STATS100B – Introduction to Mathematical Statistics
Homework 1

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Exercise 1

Solution:

$X \sim \Gamma(\alpha, \beta)$, so

$$\begin{aligned} f(x|(\alpha,\beta)) &= \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha\Gamma(\alpha)} \\ &= \frac{1}{\beta^\alpha\Gamma(\alpha)}e^{(\alpha-1)lnx}e^{-\frac{x}{\beta}} \\ &= \frac{1}{\beta^\alpha\Gamma(\alpha)}exp\{(\alpha-1)lnx - \frac{1}{\beta}x\} \end{aligned}$$

where $h(x) = 1,\ c(\boldsymbol{\theta}) = \dfrac{1}{\beta^\alpha\Gamma(\alpha)},\ \sum_{i=1}^{k}(w_i(\boldsymbol{\theta})t_i(x)) = (\alpha-1)lnx - \frac{1}{\beta}x$.

Exercise 2

Solution:

Let $F_Y(y)$ be the cdf of Y and suppose that g is a monotonic reversible function.

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(g(X) \le y) \\ &= P(X \le g^{-1}(y)) \\ &= \int_0^{g^{-1}(y)} \frac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}[g^{-1}(y)]^2}\,dx \end{aligned}$$

$$p_Y(y) = F_Y(y)' = \frac{2}{\sqrt{2\pi}}\frac{\mathrm{d}g^{-1}(y)}{\mathrm{d}y}e^{-\frac{1}{2}[g^{-1}(y)]^2}$$

Because the pdf of gamma distribution is

$$p_Y(y) = \frac{y^{\alpha-1}e^{-\frac{y}{\beta}}}{\beta^\alpha\Gamma(\alpha)},$$

let $g^{-1}(y) = c\sqrt{y}$, where c is constant.

Therefore,

$$-\frac{y}{\beta} = -\frac{1}{2}[g^{-1}(y)]^2 = -\frac{1}{2}c^2y$$

$$\beta = \frac{2}{c^2}$$

$$\begin{aligned} p_Y(y) &= \frac{2}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} e^{-\frac{1}{2}[c\sqrt{y}]^2} \\ &= \frac{y^{-\frac{1}{2}} e^{-\frac{c^2 y}{2}}}{\sqrt{2}\sqrt{\pi}} \\ &= \frac{y^{-\frac{1}{2}} e^{-\frac{y}{\beta}}}{\sqrt{2}\sqrt{\pi}} \end{aligned}$$

Let $\alpha = \dfrac{1}{2}, \beta = 2$ which means $c = 1$ at the same time.

$$\begin{aligned} p_Y(y) &= \frac{y^{-\frac{1}{2}} e^{-\frac{y}{\beta}}}{\sqrt{2}\sqrt{\pi}} \\ &= \frac{y^{-\frac{1}{2}} e^{-\frac{y}{2}}}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})} \end{aligned}$$

$\therefore$ When $Y = g(X) = \dfrac{1}{c^2}x^2 = x^2$, $Y \sim \Gamma(\frac{1}{2}, 2)$.

Actually, there are infinte transformations $Y = g(X)$ that can make Y a gamma distribution. When $Y = Cx^2$ where C is a constant, $Y \sim \Gamma(\dfrac{1}{2}, 2C)$.

On the condition that $\alpha = \dfrac{1}{2}, \beta = 2, Y = x^2$,

$$E(X) = E(Y^{\frac{1}{2}}) = \frac{\Gamma(\alpha + \frac{1}{2})\beta^{\frac{1}{2}}}{\Gamma(\alpha)} = \frac{\Gamma(1)2^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} = \sqrt{\frac{2}{\pi}}$$

$$E(X^2) = E(Y) = \frac{\Gamma(\alpha + 1)\beta^1}{\Gamma(\alpha)} = \frac{\Gamma(\frac{3}{2})2^1}{\Gamma(\frac{1}{2})} = 1$$

$$var(X) = E(X^2) - (EX)^2 = 1 - \frac{2}{\pi}$$