STATS100B – Introduction to Mathematical Statistics
Homework 2

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Exercise 1

Solution:

$$
\begin{aligned}
P(X=x) &= \binom{n}{x}\theta^x(1-\theta)^{n-x} \\
&= \binom{n}{x}(1-p)^n exp\{ln\frac{p}{1-p}x\}
\end{aligned}
$$

so we get

$$c(\boldsymbol{\theta}) = (1-p)^n,\ \sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x) = ln\frac{p}{1-p}x$$

$$\sum_{i=1}^{k}\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(x) = \frac{1}{p(1-p)}x,\ \sum_{i=1}^{k}\frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial\theta_j^2}t_i(x) = \frac{2p-1}{p^2(1-p)^2}x$$

$$\frac{\partial^2}{\partial\theta_j^2}logc(\boldsymbol{\theta}) = -\frac{n}{(1-p)^2}$$

Use the second theorem, we can get the $var(X)$.

$$var(\frac{1}{p(1-p)}X) = \frac{n}{(1-p)^2} - E(\frac{2p-1}{p^2(1-p)^2}X)$$

$$\frac{1}{p^2(1-p)^2}var(X) = \frac{n}{(1-p)^2} - \frac{2p-1}{p^2(1-p)^2}E(X) = \frac{n}{(1-p)^2} - \frac{2p-1}{p^2(1-p)^2}np$$

$$\therefore var(X) = np(1-p)$$

Exercise 2

Solution:

$$\int_x h(x)c(\boldsymbol{\theta})exp\{\sum_{i=1}^{k}w_i(\boldsymbol{\theta})t_i(x)\}dx = 1$$

Differentiate both side w.r.t θ_j,

$$\int_x\Big[h(x)\frac{\partial c(\boldsymbol{\theta})}{\partial\theta_j}exp\big(\sum_{i=1}^{k}w_i(\boldsymbol{\theta})t_i(x)\big) + h(x)c(\boldsymbol{\theta})\sum_{i=1}^{k}\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(x)exp\big(\sum_{i=1}^{k}w_i(\boldsymbol{\theta})t_i(x)\big)\Big]dx = 0$$

$$\int_x\sum_{i=1}^{k}\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(x)\Big[h(x)c(\boldsymbol{\theta})exp\big(\sum_{i=1}^{k}w_i(\boldsymbol{\theta})t_i(x)\big)\Big]dx = -\frac{1}{c(\boldsymbol{\theta})}\frac{\partial c(\boldsymbol{\theta})}{\partial\theta_j}\int_x\Big[h(x)c(\boldsymbol{\theta})exp\big(\sum_{i=1}^{k}w_i(\boldsymbol{\theta})t_i(x)\big)\Big]dx$$

$$\int_x \left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x) \right) f(x) dx = -\frac{\partial logc(\boldsymbol{\theta})}{\partial \theta_j} \int_x f(x) dx$$

$$\because \int_x f(x) dx = 1$$

$$\therefore E\left(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x) \right) = -\frac{\partial logc(\boldsymbol{\theta})}{\partial \theta_j}$$