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$$a. y_i = \beta_1 x_i + \varepsilon_i, \varepsilon_i \sim N(0, \sigma)$$

$$\therefore y_i \sim N(\beta_1 x_i, \sigma)$$

$$f(y_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \beta_1 x_i)^2}$$

$$L = (2\pi\sigma^2)^{-\frac{n}{2}} \cdot e^{-\frac{1}{2\sigma^2} \sum (y_i - \beta_1 x_i)^2}$$

$$\therefore \ln L = -\frac{n}{2} \ln 2\pi\sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \beta_1 x_i)^2$$

$$\begin{cases} \frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2} \cdot \frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (y_i - \beta_1 x_i)^2 = 0 \\ \therefore \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{\beta}_1 x_i)^2 = \frac{1}{n} \sum_{i=1}^n e_i^2 \\ \frac{\partial \ln L}{\partial \beta_1} = -\frac{1}{2\sigma^2} \cdot \sum_{i=1}^n 2(y_i - \beta_1 x_i) \cdot x_i = 0 \end{cases}$$

$$\therefore \hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2} \quad E(\hat{\beta}_1) = \frac{\sum_{i=1}^n x_i E y_i}{\sum_{i=1}^n x_i^2} = \beta_1$$

$$E(\hat{\sigma}^2) = E\left(\frac{1}{n} \sum_{i=1}^n e_i^2\right) = \frac{1}{n} \sum_{i=1}^n E(e_i^2) = \frac{1}{n} \sum_{i=1}^n [\text{var}(e_i) + (E e_i)^2]$$

$$E(e_i) = E(y_i - \hat{\beta}_1 x_i) = E(y_i) - x_i E(\hat{\beta}_1) = \beta_1 x_i - \beta_1 x_i = 0$$

$$\text{var}(e_i) = \text{var}(y_i - \hat{\beta}_1 x_i)$$

$$= \text{var}(y_i) + x_i^2 \text{var}(\hat{\beta}_1) - 2 \text{cov}(y_i, \hat{\beta}_1 x_i)$$

$$= \sigma^2 + x_i^2 \cdot \frac{\sum_{i=1}^n x_i^2 \text{var}(y_i)}{\left(\sum_{i=1}^n x_i^2\right)^2} - 2 x_i \text{cov}\left(y_i, \frac{x_i y_i}{\sum_{i=1}^n x_i^2}\right)$$

$$= \sigma^2 + \frac{x_i^2 \sigma^2}{\sum_{i=1}^n x_i^2} - 2 x_i^2 \cdot \frac{1}{\sum_{i=1}^n x_i^2} \sigma^2 = \sigma^2 - \frac{x_i^2}{\sum_{i=1}^n x_i^2} \sigma^2$$

$$\therefore E(\hat{\sigma}^2) = \frac{1}{n} \sum_{i=1}^n \left[\sigma^2 - \frac{x_i^2}{\sum_{i=1}^n x_i^2} \sigma^2 \right] = \frac{n-1}{n} \sigma^2$$

$\therefore \frac{n}{n-1} \hat{\sigma}^2$ is an unbiased estimator of σ^2

(b)

$$y_i \sim N(\beta_1 x_i, \sigma) \quad \therefore \sum_{i=1}^n \left(\frac{y_i - \beta_1 x_i}{\sigma} \right)^2 \sim \chi_n^2$$

$$\sum_{i=1}^n \left(\frac{y_i - \hat{\beta}_1 x_i}{\sigma} \right)^2 = \sum_{i=1}^n \left(\frac{y_i - \hat{\beta}_1 x_i + (\hat{\beta}_1 - \beta_1) x_i}{\sigma} \right)^2$$

χ_n^2

$$= \sum_{i=1}^n \left(\frac{e_i}{\sigma} + \frac{(\hat{\beta}_1 - \beta_1) x_i}{\sigma} \right)^2$$

$$= \sum_{i=1}^n \frac{e_i^2}{\sigma^2} + \frac{(\hat{\beta}_1 - \beta_1)^2 \sum_{i=1}^n x_i^2}{\sigma^2} + \frac{2(\hat{\beta}_1 - \beta_1) \sum_{i=1}^n x_i e_i}{\sigma^2}$$

$$(1) \quad \sum_{i=1}^n \frac{e_i^2}{\sigma^2} = \frac{n \hat{\sigma}^2}{\sigma^2} = \frac{(n-1) S_e^2}{\sigma^2}$$

$$(2) \quad \hat{\beta}_1 = \frac{\sum x_i y_i}{\sum x_i^2} \quad E(\hat{\beta}_1) = \beta_1 \quad \text{var}(\hat{\beta}_1) = \frac{\sum x_i^2 \cdot \sigma^2}{(\sum x_i^2)^2} = \frac{\sigma^2}{\sum x_i^2}$$

$$\therefore \frac{(\hat{\beta}_1 - \beta_1)^2 \sum_{i=1}^n x_i^2}{\sigma^2} = \left(\frac{\hat{\beta}_1 - \beta_1}{\sigma / \sqrt{\sum_{i=1}^n x_i^2}} \right)^2 \sim \chi_1^2$$

$$(3) \quad \sum_{i=1}^n x_i e_i = \sum_{i=1}^n x_i (y_i - \hat{\beta}_1 x_i) = \sum_{i=1}^n x_i y_i - \hat{\beta}_1 \sum_{i=1}^n x_i^2 = 0$$

Notice that $\text{cov}(e_i, \hat{\beta}_1) = \text{cov}(y_i - \hat{\beta}_1 x_i, \hat{\beta}_1) = \text{cov}(y_i, \frac{x_i y_i}{\sum x_i^2}) - x_i \text{var}(\hat{\beta}_1)$

$$= \frac{x_i}{\sum x_i^2} \text{var}(y_i^2) - x_i \cdot \frac{\sigma^2}{\sum x_i^2} = 0.$$

$$\therefore \frac{(n-1) S_e^2}{\sigma^2} \quad \text{and} \quad \left(\frac{\hat{\beta}_1 - \beta_1}{\sigma / \sqrt{\sum x_i^2}} \right)^2 \quad \text{are independent}$$

$$\therefore \frac{(n-1) S_e^2}{\sigma^2} \sim \chi_{n-1}^2$$

$$c. \quad \frac{(n-1)s_e^2}{\sigma^2} \sim \chi_{n-1}^2$$

$$M_{\frac{(n-1)s_e^2}{\sigma^2}}(t) = (1-2t)^{-\frac{(n-1)}{2}}$$

$$\therefore M_{s_e^2}(t) = M_{\frac{(n-1)s_e^2}{\sigma^2}}\left(\frac{\sigma^2}{n-1}t\right) = \left(1 - \frac{2\sigma^2}{n-1}t\right)^{-\frac{(n-1)}{2}}$$

$$\therefore s_e^2 \sim \Gamma\left(\frac{n-1}{2}, \frac{2\sigma^2}{n-1}\right)$$

$$\begin{aligned} d. \quad E(X) &= \int_0^3 x \cdot \frac{\alpha}{3^\alpha} x^{\alpha-1} dx = \frac{\alpha}{3^\alpha} \int_0^3 x^\alpha dx \\ &= \frac{\alpha}{3^\alpha} \cdot \frac{1}{\alpha+1} x^{\alpha+1} \Big|_0^3 \\ &= \frac{3\alpha}{\alpha+1} \end{aligned}$$

$$\therefore \bar{X} = \frac{3\alpha}{\alpha+1} \Rightarrow \hat{\alpha} = \frac{\bar{X}}{3-\bar{X}}$$

$$L = \alpha^n \cdot 3^{-n\alpha} \cdot \left(\prod x_i\right)^{\alpha-1}$$

$$\ln L = n \ln \alpha - n\alpha \ln 3 + (\alpha-1) \sum_{i=1}^n \ln x_i$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - n \ln 3 + \sum_{i=1}^n \ln x_i = 0$$

$$\therefore \hat{\alpha} = \frac{n}{n \ln 3 - \sum_{i=1}^n \ln x_i}$$

$$e. \quad y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$$

$$f(y_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2}$$

$$\therefore L = (2\pi\sigma^2)^{-\frac{n}{2}} \cdot e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2}$$

$$\ln L = -\frac{n}{2} \ln 2\pi\sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\partial \ln L}{\partial \beta_0} = \frac{1}{\sigma^2} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i), \quad \frac{\partial \ln L}{\partial \beta_1} = \frac{1}{\sigma^2} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) x_i$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2} \cdot \frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\therefore I(\theta) = -E \begin{pmatrix} \frac{\partial^2 \ln L}{\partial \beta_0^2} & \frac{\partial^2 \ln L}{\partial \beta_0 \partial \beta_1} & \frac{\partial^2 \ln L}{\partial \beta_0 \partial \sigma^2} \\ \frac{\partial^2 \ln L}{\partial \beta_1 \partial \beta_0} & \frac{\partial^2 \ln L}{\partial \beta_1^2} & \frac{\partial^2 \ln L}{\partial \beta_1 \partial \sigma^2} \\ \frac{\partial^2 \ln L}{\partial \sigma^2 \partial \beta_0} & \frac{\partial^2 \ln L}{\partial \sigma^2 \partial \beta_1} & \frac{\partial^2 \ln L}{\partial \sigma^2 \partial \sigma^2} \end{pmatrix}$$

$$= -E \begin{pmatrix} -\frac{n}{\sigma^2} & -\frac{\sum_{i=1}^n x_i}{\sigma^2} & -\frac{1}{\sigma^4} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) \\ -\frac{\sum_{i=1}^n x_i}{\sigma^2} & -\frac{\sum_{i=1}^n x_i^2}{\sigma^2} & -\frac{1}{\sigma^4} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) x_i \\ -\frac{1}{\sigma^4} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) & -\frac{1}{\sigma^4} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) x_i & \frac{n}{2} \cdot \frac{1}{\sigma^4} - \frac{1}{\sigma^6} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{n}{\sigma^2} & \frac{\sum_{i=1}^n x_i}{\sigma^2} & 0 \\ \frac{\sum_{i=1}^n x_i}{\sigma^2} & \frac{\sum_{i=1}^n x_i^2}{\sigma^2} & 0 \\ 0 & 0 & \frac{n}{2\sigma^4} \end{pmatrix}$$

$$\therefore J^{-1}(\theta) = \begin{pmatrix} \frac{\sigma^2 \sum X_i^2}{n \sum (X_i - \bar{X})^2} & \frac{\sigma^2 \sum X_i}{n \sum (X_i - \bar{X})^2} & 0 \\ \frac{\sigma^2 \sum X_i}{n \sum (X_i - \bar{X})^2} & \frac{\sigma^2}{\sum (X_i - \bar{X})^2} & 0 \\ 0 & 0 & \frac{2\sigma^4}{n} \end{pmatrix}$$

$$\begin{aligned} \therefore \text{var}(\hat{\beta}_0) &= \sigma^2 \cdot \left(\frac{1}{n} + \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2} \right) \\ &= \sigma^2 \cdot \frac{\sum (X_i - \bar{X})^2 + n\bar{X}^2}{n \sum (X_i - \bar{X})^2} \\ &= \frac{\sigma^2 \cdot \sum X_i^2}{n \sum (X_i - \bar{X})^2} \end{aligned}$$

$\therefore \hat{\beta}_0$ is efficient

$$\therefore \text{var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum (X_i - \bar{X})^2}$$

$\therefore \hat{\beta}_1$ is efficient.