University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 8

Answer the following questions:

a. Consider the regression model through the origin $y_i = \beta_1 x_i + \epsilon_i$, where $\epsilon_i \sim N(0, \sigma)$. It is assumed that the regression line passes through the origin $(0, 0)$. Find the MLE of σ^2, its expectation, and finally adjust it to be unbiased.

b. Refer to question (a). Show that $\frac{(n-1)s_e^2}{\sigma^2} \sim \chi^2_{n-1}$, where s_e^2 is the unbiased estimator of σ^2 from question (a).

c. Refer to question (a). Find the distribution of s_e^2.

d. Let $X_1, X_2, \cdots, X_n$ denote an i.i.d. random sample from the following distribution $(\alpha > 0)$.

$$f(x) = \begin{cases} \frac{\alpha x^{\alpha-1}}{3^a}, & 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Find the expected value of X.
Derive the method of moments estimator of α.
Derive the method of maximum likelihood estimate of α.

e. Consider the regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, where $\epsilon_i \sim N(0, \sigma)$. Find the Fisher information matrix $I(\beta_0, \beta_1, \sigma^2)$. Are $\hat{\beta}_0$ and $\hat{\beta}_1$ efficient estimators of β_0 and β_1?

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a. $y_i = \beta_1 x_i + \varepsilon_i$, $\varepsilon_i \sim N(0, \sigma)$

$\therefore y_i \sim N(\beta_1 x_i, \sigma)$

$$f(y_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \beta_1 x_i)^2}$$

$$L = (2\pi\sigma^2)^{-\frac{n}{2}} \cdot e^{-\frac{1}{2\sigma^2}\sum(y_i - \beta_1 x_i)^2}$$

$$\therefore \ln L = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 x_i)^2$$

$$\begin{cases} \frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2}\cdot\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i - \beta_1 x_i)^2 = 0 \\ \therefore \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{\beta}_1 x_i)^2 = \frac{1}{n}\sum_{i=1}^{n} e_i^2 \\ \frac{\partial \ln L}{\partial \beta_1} = -\frac{1}{2\sigma^2}\cdot\sum_{i=1}^{n} 2(y_i - \beta_1 x_i)\cdot x_i = 0 \end{cases}$$

$$\therefore \hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} \quad . \quad E(\hat{\beta}_1) = \frac{\sum_{i=1}^{n} x_i E y_i}{\sum_{i=1}^{n} x_i^2} = \beta_1$$

$$E(\hat{\sigma}^2) = E\left(\frac{1}{n}\sum_{i=1}^{n} e_i^2\right) = \frac{1}{n}\sum_{i=1}^{n} E(e_i^2) = \frac{1}{n}\sum_{i=1}^{n}\left[var(e_i) + (E e_i)^2\right]$$

$$E(e_i) = E(y_i - \hat{\beta}_1 x_i) = E(y_i) - x_i E(\hat{\beta}_1) = \beta_1 x_i - \beta_1 x_i = 0$$

$$var(e_i) = var(y_i - \hat{\beta}_1 x_i)$$

$$= var(y_i) + x_i^2 var(\hat{\beta}_1) - 2cov(y_i, \hat{\beta}_1 x_i)$$

$$= \sigma^2 + x_i^2 \cdot \frac{\sum_{i=1}^{n} x_i^2 var(y_i)}{\left(\sum_{i=1}^{n} x_i^2\right)^2} - 2x_i cov\left(y_i, \frac{x_i y_i}{\sum_{i=1}^{n} x_i^2}\right)$$

$$= \sigma^2 + \frac{x_i^2 \sigma^2}{\sum_{i=1}^{n} x_i^2} - 2x_i^2 \cdot \frac{1}{\sum_{i=1}^{n} x_i^2}\sigma^2 = \sigma^2 - \frac{x_i^2}{\sum_{i=1}^{n} x_i^2}\sigma^2$$

$$\therefore E(\hat{\sigma}^2) = \frac{1}{n}\sum_{i=1}^{n}\left[\sigma^2 - \frac{x_i^2}{\sum_{i=1}^{n} x_i^2}\sigma^2\right] = \frac{n-1}{n}\sigma^2$$

$\therefore \frac{n}{n-1}\hat{\sigma}^2$ is an unbiased estimator of σ^2

(b)

$$y_i \sim N(\beta_1 x_i, \sigma) \qquad \therefore \sum_{i=1}^{n}\left(\frac{y_i - \beta_1 x_i}{\sigma}\right)^2 \sim \chi_n^2$$

$$\underbrace{\sum_{i=1}^{n}\left(\frac{y_i-\beta_1 x_i}{\sigma}\right)^2}_{\chi_n^2} = \sum_{i=1}^{n}\left(\frac{y_i - \hat{\beta}_1 x_i + (\hat{\beta}_1 - \beta_1)x_i}{\sigma}\right)^2$$

$$= \sum_{i=1}^{n}\left(\frac{e_i}{\sigma} + \frac{(\hat{\beta}_1-\beta_1)x_i}{\sigma}\right)^2$$

$$= \sum_{i=1}^{n}\frac{e_i^2}{\sigma^2} + \underbrace{\frac{(\hat{\beta}_1-\beta_1)^2\sum_{i=1}^{n} x_i^2}{\sigma^2}}_{\chi_1^2} + \frac{2(\hat{\beta}_1-\beta_1)\sum_{i=1}^{n} x_i e_i}{\sigma^2}$$

(1) $\sum_{i=1}^{n}\frac{e_i^2}{\sigma^2} = \frac{n\hat{\sigma}^2}{\sigma^2} = \frac{(n-1)S_e^2}{\sigma^2}$

(2) $\hat{\beta}_1 = \frac{\sum x_i y_i}{\sum x_i^2}$. $E(\hat{\beta}_1) = \beta_1$. $var(\hat{\beta}_1) = \frac{\sum x_i^2 \cdot \sigma^2}{(\sum x_i^2)^2} = \frac{\sigma^2}{\sum x_i^2}$

$$\therefore \frac{(\hat{\beta}_1-\beta_1)^2\sum_{i=1}^{n} x_i^2}{\sigma^2} = \left(\frac{\hat{\beta}_1 - \beta_1}{\sigma/\sqrt{\sum_{i=1}^{n} x_i^2}}\right)^2 \sim \chi_1^2$$

(3) $\sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i(y_i - \hat{\beta}_1 x_i) = \sum_{i=1}^{n} x_i y_i - \hat{\beta}_1\sum_{i=1}^{n} x_i^2 = 0$

Notice that $cov(e_i, \hat{\beta}_1) = cov(y_i - \hat{\beta}_1 x_i, \hat{\beta}_1) = cov(y_i, \frac{x_i y_i}{\sum x_i^2}) - x_i\, var(\hat{\beta}_1)$

$$= \frac{x_i}{\sum x_i^2} var(y_i^2) - x_i \cdot \frac{\sigma^2}{\sum x_i^2} = 0.$$

$\therefore \frac{(n-1)S_e^2}{\sigma^2}$ and $\left(\frac{(\hat{\beta}_1-\beta_1)}{\sigma/\sqrt{\sum x_i^2}}\right)^2$ are independent

$$\therefore \frac{(n-1)S_e^2}{\sigma^2} \sim \chi_1^2$$

c. $\frac{(n-1)S_e^2}{\sigma^2} \sim \chi^2_{n-1}$

$$M_{\frac{(n-1)S_e^2}{\sigma^2}}(t) = (1-2t)^{-\frac{(n-1)}{2}}$$

$$\therefore M_{S_e^2}(t) = M_{\frac{(n-1)S_e^2}{\sigma^2}}\left(\frac{\sigma^2}{n-1}t\right) = \left(1-\frac{2\sigma^2}{n-1}t\right)^{-\frac{(n-1)}{2}}$$

$$\therefore S_e^2 \sim \Gamma\left(\frac{n-1}{2}, \frac{2\sigma^2}{n-1}\right)$$

d. $E(X) = \int_0^3 x \cdot \frac{\alpha}{3^\alpha} x^{\alpha-1}\, dx = \frac{\alpha}{3^\alpha}\int_0^3 x^\alpha\, dx$

$$= \frac{\alpha}{3^\alpha} \cdot \frac{1}{\alpha+1} x^{\alpha+1} \Big|_0^3$$

$$= \frac{3\alpha}{\alpha+1}$$

$$\therefore \bar{X} = \frac{3\alpha}{\alpha+1} \Rightarrow \hat{\alpha} = \frac{\bar{X}}{3-\bar{X}}$$

$$L = \alpha^n \cdot 3^{-n\alpha} \cdot \left(\prod X_i\right)^{\alpha-1}$$

$$\ln L = n\ln\alpha - n\alpha\ln 3 + (\alpha-1)\sum_{i=1}^n \ln X_i$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - n\ln 3 + \sum_{i=1}^n \ln X_i = 0$$

$$\therefore \hat{\alpha} = \frac{n}{n\ln 3 - \sum_{i=1}^n \ln X_i}$$

e. $y_i \sim N(\beta_0+\beta_1 x_i, \varepsilon_i)$

$$f(y_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\beta_0-\beta_1 x_i)^2}$$

$$\therefore L = (2\pi\sigma^2)^{-\frac{n}{2}} \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)^2}$$

$$\ln L = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)^2$$

$$\frac{\partial \ln L}{\partial \beta_0} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i), \quad \frac{\partial \ln L}{\partial \beta_1} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)x_i$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2}\cdot\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)^2$$

$$\therefore I(\theta) = -E\begin{pmatrix} \frac{\partial^2 \ln L}{\partial \beta_0^2} & \frac{\partial^2 \ln L}{\partial \beta_0 \beta_1} & \frac{\partial^2 \ln L}{\partial \beta_0 \partial \sigma^2} \\ \frac{\partial^2 \ln L}{\partial \beta_1 \partial \beta_0} & \frac{\partial^2 \ln L}{\partial \beta_1^2} & \frac{\partial^2 \ln L}{\partial \beta_1 \partial \sigma^2} \\ \frac{\partial^2 \ln L}{\partial \sigma^2 \partial \beta_0} & \frac{\partial^2 \ln L}{\partial \sigma^2 \partial \beta_1} & \frac{\partial^2 \ln L}{\partial \sigma^{2(2)}} \end{pmatrix}$$

$$= -E\begin{pmatrix} \frac{-n}{\sigma^2} & \frac{-\sum_{i=1}^{n} x_i}{\sigma^2} & -\frac{1}{\sigma^4}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i) \\ \frac{-\sum_{i=1}^{n} x_i}{\sigma^2} & \frac{-\sum_{i=1}^{n} x_i^2}{\sigma^2} & -\frac{1}{\sigma^4}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)x_i \\ -\frac{1}{\sigma^4}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i) & -\frac{1}{\sigma^4}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)x_i & \frac{n}{2}\cdot\frac{1}{\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1 x_i)^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{n}{\sigma^2} & \frac{\sum_{i=1}^{n} x_i}{\sigma^2} & 0 \\ \frac{\sum_{i=1}^{n} x_i}{\sigma^2} & \frac{\sum_{i=1}^{n} x_i^2}{\sigma^2} & 0 \\ 0 & 0 & \frac{n}{2\sigma^4} \end{pmatrix}$$

$$\therefore I^{-1}(\theta) = \begin{pmatrix} \frac{\sigma^2 \sum X_i^2}{n \sum (X_i - \bar{X})^2} & \frac{\sigma^2 \sum X_i}{n \sum (X_i - \bar{X})^2} & 0 \\ \frac{\sigma^2 \sum X_i}{n \sum (X_i - \bar{X})^2} & \frac{\sigma^2}{\sum (X_i - \bar{X})^2} & 0 \\ 0 & 0 & \frac{2\sigma^4}{n} \end{pmatrix}$$

$$\because var(\hat{\beta}_0) = \sigma^2 \cdot \left(\frac{1}{n} + \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2} \right)$$

$$= \sigma^2 \cdot \frac{\sum (X_i - \bar{X})^2 + n\bar{X}^2}{n \sum (X_i - \bar{X})^2}$$

$$= \frac{\sigma^2 \cdot \sum X_i^2}{n \sum (X_i - \bar{X})^2}$$

$\therefore \hat{\beta}_0$ is efficient

$$\because var(\hat{\beta}_1) = \frac{\sigma^2}{\sum (X_i - \bar{X})^2}$$

$\therefore \hat{\beta}_1$ is efficient.