Underactuated Robotics Report

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1 Introduction

For this project, we were tasked with the selection of an underactuated robotic system, the derivation of its model and its simulation in a programming platform. We needed to explore and implement 2 different nonlinear controllers that can control the robot successfully.

We decided to go with the inverted pendulum-cart robot, which can control the position of the pendulum only by controlling the position of the cart. The aim of the following controllers is to successfully stabilize the pendulum in the upright position. We will firstly explore the uncontrolled case, followed by a linear controller to set the base for our experiment. After that, we will implement and test two non-linear controllers for two different use-cases.

2 System Model

We modeled the robot as a 2D inverted pendulum on top of a 2D cart that sits on top of an immovable surface. The pendulum can only rotate around its base, the cart can only move on the x axis, and the controlling force 'u' is applied onto the cart directly.

The following is the diagram of the robot and a description of all the variables and parameters:

x: horizontal displacement of the cart.
θ: angle of the pendulum from the vertical.
M: mass of the cart.
m: mass of the pendulum bob.
l: length to pendulum's center of mass.
u: horizontal force applied to the cart (our control input).
g: gravitational acceleration.
With generalized coordinates and state vector:

$$q = (x, \theta) \quad , \qquad X = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

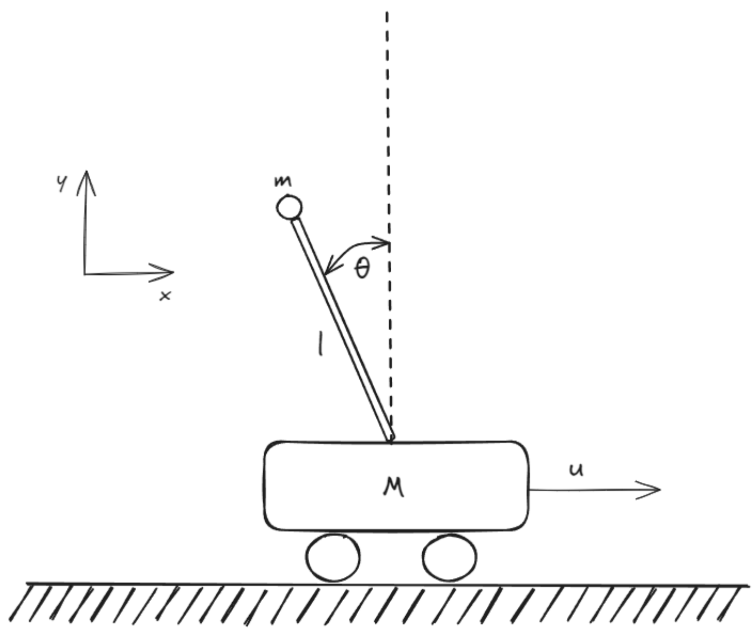

Figure 1: Diagram of the inverted pendulum-cart robot.

2.1 Euler-Lagrange Modeling

In this section, we perform the Euler-Lagrange modeling for the inverted pendulum-cart robot.

1. Cart Kinetic Energy:
$$T_{cart} = \frac{1}{2}M\dot{x}^2$$
2. Pendulum Kinetic Energy:
The pendulum bob's velocity has two components:
$$v_x = \dot{x} + l\dot{\theta}\cos\theta \qquad , \qquad v_y = l\dot{\theta}\sin\theta$$
so:
$$T_{pend} = \frac{1}{2}m({v_x}^2 + {v_y}^2) = \frac{1}{2}m(\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2)$$
3. Total Kinetic Energy:
$$T = T_{cart} + T_{pend} = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2$$
4. Potential Energy:

We consider zero potential at the pivot level (so that $\theta = 0$ hangs directly down), the center of mass of the pendulum has vertical displacement $y = -l\cos\theta$. So:

$$V = mg(l - l\cos\theta) = mgl(1 - \cos\theta)$$

5. The Lagrangian: we assume $\theta = 0$ hangs directly down, so the center of mass of the pendulum has vertical displacement $y = -l\cos\theta$. So:

$$L = T - V = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

6. Euler–Lagrange for generalized coordinate:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = Q_i,$$

with $Q_x = u$ being the only external generalized force.
For $q_1 = x$ and $q_2 = \theta$ we get two equations:
Cart equation:

$$\frac{d}{dt}[(M+m)\dot{x} + ml\dot{\theta}\cos\theta] - ml\dot{x}\dot{\theta}(-\sin\theta) = u$$

because no direct x dependence $\frac{\partial L}{\partial x} = 0$, expanding:

$$u = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

Pendulum equation:

$$\frac{d}{dt}[ml\dot{x}\cos\theta + ml^2\dot{\theta}] - [-ml\dot{x}\dot{\theta}\sin\theta + mgl\sin\theta] = 0$$

expanding:

$$ml\ddot{x}\cos\theta + ml^2\ddot{\theta} - mgl\sin\theta = 0$$

Collecting the two equations:

$$\begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} -ml\theta^2\sin\theta \\ -mgl\sin\theta \end{bmatrix}$$

which corresponds to the Inertia tensor:

$$B(\theta) = \begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix}$$

and the Coriolis and centrifugal terms:

$$C(\theta, \dot{\theta}) = \begin{bmatrix} -ml\theta^2\sin\theta \\ -mgl\sin\theta \end{bmatrix}$$

And finally, if we solve for $\ddot{x}$ and $\ddot{\theta}$ explicitly, we get:

$$\ddot{x} = \frac{u + ml\sin\theta\dot{\theta}^2 - mg\sin\theta\cos\theta}{M + m\sin^2\theta}$$

,

$$\ddot{\theta} = \frac{-u\cos\theta - ml\dot{\theta}^2\cos\theta\sin\theta + (M+m)g\sin\theta}{l(M + m\sin^2\theta)}$$

2.2 Analysis of Model

In this part we are going to linearize the inverted pendulum cart model and we will determine if it is controllable.

2.2.1 Linearization

We select as the equilibrium point the position where the cart is at the origin with no velocity, the pendulum is standing strait up also without any motion, and there is no input force to the system. So:

$$x_{eq} = 0 \quad , \quad \dot{x}_{eq} = 0 \quad , \quad \theta_{eq} = \pi \quad , \quad \dot{\theta}_{eq} = 0 \quad , \quad u_{eq} = 0$$

We define a small angular deviation:

$$\phi \;\equiv\; \theta - \pi,$$

so that $\phi = 0$ corresponds to the pendulum being exactly upright (inverted).

From 2.1, the exact nonlinear equations are:

$$\ddot{x} = \frac{u + m\,l\,\sin\theta\,\dot{\theta}^2 - m\,g\,\sin\theta\,\cos\theta}{M + m\,\sin^2\theta},$$

$$\ddot{\theta} = \frac{-\,u\,\cos\theta - m\,l\,\dot{\theta}^2\,\cos\theta\,\sin\theta + (M+m)\,g\,\sin\theta}{l\,(M + m\,\sin^2\theta)}$$

We now substitute $\theta = \pi + \phi$, and assume an approximations valid for small ϕ:

$$\sin(\pi + \phi) = -\,\sin\phi \;\approx\; -\phi \quad , \quad \cos(\pi + \phi) = -\,\cos\phi \;\approx\; -1.$$

Additionally, we assume $\dot{\phi}$ is small, so we may drop all $\dot{\phi}^2$ terms.

(a) Linearizing $\ddot{x}$

Starting from the nonlinear expression:

$$\ddot{x} = \frac{u + m\,l\,\sin\theta\,\dot{\theta}^2 - m\,g\,\sin\theta\,\cos\theta}{M + m\,\sin^2\theta},$$

we replace $\theta = \pi + \phi$, and apply the small-angle approximations:

$$\sin\theta \approx -\,\phi \quad , \quad \cos\theta \approx -\,1 \quad , \quad \dot{\theta}^2 \approx \dot{\phi}^2 \approx 0 \quad , \quad \sin^2\theta \approx \phi^2 \approx 0$$

Substituting into the equation:

$$\ddot{x} \approx \frac{u - m\,g\,\phi}{M} = -\frac{m\,g}{M}\,\phi + \frac{1}{M}\,u$$

(b) Linearizing $\ddot{\theta}$

From:

$$\ddot{\theta} = \frac{-u\cos\theta - ml\dot{\theta}^2\cos\theta\sin\theta + (M+m)g\sin\theta}{l(M+m\sin^2\theta)}$$

we substitute $\theta = \pi + \phi$ and neglect all $\dot{\theta}^2$ and $\sin^2\theta$ terms. Using the approximations:

$$\cos\theta \approx -1 \quad , \quad \sin\theta \approx -\phi \quad , \quad l(M+m\sin^2\theta) \approx lM$$

So:

$$\ddot{\theta} \approx \frac{-u(-1) + (M+m)g(-\phi)}{lM} = \frac{u - (M+m)g\phi}{lM}$$

$$\Rightarrow \ddot{\theta} \approx -\frac{(M+m)\,g}{lM}\,\phi + \frac{1}{lM}\,u$$

2.2.2 State Space Matrices A and B

We are going to need to calculate the controllability matrix in the next step, so starting with the state space matrices by collecting all four first-order derivatives:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\dfrac{m\,g}{M}\,x_3 + \dfrac{1}{M}\,u, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = -\dfrac{(M+m)\,g}{l\,M}\,x_3 + \dfrac{1}{l\,M}\,u \end{cases}$$

which can be expressed in matrix form as:

$$\dot{\mathbf{x}} = A\,\mathbf{x} + B\,u$$

with state vector:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix}$$

and:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{m\,g}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{(M+m)\,g}{l\,M} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ \dfrac{1}{l\,M} \end{bmatrix}$$

2.2.3 Controllability check

Our system is controllable if and only if the 4×4 controllability matrix:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

has full rank.

We now construct each block:

AB:

$$AB = A \cdot B = \begin{bmatrix} \frac{1}{M} \\ 0 \\ \frac{1}{l\,M} \\ 0 \end{bmatrix}$$

$A^2B = A(AB)$:

$$A^2B = A(AB) = A \cdot \begin{bmatrix} \frac{1}{M} \\ 0 \\ \frac{1}{l\,M} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{mg}{M^2} \\ 0 \\ -\frac{(M+m)\,g}{l^2 M^2} \end{bmatrix}$$

$A^3B = A(A^2B)$:

$$A^3B = A \cdot \begin{bmatrix} 0 \\ -\frac{mg}{M^2} \\ 0 \\ -\frac{(M+m)\,g}{l^2 M^2} \end{bmatrix} = \begin{bmatrix} -\frac{mg}{M^2} \\ 0 \\ -\frac{(M+m)\,g}{l^2 M^2} \\ 0 \end{bmatrix}$$

So the final controllability matrix:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{M} & 0 & -\frac{mg}{M^2} \\ \frac{1}{M} & 0 & -\frac{mg}{M^2} & 0 \\ 0 & \frac{1}{lM} & 0 & -\frac{(M+m)\,g}{l^2 M^2} \\ \frac{1}{lM} & 0 & -\frac{(M+m)\,g}{l^2 M^2} & 0 \end{bmatrix}$$

And by a symbolic rank calculation we find:

$$\operatorname{rank}(\mathcal{C}) = 4$$

Thus, all four states $(x, \dot{x}, \phi, \dot{\phi})$ are reachable from the single input u and the linearized system is controllable.

3 Controller Design

Before we implement any controller, we first we examined the uncontrolled case, where the only force applied to the pendulum and the cart is gravity.

The pendulum was set almost at the top position and then left to fall. We observed the expected behavior of a sinusoidal movement of the pendulum that corresponded to a smaller sinusoidal movement of the cart. The only stable position in this case is when the pendulum is at the down position, since at the perfectly upright position, any infinitesimal disturbance is enough to destabilize the pendulum-cart system.

In the next step we decided to implement a linear controller in order to test some basic control of the robot before the more complicated non linear methods. We decided to go with the Linear Quadratic Regulator (LQR) controller, which stabilizes the system when it is in its linearized region, around the upright position [1]. This controller is also going to come in use in the future non linear controllers.

3.0.1 LQR

We choose the quadratic cost function:

$$J = \int_0^{\infty} \left(\mathbf{x}^\top Q \, \mathbf{x} + u^\top R \, u \right) \, dt$$

with the weighting matrices:

$$Q = \mathrm{diag}(10,\ 1,\ 100,\ 1), \qquad R = 0.1$$

These weights penalize the cart-position error, cart-velocity, pendulum-angle deviation , and pendulum-angular velocity respectively, while R limits control effort.
The optimal gain matrix is computed in MATLAB using:

```
K = lqr(A, B, Q, R);
```

which solves the continuous-time algebraic Riccati equation. Thus, the optimal state-feedback control law is:

$$u = -K \, \mathbf{x} = -\begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix}$$

As we can see from the following Figure 2, the LQR controller successfully manages to stabilize the pendulum in the upright position, while in the uncontrolled case the pendulum is continuously oscillating.

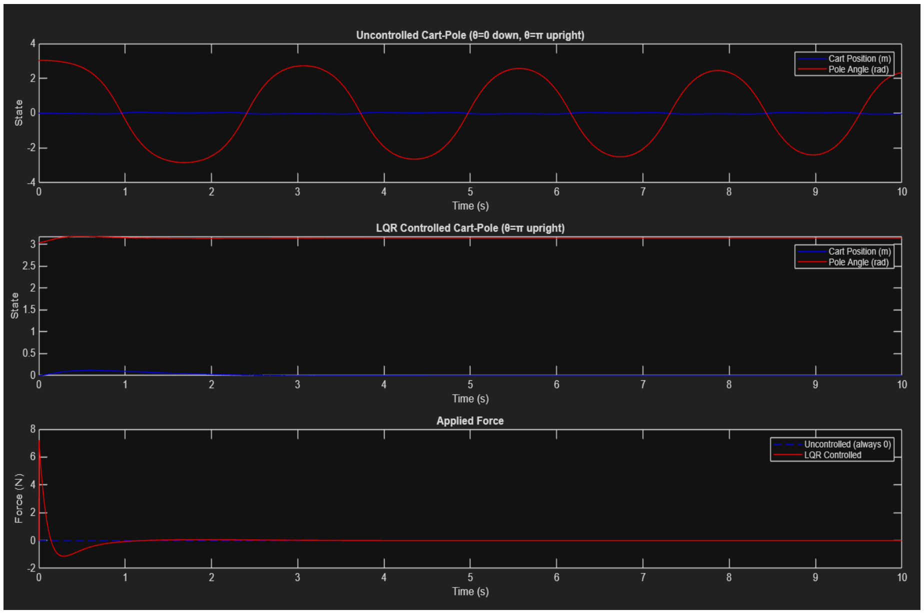

Figure 2: Uncontrolled and LQR controlled cases.

3.1 Non Linear Controller Design Method 1

For the first non linear controller we decided to go with a controller that can operate away from the linearized point. So we implemented an energy based swing up controller [2]. This controller is designed to move the pendulum from a position close to the down resting point, to a position very close to the upright position. It is not suitable to stabilize the pendulum at that point, so we are calling on the LQR controller when we are sufficiently close to that position. The final design is still non linear since combining a linear and a non linear controller is still considered a non linear approach.

3.1.1 Energy Based Swing up Controller

The total pendulum energy is given by:

$$E_{\text{total}}(\theta, \dot{\theta}, \dot{x}) = T_{\text{pend}} + V_{\text{pend}} = \frac{1}{2} m_p \left[\dot{x}^2 + 2\,l\,\cos\theta\,\dot{x}\,\dot{\theta} + l^2\,\dot{\theta}^2 \right] + m_p\,g\,l\,(1 - \cos\theta).$$

However, in the energy shaping swing up controller, we focus only on the pendulum's energy relative to its pivot, effectively subtracting out any kinetic energy due solely to cart velocity $\dot{x}$. In other words, we assume the pivot is momentarily stationary and define:

- **Pendulum relative kinetic energy:**

$$T_{\text{rel}}(\dot{\theta}) = \tfrac{1}{2}\,m_p\,(l\,\dot{\theta})^2$$

- **Pendulum potential energy:**

$$V_{\text{rel}}(\theta) = m_p\, g\, l\,(1 - \cos\theta)$$

So, the energy for swing up is:

$$E(\theta, \dot{\theta}) = T_{\text{rel}} + V_{\text{rel}} = \frac{1}{2}\, m_p\, l^2\, \dot{\theta}^2 + m_p\, g\, l\,(1 - \cos\theta).$$

The standard energy-shaping control law targets only the pendulum's motion about its pivot, making this relative energy the appropriate quantity for feedback.

Equilibrium Points:

- Hanging equilibrium: $\theta = 0,\ \dot{\theta} = 0$

$$E(0, 0) = \tfrac{1}{2}\, m_p\, l^2 \cdot 0^2 + m_p\, g\, l\,(1 - 1) = 0$$

- Inverted equilibrium: $\theta = \pi,\ \dot{\theta} = 0$

$$E(\pi, 0) = \tfrac{1}{2}\, m_p\, l^2 \cdot 0^2 + m_p\, g\, l\,(1 - (-1)) = 2\, m_p\, g\, l$$

We define the desired energy level as:

$$E_{\text{des}} = 2\, m_p\, g\, l$$

Energy Error:

$$\Delta E = E_{\text{des}} - E(\theta, \dot{\theta}) = 2\, m_p\, g\, l - \Big[\tfrac{1}{2}\, m_p\, l^2\, \dot{\theta}^2 + m_p\, g\, l\,(1 - \cos\theta)\Big]$$

Simplifying:

$$\begin{aligned}
\Delta E &= 2\, m_p\, g\, l - \tfrac{1}{2}\, m_p\, l^2\, \dot{\theta}^2 - m_p\, g\, l + m_p\, g\, l\, \cos\theta \\
&= m_p\, g\, l - \tfrac{1}{2}\, m_p\, l^2\, \dot{\theta}^2 + m_p\, g\, l\, \cos\theta
\end{aligned}$$

Switching Sign Term:

$$S(\theta, \dot{\theta}) = \operatorname{sign}\big(\dot{\theta}\, \cos\theta\big), \qquad S = 1 \quad \text{if } \dot{\theta}\, \cos\theta = 0$$

Energy-Shaping Torque:

$$u_{\text{swing}} = -\, k_{\text{swing}}\, \Delta E\; S(\theta, \dot{\theta}), \qquad k_{\text{swing}} > 0$$

From [3] we get:

$$\frac{dE}{dt} = -\,u\,\dot{\theta}\,\cos\theta$$

Substituting the swing-up control law:

$$u_{\text{swing}} = -k_{\text{swing}}\,\Delta E\,S$$

yields:

$$\frac{dE}{dt} = -\big[-k_{\text{swing}}\,\Delta E\,S\big]\,\dot{\theta}\,\cos\theta = k_{\text{swing}}\,\Delta E\,|\dot{\theta}\,\cos\theta|$$

Hence:

- If $\Delta E > 0$, then $\frac{dE}{dt} > 0$, so E increases toward E_{des}.
- If $\Delta E < 0$, then $\frac{dE}{dt} < 0$, so E decreases toward E_{des}.

So, the $E(\theta, \dot{\theta})$ converges to $2\,m_p\,g\,l$, ensuring the pendulum reaches $\theta = \pi$ with $\dot{\theta} \approx 0$.

3.1.2 Hybrid Switching to LQR

Once the pendulum gets near the upright position (arbitrarily chosen as ¡0.3 rad) such that:

$$|\,\theta - \pi\,| < \theta_{\text{enter}} = 0.3 \text{ rad}$$

we switch to the LQR law. If the pendulum deviates beyond:

$$|\,\theta - \pi\,| > \theta_{\text{exit}} = 0.5 \text{ rad}$$

we revert to the swing up control u_{swing}.

3.1.3 Controller Test

To test the controller we set an initial position of the pendulum close to downright position and give it a small initial velocity to help with the energy conversion.

As we observe in Figure 3 the pendulum reaches a position close to upright, where the LQR controller takes over, and after a small overshoot it stabilizes at -pi rad. The cart is also moving at the start in order to push enough energy into the pendulum for it to climb to the upright position. After that it returns and stabilizes at its initial position.

As of the forces, two spikes are observed at the switching points of the swing up controller (the start and the LQR switch point) in Figure 4. We also see that after the pendulum is stabilized, the controller applies no force to the cart. So in total, the controller successfully stabilized our system.

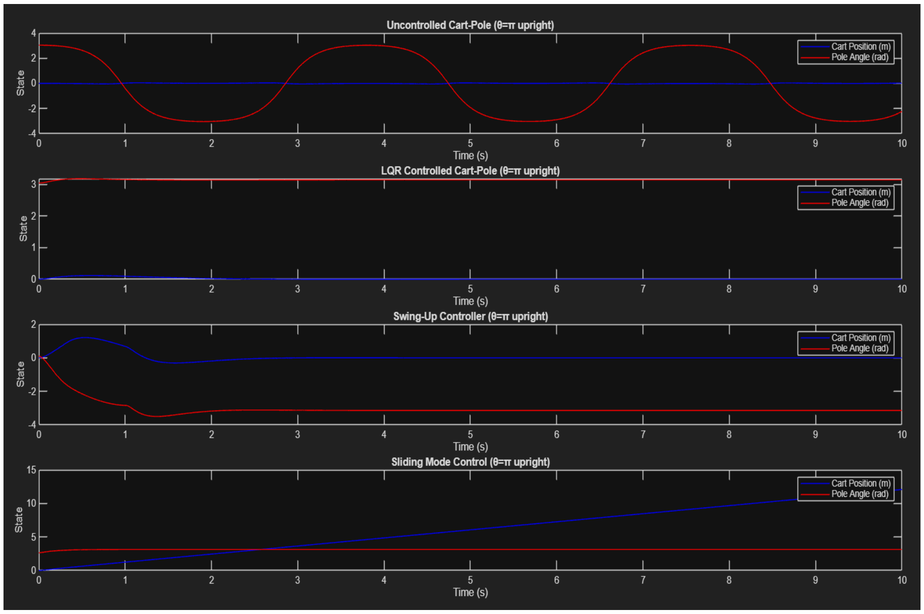

Figure 3: Position response of all the controllers.

3.2 Non Linear Controller Design Method 2

Since we explored a non linear controller that can swing the pendulum all the way to the upright position, for the second controller we decided to go with the Sliding Mode Controller (SMC) which is usually used as a local stabilizer for initial positions around the upright one. [4][5][6]

The goal of the sliding mode controller is to drive the pendulum angle to the inverted position $\theta = \pi$ by enforcing a "sliding surface" on which the dynamics have desirable convergence properties.

So, we start by defining the angle error:

$$\phi = \theta - \pi, \qquad \dot{\phi} = \dot{\theta}$$

We choose the sliding variable:

$$s(\mathbf{x}) = \dot{\phi} + \lambda\,\phi = \dot{\theta} + \lambda\,(\theta - \pi)$$

where $\lambda > 0$ is a design parameter.

On the sliding goal $s(\mathbf{x}) = 0$, we have:

$$\dot{\theta} = -\,\lambda\,(\theta - \pi)$$

This implies that $\theta(t)$ converges exponentially to π. Indeed, since

$$\dot{\phi} + \lambda\,\phi = 0 \quad \Rightarrow \quad \phi(t) = \phi(0)\,e^{-\lambda t}$$

we conclude:

$$\theta(t) \to \pi \quad \text{exponentially as } t \to \infty$$

To enforce $s = 0$ in finite time, the SMC applies a discontinuous "switching" control on top of an "equivalent" (continuous) term. We compute u_{eq} by requiring $\dot{s} = 0$ when $s = 0$.
We compute $\dot{s}$:

$$s = \dot{\theta} + \lambda(\theta - \pi),$$
$$\dot{s} = \ddot{\theta} + \lambda\,\dot{\theta}$$

We substitute $\ddot{\theta}$ from the previous system dynamics, so:

$$\dot{s} = \frac{-\,u\,\cos\theta - m_p\,l\,\dot{\theta}^2\,\cos\theta\,\sin\theta + (m_c + m_p)\,g\,\sin\theta}{l\left(m_c + m_p\,\sin^2\theta\right)} + \lambda\,\dot{\theta}$$

We solve $\dot{s} = 0$ for u when $s = 0$ using trigonometric identities:

$$\begin{aligned} 0 &= \frac{-\,u_{\mathrm{eq}}\,(-\cos\phi) - m_p\,l\,(\lambda^2\,\phi^2)(-\cos\phi)(-\sin\phi) + (m_c + m_p)\,g\,(-\sin\phi)}{l\left(m_c + m_p\,\sin^2\phi\right)} + \lambda\,(-\lambda\,\phi) \\ &= \frac{u_{\mathrm{eq}}\,\cos\phi - m_p\,l\,\lambda^2\,\phi^2\,\cos\phi\,\sin\phi - (m_c + m_p)\,g\,\sin\phi}{l\left(m_c + m_p\,\sin^2\phi\right)} - \lambda^2\,\phi \end{aligned}$$

multiply both sides by $l\left(m_c + m_p\,\sin^2\phi\right)$:

$$0 = u_{\mathrm{eq}}\,\cos\phi - m_p\,l\,\lambda^2\,\phi^2\,\cos\phi\,\sin\phi - (m_c + m_p)\,g\,\sin\phi - l\,\lambda^2\,\phi\left(m_c + m_p\,\sin^2\phi\right)$$

and finally, solving for u_{eq}:

$$\boxed{u_{\mathrm{eq}} = \frac{m_p\,l\,\lambda^2\,\phi^2\,\cos\phi\,\sin\phi + (m_c + m_p)\,g\,\sin\phi + l\,\lambda^2\,\phi\left(m_c + m_p\,\sin^2\phi\right)}{\cos\phi}} \quad \text{with } \phi = \theta - \pi$$

In terms of stability, to maintain $\dot{s} = 0$ using u_{eq}, we add a discontinuous "switching" term to guarantee finite-time attraction to $s = 0$. Choose:

$$u = u_{\mathrm{eq}} - k\,\operatorname{sign}(s(\mathbf{x})), \quad k > 0.$$

Additionally, to reduce chattering, we often use a boundary-layer saturation function $\operatorname{sat}(\cdot)$:

$$u = u_{\mathrm{eq}} - k\,\operatorname{sat}\left(\frac{s}{\varepsilon}\right), \quad \operatorname{sat}(\sigma) = \begin{cases} +1, & \sigma > 1, \\ \sigma, & -1 \le \sigma \le 1, \\ -1, & \sigma < -1, \end{cases}$$

where $\varepsilon > 0$ is a small thickness parameter.
Once $|s| \le \varepsilon$, the dynamics are governed approximately by:

$$s = \dot{\theta} + \lambda(\theta - \pi) \approx 0 \quad \Longrightarrow \quad \dot{\theta} = -\,\lambda\,(\theta - \pi).$$

Solving this gives exponential convergence:

$$\theta(t) - \pi = (\theta(t_0) - \pi)\, e^{-\lambda\,(t - t_0)}, \quad \dot{\theta}(t) = -\lambda\,(\theta(t) - \pi).$$

Meanwhile, the cart position x and velocity $\dot{x}$ evolve according to the first equation of motion with $u \approx u_{\mathrm{eq}}$. Substituting $\theta(t) \approx \pi$ and $\dot{\theta} \approx -\lambda(\theta - \pi) \approx 0$, one finds:

$$\ddot{x} \approx 0,$$

so x remains bounded.

Noteworthy is the fact that, the SMC law only forces the pendulum angle onto its sliding surface, it does not directly regulate the cart's position. Once θ is locked at π, the SMC "equivalent" control exactly balances gravity on the pendulum, so any residual cart velocity is neither driven to zero nor opposed by the switching term. This is the reason we see the cart in our implementation, drift with a constant speed after the the pendulum is stabilized at the upright position.

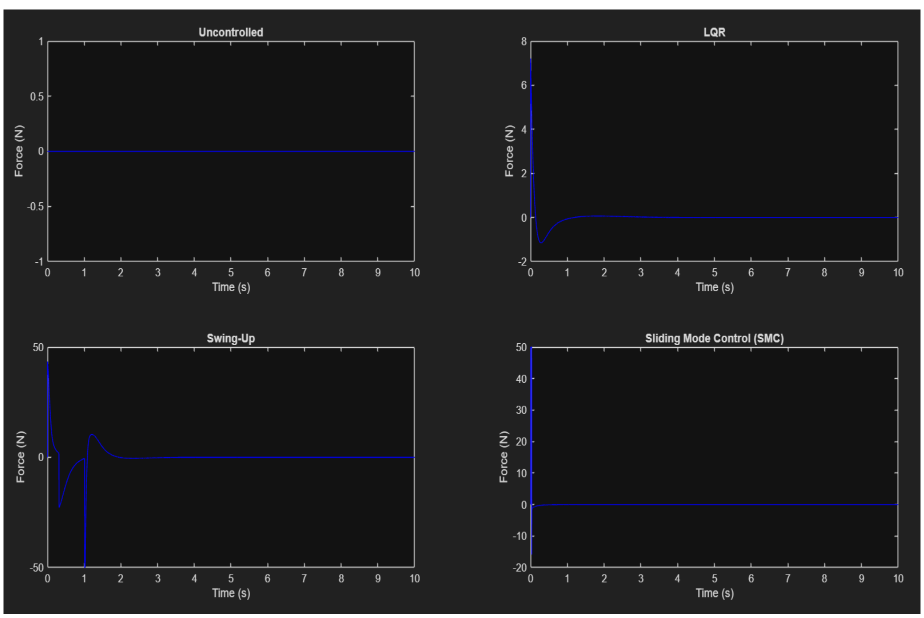

Figure 4: Force response of all the controllers.

4 Simulation Results

In the simulated results we observe that all the controllers succeeded into stabilizing the pendulum at π or $-\pi$ rad from the bottom. Some of them work as local stabilizers, needing the pendulum to be withing a small region from the stabilizing position, and some others can swing the pendulum all the way from the bottom resting point.

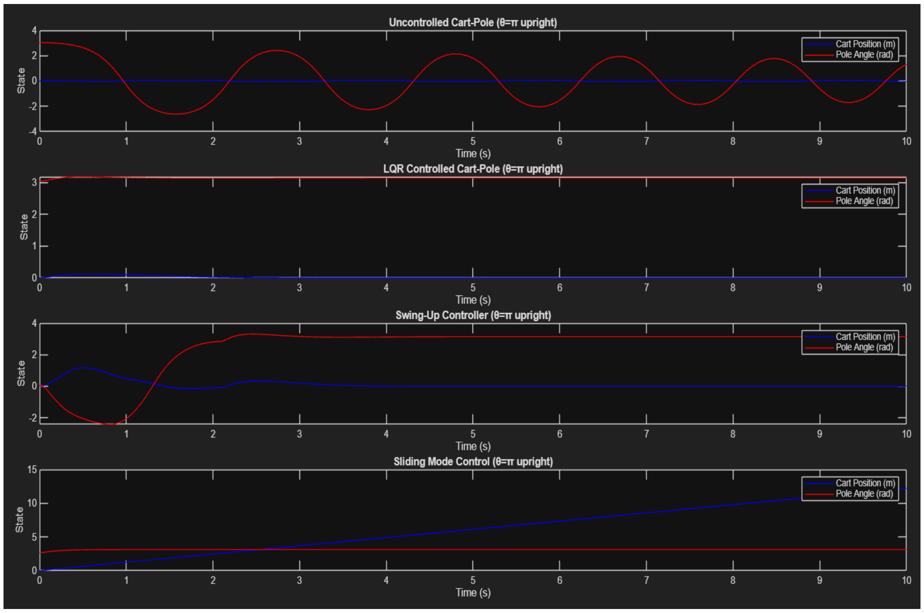

Figure 5: Force response of all the controllers with added friction.

In all of the cases that we observe an initial stabilization period, where the forces applied to the cart resemble spikes and oscillations, but after that period the force required and applied returns to zero. All of the tests and simulation has been done assuming zero friction in the system, but an additional set of tests show that even with added friction on the cart and at the pivot point of the pendulum, the controllers behave the same way (see Figure 5 and 6). The friction we tested is 0.5 N*s/m for the cart and 0.03 N*s/m for the pendulum.

5 Conclusions

To conclude, we explored, implemented and demonstrated two non linear controllers, as we as the uncontrolled and linearly controlled case, for the inverted pendulum-cart robotic system.

In all of the cases the robot behaved as expected and the controllers succeeded in their goal. In the future we would like to improve the SMC controller

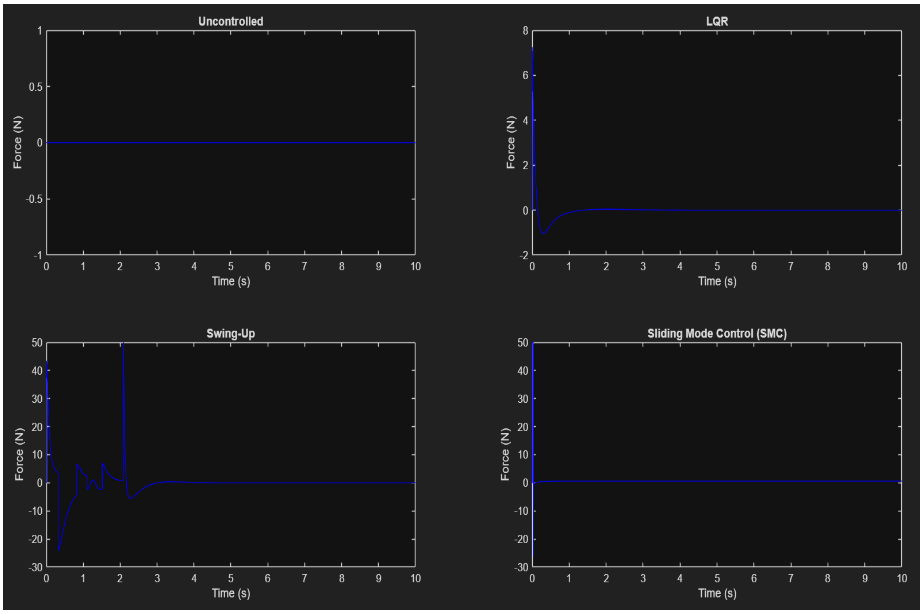

Figure 6: Force response of all the controllers with added friction.

so that it also stabilizes the cart at its original position, and also implement a combined swing-up-SMC controller in order to explore the complete stabilization of the pendulum , starting from the downward position, but with completely non linear controllers.

References

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Appendix

A Euler–Lagrange Equations

The Euler–Lagrange equation for each generalized coordinate q_i is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i.$$

We compute this for $q_1 = x$ and $q_2 = \theta$.

A.1 Equation for x

1. Compute $\dfrac{\partial L}{\partial \dot{x}}$:

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left[\tfrac{1}{2}(M+m)\,\dot{x}^2 + m\,l\,\cos\theta\,\dot{x}\,\dot{\theta}\right] = (M+m)\,\dot{x} + m\,l\,\cos\theta\,\dot{\theta}.$$

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2. Take the time derivative:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left[(M+m)\,\dot{x} + m\,l\,\cos\theta\,\dot{\theta}\right].$$

Since M, m, and l are constants:

$$\begin{aligned}\frac{d}{dt}\left[(M+m)\,\dot{x} + m\,l\,\cos\theta\,\dot{\theta}\right] &= (M+m)\,\ddot{x} + m\,l\,\frac{d}{dt}\left(\cos\theta\,\dot{\theta}\right)\\ &= (M+m)\,\ddot{x} + m\,l\,\left(-\sin\theta\,\dot{\theta}\,\dot{\theta} + \cos\theta\,\ddot{\theta}\right)\\ &= (M+m)\,\ddot{x} + m\,l\,\cos\theta\,\ddot{\theta} - m\,l\,\sin\theta\,\dot{\theta}^2.\end{aligned}$$

3. Compute $\dfrac{\partial L}{\partial x}$. Since L does not depend explicitly on x (only on $\dot{x}$, θ, and $\dot{\theta}$):

$$\frac{\partial L}{\partial x} = 0.$$

4. Euler–Lagrange equation for $q_1 = x$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q_x \quad \Longrightarrow \quad (M+m)\,\ddot{x} + m\,l\,\cos\theta\,\ddot{\theta} - m\,l\,\sin\theta\,\dot{\theta}^2 = u.$$

$$\boxed{(M+m)\,\ddot{x} + m\,l\,\cos\theta\,\ddot{\theta} - m\,l\,\sin\theta\,\dot{\theta}^2 = u} \tag{A.1}$$

A.2 Equation for θ

1. Compute $\dfrac{\partial L}{\partial \dot\theta}$:

$$\frac{\partial L}{\partial \dot\theta} = \frac{\partial}{\partial \dot\theta}\left[m\,l\,\cos\theta\,\dot x\,\dot\theta + \tfrac{1}{2}\,m\,l^2\,\dot\theta^2\right] = m\,l\,\cos\theta\,\dot x + m\,l^2\,\dot\theta.$$

2. Take the time derivative:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{d}{dt}\left[m\,l\,\cos\theta\,\dot x + m\,l^2\,\dot\theta\right] &= m\,l\,\frac{d}{dt}(\cos\theta\,\dot x) + m\,l^2\,\ddot\theta \\ &= m\,l\,\left(-\sin\theta\,\dot\theta\,\dot x + \cos\theta\,\ddot x\right) + m\,l^2\,\ddot\theta \\ &= m\,l\,\cos\theta\,\ddot x - m\,l\,\sin\theta\,\dot x\,\dot\theta + m\,l^2\,\ddot\theta.\end{aligned}$$

3. Compute $\dfrac{\partial L}{\partial \theta}$:

   The only dependence for the Lagrangian for θ enters in the terms $m\,l\,\cos\theta\,\dot x\,\dot\theta$ and $-m\,g\,l\,(1-\cos\theta)$. Thus:

$$\begin{aligned}\frac{\partial L}{\partial \theta} &= \frac{\partial}{\partial \theta}\big[m\,l\,\cos\theta\,\dot x\,\dot\theta\big] - \frac{\partial}{\partial \theta}\big[m\,g\,l\,(1-\cos\theta)\big] \\ &= -m\,l\,\sin\theta\,\dot x\,\dot\theta + m\,g\,l\,\sin\theta.\end{aligned}$$

4. Euler–Lagrange equation for $q_2 = \theta$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = Q_\theta = 0 \Rightarrow$$

$$\Rightarrow \left[m\,l\,\cos\theta\,\ddot x - m\,l\,\sin\theta\,\dot x\,\dot\theta + m\,l^2\,\ddot\theta\right] - \left[-m\,l\,\sin\theta\,\dot x\,\dot\theta + m\,g\,l\,\sin\theta\right] = 0 \Rightarrow$$

$$\Rightarrow m\,l\,\cos\theta\,\ddot x - m\,l\,\sin\theta\,\dot x\,\dot\theta + m\,l^2\,\ddot\theta + m\,l\,\sin\theta\,\dot x\,\dot\theta - m\,g\,l\,\sin\theta = 0 \Rightarrow$$

$$\Rightarrow m\,l\,\cos\theta\,\ddot x + m\,l^2\,\ddot\theta - m\,g\,l\,\sin\theta = 0.$$

   So:

$$\boxed{m\,l\,\cos\theta\,\ddot x + m\,l^2\,\ddot\theta - m\,g\,l\,\sin\theta = 0} \tag{A.2}$$

B. Solving for $\ddot{x}$ and $\ddot{\theta}$ explicitly

We start by inverting the matrix $B(\theta)$. Its determinant is:

$$\det B = (M+m)\,(m\,l^2) - (m\,l\,\cos\theta)^2 = m\,l^2\,(M + m\,\sin^2\theta).$$

The inverse is:

$$B^{-1}(\theta) = \frac{1}{\det B}\begin{bmatrix} m\,l^2 & -\,m\,l\,\cos\theta \\ -\,m\,l\,\cos\theta & M+m \end{bmatrix} = \frac{1}{m\,l^2\,(M + m\,\sin^2\theta)}\begin{bmatrix} m\,l^2 & -\,m\,l\,\cos\theta \\ -\,m\,l\,\cos\theta & M+m \end{bmatrix}.$$

Multiply both sides of equation:

$$\begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} -ml\theta^2\sin\theta \\ -mgl\sin\theta \end{bmatrix}$$

by $B^{-1}(\theta)$ and:

$$W = B^{-1}(\theta)\,\left[\begin{bmatrix} u \\ 0 \end{bmatrix} - C(\theta,\dot{\theta})\right].$$

Then

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = W.$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = W = B^{-1}(\theta)\,\begin{bmatrix} u + m\,l\,\sin\theta\,\dot{\theta}^2 \\ m\,g\,l\,\sin\theta \end{bmatrix}.$$

B.1. Compute $\ddot{x}$:

$$\begin{aligned}
\ddot{x} &= \frac{1}{\det B}\left[m\,l^2\left(u + m\,l\,\sin\theta\,\dot{\theta}^2\right) - (m\,l\,\cos\theta)(m\,g\,l\,\sin\theta)\right] \\
&= \frac{m\,l^2\,u + m^2\,l^3\,\sin\theta\,\dot{\theta}^2 - m^2\,g\,l^2\,\sin\theta\,\cos\theta}{m\,l^2\,(M + m\,\sin^2\theta)} \\
&= \frac{u + m\,l\,\sin\theta\,\dot{\theta}^2 - m\,g\,\sin\theta\,\cos\theta}{M + m\,\sin^2\theta}.
\end{aligned}$$

B.2. Compute $\ddot{\theta}$:

$$\begin{aligned}
\ddot{\theta} &= \frac{1}{\det B}\left[-\,m\,l\,\cos\theta\left(u + m\,l\,\sin\theta\,\dot{\theta}^2\right) + (M+m)\,m\,g\,l\,\sin\theta\right] \\
&= \frac{-\,m\,l\,\cos\theta\,u - m^2\,l^2\,\cos\theta\,\sin\theta\,\dot{\theta}^2 + (M+m)\,m\,g\,l\,\sin\theta}{m\,l^2\,(M + m\,\sin^2\theta)} \\
&= \frac{-\,u\,\cos\theta - m\,l\,\dot{\theta}^2\,\cos\theta\,\sin\theta + (M+m)\,g\,\sin\theta}{l\,\left(M + m\,\sin^2\theta\right)}.
\end{aligned}$$

So, the fully explicit form is:

$$\boxed{\begin{aligned} \ddot{x} &= \frac{u + m\,l\,\sin\theta\,\dot{\theta}^2 \;-\; m\,g\,\sin\theta\,\cos\theta}{M + m\,\sin^2\theta}, \\[6pt] \ddot{\theta} &= \frac{-\,u\,\cos\theta \;-\; m\,l\,\dot{\theta}^2\,\cos\theta\,\sin\theta \;+\; (M+m)\,g\,\sin\theta}{l\,\left(M + m\,\sin^2\theta\right)}. \end{aligned}} \tag{A.3}$$