STATS100B – Introduction to Mathematical Statistics
Homework 7

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Question a

Solution :

$$L=\frac{1}{(2\pi)^{\frac{n}{2}}}|\boldsymbol{\Sigma}|^{-\frac{1}{2}}e^{\frac{1}{2}(\boldsymbol{Y}-\boldsymbol{\mu}\mathbf{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y}-\boldsymbol{\mu}\mathbf{1})}$$

$$lnL=-\frac{n}{2}ln(2\pi\sigma^2)-\frac{1}{2}ln|V|-\frac{1}{2}\sigma^2(Y-\mu)V^{-1}(Y-\mu)$$

$$\frac{\partial\ln L}{\partial\mu}=-\frac{1}{2\sigma^2}\left[-Y'V^{-1}1-1'V^{-1}Y+2\mu1'V^{-1}1\right]=0$$

$$\therefore\hat{\mu}=\frac{1'V^{-1}Y}{1'V^{-1}1}$$

$$\frac{\partial\ln L}{\partial\sigma^2}=-\frac{n}{2\sigma^2}+\frac{1}{\sigma^4}\left(Y-\mu1\right)'V^{-1}\left(Y-\mu1\right)=0$$

$$\therefore\hat{\sigma^2}=\frac{\left(Y-\hat{\mu}1\right)'V^{-1}\left(Y-\hat{\mu}1\right)}{n}$$

Question b

Solution :

$$E(\hat{\mu})=E\left(\frac{1'V^{-1}Y}{1'V^{-1}1}\right)=\frac{1'V^{-1}E(Y)}{1'V^{-1}1}=\frac{1'V^{-1}\mu1}{1'V^{-1}1}=\mu$$

Question c

Solution:

$$\boldsymbol{I}(\boldsymbol{\theta})=-E\begin{pmatrix}\frac{\partial^2\ln L}{\partial\mu^2} & \frac{\partial^2\ln L}{\partial\mu\partial\sigma^2}\\ \\ \frac{\partial^2\ln L}{\partial\sigma^2\partial\mu} & \frac{\partial\ln L}{\partial\sigma^{2(2)}}\end{pmatrix}$$

$$\frac{\partial^2\ln L}{\partial\mu^2}=-\frac{1'V^{-1}1}{\sigma^2}$$

$$\frac{\partial^2\ln L}{\partial\mu\partial\sigma^2}=\frac{1}{\sigma^4}\left[\mu1'V^{-1}1-1'V^{-1}Y\right]$$

$$\frac{\partial^2 \ln L}{\partial \sigma^2 \partial \mu} = \frac{1}{\sigma^4}\left[\mu 1' V^{-1} 1 - 1' V^{-1} Y\right]$$

$$\frac{\partial \ln L}{\partial {\sigma^2}^{(2)}} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\left(Y - \mu 1\right)' V^{-1}\left(Y - \mu 1\right)$$

$$\therefore \boldsymbol{I}(\boldsymbol{\theta}) = \left(-\frac{1' V^{-1} 1}{\sigma^2}\right)$$

Question d

Solution:

$$\begin{aligned}
\sum_{i=1}^{4}\left(X_i - \bar{X}\right)^2 &= \sum_{i=1}^{4} X_i^2 - 4\bar{X}^2 \\
&= \sum_{i=1}^{4} X_i^2 - 4\Big[\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\Big]^2 \\
&= \sum_{i=1}^{4} X_i^2 - \frac{1}{2}\left(\sum_{i=1}^{4} X_i^2 + 2\sum_{1 \le i < j \le 4} X_i X_j\right) \\
&= \frac{3}{4}\sum_{i=1}^{4} X_i^2 - \frac{1}{2}\sum_{1 \le i < j \le 4} X_i X_j
\end{aligned}$$

$$\begin{aligned}
RHS &= \Big(\frac{1}{2}X_1^2 - X_1X_2 + \frac{1}{2}X_2^2\Big) + \Big(\frac{2}{3}X_3^2 + \frac{1}{6}\left(X_1 + X_2\right)^2 - \frac{2}{3}X_3\left(X_1 + X_2\right)\Big) + \frac{3}{4}\Big(X_4 - \frac{1}{3}(X_1 + X_2 + X_3)\Big)^2 \\
&= \cdots\cdots (simple\, but\, tedious\, simplifications) \\
&= \frac{3}{4}\sum_{i=1}^{4} X_i^2 - \frac{1}{2}\sum_{1 \le i < j \le 4} X_i X_j
\end{aligned}$$

$$\therefore \sum_{i=1}^{4}\left(X_i - \bar{X}\right)^2 = \frac{\left(X_1 - X_2\right)^2}{2} + \frac{\Big[X_3 - \frac{(X_1+X_2)}{2}\Big]^2}{\frac{3}{2}} \frac{\Big[X_4 - \frac{(X_1+X_2+X_3)}{3}\Big]^2}{\frac{4}{3}}$$

$$\because X_1 - X_2 \sim N(0, \sqrt{2})$$

$$\therefore \frac{(X_1 - X_2)^2}{2} \sim \chi_1^2$$

$$\because X_3 - \frac{X_1 + X_2}{2} \sim N(0, \sqrt{\frac{3}{2}})$$

$$\therefore \frac{(X_3 - \frac{X_1+X_2}{2})^2}{\frac{3}{2}} \sim \chi_1^2$$

$$\because X_4 - \frac{X_1 + X_2 + X_3}{3} \sim N(0, \sqrt{\frac{4}{3}})$$

$$\therefore \frac{(X_4 - \frac{X_1+X_2+X_3}{3})^2}{\frac{4}{3}} \sim \chi_1^2$$

$$\boldsymbol{X} = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \end{pmatrix}'$$

$$\begin{pmatrix} X_1 - X_2 \\ X_3 - \frac{X_1+X_2}{2} \\ X_4 - \frac{X_1+X_2+X3}{3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \boldsymbol{AX}$$

$$var(\boldsymbol{AX}) = \boldsymbol{A}var(\boldsymbol{X})\boldsymbol{A}' = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} \\ -1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

So the three terms in the RHS are independent each with a χ_1^2 distribution.