Derivatives of Trigonometric Functions

The basic trigonometric limit:

Theorem: $$\lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin x}$$ (x in radians)

> *Note*: In calculus, unless otherwise noted, all angles are measured in radians, and not in degrees.

This theorem is sometimes referred to as the *small-angle approximation* because it really says that, for very small angles x, $\sin x \approx x$.

> *Note*: Cosine behaves even better near 0, where $\lim_{x \to 0} \cos x = 1$.

ex. Show that $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim_{x \to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x \to 0} \frac{-\sin^2 x}{x(\cos x + 1)}$$

$$= \lim_{x \to 0} \frac{-\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{\cos x + 1} = -\lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{\cos x + 1} = -(1)\left(\frac{0}{1+1}\right) = 0$$

ex. Evaluate $\lim_{x\to 0} \frac{\sin 2x}{5x}$

$$\lim_{x\to 0} \frac{\sin 2x}{5x} = \frac{1}{5}\lim_{x\to 0} \frac{\sin 2x}{x} \cdot \frac{2}{2} = \frac{2}{5}\lim_{x\to 0} \frac{\sin 2x}{2x}$$

The idea above is to match the angle in the sine function with the denominator. We'll then apply the basic trigonometric limit. To do so, first we substitute $\theta = 2x$. Note that as x approaches 0, so does θ. Hence,

$$\frac{2}{5}\lim_{x\to 0} \frac{\sin 2x}{2x} = \frac{2}{5}\lim_{\theta\to 0} \frac{\sin \theta}{\theta} = \frac{2}{5}\cdot 1 = \frac{2}{5}$$

ex. Evaluate $\lim_{x\to 0} \frac{\sin 4x}{\sin 3x}$

$$\lim_{x\to 0} \frac{\sin 4x}{\sin 3x} = \lim_{x\to 0} \frac{\sin 4x}{\sin 3x}\cdot\frac{x}{x} = \lim_{x\to 0} \frac{\sin 4x}{x}\cdot\frac{x}{\sin 3x} = \lim_{x\to 0} \frac{\sin 4x}{x}\cdot\lim_{x\to 0} \frac{x}{\sin 3x}$$

Repeat the same trick as in the previous example, let $\theta = 4x$ and $\lambda = 3x$. Both θ and λ approach 0 when x does. Then apply the theorem twice.

$$= \lim_{x\to 0} \frac{\sin 4x}{x}\cdot\frac{4}{4}\cdot\lim_{x\to 0} \frac{x}{\sin 3x}\cdot\frac{3}{3} = \frac{4}{3}\lim_{x\to 0} \frac{\sin 4x}{4x}\cdot\lim_{x\to 0} \frac{3x}{\sin 3x}$$

$$= \frac{4}{3}\lim_{\theta\to 0} \frac{\sin \theta}{\theta}\cdot\lim_{\lambda\to 0} \frac{\lambda}{\sin \lambda} = \frac{4}{3}\cdot 1\cdot 1 = \frac{4}{3}$$

In fact, after doing a few examples like those, we can see a (very nice) pattern. To sum it up:

Suppose m and n are nonzero real numbers, then

$$\lim_{x\to 0} \frac{\sin mx}{nx} = \frac{m}{n}$$

$$\lim_{x\to 0} \frac{mx}{\sin nx} = \frac{m}{n}$$

$$\lim_{x\to 0} \frac{\sin mx}{\sin nx} = \frac{m}{n}$$

(Trivially, we also have:

$$\lim_{x\to 0} \frac{mx}{nx} = \frac{m}{n}.)$$

ex. Evaluate $\lim_{x\to 0} \frac{\tan 7x}{2x}$

$$\lim_{x\to 0} \frac{\tan 7x}{2x} = \lim_{x\to 0} \frac{1}{2x} \cdot \frac{\sin 7x}{\cos 7x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin 7x}{x} \cdot \frac{1}{\cos x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin 7x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x}$$

$$= \frac{1}{2} \cdot \frac{7}{1} \cdot \frac{1}{1} = \frac{7}{2}$$

Recall that since $\cos x$ is continuous everywhere, the *direct substitution property* applies, therefore,

$$\lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{\lim_{x\to 0} \cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

Now, the main topic --

<u>Derivatives of Trigonometric Functions</u>

ex. What is the derivative of $\sin x$?

Start with the limit definition of derivative:

$$\frac{d}{dx} \sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{[\sin x \cos h + \sin h \cos x] - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h\to 0} \frac{\sin h \cos x}{h} = \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\sin h}{h} \cdot \cos x$$

$$= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \cos x = \sin x \cdot (0) + (1)\cos x = \cos x$$

Therefore, $\frac{d}{dx} \sin x = \cos x$

ex. Find the derivative of $\csc x$.

$$\frac{d}{dx}\csc x = \frac{d}{dx}\frac{1}{\sin x} = \frac{\sin x\left(\frac{d}{dx}1\right) - 1\left(\frac{d}{dx}\sin x\right)}{(\sin x)^2} = \frac{\sin x \cdot 0 - 1 \cdot \cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} = -\csc x \cot x$$

Therefore, $\frac{d}{dx}\csc x = -\csc x \cot x$

The complete list of derivatives of trigonometric functions:

1. $\frac{d}{dx}\sin x = \cos x$

2. $\frac{d}{dx}\cos x = -\sin x$

3. $\frac{d}{dx}\tan x = \sec^2 x$

4. $\frac{d}{dx}\sec x = \sec x \tan x$

5. $\frac{d}{dx}\cot x = -\csc^2 x$

6. $\frac{d}{dx}\csc x = -\csc x \cot x$

ex. Differentiate $f(x) = \sec x + 5 \csc x$

$f'(x) = \sec x \tan x + 5(-\csc x \cot x) = \sec x \tan x - 5 \csc x \cot x$

ex. Differentiate $f(x) = x^2 \cos x - 2x \sin x - 3 \cos x$

$f'(x) = [x^2(-\sin x) + (2x) \cos x] - 2[x (\cos x) + (1)\sin x] - 3(-\sin x)$

$= -x^2 \sin x + 2x \cos x - 2x \cos x - 2\sin x + 3\sin x$

$= -x^2 \sin x + \sin x$

ex. Differentiate $s(t) = \dfrac{\sin t}{1 - \cos t}$

$$s'(t) = \frac{(1-\cos t)(\cos t) - (\sin t)(0 - (-\sin t))}{(1-\cos t)^2}$$

$$= \frac{\cos t - \cos^2 t - \sin^2 t}{(1-\cos t)^2} = \frac{\cos t - (\cos^2 t + \sin^2 t)}{(1-\cos t)^2}$$

$$= \frac{\cos t - 1}{(1-\cos t)^2} = \frac{-(1-\cos t)}{(1-\cos t)^2} = \frac{-1}{1-\cos t} = \frac{1}{\cos t - 1}$$

ex. **Simple Harmonic Motion** Suppose the oscillating motion (in meters) of a weight attached to a spring is described by the displacement function

$$s(t) = 2\cos t + \sin t$$

Find its velocity and acceleration functions, and its speed and acceleration at $t = \pi/2$ sec.

Velocity: $v(t) = s'(t) = -2\sin t + \cos t$
Acceleration: $a(t) = v'(t) = -2\cos t - \sin t$

Its speed when $t = \pi/2$ is

$$|v(\pi/2)| = |-2\sin(\pi/2) + \cos(\pi/2)| = |-2 + 0| = 2 \quad \text{(m/sec)}$$

Its acceleration at the same time is

$$a(\pi/2) = -2\cos(\pi/2) - \sin(\pi/2) = 0 - 1 = -1 \quad (\text{m/sec}^2)$$