STATS100B – Introduction to Mathematical Statistics
Homework 7

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Question a

Solution:

$$L = \frac{1}{(2\pi)^{\frac{n}{2}}}|\boldsymbol{\Sigma}|^{-\frac{1}{2}}e^{\frac{1}{2}(\boldsymbol{Y}-\mu\boldsymbol{1})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y}-\mu\boldsymbol{1})}$$

$$lnL = -\frac{n}{2}ln(2\pi\boldsymbol{\Sigma}^2) - \frac{1}{2}ln|\boldsymbol{V}| - \frac{1}{2\sigma^2}(\boldsymbol{Y}-\mu)V^{-1}(\boldsymbol{Y}-\mu)$$

$$\frac{\partial \ln L}{\partial \mu} = -\frac{1}{2\sigma^2}\left[-\boldsymbol{Y'V^{-1}1} - \boldsymbol{1'V^{-1}Y} + 2\boldsymbol{\mu 1'V^{-1}1}\right] = 0$$

$$\therefore \hat{\mu} = \frac{\boldsymbol{1'V^{-1}Y}}{\boldsymbol{1'V^{-1}1}}$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{\sigma^4}(\boldsymbol{Y}-\mu\boldsymbol{1})'\boldsymbol{V}^{-1}(\boldsymbol{Y}-\mu\boldsymbol{1}) = 0$$

$$\therefore \hat{\sigma^2} = \frac{(\boldsymbol{Y}-\hat{\mu}\boldsymbol{1})'\boldsymbol{V}^{-1}(\boldsymbol{Y}-\hat{\mu}\boldsymbol{1})}{n}$$

Question b

Solution:

$$E(\hat{\mu}) = E\left(\frac{1'V^{-1}\boldsymbol{Y}}{1'V^{-1}1}\right) = \frac{1'V^{-1}E(\boldsymbol{Y})}{1'V^{-1}1} = \frac{1'V^{-1}\mu 1}{1'V^{-1}1} = \mu$$

Question c

Solution:

$$\boldsymbol{I}(\boldsymbol{\theta}) = -E\begin{pmatrix} \frac{\partial^2 \ln L}{\partial \mu^2} & \frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} \\ \\ \frac{\partial^2 \ln L}{\partial \sigma^2 \partial \mu} & \frac{\partial \ln L}{\partial \sigma^{2(2)}} \end{pmatrix}$$

$$\frac{\partial^2 \ln L}{\partial \mu^2} = -\frac{1'V^{-1}1}{\sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} = \frac{1}{\sigma^4}\left[\mu 1'V^{-1}1 - 1'V^{-1}\boldsymbol{Y}\right]$$

$$\frac{\partial^2 \ln L}{\partial \sigma^2 \partial \mu} = \frac{1}{\sigma^4}\left[\mu 1' V^{-1} 1 - 1' V^{-1} \boldsymbol{Y}\right]$$

$$\frac{\partial \ln L}{\partial {\sigma^2}^{(2)}} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\left(Y - \mu 1\right)' V^{-1}\left(\boldsymbol{Y} - \mu 1\right)$$

$$\therefore \boldsymbol{I}(\boldsymbol{\theta}) \Rightarrow ($$

Question d

Solution:

$$f\left(y_i\right) = \frac{1}{\sqrt{2\pi} i\sigma} e^{-\frac{\left(y - i\theta\right)^2}{2\left(i\sigma\right)^2}}$$

$$L = \prod_{i=1}^{n} f\left(y_i\right) = \left(2\pi\right)^{-\frac{n}{2}} \prod_{i=1}^{n} i\sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{y_i - i\theta}{i}\right)^2}$$

$$\ln L = -\frac{n}{2}\ln 2\pi + \sum_{i=1}^{n}\ln i - n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\theta - \frac{y_i}{i}\right)^2$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\theta - \frac{y_i}{i}\right) = 0$$

$$\therefore \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{i}$$

$$var\left(\hat{\theta}\right) = \frac{1}{n^2}\sum\frac{var\left(y_i\right)}{i^2} = \frac{1}{n^2}\sum_{i=1}^{n}\frac{i^2\sigma^2}{i^2} = \frac{\sigma^2}{n}$$

$$\because \frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{\sigma^2}$$

$$var\left(\hat{\theta}\right) \geq -\frac{1}{E\left(\dfrac{\partial^2 \ln L}{\partial \theta^2}\right)} = \frac{\sigma^2}{n}$$

$$\therefore \hat{\theta}\ is\ an\ efficient\ estimator.$$

$$\begin{aligned}
\sum_{i=1}^{4}\left(X_i - \bar{X}\right)^2 &= \sum_{i=1}^{4} X_i^2 - 4\bar{X}^2 \\
&= \sum_{i=1}^{4} X_i^2 - 4\Big[\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\Big]^2 \\
&= \sum_{i=1}^{4} X_i^2 - \frac{1}{2}\left(\sum_{i=1}^{4} X_i^2 + 2\sum_{1\leq i<j\leq 4} X_i X_j\right) \\
&= \frac{3}{4}\sum_{i=1}^{4} X_i^2 - \frac{1}{2}\sum_{1\leq i<j\leq 4} X_i X_j
\end{aligned}$$

$$\begin{aligned} RHS &= \Big(\frac{1}{2}X_1^2 - X_1X_2 + \frac{1}{2}X_2^2\Big) + \Big(\frac{2}{3}X_3^2 + \frac{1}{6}\left(X_1+X_2\right)^2 - \frac{2}{3}X_3\left(X_1+X_2\right)\Big) + \frac{3}{4}\Big(X_4 - \frac{1}{3}(X_1+X_2+X_3)\Big)^2 \\ &= \cdots\cdots (simple\, but\, tedious\, simplifications) \\ &= \frac{3}{4}\sum_{i=1}^{4} X_i^2 - \frac{1}{2}\sum_{1\le i<j\le 4} X_iX_j \end{aligned}$$

$$\therefore \sum_{i=1}^{4}\left(X_i - \bar{X}\right)^2 = \frac{\left(X_1-X_2\right)^2}{2} + \frac{\left[X_3 - \frac{(X_1+X_2)}{2}\right]^2}{\frac{3}{2}} \frac{\left[X_4 - \frac{(X_1+X_2+X_3)}{3}\right]^2}{\frac{4}{3}}$$

$$\because X_1 - X_2 \sim N(0, \sqrt{2})$$
$$\therefore \frac{(X_1-X_2)^2}{2} \sim \chi_1^2$$
$$\because X_3 - \frac{X_1+X_2}{2} \sim N(0, \sqrt{\frac{3}{2}})$$
$$\therefore \frac{(X_3 - \frac{X_1+X_2}{2})^2}{\frac{3}{2}} \sim \chi_1^2$$
$$\because X_4 - \frac{X_1+X_2+X_3}{3} \sim N(0, \sqrt{\frac{4}{3}})$$
$$\therefore \frac{(X_4 - \frac{X_1+X_2+X_3}{3})^2}{\frac{4}{3}} \sim \chi_1^2$$

$$\boldsymbol{X} = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \end{pmatrix}'$$

$$\begin{pmatrix} X_1 - X_2 \\ X_3 - \frac{X_1+X_2}{2} \\ X_4 - \frac{X_1+X_2+X3}{3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \boldsymbol{AX}$$

$$var(\boldsymbol{AX}) = \boldsymbol{A}var(\boldsymbol{X})\boldsymbol{A}' = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} \\ -1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

So the three terms in the RHS are independent each with a χ_1^2 distribution.

Question g

$$E\left(T-\theta\right)^2 = var\left(T\right) + B^2$$

$$\begin{aligned} var(T) &= var(\alpha_1\bar{X} + \alpha_2 cS) \\ &= \alpha_1^2 var(\bar{X}) + \alpha_2^2 var(cS) \\ &= \alpha_1^2\frac{\theta^2}{n} + \alpha_2^2(c^2-1)\theta^2 \end{aligned}$$

$$B = E(T) - \theta = (\alpha_1 + \alpha_2 - 1)\theta$$

$$\therefore E\left(T-\theta\right)^2 = \left[\frac{\alpha_1^2}{n} + \left(c^2-1\right)\alpha_2^2 + \left(\alpha_1+\alpha_2-1\right)^2\right]\theta^2$$

$$\begin{cases} \dfrac{\partial E\left(T-\theta\right)^2}{\partial \alpha_1} = \dfrac{2\alpha_1}{n} + 2\left(\alpha_1+\alpha_2-1\right) = 0 \\ \dfrac{\partial E\left(T-\theta\right)^2}{\partial \alpha_2} = 2\left(c^2-1\right)\alpha_2 + 2\left(\alpha_1+\alpha_2-1\right) = 0 \end{cases}$$

$$\therefore \begin{cases} \alpha_1 = \dfrac{n(c^2-1)}{(n+1)(c^2-1)+1} \\ \alpha_2 = \dfrac{1}{(n+1)(c^2-1)+1} \end{cases}$$

$$A = \frac{\partial^2 E}{\partial \alpha_1^2} = \frac{2}{n} + 2, B = \frac{\partial^2 E}{\partial \alpha_1 \partial \alpha_2} = 2, C = \frac{\partial^2 E}{\partial \alpha_2^2} = 2c^2$$

$$\because A < 0,\ B^2 - AC = 4\left[1 - \frac{n+1}{n}c^2\right] < 0\,(Using\ WolframAlpha)$$

$$\therefore E\left(T-\theta\right)^2\, gets\ the\ minimum\ when\ \alpha_1\ and\ \alpha_2\ equal\ the\ above\ values.$$

$$\therefore T = \frac{n(c^2-1)}{(n+1)(c^2-1)+1}\bar{X} + \frac{1}{(n+1)(c^2-1)+1}cS\ is\ the\ estimator\ that\ minimizes\ E\left(T-\theta\right)^2$$