

STATS100B – Introduction Mathematical Statistics

Homework 1

Feng Shiwei UID:305256428

Exercise 1

Solution:

$$\begin{aligned} P(X = x) &= \binom{n}{x} \theta^x (1 - \theta)^{n-x} \\ &= \binom{n}{x} (1 - p)^n \exp\left\{\ln \frac{p}{1-p} x\right\} \end{aligned}$$

so we get

$$\begin{aligned} c(\boldsymbol{\theta}) &= (1 - p)^n, \sum_{i=1}^k w_i(\boldsymbol{\theta}) t_i(x) = \ln \frac{p}{1-p} x \\ \sum_{i=1}^k \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x) &= \frac{1}{p(1-p)} x, \sum_{i=1}^k \frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(x) = \frac{2p-1}{p^2(1-p)^2} x \\ \frac{\partial^2}{\partial \theta_j^2} \log c(\boldsymbol{\theta}) &= -\frac{n}{(1-p)^2} \end{aligned}$$

Use the second theorem, we can get the $\text{var}(X)$.

$$\begin{aligned} \text{var}\left(\frac{1}{p(1-p)} X\right) &= \frac{n}{(1-p)^2} - E\left(\frac{2p-1}{p^2(1-p)^2} X\right) \\ \frac{1}{p^2(1-p)^2} \text{var}(X) &= \frac{n}{(1-p)^2} - \frac{2p-1}{p^2(1-p)^2} E(X) = \frac{n}{(1-p)^2} - \frac{2p-1}{p^2(1-p)^2} np \\ \therefore \text{var}(X) &= np(1-p) \end{aligned}$$

Exercise 2

Solution:

$$\int_x h(x) c(\boldsymbol{\theta}) \exp\left\{\sum_{i=1}^k w_i(\boldsymbol{\theta}) t_i(x)\right\} dx = 1$$

Differentiate both side w.r.t θ_j ,

$$\begin{aligned} \int_x \left[h(x) \frac{\partial c(\boldsymbol{\theta})}{\partial \theta_j} \exp\left(\sum_{i=1}^k w_i(\boldsymbol{\theta}) t_i(x)\right) + h(x) c(\boldsymbol{\theta}) \sum_{i=1}^k \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x) \exp\left(\sum_{i=1}^k w_i(\boldsymbol{\theta}) t_i(x)\right) \right] dx &= 0 \\ \int_x \sum_{i=1}^k \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x) \left[h(x) c(\boldsymbol{\theta}) \exp\left(\sum_{i=1}^k w_i(\boldsymbol{\theta}) t_i(x)\right) \right] dx &= -\frac{1}{c(\boldsymbol{\theta})} \frac{\partial c(\boldsymbol{\theta})}{\partial \theta_j} \int_x \left[h(x) c(\boldsymbol{\theta}) \exp\left(\sum_{i=1}^k w_i(\boldsymbol{\theta}) t_i(x)\right) \right] dx \end{aligned}$$

$$\int_x \left(\sum_{i=1}^k \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x) \right) f(x) dx = - \frac{\partial \log c(\boldsymbol{\theta})}{\partial \theta_j} \int_x f(x) dx$$

$$\because \int_x f(x) dx = 1$$

$$\therefore E \left(\sum_{i=1}^k \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(x) \right) = - \frac{\partial \log c(\boldsymbol{\theta})}{\partial \theta_j}$$