FSPL – Assignment 3

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1 Problem 2.1

$$Syntax : < comm > ::= < var >, < var > := < intexp >, < intexp >$$

$$Semantic : [\![v_1, v_2 := e_1, e_2]\!]_{comm}\sigma = [\sigma | v_1 : [\![e_1]\!]_{intexp}\sigma,\ v_2 : [\![e_2]\!]_{intexp}]\sigma$$

2 Problem 2.2

2.1 a

$$Syntax : < comm > ::= \textbf{\textit{repeat}} \ \ < comm > \ \ \textbf{\textit{until}} \ \ < boolexp >$$

$$Semantic : [\![\textbf{\textit{repeat}} \ c \ \textbf{\textit{until}} \ b]\!]_{comm}\sigma = Y_{\Sigma \to \Sigma_\perp} F$$

$$where \ F \ f \, \sigma = \Big(\textbf{\textit{if}} \ \neg b \ \ \textbf{\textit{then}} \, f \ \ \textbf{\textit{else}} \ \ \textbf{\textit{skip}} \Big)_{\perp\!\!\!\perp} ([\![c]\!]_{comm}\sigma)$$

2.2 b

$$Syntactic \ sugar : \ \ \textbf{\textit{repeat}} \ \ c \ \ \textbf{\textit{until}} \ \ b \stackrel{def}{=} c \, ; \textbf{\textit{while}} \neg \ b \ \ \textbf{\textit{do}} \, c$$

2.3 c

For all $\sigma \in \Sigma$, according to the definition in (a),

$$\begin{aligned}
[\![\textbf{\textit{repeat}} \ c \ \textbf{\textit{until}} \ b]\!]_{comm}\sigma &= F [\![\textbf{\textit{repeat}} \ c \ \textbf{\textit{until}} \ b]\!]_{comm}\sigma \\
&= \Big(\textbf{\textit{if}} \ \neg b \ \ \textbf{\textit{then}} \ (\textbf{\textit{repeat}} \ c \ \textbf{\textit{until}} \ b) \ \ \textbf{\textit{else}} \ \ \textbf{\textit{skip}} \Big)_{\perp\!\!\!\perp} ([\![c]\!]_{comm}\sigma) \\
&= \textbf{\textit{if}} \ \neg([\![b]\!]_{boolexp\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma)) \ \textbf{\textit{then}} \ (\textbf{\textit{repeat}} \ c \ \textbf{\textit{until}} \ b) \ {\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma) \ \textbf{\textit{else}} \ ([\![c]\!]_{comm}\sigma)
\end{aligned}$$

According to the definition in (b),

$$\begin{aligned}
[\![\textbf{\textit{repeat}} \ c \ \textbf{\textit{until}} \ b]\!]_{comm}\sigma &= [\![c \, ; \textbf{\textit{while}} \neg \ b \ \ \textbf{\textit{do}} \, c]\!]_{comm}\sigma \\
&= [\![\textbf{\textit{while}} \neg b \ \ \textbf{\textit{do}} \ \ c]\!]_{comm\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma) \\
&= \textbf{\textit{if}} \ \neg([\![b]\!]_{boolexp\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma)) \ \textbf{\textit{then}} \ [\![c \, ; \textbf{\textit{while}} \neg \ b \ \ \textbf{\textit{do}} \, c]\!]_{comm\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma) \ \textbf{\textit{else}} \ ([\![c]\!]_{comm}\sigma) \\
&= \textbf{\textit{if}} \ \neg([\![b]\!]_{boolexp\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma)) \ \textbf{\textit{then}} \ (\textbf{\textit{repeat}} \ c \ \textbf{\textit{until}} \ b) \ {\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma) \ \textbf{\textit{else}} \ ([\![c]\!]_{comm}\sigma)
\end{aligned}$$

Therefore, the two definitions are equivalent.

3 Problem 2.3

We can divide σx into three conditions and discuss them respectively.

(1) When σx is even and $\sigma x < 0$, $\sigma x - 2 < 0$.

So it will never terminates. So when $\sigma x < 0$, $[\![\textbf{\textit{while}}\ x \neq 0\ \ \textbf{\textit{do}}\ x := x - 2]\!]_{comm}\sigma = \perp$.

(2) When σx is odd, $\sigma x - 2$ is always add. But 0 is an even number, so x will never equal 0. So it will never terminates. So when $\sigma x < 0$, $[\![\textbf{\textit{while}}\ x \neq 0\ \ \textbf{\textit{do}}\ x := x - 2]\!]_{comm}\sigma = \perp$.

(3) When σx is even and $\sigma x \geq 0$, x must be in the form of $2k$ where $k \geq 0$. Let's do the induction.

When $\sigma x = 0$, it will terminate directly and go to the state $[\sigma | x : 0]$.

Suppose that when $\sigma x = 2k$, it will terminate in the state of $[\sigma | x : 0]$.

$$[\![\textbf{\textit{while}}\ x \neq 0\ \ \textbf{\textit{do}}\ x := x - 2]\!]_{comm}[\sigma | x : 2k + 2] \to^{*} [\![\textbf{\textit{while}}\ x \neq 0\ \ \textbf{\textit{do}}\ x := x - 2]\!]_{comm}[\sigma | x : 2k]$$

According the induction hypothesis, $[\![\textbf{\textit{while}}\ x \neq 0\ \ \textbf{\textit{do}}\ x := x - 2]\!]_{comm}[\sigma | x : 2k]$ will terminate in the state of $[\sigma | x : 0]$. So we can get that $[\![\textbf{\textit{while}}\ x \neq 0\ \ \textbf{\textit{do}}\ x := x - 2]\!]_{comm}[\sigma | x : 2k + 2]$ will terminate in the state of $[\sigma | x : 0]$.

Now we come to the conclusion that for all $k \geq 0$, $[\![\textbf{\textit{while}}\ x \neq 0\ \ \textbf{\textit{do}}\ x := x - 2]\!]_{comm}[\sigma | x : 2k]$ will terminate in the state of $[\sigma | x : 0]$.

In conlusion, by combining(1)(2)(3) together we can prove the equation in Problem2.3.

4 Problem 2.4

$$F\ f\sigma = \textbf{\textit{if}}\ [\![b]\!]_{boolexp}\sigma\ \textbf{\textit{then}}\ f_{\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma)\ \textbf{\textit{else}}\ \sigma$$

First we have to prove that F is monotone. Consider two functions f and h, where $f \sqsubseteq h$.

For all $\sigma \in \Sigma$, if $[\![b]\!]_{boolexp}\sigma = \textbf{\textit{false}}$, or if $[\![b]\!]_{boolexp}\sigma = \textbf{\textit{true}}$ and $([\![c]\!]_{comm}\sigma) = \perp$, it is obvious that $Ff\sigma = Fh\sigma$, which means $Ff\sigma \sqsubseteq Fh\sigma$. If $[\![b]\!]_{boolexp}\sigma = \textbf{\textit{true}}$ and $([\![c]\!]_{comm}\sigma) \neq \perp$, $Ff\sigma = f_{\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma), Fh\sigma = h_{\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma)$. Because $f \sqsubseteq h$, we have that for all x, $fx \sqsubseteq hx$. So $f_{\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma) \sqsubseteq h_{\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma)$ which means $Ff\sigma \sqsubseteq Fh\sigma$.

Next we have to prove that F is continous. Consider an interesting chain of functions $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq .....$ and $g = \sqcup_{n=0}^{\infty} f_n$.

For all $\sigma \in \Sigma$, if $[\![b]\!]_{boolexp}\sigma = \textbf{\textit{false}}$, or if $[\![b]\!]_{boolexp}\sigma = \textbf{\textit{true}}$ and $([\![c]\!]_{comm}\sigma) = \perp$, it is obvious that $F(\sqcup_{n=0}^{\infty} f_n)\sigma = (\sqcup_{n=0}^{'\infty} Ff_n)\sigma$, which means $F(\sqcup_{n=0}^{\infty} f_n)\sigma \sqsubseteq \sqcup_{n=0}^{'\infty}(Ff_n)\sigma$.

If $[\![b]\!]_{boolexp}\sigma = \textbf{\textit{true}}$ and $([\![c]\!]_{comm}\sigma) \neq \perp$,

$$F(\sqcup_{n=0}^{\infty} f_n)\sigma = Fg\sigma = g([\![c]\!]_{comm}\sigma)$$

$$\sqcup_{n=0}^{'\infty}(Ff_n)\sigma = \sqcup_{n=0}^{'\infty} f_n([\![c]\!]_{comm}\sigma) \sqsubseteq g([\![c]\!]_{comm}\sigma)$$

$$\therefore F(\sqcup_{n=0}^{\infty} f_n)\sigma \sqsubseteq \sqcup_{n=0}^{'\infty}(Ff_n)\sigma$$

In conclusion, F in the semantic equation for the *while* command is continous.

5 Problem 2.5

Define a series of commands: $w_0, w_1, w_2.....$

$$w_0 \stackrel{def}{=} \textbf{\textit{while true do skip}}$$

$$w_{i+1} \stackrel{def}{=} \textbf{\textit{if}}\ b\ \textbf{\textit{then}}\ (c; w_i)\ \textbf{\textit{else skip}}$$

Define the function F:

$$F\ f\,\sigma = \textbf{\textit{if}}\ [\![b]\!]_{boolexp}\sigma\ \textbf{\textit{then}}\ f_{\perp\!\!\!\perp}([\![c]\!]_{comm}\sigma)\ \textbf{\textit{else}}\ \sigma$$

$$where\ F[\![\ \textbf{\textit{while}}\ b\ \textbf{\textit{do}}\ c]\!]_{comm} = [\![\ \textbf{\textit{while}}\ b\ \textbf{\textit{do}}\ c]\!]_{comm}$$

For all $\sigma \in \Sigma$, consider the left side of the equation:

$$\begin{aligned}[\![\ \textbf{\textit{while}}\ b\ \textbf{\textit{do}}\ c]\!]_{comm}\sigma &= \sqcup_{n=0}^{\infty}[\![w_n]\!]_{comm}\sigma \\ &= \sqcup_{n=0}^{\infty}F^n\perp\sigma\end{aligned}$$

Consider the right side of the equation:

$$[\![\ \textbf{\textit{while}}\ b\ \textbf{\textit{do}}\ (c;\ \textbf{\textit{if}}\ b\ \textbf{\textit{then}}\ c\ \textbf{\textit{else skip}}\ )]\!]_{comm}\sigma = [\![\ \textbf{\textit{while}}\ b\ \textbf{\textit{do}}\ (c; w_1)]\!]_{comm}\sigma$$

Suppose $\textbf{\textit{while}}\ b\ \textbf{\textit{do}}\ (c; w_1)$ terminates after $\textbf{\textit{while}}$ testing b exactly $n(>0)$ times. It indicates that $\textbf{\textit{while}}\ b\ \textbf{\textit{do}}\ (c; w_1)$ terminates after testing b $2n-2$ or $2n-1$ times because both $\textbf{\textit{while}}$ and w_1 test b and it can terminates at both places. Therefore we can operate on the equation above:

$$\begin{aligned}[\![\ \textbf{\textit{while}}\ b\ \textbf{\textit{do}}\ (c;\ \textbf{\textit{if}}\ b\ \textbf{\textit{then}}\ c\ \textbf{\textit{else skip}}\ )]\!]_{comm}\sigma &= [\![\ \textbf{\textit{while}}\ b\ \textbf{\textit{do}}\ (c; w_1)]\!]_{comm}\sigma \\ &= \sqcup_{n=1}^{\infty}(F^{2n-2}\perp\sigma\ \sqcup\ F^{2n-1}\perp\sigma) \\ &= \sqcup_{n=0}^{\infty}F^n\perp\sigma\end{aligned}$$

$$\therefore [\![\ \textbf{\textit{while}}\ b\ \textbf{\textit{do}}\ c]\!]_{comm} = [\![\ \textbf{\textit{while}}\ b\ \textbf{\textit{do}}\ (c;\ \textbf{\textit{if}}\ b\ \textbf{\textit{then}}\ c\ \textbf{\textit{else skip}}\ )]\!]_{comm}$$

6 Problem 2.9

$$\textbf{\textit{for}}\ v := e_0\ \textbf{\textit{to}}\ e_1\ \textbf{\textit{do}}\ c \stackrel{def}{=} \textbf{\textit{newvar}}\ w := e_1\ \textbf{\textit{in}}\ \textbf{\textit{newvar}}\ v := e_0\ \textbf{\textit{in}}$$

$$(\textbf{\textit{while}}\ v < w\ \textbf{\textit{do}}\ (c; v := v+1));\ \textbf{\textit{if}}\ v = w\ \textbf{\textit{then}}\ c\ \textbf{\textit{else skip}}$$

7 Problem 2.10

If c doesn't contain occurence of $\textbf{\textit{dotwice}}$,

$$\textbf{\textit{dotwice}}\ c = c; c$$

If c contaions occurences of $\textbf{\textit{dotwice}}$, $c = \textbf{\textit{dotwice}}\ d$

$$\textbf{\textit{dotwice}}\ c = \textbf{\textit{dotwice}}\ (\textbf{\textit{dotwice}}\ d) = \textbf{\textit{dotwice}}\ (d; d) = d; d; d; d$$

The right side will eventually contain no occurence of $\textbf{\textit{dotwice}}$. If the d mentioned above also contains the occurence of $\textbf{\textit{dotwice}}$, the length of final result will just explode in an exponential way.