STATS100B – Introduction to Mathematical Statistics
Homework 5

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Exercise a

(1) Solution:

$$X \sim F_{m,n},\, Y = \frac{1}{X} \sim F_{n,m}$$
$$P(X < F_{\alpha;m,n}) = \alpha$$
$$P(\frac{1}{X} > \frac{1}{F_{\alpha;m,n}}) = \alpha$$
$$P(Y > \frac{1}{F_{\alpha;m,n}}) = 1-\alpha$$
$$P(Y < \frac{1}{F_{\alpha;m,n}}) = \alpha$$
$$\therefore \frac{1}{F_{\alpha;m,n}} = F_{1-\alpha;n,m},$$

which means

$$F_{\alpha;m,n} = \frac{1}{F_{1-\alpha;n,m}}$$

(2) Solution:

$$Y \sim t_n,\, X = Y^2 \sim F_{1,n}$$
$$P(-t_{1-\frac{\alpha}{2},n} < Y < t_{1-\frac{\alpha}{2},n}) = 1-\alpha$$
$$P(Y^2 < t^2_{1-\frac{\alpha}{2},n}) = 1-\alpha$$
$$P(X < t^2_{1-\frac{\alpha}{2},n}) = 1-\alpha$$
$$\therefore t^2_{1-\frac{\alpha}{2},n} = F_{1-\alpha;1,n}$$

Exercise b

Solution:

$$\bar{X} \sim N(\mu_1, \frac{\sigma_1}{\sqrt{13}}),\, \bar{Y} \sim N(\mu_2, \frac{\sigma_2}{\sqrt{16}})$$
$$\overline{X} + \overline{Y} \sim N\left(\mu_1 + \mu_2, \sqrt{\frac{\sigma_1^2}{13} + \frac{\sigma_2^2}{16}}\right)$$

$$\frac{\left(\overline{X}+\overline{Y}\right)-(\mu_1+\mu_2)}{\sqrt{\dfrac{\sigma_1^2}{13}+\dfrac{\sigma_2^2}{16}}} \sim N(0,1)$$

$$\frac{12S_X^2}{\sigma_1} \sim \chi_{12}^2,\ \frac{15S_Y^2}{\sigma_2} \sim \chi_{15}^2$$

$$\frac{12S_X^2}{\sigma_1}+\frac{15S_Y^2}{\sigma_2} \sim \chi_{27}^2$$

$$\frac{\dfrac{\left(\overline{X}+\overline{Y}\right)-(\mu_1+\mu_2)}{\sqrt{\frac{\sigma_1^2}{13}+\frac{\sigma_2^2}{16}}}}{\sqrt{\dfrac{\frac{12S_X^2}{\sigma_1}+\frac{15S_Y^2}{\sigma_2}}{27}}} \sim t_{27}$$

Exercise c

Solution:

$$\begin{aligned}
E(X) &= E\left[\frac{\chi_n^2/n}{\chi_m^2/m}\right] \\
&= \frac{m}{n}E\left[\chi_n^2\right]E\left[(\chi_m^2)^{-1}\right] \\
&= \frac{m}{n}\cdot n\cdot\frac{\Gamma(\dfrac{m}{2}-1)\cdot 2^{-1}}{\Gamma(\dfrac{m}{2})} \\
&= m\cdot\frac{1}{2(\dfrac{m}{2}-1)} \\
&= \frac{m}{m-2}
\end{aligned}$$

$$\begin{aligned}
var(X) &= var\left[\frac{\chi_n^2/n}{\chi_m^2/m}\right] \\
&= E\left[\left(\frac{\chi_n^2/n}{\chi_m^2/m}\right)^2\right]-\left(E\left(\frac{\chi_n^2/n}{\chi_m^2/m}\right)\right)^2 \\
&= \frac{m^2}{n^2}\cdot E\Big[(\chi_n^2)^2\Big]\cdot E\Big[(\chi_m^2)^{-2}\Big]-\frac{m^2}{n^2}\cdot\Big(E(\chi_n^2)\Big)^2\cdot\Big(E\left[(\chi_m^2)^{-1}\right]\Big)^2 \\
&= \frac{m^2}{n^2}\times\frac{\Gamma(\dfrac{n}{2}+2)\cdot 2^2}{\Gamma(\dfrac{n}{2})}\times\frac{\Gamma(\dfrac{m}{2}-2)\cdot 2^{-2}}{\Gamma(\dfrac{m}{2})}-\frac{m^2}{n^2}\times n^2\times\Big(\frac{\Gamma(\dfrac{m}{2}-1)\cdot 2^{-1}}{\Gamma(\dfrac{m}{2})}\Big)^2 \\
&= \frac{m^2}{n^2}\times n(n+2)\times\frac{1}{(m-2)(m-4)}-\frac{m^2}{n^2}\times n^2\times\frac{1}{(m-2)^2} \\
&= \frac{2m^2(m+n-2)}{n(m-2)^2(m-4)}
\end{aligned}$$

Exercise d

Exercise e

$$\boldsymbol{Y}'\boldsymbol{\Sigma}\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \end{pmatrix}$$
$$= \frac{\sigma_2^2 Y_1^2 - 2\sigma_{12}Y_1Y_2 + \sigma_1^2 Y_2^2}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}$$

$$\boldsymbol{Y}'\boldsymbol{\Sigma}\boldsymbol{Y} - \frac{Y_1^2}{\sigma_1^2} = \frac{\sigma_2^2 Y_1^2 - 2\sigma_{12}Y_1Y_2 + \sigma_1^2 Y_2^2 - \sigma_1^2\sigma_2^2\left(1-\rho^2\right)\frac{Y_1^2}{\sigma_1^2}}{\sigma_1^2\sigma_2^2\left(1-\rho^2\right)}$$
$$= \frac{\frac{Y_1^2}{\sigma_1^2} - 2\rho\frac{y_1}{\sigma_1}\frac{Y_2}{\sigma_2} + \frac{Y_2^2}{\sigma_2^2} - \left(1-\rho^2\right)\frac{Y_1^2}{\sigma_1^2}}{1-\rho}$$
$$= \frac{\left(\rho\frac{Y_1}{\sigma_1} - \frac{Y_2}{\sigma_2}\right)^2}{1-\rho^2}$$

$$\because \begin{pmatrix} \rho\frac{Y_1}{\sigma_1} & -\frac{Y_2}{\sigma_2} \end{pmatrix} = \begin{pmatrix} \frac{\rho}{\sigma_1} & -\frac{1}{\sigma_2} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$var\left(\begin{pmatrix} \rho\frac{Y_1}{\sigma_1} & -\frac{Y_2}{\sigma_2} \end{pmatrix}\right) = \begin{pmatrix} \frac{\rho}{\sigma_1} & -\frac{1}{\sigma_2} \end{pmatrix} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix} \begin{pmatrix} \frac{\rho}{\sigma_1} \\ -\frac{1}{\sigma_2} \end{pmatrix} = 1-\rho^2$$

$$E\left(\begin{pmatrix} \rho\frac{Y_1}{\sigma_1} & -\frac{Y_2}{\sigma_2} \end{pmatrix}\right) = \frac{\rho}{\sigma_1}\times 0 - \frac{1}{\sigma_2}\times 0 = 0$$

$$\therefore \frac{\rho\frac{Y_1}{\sigma_1} - \frac{Y_2}{\sigma_2}}{\sqrt{1-\rho^2}} \sim N(0,1)$$

$$\boldsymbol{Y}'\boldsymbol{\Sigma}\boldsymbol{Y} - \frac{Y_1^2}{\sigma_1^2} = \frac{\left(\rho\frac{Y_1}{\sigma_1} - \frac{Y_2}{\sigma_2}\right)^2}{1-\rho^2} \sim \chi_1^2$$

Exercise 4

$$
\begin{aligned}
E(X^4) &= E[X^3(X-\mu+\mu)] \\
&= E[X^3(X-\mu)] + \mu E(X^3) \\
&= \sigma^2 E(X^2) + \mu\Big[E[(X^2)(X-\mu)] + \mu E(X^2)\Big] \\
&= \sigma^2[(EX)^2 + var(X)] + \mu\Big[2\sigma^2 E(X) + \mu[(EX)^2 + var(X)]\Big] \\
&= \sigma^2[\mu^2+\sigma^2] + \mu\Big[2\mu\sigma + \mu[\mu^2+\sigma^2]\Big] \\
&= \mu^2 + 6\mu^2\sigma^2 + 3\sigma^4
\end{aligned}
$$