University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Homework 10

EXERCISE 1
A coin is thrown independently 10 times to test that the probability of heads is $\frac{1}{2}$ against the alternative that the probability is not $\frac{1}{2}$. The test rejects H_0 if either 0 or 10 heads are observed.

a. What is the significance level α of the test?

b. If in fact the probability of heads is 0.1, what is the power of the test?

EXERCISE 2
The output voltage of a certain electric circuit is specified to be 130 volts. The population standard deviation is known to be $\sigma = 3.0$ volts. A sample of 40 readings on the voltage of this circuit gave a sample mean of 128.6 volts.

a. Test the hypothesis that the mean output voltage is 130 volts against the alternative that it is less than 130 volts. Use $\alpha = 0.05$.

b. Based on your answer to (a), is it possible that the mean output voltage is still 130 volts? Explain.

c. If the true population mean output voltage is 128.6 volts, compute the probability of a type II error (β) and the power of the test $(1-\beta)$ when $\alpha = 0.05$.

d. For this part you do not have to show any calculations.
How would the type II error β be affected if:

   i. The type I error α decreases to 0.01?
   ii. The true population mean is 129.6 volts?

EXERCISE 3
Answer the following questions:

a. The lifetime of certain batteries are supposed to have a variance of 150 hours^2. Using $\alpha = 0.05$ test the following hypothesis
$H_0: \sigma^2 = 150$
$H_a: \sigma^2 > 150$
if the lifetimes of 15 of these batteries (which constitutes a random sample from a normal population) have:

$$\sum_{i=1}^{15} x_i = 250, \quad \sum_{i=1}^{15} x_i^2 = 8000.$$

where X denotes the lifetime of a battery.

b. A confidence interval is *unbiased* if the expected value of the interval midpoint is equal to the estimated parameter. For example the midpoint of the interval $\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ is $\bar{x}$, and $E(\bar{x}) = \mu$. Now consider the confidence interval for σ^2. Show that the expected value of the midpoint of this confidence interval is not equal to σ^2.

EXERCISE 4
Let X be a uniform random variable on $(0, \theta)$. You have exactly one observation from this distribution and you want to test the null hypothesis $H_0: \theta = 10$ against the alternative $H_a: \theta > 10$, and you want to use significance level $\alpha = 0.10$. Two testing procedures are being considered:
Procedure G rejects H_0 if and only if $X \geq 9$.
Procedure K rejects H_0 if either $X \geq 9.5$ or if $X \leq 0.5$.

a. Confirm that Procedure G has a Type I error probability of 0.10.

b. Confirm that Procedure K has a Type I error probability of 0.10.

c. Find the power of Procedure G when $\theta = 12$.

d. Find the power of Procedure K when $\theta = 12$.

EXERCISE 5

Suppose that the length in millimeters of metal fibers produced by a certain process follow the normal distribution with mean μ and standard deviation σ (both are unknown). We will test:
$H_0 : \mu = 5.2$
$H_a : \mu \neq 5.2$
A sample size of $n = 15$ metal fibers was selected and was found that $\bar{x} = 5.4$ and $s = 0.4266$.

a. Approximate the $p-$value using only your t table and use it to test this hypothesis. Assume $\alpha = 0.05$.

b. Assume now that the population standard deviation is known and it is equal to $\sigma = 0.4266$. Compute the power of the test when the actual mean is $\mu_a = 5.35$ and you can accept $\alpha = 0.05$.

c. On the previous page draw the two distributions (under H_0 and under H_a) and show the Type I error and the Type II error on them.

d. Assume now that the hypothesis we are testing is
$H_0 : \mu = 5.2$
$H_a : \mu > 5.2$
Determine the sample size needed in order to detect with probability 95% a shift from $\mu_0 = 5.2$ to $\mu_a = 5.3$ if you are willing to accept a Type I error $\alpha = 0.05$. Assume $\sigma = 0.4266$.

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Ex 1. (a) $\alpha = P(\text{falsely reject } H_0)$

$= P(X=0 \text{ or } X=10 \mid p=\frac{1}{2}) = 2 \times \frac{1}{2^{10}} = \frac{1}{512}$

(b) $1-\beta = P(X=0 \text{ or } X=10 \mid p=0.1)$

$= C_{10}^{0}\, 0.1^{0}\, 0.9^{10} + C_{10}^{0}\, 0.1^{10}\, 0.9^{0} = 0.3487$

Ex 2. (a) $H_0: \mu = 130$, $H_1: \mu < 130$.

$p\text{-value} = P(\bar{X} < 128.6) = P\left(Z < \frac{128.6 - 130}{3/\sqrt{40}}\right)$

$= P(Z < -2.951)$

$\approx 0.0016 < \alpha$

$\therefore H_0$ is rejected.

(b) Yes. We may contain a Type I error. (Falsely reject H_0)

(c) $1-\beta = P\left(\bar{X} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} \,\middle|\, \mu = \mu_a\right) = P\left(Z < \frac{\mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_a}{\sigma/\sqrt{n}}\right)$

$= P(Z < 1.306) \approx 0.904$

$\therefore \beta \approx 0.096$

(d)(i). α decreases $\rightarrow$ β decreases

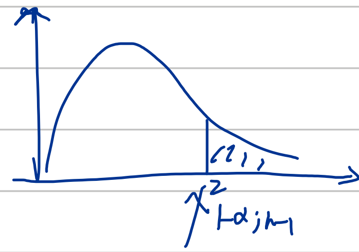
$\beta = P\left(Z > \frac{\mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_a}{\sigma/\sqrt{n}}\right)$

μ_a increases $\rightarrow$ β increases

Ex 3.

(a) $H_0: \sigma^2 = 150.$

$H_a: \sigma^2 > 150$



$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Reject H_0 if $\frac{(n-1)S^2}{\sigma^2} > \chi^2_{1-\alpha;n-1}$

$$S^2 = \frac{1}{15-1}\left(\sum x_i^2 - \frac{1}{15}\left(\sum x_i\right)^2\right) = 273.8$$

$$\frac{(n-1)S^2}{\sigma^2} = \frac{14 \times 273.8}{150} = 25.55 > 23.68 = \chi^2_{0.95;14} \quad \therefore \text{reject } H_0$$

(b) The confident interval is $\left(\frac{(n-1)S^2}{\chi^2_{1-\alpha/2;n-1}}, \frac{(n-1)S^2}{\chi^2_{\alpha/2;n-1}}\right)$

$$E\left(\frac{1}{2}\left[\frac{(n-1)S^2}{\chi^2_{1-\alpha/2;n-1}} + \frac{(n-1)S^2}{\chi^2_{\alpha/2;n-1}}\right]\right) = \sigma^2 E\left(\frac{1}{2}\left(\frac{1}{\chi^2_{1-\alpha/2;n-1}} + \frac{1}{\chi^2_{\alpha/2;n-1}}\right)\right)$$

$$\neq \sigma^2$$

Ex 4.

(a) $\alpha = P(X \geq 9 \mid \theta = 10) = \int_9^{10} \frac{1}{10}\,dx = 0.10$

(b) $\alpha = P(X \leq 0.5 \text{ or } X \geq 9.5 \mid \theta = 10)$

$$= \int_0^{0.5} \frac{1}{10}\,dx + \int_{9.5}^{10} \frac{1}{10}\,dx$$

$$= 0.10$$

(c) $1-\beta = P(X \geq 9 \mid \theta = 12) = \int_9^{12} \frac{1}{12}\,dx = 0.25$

(d) $1-\beta = P(X \leq 0.5 \text{ or } X \geq 9.5 \mid \theta = 12) = \int_0^{0.5} \frac{1}{12}\,dx + \int_{9.5}^{12} \frac{1}{12}\,dx = 0.25$

Ex5. (a) $\dfrac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$

$$\text{p-value} = 2P(\bar{X} > 5.4)$$

$$= 2P(t_{14} > 1.816) \qquad t_{14;0.95} = 1.761,\ t_{14;0.975} = 2.145$$

$$> 2 \times 0.025 = 0.05 = \alpha$$

$\therefore H_0$ is not rejected

(b) $$\beta = P\left(\mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X} < \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \,\middle|\, \mu = \mu_a\right)$$

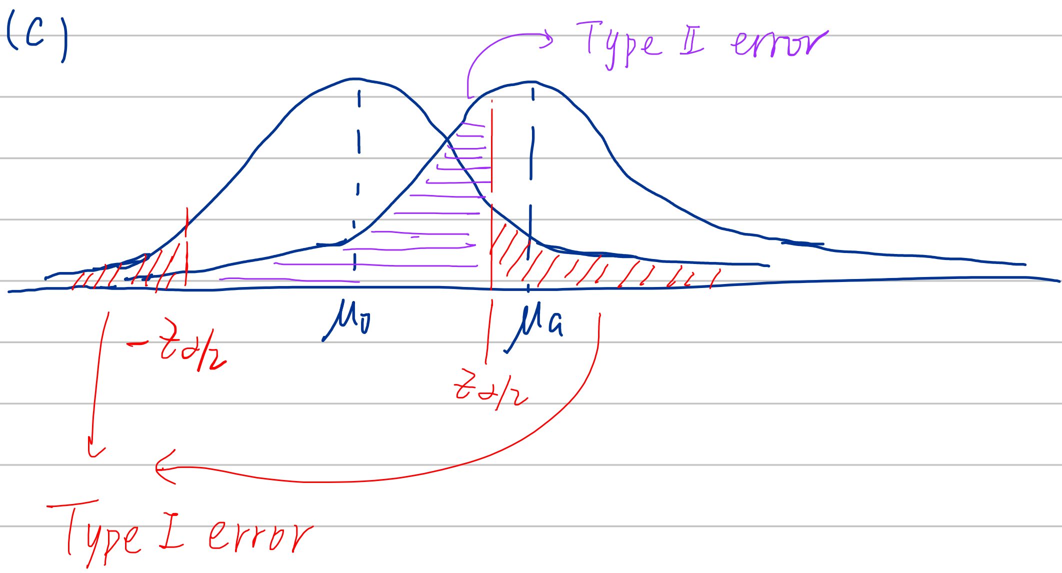
$$= P\left(\frac{\mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} - \mu_a}{\frac{\sigma}{\sqrt{n}}} < Z < \frac{\mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} - \mu_a}{\sigma/\sqrt{n}}\right)$$

$$= P(-3.32 < Z < 0.598)$$

$$\approx 0.7219$$

$\therefore 1-\beta \approx 0.2781$

(c)



d. $$\begin{cases} C = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \\ C = \mu_a - z_\beta \frac{\sigma}{\sqrt{n}} \end{cases}$$

$$\therefore n = \frac{\sigma^2 (z_\alpha + z_\beta)^2}{(\mu_a - \mu_0)^2}$$

$\sigma = 0.4266.$ $\alpha = 0.05.$ $\beta = 0.05.$ $\mu_a = 5.3.$ $\mu_0 = 5.2$

$$\therefore n = \frac{0.4266^2 \cdot (2 \times 1.645)^2}{(0.1)^2} \approx 197$$