Game Theory – Homework 1

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Exercise a

Solution:

$$\because S_{n+1} \geq 2S_n$$

$$\therefore a_{n+1} \geq S_n$$

$$\therefore a_n \geq S_{n-1} \geq 2^1 S_{n-2} \geq 2^2 S_{n-3} \geq \cdots \geq 2^{n-2} S_1, n \geq 2$$

$$\therefore \frac{a_n}{2^n} \geq \frac{S_1}{4} = \frac{a_1}{4}$$

There exists a constant $c \in (0, \frac{a_1}{4}]$, such that $a_n \geq 2^n c$ for every positive n.

Exercise b

Suppose that the eigenvalue is λ.

$$\begin{pmatrix} 2 & -1 & b \\ 5 & a & 3 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} 2 - 1 - b = \lambda \\ 5 + a - 3 = \lambda \\ -1 + 2 + 1 = -\lambda \end{cases}$$

$$\therefore a = -4,\ b = 3,\ \lambda = -4$$

Exercise c

$$e^{-\frac{\epsilon^2-\epsilon^3}{2}-1+\sqrt{1+4\epsilon^2}} \geq -\frac{\epsilon^2-\epsilon^3}{2} + \sqrt{1+4\epsilon^2} = \frac{1}{2}\sqrt{1+4\epsilon^2} + \frac{1}{2} + \frac{1}{2}\sqrt{1+4\epsilon^2} - \frac{\epsilon^2-\epsilon^3+1}{2}$$

Next, prove that $\frac{1}{2}\sqrt{1+4\epsilon^2} \geq \frac{\epsilon^2-\epsilon^3+1}{2}$.

$$\begin{aligned} \frac{1}{2}\sqrt{1+4\epsilon^2} - \frac{\epsilon^2-\epsilon^3+1}{2} &\Leftrightarrow \sqrt{1+4\epsilon^2} \geq \epsilon^2 - \epsilon^3 + 1 \\ &\Leftrightarrow \sqrt{1+4\epsilon^2} - 1 \geq \epsilon^2 - \epsilon^3 \\ &\Leftrightarrow \frac{4\epsilon^2}{\sqrt{1+4\epsilon^2}+1} \geq \epsilon^2 - \epsilon^3 \\ &\Leftrightarrow \frac{4}{\sqrt{1+4\epsilon^2}+1} \geq 1 - \epsilon \end{aligned}$$

$$\because \frac{4}{\sqrt{1+4\epsilon^2}+1} \geq \frac{4}{\sqrt{5}+1} > 1 > 1-\epsilon$$
$$\therefore \frac{1}{2}\sqrt{1+4\epsilon^2} \geq \frac{\epsilon^2-\epsilon^3+1}{2}$$
$$\therefore e^{-\frac{\epsilon^2-\epsilon^3}{2}-1+\sqrt{1+4\epsilon^2}} \geq \frac{1}{2}\sqrt{1+4\epsilon^2}+\frac{1}{2}$$
$$\therefore \frac{1}{2}(1+\sqrt{1+4\epsilon^2})e^{1-\sqrt{1+4\epsilon^2}} \leq e^{-\frac{\epsilon^2-\epsilon^3}{2}}$$

Exercise d

Prove that in n-Cournet case, the Nash Equilibria is given by

$$\Big\{\Big(\frac{a-c}{(n+1)b}, \cdots, \frac{a-c}{(n+1)b}\Big)\Big\}$$

证明. Assmue that $(q_1^*, q_2^*, \cdots, q_n^*)$ is a Nash equilibrium.
First, prove that $q_i^* > 0$ by contradiction. Suppose that $q_i^* = 0$, for $i = 1, 2, \cdots, n$. Then we get $u(q_1^*, q_2^*, \cdots, q_n^*) = (a - b(q_1^* + q_2^* + \cdots + q_n^*) - c)q_i^* = 0$, for $i = 1, 2, \cdots, n$. So, q_i^* can not equal 0 at the same time.
Without the loss of generality, assume that $q_1^* = 0$, while $q_i > 0$, for $i = 2, 3 \cdots, n$.
From

$$\begin{cases} q_1^* = 0 \\ \dfrac{\partial u_i(q_1^*, q_2^*, \cdots, q_n^*)}{\partial q_i} = 0 \quad i = 2, 3, \cdots, n \end{cases}$$

We get

$$q_i^* = \frac{a-c-b\sum_{k=2,k\neq i}^n q_k^*}{2b} > 0, i = 2, 3, \cdots, n$$

and from this equation, we can get

$$bq_i^* = a-c-b\sum_{k=2}^n q_k^* > 0, i = 2, 3, \cdots, n$$

By definition of Nash equilibrium,

$$q_1^* = max\Big\{0, \frac{a-c-b\sum_{i=2}^n q_i^*}{2b}\Big\} = max\Big\{0, \frac{bq_i^*}{2b}\Big\} = \frac{q_i^*}{2}, i = 2, 3, \cdots, n$$
$$\therefore q_1^* \neq 0$$

We get the contradiction, so $q_i^* > 0$, for $i = 1, 2, \cdots, n$.

Second, since $q_i^* > 0$, $i = 1, 2, \cdots, n$,we have n equations.

$$q_i^* = \frac{a-c-b\sum_{k=1,k\neq i}^n q_k^*}{2b}, \ i = 1, 2, \cdots, n.$$
$$\therefore q_i^* = \frac{a-c-b\sum_{k=1}^n q_k^*}{b}, \ i = 1, 2, \cdots, n.$$
$$\therefore \sum_{i=1}^n q_i^* = \frac{n(a-c)}{b} - n\sum_{k=1}^n q_k^*$$
$$\sum_{i=1}^n q_i^* = \frac{n(a-c)}{(n+1)b}$$

$$\therefore q_i^* = \frac{a-c}{b} - \frac{n(a-c)}{(n+1)b} = \frac{a-c}{(n+1)b},\ i = 1, 2, \cdots, n$$

In conclusion, in n-Cournet case the Nash Equilibria is given by

$$\Big\{\Big(\frac{a-c}{(n+1)b}, \cdots, \frac{a-c}{(n+1)b}\Big)\Big\}$$

$\square$

Exercise e

Solution:

$$B_1(h) = \{c\}, B_1(i) = \{e\}, B_1(j) = \{e\}, B_1(k) = \{b, c\}, B_1(l) = \{e\}, B_1(m) = \{e\}$$

$$B_2(a) = \{i, l\}, B_2(b) = \{h\}, B_2(c) = \{m\}, B_2(d) = \{m\}, B_2(e) = \{l\}$$

Therefore, a pure Nash Equilibrium is $\{(e, l)\}$.

Exercise f

Let player1 denotes the King, player2 denotes Tian Ji. Suppose that the distribution of player1 is $(a, b, c, d, e, 1-a-b-c-d-e)$. We have

$$\begin{cases} u_2(1, p_{-1}) = -3a - b - c + d - e - (1-a-b-c-d-e) \\ u_2(2, p_{-1}) = -a - 3b + c - d - e - (1-a-b-c-d-e) \\ u_2(3, p_{-1}) = -a - b - 3c - d - e + (1-a-b-c-d-e) \\ u_2(4, p_{-1}) = -a - b - c - 3d + e - (1-a-b-c-d-e) \\ u_2(5, p_{-1}) = a - b - c - d - 3e - (1-a-b-c-d-e) \\ u_2(6, p_{-1}) = -a + b - c - d - e - 3(1-a-b-c-d-e) \\ u_2(1, p_{-1}) = u_2(2, p_{-1}) = u_2(3, p_{-1}) = u_2(4, p_{-1}) = u_2(5, p_{-1}) = u_2(6, p_{-1}) \end{cases}$$

$$\Rightarrow \begin{cases} a = d = e \\ b = c \\ a + b = \dfrac{1}{3} \end{cases}$$

Let $a = b = c = d = e = 1 - a - b - c - d - e = \frac{1}{6}$.

By symmetricity, $\Big\{\Big(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\Big), \Big(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\Big)\Big\}$ can be a mixed Nash Equilibrium.