Ψηφιακή Επεξεργασία Εικόνας (ΨΕΕ) – MYE037

Εαρινό εξάμηνο 2023-2024

Filtering in the Frequency Domain (Fundamentals)

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Outline

- *We will see how we can analyze the frequency content of the image. Specifically, we will first present the 1D versions for:*

- *Continuous Fourier Transform*
- *Discrete Fourier Transform (DFT)*

...and we will see how they are generalized for 2D discrete signals, such as digital images.

Filtering in the Frequency Domain

Filter: A device or material for suppressing or minimizing waves or oscillations of certain frequencies.

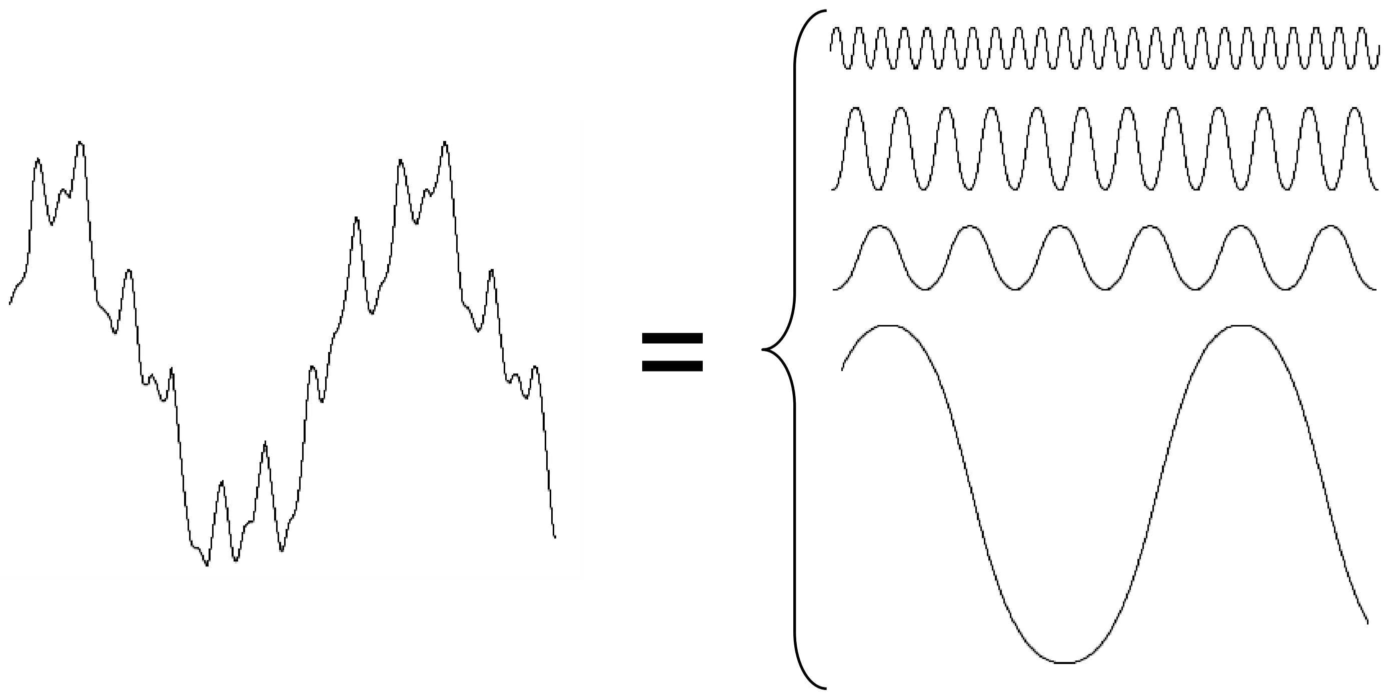
Frequency: The number of times that a periodic function repeats the same sequence of values during a unit variation of the independent variable.

Webster's New Collegiate Dictionary

Jean Baptiste Joseph Fourier

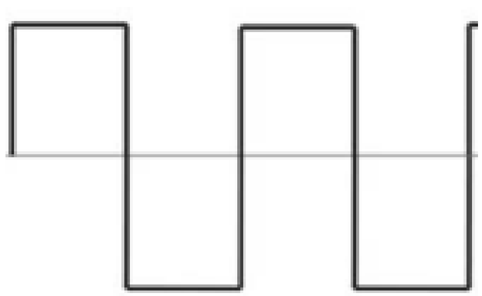
- Fourier was born in Auxerre, France in 1768.
  - Most famous for his work "*La Théorie Analytique de la Chaleur*" published in 1822.
  - Translated into English in 1878: "*The Analytic Theory of Heat*".
- Nobody paid much attention when the work was first published.
- One of the most important mathematical theories in modern engineering.

The Big Idea

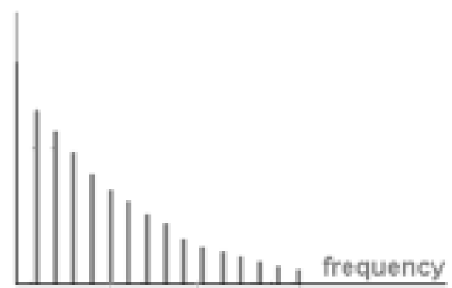


Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a *Fourier series*

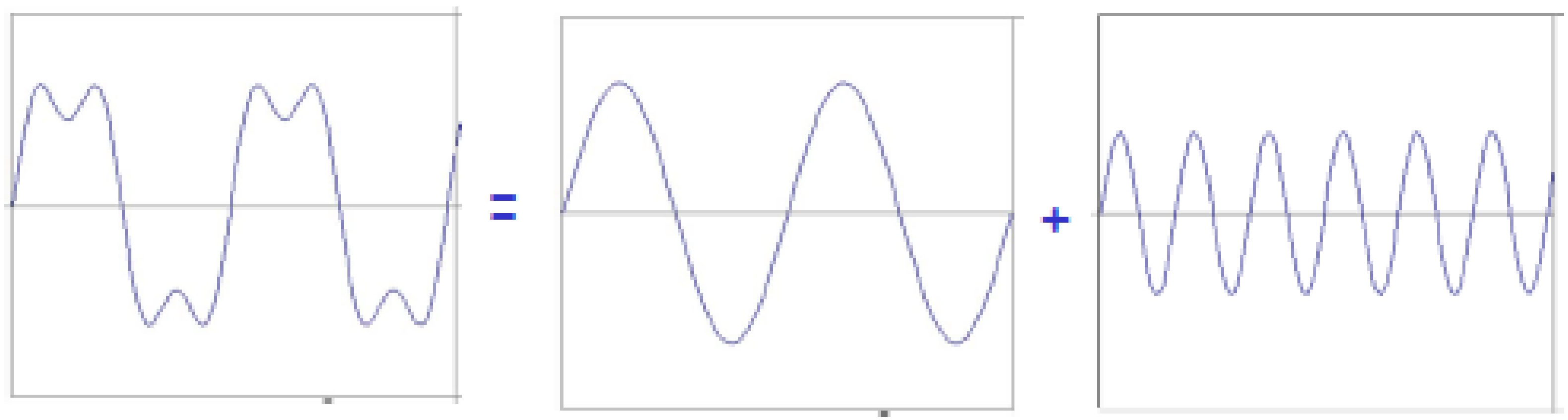
The Big Idea (cont..)



$$= \quad f(x) = \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin nx$$

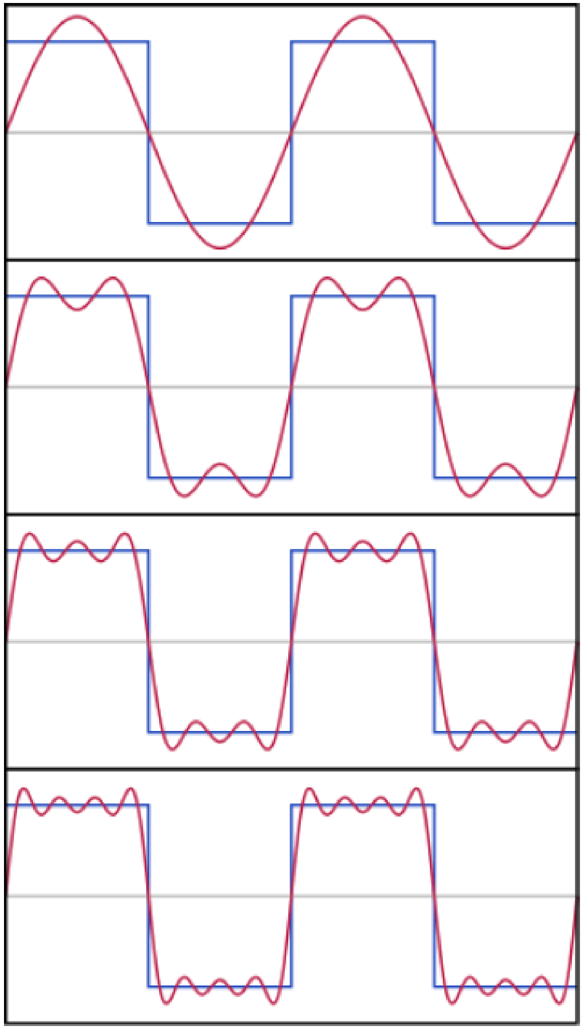


Fourier series



$$f(x) = \sin x + \frac{1}{3}\sin 3x + \ldots$$

Fourier series



Fourier series (analytic expression)

- c_n represents the contribution of the $n-th$ frequency component to the Fourier series of $f(t)$.

$$c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} f(t)\, e^{-j2\pi \frac{n}{\Delta T} t} dt$$

$$\mathrm{f(t)} = \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi \frac{n}{\Delta T} t}$$

Reminder – Euler's formula approximation using Taylor series

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad f(x) = \sum_{n}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

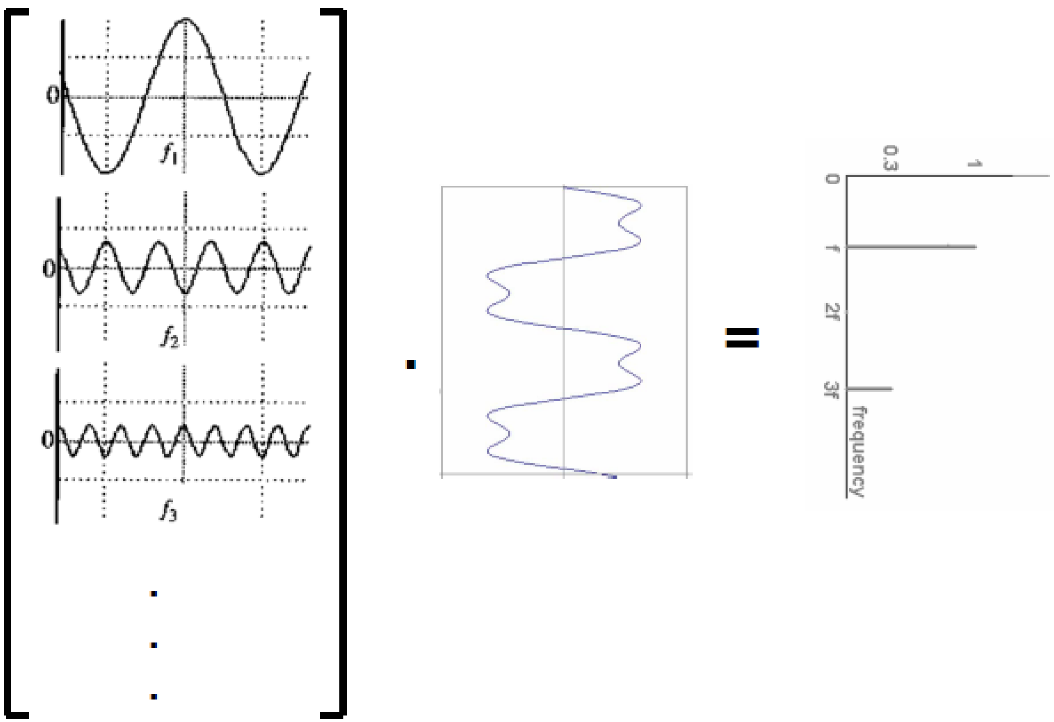
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$e^{jx} = 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} - \frac{x^6}{6!} - j\frac{x^7}{7!} + \ldots$$

A different representation (change of basis)

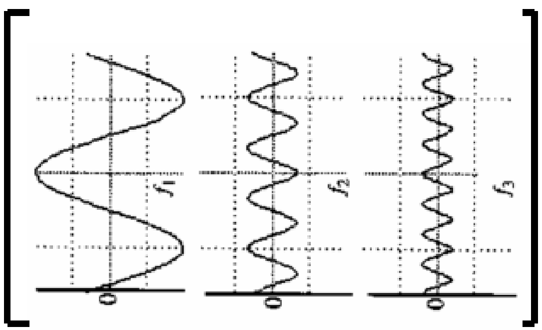
- This representation is akin to a change of basis, where we express the function in terms of a new set of basis functions.
- Trigonometric basis $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \dots\}$ is the basis in which the Fourier series expresses f(x).



$$c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} f(t)\, e^{-j2\pi \frac{n}{\Delta T} t}\, dt$$

A different representation (change of basis)

- Same holds for the reverse operation.
- Trigonometric basis changes to the exponential basis $\left\{e^{inx}\right\}_{n=-\infty}^{\infty}$



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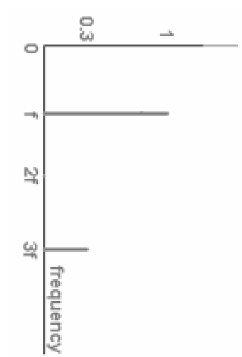

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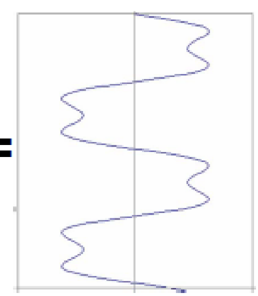


$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi \frac{n}{\Delta T} t}$$

$\vdots$

Transition from discrete to 1D continuous signals

- The Fourier series expansion of a periodic signal $f(t)$.

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j\frac{2\pi}{T}nt}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi}{T}nt} dt$$

1D continuous signals

- The Fourier transform of a continuous signal $f(t)$

$$F(\mu) = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi\mu t} dt$$

- The reverse Fourier transform

$$f(t) = \int_{-\infty}^{+\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

1D continuous FT

- A 1-1 mapping between functions $f(t)$ and $F(\mu)$

$$f \leftrightarrow \mathrm{F}$$

$$\mathrm{F}(\mu) = F\{\mathrm{f}(\mathrm{t})\}$$

$$\mathrm{f}(\mathrm{t}) = F^{-1}\{\mathrm{F}(\mu)\}$$

1D continuous signals (cont.)

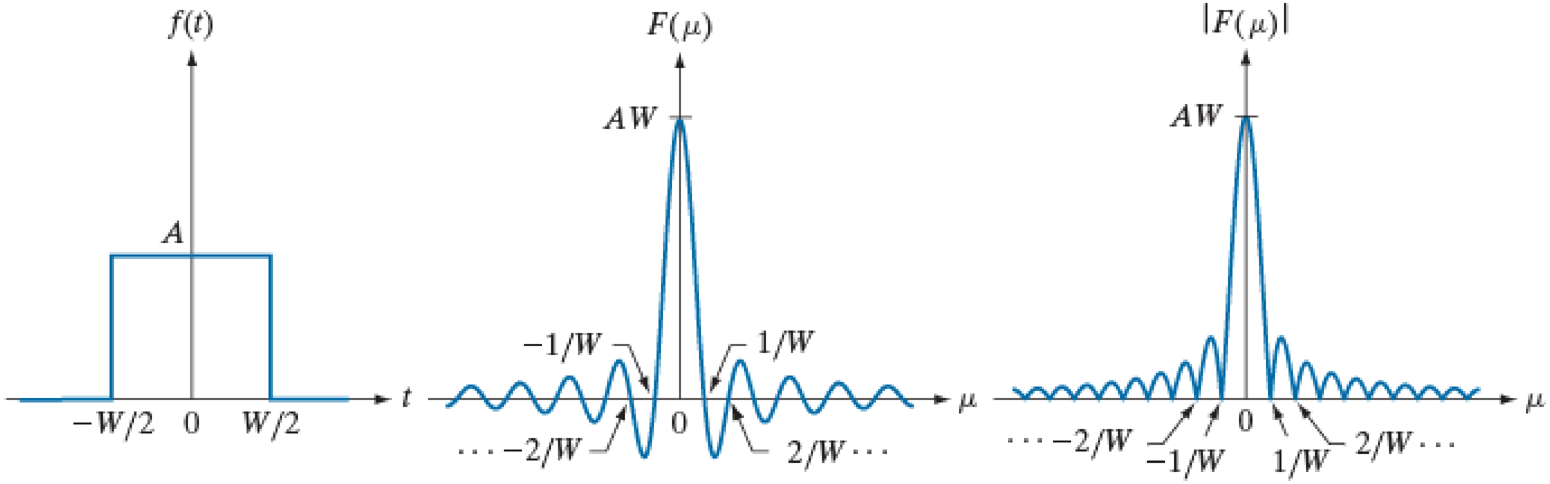


FIGURE 4.4 (a) A box function, (b) its Fourier transform, and (c) its spectrum. All functions extend to infinity in both directions. Note the inverse relationship between the width, W, of the function and the zeros of the transform.

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t} dt = \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2}$$

$$= \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right] = \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right] = AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

trigonometric identity $\sin\theta = (e^{j\theta} - e^{-j\theta})/2j$.

1D continuous signals (cont.)

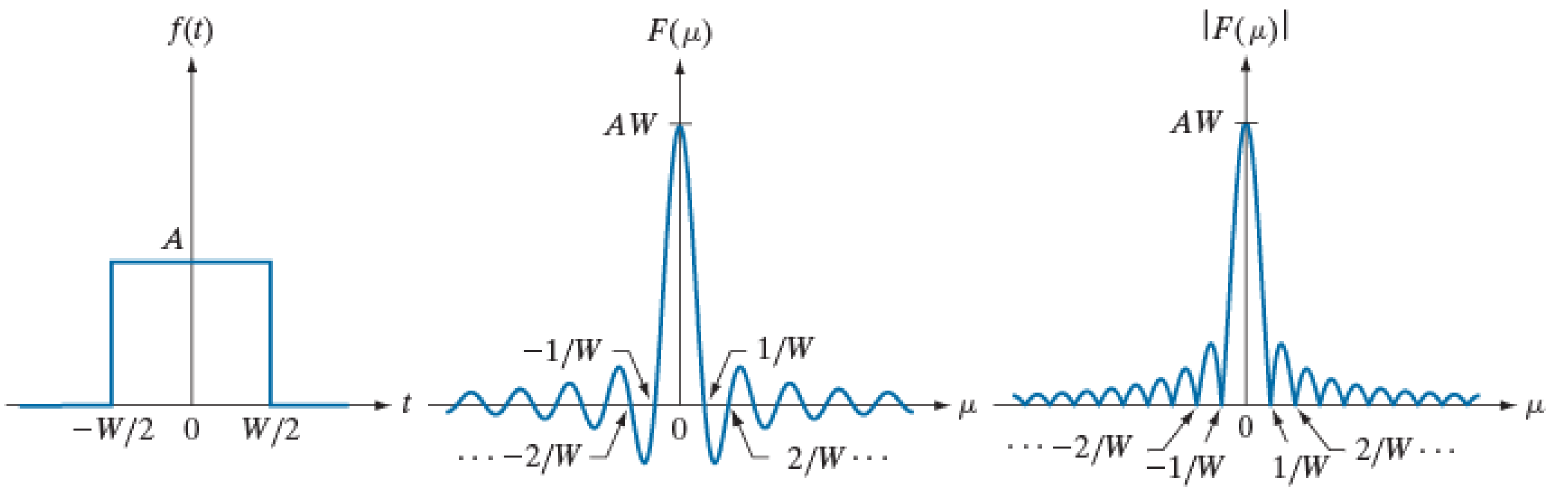


FIGURE 4.4 (a) A box function, (b) its Fourier transform, and (c) its spectrum. All functions extend to infinity in both directions. Note the inverse relationship between the width, W, of the function and the zeros of the transform.

$$f(t) = \mathrm{A}P_{W/2}(t) \leftrightarrow F(\mu) = \mathrm{A}W \frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

1D continuous signals (cont.)

- Convolution property of the FT.
  - convolution in the time domain corresponds to multiplication in the frequency domain, and vice versa.

$$f(t)*h(t)=\int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$$

$$f(t)*h(t) \leftrightarrow F(\mu)H(\mu)$$

$$f(t)h(t) \leftrightarrow F(\mu)*H(\mu)$$

1D continuous signals

- Dirac delta function may be considered both as continuous and discrete.
- Useful for the representation of discrete signals through sampling of continuous signals.
- The property of "selection" refers to the fact that the delta function acts as a "filter" or "switch" that selects or "marks" a specific point in time or space.

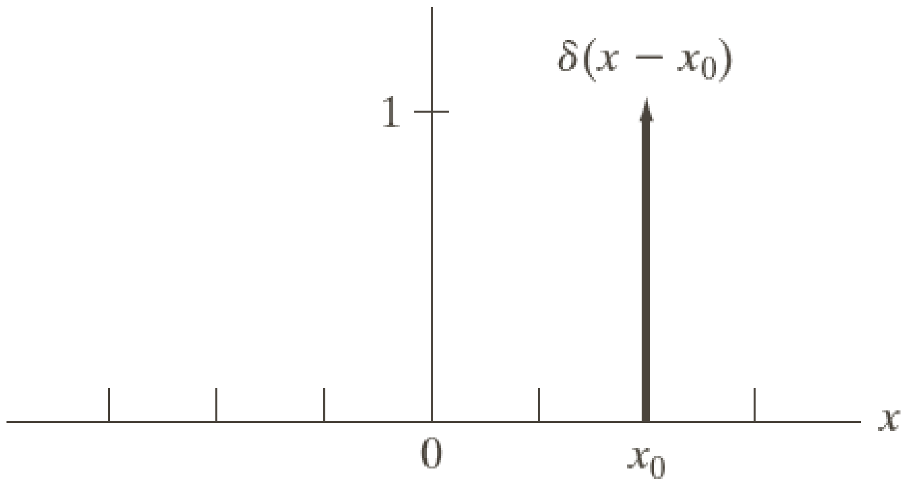
$$\delta(x - x_0) = \begin{cases} +\infty, & x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x) = 1$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x - x_0)dx = f(x_0)$$

1D continuous signals

- Discrete version of “Delta”



$$\delta(x - x_0) = 0, \alpha\nu \; x \neq x_0$$

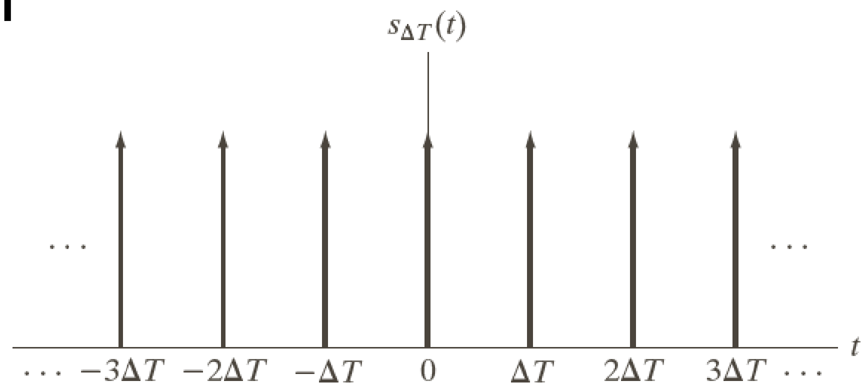
$$\delta(x - x_0) = 1, \alpha\nu \; x = x_0$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x - x_0)dx = f(x_0)$$

1D continuous signals (cont.)

Impulse train function

$$S_{\Delta T}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n\Delta T)$$



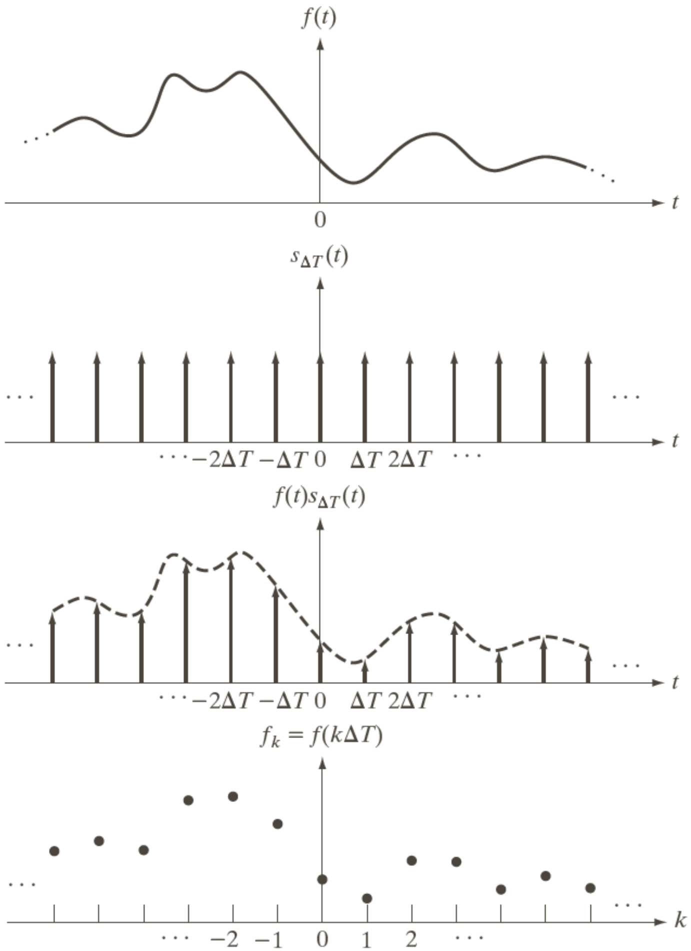
$$x[n] = x(t)S_{\Delta T}(t) = \sum_{n=-\infty}^{+\infty} x(t)\delta(t - n\Delta T) = \sum_{n=-\infty}^{+\infty} x(n\Delta T)\delta(t - n\Delta T)$$

1D continuous signals (from continuous to discrete - sampling)

$$x[n] = x(t)S_{\Delta T}(t)$$

$$= \sum_{n=-\infty}^{+\infty} x(t)\delta(t - n\Delta T)$$

$$= \sum_{n=-\infty}^{+\infty} x(n\Delta T)\delta(t - n\Delta T)$$



Continuous Fourier Transform of a discrete signal

- After expressing the sampling as

$$\widetilde{f}(t) = f(t)s_{\Delta T}(t)$$

- We compute the Fourier Transform of the discrete signal.

$$\widetilde{F}(\mu) = F(\mu) * S(\mu) = \int_{-\infty}^{\infty} F(\tau)S(\mu-\tau)d\tau$$

- It can be proven that the **Fourier transform** of the **periodic impulse train**, is:

$$S(\mu) = \Im\{s_{\Delta T}(t)\} = \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

Continuous Fourier Transform of a discrete signal

- We compute the Fourier Transform of the discrete signal.

$$\tilde{F}(\mu) = F(\mu) * S(\mu) = \int_{-\infty}^{\infty} F(\tau) S(\mu - \tau) d\tau =$$

$$S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

$$= \frac{1}{\Delta \mathrm{T}} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta(\mu - \tau - \frac{n}{\Delta \mathrm{T}}) d\tau =$$

$$= \frac{1}{\Delta \mathrm{T}} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau) \delta(\mu - \tau - \frac{n}{\Delta \mathrm{T}}) d\tau =$$

$$\int_{-\infty}^{+\infty} f(x) \delta(x - x_0) dx = f(x_0)$$

$$= \frac{1}{\Delta \mathrm{T}} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta \mathrm{T}})$$

1D continuous signals (cont.)

- Intermediate result
  - The Fourier transform of the impulse train.

$$\sum_{n=-\infty}^{+\infty} \delta(t - n\Delta T) \leftrightarrow \frac{1}{\Delta T} \sum_{n=-\infty}^{+\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

- It is also an impulse train in the frequency domain.
- Impulses are equally spaced every $1/\Delta T$.

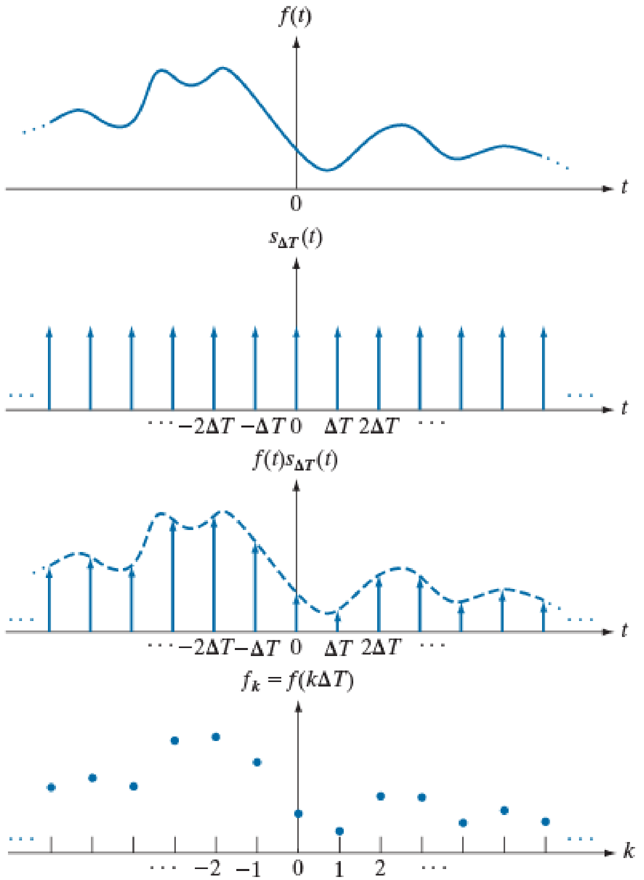
1D continuous signals (cont.)

Sampling

$$x[n] = x(t)S_{\Delta T}(t)$$

$$= \sum_{n=-\infty}^{+\infty} x(t)\delta(t - n\Delta T)$$

$$= \sum_{n=-\infty}^{+\infty} x(n\Delta T)\delta(t - n\Delta T)$$



1D continuous signals (cont.)

- Sampling
  - The spectrum of the discrete signal consists of repetitions of the spectrum of the continuous signal every $1/\Delta T$.
  - The Nyquist criterion should be satisfied.

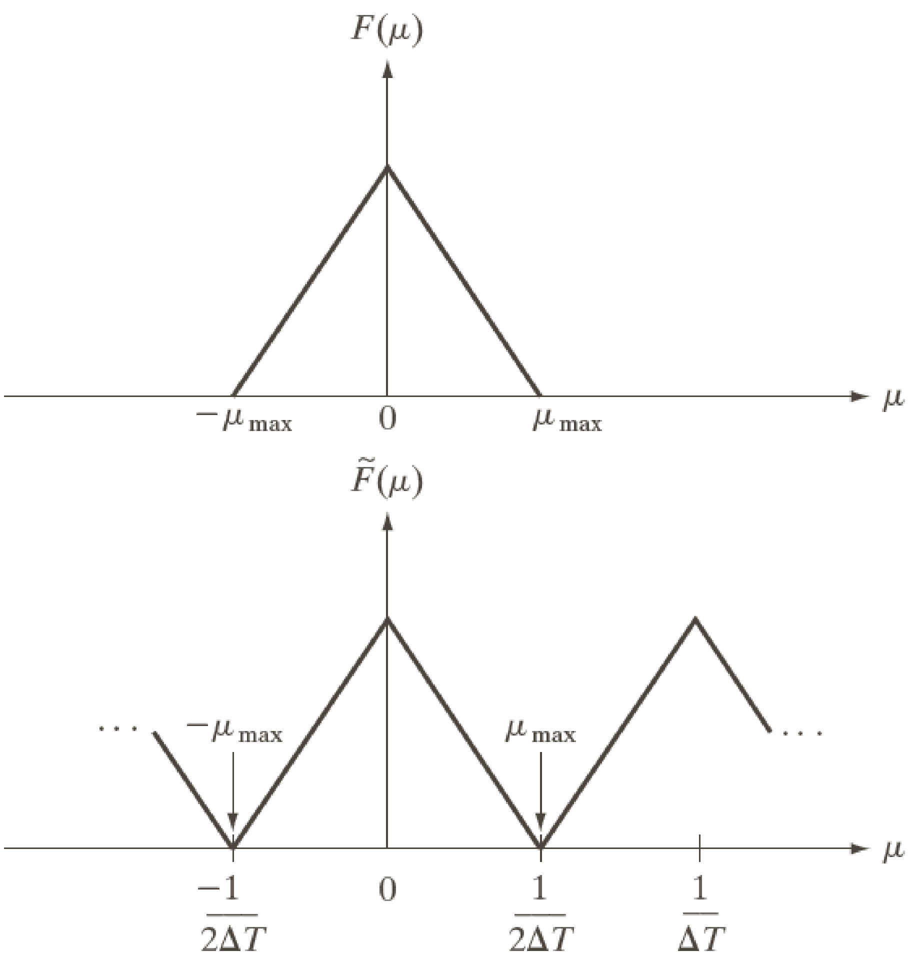
$$f(t) \leftrightarrow F(\mu)$$

$$\tilde{f}(n\Delta T) = f[n] \leftrightarrow \tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{+\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

1D continuous signals (cont.)

Nyquist theorem

$$\frac{1}{\Delta T} \geq 2\mu_{\max}$$



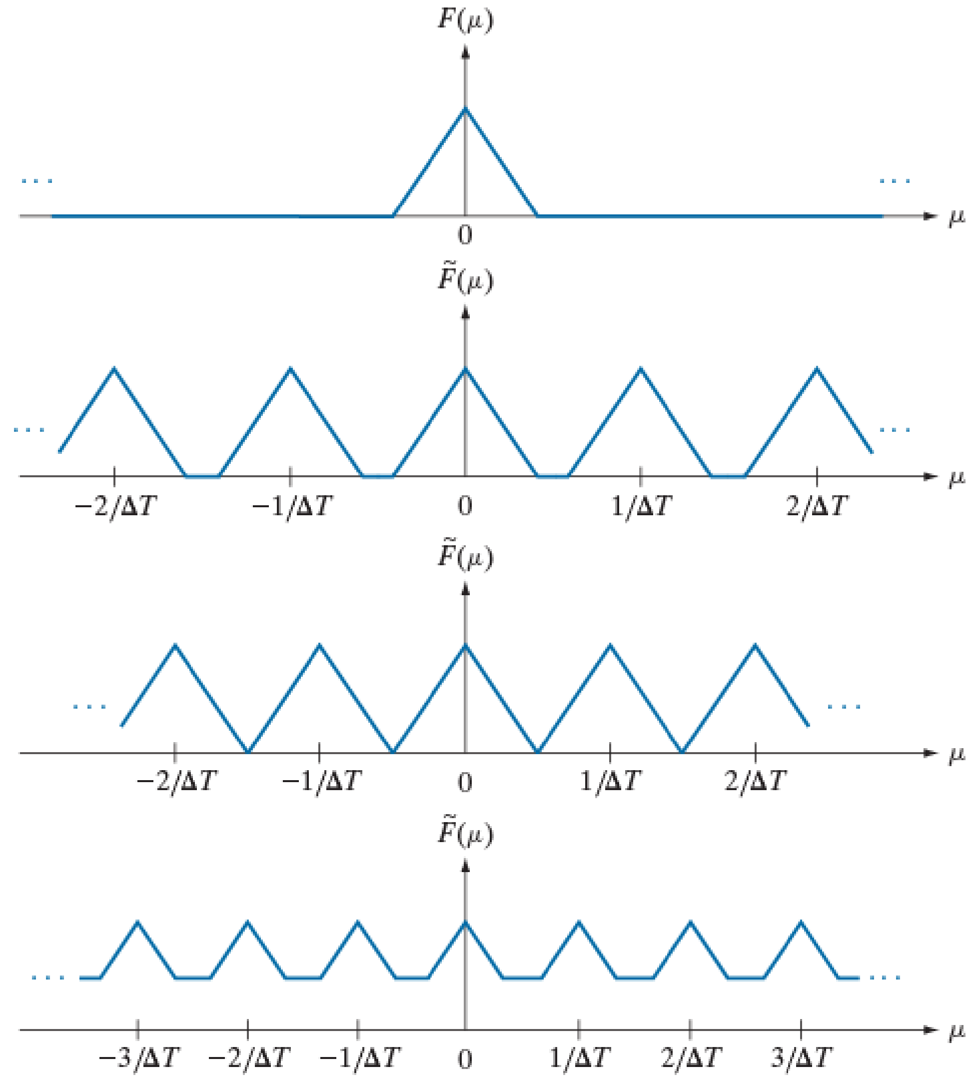
1D continuous signals (cont.)

FT of a continuous signal

Oversampling

Critical sampling with the Nyquist frequency

Undersampling
Aliasing appears



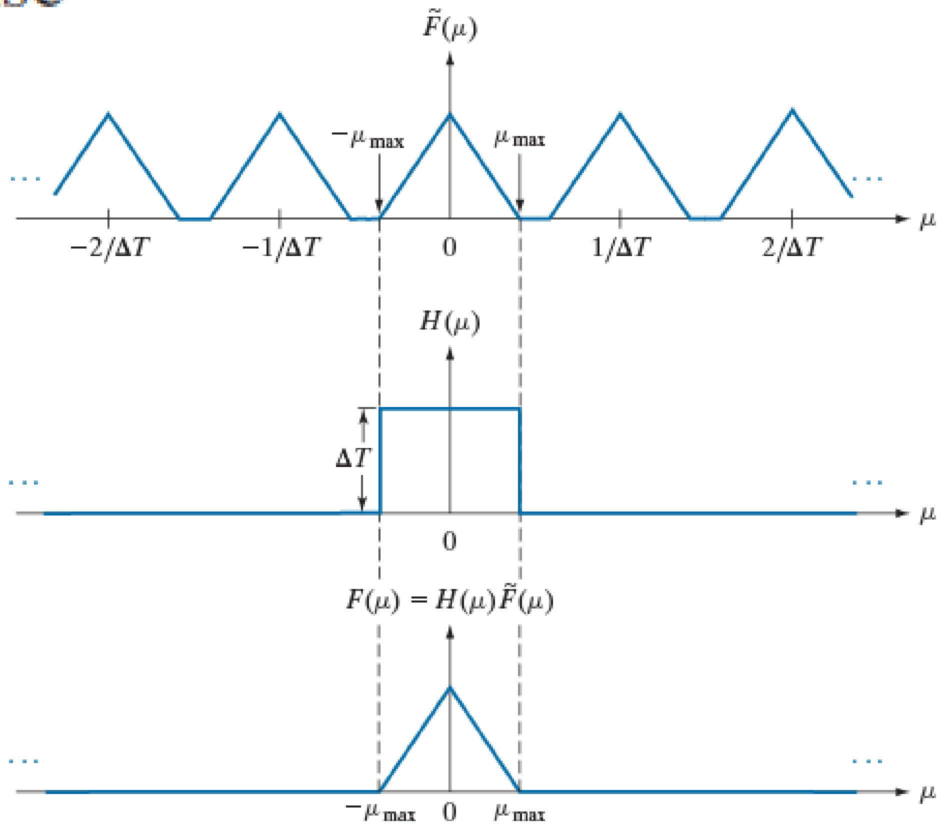
Reconstruction (correct sampling)

$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \le \mu \le \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$

$$F(\mu) = \tilde{F}(\mu) H(\mu)$$

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

$$f(t) = \tilde{f}(t) * T \sin c\left(\frac{t}{\Delta T}\right)$$



1D continuous signals (cont.)

- Reconstruction
  - Provided a correct sampling, the continuous signal may be perfectly reconstructed by its samples using inverse transform:

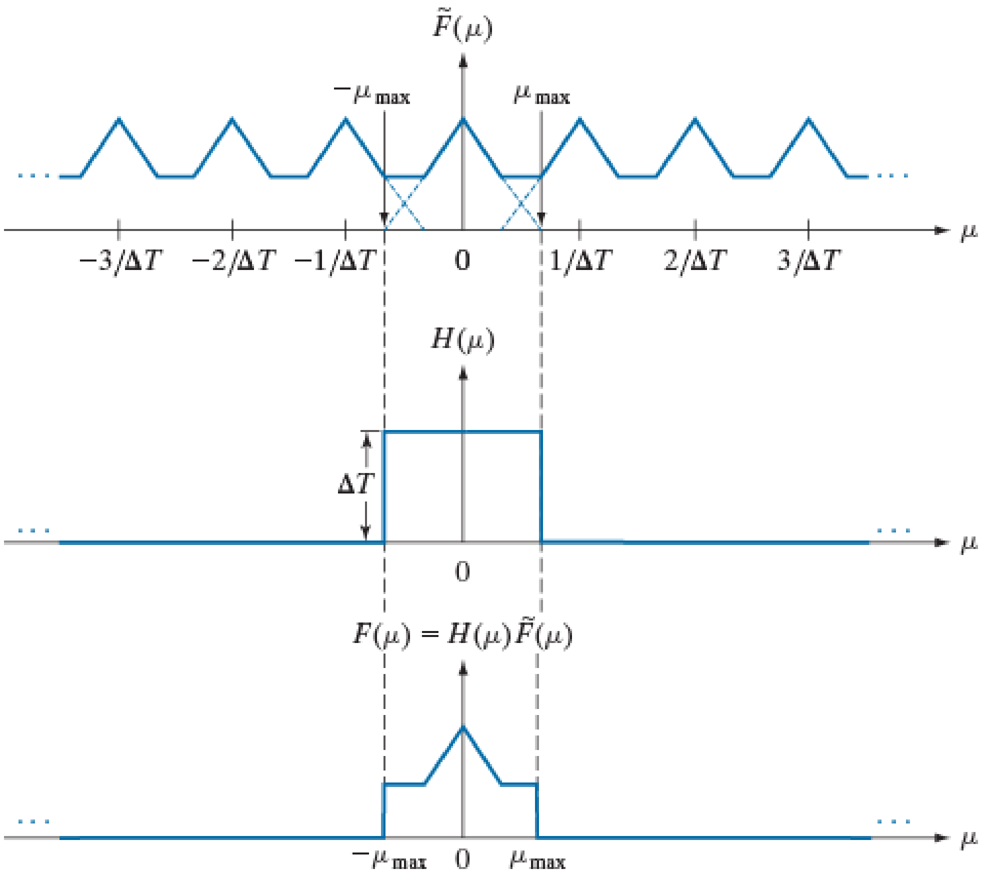
$$f(t) = \Im^{-1}\{F(\mu)\} = \Im^{-1}\left\{H(\mu)\tilde{F}(\mu)\right\}$$

$$= h(t) \star \tilde{f}(t)$$

$$f(t) = \sum_{n=-\infty}^{+\infty} f(n\Delta T)\ \mathrm{sinc}\left[\frac{(t - n\Delta T)}{n\Delta T}\right]$$

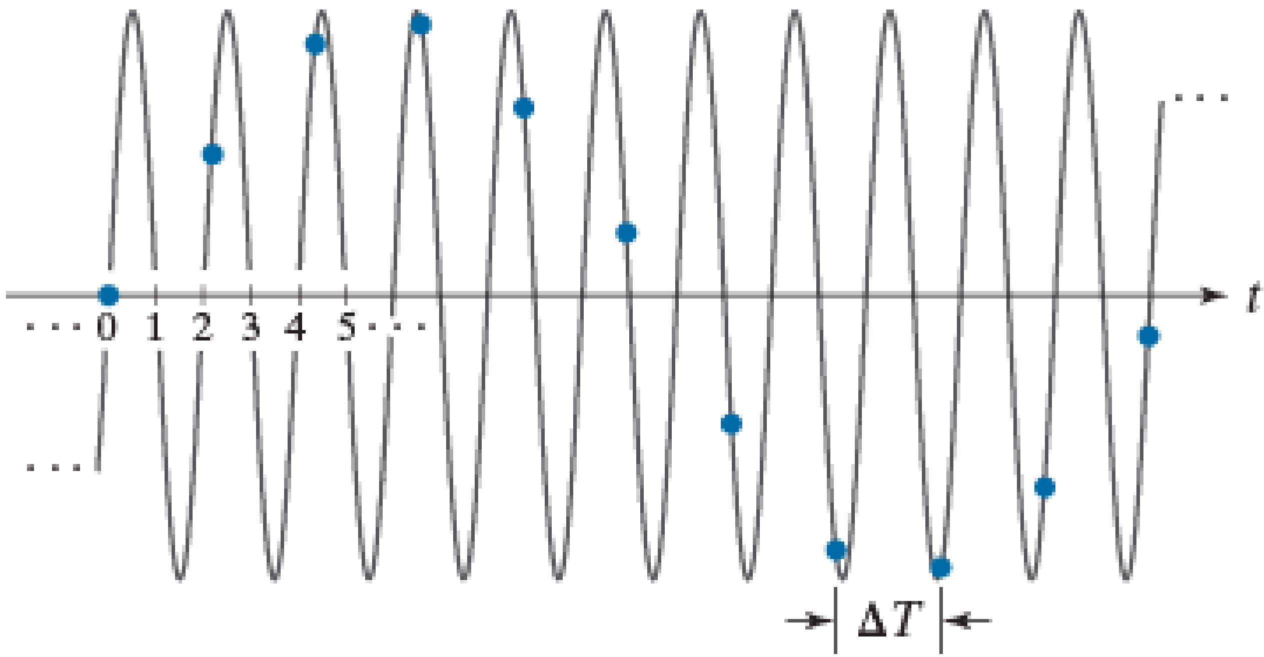
1D continuous signals (cont.)

- Aliasing, the reconstruction of the continuous signal is not correct.



1D continuous signals (cont.)

Aliased signal



The Discrete Fourier Transform

- The Fourier transform of a sampled (discrete) signal is a continuous function of the frequency.

$$\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{+\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

- For a N-length discrete signal, taking N samples of its Fourier transform at frequencies:

$$\mu_k = \frac{k}{N\Delta T}, \; k = 0, 1, .., N-1$$

provides the discrete Fourier transform (DFT) of the signal.

The Discrete Fourier Transform (cont..)

- This is possible since:

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt =$$

$$= \int_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f(t)\delta(t - n\Delta \mathrm{T}) e^{-j2\pi\mu t} dt =$$

$$= \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta \mathrm{T}) e^{-j2\pi\mu t} dt =$$

$$= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta \mathrm{T}}$$

$$\mu_k = \frac{k}{N\Delta T}, \; k = 0, 1, .., N-1$$

The Discrete Fourier Transform (cont.)

- DFT pair of signal $f[n]$ of length N.

$$F[k]=\sum_{n=0}^{N-1} f[n]e^{-j\frac{2\pi nk}{N}}, \qquad 0\le k\le N-1$$

$$f[n]=\frac{1}{N}\sum_{n=0}^{N-1} F[k]e^{j\frac{2\pi nk}{N}}, \qquad 0\le n\le N-1$$

The Discrete Fourier Transform (cont.)

- Property
  - The DFT of a N-length $f[n]$ signal is periodic with period N.

$$F[k+N]=F[k]$$

  - This is due to the periodicity of the complex exponential:

$$F[k]=\sum_{n=0}^{N-1} f[n]e^{-j\frac{2\pi nk}{N}}$$

$$e^{-j\frac{2\pi n(k+N)}{N}}=e^{-j\frac{2\pi nk}{N}-j2\pi n}=e^{-j\frac{2\pi nk}{N}}e^{-j2\pi n}=e^{-j\frac{2\pi nk}{N}}$$

The Discrete Fourier Transform (cont.)

- DFT of a N-length $f[n]$ signal is periodic with period N.

$$F[k+N] = \sum_{n=0}^{N-1} f[n] \cdot e^{-j\frac{2\pi kn}{N}} \cdot e^{-j2\pi n}$$

$e^{-j2\pi n}$ is periodic with period N. So we have:

$$F[k+N] = F[k]$$

- Which can be simplified assuming:

$$(w_N)^n = \left(e^{-j\frac{2\pi}{N}}\right)^n \Leftrightarrow w_N^n = e^{-j\frac{2\pi n}{N}}$$

The Discrete Fourier Transform (cont.)

- DFT pair of signal $f[n]$ of length N may be expressed in matrix-vector form.

$$F[k] = \sum_{n=0}^{N-1} f[n] w_N^{nk}, \qquad 0 \le k \le N-1$$

$$f[n] = \frac{1}{N} \sum_{n=0}^{N-1} F[k] w_N^{-nk}, \qquad 0 \le n \le N-1$$

$$w_N = e^{-j\frac{2\pi}{N}}$$

The Discrete Fourier Transform (cont.)

$$\mathbf{F} = \mathbf{A}\mathbf{f} \qquad w_N^{nk} = e^{-j\frac{2\pi kn}{N}}$$

$$\mathbf{A} = \begin{bmatrix} \left(w_N^0\right)^0 & \left(w_N^0\right)^1 & \left(w_N^0\right)^2 & \cdots & \left(w_N^0\right)^{N-1} \\ \left(w_N^1\right)^0 & \left(w_N^1\right)^1 & \left(w_N^1\right)^2 & \cdots & \left(w_N^1\right)^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(w_N^{N-1}\right)^0 & \left(w_N^{N-1}\right)^1 & \left(w_N^{N-1}\right)^2 & \cdots & \left(w_N^{N-1}\right)^{N-1} \end{bmatrix}$$

$$\mathbf{f} = \left[f[0], f[1], \ldots, f[N-1]\right]^T, \; \mathbf{F} = \left[F[0], F[1], \ldots, F[N-1]\right]^T$$

The Discrete Fourier Transform (cont.)

Example for N=4 (for $k = 0, e^{-j\frac{2\pi(0)n}{N}} = e^0 = 1$)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

f[n] = [1, 2, 3, 4]

$$F = [10,*,*,*]$$

The Discrete Fourier Transform (cont.)

The inverse DFT is then expressed by:

$$\mathbf{f} = \mathbf{A}^{-1}\mathbf{F}$$

$$\mathbf{A}^{-1} = \frac{1}{N}\left(\mathbf{A}^{*}\right)^{T} = \frac{1}{N}\left(\begin{bmatrix} \left(w_N^0\right)^0 & \left(w_N^0\right)^1 & \left(w_N^0\right)^2 & \cdots & \left(w_N^0\right)^{N-1} \\ \left(w_N^1\right)^0 & \left(w_N^1\right)^1 & \left(w_N^1\right)^2 & \cdots & \left(w_N^1\right)^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(w_N^{N-1}\right)^0 & \left(w_N^{N-1}\right)^1 & \left(w_N^{N-1}\right)^2 & \cdots & \left(w_N^{N-1}\right)^{N-1} \end{bmatrix}^{*}\right)^{T}$$

This is derived by the complex exponential sum property.

Also, since A is symmetric: $A^{-1} = \frac{1}{N} A^{*}$

Linear convolution

$$f[n]=\{\underline{1},\ 2,\ 2\},\ \ h[n]=\{\underline{1},\ -1\},\ \ N_1=3, N_2=2$$

$$g[n]=f[n]*h[n]=\sum_{m=-\infty}^{+\infty} f[m]h[n-m]$$

is of length N=N_1+N_2-1=4

Linear convolution (cont.)

$$f[n]=\{\underline{1},\ 2,\ 2\},\ \ h[n]=\{\underline{1},\ -1\},\ \ N_1=3, N_2=2$$

$$g[n]=f[n]*h[n]=\sum_{m=-\infty}^{+\infty} f[m]h[n-m]$$

	$f[m]$							$g[n]$
$n=0$	$h[0-m]$	-1	1				$\rightarrow$	1
$n=1$	$h[1-m]$		-1	1			$\rightarrow$	1
$n=2$	$h[2-m]$			-1	1		$\rightarrow$	0
$n=3$	$h[3-m]$				-1	1	$\rightarrow$	-2

$$g[n]=\{\underline{1},\ 1,\ 0,\ -2\}$$

Circular shift

- Signal $x[n]$ of length N.
- A circular shift ensures that the resulting signal will keep its length N.
- It is a shift modulo N denoted by

$$x[(n-m)_N] = x[(n-m) \bmod N]$$

- Example: $x[n]$ is of length N=8.

$$x[(-2)_N] = x[(-2)_8] = x[6]$$
$$x[(10)_N] = x[(10)_8] = x[2]$$

Circular convolution

$$f[n]=\{\underline{1},\ 2,\ 2\},\ \ h[n]=\{\underline{1},\ -1\},\ \ N_1=3, N_2=2$$

$$g[n]=f[n] \circledast h[n]=\sum_{m=-\infty}^{+\infty} f[m]h[(n-m)_N]$$

Circular shift modulo N

The result is of length $N=\max\{N_1,N_2\}=3$

Circular convolution (cont.)

$$f[n]=\{\underline{1},\ 2,\ 2\},\ \ h[n]=\{\underline{1},\ -1\},\ \ N_1=3, N_2=2$$

$$g[n]=f[n]✪h[n]=\sum_{m=-\infty}^{+\infty} f[m]h[(n-m)_N]$$

	$f[m]$		1	2	2	$g[n]$
$n=0$	$h[(0-m)_N]$	-1	1		-1	-1
$n=1$	$h[(1-m)_N]$		-1	1		1
$n=2$	$h[(2-m)_N]$			-1	1	0

$$g[n]=\{-\underline{1},\ 1,\ 0,\}$$

DFT and convolution

$$g[n] = f[n] \circledast h[n] \leftrightarrow G[k] = F[k]H[k]$$

- The property holds for the circular convolution.
- In signal processing we are interested in linear convolution.
- Is there a similar property for the linear convolution?

DFT and convolution (cont.)

$$g[n]=f[n] \circledast h[n] \leftrightarrow G[k]=F[k]H[k]$$

- Let $f[n]$ be of length N_1 and $h[n]$ be of length N_2.
- Then $g[n]=f[n]*h[n]$ is of length N_1+N_2-1.
- If the signals are zero-padded to length $N=N_1+N_2-1$ then their circular convolution will be the same as their linear convolution:

$$\tilde{g}[n]=\tilde{f}[n]*\tilde{h}[n] \leftrightarrow \tilde{G}[k]=\tilde{F}[k]\tilde{H}[k]$$

Zero-padded signals

DFT and convolution (cont.)

$$f[n]=\{\underline{1},\ 2,\ 2\},\ \ h[n]=\{\underline{1},\ -1\},\ \ N_1=3, N_2=2$$

Zero-padding to length $N=N_1+N_2-1=4$

$$\tilde{f}[n]=\{\underline{1},\ 2,\ 2,\ 0\},\ \ \tilde{h}[n]=\{\underline{1},\ -1,\ 0,\ 0\}$$

$f[m]$				1	2	2	0	$g[n]$
$h[(n-0)_4]$	0	0	-1	1	0	0	-1	1
$h[(n-1)_4]$		0	0	-1	1	0	0	1
$h[(n-2)_4]$			0	0	-1	1	0	0
$h[(n-3)_4]$				0	0	-1	1	-2

The result is the same as the linear convolution.

DFT and convolution (cont.)

Verification using DFT

$$\tilde{\mathbf{F}} = \mathbf{A}\tilde{\mathbf{f}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -1-j2 \\ 1 \\ -1+j2 \end{bmatrix}$$

$$\tilde{\mathbf{H}} = \mathbf{A}\tilde{\mathbf{h}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1+j \\ 2 \\ 1-j \end{bmatrix}$$

DFT and convolution (cont.)

$$\tilde{G}[k] = \tilde{F}[k]\,\tilde{H}[k]$$

Element-wise multiplication

$$\tilde{\mathbf{G}} = \tilde{\mathbf{F}} \times \tilde{\mathbf{H}} = \begin{bmatrix} 5 \times 0 \\ (-1 - j2) \times (1 + j) \\ 1 \times 2 \\ (-1 + j2) \times (1 - j) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 - j3 \\ 2 \\ 1 + j3 \end{bmatrix}$$

DFT and convolution (cont.)

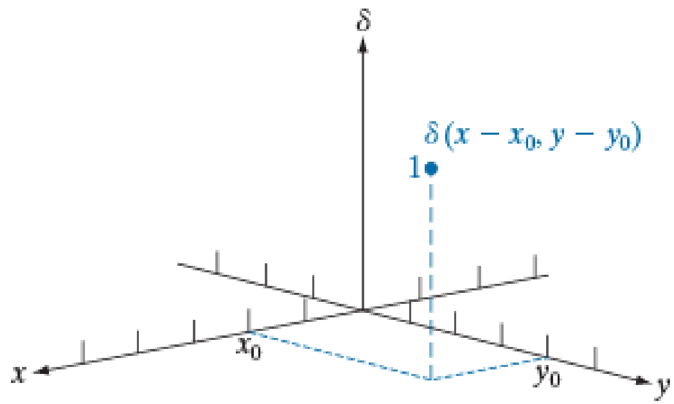
Inverse DFT of the result

$$\tilde{g} = A^{-1}\tilde{G} = \frac{1}{4}\left(A^{*}\right)^{\mathrm{T}}\tilde{G} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 0 \\ 1-3j \\ 2 \\ 1+3j \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}$$

The same result as their linear convolution.

2D continuous signals

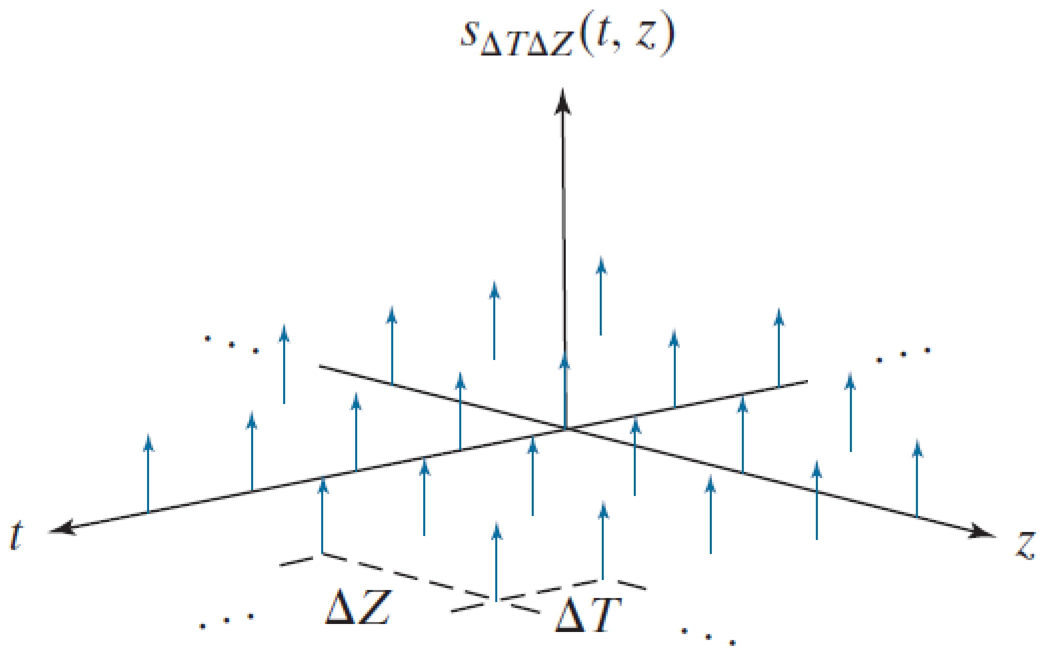
$$\delta(x-x_0, y-y_0)=\begin{cases}+\infty, & x=x_0, y=y_0 \\ 0 & \text{otherwise}\end{cases}$$



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(x-x_0, y-y_0)dydx = f(x_0, y_0) \quad \text{Selection prop.}$$

$$\delta(x-x_0, y-y_0)=\delta(x-x_0)\delta(y-y_0) \quad \text{seperability}$$

2D continuous signals (cont.)



The 2D impulse train is also separable:

$$\boldsymbol{S}_{\Delta X \Delta Y}(x, y) = \boldsymbol{S}_{\Delta X}(x)\boldsymbol{S}_{\Delta Y}(y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta X, y - nDY)$$

2D continuous signals (cont.)

- The Fourier transform of a continuous 2D signal $f(x,y)$.

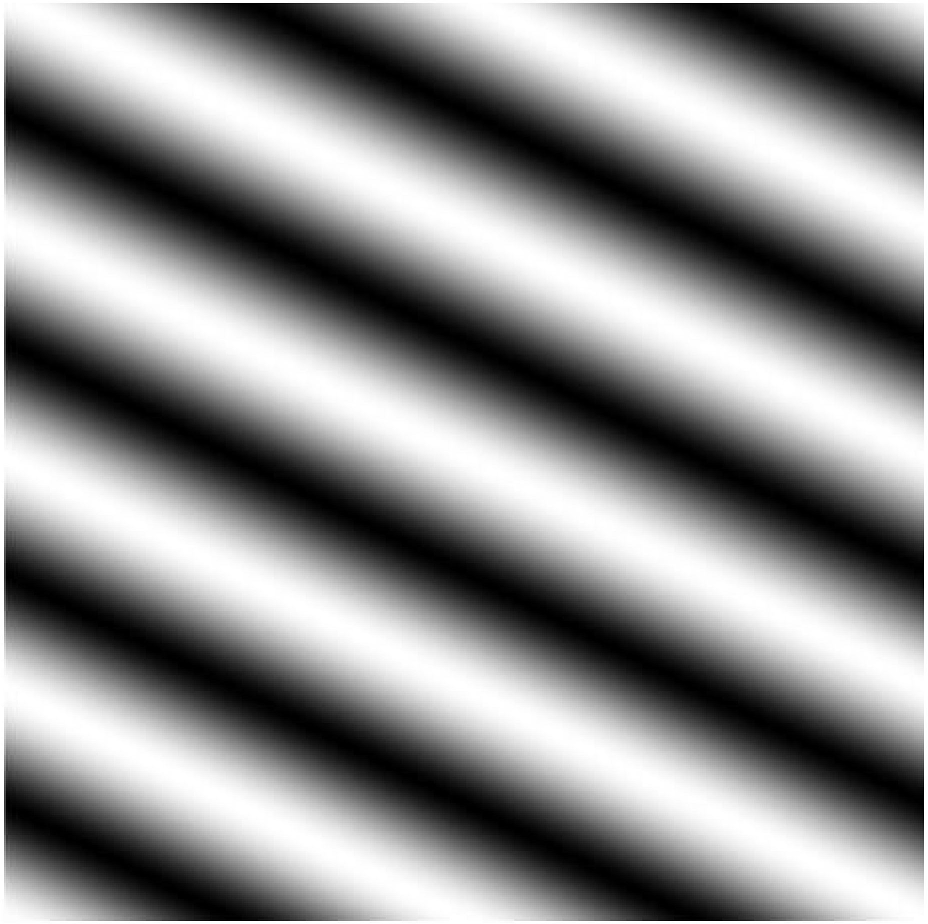
$$F(\mu,\nu)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi(\mu x+\nu y)}dydx$$

$$f(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\mu,\nu)e^{j2\pi(\mu x+\nu y)}d\nu d\mu$$

2D basis functions (example)

- Basis functions example of 2D continuous FT (real part)

$$e^{-j2\pi(\mu x+vy)}$$



2D basis functions (example)

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2D basis functions (example)

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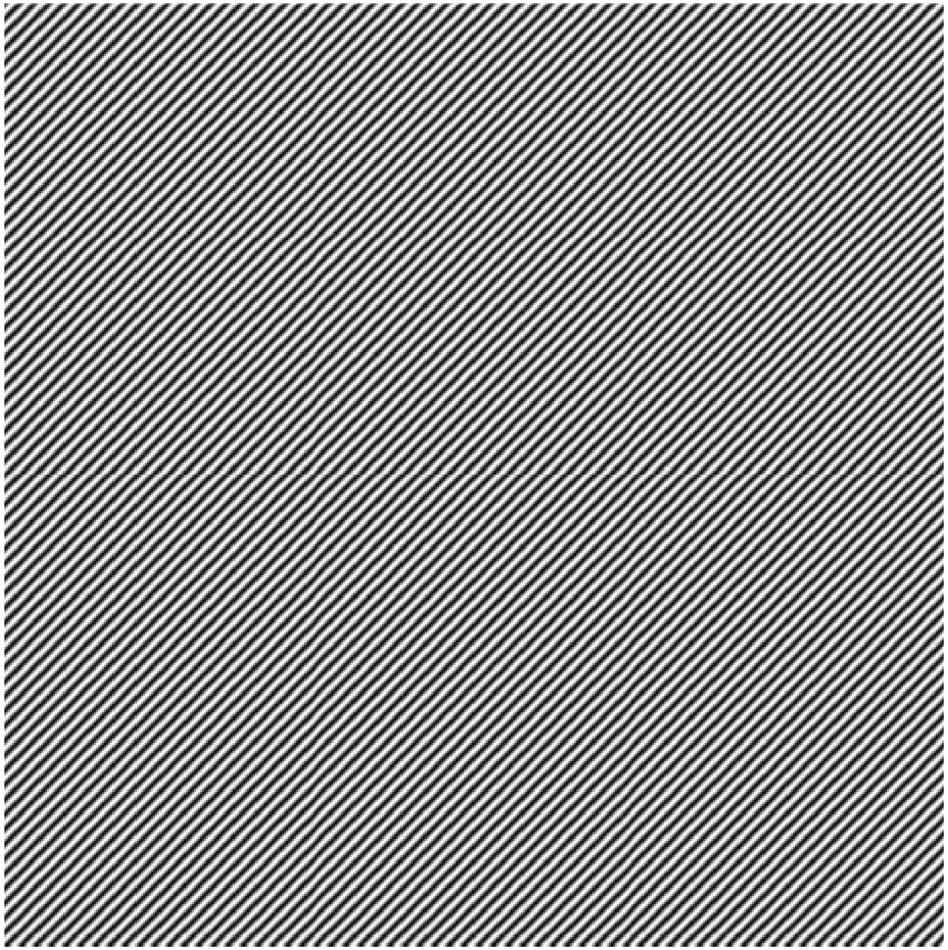
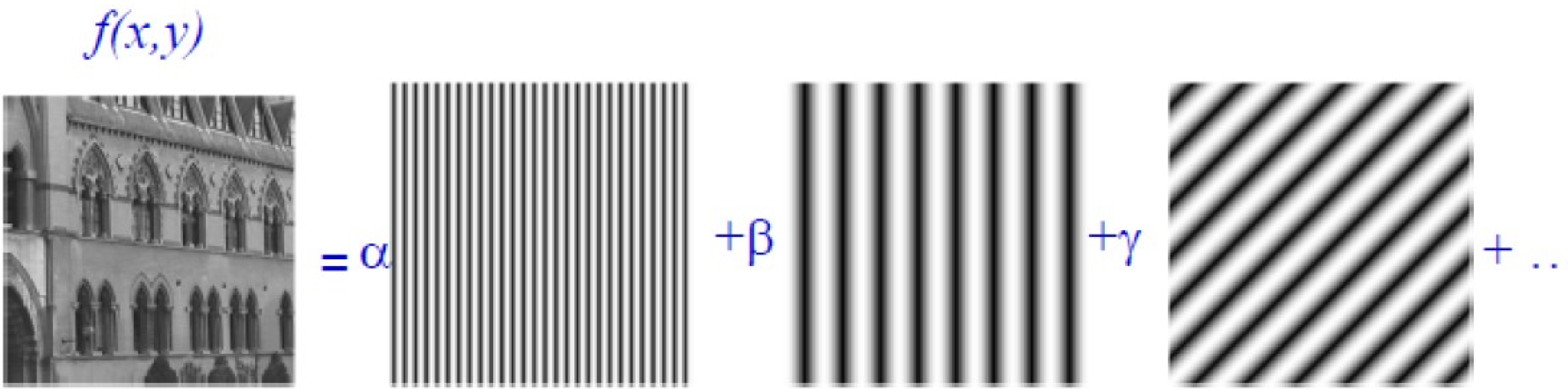


2D basis functions (example)

- Basis functions example of 2D continuous FT (real part)



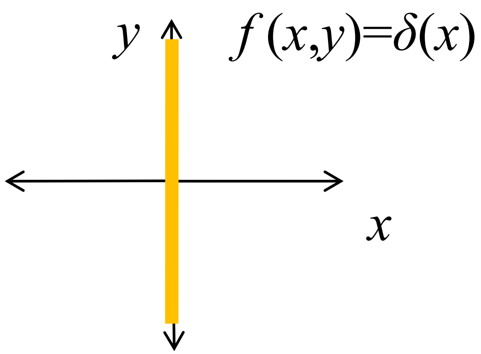
2D continuous signals (cont.)

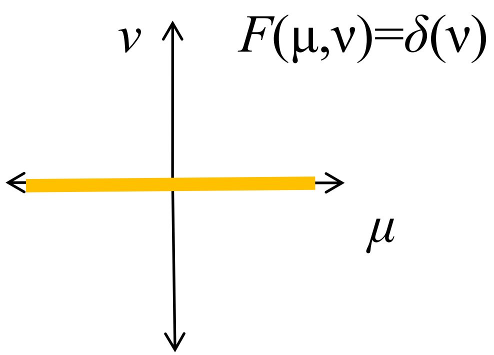
- Example: FT of $f(x,y)=\delta(x)$

$$F(\mu,\nu)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\delta(x)e^{-j2\pi(\mu x+\nu y)}dydx$$

$$=\int_{-\infty}^{+\infty}\delta(x)e^{-j2\pi\mu x}dx\int_{-\infty}^{+\infty}e^{-j2\pi\nu y}dy$$

$$=\int_{-\infty}^{+\infty}e^{-j2\pi\nu y}dy \;=\delta(\nu)$$





2D continuous signals (cont.)

- Reminder

$$\Im\{\delta(t)\} = F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt = \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t)\, dt = e^{-j2\pi\mu 0}$$

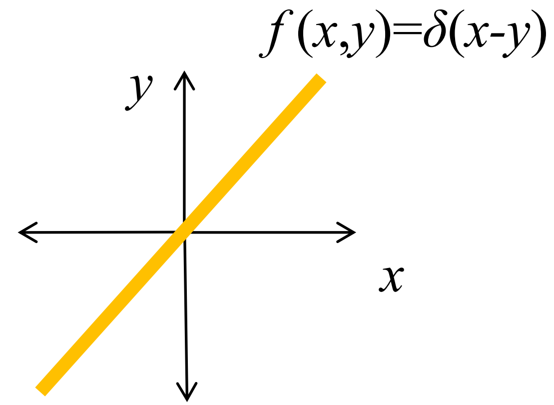
$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0)$$

- Hence, the term that follows is '1'.

$$\int_{-\infty}^{+\infty} \delta(x) e^{-j2\pi\mu x} dx$$

2D continuous signals (cont.)

- Example: FT of $f(x,y)=\delta(x-y)$

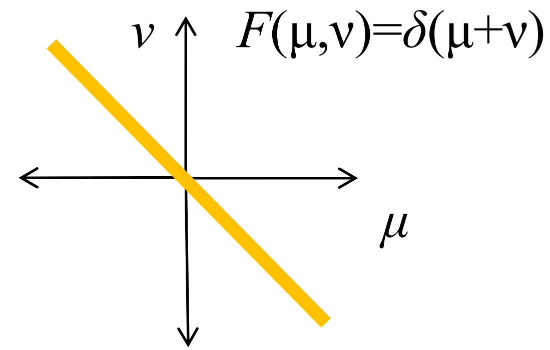


$$F(\mu,\nu)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\delta(x-y)e^{-j2\pi(\mu x+\nu y)}dydx$$

$$=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\delta(x-y)e^{-j2\pi\mu x}dx\right]e^{-j2\pi\nu y}dy$$

$$=\int_{-\infty}^{+\infty}e^{-j2\pi\mu y}e^{-j2\pi\nu y}dy=\int_{-\infty}^{+\infty}e^{-j2\pi(\mu+\nu)y}dy$$

$$=\delta(\mu+\nu)$$



2D continuous signals (cont.)

- Reminder

$$\Im\{\delta(t-t_0)\} = F(\mu) = \int_{-\infty}^{\infty} \delta(t-t_0)e^{-j2\pi\mu t}dt = \int_{-\infty}^{\infty} e^{-j2\pi\mu t}\,\delta(t-t_0)\,dt = e^{-j2\pi\mu t_0}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

- Hence, the term that follows is:

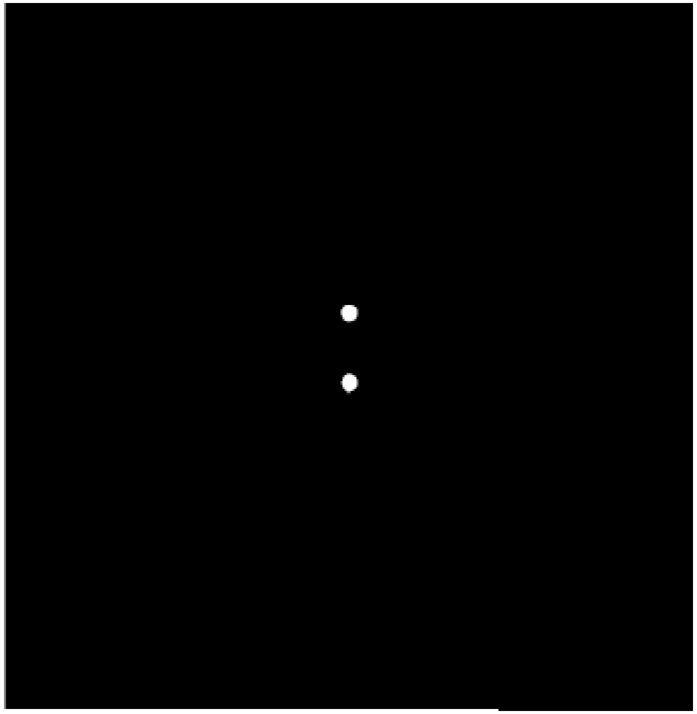
$$\left[\int_{-\infty}^{+\infty} \delta(x-y)e^{-j2\pi\mu x}dx\right] = e^{-j2\pi\mu y}$$

2D continuous signals (cont.)

- Example

$$f(x,y) = \delta(x, y-a) + \delta(x, y+a)$$

$f(x,y)$

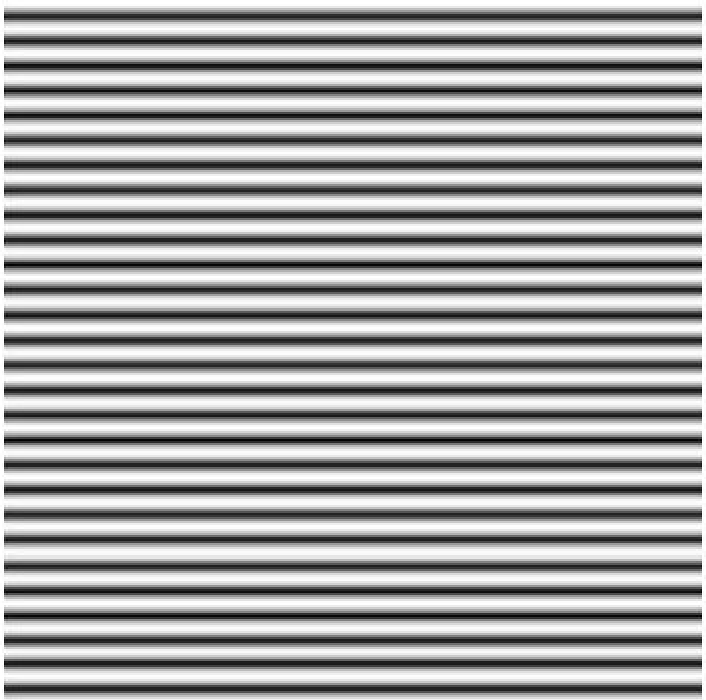


2D continuous signals (cont.)

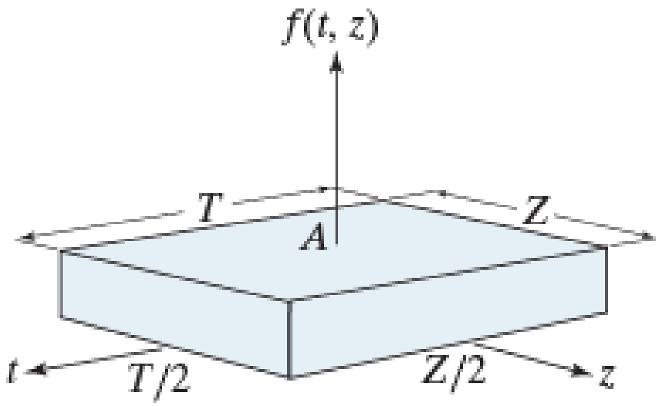
$$F(\mu, v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\delta(x, y-a) + \delta(x, y+a)] e^{-j2\pi(\mu+vy)} dy dx$$

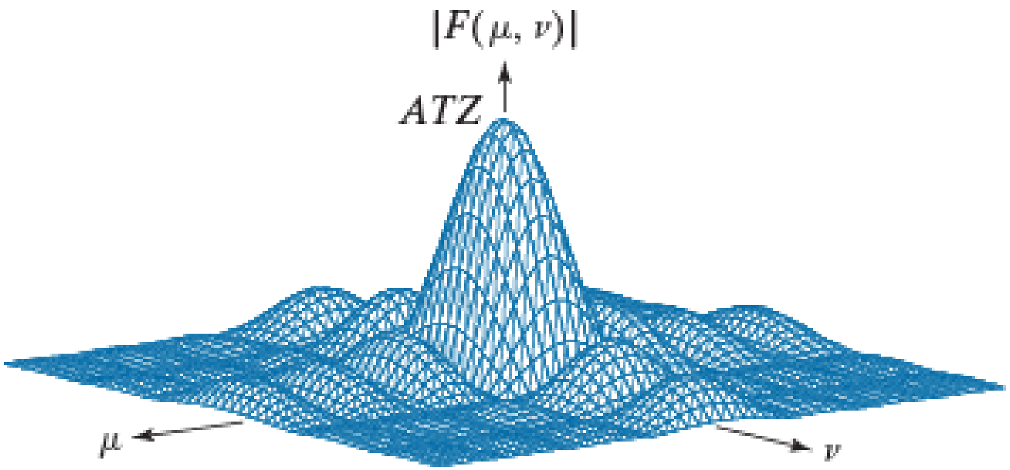
$$= e^{-j2\pi a v} + e^{j2\pi a v} = 2cos2\pi\alpha v$$

$F(\mu, v)$



2D continuous signals (cont.)





$$f(x,y) = \mathrm{A}\,P_{W/2,W/2}(x,y) \leftrightarrow F(\mu,\nu) = \mathrm{A}W^2 \frac{\sin(\pi\mu W)}{(\pi\mu W)} \frac{\sin(\pi\nu W)}{(\pi\nu W)}$$

2D continuous signals (cont.)

- 2D continuous convolution

$$f(x,y)*h(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x-\alpha,y-\beta)h(\alpha,\beta)\,d\alpha\, d\beta$$

- We will examine the discrete convolution in more detail.
- Convolution property

$$f(x,y)*h(x,y) \leftrightarrow F(\mu,\nu)H(\mu,\nu)$$

2D continuous signals (cont.)

- 2D sampling is accomplished by

$$\boldsymbol{s}_{\Delta X \Delta Y}(x, y) = \boldsymbol{s}_{\Delta X}(x)\boldsymbol{s}_{\Delta Y}(y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta X, y - nDY)$$

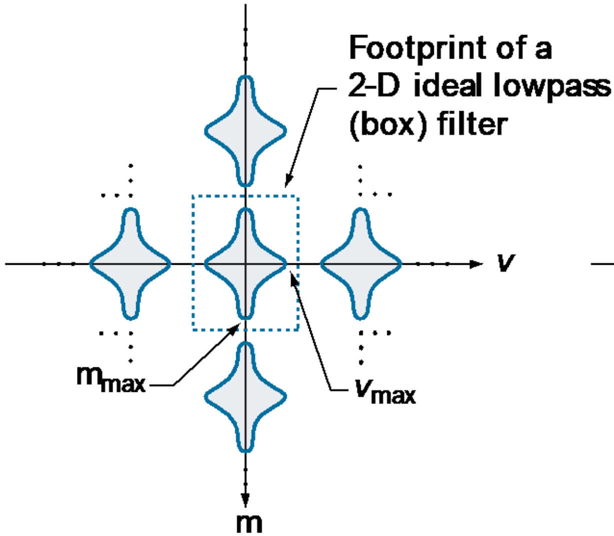
- The FT of the sampled 2D signal consists of repetitions of the spectrum of the 1D continuous signal.

$$\tilde{F}(\mu, \nu) = \frac{1}{\Delta X}\frac{1}{\Delta Y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathrm{F}(\mu - \frac{m}{\Delta X}, \nu - \frac{n}{\Delta X})$$

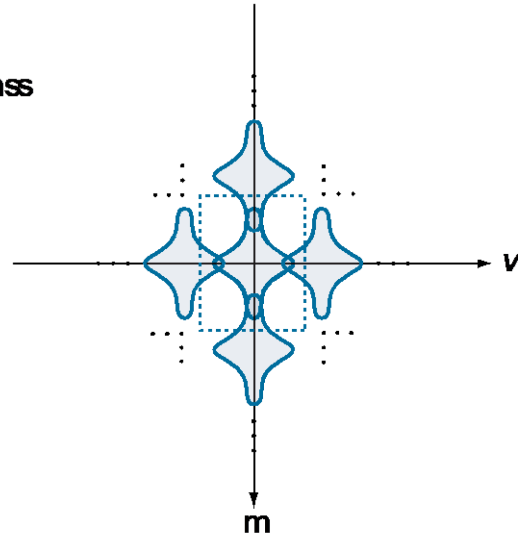
2D continuous signals (cont.)

- The Nyquist theorem involves both the horizontal and vertical frequencies.

$$\frac{1}{\Delta X} \geq 2\mu_{\max}, \quad \frac{1}{\Delta Y} \geq 2v_{\max}$$



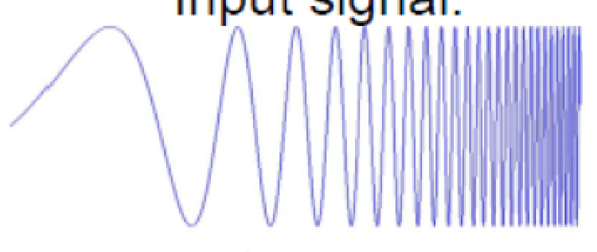
Over-sampled



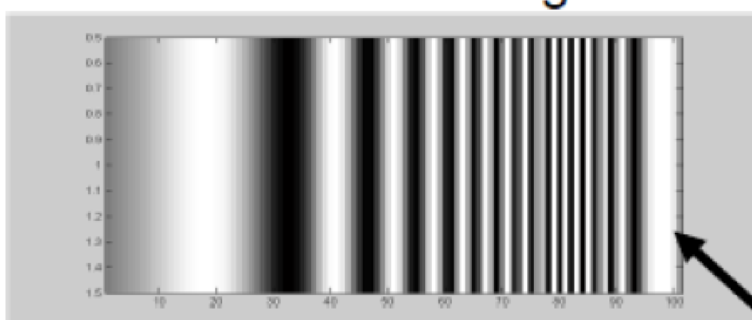
Under-sampled

Aliasing

Input signal:



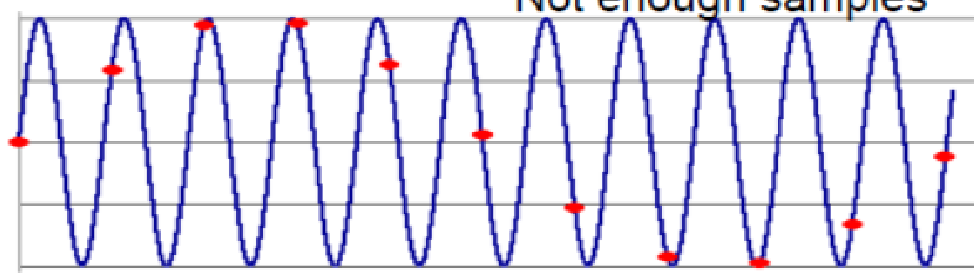
Plot as image:



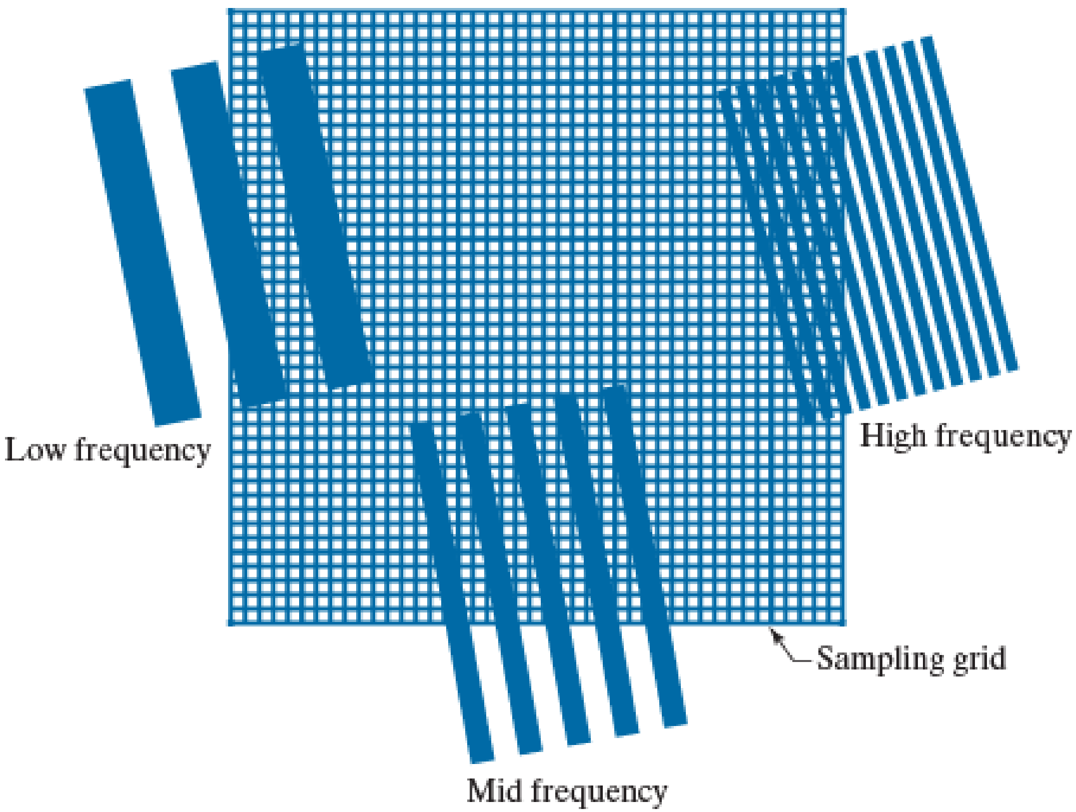
x = 0:.05:5; imagesc(sin((2.^x).*x))

Aliasing

Not enough samples



Aliasing



Aliasing and image resampling

mages taken from Gonzalez & Woods, Digital Image Processing (2002)

a b c

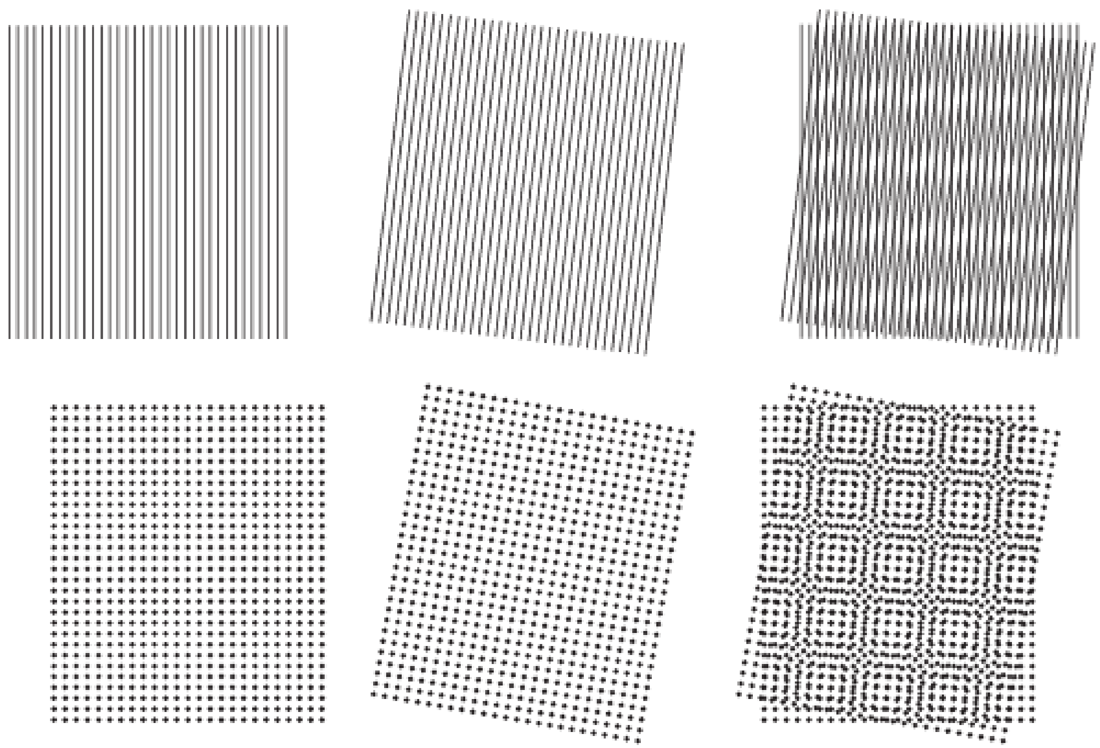
FIGURE 4.19 Illustration of aliasing on resampled natural images. (a) A digital image of size 772 × 548 pixels with visually negligible aliasing. (b) Result of resizing the image to 33% of its original size by pixel deletion and then restoring it to its original size by pixel replication. Aliasing is clearly visible. (c) Result of blurring the image in (a) with an averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Aliasing - Moiré Patterns

- Effect of sampling a scene with periodic or nearly periodic components (e.g. overlapping grids, TV raster lines and stripped materials).
- In image processing the problem arises when scanning media prints (e.g. magazines, newspapers).
- The problem is more general than sampling artifacts.

Aliasing - Moiré Patterns (cont.)

- Superimposed grid drawings (not digitized) produce the effect of new frequencies not existing in the original components.



Aliasing - Moiré Patterns (cont.)

- In the printing industry the problem comes when scanning photographs from the superposition of:
  - The sampling lattice (usually horizontal and vertical).
  - Dot patterns on the newspaper image.

Aliasing - Moiré Patterns (cont.)

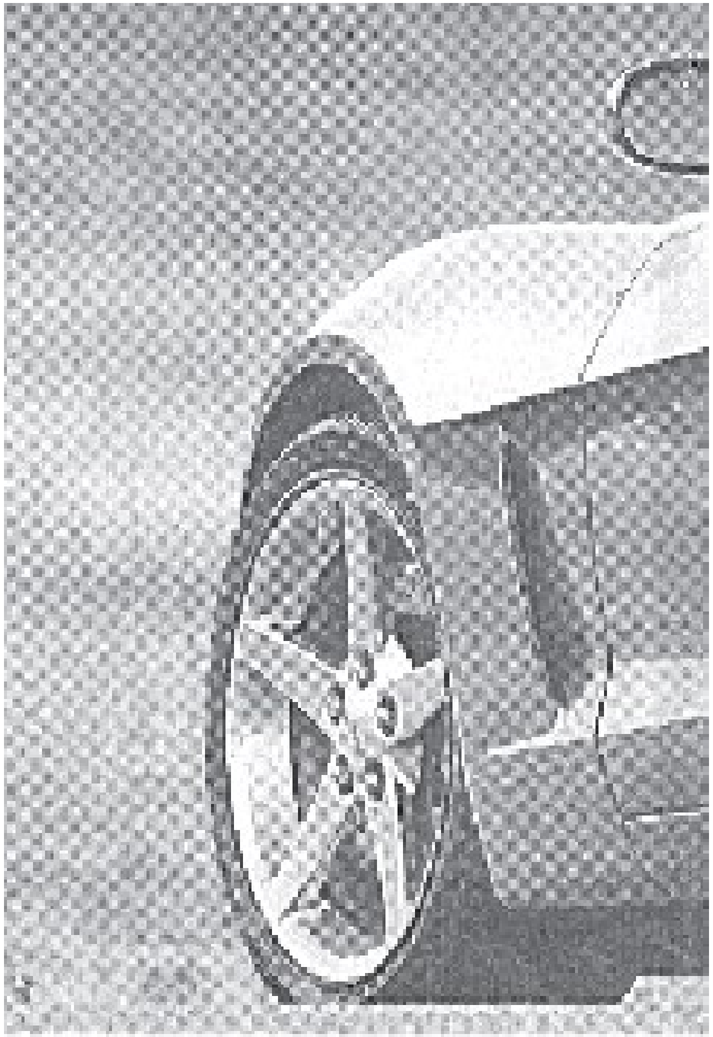
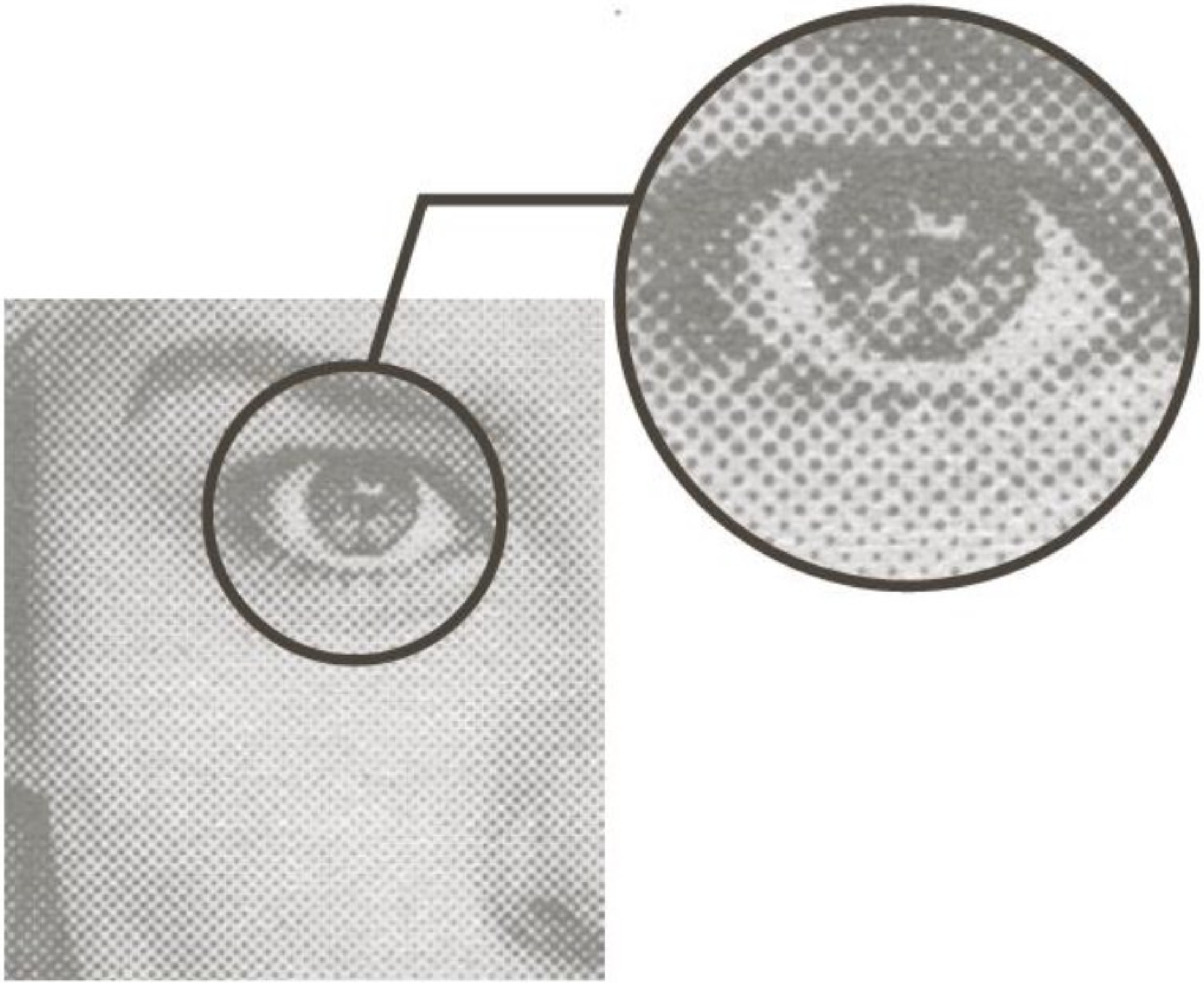


FIGURE 4.21 A newspaper image of size 246×168 pixels sampled at 75 dpi showing a moiré pattern. The moiré pattern in this image is the interference pattern created between the $\pm 45°$ orientation of the halftone dots and the north–south orientation of the sampling grid used to digitize the image.

Aliasing - Moiré Patterns (cont.)



2D Discrete Fourier Transform (2D DFT)

- 2D DFT pair of image $f[m,n]$ of size MxN.

$$F[k,l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

$$f[m,n] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

$$\begin{cases} 0 \le k \le M-1 \\ 0 \le l \le N-1 \end{cases}, \quad \begin{cases} 0 \le m \le M-1 \\ 0 \le n \le N-1 \end{cases}$$

2D Discrete Fourier Transform (2D DFT)

Separability of the 2D DFT:

- We can express the 2D DFT as two 1D DFTs:
- First, perform a 1D DFT along the columns and then along the rows (or vice versa).

2D Discrete Fourier Transform (2D DFT)

2D DFT can be represented in matrix form:

- Reminder for 1D: $\mathbf{F} = \mathbf{A}\mathbf{f}$ $\qquad w_N^{nk} = e^{-j\frac{2\pi kn}{N}}$

$$\mathbf{A} = \begin{bmatrix} \left(w_N^0\right)^0 & \left(w_N^0\right)^1 & \left(w_N^0\right)^2 & \cdots & \left(w_N^0\right)^{N-1} \\ \left(w_N^1\right)^0 & \left(w_N^1\right)^1 & \left(w_N^1\right)^2 & \cdots & \left(w_N^1\right)^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(w_N^{N-1}\right)^0 & \left(w_N^{N-1}\right)^1 & \left(w_N^{N-1}\right)^2 & \cdots & \left(w_N^{N-1}\right)^{N-1} \end{bmatrix}$$

2D Discrete Fourier Transform (2D DFT)

2D DFT can be represented in matrix form:

- In a similar fashion for 2D we employ the same matrix A:

$$F = \mathrm{A}f\mathrm{A}^{\mathrm{T}}$$

- Where now *F, f* are now NxN matrices:
    - Equivalent: $F = AfA$, since $A = A^T$

The 2D DFT (cont.)

- All of the properties of 1D DFT hold.
- Particularly:
  - Let $f[m,n]$ be of size M_1xN_1 and $h[m,n]$ of size M_2xN_2 .
  - If the signals are zero-padded to size (M_1+M_2-1)x(N_1+N_2-1) then their circular convolution will be the same as their linear convolution and:

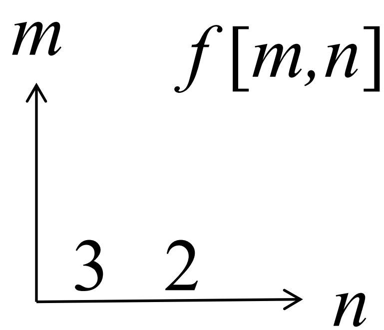
$$\tilde{g}[m,n] = \tilde{f}[m,n] * \tilde{h}[m,n] \leftrightarrow \tilde{G}[k,l] = \tilde{F}[k,l]\tilde{H}[k,l]$$

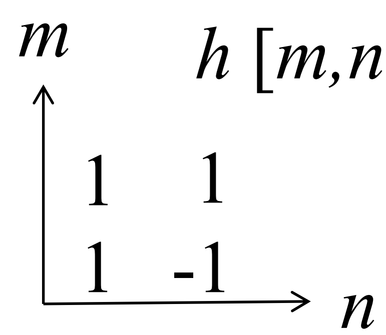
Fast Fourier Transform (FFT)

- Another reason the Fourier Transform is used in Digital Image Processing is the Fast Fourier Transform (FFT) algorithm.

- Reduces the complexity from $O(\,N^4)\ to$ $O(N^2 \log N^2)$

2D discrete convolution

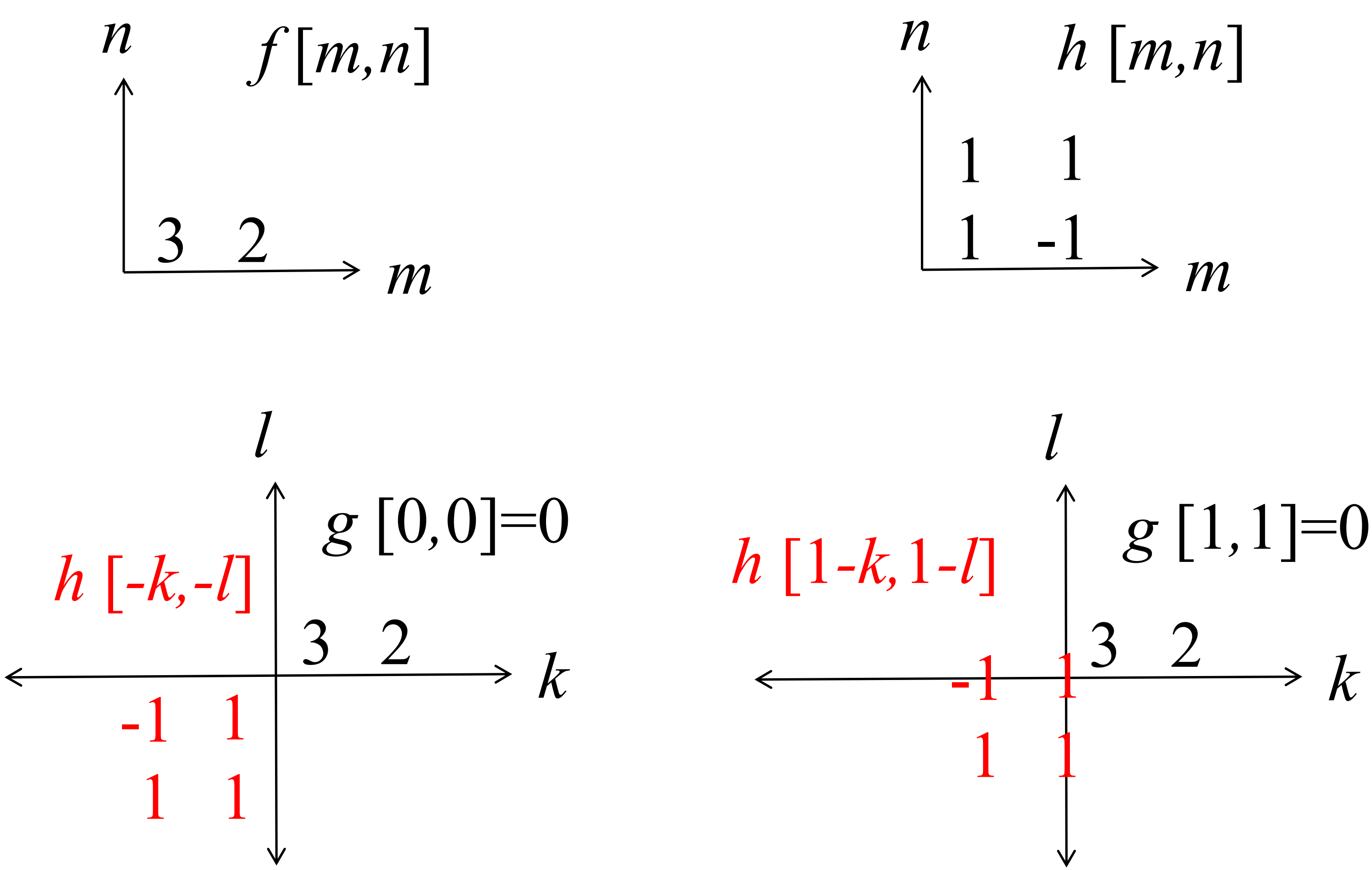




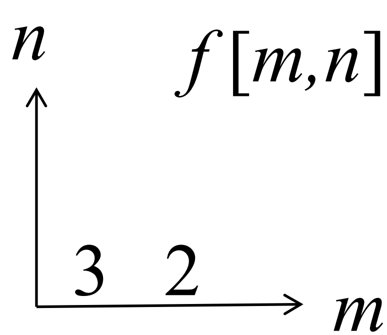
$$g[m,n] = f[m,n] * h[m,n] = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} f[k,l]h[m-k,n-l]$$

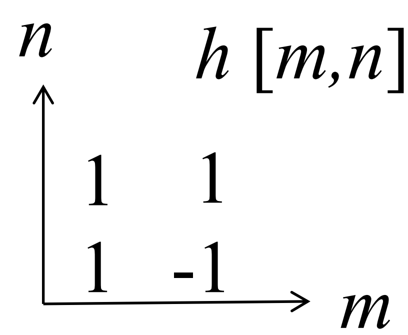
- Take the symmetric of one of the signals with respect to the origin.
- Shift it and compute the sum at every position $[m,n]$.

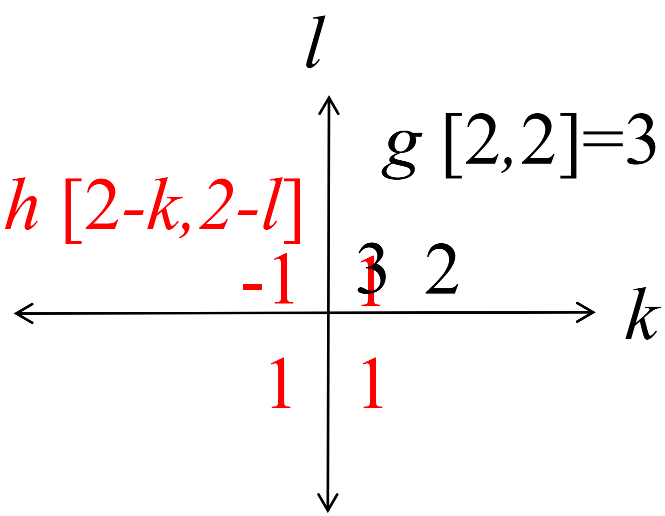
2D discrete convolution (cont.)

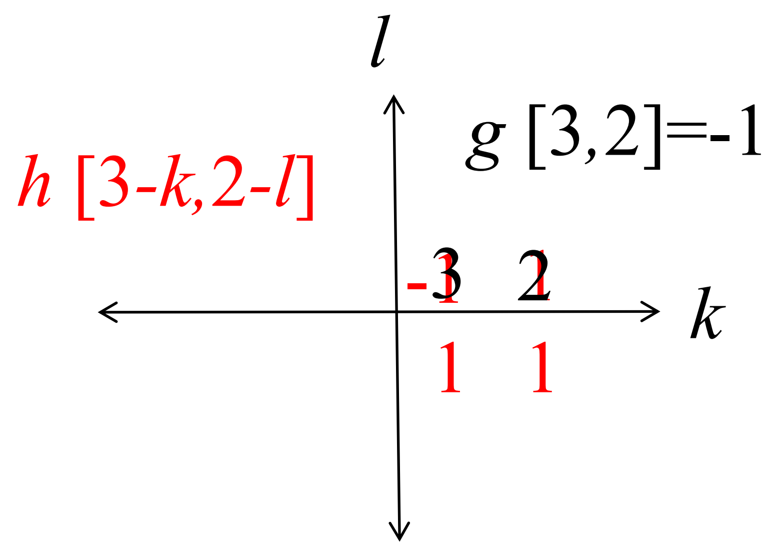


2D discrete convolution (cont.)

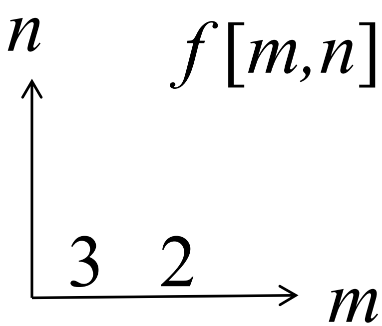


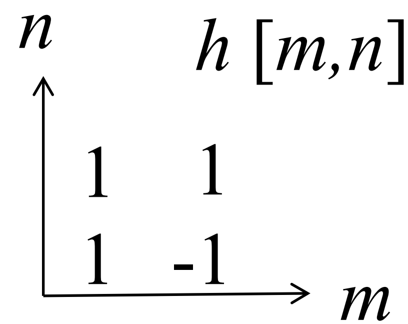


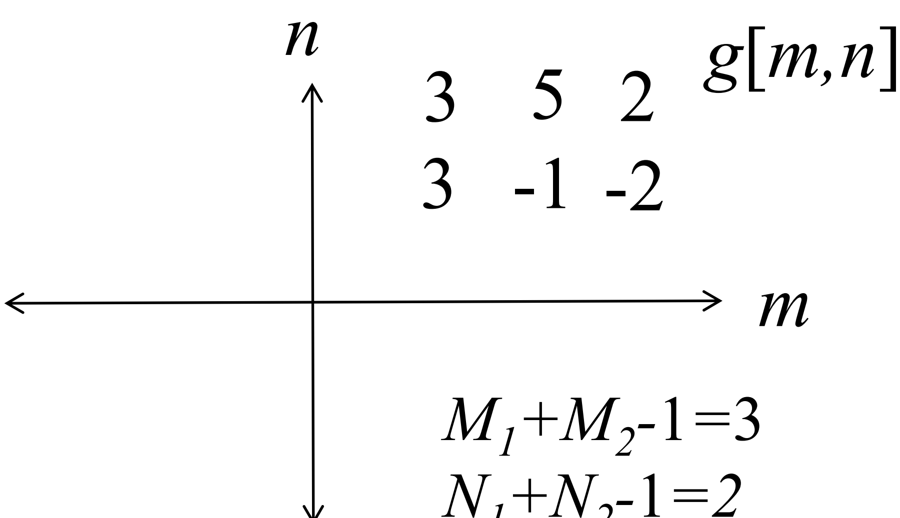




2D discrete convolution (cont.)







$M_1+M_2-1=3$

$N_1+N_2-1=2$

Synopsis so far

The key points of this lecture were:

- The Fourier Transform as a change of basis.
- Generalization to 2D signals.
- The convolution theorem.
- Nyquist criterion theorem.
- The Discrete Fourier Transform (DFT) and its inverse.
- Representation of the DFT as matrix multiplication.