SMA 2172 Calculus I
Tutorial I: Functions

Topics: functions, domain, codomain, range, properties of functions, composition of functions, and Inverse functions.

1 Domains, Co-domains and Range of Functions

1. Determine the largest domain and range for the following functions:
   (a) $y = x^2 + 1$
   (b) $y = 1 - x^2$
   (c) $y = \sqrt{x^2 - 64}$
   (d) $y = \dfrac{1}{1-x}$
   (e) $y = (x+1)(x+4)$
   (f) $y = \dfrac{4x-7}{x^2-6x+8}$
   (g) $y = \dfrac{x}{6} + x^2$
   (h) $y = \sqrt{15-3x}$
   (i) $y = \dfrac{(x-1)(x+2)}{x-3}$
   (j) $y = \dfrac{x+1}{x^2-4}$
   (k) $y = \sqrt{9-x^2}$
   (l) $x^2 + 4.5x + 2$
2. Let $f(x) = \sqrt{3x+12}$ and $g(x) = \sqrt{12-3x}$. Determine the largest domain and range of the functions: (a). $f(x) + g(x)$ (b). $\dfrac{f(x)}{g(x)}$.
3. Let $f(x) = \sqrt{1-x}$ and $g(x) = \sqrt{x^2-4}$. Determine the largest domain and range of the functions: (a) $f(x) + g(x)$ (b). $f(x) \cdot g(x)$. (c). $\dfrac{f(x)}{g(x)}$.
4. Let $f(x) = \sqrt{2+x} \cdot \sqrt{9-x^2}$ and $g(x) = \sqrt{(2+x)(9-x^2)}$. Show that, as functions
$$\sqrt{2+x} \cdot \sqrt{9-x^2} \neq \sqrt{(2+x)(9-x^2)}.$$
5. For the following functions, find the maximal domains of:
$(f+g)(x), (f-g)(x), (fg) \cdot (x), \left(\dfrac{f}{g}\right)(x)$.
   (a) $f(x) = \dfrac{1}{x+1}$; $\quad g(x) = \dfrac{1}{x+2}$.
   (b) $f(x) = \dfrac{1}{\sqrt{x+1}}$; $\quad g(x) = \sqrt{12-3x}$.
   (c) $f(x) = x^2 - 1$; $\quad g(x) = \dfrac{1}{x+1}$.

(d) $f(x) = \sqrt{3x+12}$; $\quad g(x) = \sqrt{3x^2+12x}$.

6. Find the domain of $f(x) = \sqrt{\dfrac{x}{x+1}}$

7. Find the domain and range of the function $f(x) = 2\sqrt{4-x^2} - 3$.

8. Let $f(x) = \dfrac{\sqrt{x^2-4}}{5-\sqrt{36-x^2}}$. Then, in interval notation, that part of the domain of f which is to the right of the origin is $[2, a) \cup (a, b]$. Find the values of a and b.

9. Let $f(x) = (-x^2 - 7x - 10)^{-1/2}$
   Find

   (a) $f(-3)$

   (b) the domain of f.

10. Find the domain of the function $f(x) = \sqrt{1+\sin x}$.

11. Find the range of the function $f(x) = 4\sin(2x) + 1$.

12. Find the domain of $f(x) = \sqrt{\dfrac{x}{x+1}}$

13. Find the domain and range of the function $f(x) = 2\sqrt{4-x^2} - 3$.

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2 Composition of Functions

1. For $f(x) = 2x+3$ and $g(x) = -x^2+1$, find the composite function defined by $(f \circ g)(x)$.

2. Given $f(2) = 3$, $g(3) = 2$, $f(3) = 4$ and $g(2) = 5$, evaluate $f(\circ g)(3)$.

3. Functions f and g are given by $f(x) = \sqrt{x+2}$ and $g(x) = \ln(1-x)$. Find the composite function defined by $(g \circ f)(x)$ and describe its domain and range.

4. Find the range of the composite function $f(g(x))$ given that $f(x) = x+4$ and $g(x) = x^2+2$.

5. Find the domain of the composite function $g \circ f$ if $f(x) = \sqrt{x}$ and $g(x) = x^2+2$.

6. Find $(f \circ g)(x)$ and the domain of $f \circ g$, given $f(x) = (x-1)/(x+2)$, $g(x) = (x+1)/(x-2)$.

7. Find the composition $(f \circ g)(x)$ and the domain of $f \circ g)$,given$f(x) = x^2 + 2$, $g(x) = \sqrt{x-2}$.
8. Let $f(x) = x + 5$, $g(x) = \sqrt{x}$, and $h(x) = x^2$. Find $(g \circ (h - (g \circ f)))(4)$.
9. Let $h(x) = \dfrac{1}{\sqrt{x+6}}$, $f(x) = \frac{1}{x}$ and $g(x) = 5 - x^2$. Calculate
   (a) $(h \circ g)(2)$
   (b) $(g \circ f)(2)$
   (c) $(f \circ f)(2)$
   (d) $(g \circ f \circ h)(5)$
10. Find the composition $(f \circ g)(x)$ and its domain given f and g below.
   (a) $f(x) = 2x^3 + x - 1$, $g(x) = x^2$
   (b) $f(x) = |x^2 - 4|$, $g(x) = x - 1$
   (c) $f(x) = x^2 - 5$, $g(x) = \sqrt{x+5}$
   (d) $f(x) = \ln x$, $g(x) = \sqrt{x+5}$
   (e) $f(x) = \sin x$, $g(x) = x - 2$.

3 Injective, Surjective and Bijective functions

1. If $f : R \to R$ is given by $f(x) = 3x + 7$, prove it is one-to-one.
2. Why is $f : R \to R$ given by $f(x) = x^2$ not $1 - 1$?
3. Let $f : R^+ \to R$ be defined by $f(x) = x^2 - 4x + 5$. What is the largest codomain so that f is subjective (onto)?
   Answer: The codomain is the set B say where $B = \{x | x \in \mathbb{R} \; and \, x \geq 1\}$.
4. Which of the following functions are injective, surjective, and bijective on their respective domains. Take the domains of the functions as those values of x for which the function is well defined
   (a) $f(x) = x^3 - 2x + 1$
   (b) $f(x) = \dfrac{x+1}{x-1}$
   (c) $f(x) = \begin{cases} x^2 & x \leq 0 \\ x+1 & x > 0 \end{cases}$
   (d) $f(x) = \dfrac{1}{x^2 - 1}$
5. For the function $f : (1, \infty) \to (0, 1)$ defined ny
$$f(x) = \frac{x-1}{x+1}, \quad x > 1$$
   (a) Draw the graph of the function
   (b) Prove that the function is $1 - 1$
   (c) Find the inverse of the function
   (d) Find the domain and range of the inverse function
   (e) Draw the graph of the inverse function

4 Inverse functions

1. If $f(x) = x^5 - 1$, find the inverse function.

2. Let $f(x) = 2x + \cos x + \sin^2 x$ for $-10 \leq x \leq 10$. Show t hat f has an inverse.

3. Let $f(x) = \dfrac{4x+3}{x+2}$.

   (a) Show that f is one-to-one

   (b) Find $f^{-1}(-2)$

   (c) Find dom f^{-1}

4. Find the inverse of each of the following functions

   (a) $f(x) = (x-2)^5 + 3$

   (b) $g(x) = \dfrac{-2x-2}{x+2}$

   (c) $g(x) = \sqrt[5]{\dfrac{-x+2}{2}}$

   (d) $f(x) = \dfrac{7-3x}{x-2}$

   (e) $f(x) = \dfrac{3}{4x-4}\ x > 1$

   (f) $f(x) = \dfrac{2x-7}{3x+5}$

   (g) $f(x) = \dfrac{4x-3}{2x+1}$

   (h) $f(x) = \dfrac{5}{6}x - \dfrac{3}{4}$

   (i) $f(x) = \dfrac{2x-5}{3}$

   (j) $f(x) = \dfrac{x+1}{x-3}$

   (k) $f(x) = \dfrac{2x+5}{3x-2}$