STATS100B – Introduction Mathmatical Statistics

Homework 3

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Exercise 1

(a) Solution:

$$M_X(t) = exp\{\frac{\sigma^2}{2}t^2 + \mu t\}$$

$$M_{aX+b}(t) = e^{bt} M_X(at)$$

$$= e^{bt} exp\left\{\frac{\sigma^2}{2}a^2t + \mu at\right\}$$

$$= exp\left\{\frac{(a\sigma)^2}{2}t^2 + (a\mu + b)t\right\}$$

$$\therefore aX + b \sim N(a\mu + b, a\sigma)$$

(b) Solution:

$$F_Y(y) = P(aX + b \le y)$$

$$= P(X \le \frac{y - b}{a})$$

$$= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$p_Y(y) = F'_Y(y)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left[\frac{y-b}{a} - \mu\right]^2}{2\sigma^2}} \frac{1}{a}$$

$$= \frac{1}{\sqrt{2\pi}(a\sigma)} e^{-\frac{1}{2(a\sigma)^2}[y-(a\mu+b)]^2}$$

$$\therefore aX + b \sim N(a\mu + b, a\sigma)$$

Exercise 2

Solution:

By definiation,

$$M_{lnx}(t) = E(e^{tlnx}) = E(X^t)$$

$$\because lnX \sim N(\mu, \sigma)$$

$$\therefore M_{lnx}(t) = exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$$

$$\therefore E(X^t) = exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$$

$$E(X) = exp\{\mu + \frac{1}{2}\sigma^2\}, \ E(X^2) = exp\{2\mu + 2\sigma^2\}$$

$$var(X) = E(X^2) - [E(X)]^2 = e^{\sigma^2 + 2\mu}(e^{\sigma^2} - 1)$$

Exercise 3

$$M_X(t) = (1 - \beta t)^{-\alpha}$$

$$M_T(t) = \prod_{i=1}^{n} M_{X_i}(t) = (1 - \beta t)^{-n\alpha}$$

$$\therefore T \sim \Gamma(n\alpha, \beta)$$

$$M_{\bar{X}}(t) = \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{n}\right) = \left(1 - \frac{\beta}{n}t\right)^{-n\alpha}$$

$$\therefore T \sim \Gamma\left(n\alpha, \frac{\beta}{n}\right)$$

Exercise 4

$$E(X^4) = E[X^3(X - \mu + \mu)]$$

$$= E[X^3(X - \mu)] + \mu E(X^3)$$

$$= \sigma^2 E(X^2) + \mu\Big[E[(X^2)(X - \mu)] + \mu E(X^2)\Big]$$

$$= \sigma^2[(EX)^2 + var(X)] + \mu\Big[2\sigma^2 E(X) + \mu[(EX)^2 + var(X)]\Big]$$

$$= \sigma^2[\mu^2 + \sigma^2] + \mu\Big[2\mu\sigma + \mu[\mu^2 + \sigma^2]\Big]$$

$$= \mu^2 + 6\mu^2\sigma^2 + 3\sigma^4$$

Exercise 5

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \sigma_{12} \\ \sigma_{21} & x_2 \end{pmatrix} \right)$$

$$M_X(t) = M_{X_1,X_2}(t_1, t_2) = E(e^{t_1 X_1 + t_2 X_2}) = exp\left\{ t_1\mu_1 + t_2\mu_2 + \frac{1}{2}\sigma_1^2 t_1^2 + \frac{1}{2}\sigma_2^2 t_2^2 + t_1 t_2 \sigma_{12} \right\}$$

$$\frac{\partial^2 \Psi(t_1, t_2)}{\partial t_1 \partial t_2} = \sigma_{12}$$

$$\therefore \frac{\partial^2 \Psi(0,0)}{\partial t_1 \partial t_2} = \sigma_{12} = cov(X_1, X_2)$$

$$\therefore \frac{\partial^2 M_{X_1,X_2}(t_1, t_2)}{\partial t_1 \partial t_2} = \sigma_{12} e^{\Psi(t_1,t_2)} + \left(\mu_1 + \sigma_1^2 t_1 + t_1 \sigma_{12} \right)\left(\mu_2 + \sigma_2^2 t_2 + t_1 \sigma_{12} \right)$$

$$\frac{\partial M_{X_1,X_2}(t_1, t_2)}{\partial t_1} = \left(\mu_1 + \sigma_1^2 t_1 + t_1 \sigma_{12} \right) e^{\Psi(t_1,t_2)}$$

$$\frac{\partial M_{X_1,X_2}(t_1, t_2)}{\partial t_2} = \left(\mu_2 + \sigma_2^2 t_2 + t_2 \sigma_{12} \right) e^{\Psi(t_1,t_2)}$$

$$\Psi(0,0) = 0$$

$$\therefore \frac{\partial^2 M_{X_1,X_2}(0,0)}{\partial t_1 \partial t_2} - \frac{\partial M_{X_1,X_2}(0,0)}{\partial t_1} \times \frac{\partial M_{X_1,X_2}(0,0)}{\partial t_2} = (\sigma_{12} + \mu_1\mu_2) - \mu_1 \times \mu_2$$

$$= \sigma_{12}$$

$$= cov(X_1, X_2)$$