STATS100B – Introduction to Mathematical Statistics
Homework 7

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Question a

Solution:

$$E(\bar{X}) = \mu = \lambda,\ E(S^2) = \sigma^2 = \lambda$$
$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$
$$\ln f(x) = x \ln \lambda - \lambda - \ln x!$$
$$\frac{\partial \ln f(x)}{\partial \lambda} = \frac{x}{\lambda} - 1$$
$$\frac{\partial^2 \ln f(x)}{\partial \lambda^2} = -\frac{x}{\lambda^2}$$

Let's find the Cramer-Rao Lower Bound.

$$var\left(\hat{\lambda}\right) \geq -\frac{1}{nE\left(-\frac{x}{\lambda^2}\right)} = \frac{\lambda^2}{n\lambda} = \frac{\lambda}{n}$$
$$\because var(\bar{X}) = \frac{\sigma^2}{n} = \frac{\lambda}{n}$$
$$\therefore var(\sigma^2) \geq var(\bar{X})$$

So $\bar{X}$ is a better estimator.

Question b

Solution:

$$E\left(\alpha\bar{X} + (1-\alpha)\,cS\right) = \alpha E\left(\bar{X}\right) + (1-\alpha)\,E\,(cS) = \alpha\theta + (1-\alpha)\,\theta = \theta$$
$$var\left(\alpha\bar{X} + (1-\alpha)\,cS\right) = \alpha \cdot var\left(\bar{X}\right) + (1-\alpha)^2 \cdot var\,(cS)$$
$$\frac{\mathrm{d}var\left(\alpha\bar{X} + (1-\alpha)\,cS\right)}{\mathrm{d}\alpha} = 2\alpha \cdot var\left(\bar{X}\right) + 2\,(\alpha-1) \cdot var\,(S) = 0$$
$$\alpha = \frac{var\,(cS)}{var\,(cS) + var\left(\bar{X}\right)}$$
$$\because var(\bar{X}) = \frac{\theta^2}{n},\ var(cS) = E(c^2S^2) - (E(cS))^2 = c^2\theta^2 - \theta^2 = (c^2-1)\theta^2$$
$$\therefore \alpha = \frac{n(c^2-1)}{n(c^2-1)+1}$$

Question c

Solution:

$$X \sim (\alpha, \beta),\ \bar{X} \sim (n\alpha, \frac{\beta}{n})$$

$$E\Big(\frac{1}{\bar{X}}\Big) = \frac{\Gamma(n\alpha - 1)\Big(\frac{\beta}{n}\Big)^{-1}}{\Gamma(n\alpha)} = \frac{n}{(n\alpha - 1)\beta}$$

$$\therefore \hat{\theta} = \frac{n\alpha - 1}{n\bar{X}},\ E(\hat{\theta}) = \frac{n\alpha - 1}{n} E\Big(\frac{1}{\bar{X}}\Big) = \frac{1}{\beta}$$

Question d

Solution:

$$\begin{aligned}
\sum_{i=1}^{4} \left(X_i - \bar{X}\right)^2 &= \sum_{i=1}^{4} X_i^2 - 4\bar{X}^2 \\
&= \sum_{i=1}^{4} X_i^2 - 4\Big[\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\Big]^2 \\
&= \sum_{i=1}^{4} X_i^2 - \frac{1}{2}\left(\sum_{i=1}^{4} X_i^2 + 2\sum_{1 \le i < j \le 4} X_i X_j\right) \\
&= \frac{3}{4}\sum_{i=1}^{4} X_i^2 - \frac{1}{2}\sum_{1 \le i < j \le 4} X_i X_j
\end{aligned}$$

$$\begin{aligned}
RHS &= \Big(\frac{1}{2}X_1^2 - X_1X_2 + \frac{1}{2}X_2^2\Big) + \Big(\frac{2}{3}X_3^2 + \frac{1}{6}\left(X_1 + X_2\right)^2 - \frac{2}{3}X_3\left(X_1 + X_2\right)\Big) + \frac{3}{4}\Big(X_4 - \frac{1}{3}(X_1 + X_2 + X_3)\Big)^2 \\
&= \cdots\cdots (simple\, but\, tedious\, simplifications) \\
&= \frac{3}{4}\sum_{i=1}^{4} X_i^2 - \frac{1}{2}\sum_{1 \le i < j \le 4} X_i X_j
\end{aligned}$$

$$\therefore \sum_{i=1}^{4} \left(X_i - \bar{X}\right)^2 = \frac{\left(X_1 - X_2\right)^2}{2} + \frac{\left[X_3 - \frac{(X_1 + X_2)}{2}\right]^2}{\frac{3}{2}} \frac{\left[X_4 - \frac{(X_1 + X_2 + X_3)}{3}\right]^2}{\frac{4}{3}}$$

$$\because X_1 - X_2 \sim N(0, \sqrt{2})$$

$$\therefore \frac{(X_1 - X_2)^2}{2} \sim \chi_1^2$$

$$\because X_3 - \frac{X_1 + X_2}{2} \sim N(0, \sqrt{\frac{3}{2}})$$

$$\therefore \frac{(X_3 - \frac{X_1 + X_2}{2})^2}{\frac{3}{2}} \sim \chi_1^2$$

$$\because X_4 - \frac{X_1 + X_2 + X_3}{3} \sim N(0, \sqrt{\frac{4}{3}})$$

$$\therefore \frac{(X_4 - \frac{X_1 + X_2 + X_3}{3})^2}{\frac{4}{3}} \sim \chi_1^2$$

$$\boldsymbol{X} = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \end{pmatrix}'$$

$$\begin{pmatrix} X_1 - X_2 \\ X_3 - \frac{X_1+X_2}{2} \\ X_4 - \frac{X_1+X_2+X3}{3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \boldsymbol{AX}$$

$$var(\boldsymbol{AX}) = \boldsymbol{A}var(\boldsymbol{X})\boldsymbol{A}' = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} \\ -1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

So the three terms in the RHS are independent each with a χ_1^2 distribution.