**University of California, Los Angeles**
**Department of Statistics**

**Statistics 100B** **Instructor: Nicolas Christou**

**Homework 7**

Answer the following questions:

a. Suppose $Y_1, Y_2, \ldots, Y_n$ follow multivariate normal with mean $\mu\mathbf{1}$ and variance covariance matrix $\sigma^2\mathbf{V}$, where $\mathbf{V}$ is an $n \times n$ symmetric matrix of known constants. Show that the maximum likelihood estimates of $\mu$ and $\sigma^2$ are $\hat{\mu} = \frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{Y}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}$ and $\hat{\sigma}^2 = \frac{(\mathbf{Y}-\hat{\mu}\mathbf{1})'\mathbf{V}^{-1}(\mathbf{Y}-\hat{\mu}\mathbf{1})}{n}$.

b. Refer to question (a). Find $E(\hat{\mu})$ and $E(\hat{\sigma}^2)$. ?

c. Refer to question (a). Find the Fisher information matrix $\mathbf{I}(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\mu, \sigma^2)'$. Is $\hat{\mu}$ an efficient estimator of $\mu$?

d. Let $Y_1, Y_2, \ldots, Y_n$ independent random variables, and let $Y_i \sim N(i\theta, i\sigma)$, i.e. $E(Y_i) = i\theta$ and $var(Y_i) = i^2\sigma^2$, for $i = 1, 2, \ldots, n$. Find the maximum likelihood estimator of $\theta$. Is this estimator efficient estimator of $\theta$?

e. Suppose that the radius of a circle is measured with an error $\epsilon \sim N(0, \sigma)$. If $n$ independent measurements are made find an unbiased estimator of the area of the circle.

f. Let $Y_1, \ldots, Y_n$ be i.i.d. random variables from the Weibull distribution $f(y|\theta) = (\frac{2y}{\theta})exp(-\frac{y^2}{\theta}), y > 0$. Show that $\hat{\theta} = \frac{\sum_{i=1}^n Y_i^2}{n}$ is unbiased estimator of $\theta$. Is $\hat{\theta}$ an efficient estimator of $\theta$?

g. Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, \theta), \theta > 0$. For this model both $\bar{X}$ and $cS$ are unbiased estimators of $\theta$, where $c = \frac{\sqrt{n-1}\Gamma(\frac{n-1}{2})}{\sqrt{2}\Gamma(\frac{n}{2})}$. Define the estimator $T = \alpha_1\bar{X} + \alpha_2(cS)$, where we do not assume that $\alpha_1 + \alpha_2 = 1$. Find the estimator that minimizes $E(T-\theta)^2$.

a. $f(Y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \cdot |\Sigma|^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}(Y-\mu\mathbf{1})'\Sigma^{-1}(Y-\mu\mathbf{1})}$

$\ln L = -\frac{n}{2}\ln 2\pi - \frac{1}{2}\ln|\sigma^2 V| - \frac{1}{2}(Y-\mu\mathbf{1})'(\sigma^2 V)^{-1}(Y-\mu\mathbf{1})$

$= -\frac{n}{2}\ln 2\pi - \frac{1}{2}\ln(\sigma^2)^n - \frac{1}{2}\ln|V| - \frac{1}{2}\cdot\frac{1}{\sigma^2}\cdot(Y'-\mu\mathbf{1}')V^{-1}(Y-\mu\mathbf{1})$

$-\frac{1}{2}\frac{1}{\sigma^2}(Y'V^{-1}Y - \mu Y'V^{-1}\mathbf{1} - \mu\mathbf{1}'V^{-1}Y + \mu^2\mathbf{1}'V^{-1}\mathbf{1})$

$\frac{\partial \ln L}{\partial\mu} = -\frac{1}{2\sigma^2}\Big[-\underbrace{Y'V^{-1}\mathbf{1}}_{1\times n\ n\times n} = \underbrace{-\mathbf{1}'V^{-1}Y}_{1\times n\ n\times n} + 2\mu\mathbf{1}'V^{-1}\mathbf{1}\Big] = 0$

$\therefore \hat{\mu} = \frac{\mathbf{1}'V^{-1}Y}{\mathbf{1}'V^{-1}\mathbf{1}}$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2}\cdot\frac{1}{\sigma^2} - \frac{1}{2}\cdot\frac{-2}{\sigma^4}(Y-\mu 1)'V^{-1}(Y-\mu 1) = 0$$

$$\therefore \sigma^2 = \frac{(Y-\mu 1)'V^{-1}(Y-\mu 1)}{n}.$$

$$\therefore \hat{\sigma}^2 = \frac{(Y-\hat{\mu} 1)'V^{-1}(Y-\hat{\mu} 1)}{n}$$

b. $E(\hat{\mu}) = E\left(\frac{1'V^{-1}Y}{1'V^{-1}1}\right) = \frac{1}{1'V^{-1}1}\cdot 1'V^{-1}E(Y) = \frac{1}{1'V^{-1}1}\, 1'V^{-1}\cdot\mu 1 = \mu.$

$$E(\hat{\mu}^2) = E\left(\frac{(1'V^{-1}Y)^2}{(1'V^{-1}1)^2}\right)$$

$$E(\hat{\sigma}^2) = \frac{1}{n}E\left(Y'V^{-1}Y - \hat{\mu}\, Y'V^{-1}1 - \hat{\mu}\, 1'V^{-1}Y + \hat{\mu}^2 1'V^{-1}1\right)$$

$$E(\hat{\sigma}^2) = \frac{1}{n}E(Y-\hat{\mu}1)'V^{-1}(Y-\hat{\mu}1)$$

$$= \frac{1}{n}E\left[tr(Y-\hat{\mu}1)'V^{-1}(Y-\hat{\mu}1)\right]$$

$$= \frac{1}{n}tr\, E\left(V^{-1}(Y-\hat{\mu}1)(Y-\hat{\mu}1)'\right)$$

$$= \frac{1}{n}tr\, V^{-1}E(Y-\hat{\mu}1)(Y-\hat{\mu}1)'$$

$$= \frac{1}{n}tr\, V^{-1}\left[var(Y-\hat{\mu}1) + \underbrace{E(Y-\hat{\mu}1)}_{=0}\underbrace{E(Y-\hat{\mu}1)'}_{=0}\right]$$

$$= \frac{1}{n}tr\, V^{-1}var(Y-\hat{\mu}1)$$

$$var(Y-\hat{\mu}1) = var\left(Y - \frac{1 1'V^{-1}Y}{1'V^{-1}1}\right)$$

$$= var\left(\left(I - \frac{1 1'V^{-1}}{1'V^{-1}1}\right)Y\right)$$

$$= \left(I - \frac{1 1'V^{-1}}{1'V^{-1}1}\right)\sigma^2 I\left(I - \frac{1 1'V^{-1}}{1'V^{-1}1}\right)'$$

$$= \sigma^2 \left( V - \frac{\mathbf{1}\mathbf{1}'}{\mathbf{1}'V^{-1}\mathbf{1}} \right)$$

$$\therefore E(\hat{\sigma}^2) = \frac{1}{n} \operatorname{tr} V^{-1} \left( \sigma^2 \left( V - \frac{\mathbf{1}\mathbf{1}'}{\mathbf{1}'V^{-1}\mathbf{1}} \right) \right)$$

$$= \frac{1}{n} \sigma^2 \left[ \operatorname{tr} V^{-1} V - \operatorname{tr} \frac{V^{-1}\mathbf{1}\mathbf{1}'}{\mathbf{1}'V^{-1}\mathbf{1}} \right.$$

$$= \frac{1}{n} \sigma^2 \left( n - \operatorname{tr} \frac{\mathbf{1}'V^{-1}\mathbf{1}}{\mathbf{1}'V^{-1}\mathbf{1}} \right)$$

$$= \frac{n-1}{n} \sigma^2$$

c. $I(\theta) = -E\begin{pmatrix} \frac{\partial^2 \ln L}{\partial \mu^2} & \frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} \\ \frac{\partial^2 \ln L}{\partial \sigma^2 \partial \mu} & \frac{\partial^2 \ln L}{\partial \sigma^{2(2)}} \end{pmatrix}$

$$\frac{\partial^2 \ln L}{\partial \mu^2} = -\frac{1'V^{-1}1}{\sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} = \frac{1}{2\sigma^4}\left[2\mu 1'V^{-1}1 - 2 1'V^{-1}Y\right] = \frac{1}{\sigma^4}\left[1'V^{-1}1\mu - 1'V^{-1}Y\right]$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2}\cdot\frac{1}{\sigma^2} + \frac{1}{2}\cdot\frac{1}{\sigma^4}(Y-\mu 1)'V^{-1}(Y-\mu 1)$$

$$\frac{\partial^2 \ln L}{\partial \sigma^{2(2)}} = \frac{n}{2}\cdot\frac{1}{\sigma^4} - \frac{1}{\sigma^6}(Y-\mu 1)'V^{-1}(Y-\mu 1)$$

$$I^{-1}(v) = \begin{pmatrix} \frac{\sigma^2}{1'V^{-1}1} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix}$$

$-\frac{1'V^{-1}1}{\sigma^2} \neq 0$ when $n \to \infty$ $\therefore \hat{\mu}$ is not efficient

$$E(\hat{\mu}) = \mu$$

$$\text{var}(\hat{\mu}) = \text{var}\left(\frac{1'V^{-1}Y}{1'V^{-1}1}\right) = \sigma^2 \frac{1'V^{-1}VV^{-1}1}{(1'V^{-1}1)^2} = \frac{\sigma^2}{1'V^{-1}1}$$

d. $f(y_i) = \frac{1}{\sqrt{2\pi}\, i\sigma} e^{-\frac{(y_i - i\theta)^2}{2(i\sigma)^2}}$ $\quad (2\pi)^{-\frac{1}{2}}$

$$L = \prod_{i=1}^{n} f(y_i) = (2\pi)^{-\frac{n}{2}} \cdot \prod_{i=1}^{n} i \cdot \sigma^{-n} \cdot e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(\frac{y_i - i\theta}{i}\right)^2}$$

$$\ln L = -\frac{n}{2} \ln 2\pi + \sum_{i=1}^{n} \ln i - n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(\theta - \frac{y_i}{i}\right)^2$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} \cdot 2\left(\theta - \frac{y_i}{i}\right) = 0$$

$$\therefore \hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{i}$$

$$E(\hat{\theta}) = \frac{1}{n} \sum_{i=1}^{n} E\left(\frac{y_i}{i}\right) = \frac{1}{n} \cdot \sum_{i=1}^{n} \frac{E(y_i)}{i} = \frac{1}{n} \cdot n\theta = \theta.$$

$$var(\hat{\theta}) = \frac{1}{n^2} \cdot \sum_{i=1}^{n} var\left(\frac{y_i}{i}\right) = \frac{1}{n^2} \sum_{i=1}^{n} \frac{var(y_i)}{i^2} = \frac{1}{n^2} \cdot \sum_{i=1}^{n} \frac{i^2\sigma^2}{i^2} = \frac{\sigma^2}{n} \to 0$$

when $n \to \infty$

$\therefore \hat{\theta}$ is efficient

e. Suppose the radius equals $R$.

$$R_i = R + \epsilon_i$$

$$E(\bar{R}) = \frac{1}{n} \cdot \sum_{i=1}^{n} E(R + \epsilon_i) = R.$$

$$E(\bar{R}^2) = var(\bar{R}) + E^2(\bar{R}) = \frac{1}{n^2} \sum_{i=1}^{n} var(R + \epsilon_i) + R^2 = \frac{\sigma^2}{n} + R^2$$

$$\hat{R} = \bar{R}^2 - \frac{\sigma^2}{n}$$

$$\therefore \hat{S} = \pi\left(\bar{R}^2 - \frac{\sigma^2}{n}\right)$$

f. $f(y|\theta)=\dfrac{2y}{\theta}e^{-\frac{y^2}{\theta}}$

$X=Y^2$

$X\sim exp(\frac{1}{\theta})$.

$\hat{\theta}=\frac{1}{n}\sum_{i=1}^{n}Y_i^2$

$$E(Y_i^2)=\int_0^{+\infty}y^2\cdot\frac{2y}{\theta}\cdot e^{-\frac{y^2}{\theta}}dy$$

$$=\theta\int_0^{+\infty}\frac{y^2}{\theta}\cdot e^{-\frac{y^2}{\theta}}d\left(\frac{y^2}{\theta}\right)$$

$t=\frac{y^2}{\theta}$ $$=\theta\int_0^{+\infty}te^{-t}dt$$

$$=\theta\cdot\left[-te^{-t}\Big|_0^{+\infty}-\int_0^{+\infty}-e^{-t}dt\right]$$

$$=\theta\left[-\int_0^{+\infty}e^{-t}d(-t)\right]$$

$$=\theta$$

$$\therefore E(\hat{\theta})=\frac{1}{n}\sum_{i=1}^{n}E(Y_i^2)=\frac{1}{n}\cdot n\theta=\theta$$

$$E(Y_i^4)=\int_0^{+\infty}y^4\cdot\frac{2y}{\theta}e^{-\frac{y^2}{\theta}}dy \qquad \theta^2\frac{y^4}{\theta^2}e^{-\frac{y^2}{\theta}}\cdot d\frac{y^2}{\theta}$$

$$=\theta^2\int_0^{+\infty}\left(\frac{y^2}{\theta}\right)^2e^{-\frac{y^2}{\theta}}d\left(\frac{y^2}{\theta}\right)$$

$t=\frac{y^2}{\theta}$ $$=\theta^2\int_0^{+\infty}t^2e^{-t}dt$$

$$=\theta^2\left[-t^2e^{-t}\Big|_0^{+\infty}-\int_0^{+\infty}2t(-e^{-t})dt\right]$$

$$=\theta^2\cdot\left[2\int_0^{+\infty}te^{-t}dt\right]$$

$$=2\theta^2$$

$$\therefore var(\hat{\theta}) = \frac{1}{n^2}\sum_{i=1}^{n} var(Y_i^2)$$

$$= \frac{1}{n^2}\cdot\sum_{i=1}^{n}\left[E(Y_i^4) - E^2(Y_i^2)\right]$$

$$= \frac{1}{n^2}\cdot\sum_{i=1}^{n}\left[2\theta^2 - \theta^2\right]$$

$$= \frac{\theta^2}{n} \longrightarrow 0 \quad \text{when } n\to\infty$$

$\therefore \hat{\theta}$ is efficient

g. $E(T-\theta)^2 = var(T) + B^2$

$$var(T) = Var(\alpha_1\bar{X} + \alpha_2 cS)$$

$$var(cS) = E(c^2S^2) - E^2(cS) = (c^2-1)\theta^2$$

$$= \alpha_1^2\, var(\bar{X}) + \alpha_2^2\, var(cS)$$

$$= \alpha_1^2\cdot\frac{\theta^2}{n} + \alpha_2^2\cdot(c^2-1)\theta^2$$

$$B = E(T) - \theta = (\alpha_1+\alpha_2-1)\theta$$

$$\therefore E(T-\theta)^2 = \frac{\alpha_1^2}{n}\theta^2 + \alpha_2^2(c^2-1)\theta^2 + (\alpha_1+\alpha_2-1)^2\theta^2$$

$$= \left[\frac{\alpha_1^2}{n} + (c^2-1)\alpha_2^2 + (\alpha_1+\alpha_2-1)^2\right]\theta^2$$

$$\begin{cases} \dfrac{\partial f}{\partial\alpha_1} = \dfrac{2\alpha_1}{n} + 2(\alpha_1+\alpha_2-1) = 0 \\ \dfrac{\partial f}{\partial\alpha_2} = 2(c^2-1)\alpha_2 + 2(\alpha_1+\alpha_2-1) = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 = \dfrac{n(c^2-1)}{(n+1)(c^2-1)+1} \\ \alpha_2 = \dfrac{1}{(n+1)(c^2-1)+1} \end{cases}$$

$$A = \frac{\partial^2 f}{\partial\alpha_1^2} = \frac{2}{n}+2, \quad B = \frac{\partial^2 f}{\partial\alpha_1\partial\alpha_2} = 2, \quad C = \frac{\partial^2 f}{\partial\alpha_2^2} = 2(c^2-1)+2 = 2c^2$$

$$A>0.\ B^2 - AC = 4 - \left(\frac{2}{n}+2\right)(2c^2) = 4 - 4\left(\frac{1}{n}+1\right)c^2$$

$$4\left[1-\left(\frac{1}{n}+1\right)\cdot c^2\right]$$

$$1-\frac{n+1}{n}\cdot\frac{n-1}{2}\cdot\frac{\Gamma^2\left(\frac{n-1}{2}\right)}{\Gamma^2\left(\frac{n}{2}\right)}$$