STATS100B – Introduction to Mathematical Statistics
Homework 3

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Exercise 1

(a) Solution:

$$M_X(t) = exp\{\frac{\sigma^2}{2}t^2 + \mu t\}$$

$$\begin{aligned} M_{aX+b}\left(t\right) &= e^{bt}M_X\left(at\right) \\ &= e^{bt}exp\left\{\frac{\sigma^2}{2}a^2t + \mu at\right\} \\ &= exp\left\{\frac{\left(a\sigma\right)^2}{2}t^2 + \left(a\mu + b\right)t\right\} \\ \therefore aX + b &\sim N(a\mu + b, a\sigma) \end{aligned}$$

(b) Solution:

$$\begin{aligned} F_Y\left(y\right) &= P\left(aX + b \leq y\right) \\ &= P(X \leq \frac{y-b}{a}) \\ &= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(x-\mu\right)^2}{2\sigma^2}} dx \end{aligned}$$

$$\begin{aligned} p_Y\left(y\right) &= F_Y'\left(y\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left[\frac{y-b}{a} - \mu\right]^2}{2\sigma^2}} \frac{1}{a} \\ &= \frac{1}{\sqrt{2\pi}\left(a\sigma\right)} e^{-\frac{1}{2\left(a\sigma\right)^2}\left[y-\left(a\mu+b\right)\right]^2} \\ \therefore aX + b &\sim N(a\mu + b, a\sigma) \end{aligned}$$

Exercise 2

Solution:

By definiation,

$$M_{lnx}(t) = E(e^{tlnx}) = E(X^t)$$

$$\because lnX \sim N(\mu, \sigma)$$

$$\therefore M_{lnx}(t) = exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$$

$$\therefore E(X^t) = exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$$

$$E(X) = exp\{\mu + \frac{1}{2}\sigma^2\},\ E(X^2) = exp\{2\mu + 2\sigma^2\}$$

$$var(X) = E(X^2) - [E(X)]^2 = e^{\sigma^2+2\mu}(e^{\sigma^2} - 1)$$

Exercise 3

$$M_X(t) = (1 - \beta t)^{-\alpha}$$

$$M_T(t) = \prod_{i=1}^{n} M_{X_i}(t) = (1 - \beta t)^{-n\alpha}$$

$$\therefore T \sim \Gamma(n\alpha, \beta)$$

$$M_{\bar{X}}(t) = \prod_{i=1}^{n} M_{X_i}\Big(\frac{t}{n}\Big) = \Big(1 - \frac{\beta}{n}t\Big)^{-n\alpha}$$

$$\therefore T \sim \Gamma\Big(n\alpha, \frac{\beta}{n}\Big)$$

Exercise 4

$$\begin{aligned}
E(X^4) &= E[X^3(X - \mu + \mu)] \\
&= E[X^3(X - \mu)] + \mu E(X^3) \\
&= \sigma^2 E(X^2) + \mu\Big[E[(X^2)(X - \mu)] + \mu E(X^2)\Big] \\
&= \sigma^2[(EX)^2 + var(X)] + \mu\Big[2\sigma^2 E(X) + \mu[(EX)^2 + var(X)]\Big] \\
&= \sigma^2[\mu^2 + \sigma^2] + \mu\Big[2\mu\sigma + \mu[\mu^2 + \sigma^2]\Big] \\
&= \mu^2 + 6\mu^2\sigma^2 + 3\sigma^4
\end{aligned}$$

Exercise 5

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N\Big(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \sigma_{12} \\ \sigma_{21} & x_2 \end{pmatrix} \Big)$$

$$M_X(t) = M_{X_1,X_2}(t_1,t_2) = E(e^{t_1X_1+t_2X_2}) = exp\Big\{ t_1\mu_1 + t_2\mu_2 + \frac{1}{2}\sigma_1^2 t_1^2 + \frac{1}{2}\sigma_2^2 t_2^2 + t_1t_2\sigma_{12} \Big\}$$

$$\frac{\partial^2 \Psi\,(t_1,t_2)}{\partial t_1 \partial t_2} = \sigma_{12}$$

$$\therefore \frac{\partial^2 \Psi\,(0,0)}{\partial t_1 \partial t_2} = \sigma_{12} = cov(X_1,X_2)$$

$$\because \frac{\partial^2 M_{X_1,X_2}\,(t_1,t_2)}{\partial t_1 \partial t_2} = \sigma_{12} e^{\Psi(t_1,t_2)} + \big(\mu_1 + \sigma_1^2 t_1 + t_1\sigma_{12})(\mu_2 + \sigma_2^2 t_2 + t_1\sigma_{12}\big)$$

$$\frac{\partial M_{X_1,X_2}\,(t_1,t_2)}{\partial t_1} = \big(\mu_1 + \sigma_1^2 t_1 + t_1\sigma_{12}\big)\, e^{\Psi(t_1,t_2)}$$

$$\frac{\partial M_{X_1,X_2}\,(t_1,t_2)}{\partial t_2} = \big(\mu_2 + \sigma_2^2 t_2 + t_2\sigma_{12}\big)\, e^{\Psi(t_1,t_2)}$$

$$\Psi\,(0,0) = 0$$

$$\begin{aligned} \therefore \frac{\partial^2 M_{X_1,X_2}\,(0,0)}{\partial t_1 \partial t_2} - \frac{\partial M_{X_1,X_2}\,(0,0)}{\partial t_1} \times \frac{\partial M_{X_1,X_2}\,(0,0)}{\partial t_2} &= (\sigma_{12} + \mu_1\mu_2) - \mu_1 \times \mu_2 \\ &= \sigma_{12} \\ &= cov(X_1,X_2) \end{aligned}$$