

STATS100B – Introduction to Mathematical Statistics

Homework 4

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Question a

Solution:

$$\begin{aligned}M_{\bar{X}}(t) &= \left(1 - \frac{\beta}{n}t\right)^{-n\alpha} \\M_{\bar{X}}\left(\frac{2n}{\beta}t\right) &= \left(1 - \frac{\beta}{n} \times \frac{2n}{\beta}t\right)^{-n\alpha} \\&= (1 - 2t)^{-\frac{2n\alpha}{2}} \\&= M_{\frac{2n}{\beta}\bar{X}}(t)\end{aligned}$$

So the transformation $\frac{2n}{\beta}\bar{X}$ follows χ^2 distribution. The degree of freedom is $2n\alpha$.

Question b

Solution:

$$X \sim U(0, 1), E(X) = \frac{1}{2}, \text{var}(X) = \frac{1}{12}.$$

$$E\begin{pmatrix} X \\ X^2 \end{pmatrix} = \begin{pmatrix} E(X) \\ E(X^2) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ E(X^2) + \text{var}(X) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}$$

$$\text{var}\begin{pmatrix} X \\ X^2 \end{pmatrix} = \begin{pmatrix} \text{var}(X) & \text{cov}(X, X^2) \\ \text{cov}(X, X^2) & \text{var}(X^2) \end{pmatrix}$$

$$\begin{aligned}\text{var}(X^2) &= E[(X^2)^2] - [E(X^2)]^2 \\&= \int_0^1 x^4 dx - \left(\frac{1}{3}\right)^2 \\&= \frac{4}{45}\end{aligned}$$

$$\begin{aligned}\text{cov}(X, X^2) &= E[X(X^2)] - E(X)E(X^2) \\&= \int_0^1 x^3 dx - \frac{1}{2} \times \frac{1}{3} \\&= \frac{1}{12}\end{aligned}$$

$$\therefore \text{var} \begin{pmatrix} X \\ X^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{4}{45} \end{pmatrix}$$

Question c

Solution:

$$\begin{aligned} E(Y) &= E(2\sqrt{X_1 X_2}) \\ &= 2E(\sqrt{X_1})E(\sqrt{X_2}) \\ \therefore E(X^k) &= \frac{\Gamma(\alpha + k)\beta^k}{\Gamma(\alpha)}, X \sim \Gamma(\alpha, \beta) \\ \therefore E(\sqrt{X_1}) &= \frac{\Gamma\left(\alpha + \frac{1}{2}\right) \cdot 1^k}{\Gamma(\alpha)} = \sqrt{\pi} \\ E(\sqrt{X_2}) &= \frac{\Gamma(\alpha + 1) \cdot 1^k}{\Gamma\left(\alpha + \frac{1}{2}\right)} = \frac{\alpha!}{(\alpha - 1)!\sqrt{\pi}} = \frac{\alpha}{\sqrt{\pi}} \\ \therefore E(Y) &= 2 \times \sqrt{\pi} \times \frac{\alpha}{\pi} = 2\alpha \end{aligned}$$

$$\begin{aligned} \text{var}(Y) &= \text{var}(2\sqrt{X_1 X_2}) \\ &= E(Y^2) - (E(Y))^2 \\ &= E(4X_1 X_2) - (E(Y))^2 \\ &= 4E(X_1)E(X_2) - (E(Y))^2 \\ &= 4 \times \left(\alpha \times 1\right) \times \left(\left(\alpha + \frac{1}{2}\right) \times 1\right) - (2\alpha)^2 \\ &= 2\alpha \end{aligned}$$

Question d

Solution:

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \sim N_2 \left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right), \mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\begin{aligned} M_{(\bar{X}, \bar{Y})}(t_1, t_2) &= E(e^{t_1 \bar{X} + t_2 \bar{Y}}) \\ &= E\left(e^{t_1 \cdot \frac{1}{n} \sum_{i=1}^n X_i + t_2 \cdot \frac{1}{n} \sum_{j=1}^n Y_j}\right) \\ &= \prod_{i=1}^n E\left(e^{\frac{t_1}{n} X_i + \frac{t_2}{n} Y_i}\right) \\ &= \left(e^{\frac{t_1'}{n} \mu + \frac{1}{2} \frac{t_1'}{n} \mathbf{\Sigma} \frac{t_1}{n}}\right)^n \\ &= e^{t_1' \mu + \frac{1}{2} t_1' \mathbf{\Sigma} t_1} \end{aligned}$$

$$\therefore \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \sim N_2 \left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \frac{\mathbf{\Sigma}}{n} \right)$$

$$\therefore n(\bar{X} - \mu_1, \bar{Y} - \mu_2) \mathbf{\Sigma}^{-1} \begin{pmatrix} \bar{X} - \mu_1 \\ \bar{Y} - \mu_2 \end{pmatrix} = (\bar{X} - \mu_1, \bar{Y} - \mu_2) \left(\frac{\mathbf{\Sigma}}{n} \right)^{-1} \begin{pmatrix} \bar{X} - \mu_1 \\ \bar{Y} - \mu_2 \end{pmatrix} \sim \chi_2^2$$