

STATS100B – Introduction to Mathematical Statistics

Homework 7

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Question a

Solution:

$$\begin{aligned} L &= \frac{1}{(2\pi)^{\frac{n}{2}}} |\Sigma|^{-\frac{1}{2}} e^{\frac{1}{2}(\mathbf{Y}-\mu\mathbf{1})'\Sigma^{-1}(\mathbf{Y}-\mu\mathbf{1})} \\ \ln L &= -\frac{n}{2}\ln(2\pi\Sigma^2) - \frac{1}{2}\ln|\mathbf{V}| - \frac{1}{2\sigma^2}(\mathbf{Y}-\mu)\mathbf{V}^{-1}(\mathbf{Y}-\mu) \\ \frac{\partial \ln L}{\partial \mu} &= -\frac{1}{2\sigma^2} [-\mathbf{Y}'\mathbf{V}^{-1}\mathbf{1} - \mathbf{1}'\mathbf{V}^{-1}\mathbf{Y} + 2\mu\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}] = 0 \\ \therefore \hat{\mu} &= \frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} \\ \frac{\partial \ln L}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{\sigma^4} (\mathbf{Y}-\mu\mathbf{1})'\mathbf{V}^{-1}(\mathbf{Y}-\mu\mathbf{1}) = 0 \\ \therefore \hat{\sigma}^2 &= \frac{(\mathbf{Y}-\hat{\mu}\mathbf{1})'\mathbf{V}^{-1}(\mathbf{Y}-\hat{\mu}\mathbf{1})}{n} \end{aligned}$$

Question b

Solution:

$$E(\hat{\mu}) = E\left(\frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\right) = \frac{\mathbf{1}'\mathbf{V}^{-1}E(\mathbf{Y})}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} = \frac{\mathbf{1}'\mathbf{V}^{-1}\mu\mathbf{1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} = \mu$$

Question c

Solution:

$$\begin{aligned} \mathbf{I}(\boldsymbol{\theta}) &= -E\begin{pmatrix} \frac{\partial^2 \ln L}{\partial \mu^2} & \frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} \\ \frac{\partial^2 \ln L}{\partial \sigma^2 \partial \mu} & \frac{\partial^2 \ln L}{\partial \sigma^2 \partial \sigma^2} \end{pmatrix} \\ \frac{\partial^2 \ln L}{\partial \mu^2} &= -\frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}{\sigma^2} \\ \frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} &= \frac{1}{\sigma^4} [\mu\mathbf{1}'\mathbf{V}^{-1}\mathbf{1} - \mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}] \end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \sigma^2 \partial \mu} &= \frac{1}{\sigma^4} [\mu \mathbf{1}' V^{-1} \mathbf{1} - \mathbf{1}' V^{-1} \mathbf{Y}] \\ \frac{\partial \ln L}{\partial \sigma^2} &= \frac{n}{2\sigma^4} - \frac{1}{\sigma^6} (\mathbf{Y} - \mu \mathbf{1})' V^{-1} (\mathbf{Y} - \mu \mathbf{1}) \\ \therefore \mathbf{I}(\boldsymbol{\theta}) &\Rightarrow (\end{aligned}$$

Question d

Solution:

$$\begin{aligned}f(y_i) &= \frac{1}{\sqrt{2\pi}i\sigma} e^{-\frac{(y-i\theta)^2}{2(i\sigma)^2}} \\ L &= \prod_{i=1}^n f(y_i) = (2\pi)^{-\frac{n}{2}} \prod_{i=1}^n i\sigma^{-n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n \left(\frac{y_i - i\theta}{i}\right)^2} \\ \ln L &= -\frac{n}{2} \ln 2\pi + \sum_{i=1}^n \ln i - n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^n \left(\theta - \frac{y_i}{i}\right)^2 \\ \frac{\partial \ln L}{\partial \theta} &= -\frac{1}{\sigma^2} \sum_{i=1}^n \left(\theta - \frac{y_i}{i}\right) = 0 \\ \therefore \hat{\theta} &= \frac{1}{n} \sum_{i=1}^n \frac{y_i}{i} \\ \text{var}(\hat{\theta}) &= \frac{1}{n^2} \sum \frac{\text{var}(y_i)}{i^2} = \frac{1}{n^2} \sum_{i=1}^n \frac{i^2 \sigma^2}{i^2} = \frac{\sigma^2}{n} \\ \therefore \frac{\partial^2 \ln L}{\partial \theta^2} &= -\frac{n}{\sigma^2} \\ \text{var}(\hat{\theta}) &\geq -\frac{1}{E\left(\frac{\partial^2 \ln L}{\partial \theta^2}\right)} = \frac{\sigma^2}{n} \\ \therefore \hat{\theta} &\text{ is an efficient estimator.}\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^4 (X_i - \bar{X})^2 &= \sum_{i=1}^4 X_i^2 - 4\bar{X}^2 \\ &= \sum_{i=1}^4 X_i^2 - 4\left[\frac{1}{4}(X_1 + X_2 + X_3 + X_4)\right]^2 \\ &= \sum_{i=1}^4 X_i^2 - \frac{1}{2} \left(\sum_{i=1}^4 X_i^2 + 2 \sum_{1 \leq i < j \leq 4} X_i X_j \right) \\ &= \frac{3}{4} \sum_{i=1}^4 X_i^2 - \frac{1}{2} \sum_{1 \leq i < j \leq 4} X_i X_j\end{aligned}$$

$$RHS = \left(\frac{1}{2}X_1^2 - X_1X_2 + \frac{1}{2}X_2^2\right) + \left(\frac{2}{3}X_3^2 + \frac{1}{6}(X_1 + X_2)^2 - \frac{2}{3}X_3(X_1 + X_2)\right) + \frac{3}{4}\left(X_4 - \frac{1}{3}(X_1 + X_2 + X_3)\right)^2$$

$= \dots\dots (simple\ but\ tedious\ simplifications)$

$$= \frac{3}{4} \sum_{i=1}^4 X_i^2 - \frac{1}{2} \sum_{1 \leq i < j \leq 4} X_i X_j$$

$$\therefore \sum_{i=1}^4 (X_i - \bar{X})^2 = \frac{(X_1 - X_2)^2}{2} + \frac{\left[X_3 - \frac{(X_1 + X_2)}{2}\right]^2}{\frac{3}{2}} + \frac{\left[X_4 - \frac{(X_1 + X_2 + X_3)}{3}\right]^2}{\frac{4}{3}}$$

$$\therefore X_1 - X_2 \sim N(0, \sqrt{2})$$

$$\therefore \frac{(X_1 - X_2)^2}{2} \sim \chi_1^2$$

$$\therefore X_3 - \frac{X_1 + X_2}{2} \sim N(0, \sqrt{\frac{3}{2}})$$

$$\therefore \frac{\left(X_3 - \frac{X_1 + X_2}{2}\right)^2}{\frac{3}{2}} \sim \chi_1^2$$

$$\therefore X_4 - \frac{X_1 + X_2 + X_3}{3} \sim N(0, \sqrt{\frac{4}{3}})$$

$$\therefore \frac{\left(X_4 - \frac{X_1 + X_2 + X_3}{3}\right)^2}{\frac{4}{3}} \sim \chi_1^2$$

$$\mathbf{X} = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \end{pmatrix}'$$

$$\begin{pmatrix} X_1 - X_2 \\ X_3 - \frac{X_1 + X_2}{2} \\ X_4 - \frac{X_1 + X_2 + X_3}{3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \mathbf{A}\mathbf{X}$$

$$var(\mathbf{A}\mathbf{X}) = \mathbf{A}var(\mathbf{X})\mathbf{A}' = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} \\ -1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

So the three terms in the RHS are independent each with a χ_1^2 distribution.

Question e

Suppose the radius equals R . $R_i = R + \epsilon_i$, $i = 1, 2, \dots, n$.

$$E(\bar{R}) = \frac{1}{n} \sum_{i=1}^n E(R + \epsilon_i) = \frac{1}{n} \sum_{i=1}^n R + E(\epsilon_i) = R$$

$$var(\bar{R}) = var(\bar{R}) + (E\bar{R})^2 = \frac{1}{n^2} \sum_{i=1}^n var(R + \epsilon_i) + R^2 = \frac{\sigma^2}{n} + R^2$$

Question g

$$E(T - \theta)^2 = \text{var}(T) + B^2$$

$$\begin{aligned} \text{var}(T) &= \text{var}(\alpha_1 \bar{X} + \alpha_2 cS) \\ &= \alpha_1^2 \text{var}(\bar{X}) + \alpha_2^2 \text{var}(cS) \\ &= \alpha_1^2 \frac{\theta^2}{n} + \alpha_2^2 (c^2 - 1) \theta^2 \end{aligned}$$

$$B = E(T) - \theta = (\alpha_1 + \alpha_2 - 1)\theta$$

$$\therefore E(T - \theta)^2 = \left[\frac{\alpha_1^2}{n} + (c^2 - 1) \alpha_2^2 + (\alpha_1 + \alpha_2 - 1)^2 \right] \theta^2$$

$$\begin{cases} \frac{\partial E(T - \theta)^2}{\partial \alpha_1} = \frac{2\alpha_1}{n} + 2(\alpha_1 + \alpha_2 - 1) = 0 \\ \frac{\partial E(T - \theta)^2}{\partial \alpha_2} = 2(c^2 - 1)\alpha_2 + 2(\alpha_1 + \alpha_2 - 1) = 0 \end{cases}$$

$$\therefore \begin{cases} \alpha_1 = \frac{n(c^2 - 1)}{(n + 1)(c^2 - 1) + 1} \\ \alpha_2 = \frac{1}{(n + 1)(c^2 - 1) + 1} \end{cases}$$

$$A = \frac{\partial^2 E}{\partial \alpha_1^2} = \frac{2}{n} + 2, B = \frac{\partial^2 E}{\partial \alpha_1 \partial \alpha_2} = 2, C = \frac{\partial^2 E}{\partial \alpha_2^2} = 2c^2$$

$$\because A < 0, B^2 - AC = 4 \left[1 - \frac{n+1}{n} c^2 \right] < 0 \text{ (Using WolframAlpha)}$$

$\therefore E(T - \theta)^2$ gets the minimum when α_1 and α_2 equal the above values.

$$\therefore T = \frac{n(c^2 - 1)}{(n + 1)(c^2 - 1) + 1} \bar{X} + \frac{1}{(n + 1)(c^2 - 1) + 1} cS \text{ is the estimator that minimizes } E(T - \theta)^2$$