THE DERIVATIVE OF FUNCTIONS

Definition: The d.erivative of the function f is the function f' defined by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(h)}{h}$$ for all x for which this limit exists.

The function f is differentiable at $x=a$ if $\lim_{x\to a} f(x) = f(a)$ exists.

The process of finding the derivative f' is called differentiation of f.

Solution:

Example 48: Apply the definition of the derivative directly to differentiate the function $f(x)=\frac{x}{x+3}$.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\frac{x+h}{x+h+3}-\frac{x}{x+3}}{h} \\
&= \lim_{h\to 0}\frac{(x+h)(x+3)-x(x+h+3)}{h(x+h+3)(x+3)} \\
&= \lim_{h\to 0}\frac{x^2+3x+hx+3h-x^2-hx-3x}{h(x+h+3)(x+3)} \\
&= \lim_{h\to 0}\frac{3h}{h(x+h+3)(x+3)} \\
&= \lim_{h\to 0}\frac{3}{(x+h+3)(x+3)} \\
&= \frac{3}{(x+3)(x+3)} = \frac{3}{(x+3)^2}
\end{aligned}$$

This process is known as differentiation from first principles.

Differentiation of Quadratic Functions

Example 49: Let $f(x)=ax^2+bx+c$,where a,b and c are constants. Show from first principles that

$$f'(x) = 2ax+b$$

Solution:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left[a(x+h)^2+b(x+h)+c\right]-\left[ax^2+bx+c\right]}{h}$$

$$=\lim_{h\to 0}\frac{\left(ax^2+2ahx+ah^2+bx+bh+c-ax^2-bx-c\right)}{h}$$

$$=\lim_{h\to 0}\frac{2ahx+ah^2+bh}{h}$$

$$=\lim_{h\to 0}\left(2ax+ah+b\right)$$

$$=2ax+b$$

Example 50: Show from first principles that If $f(x)=3x^2-7x+7$, then $f'(x)=6x-7$

Differential Notation

$$\Delta x=h\ ;\Delta y=f(x+\Delta x)-f(x);\ \frac{\Delta y}{\Delta x}=\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

If $y=f(x)$, we often write $\frac{dy}{dx}=f'(x)$ e.g. If $y=ax^2+bx+c$,then $\frac{dy}{dx}=f'(x)=2ax+b$

Examples: Find the derivatives of the following functions from first principles.

Example 51: $f(x)=x^2$

Solution:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{2hx+h^2}{h}$$

$$=\lim_{h\to 0}2x+h\ =2x$$

Example 52: $f(x)=\frac{1}{x}$

Solution: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\lim_{h\to 0}\frac{x-x-h}{h(x+h)x}$$

$$=\lim_{h\to 0}\frac{-1}{(x+h)x}=\frac{-1}{x^2}$$

Example 53: $f(x)=\sqrt{x}$

Solution: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

Exercise: Differentiate the following functions from first principles.

1) $y=x^3$ 2) $f(x)=x^2+3x-2$ 3) $f(t)=kt$

Basic Differentiation Rules

The derivative of a constant

If $f(x)=c$ (a constant) for all x, then $f'(x)=0$ for all x. That is $\frac{dc}{dx}=f'(x)=0$

Proof: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

The Power Rule

If n is a positive integer and $f(x)=x^n$, then $f'(x)=nx^{n-1}$

Proof: $f'(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

But $(x+h)^n=x^n+nx^{n-1}h+\frac{n(n-1)}{2!}x^{n-2}h^2+\cdots+h^n$

$$\therefore f'(x)=\lim_{h\to 0}\frac{x^n+nx^{n-1}h+\frac{n(n-1)}{2!}x^{n-2}h^2+\ldots+h^n-x^n}{h}$$

$$=\lim_{h\to 0}nx^{n-1}+\frac{n(n-1)}{21}x^{n-2}h+\ldots+h^{n-1} \quad =nx^{n-1}$$

Example 54: Find (a) $f'(x)$ if $f(x)=6x^5$ (b) Find $\frac{dy}{dt}$ if $y=t^{17}$

Solution:

(a) $f(x)=6x^5 \Rightarrow f'(x)=30x^4$ (b) $y=t^{17} \Rightarrow \frac{dy}{dt}=17t^{16}$

The derivative of a linear combination

If f and g are differentiable functions and a and b are fixed real numbers, then

$$\frac{d}{dx}\left[af(x)+bg(x)\right]=af'(x)+bg'(x)$$

Proof: Let $k(x)=af(x)+bg(x)$

$$\therefore k'(x)=\lim_{h\to 0}\frac{k(x+h)-k(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left[af(x+h)+bg(x+h)\right]-\left[af(x)+bg(x)\right]}{h}$$

$$=a\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]+b\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$=af'(x)+bg'(x)$$

Example 55: Let $y=36+24x+8x^5-6x^{10}$. Find $\frac{dy}{dx}$.

Solution: $y=36+24x+8x^5-6x^{10} \Rightarrow \frac{dy}{dx}=k'(x)=24+40x^4-60x^5$

The derivative of a Polynomial

Let $y=f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_2$

$$f'(x)=\frac{dy}{dx}=na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\ldots+3a_3x^2+2a_2x^2+a_1$$

if $y=f(x)=7x^3-6x^2+4x+2$ then $\frac{dy}{dx}=f'(x)=21x^2-12x+4$

5. The Product Rule and Quotient Rule

(a)The Product Rule

If f and g are differentiable at x ,then fg is differentiated at x ,then

$$\frac{d}{dx}\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x)$$

Proof: Let $k(x)=f(x)g(x)$

$$\therefore k'(x)=\lim_{h\to 0}\frac{k(x+h)-k(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

Add and subtract at $f(x)g(x+h)$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$

$$k'(x)=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h}+\lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h}$$

$$=\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)\left(\lim_{h\to 0}g(h+x)\right)+f(x)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)$$

$$=f'(x)g(x)+f(x)g'(x)$$

The product rule says that the derivative of the product of two functions is formed by multiplying the derivative of each by the other and then adding the results.

Example 56: Find the derivative of $f(x)=(1-6x^3)(4x^2-6x+2)$

Solution

$$\begin{aligned}f'(x)&=(-18x^2)(4x^2-6x+2)+(1-6x^3)(8x-12)\\&=-72x^4+108x^3-36x^2+8x-12-48x^4+36x\\&=120x^4+144x^3-36x^2+8x-12\end{aligned}$$

Now, suppose $k(x)=f_1(x)f_2(x)\ldots f_n(x)$

$$\begin{aligned}k'(x)=&f_1'(x)f_2(x)\ldots f_n(x)\\&+f_1(x)f_2'(x)\ldots f_n(x)\\&\vdots\\&+f_1(x)f_2(x)\ldots f'(x)\end{aligned}$$

Example 57: Let $k(x)=(x-2)(x^2+6)(x^4+1)$. Find $k'(x)$.

Solution:

$$\begin{aligned}\therefore k'(x)&=(1)(x^2+6)(x^4+1)+(x-2)(2x)(x^4+1)+(x-2)(x^2+6)(4x)\\&=x^6+x^2+6x^4+6+2x(x^5+x-2x^4-2)+4x(x^3+6x-2x^2-12)\\&=x^6+x^2+6x^4+6+2x^6+2x^2-4x^5-4x+4x^4+6x^2-8x^3-48x\\&=3x^6-4x^5+10x^4-8x^3+9x^2-52x+6\end{aligned}$$

(b)The Reciprocal Rule

If f is differentiable at x and $f(x)\neq 0$, then $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right)=-\dfrac{f'(x)}{[f(x)]^2}$

Proof: Let $k(x)=\frac{1}{f(x)} \Rightarrow k'(x)=\lim_{h\to 0}\frac{k(x+h)-k(x)}{h}$

$$=\lim_{h\to 0}\frac{\frac{1}{f(x+h)}-\frac{1}{f(x)}}{h}$$

$$=\lim_{h\to 0}\frac{f(x)-f(x+h)}{hf(x+h)f(x)}$$

$$=-\left(\lim_{h\to 0}\frac{1}{f(x+h)f(x)}\right)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)$$

$$=-\frac{f'(x)}{\left[f(x)\right]^2}$$

Example 58: Find $k'(x)$ if $k(x)=\frac{1}{x^2+1}$

Solution: $k'(x)=\frac{-\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}=\frac{-2x}{\left(x^2+1\right)^2}$

(c)Power rule for a negative integer n

If n is a negative integer, then $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$

Proof: Let $m=-n$, so that m is a positive integer. Then

$$\frac{d}{dx}\left(x^n\right)=\frac{d}{dx}\left(x^{-m}\right)=\frac{d}{dx}\left(\frac{1}{x^m}\right)=\frac{\frac{d}{dx}\left(x^m\right)}{\left(x^m\right)^2}\quad=-\frac{mx^{m-1}}{x^{2m}}=(-m)x^{-m-1}=nx^{n-1}$$

Example 59: Find $f'(x)$ if $f(x)=\frac{5x^4-6x+7}{2x^2}$

Solution:

$$f(x)=\frac{5x^4-6x+7}{2x^2}$$

$$=\frac{5}{2}x^2-\frac{3}{x}+\frac{7}{2x^2}$$

: $$=\frac{5}{2}x^2-3x^{-1}+\frac{7}{2}x^{-2}$$

$$\therefore f'(x)=\frac{5}{2}(2x)-3\left(-x^{-2}\right)+\frac{7}{2}\left(-2x^{-3}\right)$$

$$=5x+\frac{3}{x^2}-\frac{7}{x^3}$$

The Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$ then $\frac{f}{g}$ is differentiable x and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f^{'}(x)g(x) - f(x)g^{'}(x)}{\left[g(x)\right]^2}$$

Proof: Let $k(x) = \frac{f(x)}{g(x)}$

$$k^{'}(x) = \lim_{h \to 0} \frac{k(x+h) - k(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - g(x+h)f(x)}{hg(x+h)g(x)}$$

Add and subtract $f(x)g(x)$

$$\lim_{h \to 0} \frac{g(x)f(x+h) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h\big(g(x)g(x+h)\big)}$$

$$= \lim_{h \to 0} \frac{g(x)f(x+h) - f(x)g(x)}{h\big(g(x)g(x+h)\big)} + \lim_{h \to 0} \frac{f(x)g(x) - f(x)g(x+h)}{h\big(g(x)g(x+h)\big)}$$

$$= \lim_{h \to 0} g(x)\frac{\left[f(x+h) - f(x)\right]}{hg(x)g(x+h)} + \lim_{h \to 0} f(x)\frac{\left[g(x) - g(x+h)\right]}{h\big(g(x)g(x+h)\big)}$$

$$= \frac{g(x)\lim_{h \to 0}\frac{f(x+h) - f(x)}{h}}{\lim_{h \to 0} g(x)g(x+h)} - \frac{f(x)\lim_{h \to 0}\frac{\left[g(x+h) - g(x)\right]}{h}}{\lim_{h \to 0} g(x)g(x+h)}$$

$$= \frac{g(x)f^{'}(x)}{(g(x)]^2} - \frac{f(x)g^{'}(x)}{(g(x)]^2}$$

$$= \frac{g(x)f^{'}(x) - f(x)g^{'}(x)}{(g(x)]^2}$$

Slope of a tangent

Let M be a slope of a tangent line at point P. Then $M = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Example 60: .If $f(x) = x^2$, find the slope of tangent line at at the point $P(a, a^2)$.

Solution: $f(a+h) = (a+h)^2$

$$M = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} = \lim_{h \to 0} 2a + h = 2a$$

Example 61: Find the slope and the equation of the tangent line to a graph of $f(x) = x^3$ at the point $P(3, 27)$.

Solution:

$$M = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{(a+h)^3 - a^3}{h} = \lim_{h \to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h}$$

$$= \lim_{h \to 0} 3a^2 + 3ah + h^2$$

$$= 3a^2$$

But *a=3*

Let $a = 3 \times 3^2 = 27$; $M = \dfrac{y - y_0}{x - x_0}$; $27 = \dfrac{y - 27}{x - 3}$; $y - 27 = 27x - 81$; $y = 27x - 54$

In general, consider $y = f(x)$,the slope of the tangent line at any arbitrary point $P(x, y)$on the curve is given by $m = f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$,where $f'(x)$is a derived function of $f(x)$.

$f'(x)$is read as ñf prime of xò

Example 62: Let $f(x) = x^2 + 1$,find $f(x)$.Use this result to find the slope of the tangent line $y = x^2 + 1$at point $x = 2, x = 0$and at $x = -2$.

Solution:

$$f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 1 - x^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} = \lim_{h \to 0} 2x + h = 2x$$

Therefore $f(x) = 2x$

When $x=2$, $f^{'}(2)=4$; When $x=0$, $f^{'}(0)=0$; when $x=-2$, $f^{'}(-2)=-4$

Definition: The function $f^{'}(x)$ defined by the format $f^{'}(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ is called the derivative of $f(x)$ with respect to x.

The derivative can also be defined in various other equivalent ways e.g

$$f^{'}(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

Note:

1 .A function is said to be differentiable at a point $x=x_0$ if it has a derivative at this point, i.e $f(x_0)$ exists. If $f(x)$ is differentiable at $x=x_0$ it must be continous there.

2. If $y=f(x)$, $f'(x)=\frac{d\,f(x)}{dx}=\frac{dy}{dx}$ =derivative of y with respect to x.

   y-dependent variable

   x-independent variable

3. The process of finding a derivative is called differentiation.

4. If you ever required to differentiate a given function from first principles, you should always start the proof by quoting the formula below

$$f^{'}(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Examples

1.Find the derivative of $f(x)=3x^2-5x+4$ from first principles

$$\begin{aligned}
f^{'}(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{3(x+h)^2-5(x+h)+4-3x^2+5x-4}{h}\\
&=\lim_{h\to 0}\frac{3\left[x^2+2xh+h^2\right]-5x-5h+4-3x^2+5x-4}{h}\\
&=\lim_{h\to 0}\frac{3x^2+6xh+3h^2-5x-5h+4-3x^2+5x-4}{h}\\
&=\lim_{h\to 0}6x+3h-5\\
&=6x-5
\end{aligned}$$

Differentiate $f(x)=\frac{x}{x-9}$ from 1^{st} principles

Solution:

$$f^{'}(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+h}{x+h-9} - \frac{x}{x-9}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(x-9) - x(x+h-9)}{h(x-9)(x+h-9)}$$

$$= \lim_{h \to 0} \frac{x^2 - 9x + xh - 9h - x^2 - xh + x9}{h(x-9)(x+h-9)}$$

$$= \lim_{h \to 0} \frac{x-9-x}{(x-9)(x+h-9)} = \frac{-9}{(x-9)^2}$$

Confirm: If $y = \dfrac{x}{x-9}$, $\quad \dfrac{dy}{dx} = \dfrac{-9}{(x-9)^2}$

Differentiate (a) $\sqrt{x+2}$ (b) $f(x) = \sqrt{x-2}$ from 1st principles

Solution:

$$y = \sqrt{x+2} = f(x)$$

$$f(x^{'}) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)+2-(x+2)}{h[\sqrt{(x+h)+2} + \sqrt{x+2}]}$$

$$= \lim_{h \to 0} \frac{h}{h[\sqrt{(x+h)+2} + \sqrt{x+2}]}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{(x+h)+2} + \sqrt{x+2}}$$

$$\frac{1}{\sqrt{x+2} + \sqrt{x+2}}$$

$$= \frac{1}{2\sqrt{x+2}}$$

(b) $f(x) = \sqrt{x-2}$

Further examples

Differentiate the following with respect to x

$y = x^5$

Solution:

$$\frac{d\left(x^5\right)}{dx} = 5x^4 \quad \Rightarrow \frac{dy}{dx} = 5x^4$$

y = x^3

Solution:

$$\frac{d}{dx}\left(5x^3\right) = 5\frac{d\left(x^3\right)}{d} = 5 \times 3x^2 = 15x^2$$

3. $y = -7x^{10}$

Solution:

$$\frac{dy}{dx} = -70x^9$$

3. $y = f\left(x\right) = 24x^2$

Solution:

$$\frac{dy}{dx} = 48x\,.$$

4.

$$y = f\left(x\right) = 8x^3 - 4x^2 + x - 5$$

Solution

$$f^{'}\left(x\right) = \frac{dy}{dx} = \frac{d}{dx}\left(8x^3 - 4x^2 + x - 5\right)$$

$$= \frac{d}{dx}\left(8x^3\right) - \frac{d}{dx}\left(4x^2\right) + \frac{d}{dx}\left(x\right) - \frac{d}{dx}(5)$$

$$= 24x^2 - 8x + 1 - 0$$

$$= 24x^2 - 8x + 1$$

5.

$$f\left(x\right) = y = \left(x+3\right)^4$$

Solution

$$1x^4 + 4(3)x^3 + 6(3)^2x^2 + 4(3)^3x + 1(3)^4$$

$$= x^4 + 12x^3 + 54x^2 + 108x + 81$$

$$\frac{dy}{dx} = 4x^3 + 36x^2 + 108x + 108$$

6. $y = \dfrac{2}{x^3}$

Solution:

$$\frac{dy}{dx} = 2.-3(x^{-3-1}) = -6x^{-3-1} = -6x^{-4} = \frac{-6}{x^4}$$

7. y = $\frac{1}{\sqrt{x}}$

Solution:

$$y = x^{-\frac{1}{2}};\ \frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2}\times\frac{1}{x^{\frac{3}{2}}} = -\frac{1}{2}\frac{1}{\sqrt{x^3}}$$

8. $y = x^{\frac{1}{2}}$

Solution: $\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

9. $(a)\ y = \frac{-2}{x^2} \quad (b)\frac{1}{3x^3}$

The Chain Rule

If $y = f(u)$ where $u = g(x)$ and $g(x)$ are differentiable functions, then the composite function defined by $y = f[g(x)]$ which has a derivative given by $\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$

Examples

1. Differentiate $(3x+2)^4$

Solution:

Let $y = (3x+2)^4$ and $u = 3x+2$; then $y = u^4$; $\frac{du}{dx} = 3$; $\frac{dy}{du} = 4u^3$; But by chain rule $\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$

$= 4u^3 \times 3 = 12u^3$; But $u = 3x+2$; $\therefore \frac{dy}{dx} = 12(3x+2)^3$

2. Differentiate $(x^2+3x)^7$

Solution:

Let $y = (x^2+3x)^7$; Let $u = x^2+3x$; $\therefore y = u^7$;

$$\frac{du}{dx} = 2x+3,\quad \frac{dy}{du} = 7u^6 \quad \therefore \frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = 7u^6(2x+3)$$

$$= 7(2x+3)(x^2+3x)^6$$

$$\therefore \frac{dy}{dx} = 7(2x+3)(x^2+3x)^6$$

3.Diffrentiate $\frac{1}{1+\sqrt{x}}$

Solution:

Let $y = (1+\sqrt{x})^{-1}$ and $u = 1+\sqrt{x}$ or $1+x^{\frac{1}{2}}$

$\therefore y = u^{-1}; \quad \frac{dy}{du} = -1u^{-1-1} = -u^{-2}; \ \frac{du}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{u^2} \times \frac{1}{2\sqrt{x}} = -\frac{1}{2u^2\sqrt{x}} = -\frac{1}{2\left(1+\sqrt{x}\right)^2\sqrt{x}}$

4.Diffrentiate $\sqrt{1+x^2}$

Solution:

Let $y = \left(1+x^2\right)^{\frac{1}{2}}$ and $u = 1+x^2$;

$\therefore y = u^{\frac{1}{2}}; \quad \frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$

$\frac{du}{dx} = 2x; \quad \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times 2x = \frac{x}{\sqrt{1+x^2}}$

5. $y = \frac{1}{1+\sqrt{x}} = \left(1+\sqrt{x}\right)^{-1}$

Solution:

$u = 1+\sqrt{x} = 1+x^{\frac{1}{2}}; \ \frac{du}{dx} = \frac{1}{2}x^{\frac{1}{2}}$

$y = u^{-1}; \ \frac{dy}{du} = -1u^{-2} = \frac{-1}{u^2}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{1}{u^2} \times \frac{1}{2}x^{-\frac{1}{2}} = -\frac{1}{\left(1+\sqrt{x}\right)^2} \times \frac{1}{2}x^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2\left(1+\sqrt{x}\right)^2\sqrt{x}}$

Exercise

Differentiate

1. $\left(3x^2+5\right)^3$ 2. $\left(3x^2-5x\right)^{-\frac{2}{3}}$ 3. $\left(6x^3-4x\right)^{-2}$ 4. $\frac{1}{\left(x^2-7x\right)^3}$ 5. $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

Mixed Examples

Differentiate the expression $y = \left(x^2-3\right)\left(x+1\right)^2$ and simplify the result.

Solution: Let $u = \left(x^2-3\right)$ and let $v = \left(x+1\right)^2$; $\frac{du}{dx} = 2x$; $\frac{dv}{dx} = 2\left(x+1\right)1 = 2\left(x+1\right)$

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}=\left(x^2-3\right)(2)(x+1)+(x+1)^2 2x$$
$$=2(x+1)\left(x^2-3\right)+2x(x+1)^2$$
$$=2(x+1)\left[\left(x^2-3\right)+x(x+1)\right]$$
$$=2(x+1)\left[x^2-3+x^2+x\right]$$
$$2(x+1)(2x+3)(x-1)=2(x+1)\left[2x^2+x-3\right]$$

2.Differentiate $\left(x^2+1\right)^3\left(x^3+1\right)^2$

Let $u=\left(x^2+1\right)^3$ and $v=\left(x^3+1\right)^2$

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$
$$\frac{dy}{dx}=2\left(3x^2\right)(x+1)=6x^2\left(x^3+1\right)$$
$$\frac{du}{dx}=3(2x)\left(x^2+1\right)^2$$
$$=\left(x^2+1\right)^3 6x^2\left(x^3+1\right)+\left(x^3+1\right)^2 3(2x)\left(x^2+1\right)^2$$
$$=6x^2\left(x^2+1\right)^3\left(x^3+1\right)+6x\left(x^3+1\right)^2\left(x^2+1\right)^2$$
$$=6x\left(x^3+1\right)\left(x^2+1\right)^2\left[x\left(x^2+1\right)+\left(x^3+1\right)\right]$$
$$=6x\left(x^3+1\right)\left(x^2+1\right)^2\left[2x^3+x+1\right]$$

3. Differentiate $(x-3)^2(x+2)^{-2}$

Solution:

$$y=(x-3)^2(x+2)^{-2}$$
$$u=(x-3)^2;\quad v=(x+2)^{-2}$$

$$\frac{du}{dx}=2(x-3);\quad \frac{dv}{dx}=-2(x+2)^{-3}$$

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}=(x-3)^2(-2)(x+2)^{-3}+(x+2)^{-2}(2)(x-3)$$

$$=-2(x-3)^2(x+2)^{-3}+2(x-3)(x+2)^{-2}$$
$$=\frac{-2(x-3)^2}{(x+2)^3}+\frac{2(x-3)}{(x+2)^2}\quad=\frac{-2(x-3)^2+2(x-3)(x+2)}{(x+2)^3}$$

$$\frac{2(x-3)\left[-(x-3)+x+2\right]}{(x+2)^3}=\frac{2(x-3)(5)}{(x+2)^3}=\frac{10(x-3)}{(x+2)^3}$$

4. Differentiate $y=\dfrac{x+1}{x^2-2}$

Solution:

Let $u = x+1; \quad v = x^2 - 2$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{du}{dx} = 1; \quad \frac{dv}{dx} = 2x \quad \frac{dy}{dx} = \frac{\left(x^2-2\right)1 - \left(x+1\right)\left(2x\right)}{\left(x^2-2\right)^2}$$

$$= \frac{\left(x^2-2\right) - \left(x+1\right)2x}{\left(x^2-2\right)^2} = \frac{x^2 - 2 - 2x^2 - 2x}{\left(x^2-2\right)^2} = \frac{-x^2 - 2x - 2}{(x^2-2)^2}$$

5. Differentiate $\dfrac{x}{\sqrt{\left(1+x^2\right)}}$

Solution: Let $u = x; \quad v = \left(1+x^2\right)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{\left(1+x^2\right)^{\frac{1}{2}} - x\left(\frac{1}{2}\right)\left(2x\right)\left(1+x^2\right)^{-\frac{1}{2}}}{\left(\sqrt{\left(1+x^2\right)}\right)^2}$$

$$\frac{\left(1+x^2\right)^{\frac{1}{2}} - x^2\left(1+x^2\right)^{-\frac{1}{2}}}{1+x^2} = \frac{\left(1+x^2\right)^{\frac{1}{2}} - \dfrac{x^2}{\left(1+x^2\right)^{\frac{1}{2}}}}{1+x^2}$$

$$\frac{\left(1+x^2\right) - x^2}{\left(1+x^2\right)\left(1+x^2\right)^{\frac{1}{2}}} = \frac{1}{\left(1+x^2\right)\left(1+x^2\right)^{\frac{1}{2}}} = \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}$$

6. $y = \sqrt{\dfrac{x^2-4}{x^2+4}}$

Solution:

$$y = \frac{\left(x^2-4\right)^{\frac{1}{2}}}{\left(x^2+4\right)^{\frac{1}{2}}}; u = \left(x^2-4\right)^{\frac{1}{2}}; v = \left(x^2+4\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{8x}{\left(x^2-4\right)^{\frac{1}{2}}\left(x^2+4\right)^{\frac{3}{2}}}$$

7. $y = \sqrt{\left(\dfrac{1+x}{2+x}\right)}$

Solution:

Let $u = \frac{1+x}{2+x}$; $\frac{du}{dx} = \frac{(2+x)-(1+x)}{(2+x)^2} = \frac{1}{(2+x)^2}$;

$y = u^{\frac{1}{2}}$; $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{(1+x)(2+x)^{-1}}} \times \frac{1}{(2+x)^2} = \frac{1}{2(1+x)^{\frac{1}{2}}(2+x)^{-\frac{1}{2}}(2+x)^2} = \frac{1}{2(x+1)^{\frac{1}{2}}(2+x)^{\frac{3}{2}}}$$

$$= \frac{1}{2\sqrt{(x+1)(2+x)^3}}$$

Exercise

1. $\frac{1-x^2}{1+x^2}$ 3. $\frac{x^2}{\sqrt{1+x^2}}$ 3. $\sqrt{\frac{(x+1)^3}{x+2}}$ 4. $\frac{\sqrt{x}}{\sqrt{1+x}}$ 5. $\frac{2x^2-x^3}{\sqrt{x^2-1}}$ 6. $y = \frac{(3x-x^4)}{(x^2+1)}$