \int_0^{\sqrt{2}} \rho[z]_{rho^2}^2\ d\rho = 2\pi \int_0^{\sqrt{2}} 2\rho - \rho^3\ d\rho = 2\pi[\rho^2 - \frac{\rho^4}{4}]_0^{\sqrt{2}} = 2\pi$$

9.3 Exercise 3

Compute the following integral:

$$\int\int\int_\Omega x+y+z\ dxdydz,\ \Omega = \{0 \leq x \leq 1,\ 2x \leq y \leq x,\ 0 \leq z \leq x+y\}$$

From the definition of Ω, let $D = \{0 \leq x \leq 1,\ 2x \leq y \leq x\}$ and $z \in [0, x+y]$, resulting in:

$$\int_0^1 \int_{2x}^x \int_0^{x+y} x+y+z\ dzdydx = \int_0^1 \int_{2x}^x x(x+y)+y(x+y)+\frac{(x+y)^2}{2}\ dydx$$

Most particularly, by doing some algebra, it is possible to rewrite the integral in the following way:

$$\int_0^1 \int_{2x}^x (x+y)(\frac{3x+3y}{2})\ dydx = \int_0^1 \int_{2x}^x \frac{3(x+y)^2}{2}\ dydx$$

At this point, apply Fubini's theorem for double integrals:

$$\int_0^1 \int_{2x}^x \frac{3(x+y)^2}{2}\ dydx = \int_0^1 \frac{3}{2}[\frac{(x+y)^3}{3}]_{2x}^x\ dx = \int_0^1 -\frac{19x^3}{2}\ dx = [-\frac{x^4}{8}]_0^1 = -\frac{19}{8}$$

9.4 Exercise 4

Compute the following integral:

$$\int\int\int_\Omega y\ dxdydz,\ \Omega = \{x^2+y^2 \leq 1,\ z \geq 0,\ z \leq x\}$$

Solve the integral by switching to cylindrical coordinates:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \qquad \text{for } \rho \in [0,1],\ \theta \in [0,2\pi],\ z \in [0, \rho\cos\theta]\ \Rightarrow |det(J)| = \rho \\ z = z \end{cases}$$

Therefore, it is possible to rewrite the integral in the following way:

$$\int_0^1 \int_0^{2\pi} (\int_0^{\rho\cos\theta} \rho\sin\theta \cdot \rho\ dz)\ d\rho d\theta = \int_0^1 \int_0^{2\pi} \rho^2 \sin\theta \cdot \rho\cos\theta\ d\rho d\theta$$

Solve the integral by applying decoupling:

$$\int_0^1 \rho^3\ d\rho \int_0^{2\pi} \sin\theta\cos\theta\ d\theta = [\frac{\rho^4}{4}]_0^1 [\frac{\sin^2\theta}{2}]_0^{2\pi} = 0$$

N.B.: It is possible to directly notice that the integral is 0 either by observing the periodicity of $\sin\theta$ or by applying substitution and letting $u = \sin\theta$, with $u \in [0, 0]$.

9.5 Exercise 5

Compute the following integral:

$$\int\int\int_\Omega e^y z\ dxdydz,\ \Omega = \{0 \leq x \leq 1,\ 0 \leq y \leq x^3,\ 0 \leq z \leq x\}$$

From the definition of Ω, let $D = \{0 \leq x \leq 1,\ 0 \leq y \leq x^3\}$ and $z \in [0, x]$, resulting in:

$$\int_0^1 \int_0^{x^3} (\int_0^x e^y z\ dz)\ dydx = \int_0^1 \int_0^{x^3} e^y \frac{x^2}{2}\ dydx$$

At this point, apply Fubini's theorem for double integrals:

$$\int_0^1 \int_0^{x^3} e^y \frac{x^2}{2}\ dydx = \int_0^1 \frac{x^2}{2}[e^y]_0^{x^3}\ dx = \int_0^1 \frac{x^2}{2}(e^{x^3}-1)\ dx$$

Apply integral linearity and solve each integral separately:

1. First integral:

$$\int_0^1 \frac{x^2 e^{x^3}}{2}\ dx$$

   Apply a change of variable: let $u = x^3$, for $u \in [0,1]$, and consider $\frac{du}{dx} = 3x^2$, resulting in:

$$\frac{1}{6}\int_0^1 e^u\ du = \frac{1}{6}[e^u]_0^1 = \frac{e-1}{6}$$

2. Second integral:

$$\int_0^1 \frac{x^2}{2}\ dx = [\frac{x^3}{6}]_0^1 = \frac{1}{6}$$

Therefore, it is possible to conclude that:

$$\int\int\int_\Omega e^y z\ dxdydz = \frac{e-1}{6} - \frac{1}{6} = \frac{e-2}{6}$$

9.6 Exercise 6

Compute the following integral:

$$\int\int\int_\Omega yz\ dxdydz,\ \Omega = \{x^2+y^2 \leq 4,\ y-z+1 \geq 0,\ z \geq -4\}$$

Solve the integral by switching to cylindrical coordinates:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \qquad \text{for } \rho \in [0,2],\ \theta \in [0,2\pi],\ z \in [-4, \rho\sin\theta + 1] \ \Rightarrow |det(J)| = \rho \\ z = z \end{cases}$$

Therefore, it is possible to rewrite the integral in the following way:

$$\int_0^2 \int_0^{2\pi} (\int_{-4}^{\rho\sin\theta+1} \rho\sin\theta z\cdot\rho\ dz)\ d\rho d\theta = \int_0^2\int_0^{2\pi} \rho^2\sin\theta(\frac{\rho^2\sin^2\theta + 2\rho\sin\theta - 15}{2})\ d\rho d\theta$$

At this point, apply integral linearity and solve each integral separately:

1. First integral:
   Solve the integral by applying decoupling:
   $$\int_0^2 \frac{\rho^4}{2}\ d\rho \int_0^{2\pi} \sin^3\theta\ d\theta = 0 \text{ by periodicity of } \sin^3\theta.$$
2. Second integral:
   Given $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, solve the integral by applying decoupling:
   $$\int_0^2 \rho^3\ d\rho \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\ d\theta = [\frac{\rho^4}{4}]_0^2[\frac{\theta}{2} - \frac{\sin 2\theta}{4}]_0^{2\pi} = 4\pi$$
3. Third integral:
   Solve the integral by applying decoupling:
   $$\int_0^2 \frac{15\rho^2}{2}\ d\rho \int_0^{2\pi} \sin\theta\ d\theta = 0 \text{ by periodicity of } \sin\theta.$$

Therefore, it is possible to conclude that:

$$\int\int\int_\Omega yz\ dxdydz = 0 + 4\pi - 0 = 4\pi$$