

University of California, Los Angeles
Department of Statistics

Statistics 100B

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Homework 5

Answer the following questions:

- a. Let $X \sim F_{m,n}$. Show that $F_{\alpha;m,n} = \frac{1}{F_{1-\alpha;n,m}}$ and $t_{1-\frac{\alpha}{2};n}^2 = F_{1-\alpha;1,n}$ and draw the relevant graphs for both questions.
- b. Let X_1, X_2, \dots, X_{13} and Y_1, Y_2, \dots, Y_{16} represent two independent random samples from the respective normal distributions $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$. It is given that $\sigma_1^2 = \frac{1}{5}\sigma_2^2$, but σ_2^2 is unknown. Construct a ratio that follows the t distribution with 27 degrees of freedom.
- c. Find the mean and variance of $X \sim F_{n,m}$.

- d. Let $\mathbf{X} \sim N_n(\mu\mathbf{1}, \Sigma)$, where $\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, and Σ is the variance covariance matrix of \mathbf{X} .

$$\text{Let } \Sigma = (1-\rho)\mathbf{I} + \rho\mathbf{J}, \text{ with } \rho > -\frac{1}{n-1}, \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{J} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore, when $\rho = 0$ we have $\mathbf{X} \sim N_n(\mu\mathbf{1}, \mathbf{I})$, and in this case we showed in class that \bar{X} and $\sum_{i=1}^n (X_i - \bar{X})^2$ are independent. Are they independent when $\rho \neq 0$?

- e. If $\mathbf{Y} \sim N_2(\mathbf{0}, \Sigma)$ prove that $(\mathbf{Y}'\Sigma^{-1}\mathbf{Y} - \frac{Y_1^2}{\sigma_1^2}) \sim \chi_1^2$.

$$\text{Note: } \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}, \Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix}, \sigma_{12} = \sigma_{21}, \sigma_{12} = \rho\sigma_1\sigma_2.$$