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DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

Trigonometry is the branch of Mathematics that has made itself indispensable for other branches of higher Mathematics may it be calculus, vectors, three dimensional geometry, functions-harmonic and simple and otherwise just cannot be processed without encountering trigonometric functions. Further within the specific limit, trigonometric functions give us the inverses as well.

The question now arises : Are all the rules of finding the derivatives studied by us so far appliacable to trigonometric functions ?

This is what we propose to explore in this lesson and in the process, develop the formulae or results for finding the derivatives of trigonometric functions and their inverses . In all discussions involving the trignometric functions and their inverses, radian measure is used, unless otherwise specifically mentioned.

OBJECTIVES

After studying this lesson, you will be able to :

- find the derivative of trigonometric functions from first principle;
- find the derivative of inverse trigonometric functions from first principle;
- apply product, quotient and chain rule in finding derivatives of trigonometric and inverse trigonometric functions; and
- find second order derivative of a function.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of trigonometric ratios as functions of angles.
- Standard limits of trigonometric functions namely.

  (i) $\lim\limits_{x\to 0}\sin x = 0$ (ii) $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ (iii) $\lim\limits_{x\to 0}\cos x = 1$ (iv) $\lim\limits_{x\to 0}\frac{\tan x}{x} = 1$

- Definition of derivative, and rules of finding derivatives of function.

22.1 DERIVATIVE OF TRIGONOMETRIC FUNCTIONS FROM FIRST PRINCIPLE

(i) Let y = sin x

For a small increment δx in x, let the corresponding increment in y be δy.

$\therefore \quad y+\delta y=\sin(x+\delta x)$

and $\quad \delta y=\sin(x+\delta x)-\sin x$

$$=2\cos\left[x+\frac{\delta x}{2}\right]\sin\frac{\delta x}{2} \qquad \left[\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$\therefore \quad \frac{\delta y}{\delta x}=2\cos\left(x+\frac{\delta x}{2}\right)\frac{\sin\frac{\delta x}{2}}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\cos\left(x+\frac{\delta x}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}=\cos x.1 \qquad \left[\because \lim_{\delta x\to 0}\frac{\frac{\sin\delta x}{2}}{\frac{\delta x}{2}}=1\right]$$

Thus, $\quad \dfrac{dy}{dx}=\cos x$

i.e., $\quad \dfrac{d}{dx}(\sin x)=\cos x$

(ii) Let $\quad y = \cos x$

For a small increment δx in x, let the corresponding increment in y be δy.

$\therefore \quad y+\delta y=\cos(x+\delta x)$

and $\quad \delta y=\cos(x+\delta x)-\cos x$

$$=-2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

$$\therefore \quad \frac{\delta y}{\delta x}=-2\sin\left(x+\frac{\delta x}{2}\right)\cdot\frac{\sin\frac{\delta x}{2}}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-\lim_{\delta x\to 0}\sin\left(x+\frac{dx}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$=-\sin x\cdot 1$$

Thus, $\quad \dfrac{dy}{dx}=-\sin x$

i.e., $\frac{d}{dx}(\cos x) = -\sin x$

(iii) Let $y = \tan x$

For a small increament δx in x, let the corresponding increament in y be δy .

$\therefore$ $y + \delta y = \tan(x + \delta x)$

and $\delta y = \tan(x + \delta x) - \tan x$

$$= \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+\delta x)\cdot\cos x - \sin x.\cos(x+\delta x)}{\cos(x+\delta x)\cos x}$$

$$= \frac{\sin[(x+\delta x) - x]}{\cos(x+\delta x)\cos x}$$

$$= \frac{\sin\delta x}{\cos(x+\delta x)\cdot\cos x}$$

$\therefore$ $$\frac{\delta y}{\delta x} = \frac{\sin\delta x}{\delta x}\cdot\frac{1}{\cos(x+\delta x)\cos x}$$

or $$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}\cdot\lim_{\delta x\to 0}\frac{1}{\cos(x+\delta x)\cos x}$$

$$= 1\cdot\frac{1}{\cos^2 x} \qquad \left[\because \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x} = 1\right]$$

$$= \sec^2 x$$

Thus, $\frac{dy}{dx} = \sec^2 x$

i.e. $\frac{d}{dx}(\tan x) = \sec^2 x$

(iv) Let $y = \sec x$

For a small increament δx in x, let the corresponding increament in y be δy .

$\therefore$ $y + \delta y = \sec(x+\delta x)$

and $\delta y = \sec(x+\delta x) - \sec x$

$$= \frac{1}{\cos(x+\delta x)} - \frac{1}{\cos x}$$

$$= \frac{\cos x - \cos(x+\delta x)}{\cos(x+\delta x)\cos x}$$

$$= \frac{2\sin\left[x + \frac{\delta x}{2}\right]\sin\frac{\delta x}{2}}{\cos(x+\delta x)\cos x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{\sin\left(x+\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}\cdot\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{\sin\left(x+\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x}\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$=\frac{\sin x}{\cos^2 x}\cdot 1$$

$$=\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}=\tan x.\sec x$$

Thus, $$\frac{dy}{dx}=\sec x.\tan x$$

i.e. $$\frac{d}{dx}(\sec x)=\sec x\cdot\tan x$$

Similarly, we can show that

$$\frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$$

and $$\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x\cdot\cot x$$

Example 22.1 Find the derivative of $\cot x^2$ from first principle.

Solution : $y=\cot x^2$

For a small increament δx in x, let the corresponding increament in y be δy .

$\therefore$ $$y+\delta y=\cot(x+\delta x)^2$$

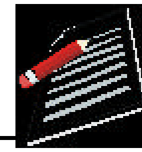
and $$\delta y=\cot(x+\delta x)^2-\cot x^2$$

$$=\frac{\cos(x+\delta x)^2}{\sin(x+\delta x)^2}-\frac{\cos x^2}{\sin x^2}$$

$$=\frac{\cos(x+\delta x)^2\sin x^2-\cos x^2\sin(x+\delta x)^2}{\sin(x+\delta x)^2\sin x^2}$$

$$=\frac{\sin[x^2-(x+\delta x)^2]}{\sin(x+\delta x)^2\sin x^2}$$

$$=\frac{\sin[-2x\delta x-(\delta x)^2]}{\sin(x+\delta x)^2\sin x^2}$$



$$= \frac{-\sin[(2x+\delta x)\delta x]}{\sin(x+\delta x)^2 \sin x^2}$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{-\sin[(2x+\delta x)\delta x]}{\delta x \sin(x+\delta x)^2 \sin x^2}$$

and $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = -\lim_{\delta x \to 0} \frac{\sin[(2x+\delta x)\delta x]}{\delta x(2x+\delta x)} \lim_{\delta x \to 0} \frac{2x+\delta x}{\sin(x+\delta x)^2 \sin x^2}$$

or $$\frac{dy}{dx} = -1 \cdot \frac{2x}{\sin x^2 . \sin x^2} \qquad \left[\because \lim_{\delta x \to 0} \frac{\sin[(2x+\delta x)\delta x]}{\delta x(2x+\delta x)} = 1\right]$$

$$= \frac{-2x}{(\sin x^2)^2} = \frac{-2x}{\sin^2 x^2}$$

$$= -2x . \operatorname{cosec}^2 x^2$$

Hence $$\frac{d}{dx}(\cot x^2) = -2x \cdot \operatorname{cosec}^2 x^2$$

Example 22.2 Find the derivative of $\sqrt{\operatorname{cosec} x}$ from first principle.

Solution : Let $y = \sqrt{\operatorname{cosec} x}$

and $$y + \delta y = \sqrt{\operatorname{cosec}(x+\delta x)}$$

$$\therefore \quad \delta y = \frac{\left[\sqrt{\operatorname{cosec}(x+\delta x)} - \sqrt{\operatorname{cosec} x}\right]\left[\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}\right]}{\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}}$$

$$= \frac{\operatorname{cosec}(x+\delta x) - \operatorname{cosec} x}{\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}}$$

$$= \frac{\dfrac{1}{\sin(x+\delta x)} - \dfrac{1}{\sin x}}{\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}}$$

$$= \frac{\sin x - \sin(x+\delta x)}{\left[\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}\right][\sin(x+\delta x)\sin x]}$$

$$= -\frac{2\cos\left(x+\dfrac{\delta x}{2}\right)\sin\dfrac{\delta x}{2}}{\left(\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}\right)[\sin(x+\delta x)\sin x]}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = -\lim_{\delta x \to 0} \frac{\cos\left(x+\dfrac{\delta x}{2}\right)}{\sqrt{\operatorname{cosec}(x+\delta x)} + \sqrt{\operatorname{cosec} x}]} \times \frac{\dfrac{\sin \delta x/2}{\delta x/2}}{[\sin(x+\delta x).\sin x]}$$

or $$\frac{dy}{dx} = \frac{-\cos x}{(2\sqrt{(\operatorname{cosec} x)}(\sin x)^2}$$

$$= -\frac{1}{2}(\text{cosec}x)^{-\frac{1}{2}}(\text{cosec}\,x\,\cot x)$$

Thus, $$\frac{d}{dx}\left(\sqrt{\text{cosec}x}\right) = -\frac{1}{2}(\text{cosec}x)^{-\frac{1}{2}}(\text{cosec}x\cot x)$$

Example 22.3 Find the derivative of $\sec^2 x$ from first principle.

Solution : Let $y = \sec^2 x$

and $$y + \delta y = \sec^2(x + \delta x)$$

then, $$\delta y = \sec^2(x + \delta x) - \sec^2 x$$

$$= \frac{\cos^2 x - \cos^2(x + \delta x)}{\cos^2(x + \delta x)\cos^2 x}$$

$$= \frac{\sin[(x + \delta x + x]\sin[(x + \delta x - x)]}{\cos^2(x + \delta x)\cos^2 x}$$

$$= \frac{\sin(2x + \delta x)\sin\delta x}{\cos^2(x + \delta x)\cos^2 x}$$

$$\frac{\delta y}{\delta x} = \frac{\sin(2x + \delta x)\sin\delta x}{\cos^2(x + \delta x)\cos^2 x.\delta x}$$

Now, $$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{\sin(2x + \delta x)\sin\delta x}{\cos^2(x + \delta x)\cos^2 x.\delta x}$$

$$\frac{dy}{dx} = \frac{\sin 2x}{\cos^2 x\cos^2 x}$$

$$= \frac{2\sin x\cos x}{\cos^2 x\cos^2 x} = 2\tan x.\sec^2 x$$

$$= 2\sec x(\sec x.\tan x)$$

$$= 2\sec x\ (\sec x \tan x)$$

CHECK YOUR PROGRESS 22.1

1. Find the derivative from first principle of the following functions with respect to x :

   (a) cosec x (b) cot x (c) cos 2 x

   (d) cot 2 x (e) $\text{cosec}x^2$ (f) $\sqrt{\sin x}$

2. Find the derivative of each of the following functions :

   (a) $2\sin^2 x$ (b) $\text{cosec}^2 x$ (c) $\tan^2 x$

22.2 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

You have learnt how we can find the derivative of a trigonometric function from first principle and also how to deal with these functions as a function of a function as shown in the alternative method. Now we consider some more examples of these derivatives.

Example 22.4 Find the derivative of each of the following functions :

(i) $\sin 2x$ (ii) $\tan\sqrt{x}$ (iii) $\operatorname{cosec}(5x^3)$

Solution : (i) Let

$$y = \sin 2x,$$
$$= \sin t, \qquad \text{where } t = 2x$$

$$\frac{dy}{dt} = \cos t \qquad \text{and} \qquad \frac{dt}{dx} = 2$$

By chain Rule, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$, we have

$$\frac{dy}{dx} = \cos t\,(2) = 2.\cos t = 2\cos 2x$$

Hence, $\frac{d}{dx}(\sin 2x) = 2\cos 2x$

(ii) Let

$$y = \tan\sqrt{x}$$
$$= \tan t \qquad \text{where } t = \sqrt{x}$$

$$\therefore \qquad \frac{dy}{dt} = \sec^2 t \qquad \text{and} \qquad \frac{dt}{dx} = \frac{1}{2\sqrt{x}}$$

By chain rule, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$, we have

$$\frac{dy}{dx} = \sec^2 t \cdot \frac{1}{2\sqrt{x}} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

Hence, $\frac{d}{dx}\left(\tan\sqrt{x}\right) = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$

Alternatively : Let $y = \tan\sqrt{x}$

$$\frac{dy}{dx} = \sec^2\sqrt{x}\,\frac{d}{dx}\sqrt{x} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

(iii) Let $y = \operatorname{cosec}(5x^3)$

$$\therefore \qquad \frac{dy}{dx} = -\operatorname{cosec}(5x^3)\cot(5x^3) \cdot \frac{d}{dx}[5x^3]$$

$$= -15x^2\operatorname{cosec}(5x^3)\cot(5x^3)$$

or you may solve it by substituting $t = 5x^3$

Example 22.5 Find the derivative of each of the following functions :

(i) $y = x^4 \sin 2x$ (ii) $y = \dfrac{\sin x}{1+\cos x}$

Solution : $y = x^4 \sin 2x$

(i) $\therefore$ $\dfrac{dy}{dx} = x^4 \dfrac{d}{dx}(\sin 2x) + \sin 2x \dfrac{d}{dx}(x^4)$ (Using product rule)

$$= x^4(2\cos 2x) + \sin 2x(4x^3)$$

$$= 2x^4 \cos 2x + 4x^3 \sin 2x$$

$$= 2x^3[x\cos 2x + 2\sin 2x]$$

(ii) $y = \dfrac{\sin x}{1+\cos x}$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$= \tan\frac{x}{2}$$

$\therefore$ $\dfrac{dy}{dx} = \sec^2\dfrac{x}{2} \cdot \dfrac{d}{dx}\left(\dfrac{x}{2}\right) = \dfrac{1}{2}\sec^2\dfrac{x}{2}$

Alternatively : You may find the derivative by using quotient rule

Let $y = \dfrac{\sin x}{1+\cos x}$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{(1+\cos x)\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$

$$= \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2}$$

$$= \frac{1}{(1+\cos x)}$$

$$= \frac{1}{2\cos^2\frac{x}{2}} = \frac{1}{2}\sec^2\frac{x}{2}$$

Example 22.6 Find the derivative of each of the following functions w.r.t. x :

(i) $\cos^2 x$ (ii) $\sqrt{\sin^3 x}$

Solution : (i) Let $y = \cos^2 x$

$= t^2$ where t = cos x

$\therefore \quad \frac{dy}{dt} = 2t$ and $\frac{dt}{dx} = -\sin x$

Using chain rule

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$, we have

$$\frac{dy}{dx} = 2\cos x.(-\sin x)$$

$$= -2\cos x \sin x = -\sin 2x$$

(ii) Let $y = \sqrt{\sin^3 x}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(\sin^3 x)^{-1/2} \cdot \frac{d}{dx}(\sin^3 x)$$

$$= \frac{1}{2\sqrt{\sin^3 x}} \cdot 3\sin^2 x \cdot \cos x$$

$$= \frac{3}{2}\sqrt{\sin x}\cos x$$

Thus, $\frac{d}{dx}\left(\sqrt{\sin^3 x}\right) = \frac{3}{2}\sqrt{\sin x}\cos x$

Example 22.7 Find $\frac{dy}{dx}$, when

(i) $y = \sqrt{\frac{1-\sin x}{1+\sin x}}$ (ii) $y = a(1-\cos t), x = a(t+\sin t)$

Solution : We have,

(i) $y = \sqrt{\frac{1-\sin x}{1+\sin x}}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left[\frac{1-\sin x}{1+\sin x}\right]^{-\frac{1}{2}} \cdot \frac{d}{dx}\left[\frac{1-\sin x}{1+\sin x}\right]$$

$$=\frac{1}{2}\sqrt{\frac{1+\sin x}{1-\sin x}}\cdot\frac{(-\cos x)(1+\sin x)-(1-\sin x)(\cos x)}{(1+\sin x)^2}$$

$$=\frac{1}{2}\sqrt{\frac{1+\sin x}{1-\sin x}}\cdot\left(\frac{-2\cos x}{(1+\sin x)^2}\right)$$

$$=-\frac{\sqrt{1+\sin x}}{\sqrt{1-\sin x}}\cdot\frac{\sqrt{1-\sin^2 x}}{(1+\sin x)^2}$$

$$=-\frac{\sqrt{1+\sin x}\sqrt{1+\sin x}}{(1+\sin x)^2}=\frac{-1}{1+\sin x}$$

Thus, $dy/dx = -\dfrac{1}{1+\sin x}$

Alternatively, it is more convenient to find the derivative of such square root function by rationalising the denominator.

$$y=\frac{\sqrt{1-\sin x}}{\sqrt{1+\sin x}}\times\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}$$

$$=\frac{1-\sin x}{\sqrt{1-\sin^2 x}}$$

$$=\frac{1-\sin x}{\cos x}$$

$$=\sec x-\tan x$$

$$\therefore \quad \frac{dy}{dx}=\sec x\tan x-\sec^2 x=\frac{\sin x}{\cos^2 x}-\frac{1}{\cos^2 x}$$

$$=\frac{\sin x-1}{1-\sin^2 x}=-\frac{1}{1+\sin x}$$

(ii) $x=a(t+\sin t),\quad y=a(1-\cos t)$

$$\therefore \quad \frac{dx}{dt}=a(1+\cos t),\quad \frac{dy}{dt}=a(\sin t)$$

Using chain rule, $\dfrac{dy}{dx}=\dfrac{dy}{dt}\cdot\dfrac{dt}{dx}$, we have

$$\frac{dy}{dx}=\frac{a(\sin t)}{a(1+\cos t)}$$

$$=\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}}=\tan\frac{t}{2}$$

Example 22.8 Find the derivative of each of the following functions at the indicated points :

(i) $y = \sin 2x + (2x-5)^2$ at $x = \frac{\pi}{2}$

(ii) $y = \cot x + \sec^2 x + 5$ at $x = \pi/6$

Solution :

(i) $y = \sin 2x + (2x-5)^2$

$$\therefore \quad \frac{dy}{dx} = \cos 2x \frac{d}{dx}(2x) + 2(2x-5)\frac{d}{dx}(2x-5)$$

$$= 2\cos 2x + 4(2x-5)$$

At $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 2\cos\pi + 4(\pi - 5)$

$$= -2 + 4\pi - 20$$

$$= 4\pi - 22$$

(ii) $y = \cot x + \sec^2 x + 5$

$$\therefore \quad \frac{dy}{dx} = -\mathrm{cosec}^2 x + 2\sec x(\sec x \tan x)$$

$$= -\mathrm{cosec}^2 x + 2\sec^2 x \tan x$$

At $x = \frac{\pi}{6}$, $\frac{dy}{dx} = -\mathrm{cosec}^2\frac{\pi}{6} + 2\sec^2\frac{\pi}{6}\tan\frac{\pi}{6}$

$$= -4 + 2\cdot\frac{4}{3}\frac{1}{\sqrt{3}}$$

$$= -4 + \frac{8}{3\sqrt{3}}$$

Example 22.9 If sin y = x sin (a+y), prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Solution : It is given that

$$\sin y = x \sin(a+y) \quad \text{or} \quad x = \frac{\sin y}{\sin(a+y)} \qquad .....(1)$$

Differentiating w.r.t. x on both sides of (1) we get

$$1 = \left[\frac{\sin(a+y)\cos y - \sin y \cos(a+y)}{\sin^2(a+y)}\right]\frac{dy}{dx}$$

or
$$1 = \left[\frac{\sin(a+y-y)}{\sin^2(a+y)}\right]\frac{dy}{dx}$$

or $$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$$

Example 22.10 If $y=\sqrt{\sin x+\sqrt{\sin x+....\text{to infinity}}}$,

prove that $$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

Solution : We are given that

$$y=\sqrt{\sin x+\sqrt{\sin x+...\text{to infinity}}}$$

or $$y=\sqrt{\sin x+y} \qquad \text{or} \qquad y^2=\sin x+y$$

Differentiating with respect to x , we get

$$2y\frac{dy}{dx}=\cos x+\frac{dy}{dx} \qquad \text{or} \qquad (2y-1)\frac{dy}{dx}=\cos x$$

Thus, $$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

CHECK YOUR PROGRESS 22.2

1. Find the derivative of each of the following functions w.r.t x :

(a) $y=3\sin 4x$ (b) $y=\cos 5x$ (c) $y=\tan\sqrt{x}$

(d) $y=\sin\sqrt{x}$ (e) $y=\sin x^2$ (f) $y=\sqrt{2}\tan 2x$

(g) $y=\pi\cot 3x$ (h) $y=\sec 10x$ (i) $y=\operatorname{cosec} 2x$

2. Find the derivative of each of the following functions :

(a) $f(x)=\frac{\sec x-1}{\sec x+1}$ (b) $f(x)=\frac{\sin x+\cos x}{\sin x-\cos x}$ (c) $f(x)=x\sin x$

(d) $f(x)=\left(1+x^2\right)\cos x$ (e) $f(x)=x\operatorname{cosec} x$ (f) $f(x)=\sin 2x\cos 3x$

(g) $f(x)=\sqrt{\sin 3x}$

3. Find the derivative of each of the following functions :

(a) $y=\sin^3 x$ (b) $y=\cos^2 x$ (c) $y=\tan^4 x$

(d) $y=\cot^4 x$ (e) $y=\sec^5 x$ (f) $y=\operatorname{cosec}^3 x$

(g) $y=\sec\sqrt{x}$ (h) $y=\sqrt{\frac{\sec x+\tan x}{\sec x-\tan x}}$

4. Find the derivative of the following functions at the indicated points :

(a) $y = \cos(2x + \pi/2), x = \dfrac{\pi}{3}$ (b) $y = \dfrac{1+\sin x}{\cos x}, x = \dfrac{\pi}{4}$

5. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + ...,}}}$ to infinity

Show that $(2y-1)\dfrac{dy}{dx} = \sec^2 x$.

6. If $\cos y = x\cos(a+y)$,

prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$.

22.3 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS FROM FIRST PRINCIPLE

We now find derivatives of standard inverse trignometric functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, by first principle.

(i) We will show that by first principle the derivative of $\sin^{-1} x$ w.r.t. x is given by

$$\mathbf{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}}}$$

Let $y = \sin^{-1} x$. Then $x = \sin y$ and so $x + \delta x = \sin(y + \delta y)$

As $\delta x \to 0$, $\delta y \to 0$.

Now, $\delta x = \sin(y+\delta y) - \sin y$

$\therefore$ $1 = \dfrac{\sin(y+\delta y) - \sin y}{\delta x}$ [On dividing both sides by δx]

or $1 = \dfrac{\sin(y+\delta y) - \sin y}{\delta y} \cdot \dfrac{\delta y}{\delta x}$

$\therefore$ $1 = \lim\limits_{\delta y \to 0} \dfrac{\sin(y+\delta y) - \sin y}{\delta y} \cdot \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$ [$\because \delta y \to 0$ when $\delta x \to 0$]

$$= \left[\lim_{\delta y \to 0} \frac{2\cos\left(y + \frac{1}{2}\delta y\right)\sin\left(\frac{1}{2}\delta y\right)}{\delta y}\right] \cdot \frac{dy}{dx}$$

$$= (\cos y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}}$$

$$\therefore \quad \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{(1-x^2)}}$$

(ii) $$\mathbf{\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{\left(1-x^2\right)}}}$$

For proof proceed exactly as in the case of $\sin^{-1} x$.

(iii) Now we show that,

$$\mathbf{\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}}$$

Let $y = \tan^{-1} x$ .Then $x = \tan y$ and so $x + \delta x = \tan(y + \delta y)$

As $\delta x \to 0$, also $\delta y \to 0$

Now, $\delta x = \tan(y+\delta y) - \tan y$

$$\therefore \quad 1 = \frac{\tan(y+\delta y) - \tan y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

$$\therefore \quad 1 = \lim_{\delta y \to 0} \frac{\tan(y+\delta y) - \tan y}{\delta y} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \qquad [\because \delta y \to 0 \text{ when } \delta x \to 0]$$

$$= \left[\lim_{\delta y \to 0} \left\{\frac{\sin(y+\delta y)}{\cos(y+\delta y)} - \frac{\sin y}{\cos y}\right\} \Big/ \delta y\right] \cdot \frac{dy}{dx}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin(y+\delta y)\cos y - \cos(y+\delta y)\sin y}{\delta y.\cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin(y+\delta y - y)}{\delta y.\cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{\sin \delta y}{\delta y} \cdot \frac{1}{\cos(y+\delta y)\cos y}\right]$$

$$= \frac{dy}{dx}\, \frac{1}{\cos^2 y} = \frac{dy}{dx} \cdot \sec^2 y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\therefore \quad \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

(iv) $$\mathbf{\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}}$$

For proof proceed exactly as in the case of $\tan^{-1} x$.

(v) We have by first principle $\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{sec}^{-1}\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{x}\sqrt{(\mathbf{x}^2 - \mathbf{1})}}$

Let $y = \sec^{-1} x$. Then x = sec y and so $x + \delta x = \sec(y + \delta y)$.

As $\delta x \to 0$, also $\delta y \to 0$.

Now $\qquad \delta x = \sec(y + \delta y) - \sec y.$

$$\therefore \qquad 1 = \frac{\sec(y+\delta y) - \sec y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

$$1 = \lim_{\delta y \to 0} \frac{\sec(y+\delta y) - \sec y}{\delta y} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \qquad [\because \delta y \to 0 \text{ when } \delta x \to 0]$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{2\sin\left(y + \frac{1}{2}\delta y\right)\sin\left(\frac{1}{2}\delta y\right)}{\delta y . \cos y \cos(y + \delta y)}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{\sin\left(y + \frac{1}{2}\delta y\right)}{\cos y \cos(y + \delta y)} \cdot \frac{\sin\left(\frac{1}{2}\delta y\right)}{\frac{1}{2}\delta y} \right]$$

$$= \frac{dy}{dx} \cdot \frac{\sin y}{\cos y \cos y} = \frac{dy}{dx} \cdot \sec y \tan y$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\sqrt{(\sec^2 y - 1)}} = \frac{1}{x\sqrt{(x^2 - 1)}}$$

$$\therefore \qquad \frac{d}{dx} = \left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2 - 1}}$$

(vi) $\frac{\mathbf{d}}{\mathbf{dx}}\left(\mathbf{cosec}^{-1}\mathbf{x}\right) = \frac{\mathbf{1}}{\mathbf{x}\sqrt{(\mathbf{x}^2 - \mathbf{1})}}.$

For proof proceed as in the case of $\sec^{-1} x$.

Example 22.11 Find derivative of $\sin^{-1}(x^2)$ from first principle.

Solution : Let $\qquad y = \sin^{-1} x^2$

$$\therefore \qquad x^2 = \sin y$$

Now, $\qquad (x + \delta x)^2 = \sin(y + \delta y)$

$$\frac{(x+\delta x)^2 - x^2}{\delta x} = \frac{\sin(y + \delta y) - \sin y}{\delta x}$$

MODULE - V
Calculus

Notes

$$\lim_{\delta x\to 0}\frac{(x+\delta x)^2-x^2}{(x+\delta x)-x}=\lim_{\delta y\leftarrow 0}\frac{2\cos\left(y+\frac{\delta y}{2}\right)}{2}\cdot\frac{\sin\frac{\delta y}{2}}{\frac{\delta y}{2}}\cdot\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$

$$\Rightarrow \qquad 2x=\cos y\cdot\frac{dy}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx}=\frac{2x}{\cos y}=\frac{2x}{\sqrt{1-\sin^2 y}}=\frac{2x}{\sqrt{1-x^4}}.$$

Example 22.12 Find derivative of $\sin^{-1}\sqrt{x}$ w.r.t. x by delta method.

Solution : Let $y=\sin^{-1}\sqrt{x}$

$$\Rightarrow \qquad \sin y=\sqrt{x} \qquad ..(1)$$

Also $$\sin(y+\delta y)=\sqrt{x+\delta x} \qquad ..(2)$$

From (1) and (2), we get

$$\sin(y+\delta y)-\sin y=\sqrt{x+\delta x}-\sqrt{x}$$

or $$2\cos\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right)=\frac{\left(\sqrt{x+\delta x}-\sqrt{x}\right)\left(\sqrt{x+\delta x}+\sqrt{x}\right)}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$=\frac{\delta x}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$\therefore \qquad \frac{2\cos\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right)}{\delta x}=\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$$

or $$\frac{\delta y}{\delta x}\cdot\cos\left(y+\frac{\delta y}{2}\right)\cdot\frac{\sin\left(\frac{\delta y}{2}\right)}{\frac{\delta y}{2}}=\frac{1}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$\therefore \qquad \lim_{\delta x\to 0}\frac{\delta y}{\delta x}\cdot\lim_{\delta y\to 0}\cos\left(y+\frac{\delta y}{2}\right)\cdot\lim_{\delta y\to 0}\frac{\sin\left(\frac{\delta y}{2}\right)}{\frac{\delta y}{2}}$$

$$=\lim_{\delta x\to 0}\frac{1}{\sqrt{x+\delta x}+\sqrt{x}} \qquad (\because\ \delta y\to 0\ \text{ as }\ \delta x\to 0)$$

or $$\frac{dy}{dx}\cos y=\frac{1}{2\sqrt{x}} \quad \text{or} \quad \frac{dy}{dx}=\frac{1}{2\sqrt{x}\cos y}=\frac{1}{2\sqrt{x}\sqrt{1-\sin^2 y}}=\frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$\therefore \qquad \frac{dy}{dx}=\frac{1}{2\sqrt{x}\ \sqrt{1-x}}$$

CHECK YOUR PROGRESS 22.3

1. Find by first principle that derivative of each of the following :

(i) $\cos^{-1} x^2$ (ii) $\frac{\cos^{-1} x}{x}$ (iii) $\cos^{-1}\sqrt{x}$

(iv) $\tan^{-1} x^2$ (v) $\frac{\tan^{-1} x}{x}$ (vi) $\tan^{-1}\sqrt{x}$

22.4 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

In the previous section, we have learnt to find derivatives of inverse trignometric functions by first principle. Now we learn to find derivatives of inverse trigonometric functions by alternative methods. We start with standard inverse trignometric functions $\sin^{-1} x, \cos^{-1} x, ...$

(i) Derivative of $\sin^{-1} x$

Solution : Let $y = \sin^{-1} x$

$\therefore$ $x = \sin y$ (i)

Differentiating w.r.t. y

$$\frac{dx}{dy} = \cos y$$

$\therefore$ $$\frac{dx}{dy} = \sqrt{1 - \sin^2 y}$$ ...[Using (i)]

or $$\frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$ $\left[\because \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}\right]$

Hence, $$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$$

Similarly we can show that

$$\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$$

(ii) Derivative of $\tan^{-1} x$

Solution : Let $\tan^{-1} x = y$

$\therefore$ $x = \tan y$

Differentiating w.r.t. y, $$\frac{dx}{dy} = \sec^2 y$$

and $$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1+\tan^2 y} \qquad [\because \text{ We have written } \tan y \text{ in terms of } x]$$

$$= \frac{1}{1+x^2}$$

Hence, $$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

Similarly, $$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}.$$

(iii) Derivative of $\sec^{-1} x$

Solution : Let $\sec^{-1} x = y$

$\therefore$ $$x = \sec y \text{ and } \frac{dx}{dy} = \sec y \tan y$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$= \frac{1}{\sec y\,[\pm\sqrt{\sec^2 y - 1}]}$$

$$= \frac{1}{\pm \sec y \sqrt{\sec^2 y - 1}}$$

$$= \frac{1}{|x|\sqrt{\sec^2 y - 1}}$$

Note : (i) When $x > 1$, sec y is + ve and tan y is + ve, $y \in \left(0, \frac{\pi}{2}\right)$

(ii) When $x < -1$, sec y is –ve and tan y is –ve, $y \in \left(\frac{\pi}{2}, \pi\right)$

Hence, $$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

Similarly $$\frac{d}{dx}(\mathrm{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2 - 1}}$$

Example 22.13 Find the derivative of each of the following :

(i) $\sin^{-1}\sqrt{x}$ (ii) $\cos^{-1} x^2$ (iii) $(\mathrm{cosec}^{-1} x)^2$

Solution :

(i) Let $y = \sin^{-1}\sqrt{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2}x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$\therefore \quad \frac{d}{dx}\sin^{-1}x = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

(ii) Let $y = \cos^{-1}x^2$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2)$$

$$= \frac{-1}{\sqrt{1-x^4}} \cdot (2x)$$

$$\therefore \quad \frac{d}{dx}\left(\cos^{-1}x^2\right) = \frac{-2x}{\sqrt{1-x^4}}$$

(iii) Let $y = (\operatorname{cosec}^{-1}x)^2$

$$\frac{dy}{dx} = 2(\operatorname{cosec}^{-1}x) \cdot \frac{d}{dx}\left(\operatorname{cosec}^{-1}x\right)$$

$$= 2(\operatorname{cosec}^{-1}x) \cdot \frac{-1}{|x|\sqrt{x^2-1}}$$

$$= \frac{-2\operatorname{cosec}^{-1}x}{|x|\sqrt{x^2-1}}$$

$$\therefore \quad \frac{d}{dx}(\operatorname{cosec}^{-1}x)^2 = \frac{-2\operatorname{cosec}^{-1}x}{|x|\sqrt{x^2-1}}$$

Example 22.14 Find the derivative of each of the following :

(i) $\tan^{-1}\dfrac{\cos x}{1+\sin x}$ (ii) $\sin(2\sin^{-1}x)$

Solution :

Let (i) $y = \tan^{-1}\dfrac{\cos x}{1+\sin x}$

$$= \tan^{-1}\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]$$

$$= \frac{\pi}{4}-\frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = -1/2$$

(ii) $y = \sin(2\sin^{-1}x)$

Let $y = \sin(2\sin^{-1}x)$

$$\therefore \quad \frac{dy}{dx} = \cos(2\sin^{-1}x)\cdot\frac{d}{dx}(2\sin^{-1}x)$$

$$\therefore \quad \frac{dy}{dx} = \cos(2\sin^{-1}x)\cdot\frac{2}{\sqrt{1-x^2}}$$

$$= \frac{2\cos(2\sin^{-1}x)}{\sqrt{1-x^2}}$$

Example 22.15 Show that the derivative of $\tan^{-1}\frac{2x}{1-x^2}$ w.r.t $\sin^{-1}\frac{2x}{1+x^2}$ is 1.

Solution : Let $y = \tan^{-1}\frac{2x}{1-x^2}$ and $z = \sin^{-1}\frac{2x}{1+x^2}$

Let $x = \tan\theta$

$$\therefore \quad y = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} \text{ and } z = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$$

$$= \tan^{-1}(\tan 2\theta) \text{ and } z = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta \quad \text{and} \quad z = 2\theta$$

$$\frac{dy}{d\theta} = 2 \quad \text{and} \quad \frac{dz}{d\theta} = 2$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dz} = 2\cdot\frac{1}{2} = 1$$ (By chain rule)

CHECK YOUR PROGRESS 22.4

Find the derivative of each of the following functions w.r.t. x and express the result in the simplest form (1-3) :

1. (a) $\sin^{-1}x^2$ (b) $\cos^{-1}\frac{x}{2}$ (c) $\cos^{-1}\frac{1}{x}$

2. (a) $\tan^{-1}(\operatorname{cosec}x - \cot x)$ (b) $\cot^{-1}(\sec x + \tan x)$ (c) $\tan^{-1}\frac{\cos x - \sin x}{\cos x + \sin x}$

3. (a) $\sin(\cos^{-1}x)$ (b) $\sec(\tan^{-1}x)$ (c) $\sin^{-1}(1-2x^2)$

(d) $\cos^{-1}(4x^3-3x)$ (e) $\cot^{-1}\left(\sqrt{1+x^2}+x\right)$

4. Find the derivative of :

$$\frac{\tan^{-1}x}{1+\tan^{-1}x} \text{ w.r.t. } \tan^{-1}x.$$

22.5 SECOND ORDER DERIVATIVES

We know that the second order derivative of a function is the derivative of the first derivative of that function. In this section, we shall find the second order derivatives of trigonometric and inverse trigonometric functions. In the process, we shall be using product rule, quotient rule and chain rule.

Let us take some examples.

Example 22.16 Find the second order derivative of

(i) $\sin x$ (ii) $x\cos x$ (iii) $\cos^{-1}x$

Solution : (i) Let y = sin x

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx}=\cos x$$

Differentiating w.r.t x both sides again, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(\cos x)=-\sin x$$

$$\therefore \quad \frac{d^2y}{dx^2}=-\sin x$$

(ii) Let y = x cos x

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx}=x(-\sin x)+\cos x.1$$

$$\frac{dy}{dx}=-x\sin x+\cos x$$

Differentiating w.r.t. x both sides again, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(-x\sin x+\cos x)$$

$$=-(x.\cos x+\sin x)-\sin x$$

$$=-x.\cos x-2\sin x$$

$$\therefore \quad \frac{d^2y}{dx^2}=-(x.\cos x+2\sin x)$$

(iii) Let $y = \cos^{-1} x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{\left(1-x^2\right)^{1/2}} = -\left(1-x^2\right)^{-\frac{1}{2}}$$

Differentiating w.r.t. x both sides, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{-1}{2}\cdot\left(1-x^2\right)^{-3/2}\cdot(-2x)\right]$$

$$= -\frac{x}{\left(1-x^2\right)^{3/2}}$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{-x}{\left(1-x^2\right)^{3/2}}$$

Example 22.17 If $y = \sin^{-1} x$, show that $\left(1-x^2\right)y_2 - xy_1 = 0$, where y_2 and y_1 respectively denote the second and first, order derivatives of y w.r.t. x.

Solution : We have, $\quad y = \sin^{-1} x$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

or $$\left(\frac{dy}{dx}\right)^2 = \frac{1}{1-x^2} \qquad \text{(squaring both sides)}$$

or $$\left(1-x^2\right)y_1^2 = 0$$

Differentiating w.r.t. x both sides, we get

$$\left(1-x^2\right)\cdot 2y_1\frac{d}{dx}(y_1) + (-2x)\cdot y_1^2 = 0$$

or $$\left(1-x^2\right)\cdot 2y_1y_2 - 2x\,y_1^2 = 0$$

or $$\left(1-x^2\right)y_2 - x\,y_1 = 0$$

CHECK YOUR PROGRESS 22.5

1. Find the second order derivative of each of the following :

   (a) $\sin(\cos x)$ $\qquad$ (b) $x^2\tan^{-1}x$

2. If $y=\frac{1}{2}\left(\sin^{-1}x\right)^2$, show that $\left(1-x^2\right)y_2-xy_1=1$.

3. If $y=\sin(\sin x)$, prove that $\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0$.

4. If $y=x+\tan x$, show that $\cos^2 x\frac{d^2y}{dx^2}-2y+2x=0$

LET US SUM UP

- (i) $\frac{d}{dx}(\sin x)=\cos x$ (ii) $\frac{d}{dx}(\cos x)=-\sin x$

  (iii) $\frac{d}{dx}(\tan x)=\sec^2 x$ (iv) $\frac{d}{dx}(\cot x)=-\mathrm{cosec}^2 x$

  (v) $\frac{d}{dx}(\sec x)=\sec x\tan x$ (vi) $\frac{d}{dx}(\mathrm{cosec}x)=-\mathrm{cosec}x\cot x$

- If u is a derivabale function of x, then

  (i) $\frac{d}{dx}(\sin u)=\cos u\frac{du}{dx}$ (ii) $\frac{d}{dx}(\cos u)=-\sin u\frac{du}{dx}$

  (iii) $\frac{d}{dx}(\tan u)=\sec^2 u\frac{du}{dx}$ (iv) $\frac{d}{dx}(\cot u)=-\mathrm{cosec}^2 u\frac{du}{dx}$

  (v) $\frac{d}{dx}(\sec u)=\sec u\tan u\frac{du}{dx}$ (vi) $\frac{d}{dx}(\mathrm{cose}u)=-\mathrm{cosec}u\cot u\frac{du}{dx}$

- (i) $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$ (ii) $\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}$

  (iii) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$ (iv) $\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}$

  (v) $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}$ (vi) $\frac{d}{dx}(\mathrm{cosec}^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}}$

- If u is a derivable function of x, then

  (i) $\frac{d}{dx}(\sin^{-1}u)=\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$ (ii) $\frac{d}{dx}(\cos^{-1}u)=\frac{-1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$

  (iii) $\frac{d}{dx}(\tan^{-1}u)=\frac{1}{1+u^2}\cdot\frac{du}{dx}$ (iv) $\frac{d}{dx}(\cot^{-1}u)=\frac{-1}{1+u^2}\cdot\frac{du}{dx}$

  (v) $\frac{d}{dx}(\sec^{-1}u)=\frac{1}{|u|\sqrt{u^2-1}}\cdot\frac{du}{dx}$ (vi) $\frac{d}{dx}(\mathrm{cosec}^{-1}u)=\frac{-1}{|u|\sqrt{u^2-1}}\cdot\frac{du}{dx}$

The second order derivative of a trignometric function is the derivative of their first order derivatives.

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

1. If $y = x^3 \tan^2 \frac{x}{2}$, find $\frac{dy}{dx}$.

2. Evaluate, $\frac{d}{dx}\sqrt{\sin^4 x + \cos^4 x}$ at $x = \frac{\pi}{2}$ and 0.

3. If $y = \frac{5x}{\sqrt[3]{(1-x)^2}} + \cos^2(2x+1)$, find $\frac{dy}{dx}$.

4. If $y = \sec^{-1}\frac{\sqrt{x}+1}{\sqrt{x}-1} + \sin^{-1}\frac{\sqrt{x}-1}{\sqrt{x}+1}$, then show that $\frac{dy}{dx} = 0$

5. If $x = a\cos^3\theta, y = a\sin^3\theta$, then find $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

6. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}$, find $\frac{dy}{dx}$.

7. Find the derivative of $\sin^{-1} x$ w.r.t. $\cos^{-1}\sqrt{1-x^2}$

8. If $y = \cos(\cos x)$, prove that

$$\frac{d^2y}{dx^2} - \cot x \cdot \frac{dy}{dx} + y.\sin^2 x = 0.$$

9. If $y = \tan^{-1} x$ show that

$$(1+x)^2 y_2 + 2xy_1 = 0.$$

10. If $y = (\cos^{-1} x)^2$, show that

$$(1-x^2)y_2 - xy_1 - 2 = 0.$$

ANSWERS

CHECK YOUR PROGRESS 22.1

(1) (a) $-\operatorname{cosec}x\cot x$ (b) $-\operatorname{cosec}^2 x$ (c) $-2\sin 2x$

(d) $-2\operatorname{cosec}^2 2x$ (e) $-2x\operatorname{cosec}x^2\cot x^2$ (f) $\dfrac{\cos x}{2\sqrt{\sin x}}$

2. (a) $2\sin 2x$ (b) $-2\operatorname{cosec}^2 x\cot x$ (c) $2\tan x\sec^2 x$

CHECK YOUR PROGRESS 22.2

1. (a) $12\cos 4x$ (b) $-5\sin 5x$ (c) $\dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$ (d) $\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$

(e) $2x\cos x^2$ (f) $2\sqrt{2}\sec^2 2x$ (g) $-3\pi\operatorname{cosec}^2 3x$

(h) $10\sec 10x\tan 10x$ (i) $-2\operatorname{cosec}2x\cot 2x$

2. (a) $\dfrac{2\sec x\tan x}{(\sec x+1)^2}$ (b) $\dfrac{-2}{(\sin x-\cos x)^2}$ (c) $x\cos x+\sin x$

(d) $2x\cos x-(1+x^2)\sin x$

(e) $\operatorname{cosec}x(1-x\cot x)$ (f) $2\cos 2x\cos 3x-3\sin 2x\sin 3x$ (g) $\dfrac{3\cos 3x}{2\sqrt{\sin 3x}}$

3. (a) $3\sin^2 x\cos x$ (b) $-\sin 2x$ (c) $4\tan^3 x\sec^2 x$ (d) $-4\cot^3 x\operatorname{cosec}^2 x$

(e) $5\sec^5 x\tan x$ (f) $-3\operatorname{cosec}^3 x\cot x$ (g) $\dfrac{\sec\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}}$

(h) $\sec x(\sec x+\tan x)$

4. (a) 1 (b) $\sqrt{2}+2$

CHECK YOUR PROGRESS 22.3

1. (i) $\dfrac{-2x}{\sqrt{1-x^4}}$ (ii) $\dfrac{-1}{x\sqrt{1-x^2}}-\dfrac{-\cos^{-1}x}{x^2}$ (iii) $\dfrac{-1}{2x^{\frac{1}{2}}\sqrt{(1-x)}}$

(iv) $\dfrac{2x}{1+x^4}$ (v) $\dfrac{1}{x(1+x^2)}-\dfrac{\tan^{-1}x}{x^2}$ (vi) $\dfrac{1}{2x^{\frac{1}{2}}(1+x)}$

CHECK YOUR PROGRESS 22.4

1. (a) $\dfrac{2x}{\sqrt{1-x^4}}$ (b) $\dfrac{-1}{\sqrt{4-x^2}}$ (c) $\dfrac{1}{x\sqrt{x^2-1}}$

2. (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) -1

3. (a) $-\frac{\cos\left(\cos^{-1}x\right)}{\sqrt{1-x^2}}$ (b) $\frac{x}{1+x^2}\cdot\sec\left(\tan^{-1}x\right)$

(c) $\frac{-2}{\sqrt{1-x^2}}$ (d) $\frac{-3}{\sqrt{1-x^2}}$ (e) $\frac{-1}{2(1+x^2)}$

4. $\frac{1}{\left(1+\tan^{-1}x\right)^2}$

CHECK YOUR PROGRESS 22.5

1. (a) $-\cos x\cos(\cos x)-\sin^2 x\sin(\cos x)$

(b) $\frac{2x(2+x^2)}{(1+x^2)^2}+2\tan^{-1}x$

TERMINAL EXERCISE

1. $x^3\tan\frac{x}{2}\sec^2\frac{x}{2}+3x^2\tan^2\frac{x}{2}$ 2. 0, 0

3. $\frac{5(3-x)}{3(1-x)^{\frac{5}{3}}}-2\sin(4x+2)$ 5. $|\sec\theta|$

6. $\frac{1}{2y-1}$ 7. $\frac{1}{2\sqrt{1-x^2}}$