

# STATS100B – Introduction to Mathematical Statistics

## Homework 7

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### Question a

Solution:

$$\begin{aligned} L &= \frac{1}{(2\pi)^{\frac{n}{2}}} |\Sigma|^{-\frac{1}{2}} e^{\frac{1}{2}(\mathbf{Y}-\mu\mathbf{1})'\Sigma^{-1}(\mathbf{Y}-\mu\mathbf{1})} \\ \ln L &= -\frac{n}{2}\ln(2\pi\Sigma^2) - \frac{1}{2}\ln|\mathbf{V}| - \frac{1}{2\sigma^2}(\mathbf{Y}-\mu)\mathbf{V}^{-1}(\mathbf{Y}-\mu) \\ \frac{\partial \ln L}{\partial \mu} &= -\frac{1}{2\sigma^2} [-\mathbf{Y}'\mathbf{V}^{-1}\mathbf{1} - \mathbf{1}'\mathbf{V}^{-1}\mathbf{Y} + 2\mu\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}] = 0 \\ \therefore \hat{\mu} &= \frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} \\ \frac{\partial \ln L}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{\sigma^4} (\mathbf{Y}-\mu\mathbf{1})'\mathbf{V}^{-1}(\mathbf{Y}-\mu\mathbf{1}) = 0 \\ \therefore \hat{\sigma}^2 &= \frac{(\mathbf{Y}-\hat{\mu}\mathbf{1})'\mathbf{V}^{-1}(\mathbf{Y}-\hat{\mu}\mathbf{1})}{n} \end{aligned}$$

### Question b

Solution:

$$\begin{aligned} E(\hat{\mu}) &= E\left(\frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\right) = \frac{\mathbf{1}'\mathbf{V}^{-1}E(\mathbf{Y})}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} = \frac{\mathbf{1}'\mathbf{V}^{-1}\mu\mathbf{1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} = \mu \\ E(\hat{\sigma}^2) &= \frac{1}{n}E(\mathbf{Y}-\hat{\mu}\mathbf{1})'\mathbf{V}^{-1}(\mathbf{Y}-\hat{\mu}\mathbf{1}) \\ &= \frac{1}{n}E[\text{tr}(\mathbf{Y}-\hat{\mu}\mathbf{1})'\mathbf{V}^{-1}(\mathbf{Y}-\hat{\mu}\mathbf{1})] \\ &= \frac{1}{n}\text{tr}E[\mathbf{V}^{-1}(\mathbf{Y}-\hat{\mu}\mathbf{1})(\mathbf{Y}-\hat{\mu}\mathbf{1})'] \\ &= \frac{1}{n}\text{tr}\mathbf{V}^{-1}E[(\mathbf{Y}-\hat{\mu}\mathbf{1})(\mathbf{Y}-\hat{\mu}\mathbf{1})'] \\ &= \frac{1}{n}\text{tr}\mathbf{V}^{-1}[\text{var}(\mathbf{Y}-\hat{\mu}\mathbf{1}) + E(\mathbf{Y}-\hat{\mu}\mathbf{1})E(\mathbf{Y}-\hat{\mu}\mathbf{1})'] \\ &= \frac{1}{n}\text{tr}[\mathbf{V}^{-1}\text{var}(\mathbf{Y}-\hat{\mu}\mathbf{1})] \end{aligned}$$

$$\begin{aligned}
\text{var}(\mathbf{Y} - \hat{\mu}\mathbf{1}) &= \text{var}\left(\mathbf{Y} - \frac{\mathbf{1}\mathbf{V}^{-1}\mathbf{Y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\mathbf{1}\right) \\
&= \text{var}\left(\mathbf{Y} - \frac{\mathbf{1}\mathbf{V}^{-1}\mathbf{Y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\right) \\
&= \text{var}\left[\left(\mathbf{I} - \frac{\mathbf{1}\mathbf{1}'\mathbf{V}^{-1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\right)\mathbf{Y}\right] \\
&= \left(\mathbf{I} - \frac{\mathbf{1}\mathbf{1}'\mathbf{V}^{-1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\right)\sigma^2\mathbf{V}\left(1 - \frac{\mathbf{1}\mathbf{1}'\mathbf{V}^{-1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\right)' \\
&= \sigma^2\left(\mathbf{V} - \frac{\mathbf{1}\mathbf{1}'}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\right)
\end{aligned}$$

$$\begin{aligned}
\therefore E(\hat{\sigma}^2) &= \frac{1}{n}\text{tr}[\mathbf{V}^{-1}\text{var}(\mathbf{Y} - \hat{\mu}\mathbf{1})] \\
&= \frac{1}{n}\sigma^2\left[\text{tr}\mathbf{V}^{-1}\mathbf{V} - \text{tr}\frac{\mathbf{V}^{-1}\mathbf{1}\mathbf{1}'}{\mathbf{1}'\mathbf{V}\mathbf{1}}\right] \\
&= \frac{1}{n}\sigma^2\left(n - \frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}'}{\mathbf{1}'\mathbf{V}\mathbf{1}}\right) \\
&= \frac{n-1}{n}\sigma^2
\end{aligned}$$

### Question c

Solution:

$$\begin{aligned}
\mathbf{I}(\boldsymbol{\theta}) &= -E\begin{pmatrix} \frac{\partial^2 \ln L}{\partial \mu^2} & \frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} \\ \frac{\partial^2 \ln L}{\partial \sigma^2 \partial \mu} & \frac{\partial^2 \ln L}{\partial \sigma^2(2)} \end{pmatrix} \\
\frac{\partial^2 \ln L}{\partial \mu^2} &= -\frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}{\sigma^2} \\
\frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} &= \frac{1}{\sigma^4}[\mu\mathbf{1}'\mathbf{V}^{-1}\mathbf{1} - \mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}] \\
\frac{\partial^2 \ln L}{\partial \sigma^2 \partial \mu} &= \frac{1}{\sigma^4}[\mu\mathbf{1}'\mathbf{V}^{-1}\mathbf{1} - \mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}] \\
\frac{\partial^2 \ln L}{\partial \sigma^2(2)} &= \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}(\mathbf{Y} - \mu\mathbf{1})'\mathbf{V}^{-1}(\mathbf{Y} - \mu\mathbf{1}) \\
\therefore \mathbf{I}^{-1}(\boldsymbol{\theta}) &= \begin{pmatrix} \frac{\sigma^2}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} & 0 \\ 0 & \frac{2\sigma^2}{n} \end{pmatrix}, \text{var}(\hat{\sigma}^2) \geq -\frac{1}{E\left(\frac{\partial^2 \ln L}{\partial \mu^2}\right)} = \frac{\sigma^2}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}
\end{aligned}$$

$$\therefore E(\hat{\mu}) = \mu$$

$$\text{var}(\hat{\mu}) = \text{var}\left(\frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\right) = \frac{\text{var}[(\mathbf{1}'\mathbf{V}^{-1})\mathbf{Y}]}{(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1})^2} = \frac{(\mathbf{1}'\mathbf{V}^{-1})\sigma^2\mathbf{V}(\mathbf{1}'\mathbf{V}^{-1})'}{(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1})^2} = \frac{\sigma^2}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}$$

$\therefore \hat{\mu}$  is efficient.

### Question d

Solution:

$$\begin{aligned}
 f(y_i) &= \frac{1}{\sqrt{2\pi i\sigma}} e^{-\frac{(y-i\theta)^2}{2(i\sigma)^2}} \\
 L &= \prod_{i=1}^n f(y_i) = (2\pi)^{-\frac{n}{2}} \prod_{i=1}^n i\sigma^{-n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n \left(\frac{y_i-i\theta}{i}\right)^2} \\
 \ln L &= -\frac{n}{2} \ln 2\pi + \sum_{i=1}^n \ln i - n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^n \left(\theta - \frac{y_i}{i}\right)^2 \\
 \frac{\partial \ln L}{\partial \theta} &= -\frac{1}{\sigma^2} \sum_{i=1}^n \left(\theta - \frac{y_i}{i}\right) = 0 \\
 \therefore \hat{\theta} &= \frac{1}{n} \sum_{i=1}^n \frac{y_i}{i} \\
 \text{var}(\hat{\theta}) &= \frac{1}{n^2} \sum \frac{\text{var}(y_i)}{i^2} = \frac{1}{n^2} \sum_{i=1}^n \frac{i^2 \sigma^2}{i^2} = \frac{\sigma^2}{n} \\
 \therefore \frac{\partial^2 \ln L}{\partial \theta^2} &= -\frac{n}{\sigma^2}, \text{var}(\hat{\theta}) \geq -\frac{1}{E\left(\frac{\partial^2 \ln L}{\partial \theta^2}\right)} = \frac{\sigma^2}{n} \\
 \therefore \hat{\theta} &\text{ is an efficient estimator.}
 \end{aligned}$$

### Question e

Suppose the radius equals  $R$ .  $R_i = R + \epsilon_i$ ,  $i = 1, 2, \dots, n$ .

$$\begin{aligned}
 E(\bar{R}) &= \frac{1}{n} \sum_{i=1}^n E(R + \epsilon_i) = \frac{1}{n} \sum_{i=1}^n R + E(\epsilon_1) = R \\
 E(\bar{R}^2) &= \text{var}(\bar{R}) + (E\bar{R})^2 = \frac{1}{n^2} \sum_{i=1}^n \text{var}(R + \epsilon_i) + R^2 = \frac{\sigma^2}{n} + R^2 \\
 \therefore \text{Define } \hat{R}^2 &= \bar{R}^2 - \frac{S^2}{n}, E(\hat{R}^2) = R^2 \\
 \therefore \text{The unbiased estimator of area is } \hat{A} &= \pi \left( \bar{R}^2 - \frac{S^2}{n} \right)
 \end{aligned}$$

### Question f

$$\begin{aligned}
 X &= Y^2 \\
 F_X(x) &= P(X \leq x) \\
 &= P(Y \leq \sqrt{x}) \\
 &= \int_0^{\sqrt{x}} \frac{2y}{\theta} e^{-\frac{y^2}{\theta}} dy
 \end{aligned}$$

$$\begin{aligned}
f(x) &= F'_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \\
\therefore Y^2 &\sim \exp\left(\frac{1}{\theta}\right) \\
\text{var}(\hat{\theta}) &= \frac{1}{n^2} \sum_{i=0}^n \text{var}(Y_i^2) = \frac{1}{n^2} n \theta^2 = \frac{\theta^2}{n} \\
\therefore \frac{\partial^2 \ln f(x)}{\partial \theta^2} &= \frac{1}{\theta^2} - \frac{2x}{\theta^3} \\
\text{var}(\hat{\theta}) &\geq -\frac{1}{nE\left(\frac{1}{\theta^2} - \frac{2x}{\theta^3}\right)} = -\frac{1}{nE\left(\frac{1}{\theta^2} - \frac{2E(x)}{\theta^3}\right)} = \frac{\theta^2}{n} \\
\therefore \hat{\theta} &\text{ is efficient}
\end{aligned}$$

### Question g

$$\begin{aligned}
E(T - \theta)^2 &= \text{var}(T) + B^2 \\
\text{var}(T) &= \text{var}(\alpha_1 \bar{X} + \alpha_2 cS) \\
&= \alpha_1^2 \text{var}(\bar{X}) + \alpha_2^2 \text{var}(cS) \\
&= \alpha_1^2 \frac{\theta^2}{n} + \alpha_2^2 (c^2 - 1) \theta^2 \\
B &= E(T) - \theta = (\alpha_1 + \alpha_2 - 1) \theta \\
\therefore E(T - \theta)^2 &= \left[ \frac{\alpha_1^2}{n} + (c^2 - 1) \alpha_2^2 + (\alpha_1 + \alpha_2 - 1)^2 \right] \theta^2 \\
\begin{cases} \frac{\partial E(T - \theta)^2}{\partial \alpha_1} = \frac{2\alpha_1}{n} + 2(\alpha_1 + \alpha_2 - 1) = 0 \\ \frac{\partial E(T - \theta)^2}{\partial \alpha_2} = 2(c^2 - 1) \alpha_2 + 2(\alpha_1 + \alpha_2 - 1) = 0 \end{cases} \\
\therefore \begin{cases} \alpha_1 = \frac{n(c^2 - 1)}{(n + 1)(c^2 - 1) + 1} \\ \alpha_2 = \frac{1}{(n + 1)(c^2 - 1) + 1} \end{cases} \\
A = \frac{\partial^2 E}{\partial \alpha_1^2} = \frac{2}{n} + 2, B = \frac{\partial^2 E}{\partial \alpha_1 \partial \alpha_2} = 2, C = \frac{\partial^2 E}{\partial \alpha_2^2} = 2c^2 \\
\therefore A < 0, B^2 - AC = 4 \left[ 1 - \frac{n + 1}{n} c^2 \right] < 0 \text{ (Using WolframAlpha)} \\
\therefore E(T - \theta)^2 &\text{ gets the minimum when } \alpha_1 \text{ and } \alpha_2 \text{ equal the above values.} \\
\therefore T &= \frac{n(c^2 - 1)}{(n + 1)(c^2 - 1) + 1} \bar{X} + \frac{1}{(n + 1)(c^2 - 1) + 1} cS \text{ is the estimator that minimizes } E(T - \theta)^2
\end{aligned}$$