STATS100B – Introduction to Mathematical Statistics
Homework 7

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Question a

Solution:

$$L = \frac{1}{(2\pi)^{\frac{n}{2}}} |\mathbf{\Sigma}|^{-\frac{1}{2}} e^{\frac{1}{2}(\boldsymbol{Y}-\mu\mathbf{1})'\mathbf{\Sigma}^{-1}(\boldsymbol{Y}-\mu\mathbf{1})}$$

$$lnL = -\frac{n}{2}ln(2\pi\mathbf{\Sigma}^2) - \frac{1}{2}ln|\boldsymbol{V}| - \frac{1}{2\sigma^2}(\boldsymbol{Y}-\mu)V^{-1}(\boldsymbol{Y}-\mu)$$

$$\frac{\partial \ln L}{\partial \mu} = -\frac{1}{2\sigma^2}\left[-\boldsymbol{Y'V^{-1}1} - \boldsymbol{1'V^{-1}Y} + 2\boldsymbol{\mu 1'V^{-1}1}\right] = 0$$

$$\therefore \hat{\mu} = \frac{\boldsymbol{1'V^{-1}Y}}{\boldsymbol{1'V^{-1}1}}$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{\sigma^4}(\boldsymbol{Y}-\mu\mathbf{1})'\boldsymbol{V}^{-1}(\boldsymbol{Y}-\mu\mathbf{1}) = 0$$

$$\therefore \hat{\sigma^2} = \frac{(\boldsymbol{Y}-\hat{\mu}\mathbf{1})'\boldsymbol{V}^{-1}(\boldsymbol{Y}-\hat{\mu}\mathbf{1})}{n}$$

Question b

Solution:

$$E(\hat{\mu}) = E\left(\frac{\boldsymbol{1'V^{-1}Y}}{\boldsymbol{1'V^{-1}1}}\right) = \frac{\mathbf{1}'\boldsymbol{V}^{-1}E(\boldsymbol{Y})}{\boldsymbol{1'V^{-1}1}} = \frac{\mathbf{1}'\boldsymbol{V}^{-1}\mu\mathbf{1}}{\boldsymbol{1'V^{-1}1}} = \mu$$

$$\begin{aligned}
E(\hat{\sigma^2}) &= \frac{1}{n}E(\boldsymbol{Y}-\hat{\mu}\mathbf{1})'\boldsymbol{V}^{-1}(\boldsymbol{Y}-\hat{\mu}\mathbf{1}) \\
&= \frac{1}{n}E\left[tr(\boldsymbol{Y}-\hat{\mu}\mathbf{1})'\boldsymbol{V}^{-1}(\boldsymbol{Y}-\hat{\mu}\mathbf{1})\right] \\
&= \frac{1}{n}trE\left[\boldsymbol{V}^{-1}(\boldsymbol{Y}-\hat{\mu}\mathbf{1})(\boldsymbol{Y}-\hat{\mu}\mathbf{1})'\right] \\
&= \frac{1}{n}tr\boldsymbol{V}^{-1}E\left[(\boldsymbol{Y}-\hat{\mu}\mathbf{1})(\boldsymbol{Y}-\hat{\mu}\mathbf{1})'\right] \\
&= \frac{1}{n}tr\boldsymbol{V}^{-1}\left[var(\boldsymbol{Y}-\hat{\mu}\mathbf{1}) + E(\boldsymbol{Y}-\hat{\mu}\mathbf{1})E(Y-\hat{\mu}1)\right] \\
&= \frac{1}{n}tr\left[\boldsymbol{V}^{-1}var(\boldsymbol{Y}-\hat{\mu}\mathbf{1})\right]
\end{aligned}$$

$$
\begin{aligned}
var\left(\boldsymbol{Y}-\hat{\mu}\boldsymbol{1}\right) &= var\left(\boldsymbol{Y}-\frac{\boldsymbol{1}\boldsymbol{V}^{-1}\boldsymbol{Y}}{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}\boldsymbol{1}\right)\\
&= var\left(\boldsymbol{Y}-\frac{\boldsymbol{1}\boldsymbol{1}\boldsymbol{V}^{-1}\boldsymbol{Y}}{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}\right)\\
&= var\left[\left(\boldsymbol{I}-\frac{\boldsymbol{1}\boldsymbol{1}'\boldsymbol{V}^{-1}}{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}\right)\boldsymbol{Y}\right]\\
&= \left(\boldsymbol{I}-\frac{\boldsymbol{1}\boldsymbol{1}'\boldsymbol{V}^{-1}}{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}\right)\sigma^2\boldsymbol{V}\left(1-\frac{\boldsymbol{1}\boldsymbol{1}'\boldsymbol{V}^{-1}}{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}\right)'\\
&= \sigma^2\left(\boldsymbol{V}-\frac{\boldsymbol{1}\boldsymbol{1}'}{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\therefore E(\hat{\sigma^2}) &= \frac{1}{n}tr\left[\boldsymbol{V}^{-1}var\left(\boldsymbol{Y}-\hat{\mu}\boldsymbol{1}\right)\right]\\
&= \frac{1}{n}\sigma^2\left[tr\boldsymbol{V}^{-1}\boldsymbol{V}-tr\frac{\boldsymbol{V}^{-1}\boldsymbol{1}\boldsymbol{1}'}{\boldsymbol{1}'\boldsymbol{V}\boldsymbol{1}}\right]\\
&= \frac{1}{n}\sigma^2\left(n-\frac{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}'}{\boldsymbol{1}'\boldsymbol{V}\boldsymbol{1}}\right)\\
&= \frac{n-1}{n}\sigma^2
\end{aligned}
$$

Question c

Solution:

$$
\boldsymbol{I}(\boldsymbol{\theta}) = -E\begin{pmatrix}\dfrac{\partial^2\ln L}{\partial\mu^2} & \dfrac{\partial^2\ln L}{\partial\mu\partial\sigma^2}\\[2ex] \dfrac{\partial^2\ln L}{\partial\sigma^2\partial\mu} & \dfrac{\partial\ln L}{\partial\sigma^{2(2)}}\end{pmatrix}
$$

$$
\frac{\partial^2\ln L}{\partial\mu^2} = -\frac{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}{\sigma^2}
$$

$$
\frac{\partial^2\ln L}{\partial\mu\partial\sigma^2} = \frac{1}{\sigma^4}\left[\mu\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}-\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{Y}\right]
$$

$$
\frac{\partial^2\ln L}{\partial\sigma^2\partial\mu} = \frac{1}{\sigma^4}\left[\mu\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}-\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{Y}\right]
$$

$$
\frac{\partial\ln L}{\partial\sigma^{2(2)}} = \frac{n}{2\sigma^4}-\frac{1}{\sigma^6}\left(\boldsymbol{Y}-\mu\boldsymbol{1}\right)'\boldsymbol{V}^{-1}\left(\boldsymbol{Y}-\mu\boldsymbol{1}\right)
$$

$$
\therefore \boldsymbol{I}^{-1}(\boldsymbol{\theta}) = \begin{pmatrix}\dfrac{\sigma^2}{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}} & 0\\ 0 & \dfrac{2\sigma^2}{n}\end{pmatrix},\ var(\hat{\sigma^2}) \geq -\frac{1}{E\left(\dfrac{\partial^2\ln L}{\partial\mu^2}\right)} = \frac{\sigma^2}{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}
$$

$$
\because E\left(\hat{\mu}\right)=\mu,\ var\left(\hat{\mu}\right) = var\left(\frac{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{Y}}{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}\right) = \frac{var\left[\left(\boldsymbol{1}'\boldsymbol{V}^{-1}\right)\boldsymbol{Y}\right]}{\left(\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}\right)^2} = \frac{\left(\boldsymbol{1}'\boldsymbol{V}^{-1}\right)\sigma^2\boldsymbol{V}\left(\boldsymbol{1}'\boldsymbol{V}^{-1}\right)'}{\left(\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}\right)^2} = \frac{\sigma^2}{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}
$$

$$
\therefore \hat{\mu}\ is\ efficient.
$$

Question d

Solution:

$$f\left(y_i\right)=\frac{1}{\sqrt{2\pi}i\sigma}e^{-\frac{(y-i\theta)^2}{2(i\sigma)^2}}$$

$$L=\prod_{i=1}^{n}f\left(y_i\right)=(2\pi)^{-\frac{n}{2}}\prod_{i=1}^{n}i\sigma^{-n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i-i\theta}{i}\right)^2}$$

$$\ln L=-\frac{n}{2}\ln 2\pi+\sum_{i=1}^{n}\ln i-n\ln\sigma-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\theta-\frac{y_i}{i}\right)^2$$

$$\frac{\partial\ln L}{\partial\theta}=-\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\theta-\frac{y_i}{i}\right)=0$$

$$\therefore\hat{\theta}=\frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{i}$$

$$var\left(\hat{\theta}\right)=\frac{1}{n^2}\sum\frac{var\left(y_i\right)}{i^2}=\frac{1}{n^2}\sum_{i=1}^{n}\frac{i^2\sigma^2}{i^2}=\frac{\sigma^2}{n}$$

$$\because\frac{\partial^2\ln L}{\partial\theta^2}=-\frac{n}{\sigma^2},\,var\left(\hat{\theta}\right)\geq-\frac{1}{E\left(\frac{\partial^2\ln L}{\partial\theta^2}\right)}=-\frac{1}{-\frac{1}{\sigma^2}n}=\frac{\sigma^2}{n}$$

$$\therefore\hat{\theta}\ is\ an\ efficient\ estimator.$$

Question e

Suppose the radius equals R. $R_i=R+\epsilon_i$, $\epsilon\sim N(0,\sigma),\ i=1,2,\cdots n$.

$$E(\bar{R})=\frac{1}{n}\sum_{i=1}^{n}E(R+\epsilon_1)=\frac{1}{n}\sum_{i=1}^{n}R+E(\epsilon_1)=R$$

$$E(\bar{R}^2)=var(\bar{R})+(E\bar{R})^2=\frac{1}{n^2}\sum_{i=1}^{n}var(R+\epsilon_1)+R^2=\frac{\sigma^2}{n}+R^2$$

$$\therefore Define\ \hat{R}^2=\bar{R}^2-\frac{S^2}{n},\,E(\hat{R}^2)=R^2$$

$$\therefore The\ unbiased\ estimator\ of\ area\ is\ \hat{A}=\pi\left(\bar{R}^2-\frac{S^2}{n}\right)$$

Question f

$$X=Y^2$$

$$\begin{aligned}F_X(x)&=P(X\leq x)\\&=P(Y\leq\sqrt{x})\\&=\int_0^{\sqrt{x}}\frac{2y}{\theta}e^{-\frac{y^2}{\theta}}\,\mathrm{d}y\end{aligned}$$

$$f(x) = F'_X(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$$

$$\therefore Y^2 \sim exp\left(\frac{1}{\theta}\right)$$

$$var(\hat{\theta}) = \frac{1}{n^2} \sum_{i=0}^{n} var(Y_i^2) = \frac{1}{n^2} n\theta^2 = \frac{\theta^2}{n}$$

$$\because \frac{\partial^2 \ln f(x)}{\partial \theta^2} = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$$

$$var(\hat{\theta}) \geq -\frac{1}{nE(\frac{1}{\theta^2} - \frac{2x}{\theta^3})} = -\frac{1}{nE\left(\frac{1}{\theta^2} - \frac{2E(x)}{\theta^3}\right)} = \frac{\theta^2}{n}$$

$$\therefore \hat{\theta}\ is\ efficient$$

Question g

$$E\left(T-\theta\right)^2 = var\left(T\right) + B^2$$

$$\begin{aligned} var(T) &= var(\alpha_1 \bar{X} + \alpha_2 cS) \\ &= \alpha_1^2 var(\bar{X}) + \alpha_2^2 var(cS) \\ &= \alpha_1^2 \frac{\theta^2}{n} + \alpha_2^2 (c^2-1)\theta^2 \end{aligned}$$

$$B = E(T) - \theta = (\alpha_1 + \alpha_2 - 1)\theta$$

$$\therefore E\left(T-\theta\right)^2 = \left[\frac{\alpha_1^2}{n} + \left(c^2-1\right)\alpha_2^2 + \left(\alpha_1+\alpha_2-1\right)^2\right]\theta^2$$

$$\begin{cases} \dfrac{\partial E\left(T-\theta\right)^2}{\partial \alpha_1} = \dfrac{2\alpha_1}{n} + 2\left(\alpha_1+\alpha_2-1\right) = 0 \\ \dfrac{\partial E\left(T-\theta\right)^2}{\partial \alpha_2} = 2\left(c^2-1\right)\alpha_2 + 2\left(\alpha_1+\alpha_2-1\right) = 0 \end{cases}$$

$$\therefore \begin{cases} \alpha_1 = \dfrac{n(c^2-1)}{(n+1)(c^2-1)+1} \\ \alpha_2 = \dfrac{1}{(n+1)(c^2-1)+1} \end{cases}$$

$$A = \frac{\partial^2 E}{\partial \alpha_1^2} = \frac{2}{n} + 2, B = \frac{\partial^2 E}{\partial \alpha_1 \partial \alpha_2} = 2, C = \frac{\partial^2 E}{\partial \alpha_2^2} = 2c^2$$

$$\because A < 0,\ B^2 - AC = 4\left[1 - \frac{n+1}{n}c^2\right] < 0\,(Using\ WolframAlpha)$$

$$\therefore E\left(T-\theta\right)^2\ gets\ the\ minimum\ when\ \alpha_1\ and\ \alpha_2\ equal\ the\ above\ values.$$

$$\therefore T = \frac{n(c^2-1)}{(n+1)(c^2-1)+1}\bar{X} + \frac{1}{(n+1)(c^2-1)+1}cS\ is\ the\ estimator\ that\ minimizes\ E\left(T-\theta\right)^2$$