STATS100B – Introduction to Mathematical Statistics
Homework 5

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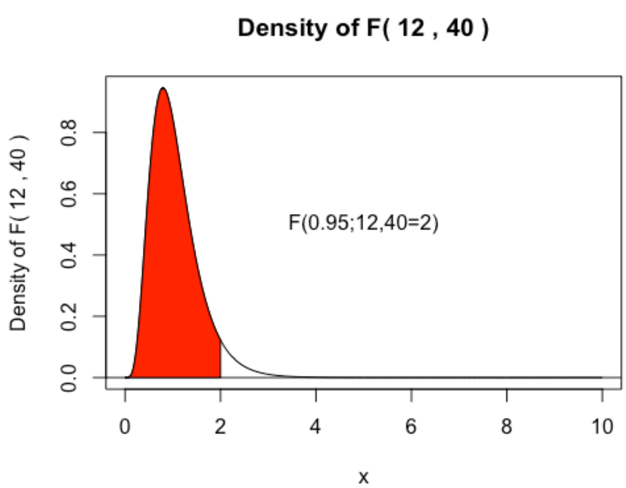
Exercise a

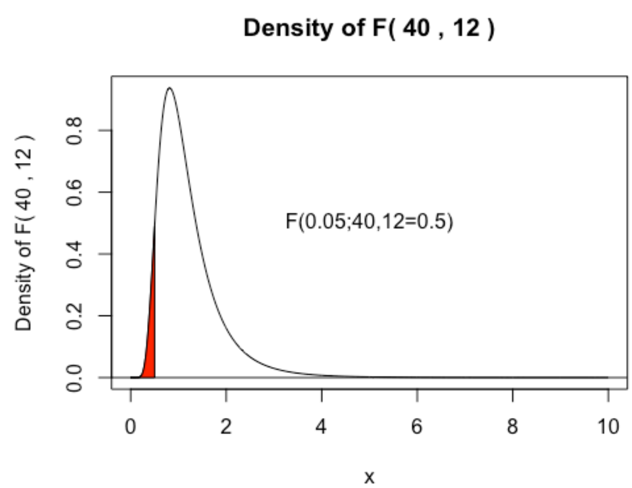
(1) Solution:

$$X \sim F_{m,n},\ Y = \frac{1}{X} \sim F_{n,m}$$
$$P(X < F_{\alpha;m,n}) = \alpha$$
$$P(\frac{1}{X} > \frac{1}{F_{\alpha;m,n}}) = \alpha$$
$$P(Y > \frac{1}{F_{\alpha;m,n}}) = 1 - \alpha$$
$$P(Y < \frac{1}{F_{\alpha;m,n}}) = \alpha$$
$$\therefore \frac{1}{F_{\alpha;m,n}} = F_{1-\alpha;n,m},$$

which means

$$F_{\alpha;m,n} = \frac{1}{F_{1-\alpha;n,m}}$$





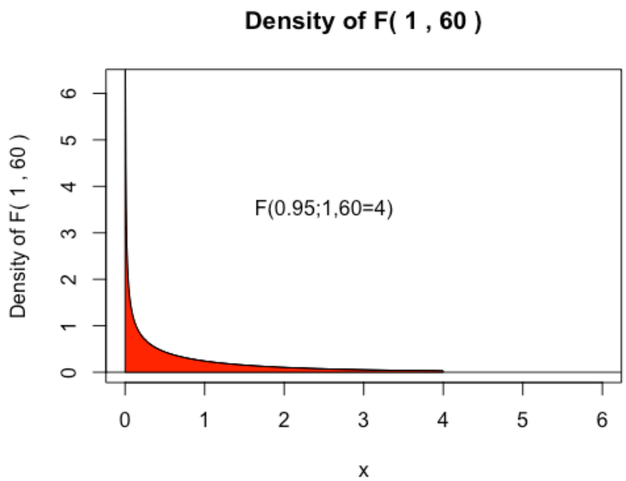
(2) Solution:

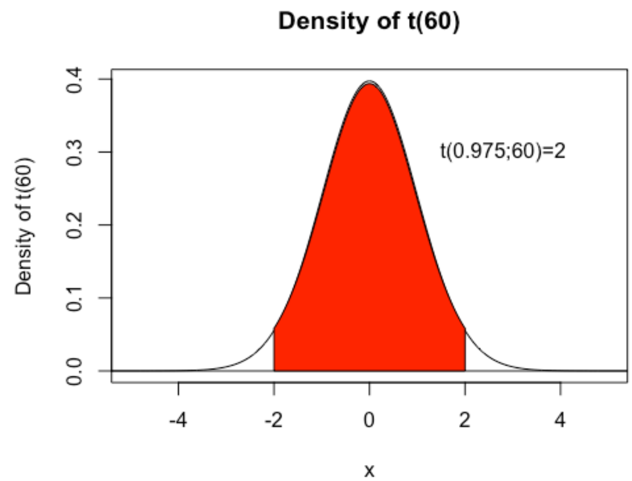
$$Y \sim t_n,\ X = Y^2 \sim F_{1,n}$$
$$P(-t_{1-\frac{\alpha}{2},n} < Y < t_{1-\frac{\alpha}{2},n}) = 1 - \alpha$$

$$P(Y^2 < t^2_{1-\frac{\alpha}{2},n}) = 1 - \alpha$$

$$P(X < t^2_{1-\frac{\alpha}{2},n}) = 1 - \alpha$$

$$\therefore t^2_{1-\frac{\alpha}{2},n} = F_{1-\alpha;1,n}$$





Exercise b

Solution:

$$\because \bar{X} \sim N(\mu_1, \frac{\sigma_1}{\sqrt{13}}), \bar{Y} \sim N(\mu_2, \frac{\sigma_2}{\sqrt{16}})$$

$$\therefore \overline{X} + \overline{Y} \sim N\left(\mu_1 + \mu_2, \sqrt{\frac{\sigma_1^2}{13} + \frac{\sigma_2^2}{16}}\right)$$

$$\frac{\left(\overline{X} + \overline{Y}\right) - (\mu_1 + \mu_2)}{\sqrt{\frac{\sigma_1^2}{13} + \frac{\sigma_2^2}{16}}} \sim N\left(0, 1\right)$$

$$\because \frac{12S_X^2}{\sigma_1} \sim \chi^2_{12}, \frac{15S_Y^2}{\sigma_2} \sim \chi^2_{15}$$

$$\frac{12S_X^2}{\sigma_1} + \frac{15S_Y^2}{\sigma_2} \sim \chi^2_{27}$$

$$\therefore \frac{\frac{\left(\overline{X} + \overline{Y}\right) - (\mu_1 + \mu_2)}{\sqrt{\frac{\sigma_1^2}{13} + \frac{\sigma_2^2}{16}}}}{\sqrt{\frac{\frac{12S_X^2}{\sigma_1} + \frac{15S_Y^2}{\sigma_2}}{27}}} \sim t_{27}$$

Exercise c

Solution:

$$
\begin{aligned}
E(X) &= E\left[\frac{\chi_n^2/n}{\chi_m^2/m}\right] \\
&= \frac{m}{n} E\left[\chi_n^2\right] E\left[(\chi_m^2)^{-1}\right] \\
&= \frac{m}{n} \cdot n \cdot \frac{\Gamma(\frac{m}{2}-1)\cdot 2^{-1}}{\Gamma(\frac{m}{2})} \\
&= m \cdot \frac{1}{2(\frac{m}{2}-1)} \\
&= \frac{m}{m-2}
\end{aligned}
$$

$$
\begin{aligned}
var(X) &= var\left[\frac{\chi_n^2/n}{\chi_m^2/m}\right] \\
&= E\left[\left(\frac{\chi_n^2/n}{\chi_m^2/m}\right)^2\right] - \left(E\left(\frac{\chi_n^2/n}{\chi_m^2/m}\right)\right)^2 \\
&= \frac{m^2}{n^2} \cdot E\Big[(\chi_n^2)^2\Big] \cdot E\Big[(\chi_m^2)^{-2}\Big] - \frac{m^2}{n^2} \cdot \Big(E(\chi_n^2)\Big)^2 \cdot \Big(E\left[(\chi_m^2)^{-1}\right]\Big)^2 \\
&= \frac{m^2}{n^2} \times \frac{\Gamma(\frac{n}{2}+2)\cdot 2^2}{\Gamma(\frac{n}{2})} \times \frac{\Gamma(\frac{m}{2}-2)\cdot 2^{-2}}{\Gamma(\frac{m}{2})} - \frac{m^2}{n^2} \times n^2 \times \Big(\frac{\Gamma(\frac{m}{2}-1)\cdot 2^{-1}}{\Gamma(\frac{m}{2})}\Big)^2 \\
&= \frac{m^2}{n^2} \times n(n+2) \times \frac{1}{(m-2)(m-4)} - \frac{m^2}{n^2} \times n^2 \times \frac{1}{(m-2)^2} \\
&= \frac{2m^2(m+n-2)}{n(m-2)^2(m-4)}
\end{aligned}
$$

Exercise d

$$
\bar{X} = \frac{1}{n}\mathbf{1}'\boldsymbol{X},\ \mathbf{1} = (1\,1\,\cdots 1)'
$$

$$
\begin{pmatrix} \bar{X} \\ X_1 - \bar{X} \\ X_2 - \bar{X} \\ \cdots \\ X_n - \bar{X} \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\mathbf{1}' \\ \boldsymbol{I} - \frac{1}{n}\mathbf{1}\mathbf{1}' \end{pmatrix} = \boldsymbol{AX}
$$

$$
var(\boldsymbol{AX}) = \begin{pmatrix} \frac{1}{n}\mathbf{1}' \\ \boldsymbol{I} - \frac{1}{n}\mathbf{1}\mathbf{1}' \end{pmatrix} \Big((1-\rho)\boldsymbol{I} - \rho\boldsymbol{J}\Big) \begin{pmatrix} \frac{1}{n}\mathbf{1}' & \boldsymbol{I} - \frac{1}{n}\mathbf{1}\mathbf{1}' \end{pmatrix}
$$

Notice that

$$\mathbf{1}'\mathbf{1} = n,\ \mathbf{1}\mathbf{1}' = \boldsymbol{J},\ \mathbf{1}'\boldsymbol{J} = n\mathbf{1}',\ \boldsymbol{J}\mathbf{1} = n\mathbf{1}$$

$$\therefore var(\boldsymbol{AX}) = \begin{pmatrix} (1-\rho)\frac{1}{n} + \rho & \mathbf{0} \\ \mathbf{0} & (1-\rho)(\boldsymbol{I} - \frac{1}{n}\mathbf{1}\mathbf{1}') \end{pmatrix}$$

$$\therefore \bar{X}\ and \begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \cdots \\ X_n - \bar{X} \end{pmatrix}\ are\ independent.$$

Exercise e

$$\begin{aligned} \boldsymbol{Y}'\boldsymbol{\Sigma}\boldsymbol{Y} &= \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} \\ &= \frac{\sigma_2^2 Y_1^2 - 2\sigma_{12}Y_1Y_2 + \sigma_1^2 Y_2^2}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \end{aligned}$$

$$\begin{aligned} \boldsymbol{Y}'\boldsymbol{\Sigma}\boldsymbol{Y} - \frac{Y_1^2}{\sigma_1^2} &= \frac{\sigma_2^2 Y_1^2 - 2\sigma_{12}Y_1Y_2 + \sigma_1^2 Y_2^2 - \sigma_1^2\sigma_2^2\left(1-\rho^2\right)\frac{Y_1^2}{\sigma_1^2}}{\sigma_1^2\sigma_2^2\left(1-\rho^2\right)} \\ &= \frac{\frac{Y_1^2}{\sigma_1^2} - 2\rho\frac{y_1}{\sigma_1}\frac{Y_2}{\sigma_2} + \frac{Y_2^2}{\sigma_2^2} - \left(1-\rho^2\right)\frac{Y_1^2}{\sigma_1^2}}{1-\rho} \\ &= \frac{\left(\rho\frac{Y_1}{\sigma_1} - \frac{Y_2}{\sigma_2}\right)^2}{1-\rho^2} \end{aligned}$$

$$\because \begin{pmatrix} \rho\frac{Y_1}{\sigma_1} & -\frac{Y_2}{\sigma_2} \end{pmatrix} = \begin{pmatrix} \frac{\rho}{\sigma_1} & -\frac{1}{\sigma_2} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$var\left(\begin{pmatrix} \rho\frac{Y_1}{\sigma_1} & -\frac{Y_2}{\sigma_2} \end{pmatrix}\right) = \begin{pmatrix} \frac{\rho}{\sigma_1} & -\frac{1}{\sigma_2} \end{pmatrix} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix} \begin{pmatrix} \frac{\rho}{\sigma_1} \\ -\frac{1}{\sigma_2} \end{pmatrix} = 1 - \rho^2$$

$$E\left(\begin{pmatrix} \rho\frac{Y_1}{\sigma_1} & -\frac{Y_2}{\sigma_2} \end{pmatrix}\right) = \frac{\rho}{\sigma_1} \times 0 - \frac{1}{\sigma_2} \times 0 = 0$$

$$\therefore \frac{\rho\frac{Y_1}{\sigma_1} - \frac{Y_2}{\sigma_2}}{\sqrt{1-\rho^2}} \sim N(0,1)$$

$$\boldsymbol{Y}'\boldsymbol{\Sigma}\boldsymbol{Y} - \frac{Y_1^2}{\sigma_1^2} = \frac{\left(\rho\frac{Y_1}{\sigma_1} - \frac{Y_2}{\sigma_2}\right)^2}{1-\rho^2} \sim \chi_1^2$$