Logical-geometric structures in $\mathbb{R}$-enriched adjunctions

Juan Luis Gastaldi[1*], Samantha Jarvis[2*], Luc Pellisier[3*], Thomas Seiller[4*] and John Terilla[5*]

Abstract

We identify a previously unobserved correspondence between the fixed points of an Isbell adjunction enriched over the extended reals and the types in a particular model of linear logic coming from a realizability situation. This alignment places a geometric structure on the space of types and, in turn, equips the categorical fixed points with new connective operations.

On the linear-logic side, the correspondence endows the type space with a cell decomposition by convex polytopes. Cells carry labels that integrate geometry with type theory and facilitate wall-crossing formulas. Over each point lies a tower of lattices indexed by nonnegative real numbers, locally constant on intervals; the breakpoints, where the lattices change, have geometric meaning and detect cells at prescribed distances from the chosen basepoint.

On the categorical side, a dynamic "execution product" builds types and guides the addition of structure on the categorical side so that the fixed points of the enriched Isbell adjunction inherit new connectives. This additional structure allows fixed points of the adjunction to in turn, generate derived adjunctions with fixed points of their own. The connectives integrated the derived fixed points and the resulting interactions assemble into a coherent structure reminiscent of a closed monoidal category, albeit with interesting differences that we spell out.

Our second aim is motivational. Distributional information from a text corpus naturally realizes the categorical construction and its geometric features — cell decomposition, wall-crossing, and towers of lattices — in a way related to, but distinct from, common NLP structures such as vector embeddings of tokens. The corpus example naturally supplies the extra realizability structure needed for the new connectives. We propose the theory of distributional types that emerges as a framework for how logical structure can emerge from distributional data: a corpus carries its own meaning and logic, and the present work offers a starting point for making this precise.

Keywords: keyword1, Keyword2, Keyword3, Keyword4

Mathematics Subject Classification: 03B47 , 18D60 , 03B65 , 68T50

Contents

1 Introduction

This paper has two aims. First, we establish a connection between a known construction in category theory and an established practice in linear logic: the fixed points of an Isbell adjunction enriched over the extended reals are aligned with the set of types in a particular model of linear logic. To the best of our knowledge, this correspondence has not been observed before, and it opens new directions in both areas.

Let $\mathcal{C}$ be a category enriched over $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and let $M : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \overline{\mathbb{R}}$ be a functor. The Isbell adjunction defines a pair of adjoint functors $M^* : [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}] \leftrightarrows [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}} : M_*$ defined by

$$M^*(f)(d) = \inf_{c \in \mathcal{C}} (M(c,d) - f(c)) \text{ and } M_*(g)(c) = \inf_{d \in \mathcal{D}} (M(c,d) - g(d)) \tag{1}$$

The fixed points of this adjunction consists of pairs (f, g) with $M^* f = g$ and $M_* g = f$ and is called the *nucleus* of M. The nucleus of M is itself a partially ordered set with $(f, g) \leq (f', g') \iff f \leq f' \iff g' \leq g$. It is also a category enriched over $\overline{\mathbb{R}}$ with enriched morphisms defined by $[(f, f'), (g, g')] = \inf_{c \in \mathcal{C}} (f'(c) - f(c)) = \inf_{d \in \mathcal{D}} (g(d) - g'(d)) \in \overline{\mathbb{R}}$. The associativity for the enriched morphisms is a kind of reversed, antisymmetric triangle inequality, which, after modding out by the action of $\mathbb{R}$ on the nucleus by $(f, g) \mapsto (f + \lambda, g - \lambda)$, defines a metric on the nucleus.

1.1 An example

For a simple example, consider sets $\mathcal{C} = \{c_0, c_1, c_2\}$ and $\mathcal{D} = \{d_1, d_2, d_3, d_4\}$ viewed as trivial $\overline{\mathbb{R}}$ categories, with no morphisms but identities. Let $M(c_i, d_j)$ be the i, j-th entry of the matrix

$$M = \begin{bmatrix} 0.7 & 1.5 & 1.7 & -1.3 \\ 1.2 & 2.6 & 0.1 & 2.2 \\ 2 & -1.6 & 2 & -2.9 \end{bmatrix}.$$

Modding out by the $\mathbb{R}$ action by taking the slice $c_0 = 0$, each point (f, g) in the nucleus of M is determined by a pair (c_1, c_2) and we obtain a picture of the nucleus of M as pictured in Figure 6.5. The natural metric on the Nucleus is the max-spread defined by

$$d((c_1, c_2), (c_1', c_2')) = \max(c_1 - c_1', c_2 - c_2', 0) - \min(c_1 - c_1', c_2 - c_2', 0).$$

The structure of $\overline{\mathbb{R}}$ used to make it an enriching category involes $\leq$ for morphisms and $+$ as a monoidal product. This then connects the nucleus with a tropical-geometric structure which endows the nucleus with a cell decomposition consisting of convex polytopes. By definition, a point (f, g) is in the nucleus means that $g(d) = \min_c (M(c, d) - f(c))$ and $f(c) = \min_d (M(c, d) - g(d))$. Call c' a witness for g at d if $g(c, d) = M(c', d) - f(c')$ meaning that the minimum defining $g(d)$ occurs at c'. Likewise, one defines d' to be a witness for f at c if $f(c) = M(c, d') - g(d')$. For points (f, g) in the nucleus, the witnesses come in pairs (c', d') meaning that c' is a witness for g at d' iff d' is a witness for g at c' and it is these witness pairs that define the cell structure of the nucleus: all the points (f, g) with the same witness pairs define a convex polytope. Regions where the witnesses are generic form cells of the highest dimension, in this example, the highest dimensional cells are the faces F_1, F_2, and F_3. For example, in the example F_3 consists of all points (f, g) whose witness pairs are (c_0, d_1),(c_2, d_2),(c_1, d_3), (c_2, d_4).

Points where there are extra witness pairs form lower dimensional cells, for example if g is in F_3 and in addition to $g(d_1) = M(c_0, d_1) - f(c_0)$, it also happens that

$g(d_1) = M(c_1, d_1) - f(c_1)$, then (f, g) lies on the edge e_2. In the example, there are twelve one cells defined by the edges $e_1, \ldots, e_{12}$. There are also ten zero dimensional cells. Some of these vertices are "anchor points" A_1, A_2, A_3, and A_4 that happen to be (f, g) pairs where g is a representable copresheaf, which are 0-cells. In the example, there are ten vertices consisting of the four anchor points and points that are the intersections of edges.

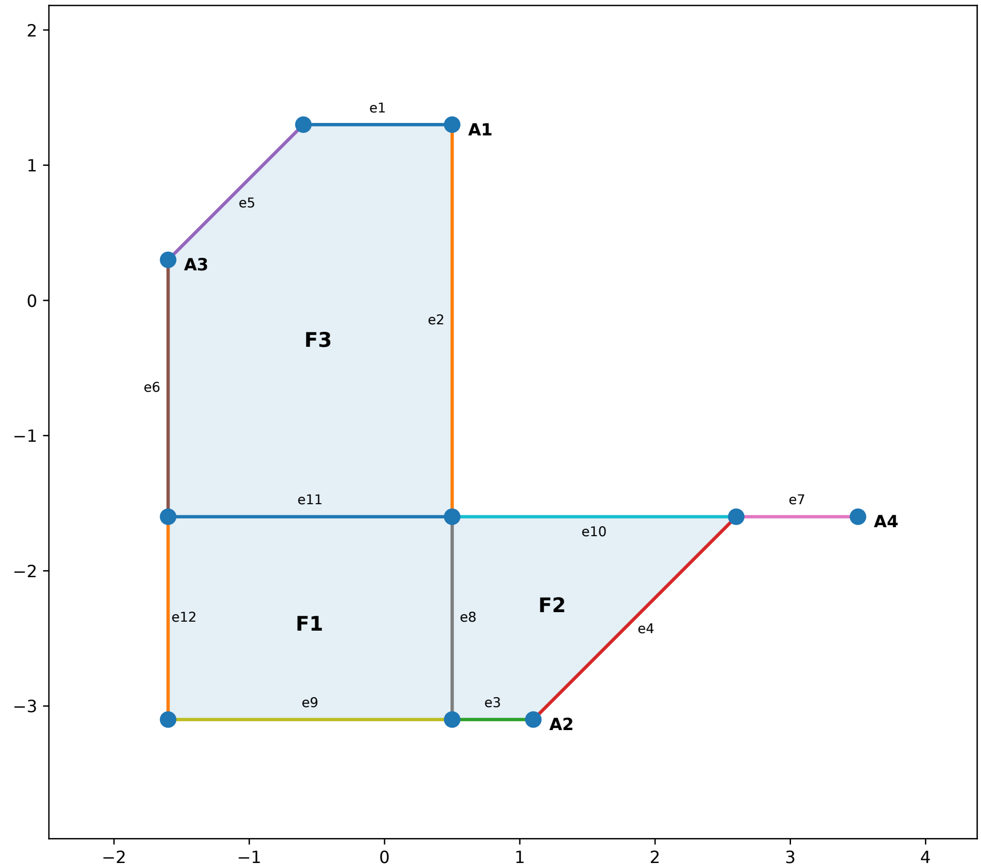


Fig. 1 Witness cell complex for the example matrix M.

1.2 Additional geometric structure

A geometric picture of the nucleus like the one just described can be understood from tropical geometry — it is in fact the tropical hyperplane arrangement determined by the columns of M. But there are small differences between our setup and established conventions in tropical geometry — for example, unlike the tropical semiring, $\overline{\mathbb{R}}$ includes both $-\infty$ and $+\infty$; also, there's a minus sign in the definition of M_* and

M^*. So we work from scratch and in Section 2 on the *Nucleus of an $\overline{\mathbb{R}}$ entriched profunctor* and Section 6 on *The Geometry of the Nucleus*, we spell out all of the details, establishing facts about the enriched adjunction and building the geometric picture from sequences of simple statements.

One of our main tools that we use to build the geometric picture is what we call the *gap matrix*, which arises naturally from our point of view. Given a functor $M : \mathcal{C}^{\text{op}} \times \mathcal{D} \to \overline{\mathbb{R}}$ and a point (f, g) in the nucleus of M, one can "gauge shift" M to a new matrix δ defined by

$$\delta(c, d) := M(c, d) - f(c) - g(d).$$

While the cell decomposition via hyperplane arrangements is known in tropical convexity, our use of the gap matrix leads to more precise metric geometry and an order-chamber stratification that we have not found in the tropical geometry literature.

The entries of δ are non-negative with zeroes at exactly the witness pairs of (f, g). Each nonzero entry $\lambda = \delta(c_i, d_j)$, is the exact threshold at which (c_i, d_j) becomes a witness — when a wall is met. More specifically, there is a point (f', g') in the nucleus of M whose distance is exactly distance λ from (f, g) and has (c_i, d_j) as a witness pair, and therefore lies in a different (lower dimensional) cell.

To continue the illustration using the example from Section 1.1, choosing the basepoint (f, g) where $f = (0, 0, 0)$ and $g = (0.7, -1.6, 0.1, -2.9)$ results in the gap matrix

$$\delta = \begin{bmatrix} 0 & 3.1 & 1.6 & 1.6 \\ 0.5 & 4.2 & 0 & 5.1 \\ 1.3 & 0 & 1.9 & 0 \end{bmatrix}.$$

Every entry of the gap matrix gives a road map to finding cells. The 0.5 in the (c_1, d_1) entry indicates that moving 0.5 units in the c_1 direction (right) from $(0, 0)$ will locate a point in a new cell, in this case the edge e_2. The 1.9 in the (c_2, d_3) entry indicates that moving 1.9 units in the c_2 direction (up) (and then applying M_*M^*) locates a new cell, in this case the 0-cell at the intersection of e_1 and e_5. This is illustrated in Figure 5. To summarize, a point (f, g) in the nucleus of M determines a gap matrix whose zeros describe the cell where (f, g) lies. This then describes a global geometric picture of a structure over the nucleus in which the information of the ordered entries of the gap matrix $\delta^{(f,g)}$ and their row-column locations lies over the point (f, g) as a fiber. One way to organize the information in the fiber is by sorting the entries of $\delta^{(f,g)}$ to obtains the ordered list of radii at which new witness relations appear, which can be as a tower of lattices indexed by non-negative real numbers. This is illustrated in the appendix. This more refined picture of the geomtry suggests a refinement of the witness cell structure by order chambers defined as sets where the entries of the gap matrices do not change order, regions where the bundles of lattices are constant. This refinement is illustrated in Figure **??**.

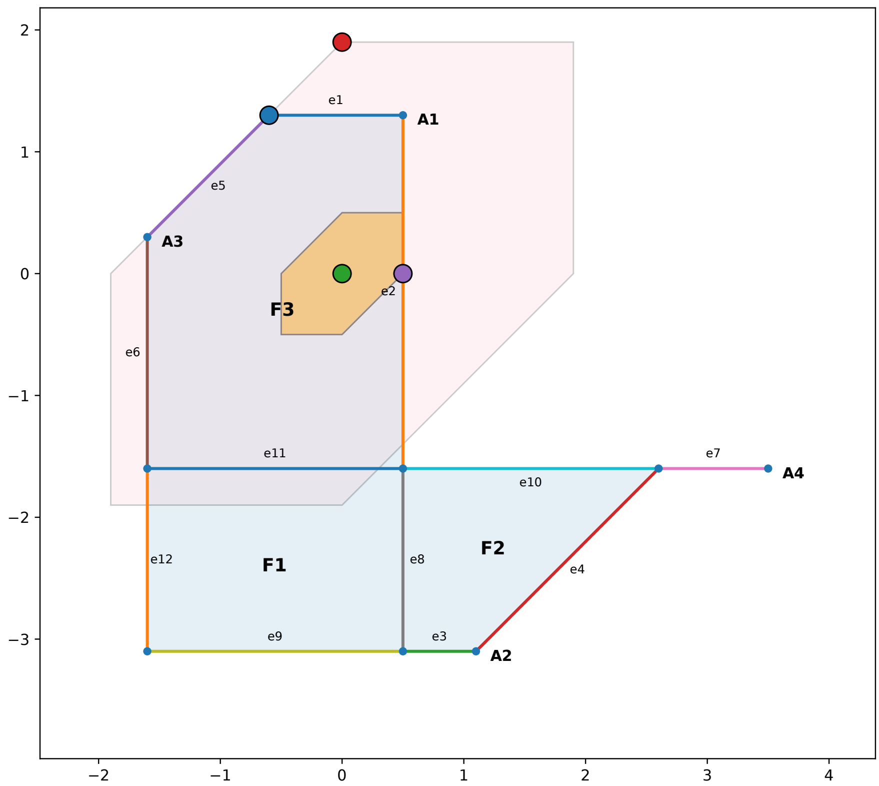


Fig. 2 The green point has coordinates $(c_1, c_2) = (0, 0)$ in the picture and represents the pair (f, g) where $f = (0, 0, 0)$ and $g = (0.7, -1.6, 0.1, -2.9)$. It is the base point of this picture. Balls of radius $\epsilon_1 = 0.5$ and $\epsilon_2 = 1.9$ centered at the green point are shown. The purple point lies in the edge e_2 and is located a distance 0.5 from the green point. The red point is a point distance 1.9 from green point but lies outside the nucleus. By applying $M_* M^*$, the red point is mapped to the blue point, which lies in the Nucleus at at the intersection of e_1 and e_5.

1.3 Linear realizability

As part of the work on linear logic and the proofs-as-programs correspondance, several models were introduced in which the formulas (or types) are reconstructed from a model of computation. Those constructions were abstracted and generalised by one of the authors under the name of linear realizability, to stress their connection with realisability techniques. A (real valued) *linear realizability situatoion* is defined to be a set $\mathcal{C}$, a measurement $p : \mathcal{C} \to \mathbb{R}$, and an associative execution product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. No compatibility between the measurement and product are assumed. In a realizability situation, one defines an orthogonality relation on $\mathcal{C} \times R$ by

$$(a, \alpha) \perp (b, \beta) \iff \alpha + \beta - p(ab) + p(a) + p(b) \leq 0$$

Then subsets of $\mathcal{C} \times \mathbb{R}$ that are bi-orthogonal to themselves are defined to be *types*. Because in this paper, the execution product is not assumed to be commutative,

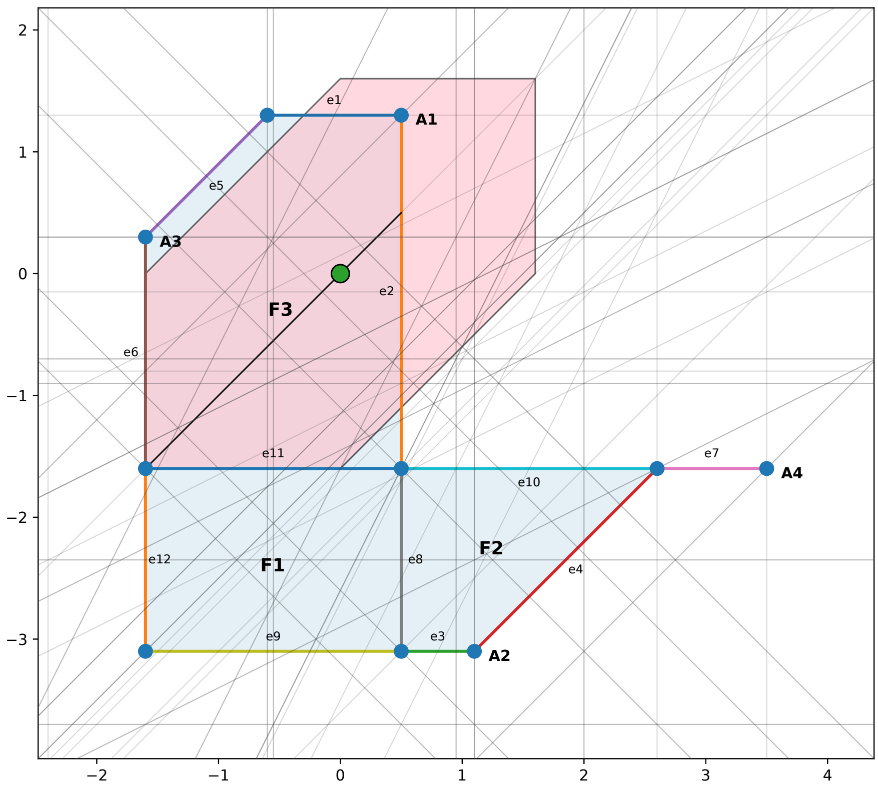


Fig. 3 The order chambers for the point (f, g) where $f = (0, 0, 0)$ and $g = M^*f = (0.7, -1.6, 0.1, -2.9)$. Notice that this point lies directly on an order chamber boundary (darkened in the picture). That is reflected in the gap matrix because there are two entries that are exactly 1.6. The ball of radius 1.6 shows that the one cells e_6 and e_{11} are precisely 1.6 units from the point (f, g). If f were chosen to be slightly to the right of the boundary, e_{11} would be closer and to the left, e_6 would be closer. Crossing other boundaries would flip other entries of the gap matrix.

orthogonality has left and right handedness; if $x \perp y$ then x is left orthogonal to y and y is right orthogonal to x. Because orthogonality has this handedness, some care is required to define types as sets that are biorthogonal to themselves and in this linear realizability situation, types come in three flavors: left, middle, and right. Once the types are carefully defined, one can use the execution product from elements of $\mathcal{C}$ to define certain connectives on the types. Among other partially defined connectives, there is an associative product $\cdot$ defined on middle types and (left and right) linear implications $\multimap_l$ and $\multimap_r$ which among other relations, satisfy (Propositions 28 and 30)

$$(A \multimap_l B) \cdot (B \multimap_l C) \subseteq A \multimap_l C$$
$$A \cdot B \multimap_l C = B \multimap_l (A \multimap_l C)$$

and

$$(B \multimap_r C) \cdot (A \multimap_r B) \subseteq A \multimap_r C$$
$$A \cdot B \multimap_r C = A \multimap_r (B \multimap_r C)$$

for middle types $A, B, C \subset \mathcal{C} \times R$. These relations suggest that types assemble into a structure that resembles a (left and right) closed monoidal category, albeit with interesting differences, which are analyzed in Section **??**.

1.4 Types are points in the nucleus

By viewing $\mathcal{C}$ as a discrete $\overline{\mathbb{R}}$ category and setting $M : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ by $M(c, d) = p(ab) - p(a) - p(b)$, one has the nucleus of M. A main result (Theorem 19) is that there is a meaningful way to identify the nucleus of M and with the set of types. This correspondence between points of the nucleus and the set of types has an impact for both. On the one hand, the metric and tropical geometry of the nucleus to be overlayed on the set of types giving a geometric structure to the logical situation and we begin to explore the interaction between the geometry and the logic.

On the other side, the way that the execution product interacts with M provides guidance for adding structure to a category $\mathcal{C}$ so that the nucleus inherits new connective operations. Specifically, if $\mathcal{C}$ has a (not-necessarily symmetric) monoidal structure $\otimes$ that is compatible with M in the following way

$$M(a \otimes b, c) + M(a, b) = M(a, b \otimes c) + M(b, c)$$

the nucleus of M inherits the connective operations that the set of types have. One can extend the profunctor M to higher powers of $\mathcal{C}$, for example

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \overline{\mathbb{R}}$$

and the points of the nucleus of this operator are (co)presheaves on $\mathcal{C} \times \mathcal{C}$, which has its own nucleus. So, the additional structure afforded by the logic allows the points of the nucleus to themselves create new derived adjunctions that in turn have their own fixed points. The connective operations explain how the fixed points and the derived fixed points interact and assemble.

1.5 Distributional type theory

Distributional information from a text corpus naturally realizes the categorical construction and its geometric features — cell decomposition, wall-crossing, and towers of lattices — in a way related to, but distinct from, common NLP structures such as vector embeddings of tokens. The corpus example naturally supplies the extra realizability structure needed for the new connectives. We propose the theory of distributional types that emerges as a framework for how logical structure can emerge from distributional data: a corpus carries its own meaning and logic, and the present work offers a starting point for making this precise.

2 The nucleus of an $\overline{\mathbb{R}}$-profunctor

Before getting to the results, some conventions about categories enriched over the extended real numbers are established. Nothing is especially original here (see [Willerton, Elliot, Lawvere] and references on Lattice Theory and Galois Connections) but minor differences in conventions can create real headaches. For the benefit of the authors as much as the reader, we include the details.

2.1 The extended real numbers $\overline{\mathbb{R}}$

Let $\overline{\mathbb{R}}$ be the category whose objects are real numbers $[-\infty, \infty]$ extended to include ∞ and $-\infty$ with a morphism $x \to y$ whenever $x \leq y$. Then, the categorical product of x and y is the minimum $x \times y = \min\{x, y\}$, and the coproduct is the maximum $x \sqcup y = \max\{x, y\}$. The initial object is $-\infty$ and ∞ is terminal. Arbitrary limits and colimits of any diagram are given by the infimum and supremum, respectively, of the objects in the diagram. Owing to the inclusion of $\pm\infty$, the category $\overline{\mathbb{R}}$ is complete and cocomplete.

The category $\overline{\mathbb{R}}$ is given the structure of a closed monoidal category as follows. The tensor product $+$ is defined by ordinary addition of real numbers, with unit given by 0, which is extended to $\pm\infty$. The only case for addition in $\overline{\mathbb{R}}$ that isn't readily settled by a common sense rule is $-\infty + \infty = \infty + (-\infty)$ which is resolved by the fact that the monoidal product is the left part of an adjunction. This implies that for any $x \in \overline{\mathbb{R}}$, the functor $+x : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ commutes with colimits. In particular, $+\infty$ must send the initial object to the initial object, implying that

$$-\infty + \infty = -\infty \text{ and by symmetry } \infty + (-\infty) = -\infty.$$

The right adjoint $[\ ,\ ]$ for $+$ satisfies $a + b \leq c$ if and only if $a \leq [b, c]$. For real numbers $[b, c] = c - b$ and more generally, $[b, c] := \sup_{t \in \overline{\mathbb{R}}}\{b + tc\}$. It's convenient to sort out the less-easy cases for $\pm\infty$ and use the subtraction sign for finite or infinite elements of $\overline{\mathbb{R}}$

$$-\infty - (-\infty) = \infty \text{ and } \infty - \infty = \infty.$$

2.2 $\overline{\mathbb{R}}$-categories and functors

A (small) $\overline{\mathbb{R}}$-*category* $\mathcal{C}$ has a set of objects (also denoted $\mathcal{C}$) and morphisms $\mathcal{C}(a, b) \in \overline{\mathbb{R}}$ satisfying

$$\mathcal{C}(a, b) + \mathcal{C}(b, c) \leq \mathcal{C}(a, c) \text{ and } \mathcal{C}(a, a) = 0 \tag{2}$$

for all $a, b, c \in \mathcal{C}$. The category $\overline{\mathbb{R}}$ is itself an $\overline{\mathbb{R}}$ category with $\overline{\mathbb{R}}$-enriched morphisms defined by $\overline{\mathbb{R}}(x, y) := [x, y] = y - x$. If $\mathcal{C}$ is an $\overline{\mathbb{R}}$ category, its *opposite category* $\mathcal{C}^{op}$ is defined to be the $\overline{\mathbb{R}}$ category with the same objects as $\mathcal{C}$ and with $\mathcal{C}^{op}(a, b) = C(b, a)$.

A functor $F : \mathcal{C} \to \mathcal{D}$ from an $\overline{\mathbb{R}}$ category $\mathcal{C}$ to an $\overline{\mathbb{R}}$ category $\mathcal{D}$ consists of a function $F : \mathcal{C} \to \mathcal{D}$ satisfying $\mathcal{C}(a, b) \leq \mathcal{D}(Fa, Fb)$ for all $a, b \in \mathcal{C}$. Two $\overline{\mathbb{R}}$ categories are isomorphic if there are invertible functors between them. The set of functors $[\mathcal{C}, \mathcal{D}]$

between two $\overline{\mathbb{R}}$ categories defines a $\overline{\mathbb{R}}$ category with morphisms

$$[\mathcal{C},\mathcal{D}](f,g) = \inf_{c\in\mathcal{C}}\{\mathcal{D}(f(c),g(c))\}$$

Unless specified otherwise, all categories in this paper will be $\overline{\mathbb{R}}$ categories, all functors will be $\overline{\mathbb{R}}$ functors, and to lighten notation, we often use $[f,f']$ for $[\mathcal{C},\mathcal{D}](f,f')$ when the context is unambiguous.

Presheaves $f : \mathcal{C}^{\mathrm{op}} \to \overline{\mathbb{R}}$ and copresheaves $g : \mathcal{D} \to \overline{\mathbb{R}}$ are $\overline{\mathbb{R}}$-functors. Their morphisms are

$$[f,f'] = \inf_{c\in\mathcal{C}}(f'(c) - f(c)) \text{ and } [g,g'] = \inf_{d\in\mathcal{D}}(g'(d) - g(d))$$

where subtraction inside the infemum is understood as the internal hom of the extended reals.

The assignment $c \mapsto \mathcal{C}(-,c)$ defines an $\overline{\mathbb{R}}$-functor $\mathcal{C} \to [\mathcal{C}^{op},\overline{\mathbb{R}}]$ called the Yoneda embedding. Similarly, the assignment $c \mapsto \mathcal{D}(c,-)$ defines a functor $\mathcal{D} \to [\mathcal{D},\overline{\mathbb{R}}]^{op}$ called the co-Yoneda embedding. The presheaf $\mathcal{C}(-,c)$ and copresheaf $\mathcal{D}(d,-)$ are *representable* and more specifically, are represented by the objects c and d respectively. Note that

$$[\mathcal{C}(-,c),\mathcal{C}(-,c')] = \mathcal{C}(c,c') \text{ and } [\mathcal{C}(c,-),\mathcal{C}(c',-)] = \mathcal{C}(c',c)$$

so the Yoneda and co-Yoneda embeddings are embeddings of $\overline{\mathbb{R}}$-enriched categories.

2.3 Tensor products and discrete categories

Given two categories $\mathcal{C}$ and $\mathcal{D}$, their *tensor product* $\mathcal{C}\otimes\mathcal{D}$ becomes a category whose objects are $\mathcal{C}\times\mathcal{D}$ and with $(\mathcal{C}\otimes\mathcal{D})((c,c'),(d,d')) = \mathcal{C}(c,c') + \mathcal{D}(d,d')$. A *profunctor* from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor $M : \mathcal{C}^{op}\otimes\mathcal{D} \to \overline{\mathbb{R}}$. That is, M is a $\mathcal{D}\times\mathcal{C}$ matrix satisfying

$$\mathcal{C}(c',c) + \mathcal{D}(d,d') \leq M(c',d') - M(c,d)$$

for all pairs $(c,d),(c',d')$ from $\mathcal{C}^{\mathrm{op}}\otimes\mathcal{D}$.

Any set $\mathcal{C}$ gives rise to a category whose objects are the elements of the set $\mathcal{C}$ and morphisms given by

$$\mathcal{C}(c,c') = \begin{cases} 0 & \text{if } c = c' \\ -\infty & \text{if } c \neq c'. \end{cases}$$

Such a category is called *discrete*. For a discrete category $\mathcal{C} = \mathcal{C}^{\mathrm{op}}$, any set function $\mathcal{C} \to \overline{\mathbb{R}}$ defines a presheaf or a copresheaf, and any set function $\mathcal{C}\times\mathcal{C} \to \overline{\mathbb{R}}$ defines a profunctor.

The nucleus of a profunctor

For any profunctor $M : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \overline{\mathbb{R}}$. The map $c \mapsto M(c, \cdot)$ defines a functor $\mathcal{C} \to [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}$ and the map $d \mapsto M(\cdot, d)$ defines a functor $\mathcal{D} \to [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$. These maps extend to functors between presheaves on $\mathcal{C}$ and copresheaves on $\mathcal{D}$.

Definition 1. . *Define functors*

$$M^* : [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}] \rightleftarrows [\mathcal{D}, \overline{\mathbb{R}}]^{op} : M_*$$

by

$$M^*(f)(d) = \inf_{c \in \mathcal{C}} \{M(c, d) - f(c)\} \tag{3}$$

$$M_*(g)(c) = \inf_{d \in \mathcal{D}} \{M(c, d) - g(d)\} \tag{4}$$

Proposition 1. *The functions M^* and M_* defined in Equation (8) and (9) form a pair of adjoint functors $M^* \dashv M_*$. That is, they are functors and they satisfy*

$$[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, g) = [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, M_* g)$$

for all $f \in [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$ and $g \in [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}$.

Proof. The statements needed to check to see that M^* and M_* are functors are $[\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, f') \leq [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, M * f')$ and $[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(g, g) \leq [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](M_* g, M_* g')$. Checking the first of these two statements (the second statement is verified similarly)

$$\begin{aligned}
[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, M^* f') &= \inf_{d \in \mathcal{D}} \{M^* f(d) - M^* f'(d)\} \\
&= \inf_{d \in \mathcal{D}} \left\{ \inf_{c \in \mathcal{C}} \{M(c, d) - f(c) - \inf_{c \in \mathcal{C}} \{M(c, d) - f'(c) \right\} \\
&\geq \inf_{d \in \mathcal{D}} \left\{ \inf_{c \in \mathcal{C}} \{M(c, d) - f(c) - (M(c, d) - f'(c)) \right\} \\
&= \inf_{c \in catC} \left\{ \inf_{d \in \mathcal{D}} \{f'(c) - f(c)) \right\} \\
&= [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, f').
\end{aligned}$$

For the adjunction part, compute left to right:

$$\begin{aligned}
[\mathcal{C}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, g) &= \inf_{d \in \mathcal{D}} \{g(d) - M^* f(d)\} \\
&= \inf_{d \in \mathcal{D}} \left\{ g(d) - \left(\inf_{c \in \mathcal{C}} M(c, d) - f(c) \right) \right\} \\
&= \inf_{c \in \mathcal{C}, d \in \mathcal{D}} \{g(d) + f(c) - M(c, d)\} \\
&= \inf_{c \in \mathcal{C}} \left\{ f(c) - \left(\inf_{d \in \mathcal{D}} M(c, d) - g(d) \right) \right\}
\end{aligned}$$

$$= \inf_{c\in\mathcal{C}} \{f(c) - M_* g(c)\}$$
$$= [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(f, M_* g).$$

□

Now we define one of the central objects in this paper.

Definition 2. *The* nucleus *of a profunctor $M : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \overline{\mathbb{R}}$ is defined to be the category whose objects are*

$$\mathrm{Nuc}(M) := \{(f, g) : M^* f = g \text{ and } M_* g = f\}$$

and whose morphisms are

$$\mathrm{Nuc}(M)((f, g), (f', g')) := [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, f') = [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(g, g').$$

The fact that both expressions for the morphisms are equal follows from the fact that fact that the functors are adjoint: if $(f, g), (f', g') \in \mathrm{Nuc}(M)$ then

$$\begin{aligned}
[\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, f') &= [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, M_* g') \\
&= [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, g') \\
&= [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(g, g').
\end{aligned}$$

It will be convenient to use the following notation: if f, f' are presheaves or copresheaves on a category $\mathcal{C}$, the expression $f \leq f'$ means that $f(c) \leq f'(c)$ for all $c \in \mathcal{C}$ and $f \geq f'$ means that $f(c) \geq f'(c)$ for all $c \in \mathcal{C}$. Notice if f and f' are presheaves then $f \leq f'$ is equivalent to $0 \leq [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, f')$. If f and f' are copresheaves then $f \leq f'$ if and only if $0 \leq [\mathcal{C}, \overline{\mathbb{R}}]^{\mathrm{op}}(f', f)$.

Lemma 2. *The maps M^* and M_* are order reversing. If $f \leq f'$ then $M^* f' \leq M^* f$ and if $g \leq g'$ then $M_* g' \leq M_* g$.*

Proof. The proof follows from the fact that M^* and M_* are functors.

$$f \leq f' \Rightarrow 0 \leq [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, f') \leq [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, M^* f') \Rightarrow M^* f' \leq M^* f$$
$$g \leq g' \Rightarrow 0 \leq [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(g', g) \leq [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](M_* g', M_* g) \Rightarrow M_* g' \leq M_* g.$$

□

Lemma 3. *The maps $M_* M^*$ and $M^* M_*$ are nondecreasing, meaning that $f \leq M_* M^* f$ and $g \leq M^* M_* g$ for $f \in [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$ and $g \in [\mathcal{C}, \overline{\mathbb{R}}]^{\mathrm{op}}$.*

Proof. Observe,

$$0 = [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, M^* f) = [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, M_* M^* f) \Rightarrow f \leq M_* M^* f.$$

Similarly,

$$0 = [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](M_* g, M_* g) = [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* M_* g, g) \Rightarrow g \leq M^* M_* g.$$

□

Lemma 4. *$M^* M_* M^* = M^*$ and $M_* M^* M_* = M_*$*

Proof. Let $f \in [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$. Since $M_* M^*$ is nondecreasing, $f \leq M_* M^* f$. Applying M^* which reverses order gives $M^* f \geq M^* M_* M^*(f)$. Applying the nondecreasing map $M^* M_*$ to $M^* f$ yields $M^* f \leq M^* M_* M^* f$. It follows that $M^* M_* M^*(f) = M^* f$. The other equation is proved similarly. □

It follows immediately from Lemma 4 that

Proposition 5. *The maps*

$$M_* M^* : [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}] \to [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}] \text{ and } M^* M_* : [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}} \to [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}$$

are projections meaning $(M_* M^*)^2 = M_* M^*$ and $(M^* M_*)^2 = M^* M_*$ whose images consist precisely of their fixed points

$$\mathrm{Fix}(M_* M^*) = \{f \in [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}] \, : \, M_* M^* f = f\}$$

$$\mathrm{Fix}(M^* M_*) = \{g \in [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}} \, : \, M^* M_* g = g\}.$$

Proposition 6. *All the categories below are isomorphic:*

$$\mathrm{Nuc}(M) \cong \mathrm{Fix}(M_* M^*) \cong \mathrm{Fix}(M^* M_*) \cong \mathrm{Im}(M^*) \cong \mathrm{Im}(M_*)$$

Proof. Recall $(f, g) \in \mathrm{Nuc}(M)$ means $M^* f = g$ and $M_* g = f$. So, the maps $(f, g) \mapsto f$ and $(f, g) \mapsto g$ define invertible functors

$$\mathrm{Nuc}(M) \to \mathrm{Fix}(M_* M^*) \text{ and } \mathrm{Nuc}(M) \to \mathrm{Fix}(M^* M_*).$$

Their inverses are given by $f \mapsto (f, M^* f)$ and $g \mapsto (M_* g, g)$. Moreover, M^* and M_* define an isomorphisms

$$M^* : \mathrm{Fix}(M_* M^*) \xrightarrow{\cong} \mathrm{Im}(M^*) \text{ and } M_* : \mathrm{Fix}(M^* M_*) \xrightarrow{\cong} \mathrm{Im}(M_*).$$

□

The fact that $\mathrm{Nuc}(M)$ is isomorphic to $\mathrm{Im}(M_*)$ and $\mathrm{Im}(M^*)$ gives an easy constructive way to find points in the Nucleus of M. Namely, for any $f \in [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$, the pair $(M_* M^* f, M^* f) \in \mathrm{Nuc}(M)$. The first entry of $(M_* M^* f, M^* f)$ is displayed as M_* of the second and applying M^* to the first gives the second. The first entry $M_* M^* f$ is a fixed point of $M_* M^*$, the second entry is a fixed point of $M^* M_*$. Likewise for any $g \in [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}$, the pair $(M_* g, M^* M_* g) \in \mathrm{Nuc}(M)$. Applying M^* to the first yields the second and applying M_* to the second yields the first. The first entry M_*g is a fixed point of M_*M^*, the second entry M^*M_*g is a fixed point of M^*M_*.

There is another characterization of the nucleus that follows from the above and the inequalities $M_*M^*f \geq f$ and $M^*M_*g \geq g$ that can be helpful.

Corollary 7. *A presheaf $f \in [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$ is a fixed point of M_*M^* iff it is the largest function that maps to M^*f under M^* and a copresheaf $g \in [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}$ is a fixed point of M^*M_* if and only if g is the largest function that maps to M_*g under M_*.*

3 Types

3.1 Linear realizability

As part of the work on linear logic and the proofs-as-programs correspondance, several models were introduced in which the formulas (or *types*) are reconstructed from a model of computation [**? ? ?**]. Those constructions were abstracted and generalised by one of the authors [**?**] under the name of *linear realizability*, to stress their connection with realisability techniques [**? ? ?**]. This construction is now recalled.

Definition 3. *A (real)* linear realizability situation *is given by a triple (C, Ex, p), where C is a set,*

$$\begin{aligned} \mathrm{Ex} : C \times C &\to C \\ (a, b) &\mapsto ab \end{aligned}$$

is an associative binary operation and $p : C \to \mathbb{R}$ is any function. The product Ex *is called the* execution product *and p is called a* measurement*.*

The measurement and the execution are assumed to have no compatibility, but the map $[\![-,-]\!] : C \times C \to \mathbb{R}$ defined by

$$[\![a, b]\!] = p(ab) - p(a) - p(b)$$

satisfies

$$[\![ab, c]\!] + [\![a, b]\!] = [\![a, bc]\!] + [\![b, c]\!] \tag{5}$$

$$\begin{aligned} [\![ab, c]\!] + [\![a, b]\!] &= p(abc) - p(ab) - p(c) + p(ab) - p(a) - p(b) \\ &= p(abc, c) - p(a) - p(b) - p(c) \\ &= p(abc) - p(a) - p(bc) + p(bc) - p(b) - p(c) \\ &= [\![a, bc]\!] + [\![b, c]\!]. \end{aligned}$$

Two elements $a, b \in C$ are defined to be orthogonal if

$$a \perp b \Leftrightarrow [\![a, b]\!] \leq 0 \Leftrightarrow p(ab) - p(a) - p(b) \leq 0.$$

Following Girard [?] and Seiller [? ?], we work with elements of $C \times \mathbb{R}$ and extend the measurement and binary products to $C \times \mathbb{R}$ as follows:

$$\begin{aligned}
[\![(a,\alpha),(b,\beta)]\!] &= \alpha + \beta - [\![a,b]\!] \\
\mathrm{Ex}((a,\alpha),(b,\beta)) &= (\mathrm{Ex}(a,b), \alpha + \beta - [\![a,b]\!])
\end{aligned}$$

Using concatanation of pairs for execution, we have

$$(a,\alpha)(b,\beta) = (ab, \alpha + \beta - [\![a,b]\!]) = (ab, \alpha + \beta - p(ab) + p(a) + p(b))$$

Orthogonality is extended as follows

Definition 4. *Let $(a,\alpha),(b,\beta) \in P \times \mathbb{R}$. Then*

$$(a,\alpha) \perp (b,\beta) \iff [\![(a,\alpha),(b,\beta)]\!] \leq 0 \iff \alpha + \beta - [\![a,b]\!] \leq 0$$

Definition 5. *Let $A, B \subseteq P \times \mathbb{R}$. Then*

$${}^{\perp}B = \{(a,\alpha) \,|\, (a,\alpha) \perp (b,\beta) \qquad \forall (b,\beta) \in B\} \subset P \times \mathbb{R}$$

$$A^{\perp} = \{(b,\beta) \,|\, (a,\alpha) \perp (b,\beta) \qquad \forall (a,\alpha) \in A\} \subset P \times \mathbb{R}$$

Without making additional assumptions on $[\![\ ,\ ]\!]$, the left- and right- orthogonal complements of a subset of $P \times \mathbb{R}$ may be distinct. We now have a handful of propositions that tell us how these orthogonal complement operators behave.

Proposition 8. *Let $A_1 \subseteq A_2 \subseteq P \times \mathbb{R}$, and $B_1 \subseteq B_2 \subseteq P \times \mathbb{R}$. Then*

$$A_2^{\perp} \subseteq A_1^{\perp} \qquad {}^{\perp}B_2 \subseteq {}^{\perp}B_1$$

Proof. We'll show the left-hand containment. Let $(b,\beta) \in A_2^{\perp}$, so $(a,\alpha) \perp (b,\beta)$ for all $(a,\alpha) \in A_2$. To see (b,β) is in $A_1^{\perp}$, note that $(a,\alpha) \perp (b,\beta)$ for all (a,α) in A_1 since $A_1 \subseteq A_2$. $\square$

The next proposition tells us that any set is contained in its double dual.

Proposition 9. *For any $A, B \subseteq P \times \mathbb{R}$,*

$$A \subseteq {}^{\perp}(A^{\perp}) \qquad B \subseteq ({}^{\perp}B)^{\perp}$$

Proof. Both containments are obvious from unpacking the definition. For example, to show that any $(a,\alpha) \in A$ is in ${}^{\perp}(A^{\perp})$, we would show that for any $(b,\beta) \in A^{\perp}$, $(a,\alpha) \perp (b,\beta)$, which is immediate. $\square$

Proposition 10. *For any A, B, the following hold:*

$${}^{\perp}\left(({}^{\perp}A)^{\perp}\right) = {}^{\perp}A \qquad \left({}^{\perp}(B^{\perp})\right)^{\perp} = B^{\perp}$$

Proof. We'll prove the left-hand equation, as the left follows a similar argument. First, note that the containment of ${}^{\perp}A$ in ${}^{\perp}(({}^{\perp}A)^{\perp})$ is immediate. If (b, β) is in ${}^{\perp}A$, then to show it is in ${}^{\perp}(({}^{\perp}A)^{\perp})$, we must show that $(b, \beta) \perp (c, \gamma)$ for all $(c, \gamma) \in ({}^{\perp}A)^{\perp}$, which is itself the right orthogonal to ${}^{\perp}A$. For the other containment, note that by Proposition 9, $A \subseteq ({}^{\perp}A)^{\perp}$. Then, again by Proposition 9, ${}^{\perp}(({}^{\perp}A)^{\perp}) \subseteq {}^{\perp}A$. $\square$

Definition 6. *We call B a right type if there exists some A so that $B = A^{\perp}$. We call A a left type if there exists some B so that $A = {}^{\perp}B$.*

Since we are not assuming that $[\![\ ,\ ]\!]$ is commutative, our results will be sensitive to left- and right- sidedness. Subsequently, many propositions and lemmas that have a left- and right- hand component, where the proofs are essentially identical. To be concise, we will often refer to a left or right type as simply a type, except where this would be confusing.

Proposition 11. *For any right type B, $({}^{\perp}B)^{\perp} = B$. For any left type A, ${}^{\perp}(A^{\perp}) = A$.*

Proof. This is basically immediate from the definition of a type and the cancellation of triple orthogonal complements. If A is a left type, then $A = {}^{\perp}B$, so then

$$\begin{aligned} {}^{\perp}(A^{\perp}) &= {}^{\perp}(({}^{\perp}B)^{\perp}) \\ &= {}^{\perp}B \\ &= A \end{aligned}$$

The argument for right types is parallel. $\square$

In fact, the condition in Proposition **??** is an equivalent definition of being a type. If, for example, ${}^{\perp}(A^{\perp}) = A$, then $A = {}^{\perp}B$ where $B = A^{\perp}$. Subsequently, we have three equivalent formulations for the set of types given by $[\![\ ,\ ]\!]$.

Definition 7. *For a fixed realizability system $(P, \mathrm{Ex}, [\![\cdot, \cdot]\!])$ Define the sets*

$${}^{\perp}\mathit{Type} = \{A \subseteq P \otimes \mathbb{R} \mid {}^{\perp}(A^{\perp}) = A\}$$

$$\begin{aligned} \mathit{Type}^{\perp} &= \{B \subseteq \mathcal{D} \times \overline{\mathbb{R}} \mid ({}^{\perp}B)^{\perp} = B\} \\ \mathit{Type} &= \{(A, B) \mid B = A^{\perp}, A = {}^{\perp}B\} \end{aligned}$$

of left types, right types, and paired types, respectively.

It is not necessarily the case that a right type will be a left type, or vice versa. However, we can always pair up left and right types, so each left type has a right type it "goes with" and vice-versa.

Proposition 12. *B is a right type with $B = A^{\perp}$ if and only if A is a left type with $A =^{\perp} B$.*

Corollary 13. *There is a bijective correspondence between the sets* ${}^{\perp}Type_M$*,* $Type_M^{\perp}$*, and* $Type_M$*, given by*

$$\begin{tikzcd} & Type \arrow[dl, "(A,B)\mapsto B"'] \arrow[dr, "(A,B)\mapsto A"] & \\ Type^{\perp} \arrow[rr, shift left, "B\mapsto {}^{\perp}B"] & & {}^{\perp}Type \arrow[ll, shift left, "A\mapsto A^{\perp}"] \end{tikzcd}$$

4 A correspondence between the nucleus and the set of types

The reader will have noticed the parallel between the construction of the nucleus via fixed points of the adjoint functors M^* and M_* and that of types defined via an orthogonality relation. We now make the connection precise.

4.1 Ω and φ

Recall first that if $\mathcal{C}$ is a discrete $\overline{\mathbb{R}}$-category, then any assignment $f : \mathcal{C} \to \overline{\mathbb{R}}$ of extended real numbers to the objects of $\mathcal{C}$ will be functorial and thus a (co)presheaf. To translate between presheaves on $\mathcal{C}$ and subsets of $\mathcal{C} \times \mathbb{R}$, we'll introduce two operators, Ω and φ. The first of these, Ω, will send a presheaf on $\mathcal{C}$ to a subset of $\mathcal{C} \times \mathbb{R}$.
Definition 8. *Given* $f : \mathcal{C} \to \overline{\mathbb{R}}$*, define a subset* Ω_f *of* $\mathcal{C} \times \mathbb{R}$ *to be*

$$\Omega_f = \{(a, \alpha) \mid \alpha \leq f(a)\}$$

Proposition 14. *Let* f, g *be presheaves on a discrete* $\overline{\mathbb{R}}$*-category* $\mathcal{C}$*. If* $\Omega_f = \Omega_g$*, then* $f = g$*.*

Proof. Since $\mathcal{C}$ is discrete, we just have to check f and g agree on objects. Let a be an object of $\mathcal{C}$. Since $f(a) \leq f(a)$, $(a, f(a)) \in \Omega_f$, so $(a, f(a)) \in \Omega_g$ as the two sets are equal, and thus $f(a) \leq g(a)$. On the other hand, $(a, g(a)) \in \Omega_g$ as $g(a) \leq g(a)$, so $(a, g(a)) \in \Omega_f$, and thus $g(a) \leq f(a)$, so $f(a) = g(a)$. □

The second operator will send subsets of $\mathcal{C} \times \mathbb{R}$ to presheaves on $\mathcal{C}$.
Definition 9. *Given a subset* $A \subseteq \mathcal{C} \times \mathbb{R}$*, define a function*

$$\varphi_A : \mathcal{C} \to \overline{\mathbb{R}}$$

by

$$a \mapsto \sup\{\alpha : (a, \alpha) \in A\}$$

where $\sup \emptyset = -\infty$*. If we are viewing* $\mathcal{C}$ *as a discrete* $\overline{\mathbb{R}}$*-category, then* φ_A *is a presheaf.*

Of course, it is not generally the case that there is a bijective correspondence between presheaves $\mathcal{C} \to \overline{\mathbb{R}}$ and subsets of $\mathcal{C} \times \mathbb{R}$. We'll restrict the subsets of $\mathcal{C} \times \mathbb{R}$ we consider to those subsets that are downwards closed and where the projection to $\mathbb{R}$ is closed in the usual topology on $\mathbb{R}$.

Definition 10. *A subset A of $\mathcal{C} \times \mathbb{R}$ is downwards closed if, for all $(a, \alpha) \in A$ and all $\beta \in \mathbb{R}$ such that $\beta \leq \alpha$, $(a, \beta) \in A$.*

Proposition 15. *For any presheaf f, Ω_f is downwards closed. Moreover, $\varphi_{\Omega_f} = f$.*

Proof. It is immediate from the definition that Ω_f is downwards closed. Let a be an object of $\mathcal{C}$. Then,

$$\begin{aligned}(\varphi_{\Omega_f})(a) &= \sup_{(a,\alpha)\in\Omega_f} \alpha \\ &= \sup\{\alpha \mid \alpha \leq f(a)\} \\ &= f(a)\end{aligned}$$

□

Proposition 16. *Let $A \subseteq \mathcal{C} \times \mathbb{R}$ be downwards closed and such that the projection to the second component is closed. Then $\Omega_{\varphi_A} = A$.*

Proof. Fix some $(a, \beta) \in A$. So

$$(a, \varphi_A(a)) = (a, \sup_{(a,\alpha)\in A} \alpha) \in A$$

Now, since $(a, \beta) \in A$, $\beta \leq \sup_{(a,\alpha)\in A} \alpha$, so $(a, \beta) \in \Omega_{\varphi_A}$. On the other hand, if $(a, \beta) \in \Omega_{\varphi_A}$, then $\beta \leq \varphi_A(a)$. Then $(a, \varphi_A(a)) \in A$. Subsequently, as A is downwards closed, $(a, \beta) \in A$. □

4.2 The nucleus is the set of types

A profunctor $M : \mathcal{C}\times\mathcal{C} \to \mathbb{R}$ can be viewed as the measurement of a realizability system $[\![a, b]\!] := M(a, b)$. The maps φ and Ω allow us to translate statements about closure under orthogonality for subsets of $\mathcal{C} \times \mathbb{R}$ into statements about functors $f : \mathcal{C} \to \overline{\mathbb{R}}$ belonging to the nucleus of M. To begin, we prove a lemma about the relationship between M_*, M^*, and taking orthogonal complements.

Lemma 17. *Let f be an element of $[\mathcal{C}^{\mathrm{op}}, \mathbb{R}]$ and let g be an element of $[\mathcal{C}, \mathbb{R}]^{op}$. Then*

$$^{\perp}\Omega_g = \Omega_{M_* g} \qquad \textit{and} \qquad \Omega_f^{\perp} = \Omega_{M^* f}$$

Proof. Let $(b, \beta) \in \Omega_f^{\perp}$, so for all (a, α) such that $\alpha \leq f(a)$,

$$0 \leq M(a, b) - (\alpha + \beta) \iff \alpha + \beta \leq M(a, b)$$

We want to show that $\beta \leq M^* f(b) = \inf_{c\in\mathcal{C}}(M(c, b) - f(c))$. Let $c \in \mathcal{C}$ be arbitrary. Then, as $f(c) \leq f(c)$, the pair $(c, f(c))$ is in Ω_f, so

$$f(c) + \beta \leq M(c, b) \iff \beta \leq M(c, b) - f(c)$$

Since this holds for all $c \in \mathcal{C}$, $\beta \leq \inf_{c\in\mathcal{C}}(M(c,b) - f(c))$, as desired. On the other hand, if we assume $(b,\beta) \in \Omega_{M^*f}$, and thus

$$\beta \leq \inf_{c\in\mathcal{C}}(M(c,b) - f(c))$$

we can show (b,β) is orthogonal to everything in Ω_f. Let $(a,\alpha) \in \Omega_f$, so $\alpha \leq f(a)$. As $\beta \in \Omega_{M^*f}$,

$$\beta \leq M(a,b) - f(a) \iff \beta + f(a) \leq M(a,b)$$

Now, since $\alpha \leq f(a)$, we have

$$\beta + \alpha \leq \beta + f(a) \leq M(a,b)$$

which is equivalent to

$$0 \leq M(a,b) - (\alpha + \beta)$$

so (b,β) is orthogonal to (a,α). The other statement is analogous.

□

Corollary 18. *Let A be a downwards closed subset of $\mathcal{C} \times \mathbb{R}$ that is closed when we project to $\mathbb{R}$. Then*

$$M^*\varphi_A = \varphi_{A^\perp} \qquad \textit{and} \qquad M_*\varphi_A = \varphi_{{}^\perp A}$$

Proof. To begin, recall that for any f, $\varphi_{\Omega_f} = f$. So, in particular, we have

$$\begin{aligned} M^*\varphi_A &= \varphi_{\Omega_{M^*\varphi_A}} \\ &= \varphi_{(\Omega_{\varphi_A})^\perp} \end{aligned}$$

We then have that $\Omega_{\varphi_A} = A$, so

$$(\Omega_{\varphi_A})^\perp = A^\perp$$

and thus

$$M^*\varphi_A = \varphi_{A^\perp}$$

The other equality is essentially the same argument. □

Theorem 19. *The operators Ω and φ give a correspondence between types and elements of the nucleus.*

a) A presheaf g is in the nucleus of M, in the sense that $M^*M_*g = g$, if and only if Ω_g is a type, and

b) If A is a type, then the presheaf induced by A, φ_A, is in the nucleus of M, in the sense that $M_*M^*\varphi_A = \varphi_A$. Conversely, assume A is downwards closed and closed when we project to $\mathbb{R}$. If φ_A is in the nucleus of M, then A is a type.

Proof. To show (a), we have:

$$\Omega_{M^*M_*g} = (\Omega_{M_*g})^\perp$$

$$= ({}^{\perp}\Omega_g)^{\perp}$$

Now, if $M^*M_*g = g$, then

$$\begin{aligned}\Omega_g &= \Omega_{M^*M_*g} \\ &= ({}^{\perp}\Omega_g)^{\perp}\end{aligned}$$

so Ω_g is a type. On the other hand, if Ω_g is a type, then

$$\begin{aligned}({}^{\perp}\Omega_g)^{\perp} &= \Omega_g \\ &= \Omega_{M^*M_*g}\end{aligned}$$

so $g = M^*M_*g$ and thus g is in the nucleus. For (b), assume A is a type. As A is downwards closed and closed when we project to $\mathbb{R}$, $\Omega_{\varphi_A} = A$. So

$$\begin{aligned}{}^{\perp}(A^{\perp}) &=^{\perp} (\Omega_{\varphi_A}^{\perp}) \\ &= \Omega_{M_*M^*\varphi_A}\end{aligned}$$

Subsequently, we have that

$$\begin{aligned}\Omega_{\varphi_A} &= A \\ &=^{\perp} (A^{\perp}) \\ &= \Omega_{M_*M^*\varphi_A}\end{aligned}$$

so by $\varphi_A = M_*M^*\varphi_A$ and thus φ_A is in the nucleus. On the other hand, if φ_A is in the nucleus of M, we already know from part (a) that Ω_{φ_A} is a type. The requirement that A is downwards closed and closed when we project to $\mathbb{R}$ then forces $A = \Omega_{\varphi_A}$, so A is a type by part (a). $\square$

5 Execution on types

Now, we consider the problem of how to extend the execution product from elements to types. The existing literature deals with linear realizability situations that satisfy certain additional conditions: for example, if execution is assumed to be commutative, or the slightly weaker assumption that $p(ab) = p(ba)$, will lead to the fact that $A^{\perp} = {}^{\perp}A$. Consequently, under either of these hypotheses, there is only one kind of type — there is no distinction between left types and right types and the execution product can be extended to an associative product on types. The general situation as presented here is a little bit more subtle and will lead to the definition of left, right, and middle types and some care is required to extend execution to types.

Lemma 20. *For elements, $x = (a, \alpha)$, $y = (b, \beta)$, $z = (c, \gamma)$,*

$$xy \perp z \iff x \perp yz$$

Proof. By definition, $xy \perp z$ is equivalent to $w \perp z$, where $w = (ab, \alpha + \beta - [\![a, b]\!])$. This, by definition, corresponds to

$$\alpha + \beta - [\![a, b]\!] + \gamma - [\![ab, c]\!] \leq 0.$$

We can now use Equation 5 to show this is equivalent to

$$\alpha + \beta + \gamma - [\![b, c]\!] - [\![a, bc]\!] \leq 0.$$

Equivalently, $x \perp u$, with $u = (\beta + \gamma - [\![b, c]\!], bc)$, that is: $x \perp yz$. $\square$

Now, define the product of two sets by

$$A \circ B := \{ab : a \in A \text{ and } b \in B\}.$$

Let us illustrate what happens if we try to define an associative product on left / right types. For left types A_1 and A_2 define

$$A_1 \cdot A_2 = {}^{\perp}\big((A_1 \circ A_2)^{\perp}\big)$$

For right types B_1 and B_2 define

$$B_1 \cdot B_2 = \big({}^{\perp}(B_1 \circ B_2)\big)^{\perp}$$

Proposition 21. *Let A, B, C be left types. Then $(A \cdot B) \cdot C \subseteq A \cdot (B \cdot C)$.*

Proof. We will prove that $(A \circ (B \cdot C))^{\perp} \subseteq ((A \cdot B) \circ C)^{\perp}$, which establishes the result by taking the left orthogonal set.

To prove this, we will first show that $z \in ((A \cdot B) \circ C)^{\perp}$ is equivalent to $z \in (A \circ (B \circ C))^{\perp}$. Let $z \in ((A \cdot B) \circ C)^{\perp}$. This means that:

$$\forall d \in A \cdot B, \forall c \in C, dc \perp z$$

This is equivalent to

$$\forall d \in A \cdot B, \forall c \in C, d \perp cz.$$

This means that $cz \in ({}^{\perp}(A \circ B)^{\perp})^{\perp}$, which is equivalent to $cz \in (A \circ B)^{\perp}$, i.e.

$$\forall ab \in A \circ B, \forall c \in C, ab \perp cz.$$

But this is equivalent to

$$\forall a \in A, \forall b \in B, \forall c \in C, (ab)c \perp z,$$

which by associativity is equivalent to

$$\forall a \in A, \forall b \in B, \forall c \in C, a(bc) \perp z,$$

that is

$$\forall a \in A, \forall bc \in B \circ C, a(bc) \perp z.$$

Hence, this is equivalent to $z \in (A \circ (B \circ C))^{\perp}$.

But $A \circ (B \circ C) \subseteq A \circ (B \cdot C)$, hence $(A \circ (B \cdot C))^{\perp} \subseteq (A \circ (B \circ C))^{\perp}$. Since we just showed that $(A \circ (B \circ C))^{\perp} = ((A \cdot B) \circ C)^{\perp}$, this means that

$$(A \circ (B \cdot C))^{\perp} \subset ((A \cdot B) \circ C)^{\perp}.$$

□

In trying to understand what is failing here, we can establish the following properties.

Proposition 22. *Let X, Y be sets, and A, B be the types generated by X, Y, respectively. Then*

$$A \cdot Y = X \cdot Y$$
$$X \cdot Y \subseteq X \cdot B$$

Proof. For the first identity:

$$\begin{aligned} z \in (A \cdot Y)^{\perp} &\Leftrightarrow z \in (A \circ Y)^{\perp} \\ &\Leftrightarrow \forall a \in A, \forall y \in Y, ay \perp z \\ &\Leftrightarrow \forall a \in A, \forall y \in Y, a \perp yz \\ &\Leftrightarrow \forall x \in X, \forall y \in Y, x \perp yz \\ &\Leftrightarrow \forall x \in X, \forall y \in Y, xy \perp z \\ &\Leftrightarrow z \in (X \circ Y)^{\perp} \end{aligned}$$

The second identity simply comes from the fact that $X \circ Y \subseteq X \circ B$, hence $(X \circ B)^{\perp} \subseteq (X \circ Y)^{\perp}$, i.e. $X \cdot Y \subseteq X \cdot B$ □

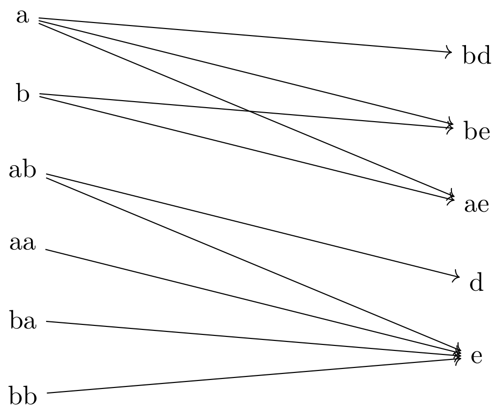
Note that if we could establish an equality instead of an inclusion in the above statement, associativity would follow. However this inclusion is, in general, strict, as illustrated by the following example.

We will consider the map p defined on $\{a, b, c, d, e\}^*$ as follows: $p(w) = 1$ for all words in the set:

$$S = \{abd, abe, aae, bbe, bae\},$$

and $p(w) = 0$ otherwise.

Then the map M can be represented as the following directed graph in the following sense: $M(w, w') = 1$ if and only if there is a directed edge from w to w' in the graph, and $M(w, w') = 0$ otherwise. We show only the part of the graph that contains edges: all vertices not shown in the picture are not involved in an edge.



We can now consider the element $(b, 0)$, and the type generated by it: $B = {}^{\perp}(\{(b, 0)\}^{\perp})$. One can check the following:

$$\begin{aligned}\{(b,0)\}^{\perp} &= \{(u,\lambda) \mid \lambda - M(b,u) \leq 0\} \\ &= \{(be,\beta) \mid \beta \leq 1\} \cup \{(ae,\beta) \mid \beta \leq 1\} \cup \{(w,\beta) \mid \beta \leq 0, w \neq ae, be\}\end{aligned}$$

and thus

$$\begin{aligned}B &= \{(u,\lambda) \mid \forall (v,\mu) \in \{(b,0)\}^{\perp}, \lambda + \mu - M(v,u) \leq 0\} \\ &= \{(b,\beta) \mid \beta \leq 0\} \cup \{(a,\beta) \mid \beta \leq 0\} \cup \{(w,\beta) \mid \beta \leq -1, w \neq a, b\}\end{aligned}$$

Now, consider the two types $F = \{(a, 0)\} \cdot \{(b, 0)\}$ and $G = \{(a, 0)\} \cdot B$. We will show that there exists elements in $F^{\perp}$ that do not belong to $G^{\perp}$, which will be enough to prove that $F \neq G$. Based on the above computations, we have:

$$\begin{aligned}F^{\perp} &= \{(ab,0)\}^{\perp} \\ &= \{(d,\mu) \mid \mu \leq 1\} \cup \{(e,\mu) \mid \mu \leq 1\} \cup \{(w,\mu) \mid \mu \leq 0, w \neq d, e\}\end{aligned}$$

while

$$\begin{aligned}G^{\perp} &= (\{(ab,\beta) \mid \beta \leq 0\} \cup \{(aa,\beta) \mid \beta \leq 0\} \cup \{(aw,\beta) \mid \beta \leq -1, w \neq a, b\})^{\perp} \\ &= \{(e,\mu) \mid \mu \leq 1\} \cup \{(w,\mu) \mid \mu \leq 0, w \neq e\}.\end{aligned}$$

More concretely, one can check directly that $(d, 1) \in F^{\perp}$ but $(d, 1) \notin G^{\perp}$. Indeed, G contains the element $(aa, 0)$ and $(aa, 0) \not\perp (d, 1)$ since this would be equivalent to $0 + 1 + M(aa, d) \leq 0$, i.e. $1 \leq 0$.

5.1 Middle types

Our analysis of the situation is that associativity does not really fail, but the operations considered above are taken without proper care. More precisely, while it makes sense to consider a left type to the left of a product $A \cdot B$ to obtain a left type, consider

B as a right type introduces discrepancy. The reason is that once expressed in terms of orthogonality between elements, the elements of b appear in between elements of a and the right-hand side of the orthogonality. While b can be moved between sides, it cannot be considered as a leftmost element. As a consequence, the use of b in the situation $ab \perp c$, need not be related in any way with the use of b in the situation $b \perp d$. While this was not a problem in earlier constructions because of the symmetry[1], the current situation requires the introduction of the notion of *middle type*.

Definition 11. *Let $a, b, c \in C \times \mathbb{R}$. We define the following orthogonality:*

$$a \perp\!\!\!\perp b, c \Leftrightarrow ba \perp c \Leftrightarrow b \perp ac.$$

Definition 12. *A* left type *is a set A such that ${}^{\perp}(A^{\perp}) = A$. A* right type *is a set C such that $({}^{\perp}C)^{\perp} = C$.*

A middle type *is a set B such that ${}^{\perp\!\!\!\perp}(B^{\perp\!\!\!\perp}) = B$. A* context type *is a set B such that $({}^{\perp\!\!\!\perp}B)^{\perp\!\!\!\perp} = B$.*

Lemma 23.

$$aa' \perp\!\!\!\perp b, c \Leftrightarrow a \perp\!\!\!\perp b, a'c \Leftrightarrow a' \perp\!\!\!\perp ba, c$$

Now, the product is defined in the following cases: the product $A \cdot B$ is well-defined when:

- A is a left type and B a middle type; $A \cdot B$ is the left type ${}^{\perp}((A \circ B)^{\perp})$;
- A is a middle type and B a middle type; $A \cdot B$ is the middle type ${}^{\perp\!\!\!\perp}((A \circ B)^{\perp\!\!\!\perp})$;
- A is a middle type and B a right type; $A \cdot B$ is the right type ${}^{\perp}((A \circ B)^{\perp})$.

We can now prove associativity.

Theorem 24. *Consider types A, B, C. Then $(A \cdot B) \cdot C$ is well defined if and only if $A \cdot (B \cdot C)$ is, and in this case:*

$$(A \cdot B) \cdot C = A \cdot (B \cdot C).$$

Proof. If A is a left type, B, C are middle types. Then:

$$\begin{aligned}
z \in ((A \cdot B) \cdot C)^{\perp} &\Leftrightarrow ((A \cdot B) \circ C)^{\perp} \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c \in C, dc \perp z \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c \in C, d \perp cz \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, ab \perp cz \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, (ab)c \perp z \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, a(bc) \perp z \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, bc \perp\!\!\!\perp a, z \\
&\Leftrightarrow \forall a \in A, \forall e \in B \cdot C, e \perp\!\!\!\perp a, z \\
&\Leftrightarrow \forall a \in A, \forall e \in B \cdot C, ae \perp z \\
&\Leftrightarrow z \in (A \circ (B \cdot C))^{\perp} \\
&\Leftrightarrow z \in (A \cdot (B \cdot C))^{\perp}
\end{aligned}$$

[1] Symmetry implies that $ab \perp c$ is equivalent to $c \perp ab$, which is equivalent to $ca \perp b$, which in turn is equivalent to $b \perp ca$.

If A, B, C are middle types. Then:

$$\begin{aligned}
z, z' \in ((A \cdot B) \cdot C)^{\perp\!\!\perp} &\Leftrightarrow ((A \cdot B) \circ C)^{\perp\!\!\perp} \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c \in C, dc \perp\!\!\perp z, z' \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c \in C, d \perp\!\!\perp z, cz' \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, ab \perp z, cz' \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, (ab)c \perp z, z' \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, a(bc) \perp z, z' \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, bc \perp\!\!\perp za, z' \\
&\Leftrightarrow \forall a \in A, \forall e \in B \cdot C, e \perp\!\!\perp za, z' \\
&\Leftrightarrow \forall a \in A, \forall e \in B \cdot C, ae \perp z, z' \\
&\Leftrightarrow z, z' \in (A \circ (B \cdot C))^{\perp\!\!\perp} \\
&\Leftrightarrow z, z' \in (A \cdot (B \cdot C))^{\perp\!\!\perp}
\end{aligned}$$

If A, B are middle types and C is a right type. Then:

$$\begin{aligned}
z \in {}^{\perp}((A \cdot B) \cdot C) &\Leftrightarrow {}^{\perp}((A \cdot B) \circ C) \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c \in C, z \perp dc \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c \in C, d \perp\!\!\perp z, c \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, ab \perp\!\!\perp z, c \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, z \perp (ab)c \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, z \perp a(bc) \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C, za \perp bc \\
&\Leftrightarrow \forall a \in A, \forall e \in B \cdot C, za \perp e \\
&\Leftrightarrow \forall a \in A, \forall e \in B \cdot C, z \perp ae \\
&\Leftrightarrow z \in {}^{\perp}(A \circ (B \cdot C)) \\
&\Leftrightarrow z \in {}^{\perp}(A \cdot (B \cdot C))
\end{aligned}$$

$\square$

Proposition 25. *Define the middle type* $1 = {}^{\perp\!\!\perp}(\{(\epsilon, p(\epsilon))\}^{\perp\!\!\perp})$. *Then* $1 \cdot B = B \cdot 1 = B$ *for all middle type* B. *Also,* $1 \cdot C = C$ *for all right type* C, *and* $A \cdot 1 = A$ *for all left type* A.

Proof. Write $\epsilon = (\epsilon, -p(\epsilon))$. Then $\epsilon a = a = a\epsilon$ for all a:

$$(\epsilon, p(\epsilon)) \cdot (a, \alpha) = (\epsilon \cdot a, \alpha - p(\epsilon) - p(\epsilon a) + p(a) + p(\epsilon)) = (a, \alpha).$$

$$(a, \alpha) \cdot (\epsilon, p(\epsilon)) = (a \cdot \epsilon, -p(\epsilon) + \alpha - p(\epsilon a) + p(a) + p(\epsilon)) = (a, \alpha).$$

Then, we simply have:

$$\begin{aligned} z, z' \in (1 \cdot B)^{\perp\!\!\perp} &\Leftrightarrow \forall u \in (1 \cdot B), u \perp\!\!\perp z, z' \\ &\Leftrightarrow \forall b \in B, \epsilon b \perp\!\!\perp z, z' \\ &\Leftrightarrow \forall b \in B, b \perp\!\!\perp z, z' \\ &\Leftrightarrow z, z' \in B^{\perp\!\!\perp} \end{aligned}$$

and

$$\begin{aligned} z, z' \in (B \cdot 1)^{\perp\!\!\perp} &\Leftrightarrow \forall u \in (B \cdot 1), u \perp\!\!\perp z, z' \\ &\Leftrightarrow \forall b \in B, b\epsilon \perp\!\!\perp z, z' \\ &\Leftrightarrow \forall b \in B, b \perp\!\!\perp z, z' \\ &\Leftrightarrow z, z' \in B^{\perp\!\!\perp} \end{aligned}$$

Similarly:

$$\begin{aligned} z \in {}^{\perp}(1 \cdot C) &\Leftrightarrow \forall u \in (1 \cdot C), z \perp u \\ &\Leftrightarrow \forall c \in C, z \perp \epsilon c \\ &\Leftrightarrow \forall c \in C, z \perp c \\ &\Leftrightarrow z \in \ \perp C \end{aligned}$$

and

$$\begin{aligned} z \in {}^{\perp}(A \cdot 1) &\Leftrightarrow \forall u \in (A \cdot a), u \perp z \\ &\Leftrightarrow \forall a \in A, \epsilon a \perp z \\ &\Leftrightarrow \forall a \in A, a \perp z \\ &\Leftrightarrow z \in A^{\perp} \end{aligned}$$

□

Note that one can define 1_l (resp. 1_r) as the left (resp. right) type defined by $(\epsilon, p(\epsilon))$. This is the only left (resp. right) type that can be considered in a product with another left (resp. right) type. Define, for A a left type and B a right type:

$$1_l \cdot A = {}^{\perp}((1_l \circ A)^{\perp}) \qquad\qquad B \cdot 1_r = ({}^{\perp}(B \circ 1_r))^{\perp}.$$

Note these are respectively left and right types. We can prove $1_l \cdot A = A$ and $B \cdot 1_r = B$.

What about $1_l \cdot C$ and $C \cdot 1_r$ with C a middle type? We have that $C \subseteq 1_l \cdot C$. Indeed, if $c \in C$, then for all $z, z' \in C^{\perp\!\!\perp}$, $c \perp\!\!\perp z, z'$. This implies that $c \perp\!\!\perp \epsilon, z'$ for all $\epsilon, z' \in C^{\perp\!\!\perp}$. But $c \perp\!\!\perp \epsilon, z'$ if and only if $\epsilon c \perp z'$.

Proposition 26. *Let A be a left type. Then $A = {}^{\perp\!\!\perp}\{(\epsilon, a') \mid a' \in A^{\perp}\}$.*

Proof. Consider $c \in {}^{\perp\!\!\perp}\{(\epsilon, a') \mid a' \in A^{\perp}\}$. Then for all $a' \in A^{\perp}$, $c \perp\!\!\perp \epsilon, a'$. This is equivalent to $\epsilon c \perp a'$ for all $a' \in A^{\perp}$, i.e. $c \perp a'$. Hence This is equivalent to $c \in A$. □

Definition 13. *Given a middle type A and a context type C, we define:*

$$A \cdot_r C = \{c, ac' \mid a \in A, c, c' \in C\},$$

$$A \cdot_l C = \{ca, c' \mid a \in A, c, c' \in C\}.$$

5.2 Linear arrows

Definition 14. *Given middle types A, B, we define:*

$$A \multimap_l B = \{f \mid \forall a \in A, af \in B\},$$

$$A \multimap_r B = \{f \mid \forall a \in A, fa \in B\}.$$

If A is a middle type, and B a left type, we can define $A \multimap_l B$ in the same way. If A is a middle type, and B a right type, we can define $A \multimap_r B$ in the same way.

Theorem 27. *If A, B are middle types:*

$$^{\perp\!\!\perp}(A \cdot_r B^{\perp\!\!\perp}) = A \multimap_r B,$$

$$^{\perp\!\!\perp}(B \cdot_l A^{\perp\!\!\perp}) = A \multimap_l B.$$

If A is a left type and B is a middle type:

$$^{\perp}(B \cdot A^{\perp}) = A \multimap_r B.$$

If A is a middle type and B is a right type:

$$(^{\perp}B \cdot A)^{\perp} = A \multimap_l B.$$

Proof. Consider A, B middle types. Then:

$$\begin{aligned} f \in A \multimap_l B &\Leftrightarrow \forall a \in A, af \in B \\ &\Leftrightarrow \forall a \in A, \forall c, c' \in B^{\perp\!\!\perp}, af \perp\!\!\perp c, c' \\ &\Leftrightarrow \forall a \in A, \forall c, c' \in B^{\perp\!\!\perp}, f \perp\!\!\perp ca, c' \\ &\Leftrightarrow f \in {}^{\perp\!\!\perp}\{ca, c' \mid a \in A, c, c' \in B^{\perp}\} \\ &\Leftrightarrow f \in {}^{\perp\!\!\perp}(A \cdot_l B^{\perp\!\!\perp}) \end{aligned}$$

$$\begin{aligned} f \in A \multimap_r B &\Leftrightarrow \forall a \in A, fa \in B \\ &\Leftrightarrow \forall a \in A, \forall c, c' \in B^{\perp\!\!\perp}, fa \perp\!\!\perp c, c' \\ &\Leftrightarrow \forall a \in A, \forall c, c' \in B^{\perp\!\!\perp}, f \perp\!\!\perp c, ac' \\ &\Leftrightarrow f \in {}^{\perp\!\!\perp}\{c, ac' \mid a \in A, c, c' \in B^{\perp}\} \\ &\Leftrightarrow f \in {}^{\perp\!\!\perp}(A \cdot_r B^{\perp\!\!\perp}) \end{aligned}$$

Consider A a left type, B a middle type. Then:

$$\begin{aligned}
f \in B \multimap_r A &\Leftrightarrow \forall b \in B, fb \in A \\
&\Leftrightarrow \forall b \in B, \forall c \in A^{\perp}, fb \perp c \\
&\Leftrightarrow \forall b \in B, \forall c \in A^{\perp\!\!\perp}, f \perp\!\!\perp bc \\
&\Leftrightarrow f \in {}^{\perp}\{bc \mid b \in B, c \in A^{\perp}\} \\
&\Leftrightarrow f \in {}^{\perp}(B \cdot A^{\perp})
\end{aligned}$$

Consider A a middle type, B a right type. Then:

$$\begin{aligned}
f \in A \multimap_l B &\Leftrightarrow \forall a \in A, af \in B \\
&\Leftrightarrow \forall a \in A, \forall c \in {}^{\perp}B, c \perp af \\
&\Leftrightarrow \forall a \in A, \forall c \in {}^{\perp}B, ca \perp f \\
&\Leftrightarrow f \in \{ca \mid a \in A, c \in {}^{\perp}B\}^{\perp} \\
&\Leftrightarrow f \in ({}^{\perp}B \cdot A)^{\perp}
\end{aligned}$$

$\square$

Proposition 28.

$$A \cdot B \multimap_l C = B \multimap_l (A \multimap_l C)$$
$$A \cdot B \multimap_r C = A \multimap_r (B \multimap_r C)$$

Proof. We prove it for middle types first.

$$\begin{aligned}
f \in A \cdot B \multimap_l C &\Leftrightarrow \forall d \in A \cdot B, df \in C \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c, c' \in C^{\perp\!\!\perp}, df \perp\!\!\perp c, c' \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c, c' \in C^{\perp\!\!\perp}, d \perp\!\!\perp c, fc' \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c, c' \in C^{\perp\!\!\perp}, ab \perp\!\!\perp c, fc' \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c, c' \in C^{\perp\!\!\perp}, abf \perp\!\!\perp c, c' \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c, c' \in C^{\perp\!\!\perp}, bf \perp\!\!\perp ca, c' \\
&\Leftrightarrow \forall b \in B, \forall c, c' \in A \cdot_l C^{\perp\!\!\perp}, bf \perp\!\!\perp c, c' \\
&\Leftrightarrow \forall b \in B, bf \in {}^{\perp\!\!\perp}(A \cdot_l C^{\perp\!\!\perp}) \\
&\Leftrightarrow \forall b \in B, bf \in A \multimap_l C \\
&\Leftrightarrow f \in B \multimap_l (A \multimap_l C)
\end{aligned}$$

$$\begin{aligned}
f \in A \cdot B \multimap_r C &\Leftrightarrow \forall d \in A \cdot B, fd \in C \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c, c' \in C^{\perp\!\!\perp}, fd \perp\!\!\perp c, c'
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \forall d \in A \cdot B, \forall c, c' \in C^{\perp\!\!\perp}, d \perp\!\!\perp cf, c' \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c, c' \in C^{\perp\!\!\perp}, ab \perp\!\!\perp cf, c' \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c, c' \in C^{\perp\!\!\perp}, fab \perp\!\!\perp c, c' \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c, c' \in C^{\perp\!\!\perp}, fa \perp\!\!\perp c, bc' \\
&\Leftrightarrow \forall a \in A, \forall c, c' \in B \cdot_r C^{\perp\!\!\perp}, fa \perp\!\!\perp c, c' \\
&\Leftrightarrow \forall a \in A, fa \in {}^{\perp\!\!\perp}(B \cdot_r C^{\perp\!\!\perp}) \\
&\Leftrightarrow \forall a \in A, fa \in A \multimap_r C \\
&\Leftrightarrow f \in A \multimap_r (B \multimap_r C)
\end{aligned}$$

Now, if A, B are middle types and C is a right type:

$$\begin{aligned}
f \in A \cdot B \multimap_r C &\Leftrightarrow \forall d \in A \cdot B, fd \in C \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c \in C^{\perp}, fd \perp c \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c \in C^{\perp}, d \perp\!\!\perp f, c \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C^{\perp}, ab \perp\!\!\perp f, c \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C^{\perp}, f(ab) \perp c \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in C^{\perp}, (fa)b \perp c \\
&\Leftrightarrow \forall a \in A, (fa) \in B \multimap_r C \\
&\Leftrightarrow f \in A \multimap_r (B \multimap_r C)
\end{aligned}$$

Now, if A, B are middle types and C is a left type:

$$\begin{aligned}
f \in A \cdot B \multimap_l C &\Leftrightarrow \forall d \in A \cdot B, df \in C \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c \in {}^{\perp}C, c \perp df \\
&\Leftrightarrow \forall d \in A \cdot B, \forall c \in {}^{\perp}C, d \perp\!\!\perp c, f \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in {}^{\perp}C, ab \perp\!\!\perp c, f \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in {}^{\perp}C, c \perp (ab)f \\
&\Leftrightarrow \forall a \in A, \forall b \in B, \forall c \in {}^{\perp}C, c \perp a(bf) \\
&\Leftrightarrow \forall b \in B, (bf) \in A \multimap_l C \\
&\Leftrightarrow f \in B \multimap_l (A \multimap_l C)
\end{aligned}$$

$\square$

Lemma 29. *Let A be the type generated by X. Then:*

$$A \multimap_r B = X \multimap_r B$$

$$A \multimap_l B = X \multimap_l B$$

Proof. We only need to prove one inclusion. Suppose that $f \in X \multimap_r B$, and consider $a \in A$. We want to prove that $fa \in B$. We know that

$$\forall x \in X, \forall z, z' \in B^{\perp\!\!\perp}, fa \perp\!\!\perp z, z'.$$

This is equivalent to

$$\forall x \in X, \forall z, z' \in B^{\perp\!\!\perp}, x \perp\!\!\perp zf, z'$$

which is equivalent to

$$\forall a \in A, \forall z, z' \in B^{\perp\!\!\perp}, a \perp\!\!\perp zf, z'$$

since A is te type generated by X. Hence

$$\forall a \in A, \forall z, z' \in B^{\perp\!\!\perp}, fa \perp\!\!\perp z, z',$$

that is $\forall a \in A, fa \in B$.

The proof for $\multimap_l$ is similar. □

Theorem 30.

$$(A \multimap_l B) \cdot (B \multimap_l C) \subseteq A \multimap_l C$$
$$(B \multimap_r C) \cdot (A \multimap_r B) \subseteq A \multimap_r C$$

Proof. Consider $gh \in (A \multimap_l B) \circ (B \multimap_l C)$. We want to show that for all $a \in A$, $a(gh) \in C$. But $a(gh) = (ag)h$. By assumption, $ag \in B$ for all $a \in A$, and thus $(ag)h \in C$ since $bh \in C$ for all $b \in B$. Hence $gh \in A \multimap_l C$. This proves that

$$(A \multimap_l B) \circ (B \multimap_l C) \subseteq A \multimap_l C$$

which implies that

$$(A \multimap_l B) \cdot (B \multimap_l C) \subseteq A \multimap_l C$$

since $A \multimap_l C$ is a type.

The proof of the second property is similar. □

Lemma 31. *Given types A, B:*

$$A \subseteq B \Leftrightarrow (\epsilon, p(\epsilon)) \in A \multimap_l B \Leftrightarrow (\epsilon, p(\epsilon)) \in A \multimap_r B$$

Definition 15. *Given two context types C, D we define:*

$$C \multimap D = \{z, z' \mid \forall c, c' \in C, zc, c'z' \in D\}.$$

Definition 16. *Given a context type C and a middle type A we define:*

$$C[A] = \{cac' \mid \forall c, c' \in C, \forall a \in A\}.$$

Theorem 32. *Given two context types C, D,*

$$C \multimap D = (C[^{\perp\!\!\perp}D])^{\perp\!\!\perp}$$

Proof. Let $z, z' \in C \multimap D$, then for all $c, c' \in C$, $zc, c'z' \in D$. I.e. for any element $u \in {}^{\perp\!\!\perp}D$, $u \perp\!\!\perp zc, c'z'$. Equivalently, for any element $u \in {}^{\perp\!\!\perp}D$, $cuc' \perp\!\!\perp z, z'$, i.e. $z, z' \in (C[{}^{\perp\!\!\perp}D])^{\perp\!\!\perp}$. □

Definition 17. *Given a type A, define $A_0 = A \cap \{(\epsilon, \lambda) \mid \lambda \in \mathbb{R}\}$.*

5.3 Categorical structure

Given the above propositions, one would expect that the category of (middle) types, with the set of morphisms from A to B defined as elements of $A \multimap_* B$, possesses the structure of a monoidal closed category. Indeed, we have shown that the dot product is associative and possesses a unit, and proposition identifies the type $A \multimap_* B$ as a good candidate for an internal hom object.

However, while the above is true in previous linear realizability constructions from the literature, it is not in the current case. The main reason being that the dot product cannot be extended to morphisms in a natural way. Indeed, consider two morphisms $f \in A \multimap_r B$ and $g \in A' \multimap_r B'$. Then how could one define a morphism $f \cdot g$ taking any element of $A\dot{A}'$ to an element of $B \cdot B'$? By definition, f and g act by concatenation on the left, and the generating set for $A \cdot A'$ contains elements of the form $a \cdot a'$ with $a \in A$ and $a' \in A'$. Then one can surely define faa', which is an element of $B \circ A'$. But applying g to a' would require to insert it at the right position, i.e. in between fa and a'. However, there is no natural way to do so.

The only case in which $f \cdot g$ can be defined is when they are of the form (ϵ, λ). Indeed, concatenating with the empty word acts as the identity, and therefore the position in which the concatenation is performed does not matter. We will thus define a category with the same objects but fewer morphisms, i.e. only those of the above form. We can then establish that this is indeed a monoidal category.

Lemma 33. *Let A, B be types. Then $A_0 \circ B_0 = (A \cdot B)_0$.*

Proof. It is clear that $A_0 \circ B_0 \subseteq (A \cdot B)_0$ since elements of $A_0 \circ B_0$ are elements of $A \circ B$, and they are all of the form (ϵ, λ). Now, if one takes an element of $(A \cdot B)_0$, it is of the form (ϵ, λ). □

Theorem 34. *The category with object middle types and morphisms $Hom(A, B)$ defined as $(A \multimap_l B)_0$ is a monoidal category.*

Proof. First, we can check this is a category. Composition is given by the dot product: if $f \in (A \multimap_l B)_0$ and $g \in (B \multimap_l C)_0$, we have seen that $fg \in (A \multimap_l C)_0$. Composition is associative since the dot product is. Moreover $(\epsilon, p(\epsilon)) \in (A \multimap A)_0$ acts as the identity. Now, we need to check that the dot product on types is a monoidal product □

Theorem 35. *The category with object middle types and morphisms $Hom(A, B)$ defined as $(A \multimap_r B)_0$ is a monoidal category.*

Proof. We first check it is a category. Composition is given by the dot product: if $f \in (A \multimap_r B)_0$ and $g \in (B \multimap_r C)_0$, we have seen that $gf \in (A \multimap_r C)_0$. This is associative and has identities, as above. The dot product is also a monoidal product in this case □

Theorem 36. *Let A, B be middle types. Consider:*

$$\lambda = \sup\{\mu \mid (\epsilon, \mu) \in A \multimap_r B\} - p(\epsilon)$$

$$\lambda' = \sup\{\mu \mid (\epsilon, \mu) \in A \multimap_l B\} - p(\epsilon)$$

Then:

$$\lambda = \lambda' = [\mathcal{C}, \overline{\mathbb{R}}](\varphi_A, \varphi_B).$$

Similarly, if A is a left type and B a middle type,

$$[\mathcal{C}, \overline{\mathbb{R}}](\varphi_A, \varphi_B) = \sup\{\mu \mid (\epsilon, \mu) \in A \multimap_r B\} - p(\epsilon).$$

Similarly, if A is a middle type and B a right type,

$$[\mathcal{C}, \overline{\mathbb{R}}](\varphi_A, \varphi_B) = \sup\{\mu \mid (\epsilon, \mu) \in A \multimap_l B\} - p(\epsilon).$$

Proof. We prove that $[\mathcal{C}, \overline{\mathbb{R}}](\varphi_A, \varphi_B) = \sup\{\mu \mid (\epsilon, \mu) \in A \multimap_r B\}$. All other equalities are obtained in a similar way.

$$[\mathcal{C}, \overline{\mathbb{R}}](\varphi_A, \varphi_B) = \sup\{\varphi_B(c) - \varphi_A(c) \mid c \in \mathcal{C}\}.$$

As seen above, $\varphi_B(c) = \sup\{\lambda \mid (c, \lambda) \in B\}$. Similarly, $\varphi_A(c) = \sup\{\lambda \mid (c, \lambda) \in A\}$. Now, suppose that $(\epsilon, \lambda) \in A \multimap_* B$. Then for all $(a, \alpha) \in A$, $(\epsilon, \lambda) \cdot (a, \alpha) \in B$, that is $(a, \alpha + \lambda + p(\epsilon)) \in B$. Hence

$$\begin{aligned}
\sup\{\lambda \mid (\epsilon, \lambda) \in A \multimap_* B\} &= \sup\{\lambda \mid \forall (a, \alpha) \in A, (a, \alpha + \lambda + p(\epsilon)) \in B\} \\
&= \sup\{\lambda \mid \forall a \in \mathcal{C}, \varphi_A(a) + \lambda + p(\epsilon)) \leq \varphi_B(a)\} \\
&= \sup\{a \in \mathcal{C} \mid \varphi_A(a) - \leq \varphi_B(a)\} - p(\epsilon) \\
&= [\mathcal{C}, \overline{\mathbb{R}}](\varphi_A, \varphi_B) - p(\epsilon)
\end{aligned}$$

□

Corollary 37. *For any types A, B we have:*

$$\sup\{\mu \mid (\epsilon, \mu) \in (A \multimap_\star B) \cdot (B \multimap_* A)\} = -d(\varphi_A, \varphi_B) + p(\epsilon),$$

where $\star, * \in \{l, r\}$.

Proof. By definition, $-d(\varphi_A, \varphi_B) = [\mathcal{C}, \overline{\mathbb{R}}](\varphi_A, \varphi_B) + [\mathcal{C}, \overline{\mathbb{R}}](\varphi_B, \varphi_A)$. Let us write $\lambda_1 = \sup\{\lambda \mid (\epsilon, \lambda) \in A \multimap_* B\}$ and $\lambda_2 = \sup\{\lambda \mid (\epsilon, \lambda) \in B \multimap_* A\}$.

As we have seen above, this is equal to

$$\sup\{\lambda \mid (\epsilon, \lambda) \in A \multimap_* B\} + \sup\{\lambda \mid (\epsilon, \lambda) \in B \multimap_* A\} - 2p(\epsilon).$$

Now, let us compute the value $\mu = \sup\{\mu \mid (\epsilon, \mu) \in (A \multimap_\star B) \cdot (B \multimap_* A)\}$. Since ϵ can be written as a dot product only as $\epsilon \cdot \epsilon$, we have:

$$\mu \geq \lambda_1 + \lambda_2 + p(\epsilon).$$

We can also show the converse inequality, since $(\epsilon, \rho) \in ((A \multimap_\star B) \circ (B \multimap_* A))^{\perp\!\!\perp}$ implies that $\rho + \lambda_1 + \lambda_2 + p(\epsilon) + p(\epsilon) \leq 0$, i.e. $\rho \leq -(\lambda_1 + \lambda_2 + 2p(\epsilon))$. Hence $\sup\{(\epsilon, \rho) \in ((A \multimap_\star B) \circ (B \multimap_* A))^{\perp\!\!\perp}\} = -(\lambda_1 + \lambda_2 + 2p(\epsilon))$.

Now, this means that if μ is such that $(\epsilon, \mu) \in (A \multimap_\star B) \cdot (B \multimap_* A)$,

$$\mu + -(\lambda_1 + \lambda_2 + 2p(\epsilon)) + p(\epsilon) \leq 0$$

Hence

$$\mu - \lambda_1 - \lambda_2 - p(\epsilon) \leq 0,$$

i.e.

$$\mu \leq \lambda_1 + \lambda_2 + p(\epsilon).$$

This proves the result. □

5.4 Derived types

Given a realizability situation $(\mathcal{C}, p, \mathrm{Ex})$, the measurement defines a profunctor $M : \mathcal{C} \times \mathcal{C} \to \overline{\mathbb{R}}$ by $(a, b) \mapsto M(a, b) = [\![a, b]\!] := p(ab) - p(a) - p(b)$. The nucleus of M consists precisely of the types associated to the realizability situation $(\mathcal{C}, p, \mathrm{Ex})$. Now, by combining execution and measurement, one can extend M to $M : \mathcal{C} \times \mathcal{C} \times \cdots \times \mathcal{C} \to \overline{\mathbb{R}}$. To be concrete, consider

$$M3 : C \times C \times C \to \overline{\mathbb{R}}$$

defined by

$$\begin{aligned} M3(a, b, c) &= M(ab, c) + M(a, b) \\ &= M(a, bc) + M(b, c) \\ &= p(abc) - p(a) - p(b) - p(c) \end{aligned}$$

By singling out the left, middle, or right factor, one can view $M3$ as a profunctor from $\mathcal{C}$ to $\mathcal{C} \times \mathcal{C}$ in several ways. For example, organizing $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ as $\mathcal{C} \times \mathcal{D}$ where $\mathcal{D} = (\mathcal{C} \times \mathcal{C})$ one has "M left"

$$ML : C \times D \to \overline{\mathbb{R}}.$$

The operator ML is just $M3$ but with domain organized as $\mathcal{C} \times (\mathcal{C} \times \mathcal{C})$. Given a presheaf $f \in [\mathcal{C}, \overline{\mathbb{R}}]$, one has the copresheaf of $\mathcal{C}$ defined by $h := M^*(f) : \mathcal{C} \to \overline{\mathbb{R}}$ and the copresheaf on $\mathcal{C} \times \mathcal{C}$ defined by $g := ML^*(f) : \mathcal{C} \times \mathcal{C} \to \overline{\mathbb{R}}$. Since M and ML are related, so are g and h.

Lemma 38. $g(b, c) = h(bc) + [\![b, c]\!]$

Proof. $h(x) = \inf_{a \in \mathcal{C}} M(a, x) - f(a)$ and

$$\begin{aligned} g(b, c) &= \inf_{a \in \mathcal{C}} ML(a, b, c) - f(a) \\ &= \inf_{a \in \mathcal{C}} M(a, bc) + M(b, c) - f(a) \end{aligned}$$

$$= M(b,c) + \inf_{a \in \mathcal{C}} M(a, bc) - f(a)$$
$$= M(b,c) + h(bc).$$

□

Now, one can view $g : \mathcal{C} \times \mathcal{C} \to \overline{\mathbb{R}}$ itself as a profunctor and consider its nucleus. Suppose $(u, v) \in \mathrm{Nuc}(g)$. Let $A = \Omega_f$, $B = \Omega_u$ and $C = \Omega_v$. This is what we mean by *derived* type: the pair (B, C) is a derived type, meaning a type derived from A meaning a type for the copresheaf $\varphi_{ML^*(A)}$ viewed as a profunctor on $\mathcal{C} \times \mathcal{C}$. Derived types are closely related to the logical structure of the linear realizability situation.

Theorem 39. *$C = B \multimap_l A^{\perp}$ and $B = C \multimap_r A^{\perp}$.*

Proof. Before we begin, we will need two inequalities. The first comes from the fact that $(f, g) \in \mathrm{Nuc}(M)$ and so

$$f(a) + g(b,c) \le M3(a,b,c) \tag{6}$$

and the second follows from $(u, v) \in \mathrm{Nuc}(g)$ and so

$$u(b) + v(c) \le g(b,c) = h(bc) + [\![b,c]\!]. \tag{7}$$

Now, to prove $C = B \multimap_l A^{\perp}$, we need to prove that $(a, \alpha) \perp (b, \beta) \cdot (c, \gamma)$ for any $(a, \alpha) \in A$, $(b, \beta) \in B$, and $(c, \gamma) \in C$. So suppose $\alpha \le f(a)$, $\beta \le u(b)$, and $\gamma \le v(c)$. We want to show $(a, \alpha) \perp (bc, \beta + \gamma - [\![b,c]\!])$ which is equivalent to showing $\alpha + \beta + \gamma - [\![b,c]\!] \le M(a, bc)$. So, look at $\alpha + \beta + \gamma - [\![b,c]\!]$:

$$\begin{aligned}
\alpha + \beta + \gamma - [\![b,c]\!] &\le f(a) + u(b) + v(c) - [\![b,c]\!] \\
&\le f(a) + h(bc) + [\![b,c]\!] - [\![b,c]\!] && \text{using (6)} \\
&= f(a) + g(b,c) - [\![b,c]\!] \\
&\le M3(a,b,c) - [\![b,c]\!] && \text{using (7)} \\
&= M(a, bc)
\end{aligned}$$

and therefore $C = B \multimap_l A^{\perp}$. The proof that $B = C \multimap_r A^{\perp}$ is similar. □

6 The geometry of the Nucleus

Let $\mathcal{C}$ and $\mathcal{D}$ be finite categories and suppose $M : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a profunctor. Recall, the formulas for $M^* : [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}] \to [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}$ and $M_* : [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}} \to [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$ are given by

$$M^* f(d) = \min_{c \in \mathcal{C}} \{M(c,d) - f(c)\} \tag{8}$$
$$M_* g(c) = \min_{d \in \mathcal{D}} \{M(c,d) - g(d)\}. \tag{9}$$

Observe that M_* and M^* are translation equivariant in the sense that for any finite λ,

$$M^*(f+\lambda) = M^*(f) - \lambda \text{ and } M^*(g-\lambda) = M_*(g) + \lambda. \tag{10}$$

It will be convenient to work with pre and copresheaves projectively.

6.1 The associated projective metric space $\mathbb{P}\mathcal{C}$

Definition 18. *Let the subscript "fs" denote the full subcategories of pre and copresheaves that are "finite somewhere". That is,*

$$[\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]_{fs} := \{f : \textit{there exists } c \in \mathcal{C} \textit{ with } f(c) \neq \pm\infty\}$$

and similarly for copresheaves.

Notice all representables are finite somewhere. Also, finite translations acts freely upon the finite somewhere pre and copresheaves.

Definition 19. *For a category $\mathcal{C}$, define the space or projective presheaves $\mathbb{P}\mathcal{C}$ to be the quotients of the finite somewhere presheaves by finite translations:*

$$\mathbb{P}\mathcal{C} := [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]_{fs}/\{f \sim f+\lambda\} \tag{11}$$

and define the space of projective copresheaves similarly.

The internal hom between pre and copresheaves is not well defined after quotienting by translations. However, with a small adjustment, the quotient $\mathbb{P}\mathcal{C}$ is naturally a metric space. This is explained next.

The $\overline{\mathbb{R}}$ version of associativity of morphisms in Equation (2) is the opposite of an asymmetric triangle inequality. Negating and symmetrizing nearly defines an $\overline{\mathbb{R}}$ valued pseudometric on any $\overline{\mathbb{R}}$-category; in particular, on the presheaves $[\mathcal{C}^{op}, \overline{\mathbb{R}}]$ let

$$d(f,f') := -[f,f'] - [f',f] \tag{12}$$

which is symmetric and satisfies the triangle inequality. However, d is not a metric since $d(f, f+\lambda) = 0$ for any finite constant λ and, if $f(c) = \pm\infty$ for all $c \in \mathcal{C}$ then $[f,f] = \infty$ and $d(f,f) = -\infty$, breaking non-negativity. Both issues disappear when working with projective pre and copresheaves and the metric has an appealing formula which can be computed using any representative. We state and prove the presheaf result, the argument is the same for copresheaves.

Proposition 40. *The distance $d(f,f')$ defined in Equation (12) defines a metric on $\mathbb{P}\mathcal{C}$ and has the formula*

$$d(f,f') = \sup_{c \notin S}(f(c) - f'(c)) - \inf_{c \notin S}(f(c) - f'(c)) \tag{13}$$

where $S = \{c \in \mathcal{C} : f(c) = f'(c) = \infty \text{ or } f(c) = f'(c) = -\infty\}$ and f, f' are representatives of translation classes of pre or copresheaves.

Proof. From the formula in the statement of the proposition, d is well defined, $d(f,f') \geq 0$ and $d(f,f') = 0$ iff $f - f' = \lambda$. It follows from the formula $d(f,f') = -[f,f'] - [f',f]$, that d is symmetric and satisfies the triangle inequality. It only remains to prove that the given formula $\sup_{c \notin S}(f(c) - f'(c)) - \inf_{c \notin S}(f(c) - f'(c))$ is equal to $-[f,f'] - [f',f]$, which is clearly symmetric and satisfies the triangle inequality.

$$\begin{aligned} -[f,f'] - [f',f] &= -\inf_{c \in \mathcal{C}}\{f'(c) - f(c)\} - \inf_{c \in \mathcal{C}}\{f(c) - f'(c)\} \\ &= -\inf_{c \notin S}\{f'(c) - f(c)\} - \inf_{c \notin S}\{f(c) - f'(c)\} \\ &= \sup_{c \notin S}(f(c) - f'(c)) - \inf_{c \notin S}(f(c) - f'(c)). \end{aligned}$$

The first line is a definition. The second follows from the fact that on S, you have $f(c) = f'(c) = +\infty$ or $f(c) = f'(c) = -\infty$, so by the subtraction rules the difference $f(c) - f'(c) = +\infty$ and so doesn't affect the infemum. The last equality follows from the fact that $(x - y) = -(y - x)$ for x, y finite, or possibly infinite but not equal. $\square$

Now, we assume M satisfies a mild non-degeneracy assumption:

Definition 20. *We say M is* nondegenerate *iff the images of M^* and M_* are finite-somewhere. That is,*

$$M^*\left([\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]_{fs}\right) \subseteq [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}_{fs} \text{ and } M_*\left([\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}_{fs}\right) \subseteq [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]_{fs}.$$

If M is nondegenerate, then the functions M^* and M_* defined in (8) and (9) descend to well defined functions between projective presheaves on $\mathcal{C}$ and projective copresheaves on $\mathcal{D}$. We define the projective nucleus of M to be the quotient of the finite-somewhere nucleus by finite translations

$$\mathbb{P}\mathrm{Nuc}(M) := \{(f,g) \in \mathrm{Nuc}(M) : f \text{ and } g \text{ are finite somewhere}\}/\{(f,g) \sim (f+\lambda, g-\lambda)\} \tag{14}$$

with the metric defined by $d((f,g),(f',g')) := \max\{d(f,f'), d(g,g')\}$. Similarly, we have the projective fixed points of $\mathbb{P}\mathrm{Fix}\, M_* M^* \subset \mathbb{P}\mathcal{C}$ and $\mathbb{P}\mathrm{Fix}\, M^* M_* \subset \mathbb{P}\mathcal{D}$.

Theorem 41. *Suppose that $M : \mathcal{C} \times \mathcal{D} \to \overline{\mathbb{R}}$ is a nondegenerate profunctor, then the functions $M^* : \mathbb{P}\mathcal{C} \to \mathbb{P}\mathcal{D}$ and $M_* : \mathbb{P}\mathcal{D} \to \mathbb{P}\mathcal{C}$ defined in (8) and (9) define contractions between projective presheaves and projective copresheaves that descend to isometries*

$$M^* : \mathbb{P}\mathrm{Fix}(M_* M^*) \leftrightarrows \mathbb{P}\mathrm{Fix}(M^* M_*) : M_*.$$

Proof. Nondegeneracy together with the translation equivariance in Equation (10) means that M^* and M_* descend to well-defined functions $M^* : \mathbb{P}\mathcal{C} \to \mathbb{P}\mathcal{D}$ and $M_* : \mathbb{P}\mathcal{D} \to \mathbb{P}\mathcal{C}$. Starting with the statement M^* is a functor:

$$[\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f,f') \leq [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, M * f') = [\mathcal{D}, \overline{\mathbb{R}}](M^* f', M * f)$$

one finds that M^* is a contraction:

$$\begin{aligned}
[f, f'] \leq [M^* f', M^* f] &\Rightarrow -[M^* f', M^* f] \leq -[f, f'] \\
&\Rightarrow -[M^* f', M^* f] - [M^* f', M^* f] \leq -[f, f'] - [f', f] \\
&\Rightarrow d(M^* f, M^* f') \leq d(f, f').
\end{aligned}$$

Similarly, $d(M_* g, M_* g') \leq d(g, g')$. Therefore, M^* and M_* are continuous contractions with respect to the metrics on $\mathbb{P}\mathcal{C}$ and $\mathbb{P}\mathcal{D}$. Since they restrict to inverse functions on $\mathrm{Fix}(M_* M^*)$ and $\mathrm{Fix}(M^* M_*)$, they must restrict to isomotries. $\square$

This implies we have the following diagram of metric spaces in which every arrow is an isometry:

$$\begin{tikzcd}
& \mathbb{P}\mathrm{Nuc}(M) \arrow[dl, bend left, "\pi_1"] \arrow[dr, bend right, "\pi_2"'] & \\
\mathbb{P}\mathrm{Fix}\, M_* M^* \arrow[ur, bend left, "i_1"] \arrow[rr, bend left, "M^*"] & & \mathbb{P}\mathrm{Fix}\, M^* M_* \arrow[ul, bend right, "i_2"'] \arrow[ll, bend left, "M_*"]
\end{tikzcd}$$

where $\pi_1(f, g) = f$ and $\pi_2(f, g) = g$, $i_1(f) = (f, M^* f)$, and $i_2(g) = (M_* g, g)$.

6.2 External gauge action

Recall the formula Equation (13) for the metric d. For a fixed constant λ, one has $d(f + \lambda, f') = d(f, f')$. Also note that for a fixed $u : \mathcal{C} \to \mathbb{R}$, one has $d(f, f') = d(f - u, f' - u)$. Moreover, translating by u and then by u' is the same as translating by $u + u'$. One finds that the sets of functions $\mathbb{R}^{\mathcal{C}} : \{\mathcal{C} \to \mathbb{R}\}$ and $\mathbb{R}^{\mathcal{D}} = \{\mathcal{D} \to \mathbb{R}\}$ are abelian groups that act on $\mathbb{P}\mathcal{C}$ and $\mathbb{P}\mathcal{D}$ by isometries.

Definition 21. *For $u \in R^{\mathcal{C}}$, let $L_u : \mathbb{P}\mathcal{C} \to \mathbb{P}\mathcal{C}$ be given by $f \mapsto f - u$ and for $v \in \mathbb{R}^{\mathcal{D}}$ let $R_v : \mathbb{P}\mathcal{D} \to \mathbb{P}\mathcal{D}$ be given by $g \mapsto g - v$. Define the gauge transform of a profunctor $M : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \overline{\mathbb{R}}$ by*

$$M^{(u,v)}(c, d) := M(c, d) - u(c) - v(d).$$

On the projective pre and copresheaves, the maps $\left(M^{(u,v)}\right)^*$ and $\left(M^{(u,v)}\right)_*$ are simply the transport of M^* and M_* via the isometries L_u and R_v:

$$\begin{tikzcd}
\mathbb{P}\mathcal{C} \arrow[rr, bend left, "M^*"] \arrow[dd, "L_u"'] & & \mathbb{P}\mathcal{D} \arrow[ll, bend left, "M_*"] \arrow[dd, "R_v"] \\
& & \\
\mathbb{P}\mathcal{C} \arrow[rr, bend right, "\left(M^{(u,v)}\right)^*"'] & & \mathbb{P}\mathcal{D} \arrow[ll, bend right, "\left(M^{(u,v)}\right)_*"']
\end{tikzcd}$$

To see that $\left(M^{(u,v)} = R_v M^* L_u^{-1}\right)$,

$$\begin{aligned}
\left(M^{(u,v)}\right)^*(f)(d) &= \min_{c\in\mathcal{C}} M^{(u,v)}(c,d) - f(c) \\
&= \min_{c\in\mathcal{C}} M(c,d) - u(c) - v(d) - f(c) \\
&= \min_{c\in\mathcal{C}} M(c,d) - (f(c) + u(c)) - v(d) \\
&= M^*(f+u)(d) - v(d)
\end{aligned}$$

and similarly $\left(M^{(u,v)}\right)_* g = M_*(g+v) - u$.
Proposition 42. *The map*

$$\Phi_{u,v} : \mathbb{P}\mathrm{Nuc}(M) \to \mathbb{P}\mathrm{Nuc}(M^{(u,v)}), \qquad (f,g) \mapsto (f-u,\ g-v)$$

is an isometry with inverse $(f',g') \mapsto (f'+u, g'+v)$*. Consequently*

$$\mathbb{P}\mathrm{Fix}(M_*M^*), \quad \mathbb{P}\mathrm{Fix}((M^{(u,v)})_*(M^{(u,v)})^*), \quad \mathbb{P}\mathrm{Nuc}(M), \ \textit{and}\ \mathbb{P}\mathrm{Nuc}(M^{(u,v)}),$$

(and dually on $\mathbb{P}\mathcal{D}$*) are all canonically isometric.*

Proof. To see $g = M^*f \Leftrightarrow g - v = \left(M^{(u,v)}\right)^*(f-u)$ notice that

$$\begin{aligned}
& g(d) = M^*(f)(d) = \min_c\left(M(c,d) - f(c)\right) \\
\iff & g(d) - v(d) = \min_c\left(M(c,d) - f(c) - v(d)\right) = \min_c\left(M(c,d) - u(c) - v(d) - (f(c) - u(c))\right) - v(d) \\
& \qquad \iff (g-v)(d) = \left(M^{(u,v)}\right)^*(f-u)(d).
\end{aligned}$$

The other parts are similar. ∎

6.3 Witness cells

In this section suppose $\mathcal{C}$ and $\mathcal{D}$ be finite categories and fix a nondegenerate profunctor $M : \mathcal{C}^{\text{op}} \to \mathcal{D}$.

Definition 22. *An element $c \in \mathcal{C}$ is a* witness *for a presheaf $f \in [\mathcal{C}^{\text{op}}, \overline{\mathbb{R}}]$ at d if and only if*

$$M^* f(d) = M(c, d) - f(c).$$

That is, c is a witness for f at d if c realizes the minimum in Equation (8). An element $d \in \mathcal{D}$ is a witness *for a copresheaf $g \in [\mathcal{D}, \overline{\mathbb{R}}]^{\text{op}}$ at c if and only if*

$$M_* g(c) = M(c, d) - g(d).$$

That is, d is a witness for g at c if d realizes the minimum in Equation (9).

The concept of a witness is invariant under finite translations. So, we can talk about witnesses for projective classes in $\mathbb{P}\mathcal{C}$ and $\mathbb{P}\mathcal{D}$. For points in $\text{Nuc}(M)$, the witnesses come in pairs and cover all of $\mathcal{C}$ and $\mathcal{D}$. To see this fact, and more generally to study the witness structure of a given a presheaf $f \in [\mathcal{C}^{\text{op}}, \overline{\mathbb{R}}]$ (or dually a copresheaf $g \in [\mathcal{D}, \overline{\mathbb{R}}]^{\text{op}}$) it is convenient to gauge-shift the profunctor M.

Definition 23. *For any $f \in \mathbb{P}\mathcal{C}$, let $g = M^*(f)$ and define the* gap matrix *of f by*

$$\delta^f(c, d) := M(c, d) - f(c) - g(d). \tag{15}$$

Dually, for any $g \in \mathbb{P}\mathcal{D}$, let $f = M_*(f)$ and define the gap matrix *of g by*

$$\delta^g(c, d) := M(c, d) - f(c) - g(d). \tag{16}$$

The fact that for any finite constant λ, $M^*(f + \lambda) = M^*(f) - \lambda$ and $M_*(g + \lambda) = M_*(g) - \lambda$ proves that $\delta^{f+\lambda} = \delta^f$ and $\delta^{g+\lambda} = \delta^g$ so the definition of the gap matrices are well defined on $\mathbb{P}\mathcal{C}$ and $\mathbb{P}\mathcal{D}$. If $(f, g) \in \text{Nuc}(M)$ then $\delta^f = \delta^g$ and we may write $\delta^{(f,g)}$ or even just δ if the point (f, g) is clear from context. The gap matrix is also a gauge-invariant notion as the next proposition shows. For what follows, we will work consistently with presheaves, but the situation for copresheaves is similar.

Proposition 43. *. Let $f \in \mathbb{P}\mathcal{C}$, $u \in \mathbb{R}^{\mathcal{C}}$ and $v \in \mathbb{R}^{\mathcal{D}}$. Externally gauging by (u, v) maps M^* to $M^{(u,v)}$ and maps f to $f - u$. The gap matrix for f (computed relative to M) and the gap matrix for $(f - u)$ (computed relative to $M^{(u,v)}$) are the same:*

$$\delta^f = \delta^{f-u}.$$

Proof. Let $g = M^*(f)$. Then $\left(M^{(u,v)}\right)^* (f - u) = g - v$. Therefore,

$$\begin{aligned} \delta^{f-u} &= M^{(u,v)}(c, d) - (f - u)(c) - (g - v)(d) \\ &= M(c, d) - u(c) - v(d) - f(c) + u(c) - g(d) + v(d) \\ &= M(c, d) - f(c) - g(d) = \delta^f(c, d). \end{aligned}$$

□

Proposition 44. *Fix $f \in \mathbb{P}\mathcal{C}$ and let δ be the gap matrix $\delta(c,d) = M(c,d) - f(c) - g(d)$. Then*

(a) $\delta(c,d) \geq 0$ for all (c,d).

(b) $\delta(c,d) = 0$ if and only if c is a witness for f at d. In particular, every column d contains a zero.

(c) If moreover $f = M_* g$ (equivalently $(f,g) \in \mathrm{Nuc}(M)$), then $\delta(c,d) = 0$ if and only if d is a witness for g at c. So every row c contains a zero.

(d) If for all $c \in \mathcal{C}$, there exists $d \in \mathcal{D}$ with $\delta(c,d) = 0$ then $f = M_* M^*(f)$.

Proof. Part (a) follows from the definition of $g(d) = M^*(f)(d) = min_{c \in \mathcal{C}} M(c,d) - f(c)$. Therefore, for any pair (c,d), we have $\delta(c,d) = M(c,d) - f(c) - g(d) \geq 0$. For (b) assume c is a witness for f at d. Then $g(d) = M(c,d) - f(c)$ hence $\delta(c,d) = 0$, and conversely: if $\delta(c,d) = 0$, then $g(d) = M(c,d) - f(c)$ so c is a witness for f at d. For (c) suppose $f = M_*(g)$. Then $f(c) = \min_{d \in \mathcal{D}} M(c,d) - g(d)$ and we see d is a witness for g at c iff $\delta(c,d) = 0$. for (d) suppose that for each $c \in \mathcal{C}$ there is a $d \in \mathcal{D}$ with $\delta(c,d) = 0$. Then we have the following, the last equality coming from $\delta(c,d) = 0$ for this pair (c,d):

$$M_* M^*(f)(c) = M_*(g)(c) \leq M(c,d) - g(d) = f(c).$$

Since $M_* M^*$ is nondecreasing (Lemma 3), we always have $f(c) \leq M_* M^*(f)(c)$, and therefore $f(c) = M_* M^* f(c)$ and we see f is a fixed point of $M_* M^*$. ☐

Corollary 45. *If $(f,g) \in \mathbb{P}\mathrm{Nuc}(M)$ then c is a witness for f at d if and only if d is a witness for g at c.*

Proof. Follows from combining parts (b) and (c) of Proposition 44. ☐

Corollary 46. *A presheaf f is a fixed point for $M_* M^*$ if and only if for every $c \in \mathcal{C}$ there exists $d \in \mathcal{D}$ so that c is a witness for f at d. A copresheaf g is a fixed point for $M^* M_*$ if and only if for every $d \in \mathcal{D}$ there exists a $c \in \mathcal{C}$ so that d is a witness for g at c.*

Proof. Follows from combining parts (b) and (d) of Proposition 44, and the dual statements for copresheaves. ☐

Definition 24. *For any $(f,g) \in \mathbb{P}\mathrm{Nuc}(M)$, let $Z(f,g)$ denote the set of witness pairs for (f,g):*

$$Z(f,g) = \{(c,d) \in \mathcal{C} \times \mathcal{D} : (c,d) \text{ is a witness pair for } (f,g)\}. \tag{17}$$

Notice $(c,d) \in Z(f,g)$ if and only if $\delta(c,d) = 0$: the witness pairs are precisely the zeros of the gap matrix and they cover all the rows and columns. The witnesses pairs partition the nucleus of M into equivalence classes: $(f,g) \sim (f',g')$ if and only if they have the same witness pairs $Z(f,g) = Z(f',g')$. For every (f,g), there must be at least one witness pair (c,d) for every row $c \in \mathcal{C}$ and every column $d \in \mathcal{D}$. As shown in the next section, the equivalence classes defined by witness pairs are convex polytopes whose codimension is determined by the number of excess witness pairs. Together, these polytopes give a cell decomposition of $\mathbb{P}\mathrm{Nuc}(M)$ providing an attractive geometric picture of $\mathbb{P}\mathrm{Nuc}(M)$.

To create this picture, situate the Nucleus as a subspace of $\mathbb{P}\mathcal{C}$. One can get a closer look at $\mathrm{Fix}(M_* M^*) = \mathrm{Im}\, M_*$ by first looking at the image of representable copresheaves. For $d \in \mathcal{D}$, the representable presheaf $g_d := \mathcal{D}(d, -) \in \mathcal{D} \to \overline{\mathbb{R}}$ is given by

$$d' \mapsto \begin{cases} 0 & \text{if } d' = d \\ -\infty & \text{otherwise} \end{cases}$$

Evaluating $M_*(g_d)$ yields

$$(M_* g_d)(c) = \min_{d' \in \mathcal{D}} \{M(c, d) - g_d(d')\} = M(c, d).$$

That is, for a fixed d, the image of the representable coresheaf g_d under M_* is the presheaf $\mathcal{C}^{\mathrm{op}} \to \overline{\mathbb{R}}$ given by $c \mapsto M(c, d)$. That is $M_*(g_d) = M(-, d)$, just the d-th column of the matrix M. Since any copresheaf g can be expressed as a (weighted) colimit of representables: just scale each representable by the value $g(d)$ and take the colimit, and M_* is cocontinuous, one has a picture of $\mathrm{Nuc}(M) \cong \mathrm{Im}(M_*)$ as consisting of all coproducts of the columns $\{M(-, d)\}$ of M. In other words, for an arbitrary copresheaf $g \in [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}$, the expression

$$M_*(g)(c) = \min_{c \in \mathcal{C}} \left(M(c, d) - g(d)\right)$$

is precisely an expression of $M_* g$ as the weighted coproduct of the columns of M with the weights $g(d) \in \overline{\mathbb{R}}$. By plotting the columns $\{M(-, d)\}$ of M in $\mathbb{P}\mathcal{C}$, one gets a picture of the extent of $\mathbb{P}\mathrm{Nuc}(M)$ and every other point in the nucleus is the weighted coproduct of these “anchor” points. This is summarized next as a proposition.

Proposition 47. *For each $d \in \mathbb{P}\mathrm{Nuc}(M)$, let $A_d := M(-, d)$ be the d-th column of M and call it the d-th anchor point. Each anchor point $A_d \in \mathrm{Fix}(M_* M^*)$. Moreover, for any $f \in \mathrm{Fix}(M_* M^*)$, there exist $\lambda_d \in \overline{\mathbb{R}}$ so that*

$$f(c) = \min_{d \in \mathcal{D}} \left(A_d(c) - \lambda_d\right).$$

Proof. This proposition follows from the paragraph that precedes it. Each anchor point $A_d = M_*(g_d)$ where g_d is the copresheaf represented by d. Since $\mathrm{Im}(M_*) = \mathrm{Fix}(M_* M^*)$, each anchor is a fixed point of $M_* M^*$. Moreover, every fixed point $f = M_*(g)$ for some g. Choose weights $\lambda_d = g(d)$ to see $f(d) = \min_{c \in \mathcal{C}} \left(M(c, d) - g(d)\right) = \min_{c \in \mathcal{C}} \left(A_d(c) - \lambda_d\right).$ ☐

For what follows, it will be helpful to assume the columns of M have finite distance from the zero presheaf in $\mathbb{P}\mathcal{C}$.

Definition 25. *Let $M : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \overline{\mathbb{R}}$. Define the norm of M to be*

$$\|M\| := \sup d(M_*(g), 0)$$

where the sup is taken over all finite-somewhere copresheaves g on $\mathcal{D}$.

Note that $|M| < \infty$ is stronger than M being nondegenerate, in the sense that $|M| < \infty$ implies the image of finite-somewhere copresheaves under M_* are not only finite-somewhere but are finite-everywhere.

Lemma 48. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are finite and $|M| < \infty$. Let $s_d = d(A_d, 0)$ be the distance from each anchor point of M to the zero presheaf in $\mathcal{C}$. Then $|M| = \max_{d\in\mathcal{D}} s_d$.*

Proof. Since each anchor point is in the image of M_*, we have $|M| \geq \max_{d\in\mathcal{D}} s_d$. For the reverse inequality, let $f = M_*(g)$. We may assume the minimum of f is zero (if it is not, just translate f by its minimum) and that $g = M^*f$ (if it is not, replace g by M^*f). Let $c_* \in \mathcal{C}$ be where $f(c^*) = 0$ and let d^* be the witness for g at c^*.

$$M^*f(d^*) = M(c^*, d^*) - f(c^*) = M(c^*, d^*).$$

Since $M^*f(d^*) = \min_{c\in\mathcal{C}} (M(c, d^*) - f(c))$, we have for any $c \in \mathcal{C}$

$$M(c^*, d^*) = M^*(f)(d^*) = M(c_*, d_*) - f(c_*) \leq M(c, d_*) - f(c)$$

hence $0 \leq f(c) \leq M(c, d_*) - M(c_*, d_*) \leq s_{d_*} \leq \max_{d\in\mathcal{D}} s_d$. Therefore $d(f, 0) \leq \max_{d\in\mathcal{D}} s_d$ and we see $|M| \leq \max_{d\in\mathcal{D}} s_d$. ☐

Corollary 49. *If $|M| < \infty$ then $\mathbb{P}\mathrm{Nuc}(M)$ is compact.*

Proof. Lemma 48 shows that $\mathbb{P}\mathrm{Nuc}(M)$ lies entirely in the ball centered at 0 of radius $|M|$, hence bounded. Closed follows from the fact that the fixed points of an operator on a Hausdorff space is always a closed set. ☐

6.4 Polyhedral structures of $\mathbb{P}\mathrm{Nuc}(M)$

Throughout this subsection $\mathcal{C}$ and $\mathcal{D}$ are finite and $M : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathbb{R}$ has real entries, which implies M has finite norm. Note that $|M| < \infty$ implies $f(c)$ and $g(d)$ are finite for every $(f, g) \in \mathbb{P}\mathrm{Nuc}(M)$. We will generally view $\mathbb{P}\mathrm{Nuc}(M) \cong \mathbb{P}\mathrm{Fix}(M_*M^*)$ as a compact subset of $\mathbb{P}\mathcal{C}$ with an anchor point A_d being the point in $\mathbb{P}\mathrm{Nuc}(M)$ furthest from the zero presheaf on $\mathcal{C}$. However, it is easier to describe the cell decomposition first in $\mathbb{P}\mathrm{Nuc}(M) \subset \mathbb{P}\mathcal{C} \times \mathbb{P}\mathcal{D}$, and more specifically to fix a base row $c_0 \in \mathcal{C}$ and work with the projective gauge slice $\mathrm{Nuc}(M)_0 \cong \mathbb{P}\mathrm{Nuc}(M)$:

$$\mathrm{Nuc}(M)_0 := \{(f, g) \in \mathrm{Nuc}(M) \ : \ f(c_0) = 0\}.$$

Definition 26. *Let $Y \subseteq \mathcal{C} \times \mathcal{D}$. We say Y covers $\mathcal{C}$ iff for every $c \in \mathcal{C}$ there exists $d \in \mathcal{D}$ with $(c, d) \in Y$ and Y covers $\mathcal{D}$ iff for every $d \in \mathcal{D}$ there exists a $c \in \mathcal{C}$ with $(c, d) \in Y$. The* witness cell *of Y in the gauge $f(c_0) = 0$ is defined to be the set of pairs $(f, g) \in \mathrm{Nuc}(M)_0$ satisfying*

$$\begin{aligned} &f(c) + g(d) \leq M(c, d) \text{ for all } (c, d) \text{ and} \\ &f(c) + g(d) = M(c, d) \text{ for all } (c, d) \in Y. \end{aligned}$$

Let $C(Y)$ denote the witness cell of Y.

Note that $C(Y)$ is a closed polyhedron. It is possible that $C(Y)$ is empty, but if there exists any point $(f,g) \in C(Y)$, then $(f,g) \in \mathrm{Nuc}(M)_0$.

Lemma 50. *If $(f,g) \in \mathcal{C}(Y)$ then $(f,g) \in \mathrm{Nuc}(M)_0$ and furthermore $Y \subseteq Z(f,g)$. Conversely, for any $(f,g) \in \mathrm{Nuc}(M)_0$ one has $(f,g) \in C(Z(f,g))$.*

Proof. If $(f,g) \in C(Y)$, then for each d the inequalities give $g(d) \leq \min_c(M(c,d) - f(c))$, while the fact that $\pi_{\mathcal{C}}(Y) = \mathcal{C}$ gives a c with $(c,d) \in Y$ and hence equality there; thus $g = M^* f$. Symmetrically, $f = M_* g$. This implies Y consists of witness pairs so $Y \subset Z(f,g)$. The converse statement that $(f,g) \in \mathrm{Nuc}_0(M)$ implies that $\delta(c,d) = M(c,d) - f(c) - g(d) \geq 0$ with equality on $Z(f,g)$ which is precisely the statement that $(f,g) \in C(Y)$. ☐

Corollary 51. *If $|M| < \infty$ then $\mathbb{P}\mathrm{Nuc}(M)$ is the union of finitely many closed, convex polytopes.*

Proof. It follows from Lemma 50 that $\mathbb{P}\mathrm{Nuc}(M) = \bigcup_Y C(Y)$. Since $\mathcal{C} \times \mathcal{D}$ is finite, there are finitely many Y. Each witness cell $C(Y)$ is a closed polyhedron, and since $|M| < \infty$ implies $\mathbb{P}\mathrm{Nuc}(M)$ is compact, each witness cell is bounded, i.e., a polytope. ☐

Corollary 52. *For any Y and Y', $C(Y) \cap C(Y') = C(Y \cup Y')$.*

Proof. The conditions to be in both $C(Y)$ and $C(Y')$ are precisely given by the union of the conditions to be in $C(Y)$ and the conditions to be in $C(Y')$. ☐

In the case M is a profunctor on discrete, finite categories with finite norm, the picture of the nucleus as a union of convex polytopes is the same as the one obtained in tropical geometry (cite: Devlin and Sturmfels). In addition to what is explained above, the combinatorics determining which sets of pairs Y yield nonempty cells $C(Y)$ is analyzed. In particular, it is proved that the combinatorial types of cell complexes are in natural bijection with the regular polyhedral subdivisions of the product of two simplices

$$\Delta_{|\mathcal{C}|-1} \times \Delta_{|\mathcal{D}|-1}.$$

Here, we illustrate with an example.

6.5 An example

To illustrate, we consider discrete categories $\mathcal{C} = \{c_0, c_1, c_2\}$, $\mathcal{D} = \{d_1, d_2, d_3, d_4\}$, and

$$M = \begin{bmatrix} 0.7 & 1.5 & 1.7 & -1.3 \\ 1.2 & 2.6 & 0.1 & 2.2 \\ 2 & -1.6 & 2 & -2.9 \end{bmatrix}.$$

Taking the projective slice $c_0 = 0$, and plotting (c_1, c_2) we obtain a picture of $\mathrm{Nuc}_0(M)$ in the plane.

We have the following four anchor points:

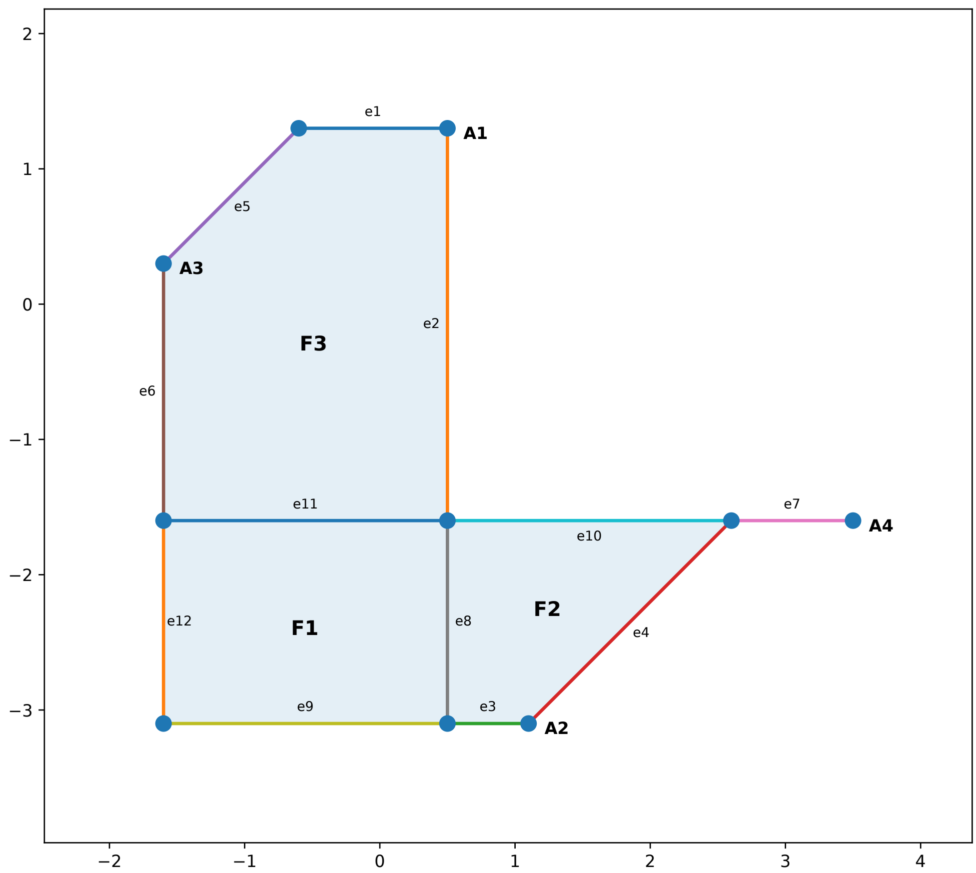


Fig. 4 Witness cell complex for the example matrix M.

$$A_1 = \begin{pmatrix} 0 \\ 0.5 \\ 1.3 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 \\ 1.1 \\ -3.1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 \\ -1.6 \\ 0.3 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 0 \\ 3.5 \\ -1.6 \end{pmatrix}$$

The point A_4 is the furthest from the origin. The entire nucleus sits inside the ball of radius $d(A_4, 0) = 5.1$ and consists of all weighted coproducts of the anchor points. There are precisely three two cells corresponding to witnesses

$$Z(F1) = \{(c_0, d_1), (c_2, d_2), (c_1, d_3), (c_2, d_4)\}$$
$$Z(F2) = \{(c_1, d_1), (c_2, d_2), (c_1, d_3), (c_2, d_4)\}$$
$$Z(F3) = \{(c_0, d_1), (c_2, d_2), (c_1, d_3), (c_0, d_4)\}$$

The two dimensional cells are the highest dimensional cells. In each two-cell, every $d \in \mathcal{D}$ has precisely one c that witnesses that d. The witness pairs must cover all of $\mathcal{C}$ so this is required, any extra witness pairs would involve an extra condition and would decrease the dimension of the cell by one. For example, the one cell $e8$ that is the shared boundary of the two cells F_2 and F_3, which is precisely the intersection $e8 = F1 \cap F_2$ of the closed cells, has witnesses

$$Z(F2 \cap F3) = Z(F2) \cup Z(F3) = \{(c_0, d_1), (c_1, d_1), (c_2, d_2), (c_1, d_3), (c_0, d_4)\}$$

One can think of moving from the interior of $F3$ to the interior of $F2$ along a path in the nucleus, and crossing e_8. On the level of witnesses, this invovles changing the witness for d_1 from c_0 to c_1. The vertex shared by all three $F1$, $F2$, and $F3$ is a zero-dimensional cell has witnesses

$$\begin{aligned} Z(F1 \cap F2 \cap F3) = Z(F1) \cup Z(F2) \cup Z(F3) \\ = \{(c_0, d_1), (c_1, d_1), (c_2, d_2), (c_1, d_3), (c_0, d_4), (c_2 d_4)\} \end{aligned}$$

6.6 Nonzero entries of the gap matrix and order chambers

Now, fix a point $(f, g) \in \mathbb{P}\mathrm{Nuc}(M)$ and let $\delta := \delta^{(f,g)}$ be the gap matrix for this point. The zero entries of δ determine the witness pairs $Z(f, g)$. That is (f, g) lies in the cell $C(Z(f, g))$. The nonzero entries of δ describe different cells located at a positive distance from (f, g). More precisely,

Theorem 53. *Let $\lambda = \delta(c_i, d_j) > 0$. Then there exists a point $(f', g') \in C(Z(f', g')) \subseteq \mathbb{P}\mathrm{Nuc}(M)$ with $d((f, g), (f', g')) = \lambda$. Furthermore $(c_i, d_j) \in Z(f', g')$ (and consequently $Z(f, g) \neq Z(f', g')$, so (f', g') lies in a different cell from (f, g).*

We give the proof and also illustrate how the theorem and proof work in an example. Here's the proof.

Proof. Fix a point $(f, g) \in \mathbb{P}\mathrm{Nuc}(M)$ and choose $\lambda = \delta(c_i, d_j) > 0$, a nonzero elements of $\delta := \delta^{(f,g)}$. Define f'' by adding λ to f in the c_i direction:

$$f''(c) = \begin{cases} f(c) & \text{if } c \neq c_i \\ f(c) + \lambda & \text{if } c = c_i. \end{cases}$$

Compute $M^*(f'')(d)$

$$\begin{aligned} M^*(f'')(d) &= \min_{c \in \mathcal{C}} M(c, d) - f''(c) \\ &= \min_{c \neq c_1} M(c, d) - f(c), M(c, d) - f(c_i) - \lambda \\ &= \min(g(d), g(d) - \delta(c_i, d) - \lambda) \\ &= g(d) - \max_{d \in \mathcal{D}}(\lambda - \delta(c_i, d), 0) \end{aligned}$$

So $g' := M^*(f'')$ has the formula

$$g'(d) = \begin{cases} g(d) & \text{if } \delta(c_i, d) \geq \lambda \\ g(d) - (\lambda - \delta(c_i, d)) & \text{if } \delta(c_i, d) < \lambda \end{cases}$$

Notice that $0 \leq g(d) - g'(d) \leq \lambda$ for all d. At d_j, we have $g'(d_j) = g(d_j)$. Since $(f, g) \in \mathrm{Nuc}(M)$, we know there exists a d_k so that $\delta(c_i, d_k) = 0$ and at this value $g(d_k) - g'(d_k) = \lambda$. Therefore

$$d(g, g') = \sup_{d \in \mathcal{D}} (g(d) - g'(d)) - \inf_{d \in \mathcal{D}} (g(d) - g'(d)) = \lambda.$$

Define $f' = M_*(g')$. Since $g, g' \in \mathrm{Im}(M^*)$ with $d(g, g') = \lambda$ and M_* is an isometry from $\mathrm{Im}(M^*) \to \mathrm{Fix}(M_* M^*)$, we have $(f', g') \in \mathrm{Nuc}(M)$ and $d((f, g), (f', g')) = \lambda$.

Finally, evaluating the new gap matrix $\delta' := \delta^{(f', g')}$ at (c_i, d_j):

$$\begin{aligned} \delta'(c_i, d_j) &= M(c_i, d_j) - f'(c_i) - g'(d_j) \\ &= M(c_i, d_j) - (f(c_i) + \lambda) - g(d_j) \\ &= (M(c_i, d_j) - f(c_i) - g(d_j)) - \lambda \\ &= \lambda - \lambda \\ &= 0 \end{aligned}$$

Since $\delta(c_i, d_j) \neq 0$ and $\delta'(c_i, d_j) = 0$ we see (f, g) and (f', g') are in different witness cells. $\square$

It may be of interest to have a formula for the presheaf $f' = M_*(g') = M_* M^*(f'')$ from the proof above. Computing:

$$\begin{aligned} f'(c) &= M_*(g') \\ &= \min_{d \in \mathcal{D}} (M(c, d) - g'(d)) \\ &= \min_{d \in \mathcal{D}} (M(c, d) - \left(g(d) - \max_{d \in \mathcal{D}} (\lambda - \delta(c_i, d), 0) \right) \\ &= f(c) + \min_{d \in \mathcal{D}} \left(\delta(c, d) + \max_{d \in \mathcal{D}} (\lambda - \delta(c_i, d), 0) \right) \end{aligned}$$

To illustrate, let us again look at the nucleus of the profunctor M from earlier. Choose $f = (0, 0, 0)$ and $g = M^* = (0.7, -1.6, 0.1, -2.9)$. In Figure 5 below, f is plotted in $\mathrm{Nuc}(M)_0$ as a green dot. The gap matrix δ is given by

$$\delta = \begin{bmatrix} 0 & 3.1 & 1.6 & 1.6 \\ 0.5 & 4.2 & 0 & 5.1 \\ 1.3 & 0 & 1.9 & 0 \end{bmatrix}.$$

The entries of δ are given by

$$0 < 0.5 < 1.3 < 1.6 < 1.9 < 3.1 < 4.2 < 5.1.$$

The ball centered at f of radius $\epsilon_1 = 0.5$ is pictured in Figure 5 in orange and the ball of radius $\epsilon_2 = 1.6$ is pictured in pink (balls in the max-spread metric on $\mathbb{P}\mathrm{Nuc}\, C \subset \mathbb{R}^2$ are hexagons). Notice that the 1 cell e_2 is located at exactly $\epsilon_1 = 0.5$ from f. To find a point in the cell e_2, we follow the steps in the proof of Theorem 53. Looking up $\epsilon_1 = 0.5$ in the matrix δ, it is in the c_1 row. So, we add 0.5 to the c_1 entry of f to get $(0, 0.5, 0)$ Keep in mind $\mathcal{C} = \{c_0, c_1, c_2\}$ and we're taking the projective slice defined by $c_0 = 0$. This point $(0, 0.5, 0)$ is in $\mathrm{Nuc}(M)$ and is plotted in purple. One can increase the radius of the ball centered at f to find new cells at every radius. The 1-cell e_1 is located $\epsilon_2 = 1.3$ units from f, the 1-cells e_6 and e_{11} are located 1.6 units from f, and so on. The fact that e_6 and e_{11} are both exactly distance 1.6 from f is related to the fact that 1.6 is a repeated entry of δ. This situation is elaborated upon in the next section 6.7 on order chambers. For now, let's take a closer look at the 1-cell e_5, which is located at exactly distance $\epsilon_4 = 1.9$ from f. To find a point in this cell, we find 1.9 is in the c_2 row of δ, so adding 1.9 to the c_2 entry of f becomes $(0, 0, 1.9)$. This point is plotted in red and is not in $\mathrm{Nuc}(M)$. Projecting to the nucleus yields $f' := M_* M^*(0, 0, 1.9) = M_*(0.1, -3.5, 0.1, -4.8) = (0.6, 0.1.9) \sim (0, -0.6, 1.3)$ which is plotted in blue. Now we've found a point f' in $\mathrm{Nuc}(M)$ exactly distance 1.9 from f. Computing the delta matrix for f' yields

$$\delta^{f'} = \begin{bmatrix} 0 & 4.4 & 1 & 2.9 \\ 1.1 & 6.1 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

notice that the c_2, d_3 entry is 0 which reflects the fact that $1.9 = \delta^f(c_2, d_3)$. The pair (c_2, d_3) was not a witness pair for f and it is a witness pair for f', which lies in a different cell from f, a fact also visible in the picture. It so happens that the c_2, d_1 entry of $\delta^{f'}$ is also zero, reflecting the fact that f' lies on the cell e_1 as well.

6.7 Order chambers

We now describe a refinement of the witness cells into order chambers. For any point $(f, g) \in \mathrm{Nuc}(M)$ in the nucleus of M the gap matrix of (f, g) defines a preorder on $\mathcal{C} \times \mathcal{D}$ by

$$(c, d) \preceq_{f,g} (c', d') \iff \delta^{(f,g)}(c, d) \leq \delta^{(f,g)}(c', d').$$

Going the other way around, let $\preceq$ be a pre-order on $\mathcal{C} \times \mathcal{D}$ and assume the minimum set Y of the preorder covers each row and column. Then $\preceq$ defines a closed subset of the witness cell $C(Y)$ defined to be the set of all $(f, g) \in \mathrm{Nuc}(M)$ satisfying:

$$\delta^{(f,g)}(c, d) \leq \delta^{(f,g)}(c', d') \text{ if and only if } (c, d) \preceq (c', d') \tag{18}$$

So, order cells refines witness cells; all points of the nucleus within an order cell have gap matrices that have zeros in the same places (so they're in the same witness

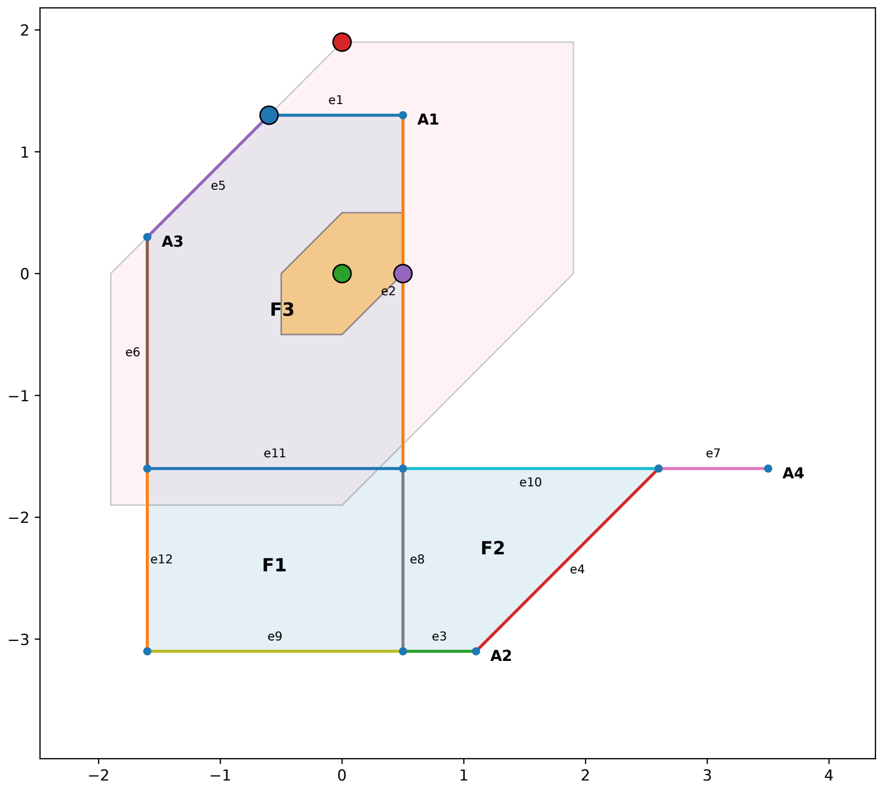


Fig. 5 Ball of radius $\epsilon_1 = 0.5$ and $\epsilon_2 = 1.9$

cell), and in addition, there are no rank-flips among the nonzero entries of their gap matrices.

Like witness cells, order cells are defined using gap matrices, and therefore they are gauge invariant notions and descend to $\mathbb{P}\mathrm{Nuc}(M)$.

6.8 Bundle of lattice-towers over the nucleus

In this next section, we discuss a kind of base change from $\overline{\mathbb{R}}$ to $\{0, 1\}$. A Boolean valued profunctor on discrete categories $\mathcal{C}$ and $\mathcal{D}$ is the same as a matrix $\mathcal{C} \times \mathcal{D} \to \{0, 1\}$ which is the same as a relation on $\mathcal{C} \times \mathcal{D}$. The nucleus of a Boolean valued profunctors is precisely the same as the lattice of formal concepts for the relation it defines.

A point $(f, g) \in \mathrm{Nuc}(M)$ defines a relation on $C \times D$ by setting $c \sim d$ iff c is a witness for f at d iff d is a witness for g at c iff $(c, d) \in Z(f, g)$. One can relax this relation "up to ε."

Definition 27. *Fix $(f, g) \in \mathrm{Nuc}(M)$. For $\varepsilon \geq 0$ define the Boolean profunctor by setting $R^{(f,g)}(c, d) = 1$ if $\delta^{(f,g)}(c, d) \leq \varepsilon$ and 0 otherwise. That is, $R^{(f,g)}_\varepsilon$ is the relation*

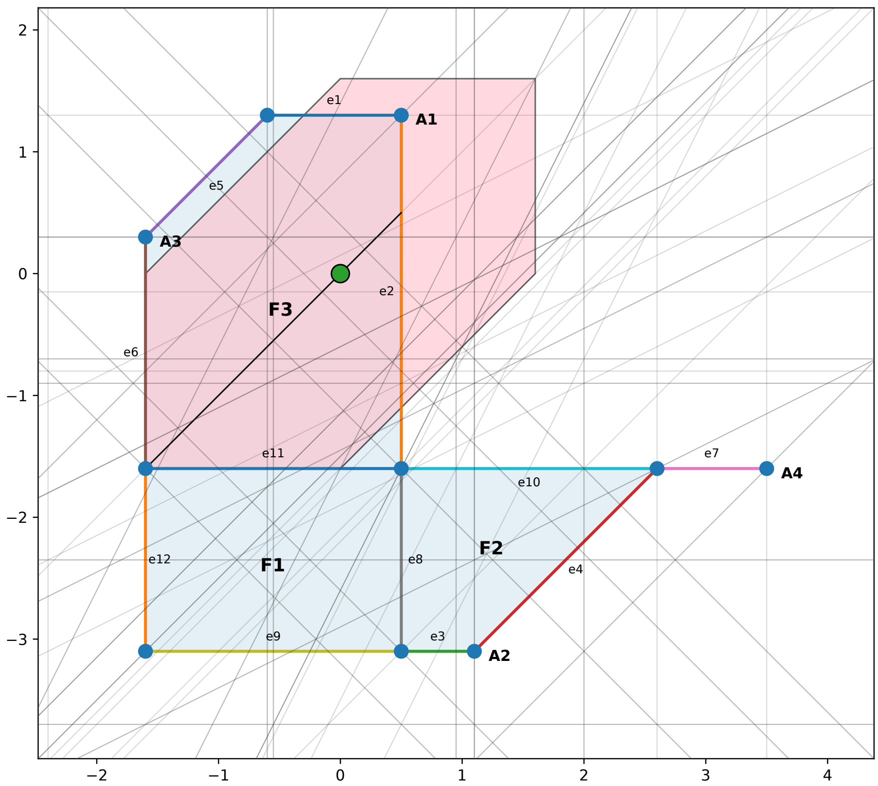


Fig. 6 The order chambers for the point (f, g) where $f = (0, 0, 0)$ and $g = M^* = (0.7, -1.6, 0.1, -2.9)$. Notice that this point lies directly on an order chamber boundary (darkened in the picture). That is reflected in the gap matrix because there are two entries that are exactly 1.6. The ball of radius 1.6 shows that the one cells e_6 and e_{11} are precisely 1.6 units from the point (f, g). If f were chosen to be slightly to the right of the boundary, e_{11} would be closer and to the left, e_6 would be closer. Crossing other boundaries would flip other entries of the gap matrix.

defined by

$$R_\varepsilon^{(f,g)}(c, d) := \begin{cases} 1 & \text{if } M(c, d) - f(c) - g(d) \le \varepsilon \\ 0 & \text{if } M(c, d) - f(c) - g(d) > \varepsilon. \end{cases}$$

Let

$$L_\varepsilon(f, g)$$

denote its Boolean nucleus, its lattice of formal concepts.

So, instead of $c \sim d$ if and only if $0 = M(c, d) - f(c) - g(d)$ as we would have for witness pairs, we have

$$c \sim_\varepsilon d \text{ if and only if } 0 \le M(c, d) - f(c) - g(d) \le \varepsilon.$$

If $\varepsilon \le \varepsilon'$ then $R_\varepsilon^{(f,g)} \subseteq R_{\varepsilon'}^{(f,g)}$. This induces two maps, a left-handed map

$$\Phi : L_\varepsilon(f, g) \to L_{\varepsilon'}(f, g)$$

defined by

$$(A,B) \mapsto (L_{\varepsilon'*}(L^*_{\varepsilon'}(A)), L^*_{\varepsilon'}(A))$$

that preserves joins and a right-handed lattice map

$$\Psi : L_\varepsilon(f,g) \to L_{\varepsilon'}(f,g)$$

defined by

$$(A,B) \mapsto (L_{\varepsilon'*}(B)), L^*_{\varepsilon'}(L_{\varepsilon'*}(B))$$

that preserves meets.

Now, let us consider the global structure of $\mathbb{P}\mathrm{Nuc}(M)$. For each point (f,g), there is a lattice of concepts $\{L_\varepsilon(f,g)\}_\varepsilon$. Inside each order chamber, the sorted order of the entries of $\delta^{(f,g)}$ is fixed, hence the filtration $R_0(f,g) \subseteq \cdots R_\varepsilon(f,g) \subseteq R_\infty(f,g)$ is fixed and the lattices $L_\varepsilon(f,g)$ are combinatorially constant (only the numeric trigger values change). When (f,g) lies on a wall, several incidences acquire the same threshold ε and enter $R_\varepsilon(f,g)$ simultaneously. Crossing the wall merely permutes these few elementary insertions; before and after the tied block the lattices agree, and the canonical map across the block is independent of the chosen linear extension of the ties by functoriality of the concept-lattice construction.

The picture of $\mathbb{P}\mathrm{Nuc}(M)$ as consisting of the union of witness cells, each a convex polytope, further refined by order chambers. Over each point (f,g) is a tower of lattices $\{L_\varepsilon(f,g)\}\varepsilon \geq 0\}$, locally constant for each ε between the trigger values which are the nonzero values of the gap matrix $\delta^{(f,g)}$. For each trigger value $\varepsilon \in \delta^{(f,g)}$, there is a witness cells of $\mathbb{P}\mathrm{Nuc}(M)$ that lie at exactly ε from (f,g), and the entire nucleus can be reconstructed from a single tower of lattices, Morse theory-style, over a single point (f,g). We illustrate with an example:

A Example lattice tower

As before, let $C = \{c_0, c_1, c_2\}$, $D = \{d_1, d_2, d_3, d_4\}$, and

$$M = \begin{bmatrix} 0.7 & 1.5 & 1.7 & -1.3 \\ 1.2 & 2.6 & 0.1 & 2.2 \\ 2.0 & -1.6 & 2.0 & -2.9 \end{bmatrix}.$$

At the point $f = (0, 0, 0)$ we have $g = M^* f = (0.7, -1.6, 0.1, -2.9)$ and the gap matrix

$$\delta \;=\; M - f - g \;=\; \begin{bmatrix} 0.0 & 3.1 & 1.6 & 1.6 \\ 0.5 & 4.2 & 0.0 & 5.1 \\ 1.3 & 0.0 & 1.9 & 0.0 \end{bmatrix}.$$

The distinct ε where $R_\varepsilon^{(f,g)}$ changes are

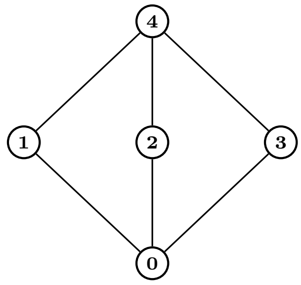
$$\{\,0.000,\ 0.500,\ 1.300,\ 1.600,\ 1.900,\ 3.100,\ 4.200,\ 5.100\,\}.$$

For each ε, we list all formal concepts (A, B) of the Boolean context $R_\varepsilon^{(f,g)} = \{(c, d) \mid \delta(c, d) \le \varepsilon\}$.

Lattices of Formal Concepts $L_\varepsilon(f, g)$

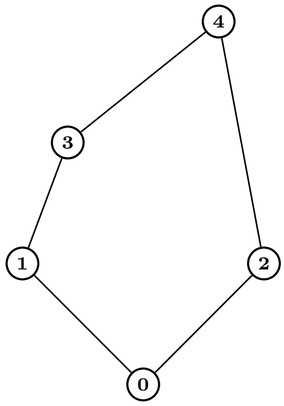
$\varepsilon = 0.000$

#	A	B
0	$\varnothing$	$\{d_1, d_2, d_3, d_4\}$
1	$\{c_0\}$	$\{d_1\}$
2	$\{c_1\}$	$\{d_3\}$
3	$\{c_2\}$	$\{d_2, d_4\}$
4	$\{c_0, c_1, c_2\}$	$\varnothing$



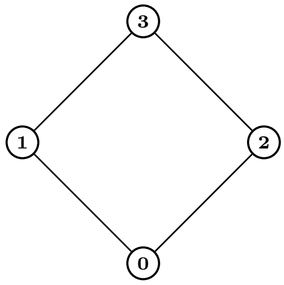
$\varepsilon = 0.500$

#	A	B
0	$\varnothing$	$\{d_1, d_2, d_3, d_4\}$
1	$\{c_1\}$	$\{d_1, d_3\}$
2	$\{c_2\}$	$\{d_2, d_4\}$
3	$\{c_0, c_1\}$	$\{d_1\}$
4	$\{c_0, c_1, c_2\}$	$\varnothing$



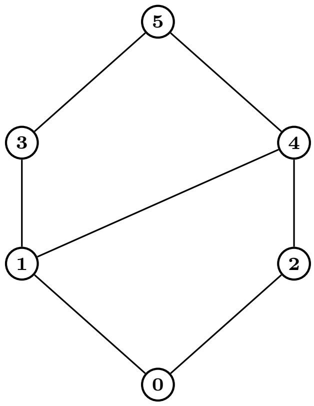
$\varepsilon = 1.300$

#	A	B
0	$\varnothing$	$\{d_1, d_2, d_3, d_4\}$
1	$\{c_1\}$	$\{d_1, d_3\}$
2	$\{c_2\}$	$\{d_1, d_2, d_4\}$
3	$\{c_0, c_1, c_2\}$	$\{d_1\}$



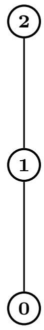
$\varepsilon = 1.600$

#	A	B
0	$\varnothing$	$\{d_1, d_2, d_3, d_4\}$
1	$\{c_0\}$	$\{d_1, d_3, d_4\}$
2	$\{c_2\}$	$\{d_1, d_2, d_4\}$
3	$\{c_0, c_1\}$	$\{d_1, d_3\}$
4	$\{c_0, c_2\}$	$\{d_1, d_4\}$
5	$\{c_0, c_1, c_2\}$	$\{d_1\}$



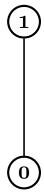
$\varepsilon = 1.900$

#	A	B
0	$\{c_2\}$	$\{d_1, d_2, d_3, d_4\}$
1	$\{c_0, c_2\}$	$\{d_1, d_3, d_4\}$
2	$\{c_0, c_1, c_2\}$	$\{d_1, d_3\}$



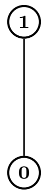
$\varepsilon = 3.100$

#	A	B
0	$\{c_0, c_2\}$	$\{d_1, d_2, d_3, d_4\}$
1	$\{c_0, c_1, c_2\}$	$\{d_1, d_3\}$



$\varepsilon = 4.200$

#	A	B
0	$\{c_0, c_2\}$	$\{d_1, d_2, d_3, d_4\}$
1	$\{c_0, c_1, c_2\}$	$\{d_1, d_2, d_3\}$



$\varepsilon = 5.100$

#	A	B
0	$\{c_0, c_1, c_2\}$	$\{d_1, d_2, d_3, d_4\}$

As mentioned in the text, from the tower of lattices it is possible to reconstruct the nucleus. Here, we use the projective slice $(c_0, c_1, c_2) \mapsto (c_1 - c_0, c_2 - c_0)$ so c_1 indicates the positive horizontal (the traditional x) direction, c_2 indicates the positive vertical (the traditional y) direction, and c_0 represents the down-left direction along the diagonal line of slope 1.

Looking at L_0. The concept $\{c_1\}, \{d_3\}$ says that the point f is to the right (positive c_1 direction) relative to the A_3 anchor. The concept $\{c_2\}, \{d_2, d_4\}$ means that f is above (positive c_2 direction) the anchors A_2 and A_4. The concept $\{c_0\}, \{d_1\}$ says geometrically that f is below and left (positive c_0 direction) of the A_1 anchor.

This orients the anchors A_1, A_2, A_3, and A_4 around the nucleus point f. This is only partial information since only the directions, not the distances, can be read from L_0.

Looking at L_1, we see the addition of (c_1, d_1) as a witness pair at distance ε_1. This says that there is a one-cell involving the anchor A_1 at distance ε_1 in the c_1 direction (right) and that this is the closest edge to f. By continuing in this fashion, looking at the new pairs that occur in the concepts in lattices L_ε as ε increases, resolves either the precises locations of the anchor points and witness cells (if the ε values are known) or relative locations (if only the lattices are known but the trigger values ε are not).