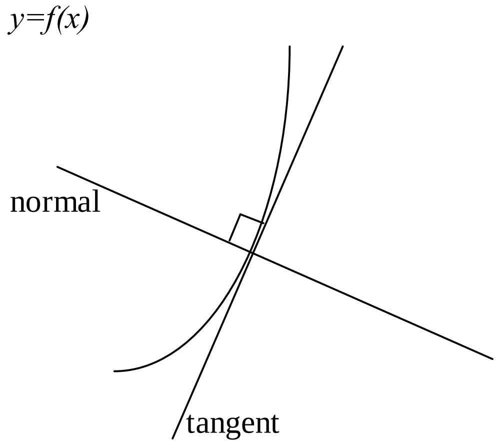
APPLICATIONS OF DIFFERENTIATION

EQUATIONS OF TANGENTS AND NORMALS

Definition: A normal to a curve at a point is the straight line through the point at right angles to the tangent at the point.



Finding the equations of tangents and normals.

Examples

1. Find the equation of the tangent to the curve $y = x^3$ at the point (2,8).

Solution:

$y = x^3$; $\therefore$ gradient of y or $\frac{dy}{dx} = 3x^2$

When $x=2$; $\frac{dy}{dx} = 3\times 2^2 = 3\times 4 = 12$

Thus the gradient of the tangent at (2, 8) is 12.

But gradient $= \frac{\Delta y}{\Delta x} = \frac{y-8}{x-2} = 12$;

$y - 8 = 12(x - 12)$

$y - 8 = 12x - 24$

$y = 12x - 24 + 8$

$y = 12x - 16$ which is the equation of the tangent.

2. Find the equation of the normal to the curve $y = (x^2 + x + 1)(x - 3)$ at the point where it cuts the x-axis.

Solution:

$y = (x^2 + x + 1)(x - 3)$. At the x-axis $y=0$.

When $y=0$, $(x^2 + x + 1)(x - 3) = 0$; $x - 3 = 0$; $x = 3$.

Or $x^2 + x + 1 = 0$; $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2}$ (No real roots).

Hence $x = 3$ and therefore the curve cuts the x-axis at (3,0).

$\therefore$ gradient $\left(\frac{dy}{dx}\right) = (2x+1)(x-3)+1(x^2+x+1) = 2x^2-6x+x-3+x^2+x+1$

$\frac{dy}{dx} = 3x^2-4x-2$; When $x=3$, $\frac{dy}{dx} = 3(3^2)-4(3)-2 = 27-12-2 = 13$

The gradient of the tangent at (3,0) is 13, $\therefore$ the gradient of the normal at (3,0) is $-\frac{1}{13}$(since for perpendicular lines with gradients m_1 and m_2, $m_1 \times m_2 = -1$)

$\frac{\Delta y}{\Delta x} = \frac{y-0}{x-3} = -\frac{1}{13}$; $13y = -x+3$; $y = -x/13 + \frac{3}{13}$ is the equation of the normal.

3. Find the slope (gradient)of the tangent to the curve $x^2+xy+y^2=7$ at the point (1,2).

Solution:

$2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx}=0$; $(x+2y)\frac{dy}{dx} = -2x-y$; $\frac{dy}{dx} = \frac{-2x-y}{x+2y}$.

At $x=1, y=2$, $\frac{dy}{dx} = \frac{-2(1)-2}{1+2(2)} = \frac{-4}{5}$.

4. Find the equation of the tangent and a normal to the curve $x^2+xy-y^2=1$at the point (2, 3).

Solution:

Equation of the tangent

$2x+y+x\frac{dy}{dx}-2y\frac{dy}{dx}=0$; $(x-2y)\frac{dy}{dx} = -2x-y$; $\frac{dy}{dx} = \frac{-2x-y}{x-2y} = \frac{-2(2)-3}{2-2(3)} = \frac{7}{4}$ at the point *x=2, y=3* .

Equation of the normal

Recall: For perpendicular lines with gradients m_1 and m_2, $m_1 \times m_2 = -1$

$\therefore$ The gradient of the normal at the point (2, 3)is $-\frac{4}{7}$.

$\frac{\Delta y}{\Delta x} = \frac{y-3}{x-2} = -\frac{4}{7}$; $7(y-3) = -4(x-2)$; $7y-21 = -4x+8$; $7y = -4x+8+21$;

$y = \frac{-4x}{7}+\frac{29}{7} = \frac{1}{7}[29-4x]$

5. Find the normal to a curve $3xy+2y^2-x^3=0$ at the point (1,2).

Solution:

$3y+3x\frac{dy}{dx}+4y\frac{dy}{dx}-3x^2=0$; $(3x+4y)\frac{dy}{dx} = 3x^2-3y$; $\frac{dy}{dx} = \frac{3x^2-3y}{3x+4y}$

At x=1, y=2, $\frac{dy}{dx} = \frac{3-6}{3+8} = \frac{-3}{11}$; $\therefore$ Gradient of the normal to the curve at (1,2) is $\frac{11}{3}$.

$\frac{\Delta y}{\Delta x} = \frac{y-2}{x-1} = \frac{11}{3}$; $3y-6 = 11x-11$; $3y = 11x-5$; $y = \frac{11x}{3}-\frac{5}{3}$

6. Find the equations of the tangent and the normal to the curve $x^2 + 2xy - y^2 = 4$ at the point $(2,4)$.

Solution:

$2x + 2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$; $(2x - 2y)\frac{dy}{dx} = -2x - 2y$; $\frac{dy}{dx} = \frac{-2x-2y}{2x-2y} = \frac{-4-8}{4-8} = \frac{-12}{-4} = 3$ at $(2,4)$.

$\therefore$ the gradient of the tangent line at $(2,4)$ is 3.

$\therefore$ gradient of the normal to the curve is $-\frac{1}{3}$.

$\frac{\Delta y}{\Delta x} = \frac{y-4}{x-2} = -\frac{1}{3}$; $3y - 12 = -x + 2$; $3y = -x + 14$; $y = \frac{1}{3}[14 - x]$

7. The parametric equations of a curve are $x = t^2 - 4$ and $y = t^3 - 4t$. Find the equation of the tangent to the curve at the point (-3,3).

Solution:

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2 - 4$; $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3t^2 - 4}{2t}$. But $x = t^2 - 4$, $y = t^3 - 4t$.

When x=-3, $-3 = t^2 - 4$, $t^2 = 1$; $t = \pm 1$

When y=3, $3 = t^3 - 4t$; $t^3 - 4t - 3 = 0$; *when* $t = -1$, $-1 + 4 - 3 = 0$ and therefore *(t+1)* is a factor of $t^3 - 4t - 3$.

$$\begin{array}{r|l} & t^2 - t - 3 \\ \hline (t+1) & t^3 - 4t - 3 \\ & \underline{t^3 + t^2} \\ & 0 - t^2 - 4t \\ & \underline{-t^2 - t} \\ & -3t - 3 \\ & \underline{-3t - 3} \\ & 0 \end{array}$$

$(t+1)(t^2 - t - 3) = 0$; $t = -1$ or $t^2 - t - 3 = 0$; $t = \frac{1 \pm \sqrt{1+12}}{2} = \frac{1 \pm \sqrt{13}}{2}$

At t=-1, $\frac{dy}{dx} = \frac{3-4}{2(-1)} = \frac{1}{2}$;

$\frac{\Delta y}{\Delta x} = \frac{y-3}{x+3} = \frac{1}{2}$; $2(y-3) = x + 3$; $2y-6=x+3$; $2y=x+9$; $y=\frac{x}{2}+\frac{9}{2}$;

Exercise

1. Find the equation of a tangent and normal to the curve $x^2 + y^2 - 6xy + 3x - 2y + 5 = 0$ at a point (3,0).

2. Find the equation of the tangent and normal to the curve $x=\frac{t}{1+t}, \quad y=\frac{t^3}{1+t}$ at the point $\left(\frac{1}{2},\frac{1}{2}\right)$.

SMALL CHANGES

Recall:

$\frac{d}{dx}=f^{'}(x)=\lim\limits_{\delta x\to o}\frac{f(x+\delta x)-f(x)}{\delta x}$ approaches the tangent line.

$\therefore$ if δx is small, then we say that $\frac{\delta y}{\partial x}\simeq\frac{dy}{dx} \qquad \Rightarrow \delta y\simeq\frac{dy}{dx}\cdot\delta x$

This approximation can be used to estimate the value of a function close to a known value.i.e $y+\delta y$ can be approximated if y is known.

Examples

1. Use $y=\sqrt{x}$ to approximate the value of $\sqrt{1\cdot 1}$.

   Solution:

Known value $\sqrt{1}=1$.

From $\sqrt{1\cdot 1}=\sqrt{1+0\cdot 1} \quad ,x=1,\delta x=0\cdot 1$

$y=\sqrt{x}; \quad \frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$

From $\delta y\simeq\frac{dy}{dx}\cdot\delta x \quad \simeq\frac{1}{2\sqrt{x}}\cdot\delta x \quad \simeq\frac{1}{2\sqrt{1}}\times 0\cdot 1 \quad \simeq 0\cdot 05 \quad \therefore \delta y\simeq 0\cdot 05$.

$\therefore \sqrt{1\cdot 1}\simeq y+\delta y \quad \simeq\sqrt{1}+0\cdot 05; \quad \sqrt{1\cdot 1}\simeq 1\cdot 05$

2. Approximate $\ln 1\cdot 1$

   Solution:

   Known value $=\ln 1=0$

Let $y=\ln x; \qquad x=1, \quad \delta x=0\cdot 1$

$\frac{dy}{dx}=\frac{1}{x}; \qquad$ But $\delta y\simeq\frac{dy}{dx}\cdot\delta x \quad \simeq\frac{1}{x}\cdot\delta x\simeq\frac{1}{1}\times 0\cdot 1\simeq 0\cdot 1$

$\therefore \ln(1\cdot 1)\simeq y+\delta y \quad \simeq \ln x+\delta y\simeq \ln 1+\delta y \quad \simeq 0+0\cdot 1 \quad \simeq 0\cdot 1 \qquad \therefore \ln 1\cdot 1\simeq 0\cdot 1$.

3. Approximate $\sqrt{101}$.

   Solution:

   Known value=100

Let $y=\sqrt{x} \quad ,x=100, \quad \delta x=1$

$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}; \quad$ But $\quad \delta y\simeq\frac{dy}{dx}\cdot\delta x \quad \simeq\frac{1}{2\sqrt{100}}\times 1 \simeq \frac{1}{20}$

$\therefore \sqrt{101} \simeq y + \delta y \simeq \sqrt{x} + \frac{1}{20} \simeq \sqrt{100} + \frac{1}{20} = 10 + 0 \cdot 05$; $\therefore \sqrt{101} = 10 \cdot 05$.

4. By taking $1^0 = 0 \cdot 0175$ radians, approximate $\sin 29^0$.

   Solution:

   Known value $\sin 30^0 = \frac{1}{2}$; Let $y = \sin x$; $x = 30^0$; $\delta x = -1$

$\frac{dy}{dx} = \cos x$; $\delta y \simeq \frac{dy}{dx} \cdot \delta x \simeq \cos x \cdot \left(-1^0\right)$

$\simeq \cos 30^0 \times \left(-1^0\right)$; But $-1^0 = -0 \cdot 0175$ radians,

$\delta y = \frac{\sqrt{3}}{2}\left(-0 \cdot 0175\right) \simeq -\frac{\sqrt{3}}{2}\left(0 \cdot 0175\right)$

$\therefore \sin 29^0 \simeq y + \delta y \simeq \sin x + \delta y \simeq \sin 30 + \delta y \simeq \frac{1}{2} - 0 \cdot 015 \simeq 0 \cdot 4848$

5. Approximate $\sqrt[3]{65}$.

   Solution:

Known value $= \sqrt[3]{64} = 4$

Let $y = 3\sqrt{x}$, $x = 64$; $\delta x = 1$

$y = x^{\frac{1}{3}}$; $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$; $\partial y \simeq \frac{dy}{dx} \times \delta x \simeq \frac{1}{3x^{\frac{2}{3}}} \times 1 \simeq \frac{1}{3\left(\sqrt[3]{64}\right)^2} \times 1 = \frac{1}{16 \times 3} = \frac{1}{48}$;

$\therefore \sqrt[3]{65} = y + \delta y \simeq \sqrt[3]{64} + \delta y \simeq 4 + \frac{1}{48} \simeq 4 \cdot 021$

6. The side of a square is 5cm. Find the increase in the area of the square when the side expands by 0.01cm.

Solution:

Let the area of the square be $A\text{cm}^2$ when the side is x cm.

Then $A = x^2$.

Now, $\delta A \simeq \frac{dA}{dx}\delta x$ $x = 5$; $\delta x = 0 \cdot 01$

$A = x^2$; $\frac{dA}{dx} = 2x$ $\therefore \delta A \simeq 2x\left(0 \cdot 01\right) \simeq 2 \times 5\left(0 \cdot 01\right) \simeq 0 \cdot 1$

$\therefore$ the increase in the area is $\simeq 0 \cdot 1$

7. Find approximation for $\sqrt{9 \cdot 01}$

   Solution:

Known value $= \sqrt{9} = 3$

Let $y = \sqrt{x}$, $x = 9$; $\delta x = 0 \cdot 01$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$; $\delta y \simeq \frac{dy}{dx} \cdot \delta x \simeq \frac{1}{2\sqrt{x}} \times 0 \cdot 01 \simeq \frac{1}{2\sqrt{9}} \times 0 \cdot 01 \simeq \frac{1}{6} \times 0 \cdot 01 = \frac{1}{600}$

$\therefore \sqrt{9\cdot 01} = y+\delta y \simeq \sqrt{x}+\delta y \simeq \sqrt{9}+\frac{1}{600} \simeq 3+\frac{1}{600} \quad \simeq 3\cdot 00167$

8. Given that $\sin 60^0 = 0\cdot 86605, \cos 60^0 = 0\cdot 50000$,and $1^0 = 0\cdot 001745$ radians, Use $\frac{\delta y}{\partial x} \approx \frac{dy}{dx}$ to calculate the value of $\sin 60\cdot 1^0$ correct to 5.d.p.

Solution:

$y=\sin x$, $x=60^0$; $\delta x = 0\cdot 1^0$

$\frac{dy}{dx}=\cos x$; $\delta y \simeq \frac{dy}{dx}\cdot \delta x \quad \simeq \cos x\left(0\cdot 1\right)^0 \simeq \cos 60^0(0\cdot 0001745) \simeq \left(0\cdot 5\right)\left(0\cdot 0001745\right)$

$\simeq 0\cdot 00008725 \therefore \sin\left(60\cdot 1^0\right) \simeq y+\delta y \simeq \sin x+\delta y \simeq \sin 60^0 + \left(0\cdot 5\right)\left(0\cdot 0001745\right)$

$\simeq 0\cdot 86605+\left(0\cdot 5\right)\left(0\cdot 0001745\right) = 0\cdot 86613725$

MIXED EXERCISE

ATTEMPT ALL QUESTIONS

1. (a)Use the linear approximation formula to approximate $\left(626\right)^{\frac{3}{4}}$.

(b)Find $\frac{dy}{dx}$ if (i) $y=\tan^{-1}\left(x^2+1\right)$ (ii) $y=\sin^{-1}x$

(c)Find $\frac{dy}{dx^2}$ and $\frac{d^2y}{dx^2}$, given $xy+x-2y-1=0$.

2.(a)If $x=\cos t$ and $y=1-\sin^2 t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b)Use logarithmic differentiation to evaluate $\frac{dy}{dx}$ if

(i) $y=\frac{\sin x\cos x\tan^3 x}{\sqrt{x}}$ (ii) $y=\frac{\left(x^2+1\right)\cot x}{3-\cot x}$

(c)Find the equation of a tangent and normal to the curve $x^2+y^2-6xy+3x-2y+5=0$ at the point (3, 0).

3.(a)Differentiate $f\left(x\right)=\cot x$ from first principles.

(b)Find $\frac{dy}{dx}$ if (i) $y=4^x$ (ii) $y=\ln\left(\cot x-\cos ecx\right)$ (iii) $y=x\sin^{-1}\left(3x\right)-\sqrt{1-9x^2}$ (iv) $y=\ln\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}$