AP Calculus	Quotient Rule

Quotient Rule

The derivative of the quotient $\frac{f}{g}$ of two differentiable functions *f* and *g* is itself differentiable at all values of *x* for which $g(x) \neq 0$. Moreover, the derivative of $\frac{f}{g}$ is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator. Symbolically, $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}, g \neq 0$.

$$OR \quad \frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\,u' - u\,v'}{v^2}, v \neq 0$$

Examples:

(a) Given: $f(x) = \frac{2x+1}{x-3}$. Find $f'(x)$.

Let $u = 2x + 1$ and $v = x - 3$

So $u' = 2$ and $v' = 1$

$$f'(x) = \frac{(x-3)(2) - (2x+1)(1)}{(x-3)^2} = \frac{2x - 6 - 2x - 1}{(x-3)^2} = \frac{-7}{(x-3)^2}$$

(b) Given: $y = \frac{x^2+6}{2x-7}$. Find y'.

$Let\ u = x^2 + 6 \ \ so \ \ u' = 2x$
$Let\ v = 2x - 7 \ \ so \ \ v' = 2$

$$y' = \frac{(2x-7)(2x) - (x^2+6)(2)}{(2x-7)^2} = \frac{4x^2 - 14x - 2x^2 - 12}{(2x-7)^2}$$
$$= \frac{2x^2 - 14x - 12}{(2x-7)^2}$$

over

(c) Write the equation of the line tangent to $f(x)=\dfrac{x}{x-1}$ at $x=2$.

$$f'(x) = \frac{(x-1)(1)-(x)(1)}{(x-1)^2} = \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$
$$f'(2) = \frac{-1}{(2-1)^2} = \frac{-1}{1^2} = -1$$

NOTE: NOT IN NOTES!!!!

To find the derivative of the sine function, use the limit process and difference quotient:

$$f(x) = \sin x$$

$$f'(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\,(\cos h-1)}{h} + \lim_{h\to 0}\frac{\cos x\sin h}{h}$$

Recall the highlighted limits are special cases and we know the answers to those:

$$f'(x) = (\sin x)(0)+(\cos x)(1) = \cos x$$

Therefore the derivative of the sine function is cosine function.

Now find the derivative of the cosine function:

$$f(x) = \cos x$$

$$f'(x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\,(\cos h-1)}{h} - \lim_{h\to 0}\frac{\sin x\sin h}{h}$$

$$= (\cos x)(0)-(\sin x)(1) = -\sin x$$

Therefore the derivative of the sine function is negative sine function.

over

Derivatives of Trigonometric Functions

Examples:

(a) Given: $f(x) = \tan x$. Find $f'(x)$.

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

$$f'(x) = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(cos^2 x)} = \frac{cos^2 x + sin^2 x}{cos^2 x} = \frac{1}{cos^2 x} = sec^2 x$$

(b) Given: $f(x) = \sec x$. Find $\frac{d}{dx} f(x)$.

$$f(x) = \frac{1}{\cos x}$$

$$\frac{d}{dx} f(x) = \frac{(\cos x)(0) - (1)(-\sin x)}{cos^2 x} = \frac{\sin x}{cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \tan x \sec x$$

(c) Given: $f(x) = \csc x$. Find $f'(x)$.

$$f(x) = \frac{1}{\sin x}$$

$$f'(x) = \frac{(\sin x)(0) - (1)(\cos x)}{sin^2 x} = \frac{-cosx}{sin^2 x} = \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$= -\cot x \csc x$$

over

(d) Given: $y = \dfrac{1-\cos x}{\sin x}$. Find y'.

$$y' = \frac{(\sin x)(\sin x) - (1-\cos x)(\cos x)}{sin^2 x}$$

$$y' = \frac{sin^2 x - (\cos x - cos^2 x)}{sin^2 x} = \frac{sin^2 x - \cos x + cos^2 x}{sin^2 x}$$

$$y' = \frac{1-\cos x}{sin^2 x} = \frac{1}{sin^2 x} - \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = csc^2 x - \cot x \csc x$$

Higher-Order Derivatives

Notation:

First	$f'(x)$	y'	$\dfrac{dy}{dx}$	$\dfrac{d}{dx} f(x)$
Second	$f''(x)$	y''	$\dfrac{d^2y}{dx^2}$	$\dfrac{d^2}{dx^2} f(x)$
Third	$f'''(x)$	y'''	$\dfrac{d^3y}{dx^3}$	$\dfrac{d^3}{dx^3} f(x)$
Fourth	$f^{(4)}(x)$	$y^{(4)}$	$\dfrac{d^4y}{dx^4}$	$\dfrac{d^4}{dx^4} f(x)$

Examples:

(a) Given: $y = x^4 + 2x^3 - 5x^2 + 7x - 10$. Find y''.

$$y' = 4x^3 + 6x^2 - 10x^1 + 7$$

$$y'' = 12x^2 + 12x - 10$$

(b) Given: $f(x) = \sin x$. Find $f^{(4)}(x)$.

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f^{(4)}(x) = \sin x$$