BSc. Actuarial Science: First Year First Semester
STA 2104 Calculus for Statistics I

By Omari C.O

STA 2104 Calculus for Statistics I is the first course in the Calculus sequence (Calculus I, II, III) offered by the Department of Statistics and Actuarial Science in DeKUT. It is a 4-credit-hour course conducted through 3 lectures and 1 tutorial class (a problem-solving class).

Objectives: To introduce students to the three main topics of Calculus; namely, Limits, Continuity, and Differentiation (methods and applications).

Learning Outcomes: After completing the course, students are expected to be able to:

1. Evaluate various limit problems both algebraically and graphically, and also evaluate limits with indeterminate forms using L'hopital's rule,

2. Check the continuity of various types of functions,

3. Differentiate various types of functions using the differentiation rules: Powers, Sum, Difference, Product, Quotient Rules, Implicit and Logarithmic Differentiation,

4. Apply differentiation to find linear approximation, extrema, monotonicity, and concavity of functions, and apply differentiation to solve some optimization problems,

5. Find antiderivative of some simple functions.

Course Outline: Functions, Limits, Continuity and differentiability. Differentiation by first principles and by rule for x^n (integral and fractional n), sums, products, quotients, chain rule, trigonometric, logarithmic and exponential functions of a single variable. Parametric differentiation. Applications: equations of tangent and normal, stationary points, rates of change, small changes and approximations. Integration: anti-derivatives and their applications to marginal and total functions e.g. marginal cost and total cost.

Teaching Methodology: The method of instruction will be lectures, interactive tutorials, and any other presentations/ demonstrations the lecturer will deem fit towards enhancing understanding of the concepts taught in class. Special attention will be given in the implementation of the methods taught using specialized software packages.

Instruction Materials/Equipment: Whiteboard, and Handouts.

Course Evaluation: The final grade for the course will be based on a final examination at the end of the semester (70%) and the Continuous Assessment Tests (CATs) (30%) which will be based on assignments throughout the semester and at least Two sit-in CATs.

Pre-requisites: All skills from Algebra I, Geometry, Algebra II, and Precalculus are assumed.

Co-requisites: SMA 2104 Mathematics for Science and STA 2100 Discrete Mathematics

Reference Books

1. S J Salas and E Hille Calculus: One and Several Variables. 7th ed. Wiley, 1995
2. Stewart. Calculus Concepts and Contexts: Multi-variable and Single Variable. Brooks/Cole Pub Co, 2004.
3. Thomas & Finney. Calculus and Analytical Geometry. 7th. Addison-Wesley, 1988
4. Edwards, C. H. Multivariable Calculus With Analytic Geometry, 5th. Prentice Hall, 1997
5. Larson, Ron; Hostetler, Robert P.; Edwards, Bruce H. Calculus With Analytic Geometry, 8th ed. Houghton Mifflin College, 2005

Make-up Policy: No makeup exams will be allowed except with proper documentation, i.e. University doctor's note, or DeKUT excused absence document.

Attendance Policy: Students must attend every class. Students are to arrive on time to class. It is the student's responsibility to find out what assignment must be made up when they are absent.

Civility Statement: Please turn off cell phones when you enter class and participate in class, active participation in this class is a vital part of your success.

Academic Integrity Policy: Each student is responsible for notice of and compliance with the provisions of the University Rules and Regulations, which are available for inspection electronically at
`http://www.dkut.ac.ke`

Week 1: Introduction of the Course/Preliminary Review: Functions

Week 2: Functions:

Week 3: Limits:

Week 4: Continuity and differentiability:

Week 5: Differentiation by first principles

Week 6: Differentiation by rule for x^n (integral and fractional n) and chain rule,

Week 7: Sums, product Rule and quotient Rule

Week 8: Trigonometric, logarithmic and exponential functions of a single variable. L'Hopital's rule.

Week 9: Parametric differentiation.

Week 10: Applications: equations of tangent and normal, rates of change and stationary points.

Week 11: Integration: anti-derivatives

Week 12: Integrals and their applications to marginal and total functions e.g. marginal cost and total cost.

Week 13: Applications

Week 14: Applications

Week 14: CAT III

Pre-Requisites: None

THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Lecture 1: Functions

By Omari C.O

In this lecture we're going to introduce functions and function notation. Both will appear in almost every section in a Calculus course and so you will need to be able to deal with them.

1 Functions

In many naturally occurring phenomena, two variables may be linked by some type of relationship. For example, the wind chill depends on the speed of the wind; the area of a circle depends on its radius; and the amount earned by your investment depends on the interest rate.

Any set of ordered pairs (x, y) is called a **relation in x and y**. Furthermore,

- The set of first components in the ordered pairs is called the **domain of the relation**.
- The set of second components in the ordered pairs is called the **range of the relation**.

Example 1.1. Find the domain and range of the relation linking the length of a woman's femur to her height

$$\{(45.5, 65.5), (48.2, 68.0), (41.8, 62.2), (46.0, 66.0), (50.4, 70.0)\}.$$

Solution.

$$\text{Domain:} \quad \{45.5, 48.2, 41.8, 46.0, 50.4\} \quad \text{Set of first coordinates}$$

$$\text{Range:} \quad \{65.5, 68.0, 62.2, 66.0, 70.0\} \quad \text{Set of second coordinates}$$

Question 1.1. Find the domain and range of the relation.

$$\left\{(0,0), (-8,4), \left(\frac{1}{2}, 1\right), (-3,4), (-8,0)\right\}$$

A relation may consist of a finite number of ordered pairs or an infinite number of ordered pairs. Furthermore, a relation may be defined by several different methods: by a list of ordered pairs, by a correspondence between the domain and range, by a graph, or by an equation.

Question 1.2. The linear equation, $y = -0.014x + 64.5$, relates the weight of a car, x, (in pounds) to its gas mileage, y, (in mpg).

a. Find the gas mileage in miles per gallon for a car weighing 2550 lb.

b. Find the gas mileage for a car weighing 2850 lb.

1.1 Definition of Function

Functions are the major objects we deal with in Calculus because they are key to describing the real world in mathematical terms.

Given a relation in x and y, we say "y **is a function of** x" if for every element x in the domain, there corresponds exactly one element y in the range.

A **function** is a relation in which for each value of the first component of the ordered pairs there is exactly one value of the second component.

A **function** is a relation in which no two different ordered pairs have the same first element.

$$F = \{(2,3),(5,7),(4,-2),(9,1),(8,4),(6,3)\}$$

A **function** can also be defined as a rule that assigns exactly one element in set B to each element in set A.

- Set A, the set of all first coordinates, is called the **domain of the function.**
- Set B, the set of all second coordinates, is called **the range of the function.**

In the function F above, the domain is the set of first elements: $\{2,5,4,9,8,6\}$ and the range is the set of second elements: $\{3,7,.2,1,4\}$.

The function F is depicted pictorially below:

The characteristic that distinguishes a function from any other relation is that there is only one output (y-value) for each input (x-value).

Example 1.2. Which of the following relations are functions?

(a) $\{(2,3),(-1,3),(5,3),(8,3),(0,3)\}$

(b) $\{(-3,4),(-2,3),(-1,2),(0,1),(1,0)\}$

(c) $\{(1,5),(2,7),(3,9),(1,4)\}$

(d) $\{(-1,6),(8,9),(-1,4),(-3,10)$

Solution. (a) Function. Although each output is the same, 3, there is only one output for each input so the relation is a function.

(b) Function. Each x is paired with only one y.

(c) Not a function. 1 is paired with two outputs: 5 and 4. We have two ordered pairs with the same first element.

Question 1.3. Let D and R be two sets of real numbers. In figure.... we show four rules that assign numbers in D to numbers in R. Which of the four rules represent functions with domain D?

(a). f_1 is not a function because it assigns two different numbers of R to one member of D. That is $f(b) = u$ and w. This contradicts the definition of a function , f_1 must assign **exactly one** member of R to each member of D.

(b). f_2 is not a function with domain D because f does not assign anything to c. In order to be a function with domain D, f must assign one member of R to each member of D.

(c). f_3 is a function because it assigns one member of R to each member of D. Despite nothing in D being assigned x, it does not contradict the definition of a function. The ran ge of f_3 is $\{u,\ v,\ w,\ y\}$ which is not equal to R, the codomain.

(d). f_4 is a function because it assigns exactly one member of R to each member of D. The fact that u is assigned three times does not contradict the definition of a function.

1.2 Function Notation

A function is defined as a relation with the added restriction that each value in the domain must have only one corresponding y-value in the range. In mathematics, functions are often given by rules or equations to define the relationship between two or more variables. For example, the equation $y = 2x + 5$ defines the set of ordered pairs such that the y-value is twice the x-value. When a function is defined by an equation, we often use **function notation**. The equation $y = 2x + 5$ can be written in function notation as

$$f(x) = 2x + 5$$

where f is the name of the function, x is an input value from the domain of the function, and $f(x)$ is the function value (or y-value) corresponding to x.

The notation $f(x)$ is read as "f of x" or "the value of the function f at x", instead of y to give the output for the input x. That is, $y = f(x)$. The use of $f(x)$ instead of y allows to state that f is a function and it gives us the input value x as well. In writing $y = f(x)$, we must distinguish between the symbol f, which stands for the function rule, and the symbol $f(x)$, which is the value of the function takes on for a given number x in the domain of f. Hence $f(x)$ is a number in the range of $f(x)$.

Input: (x)	Function Rule: $f(x) = 5x - 2$	Output: $f(x)$	Ordered Pair: $(x, f(x))$
5	$f(5) = 5(5) - 2$	23	$(5, 23)$
2	$f(2) = 5(2) - 2$	8	$(2, 8)$
1	$f(1) = 5(1) - 2$	3	$(1, 3)$
0	$f(0) = 5(0) - 2$	-2	$(0, -2)$

We write that $f(5) = 23$, meaning that the output (y) is 23 when the input (x) is 5.

The rule that defines the function can be described in several different ways, including a table, by a graph, or in a statement form. Perhaps the most common way of describing the function is an algebraic expression.

An equation will be a function if for any x in the domain of the equation (the domain is all the x's that can be plugged into the equation) the equation will yield exactly one value of y. For example, if $f(x) = 3x + 5$, then the line can be thought of as the set of all ordered pairs (or points) $(x, f(x))$ such that $y = 3x + 5$. The important fact here being that for every real number x there is a unique real number y such that $y = 3x + 5$ and the ordered pair (x, y) is on the line. This idea is generalized in the following definition.

Definition 1.1. *Let X and Y be sets of real numbers. A real-valued* ***function*** *f is a rule that assigns to each number x in X a (unique) exactly one real number $f(x)$ in Y.*

We often use the notation f or F to denote a function if there is a rule that describes the function. If there is a rule that assigns a unique member of a set Y for every member of a set X, then the function f can also be expressed as:

$$f : X \to Y$$

read as 'f is such that X maps onto Y'.

We often shorten the way we denote functions in the following manner: for example, for $f = \left\{(x, y)|y = 3x - \frac{1}{2}\right\}$, then f can simply be described by the rule given by $f(x) = 3x - \frac{1}{2}$.

A function (or mapping) is a collection of ordered pairs in which no two different pairs have the same first component. The names of functions are often given by either lowercase or uppercase letters, such as f, g, h, p, K, and M.

Example 1.3. Determine if each of the following are functions:

(a) $y = x^2 + 3$

(b) $y^2 = x + 1$

Solution. (a) The first one is a function. Given an x, there is only one way to square it and then add 3 to the result. So, no matter what value of x you put into the equation, there is only one possible value of y.

(b) This is not a function. Choose a value of x, say $x = 8$ and plug this into the equation

$$y^2 = 8 + 1 = 9.$$

Now, there are two possible values of y that we could use here. We could use $y = 3$ or $y = -3$. Since there are two possible values of y that we get from a single x, this equation isn't a function.

Example 1.4. Consider the function $f : R \to R$ given by

$$f(x) = x^2 + 2$$

We may write this as $y = x^2 + 2$ to represent this function. For each input x, the function gives exactly one output $x^2 + 2$, which is y. If $x = 3$, then $y = 11$, if $x = 6$, then $y = 38$ etc. The independent variable is x and the dependent variable is y.

1.3 Evaluating Functions

A function may be evaluated at different values of x by substituting x-values from the domain into the function. To evaluate the value of a function f, we find the output for a given input. For example, to evaluate the function defined by $f(x) = 2x^2 - 5x + 4$ at $x = -3$, substitute $x = -3$ into $f(x)$ to obtain the functional value (y-value) $f(-3)$.

$$\begin{aligned}
f(x) &= 2x^2 - 5x + 4 \\
f(-3) &= 2(-3)^2 - 5(-3) + 4 \\
&= 2(9) + 15 + 4 \\
&= 37 \\
&\Rightarrow f(-3) = 37
\end{aligned}$$

Example 1.5. Let $f(x) = x^2 - 3x + 1$. Find (a) $f(2)$ and (b) $f(-5)$.

Solution. Here,

$$\begin{aligned}
f(x) &= x^2 - 3x + 1 \\
f(2) &= 2^2 - 3(2) + 1 = -1 \\
f(-5) &= (-5)^2 - 3(-5) + 1 = 41
\end{aligned}$$

Example 1.6. Given $f(x) = -x^2 + 6x - 11$ find each of the following.

(a) $f(2)$

(b) $f(-20)$

(c) $f(t)$

(d) $f(t-3)$

(e) $f(x-2)$

(f) $f(5x-1)$

Example 1.7. Let $g(x) = x^2 - 3x + 7$. Find the following

(a) $g(10)$

(b) $g(a+2)$

(c) $g(r^2)$

(d) $g(x+h)$

(e) $\dfrac{g(x+h) - g(x)}{h}$

Question 1.4. Let $f(x) = \dfrac{x-5}{x^2+4}$. Find the following: (a). $f(2)$ (b). $f(3.5)$ (c). $f(a+5)$ (d). $f(\sqrt{a})$ (e). $f(a^2)$ (f). $f(a) + f(5)$

Question 1.5. Let $f(x) = \dfrac{x}{x+1}$ and $g(x) = \sqrt{x-1}$. Find the following:

(a) $f(1) + g(1)$

(b) $f(2) \cdot g(2)$

(c) $\dfrac{f(4)}{g(4)}$

(d) $f(a-1) + g(a+1)$

(e) $f(a^2+1) \cdot g(a^2+1)$

1.4 Domain of a Function

A function is a relation, and it is often necessary to determine its domain and range. Consider a function defined by the equation $y = f(x)$. The **domain** of a function is the set of all values that can be plugged into a function and have the function exist and have a real number for a value. So, for the domain we need to avoid division by zero, square roots of negative numbers, logarithms of zero and logarithms of negative numbers (if not familiar with logarithms we'll take a look at them a little later), etc.

Definition 1.2. ***Domain of a function:*** *The* ***domain*** *of a function is the set of all x-values that when substituted into the function, produce a real number.*

To find the domain of a function defined by $y = f(x)$ keep these guidelines in mind.

- Exclude values of x that make the denominator of a fraction zero.
- Exclude values of x that make a negative value within a square root.

Example 1.8. Let $f : \{m, n, u, v, w\} \to \{1, 2, 3\}$ be defined by $f(m) = 1$, $f(u) = 2$, $f(v) = 3$ and $f(w) = 3$. Determine the domain of this function.

Solution. The domain of the function is given by $D_f = \{m, n, u, v, w\}$.

When the domain of a function has not been specified, we assume the domain to be the largest set that the rule can be used on. This largest set is called the **maximal domain** (or the domain of definition or simply just the domain).

To determine the domain of a function if it is not given explicitly ask " **for what values of x does the rule given in the equation make sense?**", For example given $y = 1/x$, $1/x$ is not defined for $x = 0$ hence 0 is not in the domain of f. Therefore *dom* $f = \mathbb{R} \setminus \{0\}$ or $\mathbb{R} - \{0\}$ or $D_f : x \neq 0$.

Example 1.9. Determine the maximal domain of $y = x^2 + 5$.

Solution. The maximal domain of this function is $\mathbb{R}$.

Example 1.10. Let $f(x) = \dfrac{x+2}{x^2-9}$. Determine the maximal domain of the function.

Solution. A number x is in the domain of this f if and only if it does not make the denominator 0, that is, if and only if $x^2 - 9 \neq 0$. Since $x^2 - 9 = 0$ means $x = 3$ or $x = -3$, it follows that the maximal domain consists of all real numbers except $x = 3$ and $x = -3$, denoted as $D_f = \{\mathbb{R} \setminus \{-3, 3\}\}$.

Example 1.11. Find the maximal domain of the function given by $\dfrac{1}{\sqrt{x^2-16}}$.

Solution. $\sqrt{x^2-16} \neq 0$, Therefore, $x \neq \pm 4$, and $x^2 - 16 > 0$. The domain consists of all x such that $x^2 - 16 > 0$. That is $(x-4)(x+4) > 0$. Thus either $x + 4 > 0$ and $x - 4 > 0$ OR $x + 4 < 0$ and $x - 4 < 0$. Now $x + 4 > 0 \to x > -4$ and $x - 4 > 0 \to x > 4$, Therefore, $(-4, \infty) \cup (4, \infty) = (4, \infty)$.

Also $x + 4 < 0 \to x < -4$ and $x - 4 < 0 \to x < 4$, therefore, $(-\infty, -4) \cap (-\infty, 4) = (-\infty, -4)$. $x < -4$ or $x > 4$ i.e. $x \neq \pm 4$. Thus the maximal domain is $(-\infty, -4) \cup (4, \infty)$.

Example 1.12. Find the domain of $y = \sqrt{3x - 12}$.

Solution. To get the domain of the function $\sqrt{3x-12} \geq 0$ and the convention is that one takes the positive square root only,

$$\begin{aligned} 3x - 12 &\geq 0 \\ 3x &\geq 12 \\ x &\geq 4 \end{aligned}$$

Hence, the domain of this function is given by $x \in [4, \infty)$.

Example 1.13. Find the domain of the functions. Write the answers in interval notation.

(a) $f(x) = \dfrac{x+7}{2x-1}$

(b) $f(x) = \dfrac{x-4}{x^2+9}$

(c) $f(x) = \sqrt{x+4}$

(d) $f(x) = x^2 - 3x$

Solution. (a) The function will be undefined when the denominator is zero, that is, when

$$\begin{aligned} 2x - 1 &= 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \quad \text{The value } x = \frac{1}{2} \text{ must be excluded from the domain.} \end{aligned}$$

Interval notation: $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$.

(b) The quantity x^2 is greater than or equal to 0 for all real numbers x, and the number 9 is positive. Therefore, the sum $x^2 + 9$ must be positive for all real numbers x. The denominator of $f(x) = (x-4)/(x^2+9)$ will never be zero; the domain is the set of all real numbers.
Interval notation: $(-\infty, \infty)$.

(c) The function defined by $f(x) = \sqrt{x+4}$ will not be a real number when $x + 4$ is negative; hence the domain is the set of all x-values that make the radicand greater than or equal to zero:

$$\begin{aligned} x + 4 &\geq 0 \\ x &\geq -4 \end{aligned}$$

Interval notation: $[-4, \infty)$.

(d) The function defined by $g(t) = t^2 - 3t$ has no restriction on its domain because any real number substituted for t will produce a real number. The domain is the set of all real numbers.
Interval notation: $(-\infty, \infty)$

Example 1.14. Find the domain of the function f with formula

$$f(x) = \frac{1}{\sqrt{2x+4.}}$$

Solution. For the square root $\sqrt{2x+4}$ to be defined, it is necessary that $2x+4 \geq 0$. This holds when $2x \geq -4$ and thus when $x \geq -2$. For the reciprocal $1/\sqrt{2x+4}$ to be defined, we also require that $\sqrt{2x+4} \neq 0$ and thus that $x \neq -2$. Hence the domain of f is the interval $D = (-2, \infty)$.

The domain of most algebraic functions is the set of all real numbers EXCEPT:

1. any real numbers that cause the denominator of a fraction to be 0.
2. any real numbers that cause us to take the square root (or any radical with an even index) of a negative number.

There are other exceptions too, but these two are the ones we will encounter most often.

Question 1.6. Write down the domain of the functions in interval notation.

(a) $f(x) = \dfrac{2x+1}{x-9}$ $\qquad (-\infty, 0) \cup (9, \infty)$

(b) $f(x) = \dfrac{-5}{4x^2+1}$ $\qquad (-\infty, \infty)$

(c) $f(x) = \sqrt{x-2}$ $\qquad [2, \infty)$

(c) $h(x) = x + 6$ $\qquad (-\infty, \infty)$

Question 1.7. Find the domain of each of the following functions:

(a) $f(x) = \dfrac{x-4}{x^2-2x-15}$

(b) $g(t) = \sqrt{6+t-t^2}$

(c) $h(x) = \dfrac{x}{\sqrt{x^2-9}}$

1.5 Range of Functions

The **range** of f is the set of all y-values corresponding to the values of x in the domain.

The set of all values $y = f(x)$ is called the **range** of f. That is, the range of f is the set

$$\{y : y = f(x) \qquad \text{for some } x \text{ in } D\}$$

The set X is called the **domain** of the function (written as D_f or dom_f) and set Y is called the **codomain** of the function (written as cod_f).

The range of a function might not be its whole codomain. The **range** is the particular set of values in the **codomain** that the function actually maps elements of the domain to. The **codomain** is the set that the function is declared to map all domain values into.

When the domain and codomain of a function is clear we may simply describe the function by just writing the rule down. We often use the letter y to denote a function, for example

$y = 3x - \frac{1}{2}$. In this case x is called the *independent variable* and y is called the *dependent variable.*

Not all the elements of the codomain may be the result of applying some rule to the domain. The set of the actual members of the codomain that are included in the set of ordered pairs that describe a function is called the *image (or range)* of the function, (written as im_f or R_f). Simply put, the set of images of elements of x is called the range of f.

To determine the range of a function we must ask, "**What values do we obtain for y as x takes on all values in dom_f?**" For example, if f is defined by $f(x) = x^2$, then, the domain of f is $\mathbb{R}$ since any real number can be squared. The range of f is the set of nonnegative real numbers, denoted by $\mathbb{R}^+$, because the square of any real number is nonnegative, and any nonnegative real number is the square of a real number.

Example 1.15. Let $f : \{m, n, u, v, w\} \to \{1, 2, 3\}$ be defined by $f(m) = 1$, $f(u) = 2$, $f(v) = 3$ and $f(w) = 3$. Determine the range of this function.

Solution. The range of the function is given by $R_f = \{1, 2, 3\}$.

To find the range of a function $y = f(x)$, we solve the equation $y = f(x)$ to get an equation where x is now the dependent variable and y is the independent variable. If we denote the result of this by $x = g(y)$, then the **range** of $f(x)$ is precisely the domain of $g(y)$.

Steps to find range of a Function

To find the range of a function $f(x)$ described by formula, where the domain is taken to be the natural domain:

(1) Put $y = f(x)$

(2) Solve x in terms of y

(3) The range of $f(x)$ is the set of all real numbers y such that x can be solved.

Example 1.16. Find the range of $f(x) = \dfrac{1}{x}$.

Solution. Let $y = \dfrac{1}{x}$, then $x = \frac{1}{y}$, $y \neq 0$ and domain of this function (the set of all y's which are valid) is $\mathbb{R}\backslash\{0\}$. Thus range of $f(x)$ is $\mathbb{R}\backslash\{0\}$

Example 1.17. Find the range of $y = \sqrt{3x - 12}$.

Solution. As $y = \sqrt{3x - 12}$ and the convention is that one takes the positive square root only, we record that we must have $y \in [0, \infty)$ before we rearrange the equation.
Now $y = \sqrt{3x - 12} \Rightarrow y^2 = 3x - 12 \Rightarrow x = \dfrac{y^2 + 12}{3}$. The domain of this function $g(y)$ is $(-\infty, \infty)$, and so combining this with $y \in [0, \infty)$ we get range $= (-\infty, \infty) \cap [0, \infty) = [0, \infty)$.

Example 1.18. Find the range of $y = \sqrt{1 - x^2}$

Solution. If $y = \sqrt{1-x^2}$ then we note that $y \in [0, \infty)$. Now $y = \sqrt{1-x^2} \Rightarrow y^2 = 1 - x^2 \Rightarrow x = \sqrt{1-y^2}$. The domain of this function is $[-1, 1]$ and so range is $[-1, 1] \cap [0, \infty) = [0, 1]$

Example 1.19. Find the range of $y = x^2 + 5$.

Solution. Here

$$\begin{aligned} x^2 &= y - 5 \\ x &= \sqrt{y-5} \qquad \therefore y - 5 \geq 0, \quad y \geq 5. \end{aligned}$$

Hence range is $[5, , \infty)$.

Example 1.20. For each of the following functions, find its range

(a) $f(x) = x^2 + 2$

(b) $g(x) = \dfrac{1}{x-2}$

(c) $h(x) = \sqrt{1+5x}$

Solution. (a) Put $y = f(x) = x^2 + 2$. Solve for x

$$\begin{aligned} x^2 &= y - 2 \\ x &\pm \sqrt{y-2} \end{aligned}$$

Note that x can be solved if and only if $y - 2 \geq 0$.
The range of f is $\{y \in \mathbf{R} : y - 2 \geq 0\} = \{y \in R : y \geq 2\} = [2, \infty)$.

(b) Put $y = g(x) = \dfrac{1}{x-2}$.

$$\begin{aligned} \text{Solve for } x : y &= \frac{1}{x-2} \\ x - 2 &= \frac{1}{y} \\ x &= \frac{1}{y} + 2. \end{aligned}$$

Note that x can be solved if and only if $y \neq 0$. The range of g is $\{y \in \mathbb{R} : y \neq 0\} = R\{0\}$.
(c). Put $y = h(x) = \sqrt{1+5x}$. Note that y cannot be negative. Solve for x:

$$y = \sqrt{1+5x}, \quad y \geq 0 y^2 = 1 + 5x, \quad y \geq 0 x = \frac{y^2-1}{5}, \quad y \geq 0.$$

Note that x can always be solved for every $y \geq 0$. The range of h is $\{y \in \mathbb{R} : y \geq 0\} = [0, \infty)$.

Remark: $y = \sqrt{1+5x} \to y^2 = 1 + 5x$, but the converse is true only if $y \geq 0$.

Example 1.21. Find the domain and range of each of the following functions:

(a) $f(x) = 5x - 3$

(b) $g(t) = \sqrt{4 - 7t}$

(c) $h(x) = -2x^2 + 12x + 5$

(d) $f(z) = |z - 6| - 3$

(e) $g(x) = 8$

Solution. (a) We know that this is a line and that it's not a horizontal line (because the slope is 5 and not zero). This means that this function can take on any value and so the range is all real numbers. using "mathematical" notation this is:

$$\text{Range: } (-\infty, \infty).$$

The domain is also all real numbers or,

$$\text{Domain: } -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

(b) $g(t) = \sqrt{4 - 7t}$. This is a square toot and we know that square roots are always positive or zero and because we can have the square root of zero in this case

$$g\left(\frac{4}{7}\right) = \sqrt{4 - 7\left(\frac{4}{7}\right)} = \sqrt{0} = 0$$

We know then that the range will be

$$\text{Range: } [0, \infty)$$

For the domain, we need to make sure that we don't take square roots of any negative numbers and so we need to require that:

$$\begin{aligned} 4 - 7t &\geq 0 \\ t &\geq 7t \\ \frac{4}{7} &\geq t \quad \Rightarrow \quad t \leq \frac{4}{7} \end{aligned}$$

The domain is then;

$$\text{Domain: } t \leq \frac{4}{7} \quad \text{or} \quad \left(-\infty, \frac{4}{7}\right]$$

Example 1.22. Find the maximal domain and range of $y = \dfrac{1}{x^2 - 1}$.

Solution. Domain $x^2 - 1 \neq 0 \qquad \Rightarrow x^2 \neq 1 \Rightarrow x \neq \pm 1$
Domain: $\mathbb{R}\setminus\{1, -1\}$
Range:

$$y = \frac{1}{x^2 - 1} \Rightarrow x^2 y - y = 1 \Rightarrow x = \sqrt{\frac{1+y}{y}} \Rightarrow y \neq 0$$

and $\dfrac{1+y}{y} \geq 0 \Rightarrow y \neq 0$ and $y(1+y) \geq 0$. (we multiplied the inequality by y^2 so as not to change the sign.

Question 1.8. Determine the largest domain and range for the following functions

$$\begin{array}{lll}
\text{(i). } y = x^2 & \text{(ii). } y = 3x + 1 & \text{(iii). } y = x^2 + 1 \\
\text{(iv). } y = 1 - x^2 & \text{(v). } y = \sqrt{x^2 - 16} & \text{(vi). } y = \dfrac{1}{1-x} \\
\text{(vii). } y = \sqrt{4 - x^2} & \text{(viii). } y = \dfrac{1}{x^2 - 25} & \text{(ix). } y = \sqrt{9 - x^2}
\end{array}$$

Question 1.9. Is $y = f(x) = \sqrt{x}$ a function?

Note: The set of the actual members of the codomain that are included in the set of ordered pairs that describe a function is called the **image** (or **range**) of the function.

1.6 Combinations of Functions

At first we concentrate on simple functions, because many varied and complex functions can be assembled out of simple "building-block functions." Here we discuss some of the ways of combining functions to obtain new ones. Suppose that f and g be two functions with domains D_f and D_g respectively and that c is a fixed real number. The **(scalar)multiple** cf, the **sum** $f + g$, **difference** $f - g$, the **product** $f \cdot g$ and the **quotient** $\dfrac{f}{g}$ are the new functions determined by these formulas: defined by:

$$\begin{aligned}
(cf)(x) &= cf(x), \\
(f + g)(x) &= f(x) + g(x), \\
(f - g)(x) &= f(x) - g(x), \\
(f \times g)(x) &= f(x) \times g(x), \text{ and} \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \qquad \text{provided} \qquad g(x) \neq 0
\end{aligned}$$

Example 1.23. Let $f(x) = x^2 + 1$ and $g(x) = x - 1$. Then

$$\begin{aligned} 3f(x) &= 3(x^2 + 1) \\ (f+g)(x) &= (x^2+1) + (x-1) = x^2 + x, \\ (f-g)(x) &= (x^2+1) - (x-1) = x^2 - x + 2, \\ (f \cdot g)(x) &= (x^2+1)(x-1) = x^3 - x^2 + x - 1, \quad \text{and} \end{aligned}$$

$$\left(\frac{f}{g}\right)(x) = \frac{x^2+1}{x-1} \qquad (x \neq 1).$$

Example 1.24. If $f(x) = \sqrt{1-x}$ for $x \leq 1$ and $g(x) = \sqrt{1+x}$ for $x \geq -1$, then the sum and product of f and g are defined where *both* f and g are defined. Thus the domain of both

$$f(x) + g(x) = \sqrt{1-x} + \sqrt{1+x}$$

and

$$f(x) \cdot g(x) = \sqrt{1-x}\sqrt{1+x} = \sqrt{1-x^2}$$

is the closed interval $[-1, 1]$. Bur the domain of the quotient

$$\frac{f(x)}{g(x)} = \frac{\sqrt{1-x}}{\sqrt{1+x}} = \sqrt{\frac{1-x}{1+x}}$$

is the half-open interval $(-1, 1]$, because $g(-1) = 0$.

Example 1.25. Find the sum, difference, product, and quotient of the functions $f(x) = 3x - 2$ and $g(x) = x^2 - 4$.

Solution.

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) = 3x - 2 + x^2 - 4 \\ &= x^2 + 3x - 6 \end{aligned}$$

$$\begin{aligned} (f-g)(x) &= f(x) - g(x) = 3x - 2 - (x^2 - 4) \\ &= 3x - 2 - x^2 + 4 = -x^2 + 3x + 2 \end{aligned}$$

$$\begin{aligned} (f * g)(x) &= f(x)g(x) = (3x-2)(x^2-4) \\ &= 3x^3 - 2x^2 - 18x + 8 \end{aligned}$$

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{3x-2}{x^2-4}$$

Example 1.26. If $f(x) = 4x - 16$ and $g(x) = \sqrt{x+16}$, find the product and quotient of the functions and their domain of definition.

Solution. Let $f(x)\cdot g(x) = (4x-16)(\sqrt{x+16})$ The domain of $D_{f(x)}$ is $\mathbb{R}$ and $D_{g(x)}$ is $x \geq -16$, therefore $D_{f(x)g(x)}$ is $x : x \geq -16$

$D_{f(x)g(x)} = \{x : x \in D_{f(x)} \text{ and } x \in D_{g(x)}\}$ or $D_{f(x)} \cap D_{g(x)}$.

Quotient

$$\frac{f(x)}{g(x)} = \frac{4x-16}{\sqrt{x+16}}$$

Provided $x+16 > 0 \Rightarrow x > -16$. Hence the domain of definition of the quotient is the interval $(-16, +\infty)$.

Note:

$$D_{f+g} = D_f \cap D_g$$
$$D_{f-g} = D_f \cap D_g$$

For quotients, the rule is $D_{\frac{f}{g}} \subseteq D_f \cap D_g$. The following example illustrates this.

Example 1.27. Let $f(x) = 6x-12$ and $g(x) = 3-3x$. Find D_f, D_g and $D_{\frac{f}{g}}$.

Solution. $D_f = \mathbb{R}$, $D_g = \mathbb{R}$ but $D_{\frac{f}{g}} = \mathbb{R}\setminus\{1\}$, since

$$\frac{f}{g} = \frac{6x-12}{3-3x}, \qquad D_{\frac{f}{g}} \text{ is } \mathbb{R}\setminus\{1\}, \qquad 3-3x \neq 0 \quad \Rightarrow x \neq 1.$$

The domain of a function defined by, for example, addition of two functions can be smaller than expected because of the functions that make it up.

Question 1.10. If $f(x) = 2x-8$ and $g(x) = \sqrt{x+8}$, find $\dfrac{f}{g}(x)$ and its domain of definition.

Question 1.11. Let $f(x) = 2x-4$ and $g(x) = 1-x$. Find the domain of f, the domain of g and the domain of $\dfrac{f}{g}$.

Question 1.12. For the following functions find the maximal domains of the sum $(f+g)(x)$, the difference $(f-g)(x)$, product $(fg)(x)$ and quotient $\dfrac{f}{g}(x)$.

(a). $f(x) = x^2 - 1$ and $g(x) = \dfrac{1}{x+1}$

(b). $f(x) = \sqrt{3x+12}$ and $g(x) = \sqrt{3x^2+12x}$

Definition, evaluations, Domains, Codomains and Range of functions

1. Find the domain of $f(x) = \sqrt{\dfrac{x}{x+1}}$
2. Find the domain and range of the function $f(x) = 2\sqrt{4-x^2} - 3$.
3. Let $f(x) = \dfrac{\sqrt{x^2-4}}{5-\sqrt{36-x^2}}$. Then, in interval notation, that part of the domain of f which is to the right of the origin is $[2, a) \cup (a, b]$. Find the values of a and b.
4. Let $f(x) = (-x^2 - 7x - 10)^{-1/2}$
   Find
   (a) $f(-3)$
   (b) the domain of f.
5. Find the domain of the function $f(x) = \sqrt{1 + \sin x}$.
6. Find the range of the function $f(x) = 4\sin(2x) + 1$.