Lista de Exercícios

Inferência Bayesiana

SME 0809

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Problema 1

$$f(y_1, y_2, y_3) \propto \exp\left\{-(y_1 + y_2 + y_3 + \theta_{12}y_1y_2 + \theta_{23}y_2y_3 + \theta_{31}y_3y_1)\right\}$$

Resposta **(a) Densidades condicionais**

$$\begin{aligned} f(y_1 \mid y_2, y_3) &\propto \exp\left\{-(y_1 + \theta_{12}y_1y_2 + \theta_{31}y_3y_1)\right\} \\ &\propto \exp\left\{-y_1(1 + \theta_{12}y_2 + \theta_{31}y_3)\right\}, \end{aligned}$$

portanto, $Y_1 \mid (y_2, y_3) \sim Exponencial(1 + \theta_{12}y_2 + \theta_{31}y_3)$

$$\begin{aligned} f(y_2 \mid y_2, y_3) &\propto \exp\left\{-(y_2 + \theta_{12}y_1y_2 + \theta_{23}y_2y_3)\right\} \\ &\propto \exp\left\{-y_2(1 + \theta_{12}y_1 + \theta_{23}y_3)\right\}, \end{aligned}$$

portanto, $Y_2 \mid (y_1, y_3) \sim Exponencial(1 + \theta_{12}y_1 + \theta_{23}y_3)$

$$\begin{aligned} f(y_3 \mid y_1, y_1) &\propto \exp\left\{-(y_3 + \theta_{23}y_2y_3 + \theta_{31}y_3y_1)\right\} \\ &\propto \exp\left\{-y_3(1 + \theta_{23}y_2 + \theta_{31}y_1)\right\}, \end{aligned}$$

portanto, $Y_3 \mid (y_2, y_1) \sim Exponencial(1 + \theta_{23}y_2 + \theta_{31}y_1)$

(b) Gibbs Sampler

Algorithm 1: Gibbs Sampler para amostra de $f(y_1, y_2, y_3)$

Input: Chute inicial $(y_1, y_2, y_3) = (y_1^{(0)}, y_2^{(0)}, y_3^{(0)})$ e valores para os parâmtros $(\theta_{12}, \theta_{23}, \theta_{31})$

Output: Amostra simulada de $f(y_1, y_2, y_3)$

foreach $t = 1, \ldots, n$ **do**

amostre:

- 1. $y_1^{(t+1)} \sim Exponencial(1 + \theta_{12}y_2^{(t)} + \theta_{31}y_3^{(t)})$
- 2. $y_2^{(t+1)} \sim Exponencial(1 + \theta_{12}y_1^{(t+1)} + \theta_{23}y_3^{(t)})$
- 3. $y_3^{(t+1)} \sim Exponencial(1 + \theta_{23}y_2^{(t+1)} + \theta_{31}y_1^{(t+1)})$

Problema 2

- i) $Y_i \sim Poisson(\theta)$ para $i = 1, \ldots, k$,
- ii) $Y_i \sim Poisson(\lambda)$ para $i = k+1, \ldots, n = 112$

Resposta

Quantidades de Interesse

- θ : média dos primeiros k anos;
- λ : média dos últimos $n-k$ anos;
- k : ano de mudança de intensidade.

Verossimilhança

$$\begin{aligned} L(\theta, \lambda, k; y_1, \ldots, y_n) &= \prod_{i=1}^{k} \frac{\theta^{y_i}}{y_i!} e^{-\theta} \prod_{i=k+1}^{n} \frac{\lambda^{y_i}}{y_i!} e^{-\lambda} \\ &= \theta^{\sum_{i=1}^{k} y_i} e^{-k\theta} \lambda^{\sum_{i=k+1}^{n} y_i} e^{(n-k)\lambda} \prod_{i=1}^{n} \frac{1}{y_i!} \end{aligned}$$

Prioris

$$\begin{aligned} \pi(\theta) &= \frac{b_1^{a_1}}{\Gamma(b_1)} \theta^{a_1-1} e^{-b_1\theta} \\ \pi(\lambda) &= \frac{b_2^{a_2}}{\Gamma(b_2)} \lambda^{a_2-1} e^{-b_2\lambda} \\ \pi(k) &= \frac{1}{n} \mathbf{I}_{\{1,\ldots,n\}}(k) \end{aligned}$$

Priori conjunta

$$\begin{aligned} \pi(\theta, \lambda, k) &= \pi(\theta)\pi(\lambda)\pi(k) \\ &= \frac{b_1^{a_1}}{\Gamma(b_1)} \theta^{a_1-1} e^{-b_1\theta} \frac{b_2^{a_2}}{\Gamma(b_2)} \lambda^{a_2-1} e^{-b_2\lambda} \frac{1}{n} \mathbf{I}_{\{1,\ldots,n\}}(k) \end{aligned}$$

Posteriori Conjunta

$$\begin{aligned} \pi(\theta, \lambda, k \mid y_1, \ldots, y_n) &\propto L(\theta, \lambda, k; y_1, \ldots, y_n)\pi(\theta)\pi(\lambda)\pi(k) \\ &\propto \theta^{\sum_{i=1}^{k} y_i} e^{-k\theta} \lambda^{\sum_{i=k+1}^{n} y_i} e^{(n-k)\lambda} \theta^{a_1-1} e^{-b_1\theta} \lambda^{a_2-1} e^{-b_2\lambda} \frac{1}{n} \mathbf{I}_{\{1,\ldots,n\}}(k) \\ &\propto \theta^{\sum_{i=1}^{k} y_i + a_1 - 1} e^{-\theta(k+b_1)} \lambda^{\sum_{i=k+1}^{n} y_i + a_2 - 1} e^{-\lambda(n-k+b_2)} \mathbf{I}_{\{1,\ldots,n\}}(k) \end{aligned}$$

Resumo a posteriori Um resumo a posteriori para $\frac{\theta}{\lambda} = \frac{\mathbb{E}[\theta|y_1,\ldots,y_n]}{\mathbb{E}[\lambda|y_1,\ldots,y_n]}$, que pode ser interpretado como o coeficiente de variação entre os períodos 1 e 2. Um resumo para k a posteriori é dado por $\hat{k} = \arg\max_{k\in\{1,\ldots,n\}} \pi(k \mid y_1,\ldots,y_n)$, que pode ser interpretado como o ano mais provavel de mudança de intensidade.

Problema 3

$$Y_i = \alpha + \beta(x_i - \bar{x}) - \epsilon_i, \qquad \epsilon_i \sim N(0, \sigma^2)$$

Neste caso temos $\theta = (\alpha, \beta, \sigma^2)$

$$\begin{aligned}
\pi(\alpha) &= N(a_1, b_1)\\
\pi(\beta) &= N(a_2, b_2)\\
\pi(\sigma^2) &= \text{Gama Inversa}(c, d)
\end{aligned}$$

Resposta

Verossimilhança

$$\begin{aligned}
L(\alpha, \beta, \sigma^2, y_1, \ldots, y_n) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y_i - \alpha - \beta(x_i - \bar{x})^2}{2\sigma^2}\right\}\\
&= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{n/2} \exp\left\{-\sum_{i=1}^{n} \frac{(y_i - \alpha - \beta(x_i - \bar{x}))^2}{2\sigma^2}\right\}
\end{aligned}$$

Posteriori

$$\begin{aligned}
\pi(\alpha,&\beta, \sigma^2 \mid y_1, \ldots, y_n) \propto L(\alpha, \beta, \sigma^2, y_1, \ldots, y_n)\pi(\alpha)\pi(\alpha)\pi(\alpha)\\
&\propto (\sigma^2)^{-n/2} \exp\left\{-\sum_{i=1}^{n} \frac{(y_i - \alpha - \beta(x_i - \bar{x}))^2}{2\sigma^2}\right\} \exp\left\{-\frac{(\alpha - a_1)^2}{2b_1}\right\} \times\\
&\times \exp\left\{-\frac{(\beta - a_2)^2}{2b_2}\right\} (\sigma^2)^{-c-1} \exp\left\{-\frac{d}{\sigma^2}\right\}\\
&\propto (\sigma^2)^{-n/2-c-1} \exp\left\{-\frac{\sum_{i=1}^{n}(y_i - \alpha - \beta(x_i - \bar{x}))^2 - 2d}{2\sigma^2} - \frac{(\alpha - a_1)^2}{2b_1} - \frac{(\beta - a_2)^2}{2b_2}\right\}
\end{aligned}$$

Condicionais completas

$$\pi(\sigma^2 \mid \alpha, \beta, y_1, \ldots, y_n) \propto (\sigma^2)^{-n/2-c-1} \exp\left\{-\frac{\sum_{i=1}^{n}(y_i - \alpha - \beta(x_i - \bar{x}))^2 - 2d}{2\sigma^2}\right\},$$

portanto, $\sigma^2 \mid \alpha, \beta, y_1, \ldots, y_n \sim \text{Gama Inversa}\left(c + \frac{n}{2}; \frac{\sum_{i=1}^{n}(y_i - \alpha - \beta(x_i - \bar{x}))^2 + 2d}{2}\right)$.

$$
\begin{aligned}
\pi(\alpha \mid \beta, \sigma^2, y_1, \ldots, y_n) &\propto \exp\left\{-\frac{\sum_{i=1}^{n}(y_i-\alpha-\beta(x_i-\bar{x}))^2}{2\sigma^2}-\frac{(\alpha-a_1)^2}{2b_1}\right\}\\
&\propto \exp\left\{-\frac{\sum_{i=1}^{n}[2y_i(\alpha-\beta(x_i-\bar{x}))-(\alpha-\beta(x_i-\bar{x}))^2]}{2\sigma^2}+\frac{-\alpha^2+2\alpha a_1}{2b_1}\right\}\\
&\propto \exp\left\{\frac{-\alpha 2n\bar{y}-n\alpha^2+2\alpha\beta\sum_{i=1}^{n}(x_i-\bar{x})}{2\sigma^2}+\frac{-\alpha^2+2\alpha a_1}{2b_1}\right\}\\
&\propto \exp\left\{\frac{-\alpha^2(n\sigma^{-2}+b_1^{-1})+2\alpha(n\bar{y}\sigma^{-2}+a_1b_1^{-1})}{2}\right\}\\
&\propto \exp\left\{\frac{-(n\sigma^{-2}+b_1^{-1})\left[\alpha^2-2\alpha\left(\frac{\bar{y}n\sigma^{-2}+a_1b_1^{-1}}{n\sigma^{-2}+b_1^{-1}}\right)\right]}{2}\right\}\\
&\propto \exp\left\{-\frac{\left[\alpha-\left(\frac{\bar{y}n\sigma^{-2}+a_1b_1^{-1}}{n\sigma^{-2}+b_1^{-1}}\right)\right]^2}{2(n\sigma^{-2}+b_1^{-1})^{-1}}\right\}
\end{aligned}
$$

portanto, $\alpha \mid \beta, \sigma^2, y_1, \ldots, y_n \sim N\left[\frac{\bar{y}n\sigma^{-2}+a_1b_1^{-1}}{n\sigma^{-2}+b_1^{-1}}; (n\sigma^{-2}+b_1^{-1})^{-1}\right]$.

$$
\begin{aligned}
&\pi(\beta \mid \alpha, \sigma^2, y_1, \ldots, y_n) \propto\\
&\quad \exp\left\{\frac{-\sigma^2\left(\sum_{i=1}^{n}[2y_i(\alpha+\beta(x_i-\bar{x}))+(\alpha+\beta(x_i-\bar{x})^2)]\right)}{2}+\frac{-b_2^{-1}(\beta^2-2\beta a_2)}{2}\right\}\\
&\quad \propto \exp\left\{\frac{-\sigma^2\left(-2\beta\sum_{i=1}^{n}y_i(x_i-\bar{x})+2\alpha\beta\sum_{i=1}^{n}(x_i-\bar{x})+\beta^2\sum_{i=1}^{n}(x_i-\bar{x})^2\right)}{2}\right\}\times\\
&\quad \times \exp\left\{\frac{-b_2^{-1}(\beta^2-2\beta a_2)}{2}\right\}\\
&\quad \propto \exp\left\{\frac{-\sigma^{-2}\left(-2\beta\sum_{i=1}^{n}y_i(x_i-\bar{x})+\beta^2\sum_{i=1}^{n}(x_i-\bar{x})\right)-b_2^{-1}(\beta^2-2\beta a_2)}{2}\right\}\\
&\quad \propto \exp\left\{\frac{-\beta^2\left(\sigma^{-2}\sum_{i=1}^{n}(x_i-\bar{x})^2+b_2^{-1}\right)+2\beta\left(\sigma^{-2}\sum_{i=1}^{n}y_i(x_i-\bar{x})+b_2^{-1}a_2\right)}{2}\right\}\\
&\quad \propto \exp\left\{\frac{-(\sigma^{-2}\sum_{i=1}^{n}(x_i-\bar{x})^2+b_2^{-1})\left[\beta^2-2\beta\left(\frac{\sigma^{-2}\sigma_{i=1}^{n}y_i(x_i-\bar{x})+b_2^{-1}a_2}{\sigma^{-2}\sum_{i=1}^{n}(x_i-\bar{x})^2+b_2^{-1}}\right)\right]}{2}\right\}\\
&\quad \propto \exp\left\{-\frac{\left[\beta-\left(\frac{\sigma^{-2}\sum_{i=1}^{n}y_i(x_i-\bar{x})+b_2^{-1}a_2}{\sigma^{-2}\sum_{i=1}^{n}(x_i-\bar{x})+b_2^{-1}}\right)\right]^2}{2(\sigma^{-2}\sum_{i=1}^{n}(x_i-\bar{x})^2+b_2^{-1})^{-1}}\right\},
\end{aligned}
$$

portanto, $\beta \mid \alpha, \sigma^2, y_1, \ldots, y_n \sim N\left[\left(\frac{\sigma^{-2}\sum_{i=1}^n y_i(x_i-\bar{x})+b_2^{-1}a_2}{\sigma^{-2}\sum_{i=1}^n (x_i-\bar{x})^2+b_2^{-1}}\right); (\sigma^{-2}\sum_{i=1}^n (x_i-\bar{x})^2 + b_2^{-1})^{-1}\right]$.
Perceba que a distribuição condicional de β não depende de α, portanto β é condicionalmente independe de α.

Gibbs Sampler

Algorithm 2: Gibbs Sampler para amostra da posteriori $\pi(\alpha, \beta, \sigma^2 \mid y_1, \ldots, y_n)$

Input: Chute inicial $(\alpha, \beta, \sigma^2) = (\alpha^{(0)}, \beta^{(0)}, \sigma^{2(0)})$, amostra $y_1, \ldots, y_n$ e N o número de iterações
Output: Amostra simulada de $\pi(\alpha, \beta, \sigma^2 \mid y_1, \ldots, y_n)$
foreach $t = 1, \ldots, N$ **do**
amostre:

- 1. $\alpha^{(t+1)} \sim \pi(\alpha \mid \beta^{(t)}, \sigma^{2(t)}, y_1, \ldots, y_n)$
- 2. $\beta^{(t+1)} \sim \pi(\beta \mid \alpha^{(t+1)}, \sigma^{2(t)}, y_1, \ldots, y_n)$
- 3. $\sigma^{2(t+1)} \sim \pi(\sigma^2 \mid \alpha^{(t+1)}, \beta^{(t+1)}, y_1, \ldots, y_n)$

```
## Gibbs Sampler para regressao


gibbs_lm <- function(alpha0, beta0, sigma0, parametros_alpha, parametros_beta,
                     parametros_sigma y, x, N){

  amostra_gibbs <- matrix(NA, nrow = N, ncol = 3)

  a1 <- parametros_alpha[1]; b1 <- parametros_alpha[2]
  a2 <- parametros_beta[1]; b2 <- parametros_beta[2]
  c0 <- parametros_sigma[1]; d <- parametros_sigma[2]

  alpha <- alpha0
  beta <- beta0
  sigma <- sigma0
  for (ii in 1:N) {

    #atualizacao dos parametros
    mu_alpha <- (mean(y)*n*sigma^(-2) +b1^(-1)*a1)/(n*sigma^(-2) +b1^(-1))
    sigma_alpha <- (n*sigma^(-2) +b1^(-1))^(-1)

    #amostrando da dist condicional
```

```
    alpha <- rnorm(mu_alpha, sqrt(sigma_alpha), 1)

    #atualizacao dos parametros
    mu_beta <- (sigma^(-2)*t(y)%*%(x-mean(x)) +b2^(-1)*a2)
    mu_beta <- mu_beta/(sigma^(-2)*t((x-mean(x)))%*%(x-mean(x)) +b2^(-1))
    sigma_beta <- (sigma^(-2)*t((x-mean(x)))%*%(x-mean(x)) +b2^(-1))^(-1)

    #amostrando da dist condicional
    beta <- rnorm(mu_beta, sqrt(sigma_beta), 1)

    #atualizacao dos parametros
    c <- c0 + n/2
    d <- (sum(y - alpha -beta*(x -mean(x))) +2*d0)/2

    #amostrando da dist condicional
    sigma <- rinvgamma(1, c, d)

    amostra_gibbs[ii,] <- c(alpha, beta, sigma)
  }

  return(amostra_gibbs)
}
```

Problema 4

```
    # Gibbs sampling para Normal

gibbs_norm <- function(m, S, x0, N){
  amostra_gibbs <- matrix(NA, nrow = N, ncol = 2)

  x1 <- x0[1]; x2 <- x0[2]
  m1 <- m[1]; m2 <- m[2]

  s1  <- S[1,1]
  s12 <- S[1,2]
  s21 <- S[2,1]
  s2  <- S[2]
  for (ii in 1:N) {

    mu1 <- m1 +(s12/s2)*(x2-m2)
    sigma1 <- s1 - s12*s21/s2

    x1 <- rnorm(1,mu1, sqrt(sigma1))

    mu2 <- m2 + (s21/s1)*(x1-m1)
    sigma2 <- s2 -s21*s12/s1

    x2 <- rnorm(1, mu2, sqrt(sigma2))

    amostra_gibbs[ii, ] <- c(x1, x2)
  }

  return(amostra_gibbs)
}


m <- c(2,1)
S <- matrix(c(1, 0.8, 0.8, 1), 2, 2)
x0 <- c(0,0)
N <- 1000


amostra_Gibbs <- gibbs_norm(m=m, S=S, x0=x0, N=N)
```

```
# Burn-in e sub-amostra
x_Gibbs <- sample(amostra_Gibbs[-c(1:100),1], 100)
y_Gibbs <- sample(amostra_Gibbs[-c(1:100),2], 100)

### Utilizando pacote implementado no R
library(MASS)

amostra_normulti <- mvrnorm(100, m, S)

x_normulti <- amostra_normulti[,1]
y_normulti <- amostra_normulti[,2]
#### Comparação

par(mfrow=c(1,2))

hist(x_Gibbs, main = "Gibbs Sampling")
hist(x_normulti, main = "pacote R")


par(mfrow=c(1,2))

hist(y_Gibbs, main = "Gibbs Sampling")

hist(y_normulti, main = "pacote R")
```

Podemos notar com bases as figuras 1 e 2 que o algoritmo de Gibbs sampler gerou amostras com comportamento muito similar ao obtido pelo pacote R. Assim podemos concluir que o Gibbs sampler aproxima bem a normal bivariada.

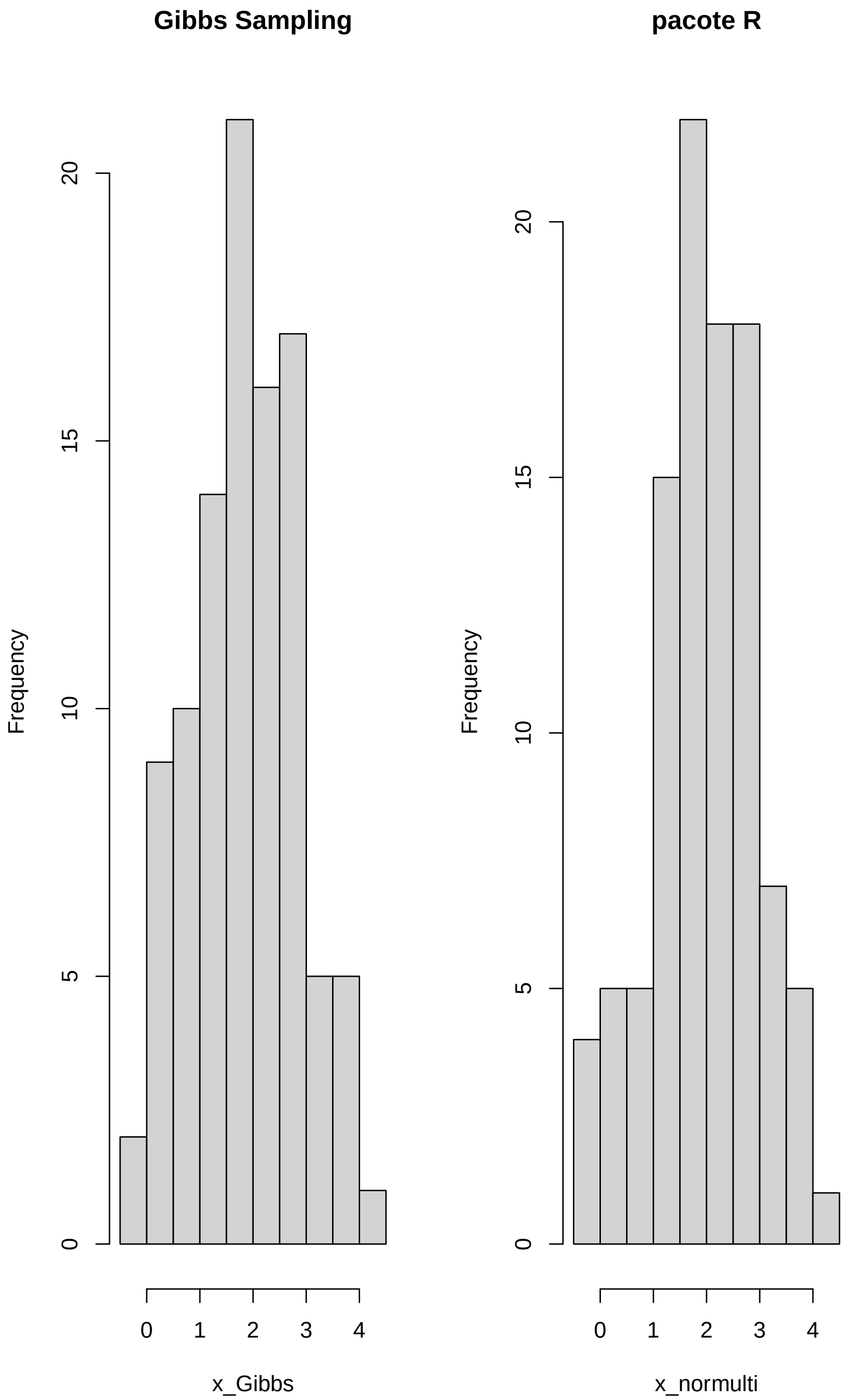

Figura 1: histograma para amostras geradas pelo Gibbs sampling e pelo Pacote MASS no R.

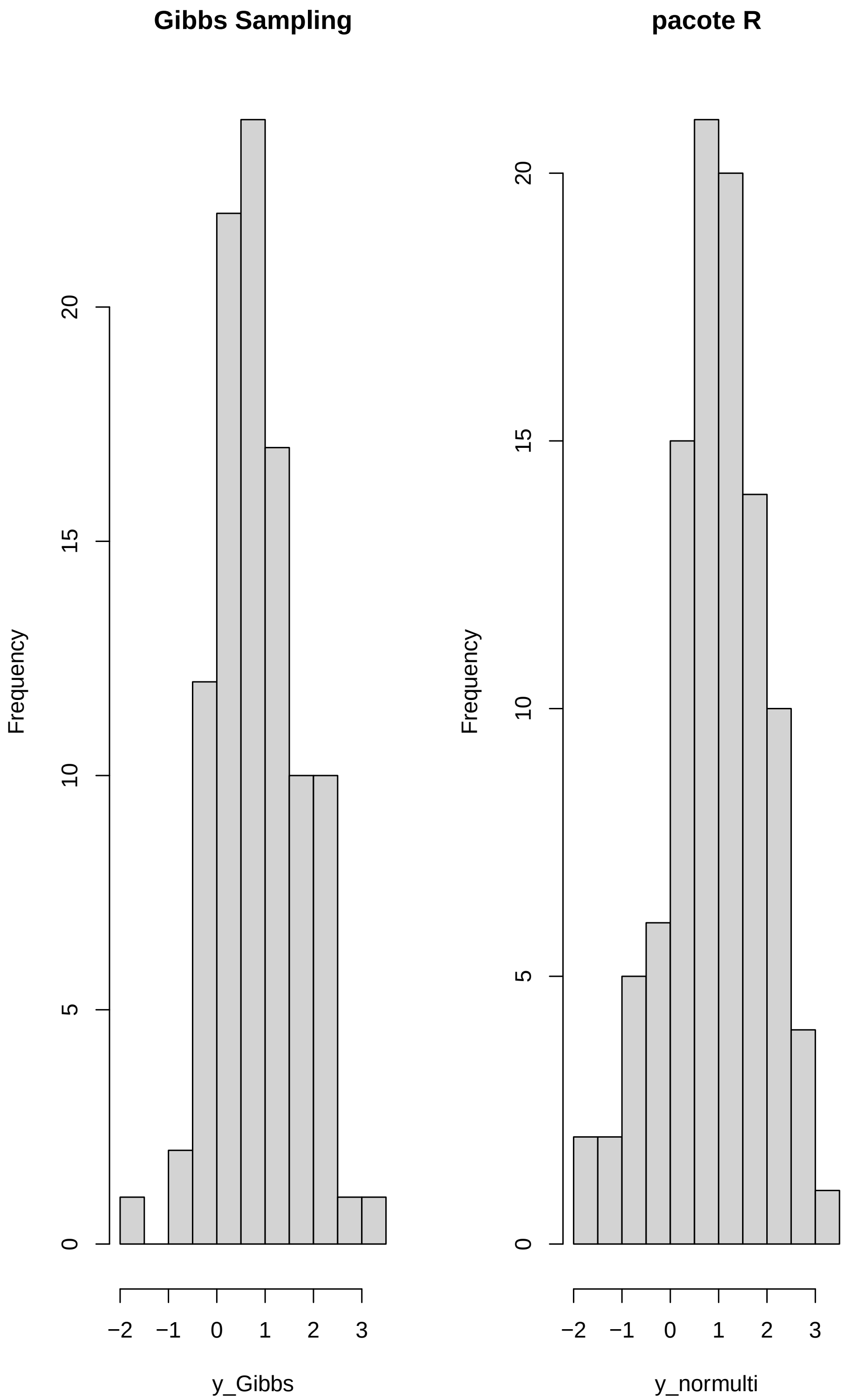

Figura 2: histograma para amostras geradas pelo Gibbs sampling e pelo Pacote MASS no R.