Critique de l'Intelligence Artificielle
Enjeux philosophiques, politiques et culturels de l'automatisation numérique
LLCP, Université Paris 8
Paris, France

Critique de la raison linguistique

De quoi les modèles neuronaux de langage sont-ils le modèle?

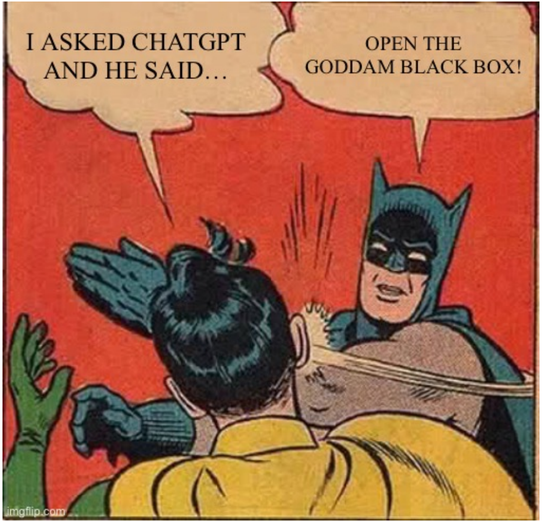
Juan Luis Gastaldi

ETH zürich

28 Mai, 2024

Plan

Plan


I ASKED CHATGPT
AND HE SAID…
OPEN THE
GODDAM BLACK BOX!
imgflip.com

Introduction

- Toute critique est incomplète sans comprehension interne des phénomèmes

- Dimension epistémologique de la critique

- Ce qu'il y a à connaître: Structure Computationnelle du Langage

Épistémè Formelle

Structure
Langages
Fornels

Chomsky

Computation
Théorie des
Automates

Langage
Grammaire
Générative

Plan

Trois Composantes Principales des Modèles de TAL

Tokenisation (Subword Tokenization)
(Sennrich et al., 2016)

Vecteurs des Mots (Word Embeddings)
(Mikolov, Sutskever, Chen, Corrado, and Dean, 2013)

Auto-Attention (Self-Attention)
(Vaswani et al., 2017)

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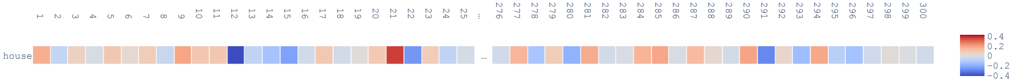
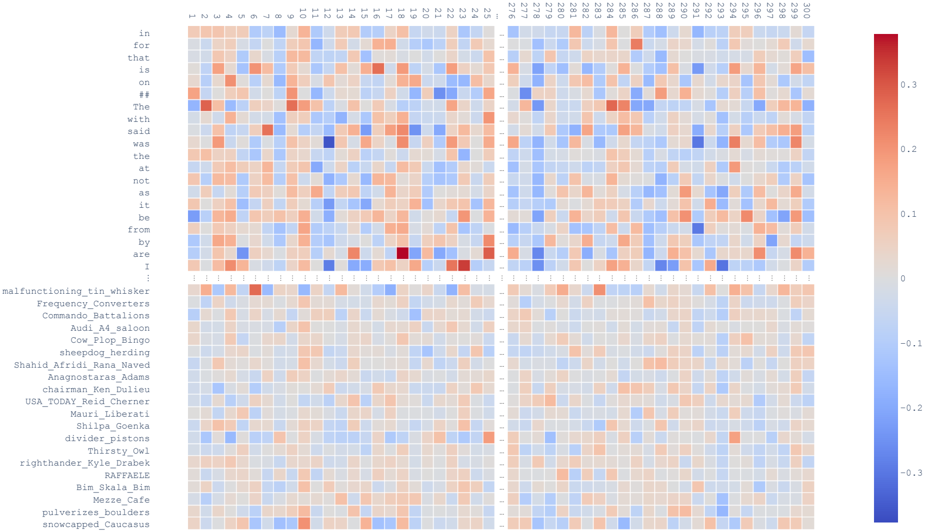
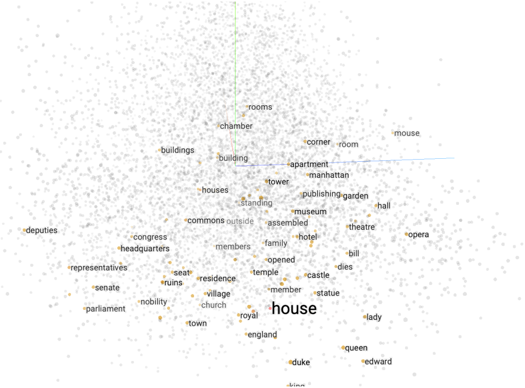
Vecteurs des Mots: Vecteur



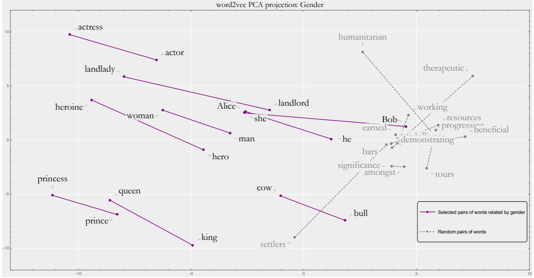
Vecteurs des Mots: Matrice



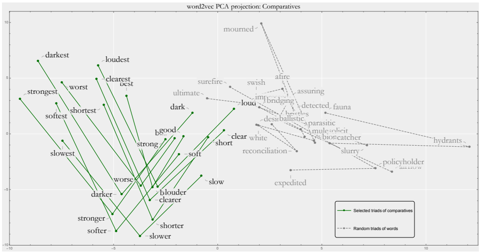
Espace de Plongement: Similarité et Analogie

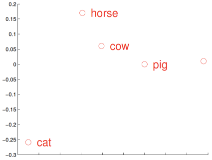


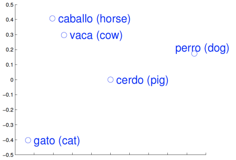
(https://projector.tensorflow.org)



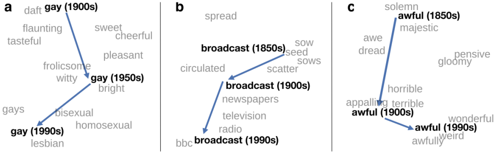
Espace de Plongement: Autres Applications



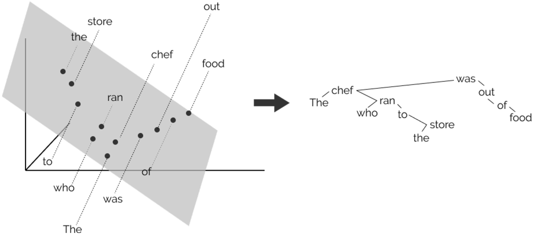




(Mikolov, Sutskever, Chen, Corrado, Dean, et al., 2013)

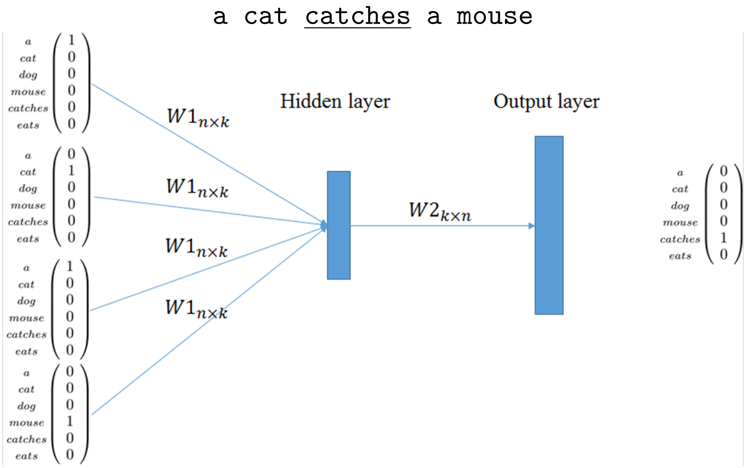


(Hamilton et al., 2016)



(https://nlp.stanford.edu/~johnhew/structural-probe.html)

Modèle word2vec



Credit: Ferrone et al., 2017

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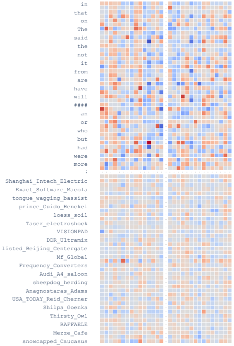
word2vec Comme Factorisation Implicite de Matrice

(Levy and Goldberg, 2014)



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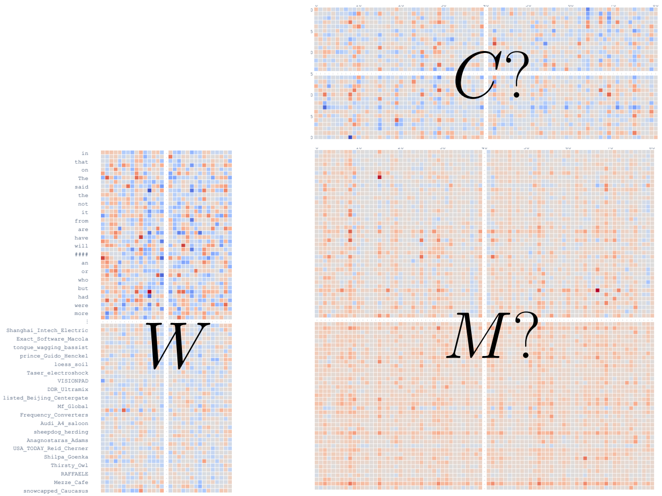

$$W \times C \approx M$$

word2vec Expliqué

(Levy and Goldberg, 2014)

$$\ell = \sum_{w \in V_w} \sum_{c \in V_c} \#(w,c)\big(\log \sigma(\vec{w} \cdot \vec{c}) + k \cdot \mathbb{E}_{c_N \sim P_D}[\log \sigma(-\vec{w} \cdot \vec{c}_N)]\big)$$

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$$\frac{\partial \ell}{\partial(\vec{w} \cdot \vec{c})} = 0 \quad \text{when} \quad \vec{w} \cdot \vec{c} = \log\left(\frac{\#(w,c) \cdot |D|}{\#(w) \cdot \#(c)}\right) - \log k$$

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Trois résultats:

- $M = PMI(w,c) - \log k$ (Pointwise Mutual Information)

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Trois résultats:

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- W est de basse dimension

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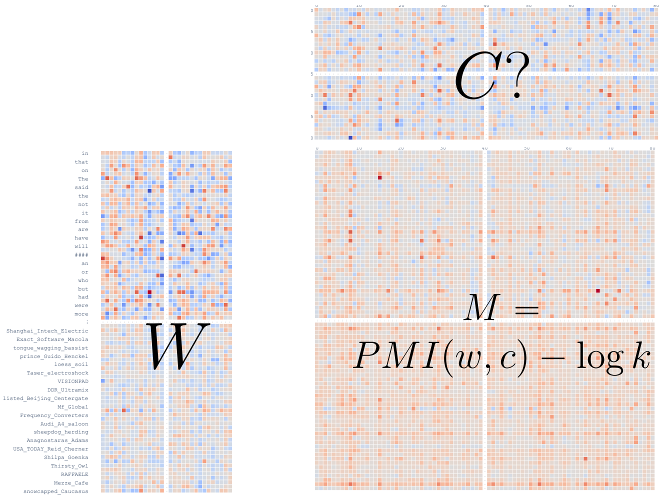
Trois résultats:

- $M = PMI(w,c) - \log k$ (Pointwise Mutual Information)
- W est de basse dimension
- La Décomposition en Valeurs Singulières (SVD) donne une solution exacte pour trouver W

Information Mutuelle Ponctuelle (PMI)



$$PMI(w,c) = \log \frac{p(w,c)}{p(w)p(c)}$$

Décomposition en Valeurs Singulières (SVD)

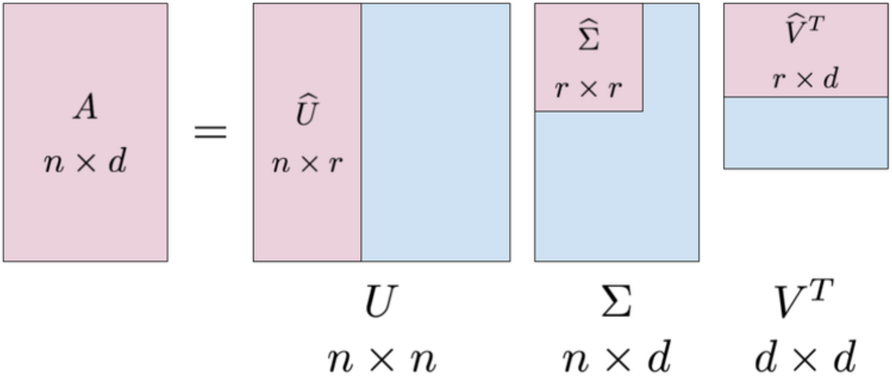
$$M = U\Sigma V^*$$

Où:

$$\begin{aligned}
M &= \text{Matrice } m \times n \text{ (réelle ou complexe)} \\
U &= \text{Matrice } m \times m \text{ unitaire} \\
\Sigma &= \text{Matrice } m \times n \text{ diagonale rectangulaire réelle non-négative} \\
V^* &= \text{Matrice } n \times n \text{ unitaire; transposée conjuguée de } V
\end{aligned}$$

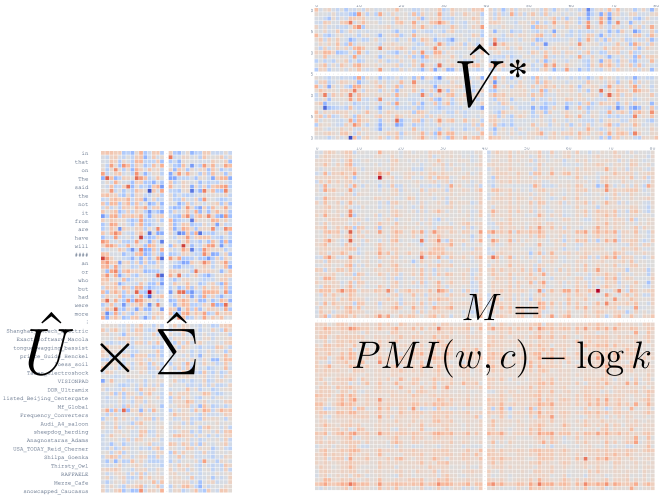
SVD Tronquée

$$M = U\Sigma V^*$$



Credit: Angela Ju

Vecteurs des Mots comme SVD Tronquée

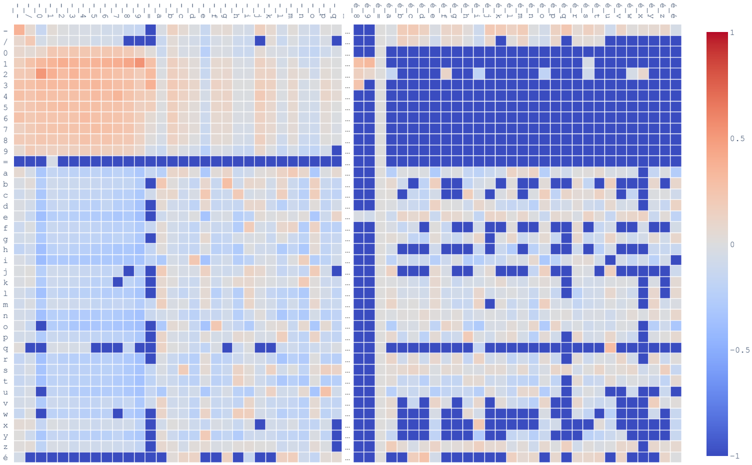


$$M \approx \hat{U} \times \hat{\Sigma} \times \hat{V}^*$$

Plan

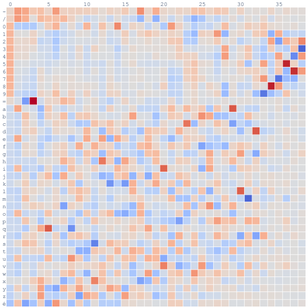
Exemple: Characters dans Wikipedia

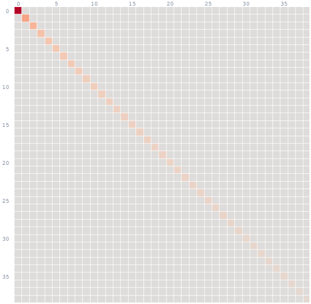
$$PMI(w,c) = \log \frac{p(w,c)}{p(w)p(c)}$$

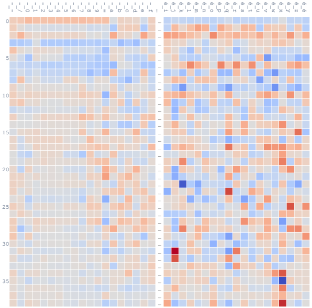


SVD d'une Matrice PMI des Characters de Wikipedia

U Σ V^*

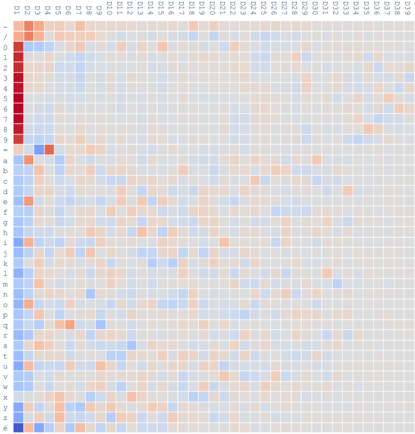






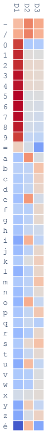
Tronquer et Plonger

$$U \times \Sigma$$



Tronquer et Plonger

$$\hat{U} \times \hat{\Sigma}$$



Tronquer et Plonger

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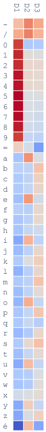

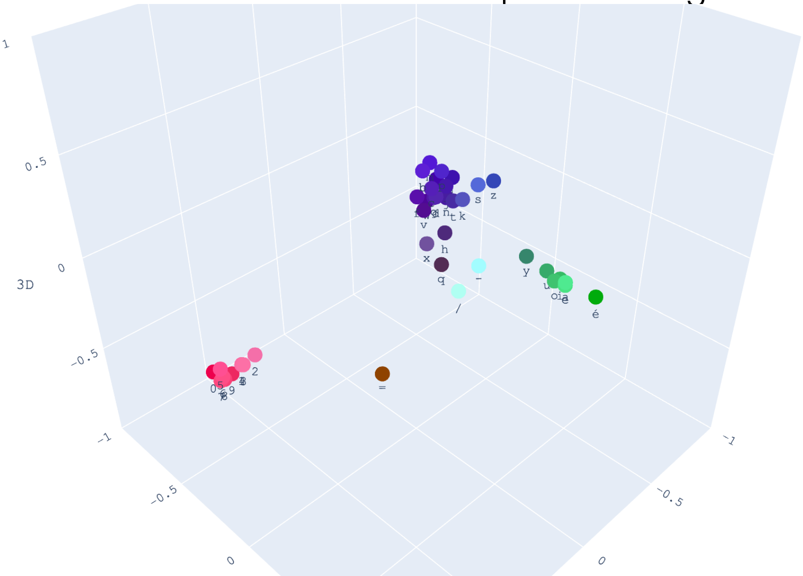

4 Why does this produce good word representations?

Good question. We don't really know.

The distributional hypothesis states that words in similar contexts have similar meanings. The objective above clearly tries to increase the quantity $v_w \cdot v_c$ for good word-context pairs, and decrease it for bad ones. Intuitively, this means that words that share many contexts will be similar to each other (note also that contexts sharing many words will also be similar to each other). This is, however, very hand-wavy.

Can we make this intuition more precise? We'd really like to see something more formal.

(Goldberg and Levy, 2014)

Plan

Décomposition en Valeurs Singulières (SVD)

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Où:

$$\begin{array}{rcl} M & = & \text{Matrice } m \times n \text{ (réelle ou complexe)} \\ U & = & \text{Matrice } m \times m \text{ unitaire} \\ \Sigma & = & \text{Matrice } m \times n \text{ diagonale rectangulaire réelle non-négative} \\ V^* & = & \text{Matrice } n \times n \text{ unitaire; transposée conjuguée de } V \end{array}$$

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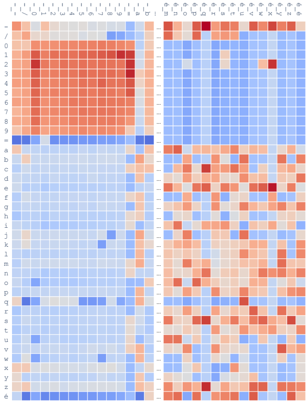
<u>En particulier</u>:

- Les colonnes de U sont des vecteurs propres de MM^*
- Les lignes de V^* sont des vecteurs propres de M^*M
- Les éléments non nuls de Σ sont les racines carrées des valeurs propres non nulles de MM^* ou M^*M

$M \times M^*$ Comme Matrice de Covariance



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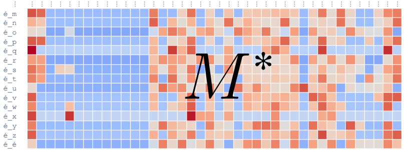

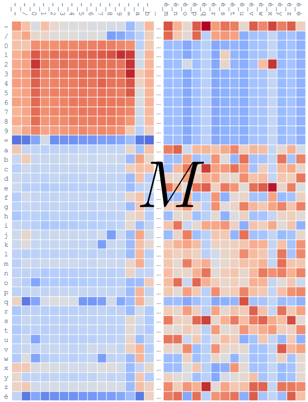

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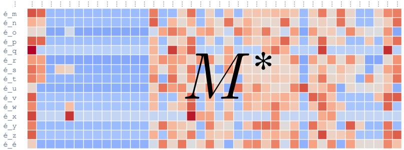
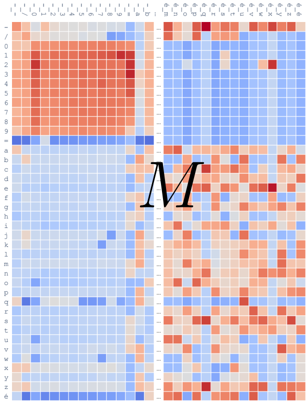
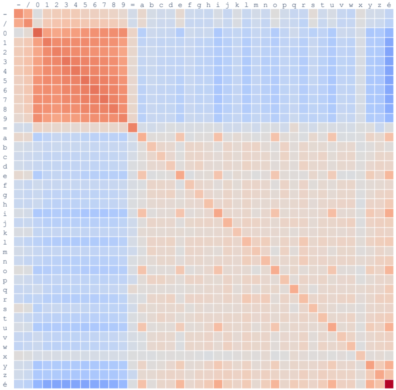
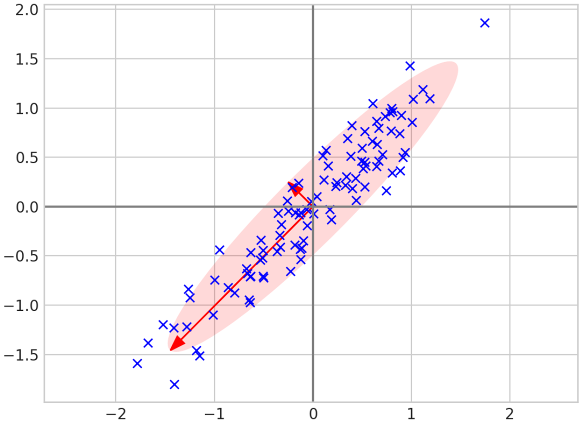




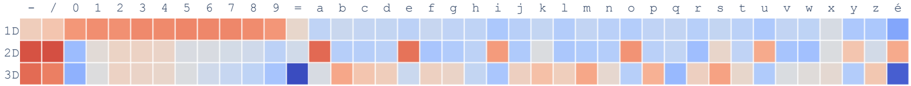


Vecteurs et Valeurs Propres



Credit: Joel Laity

Eigenvectors of $M \times M^*$:



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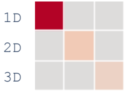
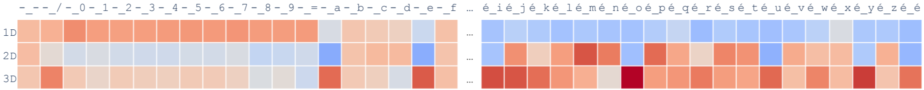


Eigenvectors of $M^* \times M$:

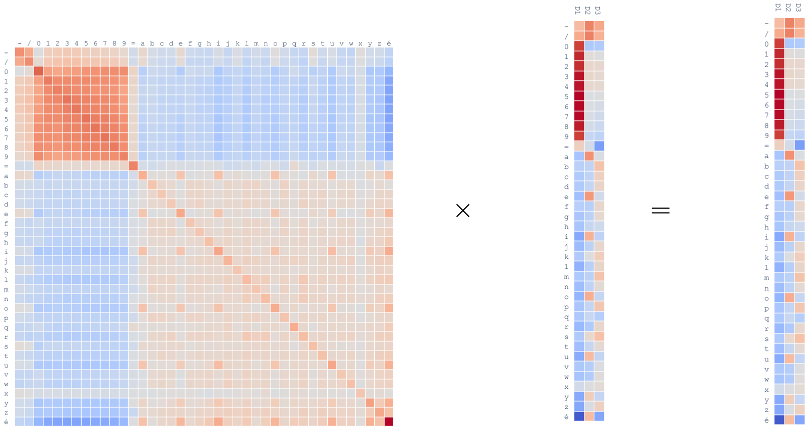


Vecteurs Propres comme Points Fixes

$$(M \times M^*)v = \lambda v$$

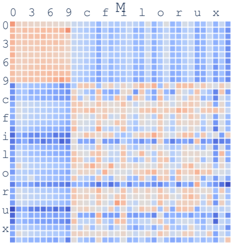
Vecteurs Propres comme Points Fixes

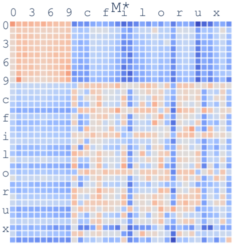
$$(M \times M^*)\textcolor{red}{v} = \lambda \textcolor{red}{v}$$

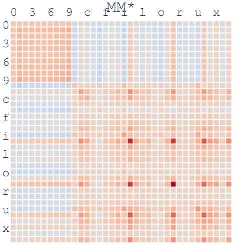


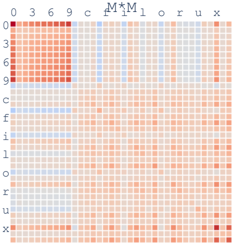
Matrices Binaires

$$(M_t \star M_t^*) \star {\color{red} v} = {\color{red} v}$$



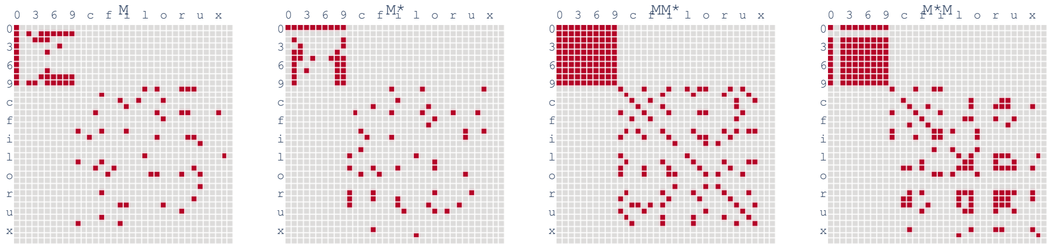




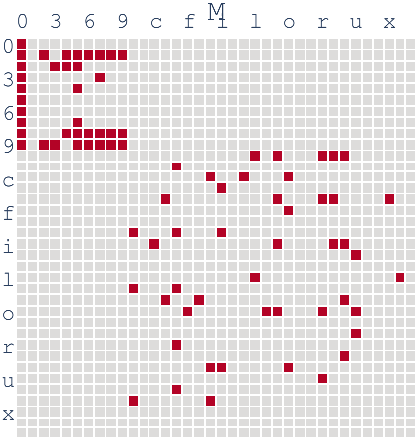


Matrices Binaires

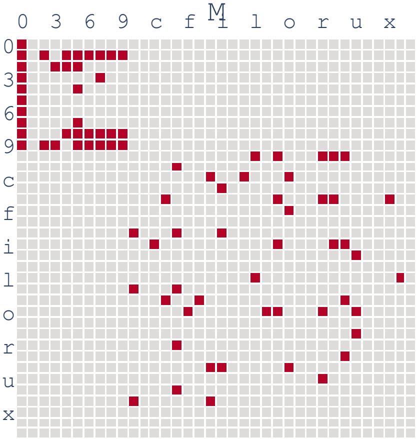
$$(M_t \star M_t^*) \star v = v$$

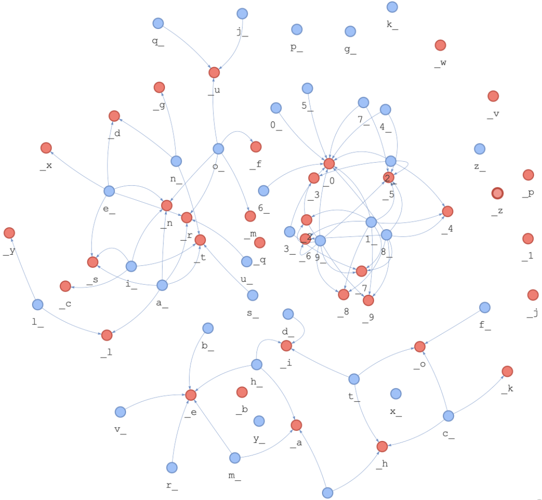


M

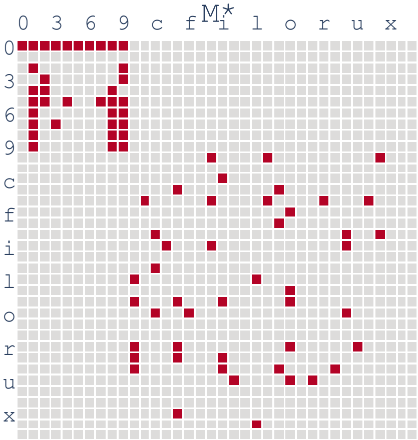


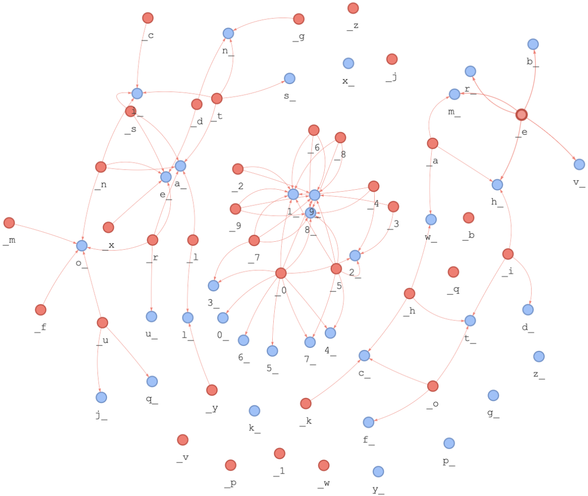
M



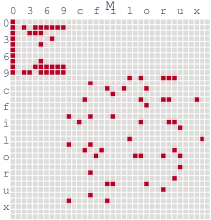


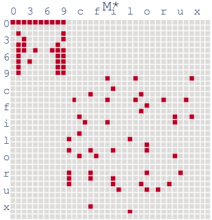
M^*

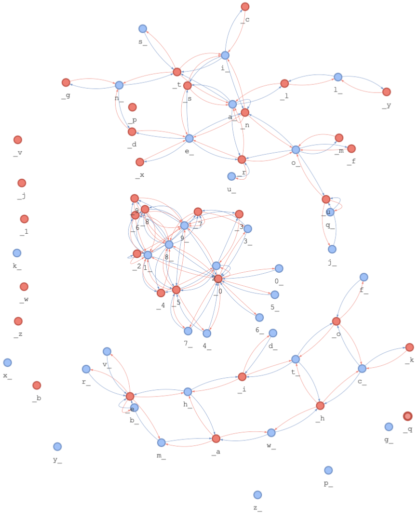




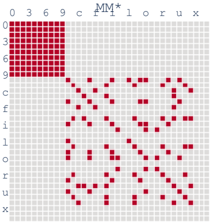
M et M^*

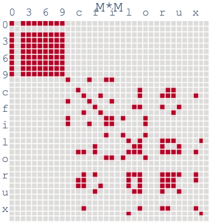

M
0 3 6 9 c f i l o r u x
0
3
6
9
c
f
i
l
o
r
u
x

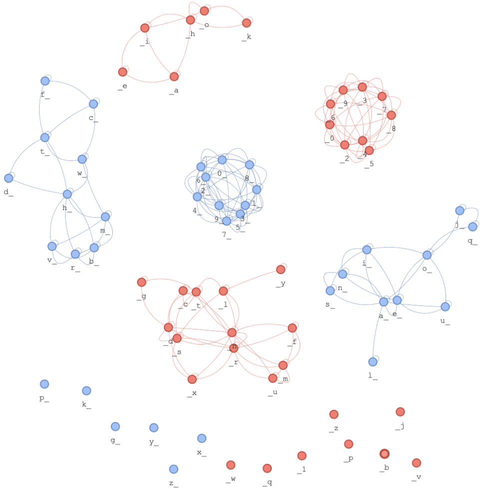

M*
0 3 6 9 c f i l o r u x
0
3
6
9
c
f
i
l
o
r
u
x



MM^* et M^*M







Quelle Structure?

Profoncteur
$f : \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$

$$\begin{tikzcd}
 & (\mathsf{Set}^{\mathsf{D}})^{op} \\
\mathsf{C} \arrow[ur] \arrow[r, hook, "Yoneda"'] & \mathsf{Set}^{\mathsf{C}^{op}} \arrow[u, dashed, "F^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathsf{Set}^{\mathsf{D}})^{op} \arrow[d, dashed, "F_*"'] & \mathsf{D} \arrow[l, hook', "Yoneda"'] \arrow[dl] \\
\mathsf{Set}^{\mathsf{C}^{op}} &
\end{tikzcd}$$

$\mathsf{C} \to (\mathsf{Set}^{\mathsf{D}})^{\mathrm{op}}$
$\mathsf{D} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$

$F^* : \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \to (\mathsf{Set}^{\mathsf{D}})^{\mathrm{op}}$
$F_* : (\mathsf{Set}^{\mathsf{D}})^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$

$F^* c_i \cong d_i$ and $F_* d_i \cong c_i$.

Isbell adjunction
$F^* : \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \leftrightarrows (\mathsf{Set}^{\mathsf{D}})^{\mathrm{op}} : F_*$

(Bradley et al., 2024)

Plan

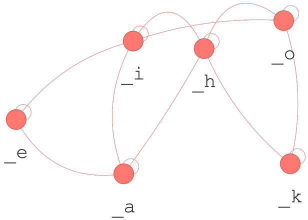
Definition (Espaces cohérents – Girard, 2006)

On appelle espace cohérent X la donnée :

Trame : Un ensemble sous-jacent, sa trame $|X|$.

Cohérence : Une relation réflexive et symétrique, sa cohérence $x \frown_X y$.

On appelle clique de X, notation $a \sqsubset X$, un sous-ensemble de $|X|$ dont les points sont deux à deux cohérents.



Théorie de Types Computationnels

Definition (Espaces cohérents – Girard, 2006)

On appelle espace cohérent X la donnée :
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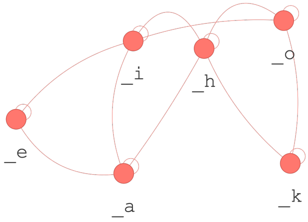

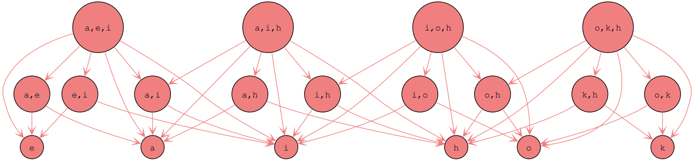

Théorie de Types Computationnels

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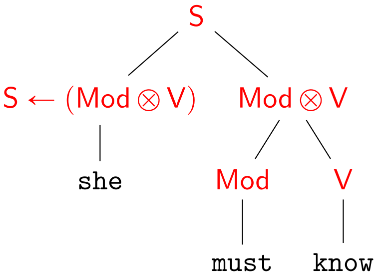
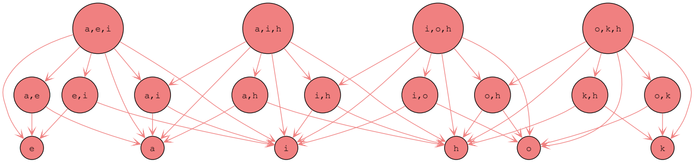




Plan

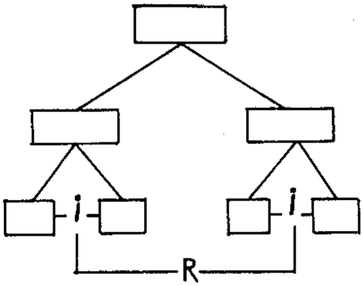
Semiotique Structurale

Une sémiotique [...] est une hiérarchie dont chacune des composantes admet une analyse ultérieure en classes définies par relation mutuelle, de telle sorte que chacune de ces classes admette une analyse en dérivés définis par mutation mutuelle.

Hjelmslev, 1975, Df. 24

Une mutation [...] est une fonction existant entre des dérivés du premier degré d'une seule et même classe, une fonction qui a une relation à une fonction entre d'autres dérivés de premier degré d'une seule et même classe et appartenant au même rang.

Hjelmslev, 1975, Df. 23



Syntagmatique et Texte

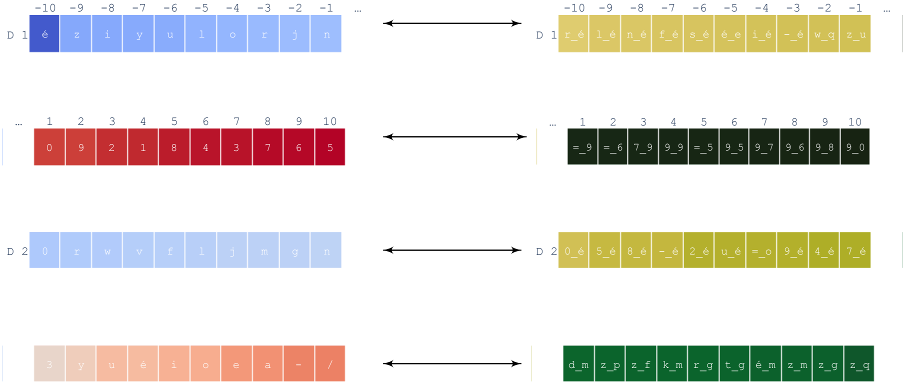
Une syntagmatique ou procès de signes [...] est un procès sémiotique.

Hjelmslev, 1975, Df. 33

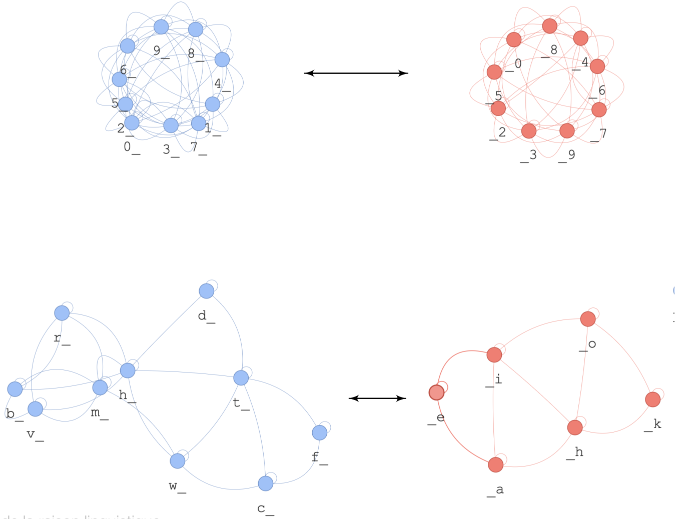
Un texte [...] est la syntagmatique d'une sémiotique dénotative dont les chaînes sont manifestés par toutes les matières.

Hjelmslev, 1975, Df. 39

Syntagmatique et Texte (Vecteurs)



Syntagmatique et Texte (Noyeaux/Types)



Paradigmatique et Langue

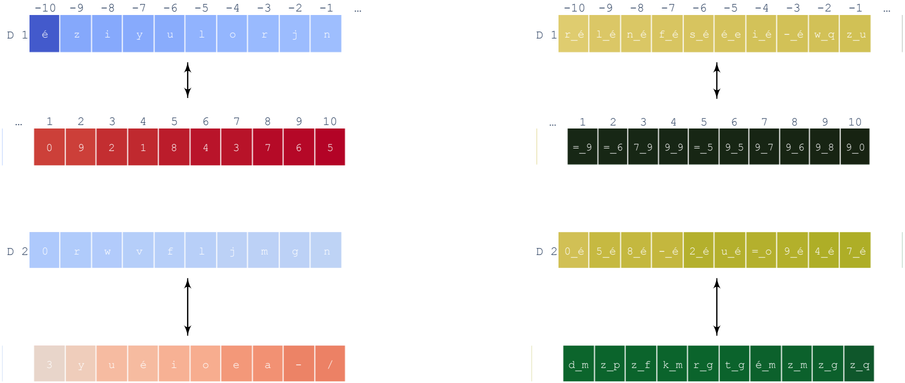
Une paradigmatique ou système de signes [...] est un système sémiotique.

Hjelmslev, 1975, Df. 35

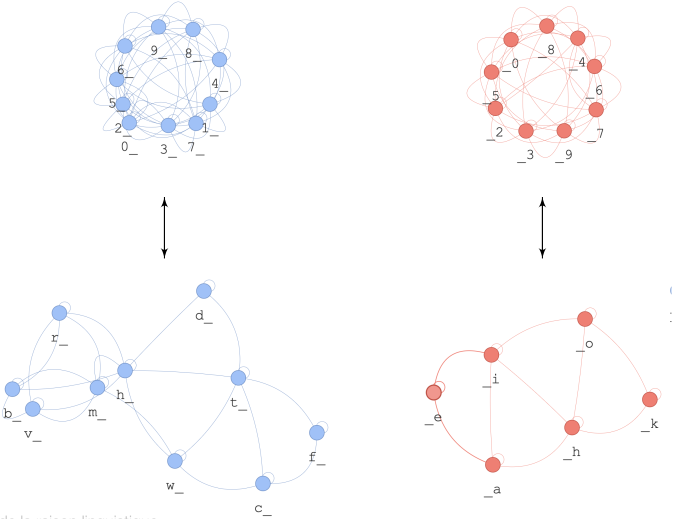
Une langue [...] est la paradigmatique d'une sémiotique dénotative dont les paradigmes sont manifestés par toutes les matières.

Hjelmslev, 1975, Df. 38

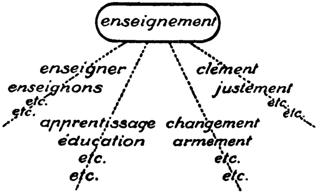
Paradigmatique et Langue (Vecteurs)



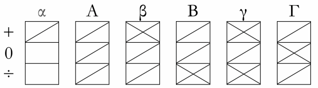
Paradigmatique et Langue (Noyaux/Types)



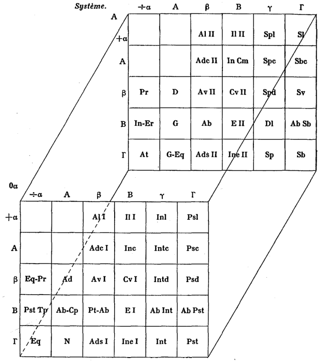
Structuralisme et Formlisme



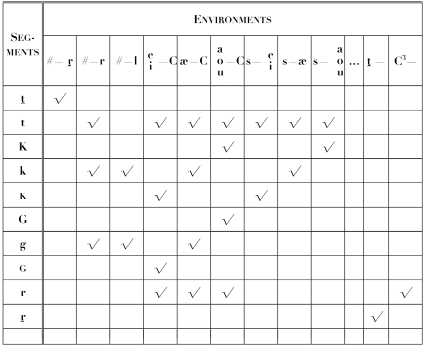
(Saussure, 1980)



(Hjelmslev, 1975)



(Hjelmslev, 1935)



Segments	Environments											
	#— r̥	#—r	#—l	e/i —C	æ—C	a/o/u —C	s— e/i	s—æ	s— a/o/u	...	t̥ —	C³—
t̥	√											
t		√		√	√	√	√	√	√			
K						√			√			
k		√	√		√			√				
ĸ				√			√					
G						√						
g		√	√		√							
ɢ				√								
r				√	√	√						√
r̥											√	

(Harris, 1960)

Structuralisme et Formlisme

	a	b	d	e	f	g	h	i
a	aa	ab	ad		af	ag	ah	
b	ba							bi
d	da			de				di
e		eb	ed			eg		
f				fe				
g								gi
h	ha							hi
i			id				ih	i

Diagram 1.

	b	d	f	g	h	a	e	i
f						**fa**	fe	
h						ha		hi
g						**ga**	**ge**	gi
b						ba	**be**	bi
d						da	de	di
a	ab	ad	af	ag	ah	aa		
e	eb	ed	**ef**	eg				
i	**ib**	id		**ig**	ih			i

Diagram 3.

		I p	I r	I s	I t	II i	II o	II u	III y	IV &
I	p r s t	—				+	+	+	—	—
II	i	+				+			+	
	o	+								
	u	+								
III	y	—				+			+	
IV	&	—								+

Diagram 2.

(SpangHanssen1959)

Structuralisme et Formlisme

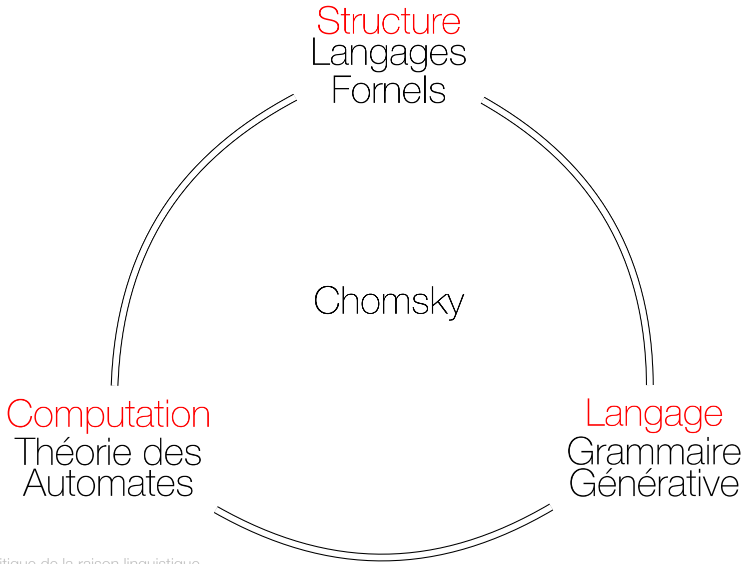
Table 8.
Vowel × binary final cluster (cf. sect. 84).

	ft	gt	ks	ds	vn	vl	dr!	mp	nk	ng	nd	nt	ns	lk	ld	lt	rk	rd	rt	rn	S	T	jC	
a	5	10	6	3	9	8	6	8	16	20	14	9	6	9	8	11	7	1	9	3	168	281	3	a
e	–	–	3	1	3	2	2	1	–	4	7	5	6	–	3	5	–	1	3	3	49	95	33	e
i	7	6	9	5	–	1	2	4	13	11	20	8	3	2	11	6	6	1	1	–	116	171	–	i
o	3	2	2	5	4	2	1	1	1	2	3	2	–	4	13	3	6	9	10	4	77	120	–	o
u	2	9	5	4	–	–	6	12	8	4	12	3	2	4	8	4	4	–	2	–	89	143	–	u
y	–	2	–	2	–	–	1	2	4	7	6	2	–	1	6	6	3	2	1	–	45	56	–	y
æ	4	11	1	–	4	4	2	2	9	11	8	1	3	2	11	4	6	6	6	4	99	145	–	æ
ø	5	2	–	–	1	4	–	–	–	–	1	2	3	–	–	–	3	–	1	6	28	47	10	ø
aa	–	–	–	1	–	–	1	–	–	–	4	–	–	–	–	–	–	2	–	1	9	11	–	aa
	26	42	26	21	21	21	21	30	51	59	75	32	23	22	60	39	35	22	33	21	680	1069	46	

(SpangHanssen1959)

Plan

Conclusion


Structure
Langages
Fornels
Chomsky
Computation
Théorie des
Automates
Langage
Grammaire
Générative

Conclusion

Structure
Espaces
Probabilistes

IA

Computation
Résaux
Neuronaux

Langage
Systèmes
d'Information

Conclusion

Structure
Catégorie de Profoncteurs

Critique

Computation
Théorie des Types

Langage
Sémiotique Structurale

Collaborations

Articles de Référence

- Gastaldi, J. L. (2020).Why can computers understand natural language?: The structuralist image of language behind word embeddings. Philosophy & Technology

- Gastaldi, J. L., & Pellissier, L. (2021).The calculus of language: Explicit representation of emergent linguistic structure through type-theoretical paradigms. Interdisciplinary Science Reviews. https://doi.org/10.1080/03080188.2021.1890484

- Bradley, T.-D., Gastaldi, J. L., & Terilla, J. (2024).The structure of meaning in language: Parallel narratives in linear algebra and category theory. Notices of the American Mathematical Society. https://api.semanticscholar.org/CorpusID:263613625

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Gastaldi, J. L., & Pellissier, L. (2021).The calculus of language: Explicit representation of emergent linguistic structure through type-theoretical paradigms. *Interdisciplinary Science Reviews*. https://doi.org/10.1080/03080188.2021.1890484

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Mikolov, T., Sutskever, I., Chen, K., Corrado, G., & Dean, J. (2013).Distributed representations of words and phrases and their compositionality. *CoRR*, *abs/1310.4546*.

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Critique de l'Intelligence Artificielle
Enjeux philosophiques, politiques et culturels de l'automatisation numérique
LLCP, Université Paris 8
Paris, France

Critique de la raison linguistique

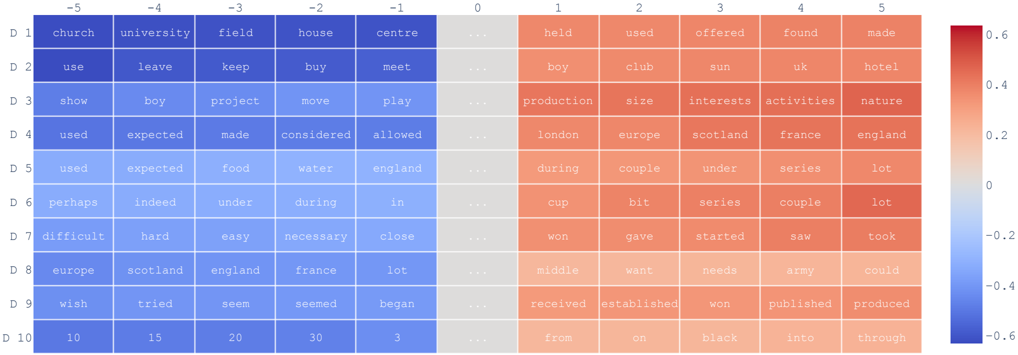
De quoi les modèles neuronaux de langage sont-ils le modèle?

Juan Luis Gastaldi

ETH zürich

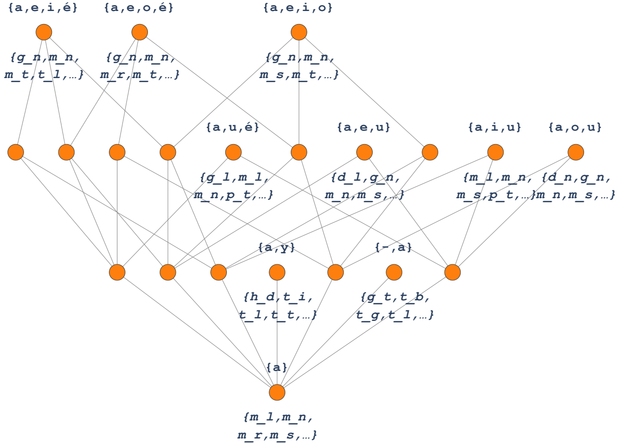
28 Mai, 2024

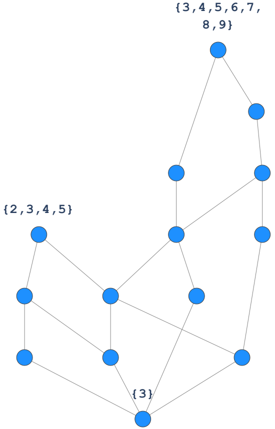
Mots



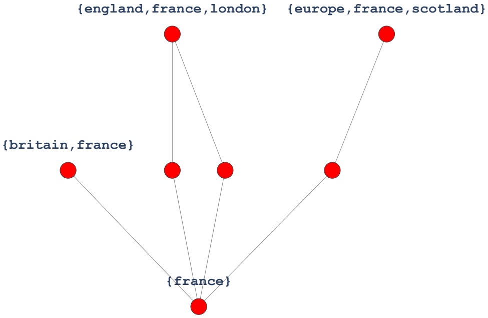
-5 -4 -3 -2 -1 0 1 2 3 4 5
D 1 church university field house centre ... held used offered found made
D 2 use leave keep buy meet ... boy club sun uk hotel
D 3 show boy project move play ... production size interests activities nature
D 4 used expected made considered allowed ... london europe scotland france england
D 5 used expected food water england ... during couple under series lot
D 6 perhaps indeed under during in ... cup bit series couple lot
D 7 difficult hard easy necessary close ... won gave started saw took
D 8 europe scotland england france lot ... middle want needs army could
D 9 wish tried seem seemed began ... received established won published produced
D 10 10 15 20 30 3 ... from on black into through
0.6
0.4
0.2
0
−0.2
−0.4
−0.6

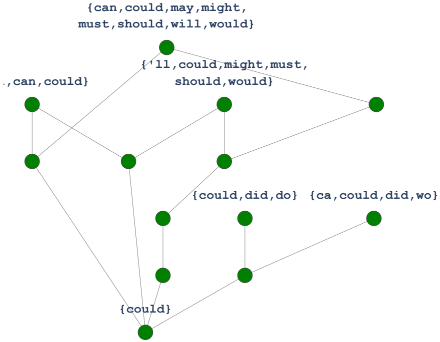
Formal Concepts



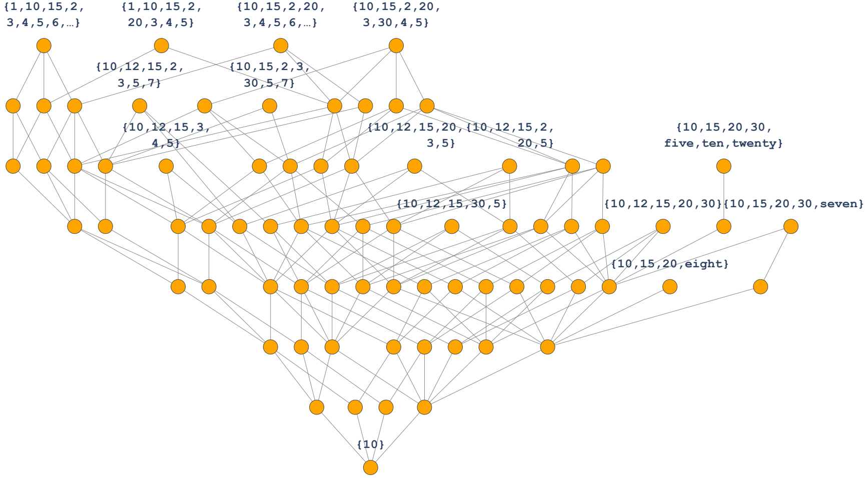


Formal Concepts (words)





Formal Concepts (words)



Produit de Matrices

$$
\begin{array}{cc}
 & \begin{array}{c} \quad\quad\quad\quad C_j \quad\quad\quad\quad \\ \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \vdots & & \vdots & & \vdots \\ & & b_{kj} & & \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nn} \end{pmatrix} \end{array} \\
\begin{array}{cc} & \\ & \\ L_i & \\ & \\ & \end{array}\begin{pmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{pmatrix} & \begin{pmatrix} & & \vdots & & \\ & & \vdots & & \\ \cdots & \cdots & c_{ij} & & \\ & & & & \\ & & & & \end{pmatrix}
\end{array}
$$

Produit de Matrices

$$
\begin{array}{cc}
 & \begin{pmatrix} \boxed{1} & 2 & 3 \\ \boxed{4} & 5 & 6 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 \\ \boxed{3 \quad 4} \\ 5 & 6 \end{pmatrix} & \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}
\end{array}
$$

Produit de Matrices

$$\begin{array}{cc} & \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} & \begin{pmatrix} & & \\ 19 & & \\ & & \end{pmatrix} \end{array}$$

Produit de Matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} & & \\ 19 & & \\ & & \end{pmatrix}$$

Produit de Matrices

$$\begin{array}{cc} & \begin{pmatrix} 1 & \boxed{2} & 3 \\ 4 & \boxed{5} & 6 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ \boxed{5 \quad 6} \end{pmatrix} & \begin{pmatrix} & & \\ 19 & & \\ & 40 & \end{pmatrix} \end{array}$$

Produit de Matrices

$$
\begin{array}{cc}
 & \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\
\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} & \begin{pmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{pmatrix}
\end{array}
$$