STATS100B – Introduction to Mathematical Statistics
Homework 4

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Question a

Solution:

$$
\begin{aligned}
M_{\bar{X}}(t) &= \left(1-\frac{\beta}{n}t\right)^{-n\alpha} \\
M_{\bar{X}}\left(\frac{2n}{\beta}t\right) &= \left(1-\frac{\beta}{n}\times\frac{2n}{\beta}t\right)^{-n\alpha} \\
&= (1-2t)^{-\frac{2n\alpha}{2}} \\
&= M_{\frac{2n}{\beta}\bar{X}}(t)
\end{aligned}
$$

So the transformation $\frac{2n}{\beta}\bar{X}$ follows χ^2 distribution. The degree of freedom is $2n\alpha$.

Question b

Solution:
$X \sim U(0,1),\ E(X)=\frac{1}{2},\ var(X)=\frac{1}{12}.$

$$
E\begin{pmatrix} X \\ X^2 \end{pmatrix} = \begin{pmatrix} E(X) \\ E(X^2) \end{pmatrix} = \begin{pmatrix} \frac{1}{12} \\ E(X^2)+var(X) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}
$$

$$
var\begin{pmatrix} X \\ X^2 \end{pmatrix} = \begin{pmatrix} var(X) & cov(X,X^2) \\ cov(X,X^2) & var(X^2) \end{pmatrix}
$$

$$
\begin{aligned}
var(X^2) &= E[(X^2)^2]-[E(X^2)]^2 \\
&= \int_0^1 x^4 dx - \left(\frac{1}{3}\right)^2 \\
&= \frac{4}{45}
\end{aligned}
$$

$$
\begin{aligned}
cov(X,X^2) &= E[X(X^2)] - E(X)E(X^2) \\
&= \int_0^1 x^3 dx - \frac{1}{2}\times\frac{1}{3} \\
&= \frac{1}{12}
\end{aligned}
$$

$$\therefore var\begin{pmatrix} X \\ X^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{4}{45} \end{pmatrix}$$

Question c

Solution:

$$\begin{aligned} E(Y) &= E(2\sqrt{X_1X_2}) \\ &= 2E(\sqrt{X_1})E(\sqrt{X_2}) \end{aligned}$$

$$\because E\left(X^k\right) = \frac{\Gamma\left(\alpha+k\right)\beta^k}{\Gamma\left(\alpha\right)},\, X \sim \Gamma(\alpha, \beta)$$

$$\therefore E\left(\sqrt{X_1}\right) = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\cdot 1^k}{\Gamma\left(\alpha\right)} = \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(\alpha\right)}$$

$$E\left(\sqrt{X_2}\right) = \frac{\Gamma\left(\alpha+1\right)\cdot 1^k}{\Gamma\left(\alpha+\frac{1}{2}\right)} = \frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)}$$

$$\therefore E(Y) = 2 \times \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(\alpha\right)} \times \frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)} = 2\alpha$$

$$\begin{aligned} var(Y) &= var(2\sqrt{X_1X_2}) \\ &= E(Y^2) - \left(E(Y)\right)^2 \\ &= E(4X_1X_2) - \left(E(Y)\right)^2 \\ &= 4E(X_1)E(X_2) - \left(E(Y)\right)^2 \\ &= 4 \times \Big(\alpha \times 1\Big) \times \Big((\alpha + \frac{1}{2}) \times 1\Big) - (2\alpha)^2 \\ &= 2\alpha \end{aligned}$$

Question d

Solution:

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right),\, \mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\begin{aligned} M_{(\bar{X},\bar{Y})}(t_1, t_2) &= E(e^{t_1\bar{X}+t_2\bar{Y}}) \\ &= E\Big(e^{t_1\cdot\frac{1}{n}\sum_{i=1}^n X_i + t_2\cdot\frac{1}{n}\sum_{j=1}^n Y_j}\Big) \\ &= \prod_{i=1}^n E\Big(e^{\frac{t_1}{n}X_i + \frac{t_2}{n}Y_i}\Big) \\ &= \Big(e^{\frac{\boldsymbol{t}'}{n}\mu + \frac{1}{2}\frac{\boldsymbol{t}'}{n}\mathbf{\Sigma}\frac{\boldsymbol{t}}{n}}\Big)^n \\ &= e^{\boldsymbol{t}'\mu + \frac{1}{2}\boldsymbol{t}'\frac{\mathbf{\Sigma}}{n}\boldsymbol{t}} \end{aligned}$$

$$\therefore \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \frac{\mathbf{\Sigma}}{n} \right)$$

$$\therefore n(\bar{X}-\mu_1, \bar{Y}-\mu_2)\mathbf{\Sigma}^{-1}\begin{pmatrix}\bar{X}-\mu_1\\ \bar{Y}-\mu_2\end{pmatrix} = (\bar{X}-\mu_1, \bar{Y}-\mu_2)\Big(\frac{\mathbf{\Sigma}}{n}\Big)^{-1}\begin{pmatrix}\bar{X}-\mu_1\\ \bar{Y}-\mu_2\end{pmatrix} \sim \chi^2_2$$