Game Theory – Homework 1

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Exercise a

Solution:

$$\because S_{n+1} \geq 2S_n$$

$$\therefore a_{n+1} \geq S_n$$

$$\therefore a_n \geq S_{n-1} \geq 2^1 S_{n-2} \geq 2^2 S_{n-3} \geq \cdots \geq 2^{n-2} S_1, n \geq 2$$

$$\therefore \frac{a_n}{2^n} \geq \frac{S_1}{4} = \frac{a_1}{4}$$

There exists a constant $c \in (0, \frac{a_1}{4}]$, such that $a_n \geq 2^n c$ for every positive n.

Exercise b

Suppose that the eigenvalue is λ.

$$\begin{pmatrix} 2 & -1 & b \\ 5 & a & 3 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} 2 - 1 - b = \lambda \\ 5 + a - 3 = \lambda \\ -1 + 2 + 1 = -\lambda \end{cases}$$

$$\therefore a = -4,\ b = 3,\ \lambda = -4$$

Exercise c

Prove that $f(\epsilon) = \frac{1}{2}(1 + \sqrt{1 + 4\epsilon^2})e^{1 - \sqrt{1+4\epsilon^2} + (\epsilon^2 - \epsilon^3)/2} \leq 1$.

$$g(\epsilon) = lnf(\epsilon) = ln(1 + \sqrt{1 + 4\epsilon^2}) - \sqrt{1 + 4\epsilon^2} + 1 - ln2 - \frac{1}{2}\epsilon^3 + \frac{1}{2}\epsilon^2$$

$$g'(\epsilon) = -\frac{4\epsilon}{\sqrt{1 + 4\epsilon^2}} - \frac{3}{2}\epsilon^2 + \epsilon$$

$$g''(\epsilon) = -\frac{4}{(1 + \sqrt{1 + 4\epsilon^2})\sqrt{1 + 4\epsilon^2}} - 3\epsilon + 1$$

Exercise d

Prove that in n-Cournet case, the Nash Equilibria is given by

$$\Big\{\Big(\frac{a-c}{(n+1)b},\cdots,\frac{a-c}{(n+1)b}\Big)\Big\}$$

证明. Assmue that $(q_1^*, q_2^*, \cdots, q_n^*)$ is a Nash equilibrium.
First, prove that $q_i^* > 0$ by contradiction. Suppose that $q_i^* = 0$, for $i = 1, 2, \cdots, n$. Then we get $u(q_1^*, q_2^*, \cdots, q_n^*) = (a - b(q_1^* + q_2^* + \cdots + q_n^*) - c)q_i^* = 0$, for $i = 1, 2, \cdots, n$. So, q_i^* can not equal 0 at the same time.
Without the loss of generality, assume that $q_1^* = 0$, while $q_i > 0$, for $i = 2, 3\cdots, n$.
From

$$\begin{cases} q_1^* = 0 \\ \dfrac{\partial u_i(q_1^*, q_2^*, \cdots, q_n^*)}{\partial q_i} = 0 \quad i = 2, 3, \cdots, n \end{cases}$$

We get

$$q_i^* = \frac{a - c - b\sum_{k=2,k\neq i}^n q_k^*}{2b} > 0, i = 2, 3, \cdots, n$$

and from this equation, we can get

$$bq_i^* = a - c - b\sum_{k=2}^n q_k^* > 0, i = 2, 3, \cdots, n$$

By definition of Nash equilibrium,

$$q_1^* = max\Big\{0, \frac{a - c - b\sum_{i=2}^n q_i^*}{2b}\Big\} = max\Big\{0, \frac{bq_i^*}{2b}\Big\} = \frac{q_i^*}{2}, i = 2, 3, \cdots, n$$

$$\therefore q_1^* \neq 0$$

We get the contradiction, so $q_i^* > 0$, for $i = 1, 2, \cdots, n$.

Second, since $q_i^* > 0,\ i = 1, 2, \cdots, n$,we have n equations.

$$q_i^* = \frac{a - c - b\sum_{k=1,k\neq i}^n q_k^*}{2b},\ i = 1, 2, \cdots, n.$$

$$\therefore q_i^* = \frac{a - c - b\sum_{k=1}^n q_k^*}{b},\ i = 1, 2, \cdots, n.$$

$$\therefore \sum_{i=1}^n q_i^* = \frac{n(a-c)}{b} - n\sum_{k=1}^n q_k^*$$

$$\sum_{i=1}^n q_i^* = \frac{n(a-c)}{(n+1)b}$$

$$\therefore q_i^* = \frac{a-c}{b} - \frac{n(a-c)}{(n+1)b} = \frac{a-c}{(n+1)b},\ i = 1, 2, \cdots, n$$

In conclusion, in n-Cournet case the Nash Equilibria is given by

$$\Big\{\Big(\frac{a-c}{(n+1)b},\cdots,\frac{a-c}{(n+1)b}\Big)\Big\}$$

□

Exercise e

Solution:

$$B_1(h) = \{c\}, B_1(i) = \{e\}, B_1(j) = \{e\}, B_1(k) = \{b, c\}, B_1(l) = \{e\}, B_1(m) = \{e\}$$

$$B_2(a) = \{i, l\}, B_2(b) = \{h\}, B_2(c) = \{m\}, B_2(d) = \{m\}, B_2(e) = \{l\}$$

Therefore, a pure Nash Equilibrium is $\{(e, l)\}$.

Exercise f