```
\documentclass[12pt,a4paper]{article}
\usepackage[utf8]{inputenc}
\usepackage[brazilian]{babel}
\usepackage{amsmath}
\usepackage{amsfonts}
\usepackage{amssymb}
\usepackage[ruled,vlined]{algorithm2e}
\renewcommand{\algorithmautorefname}{Algoritmo}% para utilizar \autoref
\usepackage[T1]{fontenc}
\usepackage{graphicx}
\usepackage[pdftex]{color,graphicx}
%\usepackage{physics}
\usepackage{siunitx}
\input{commands}

\author{Guilherme Bergamim dos Santos - 8124076, Vikson Andrade - 10733900}
\title{Lista de Exercícios - SME 0809}
\date{}
\begin{document}

%%--CABEÇALHO--%%
        \begin{center}
    {\huge Lista de Exercícios \par}
    {\LARGE Inferência Bayesiana \par}
    {\Large SME 0809 \par}
    {\Medium Guilherme Bergamim 8124076, Vikson Andrade - 10733900 \par}
        \end{center}

\problem
\begin{align*}
    f(y_{1}, y_{2}, y_{3}) \propto \exp \left\{ -(y_{1} +y_{2} +y_{3}
+\theta_{12}y_{1}y_{2} +\theta_{23}y_{2}y_{3} +\theta_{31}y_{3}y_{1})\right\}
\end{align*}

\answer
\textbf{(a) Densidades condicionais}
\begin{align*}
    f(y_{1} \mid y_{2}, y_{3}) & \propto \exp\left\{-(y_{1}
+\theta_{12}y_{1}y_{2} +\theta_{31}y_{3}y_{1})\right\} \\
                              & \propto \exp\left\{-y_{1}(1
+\theta_{12}y_{2} +\theta_{31}y_{3})\right\},
\end{align*}
portanto, $Y_{1} \mid (y_{2}, y_{3}) \sim Exponencial (1 +\theta_{12}y_{2}
+\theta_{31}y_{3})$

\begin{align*}
    f(y_{2} \mid y_{2}, y_{3}) & \propto \exp\left\{-(y_{2}
+\theta_{12}y_{1}y_{2} +\theta_{23}y_{2}y_{3})\right\} \\
                              & \propto \exp\left\{-y_{2}(1
+\theta_{12}y_{1} +\theta_{23}y_{3})\right\},
\end{align*}
portanto, $Y_{2} \mid (y_{1}, y_{3}) \sim Exponencial (1 +\theta_{12}y_{1}
+\theta_{23}y_{3})$

\begin{align*}
    f(y_{3} \mid y_{1}, y_{1}) & \propto \exp\left\{-(y_{3}
+\theta_{23}y_{2}y_{3} +\theta_{31}y_{3}y_{1})\right\} \\
```

```
                              & \propto \exp\left\{-y_{3}(1
+\theta_{23}y_{2} +\theta_{31}y_{1})\right\},
\end{align*}
portanto, $Y_{3} \mid (y_{2}, y_{1}) \sim Exponencial (1 +\theta_{23}y_{2}
+\theta_{31}y_{1})$

\textbf{(b) Gibbs Sampler}

\begin{algorithm}[H]
\caption{Gibbs Sampler para amostra de $f(y_{1}, y_{2}, y_{3})$
\label{meualgoritmo}}
    \KwIn{Chute inicial $(y_{1}, y_{2}, y_{3}) = (y_{1}^{(0)}, y_{2}^{(0)},
y_{3}^{(0)})$ e valores para os parâmtros $(\theta_{12}, \theta_{23},
\theta_{31})$}
    \KwOut{Amostra simulada de $f(y_{1}, y_{2}, y_{3})$}
           \ForEach{$t = 1, \ldots, n$}{
                amostre:
                \begin{itemize}
                    \item 1. $y_{1}^{(t+1)} \sim  Exponencial (1
+\theta_{12}y_{2}^{(t)} +\theta_{31}y_{3}^{(t)})$
                    \item 2. $y_{2}^{(t+1)} \sim  Exponencial (1
+\theta_{12}y_{1}^{(t+1)} +\theta_{23}y_{3}^{(t)})$
                    \item 3. $y_{3}^{(t+1)} \sim  Exponencial (1
+\theta_{23}y_{2}^{(t+1)} +\theta_{31}y_{1}^{(t+1)})$
                \end{itemize}
           }
\end{algorithm}


%\finalanswer{}

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\problem

\begin{itemize}
    \item i) $Y_{i} \sim Poisson(\theta)$ para $i =1, \ldots, k$,
    \item ii) $Y_{i} \sim Poisson(\lambda)$ para $i =k+1, \ldots, n = 112$
\end{itemize}

\answer

\textbf{Quantidades de Interesse}
\begin{itemize}
    \item $\theta:$ média dos primeiros $k$ anos;
    \item $\lambda:$ média dos últimos $n-k$ anos;
    \item $k:$ ano de mudança de intensidade.
\end{itemize}


\textbf{Verossimilhança}
\begin{align*}
    L(\theta, \lambda, k; y_{1}, \ldots, y_{n}) & = \prod_{i=1}^{k}
\frac{\theta^{y_{i}}}{y_{i}!} e^{-\theta} \prod_{i=k+1}^{n}
\frac{\lambda^{y_{i}}}{y_{i}!} e^{-\lambda} \\
                                            & = \theta^{\sum_{i=1}^{k}
y_{i}} e^{-k\theta}\lambda^{\sum_{i=k+1}^{n} y_{i}} e^{(n-k)\lambda}
\prod_{i=1}^{n}\frac{1}{y_{i}!}
```

```
\end{align*}

\textbf{Prioris}

\begin{align*}
    \pi(\theta) = \frac{b_{1}^{a_{1}}}{\Gamma(b_{1})} \theta^{a_{1} -1} e^{-
b_{1}\theta} \\
    \pi(\lambda) = \frac{b_{2}^{a_{2}}}{\Gamma(b_{2})} \lambda^{a_{2} -1}
e^{-b_{2}\lambda} \\
    \pi(k) = \frac{1}{n}\mathbf{I}_{\{1, \ldots, n\}} (k)
\end{align*}

Priori conjunta
\begin{align*}
    \pi(\theta, \lambda, k) &= \pi(\theta) \pi(\lambda) \pi(k) \\
                            &= \frac{b_{1}^{a_{1}}}{\Gamma(b_{1})}
\theta^{a_{1} -1} e^{-b_{1}\theta}\frac{b_{2}^{a_{2}}}{\Gamma(b_{2})}
\lambda^{a_{2} -1} e^{-b_{2}\lambda} \frac{1}{n}\mathbf{I}_{\{1, \ldots, n\}}
(k)
\end{align*}

\textbf{Posteriori Conjunta}

\begin{align*}
    \pi(\theta, \lambda, k \mid y_{1}, \ldots, y_{n}) & \propto  L(\theta,
\lambda, k; y_{1}, \ldots, y_{n})\pi(\theta) \pi(\lambda) \pi(k) \\
    & \propto \theta^{\sum_{i=1}^{k} y_{i}} e^{-
k\theta}\lambda^{\sum_{i=k+1}^{n} y_{i}} e^{(n-k)\lambda} \theta^{a_{1} -1}
e^{-b_{1}\theta} \lambda^{a_{2} -1} e^{-b_{2}\lambda}
\frac{1}{n}\mathbf{I}_{\{1, \ldots, n\}} (k) \\
    & \propto \theta^{\sum_{i=1}^{k} y_{i} + a_{1} -1} e^{-\theta(k + b_{1})}
\lambda^{\sum_{i=k+1}^{n} y_{i}+a_{2} -1}e^{-\lambda(n-k+b_{2})}
\mathbf{I}_{\{1, \ldots, n\}} (k)
\end{align*}

\textbf{Resumo a posteriori}
Um resumo a posteriori para $\frac{\theta}{\lambda} = \frac{\mathbb{E}[\theta
\mid y_{1}, \ldots, y_{n}]}{\mathbb{E}[\lambda \mid y_{1}, \ldots, y_{n}]}$,
que pode ser interpretado como o coeficiente de variação entre os períodos 1
e 2. Um resumo para $k$ a posteriori é dado por $\hat{k} = \arg \max_{k \in
\{1, \ldots, n\}} \pi(k \mid y_{1}, \ldots, y_{n})$, que pode ser
interpretado como o ano mais provavel de mudança de intensidade.

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\problem
\begin{align*}
    Y_{i} = \alpha +\beta(x_{i} -\bar{x}) - \epsilon_{i}, \hspace{1.0cm}
\epsilon_{i} \sim N(0, \sigma^2)
\end{align*}

Neste caso temos $\theta = (\alpha, \beta, \sigma^2)$

\begin{align*}
    \pi(\alpha) &= N(a_{1}, b_{1}) \\
    \pi(\beta) &= N(a_{2}, b_{2}) \\
    \pi(\sigma^2) &= \text{Gama Inversa} (c, d)
\end{align*}
```

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\answer

\textbf{Verossimilhança}

\begin{align*}
    L(\alpha, \beta, \sigma^2, y_{1}, \ldots, y_{n}) &= \prod_{i=1}^{n}
\frac{1}{\sqrt{2\pi \sigma^2}} \exp\left\{-\frac{(y_{i} -\alpha -\beta(x_{i}
-\bar{x})^2}{2\sigma^2}\right\} \\
    &= \left(\frac{1}{\sqrt{2\pi \sigma^2}}\right)^{n/2} \exp\left\{-
\sum_{i=1}^{n} \frac{(y_{i} -\alpha -\beta(x_{i} -
\bar{x}))^2}{2\sigma^2}\right\}
\end{align*}

\textbf{Posteriori}
\begin{align*}
     \pi(\alpha, & \beta, \sigma^2 \mid y_{1}, \ldots, y_{n})\propto
L(\alpha, \beta, \sigma^2, y_{1}, \ldots,
y_{n})\pi(\alpha)\pi(\alpha)\pi(\alpha) \\
    & \propto (\sigma^2)^{-n/2} \exp\left\{-\sum_{i=1}^{n} \frac{(y_{i} -
\alpha -\beta(x_{i} -\bar{x}))^2}{2\sigma^2}\right\}
    \exp\left\{-\frac{(\alpha -a_{1})^2}{2b_{1}}\right\} \times \\
    & \times \exp\left\{-\frac{(\beta -a_{2})^2}{2b_{2}}\right\}
    (\sigma^2)^{-c-1}\exp\left\{-\frac{d}{\sigma^2}\right\}  \\
    & \propto (\sigma^2)^{-n/2-c-1} \exp\left\{ -\frac{\sum_{i=1}^{n} (y_{i}
-\alpha -\beta(x_{i} -\bar{x}))^2 -2d}{2\sigma^2} -\frac{(\alpha -
a_{1})^2}{2b_{1}} -\frac{(\beta -a_{2})^2}{2b_{2}}\right\}
\end{align*}

\textbf{Condicionais completas}
\begin{align*}
    \pi(\sigma^2 \mid \alpha, \beta, y_{1}, \ldots, y_{n}) \propto
(\sigma^2)^{-n/2-c-1} \exp\left\{ -\frac{\sum_{i=1}^{n} (y_{i} -\alpha -
\beta(x_{i} -\bar{x}))^2 -2d}{2\sigma^2}\right\},
\end{align*}

portanto, $\sigma^2 \mid \alpha, \beta, y_{1}, \ldots, y_{n} \sim$ Gama
Inversa$\left(c+\frac{n}{2}; \frac{\sum_{i=1}^{n} (y_{i} -\alpha -\beta(x_{i}
-\bar{x}))^2 +2d}{2}\right)$.

\begin{align*}
    \pi(\alpha & \mid \beta, \sigma^2, y_{1}, \ldots, y_{n}) \propto \exp
\left\{ -\frac{\sum_{i=1}^{n} (y_{i} -\alpha -\beta(x_{i} -
\bar{x}))^2}{2\sigma^2} -\frac{(\alpha -a_{1})^2}{2b_{1}}\right\} \\
    & \propto \exp\left\{ -\frac{\sum_{i=1}^{n}[2y_{i}(\alpha -\beta(x_{i} -
\bar{x})) -(\alpha -\beta(x_{i} -\bar{x}))^2]}{2\sigma^2} +\frac{-\alpha^2 +
2\alpha a_{1}}{2b_{1}}\right\} \\
    & \propto \exp\left\{\frac{-\alpha2n\bar{y} -n\alpha^2
+2\alpha\beta\sum_{i=1}^{n} (x_{i} -\bar{x})}{2\sigma^2} +\frac{-\alpha^2 +
2\alpha a_{1}}{2b_{1}}\right\} \\
    & \propto \exp\left\{ \frac{-\alpha^2(n\sigma^{-2} +b_{1}^{-1}) +
2\alpha(n\bar{y}\sigma^{-2} +a_{1}b_{1}^{-1})}{2}\right\} \\
    & \propto \exp\left\{\frac{-(n\sigma^{-2} +b_{1}^{-1})\left[\alpha^2 -
2\alpha\left(\frac{\bar{y}n\sigma^{-2} +a_{1}b_{1}^{-1}}{n\sigma^{-2}
+b_{1}^{-1}}\right)\right]}{2}\right\} \\
```

```
    & \propto \exp\left\{-\frac{\left[\alpha - \left(\frac{\bar{y}n\sigma^{-
2} +a_{1}b_{1}^{-1}}{n\sigma^{-2} +b_{1}^{-1}}\right)\right]^2}{2 (n\sigma^{-
2} +b_{1}^{-1})^{-1}}\right\}
\end{align*}

portanto, $\alpha \mid \beta, \sigma^2, y_{1}, \ldots, y_{n} \sim
N\left[\frac{\bar{y}n\sigma^{-2} +a_{1}b_{1}^{-1}}{n\sigma^{-2} +b_{1}^{-1}};
(n\sigma^{-2} +b_{1}^{-1})^{-1}\right]$.

\begin{align*}
    \pi(\beta & \mid \alpha, \sigma^2, y_{1}, \ldots, y_{n}) \propto \\
    & \exp\left\{ \frac{-\sigma^{2}\left(\sum_{i=1}^{n} \left[ 2y_{i}(\alpha
+\beta(x_{i} -\bar{x})) +(\alpha +\beta(x_{i} -\bar{x})^2)\right]\right)}{2}
    +\frac{-b_{2}^{-1}(\beta^{2} -2\beta a_{2})}{2} \right\} \\
    & \propto \exp \left\{\frac{-\sigma^2 \left(-2\beta\sum_{i=1}^{n}
y_{i}(x_{i} -\bar{x}) +2\alpha \beta \sum_{i=1}^{n} (x_{i} -\bar{x})
+\beta^2\sum_{i=1}^{n} (x_{i} -\bar{x})^2\right)}{2}\right\} \times \\
    & \times \exp\left\{\frac{-b_{2}^{-1}(\beta^2-2\beta a_{2})}{2}\right\}
\\
    & \propto \exp\left\{\frac{-\sigma^{-2}\left(-2\beta\sum_{i=1}^{n}
y_{i}(x_{i} -\bar{x}) + \beta^2\sum_{i=1}^{n} (x_{i} -\bar{x})\right) -
b_{2}^{-1}(\beta^2 -2\beta a_{2})}{2}\right\} \\
    & \propto \exp\left\{\frac{-\beta^2\left(\sigma^{-2}\sum_{i=1}^{n} (x_{i}
-\bar{x})^2 +b_{2}^{-1}\right) +2\beta\left(\sigma^{-2}\sum_{i=1}^{n}
y_{i}(x_{i} -\bar{x}) +b_{2}^{-1}a_{2}\right)}{2}\right\} \\
    & \propto \exp\left\{\frac{-(\sigma^{-2}\sum_{i=1}^{n} (x_{i}-
\bar{x})^{2} +b_{2}^{-1})\left[\beta^{2}-2\beta\left(\frac{\sigma^{-
2}\sigma_{i=1}^{n} y_{i}(x_{i} -\bar{x}) +b_{2}^{-1}a_{2}}{\sigma^{-
2}\sum_{i=1}^{n}(x_{i} -\bar{x})^{2} + b_{2}^{-1}}\right)\right]}{2}\right\}
\\
    & \propto \exp \left\{-\frac{\left[\beta - \left(\frac{\sigma^{-
2}\sum_{i=1}^{n} y_{i}(x_{i} -\bar{x}) +b_{2}^{-1}a_{2}}{\sigma^{-
2}\sum_{i=1}^{n} (x_{i} -\bar{x}) +b_{2}^{-1}}\right)\right]^{2}}{2(\sigma^{-
2}\sum_{i=1}^{n} (x_{i} -\bar{x})^{2} +b_{2}^{-1})^{-1}}\right\},
\end{align*}

portanto, $\beta \mid \alpha, \sigma^2, y_{1}, \ldots, y_{n} \sim N
\left[\left(\frac{\sigma^{-2}\sum_{i=1}^{n} y_{i}(x_{i} -\bar{x}) +b_{2}^{-
1}a_{2}}{\sigma^{-2}\sum_{i=1}^{n} (x_{i} -\bar{x})^{2} +b_{2}^{-1}}\right);
(\sigma^{-2}\sum_{i=1}^{n} (x_{i} -\bar{x})^{2} +b_{2}^{-1})^{-1} \right]$.
Perceba que a distribuição condicional de $\beta$ não depende de $\alpha$,
portanto $\beta$ é condicionalmente independe de $\alpha$.

\textbf{Gibbs Sampler}

\begin{algorithm}[H]
\caption{Gibbs Sampler para amostra da posteriori $\pi( \alpha, \beta,
\sigma^2 \mid y_{1}, \ldots, y_{n})$ \label{meualgoritmo}}
    \KwIn{Chute inicial $(\alpha, \beta, \sigma^2) = (\alpha^{(0)},
\beta^{(0)}, \sigma^{2(0)})$, amostra $y_{1}, \ldots, y_{n}$ e $N$ o número
de iterações}
    \KwOut{Amostra simulada de $\pi(\alpha, \beta, \sigma^2 \mid y_{1},
\ldots, y_{n})$}
           \ForEach{$t = 1, \ldots, N$}{
                amostre:
                \begin{itemize}
```

```
                    \item 1. $\alpha^{(t+1)} \sim \pi(\alpha \mid
\beta^{(t)}, \sigma^{2(t)}, y_{1}, \ldots, y_{n})$
                    \item 2. $\beta^{(t+1)} \sim \pi(\beta \mid
\alpha^{(t+1)}, \sigma^{2(t)}, y_{1}, \ldots, y_{n})$
                    \item 3. $\sigma^{2(t+1)} \sim \pi(\sigma^2 \mid
\alpha^{(t+1)}, \beta^{(t+1)}, y_{1}, \ldots, y_{n})$
                \end{itemize}
           }
\end{algorithm}


\begin{verbatim}
## Gibbs Sampler para regressao


gibbs_lm <- function(alpha0, beta0, sigma0, parametros_alpha,
parametros_beta,
                     parametros_sigma y, x, N){

  amostra_gibbs <- matrix(NA, nrow = N, ncol = 3)

  a1 <- parametros_alpha[1]; b1 <- parametros_alpha[2]
  a2 <- parametros_beta[1]; b2 <- parametros_beta[2]
  c0 <- parametros_sigma[1]; d <- parametros_sigma[2]

  alpha <- alpha0
  beta <- beta0
  sigma <- sigma0
  for (ii in 1:N) {

    #atualizacao dos parametros
    mu_alpha <- (mean(y)*n*sigma^(-2) +b1^(-1)*a1)/(n*sigma^(-2) +b1^(-1))
    sigma_alpha <- (n*sigma^(-2) +b1^(-1))^(-1)

    #amostrando da dist condicional
    alpha <- rnorm(mu_alpha, sqrt(sigma_alpha), 1)

    #atualizacao dos parametros
    mu_beta <- (sigma^(-2)*t(y)%*%(x-mean(x)) +b2^(-1)*a2)
    mu_beta <- mu_beta/(sigma^(-2)*t((x-mean(x)))%*%(x-mean(x)) +b2^(-1))
    sigma_beta <- (sigma^(-2)*t((x-mean(x)))%*%(x-mean(x)) +b2^(-1))^(-1)

    #amostrando da dist condicional
    beta <- rnorm(mu_beta, sqrt(sigma_beta), 1)

    #atualizacao dos parametros
    c <- c0 + n/2
    d <- (sum(y - alpha -beta*(x -mean(x))) +2*d0)/2

    #amostrando da dist condicional
    sigma <- rinvgamma(1, c, d)

    amostra_gibbs[ii,] <- c(alpha, beta, sigma)
  }

  return(amostra_gibbs)
}
```

```
\end{verbatim}



\newpage
\problem

\begin{verbatim}
    # Gibbs sampling para Normal

gibbs_norm <- function(m, S, x0, N){
  amostra_gibbs <- matrix(NA, nrow = N, ncol = 2)

  x1 <- x0[1]; x2 <- x0[2]
  m1 <- m[1]; m2 <- m[2]

  s1  <- S[1,1]
  s12 <- S[1,2]
  s21 <- S[2,1]
  s2  <- S[2]
  for (ii in 1:N) {

    mu1 <- m1 +(s12/s2)*(x2-m2)
    sigma1 <- s1 - s12*s21/s2

    x1 <- rnorm(1,mu1, sqrt(sigma1))

    mu2 <- m2 + (s21/s1)*(x1-m1)
    sigma2 <- s2 -s21*s12/s1

    x2 <- rnorm(1, mu2, sqrt(sigma2))

    amostra_gibbs[ii, ] <- c(x1, x2)
  }

  return(amostra_gibbs)
}


m <- c(2,1)
S <- matrix(c(1, 0.8, 0.8, 1), 2, 2)
x0 <- c(0,0)
N <- 1000


amostra_Gibbs <- gibbs_norm(m=m, S=S, x0=x0, N=N)

# Burn-in e sub-amostra
x_Gibbs <- sample(amostra_Gibbs[-c(1:100),1], 100)
y_Gibbs <- sample(amostra_Gibbs[-c(1:100),2], 100)

### Utilizando pacote implementado no R
library(MASS)

amostra_normulti <- mvrnorm(100, m, S)
```

```
x_normulti <- amostra_normulti[,1]
y_normulti <- amostra_normulti[,2]
#### Comparação

par(mfrow=c(1,2))

hist(x_Gibbs, main = "Gibbs Sampling")
hist(x_normulti, main = "pacote R")


par(mfrow=c(1,2))

hist(y_Gibbs, main = "Gibbs Sampling")

hist(y_normulti, main = "pacote R")

\end{verbatim}

\begin{figure}[ht]
    \centering
    \includegraphics[scale=0.75]{xsamples.pdf}
    \caption{histograma para amostras geradas pelo Gibbs sampling e pelo
Pacote MASS no R.}
    \label{Fig1}
\end{figure}

\begin{figure}[ht]
    \centering
    \includegraphics[scale=0.75]{ysamples.pdf}
    \caption{histograma para amostras geradas pelo Gibbs sampling e pelo
Pacote MASS no R.}
    \label{Fig2}
\end{figure}


Podemos notar com bases as figuras \ref{Fig1} e \ref{Fig2} que o algoritmo de
Gibbs sampler gerou amostras com comportamento muito similar ao obtido pelo
pacote R. Assim podemos concluir que o Gibbs sampler aproxima bem a normal
bivariada.

\end{document}
```