Lecture 3: Limits
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1 Limits

1.1 Introduction

The concept of a limit is a central idea that distinguishes calculus from algebra and trigonometry. It is fundamental to finding the tangent to a curve or the velocity of an object. In this chapter we develop the limit, first intuitively and then formally. We use limits to describe the way a function f varies. Some functions vary continuously; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump or vary erratically. The notion of limit gives a precise way to distinguish between these behaviors. The geometric application of using limits to define the tangent to a curve leads at once to the important concept of the derivative of a function. The derivative, which we investigate thoroughly in Chapter 4, quantifies the way a function? values change.

1.2 Limits of Function Values

1.2.1 Informal Definition of a Limit

Consider the graph of $y = f(x) = x^2 + 3$

What happens to $f(x) as 4x$ gets close to the value $x = 2$?

as x gets close to $x = 2$, $f(x)$ gets close to 7. This is rather obvious, after all $f(2) = 2^2 + 3 = 7$. However not all limits are so simple. Suppose we did not know the answer, it would be appropriate to compute a few values and see what happens as x gets closer and closer to 2.

x	2.05	2.01	2.001	2.0001	2	1.9999	1.999	1.99	1.95
$f(x)$	7.2025	7.0401	7.004001	7.00040001		6.99960001	6.996001	6.9601	6.8025
				$\longrightarrow$		$\longleftarrow$			

It appears that as x gets close to 2, $f(x) = x^2 + 3$ gets close to 7.

$$\therefore \lim_{x \to 2} x^2 + 3 = 7$$

Consider the function $f(x) = \dfrac{x^2-1}{x-1}$. What happens to $f(x)$ as x approaches 1? We cannot substitute the value $x = 1$ because $f(1)$ is not defined. The table provides values of $\dfrac{x^2-1}{x-1}$ for x near 1.

x	0	.5	.9	.99	.999	.99999	1	1.00001	1.001	1.01	1.1	1.5	2
$f(x)$	1	1.5	1.9	1.99	1.999	1.99999		2.00001	2.001	2.01	2.1	2.5	3

It seems that $f(x)$ approaches 2 as x approaches 1. This makes sense because, for $x \neq 1$,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)} = x+1.$$

approaches 2 as x approaches 1. We write

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

Despite not setting $x = 1$ because that would imply division by zero, we know what happens to this function as x approaches 1. It is hence clear why we are not required to evaluate $f(x)$ at $x = 1$ when we calculate the limit as x approaches 1.

1.2.2 Definition of a Limit

Let L be a real number and suppose that $f(x)$ is defined as an open interval containing x_0 but not necessarily at x_0 itself. We say that the limit as x approaches(tends to) x_0 (denoted by $x \to x_0$) of $f(x)$ is L, written as

$$\lim_{x\to x_0} f(x) = L^1$$

In this definition f is defined on an open interval containing the number x_0 except possibly at x_0 itself. This ensures that f is defined on both sides of x_0. It is important that $f(x)$ gets close to L when x gets close to x_0 from either side

[1] This is read as 'the limit of $f(x)$ as x approaches x_0 is L'.

While we do not actually need to know what $f(x_0)$ is (in fact, $f(x_0)$ need not even exist), it is helpful to know $f(x_0)$ in the actual computation of $\lim_{x\to x_0} f(x_0)$, since it frequently happens that $\lim_{x\to x_0} f(x)$ indeed equals $f(x_0)$. However, it should be emphasized that this is not always the case. In Example 3... we showed that $\lim\limits_{x\to x_0} f(x) = 2$ even though $f(1)$ did not exist.

1.3 Elementary Properties of Limits

Let b and c be real numbers and let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \qquad \lim_{x\to c} g(x) = K$$

(a). Scalar multiple: $\lim\limits_{x\to c} [bf(x)] = bL$

(b). Sum or difference: $\lim\limits_{x\to c} [f(x) \pm g(x)] = L \pm K$

(c). Product: $\lim\limits_{x\to c} [f(x)g(x)] = LK$

(d). Quotient: $\lim\limits_{x\to c} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{L}{K}$ provided $K \neq 0$.

(e). Power: $\lim\limits_{x\to c} [f(x)]^n = L^n$

1.4 Methods of Evaluating Limits

There are various methods for evaluating the limit of a function. The most commonly used methods include

(a). Table method

(b). Direct substitution method

(c). Method of factorization(cancelation) method

(d). Method of rationalization.

1.4.1 Table method

Calculate

$$\lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{\sqrt{4+h}-2}{h}$$

where h denotes a real number.

Note that

$$f(0) = \frac{\sqrt{4+0}-2}{0} = \frac{0}{0}$$

which is an undefined expression.

Using the table

h	$f(h)$		h	$f(h)$
1	0.2360679775		-1	0.26794991924
0.5	0.2426406871		-0.5	0.2583426132
0.1	0.2484567313		-0.1	0.2515823419
0.01	0.2498439448		-0.01	0.2501564457
0.001	0.2499843740		-0.001	0.2500156290
0.0001	0.2499984200		-0.0001	0.2500015900

it appears

$$\lim_{h \to 0} \frac{\sqrt{4+h}-2}{h} = 0.25$$

Using graph of $f(x) = \sqrt{x}$, the line joining the points $(4, 2)$ and $(4 + h, \sqrt{4+h})$ is called a **secant line** to the curves. Two such secant lines for two different values of h are drawn in the Figure....

Graph of $f(x) = \sqrt{x}$ with selected tangent and secant lines.

From the graph, as $h \to 0$, (so that $(4 + h, \sqrt{4+h})$ moves along the curve towards the point $(4, 2)$, the secant lines approach the line tangent to the curve at the point $(4, 2)$.

But the slope of secant line

$$\begin{aligned} &= \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{\Delta y}{\Delta x} = \frac{\Delta y}{h} = \frac{\sqrt{4+h}-2}{h} \end{aligned}$$

and the slope of the tangent line

$$\begin{aligned} &= \lim_{h \to 0} (\text{slope of secant line}) \\ &= \lim_{h \to 0} \frac{\sqrt{4+h}-2}{h} = \frac{1}{4} \end{aligned}$$

This technique of computing a limit in order to find the slope of a line tangent to a given curve is the central technique in calculus. Compute

$$\lim_{x \to 0} \frac{\sin\theta}{\theta}$$

where θ is measured in radians. Since

$$f(0) = \frac{\sin 0}{0} = \frac{0}{0}$$

which is un defined, we use the table method.

θ	$\sin\theta$	$\sin\theta/\theta$
1	0.8414709848	0.8414709848
0.5	0.4794255386	0.9588510772
0.1	0.09983341665	0.9983341665
0.01	0.009999833334	0.9999833334
0.001	0.0009999998333	0.9999998333
0.000	0.00009999999983	0.9999999983

Since $\sin(-\theta) = -\sin\theta$, if $\theta = -0.0001$, then $\sin(-0.0001) = -\sin(0.0001)$, so that

$$\frac{\sin(-0.0001)}{-0.0001} = \frac{-0.00009999999983}{-0.0001} = 0.9999999983$$

Generally if $\theta > 0$, then

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta}$$

So, we pick only positive values for θ in the table. Therefore

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

1.4.2 Direct substitution method

Evaluate the following limits. The Limit of a Polynomial $f(x) = 4x^2 + 3$

$$\begin{aligned}\lim_{x\to 2}(4x^2+3) &= \lim_{x\to 2}(4x^2+3) = \lim_{x\to 2} 4x^2 + \lim_{x\to 2} 3 \qquad \text{property two.}\\ &= 4\left[\lim_{x\to 2} x^2\right] + \lim_{x\to 2} 3\\ &= (4^2) + 3\\ &= 19\end{aligned}$$

In the example above note that the limit (as $x \to 2$) of the *Polynomial function* $P(x) = 4x^2 + 3$ is simply the value of P at $x = 2$.

$$\lim_{x\to 2} P(x) = P(2) = 4(2^2) + 3 = 19$$

This *direct substitution* property is valid for all polynomial and rational functions with nonzero denominators. If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c)$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(x) \neq 0$, then

$$\lim_{x\to c} r(x) = \frac{p(c)}{q(c)}$$

Find $\lim\limits_{x\to 1} \dfrac{x^2+x+2}{x+1}$ Because the denominator is not 0 when $x = 1$, you can apply the theorem above to obtain

$$\lim_{x\to 1} \frac{x^2+x+2}{x+1} = \frac{1^2+1+2}{1+1} = \frac{4}{2} = 2$$

Find $\lim\limits_{x\to 2} \dfrac{x^3-5x^2+2x-4}{x^2-3x+3}$ In this case, neither the numerator nor the denominator approaches 0. In fact, $\lim\limits_{x\to 2}(x^3-5x^2+2x-4) = -12$, $\lim\limits_{x\to 2}(x^2-3x+3) = 1$. Hence, our limit is

$$\lim_{x\to 2} \frac{x^3-5x^2+2x-4}{x^2-3x+3} = \frac{2^3-5(2)^2+2(2)-4}{2^2-3(2)+3} = \frac{-12}{1} = -12$$

Find $\lim\limits_{u\to 0} \dfrac{5u^2-4}{u+1}$ Find $\lim\limits_{x\to 2} \dfrac{x^2+5x+3}{2x^3-x+4}$

1.4.3 Factorization method

Find the limit: $\lim\limits_{x\to -3} \dfrac{x^2+x-6}{x+3}$. Although you are taking the limit of a rational function, you *cannot* apply the theorem above because the limit of the denominator is 0.

$$\lim_{x\to -3} \frac{x^2+x-6}{x+3} = \frac{\lim_{x\to -3}(x^2+x-6)}{\lim_{x\to -3}(x+3)}. \qquad \text{substitution method fails.}$$

In fact the limit of the numerator is also 0. However, the numerator and denominator have a common factor $(x+3)$. Thus, for all $x \neq -3$, you can cancel this factor to obtain

$$\frac{x^2+x-6}{x+3} = \frac{(x+3)(x-2)}{x+3} = x-2, \qquad x \neq -3.$$

Then it follows that

$$\lim_{x\to -3} \frac{x^2+x-6}{x+3} = \lim_{x\to -3}(x-2) = -5$$

In the above example direct substitution produces the meaningless fractional form 0/0. Such an expression is called an **indeterminate form** because you cannot (from the form alone) determine the limit. When you try to evaluate a limit and encounter this form, remember that you rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to *cancel like factors (factorization)*, as shown in the above example. A second way is to *rationalize the numerator*, as it will be shown shortly. Find the limit: $\lim\limits_{x\to 5} \dfrac{x^2-25}{x-5}$ The numerator and denominator both approach zero. Factorizing the numerator,

$$\begin{aligned}\lim_{x\to 5} \frac{x^2-25}{x-5} &= \lim_{x\to 5} \frac{(x+5)(x-5)}{(x-5)} \\ &= \lim_{x\to 5}(x+5) = 10\end{aligned}$$

Find the limit: $\lim\limits_{x\to 1} \dfrac{x^3-1}{x-1}$ Factorizing the numerator,

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1}$$

$$= \lim_{x\to 1}(x^2+x+1) = 1+1+1 = 3$$

Find the limit: $\lim\limits_{x\to 2} \left(\dfrac{1}{x-2} - \dfrac{4}{x^2-4}\right)$ Since $x^2-4 = (x+2)(x-2)$, we can factor out $\dfrac{1}{x-2}$ and simplify:

$$\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)=\lim_{x\to 2}\frac{1}{x-2}\left(1-\frac{4}{x+2}\right)=\lim_{x\to 2}\frac{1}{x-2}\left(\frac{x+2-4}{x+2}\right)$$

$$=\lim_{x\to 2}\frac{1}{x-2}\left(\frac{x-2}{x+2}\right)=\lim_{x\to 2}\left(\frac{1}{x+2}\right)=\frac{1}{4}$$

Find the limit: $\displaystyle\lim_{x\to 1}\left(\frac{x^3+x^2-x-1}{x-1}\right)$ Factorizing the numerator:

$$\lim_{x\to 1}\frac{x^3+x^2-x+1}{x-1}=\lim_{x\to 1}\frac{x^2(x+1)-1(x+1)}{x-1}=\lim_{x\to 1}\frac{(x^2-1)(x+1)}{x-1}$$

$$=\lim_{x\to 1}\frac{(x+1)(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)^2=4$$

Find the limit $\displaystyle\lim_{x\to -2}\left(\frac{2x^2-8}{x+2}\right)$ Factorizing the numerator:

$$\lim_{x\to -2}\frac{2x^2-8}{x+2}=\lim_{x\to -2}\frac{2(x^2-4)}{x+2}=\lim_{x\to -2}\frac{2(x-2)(x+2)}{x+2}$$

$$=\lim_{x\to -2}2(x-2)=-8$$

Find the limit $\displaystyle\lim_{x\to 1}\frac{1}{x-1}-\frac{2}{x^2-1}$

$$\begin{aligned}\lim_{x\to 1}\frac{1}{x-1}-\frac{2}{x^2-1}&=\lim_{x\to 1}\frac{x^2-1-2(x-1)}{(x^2-1)(x-1)}\\&=\lim_{x\to 1}\frac{x^2-2x+1)}{(x^2-1)(x-1)}=\lim_{x\to 1}\frac{(x-1)(x-1)}{(x+1)(x-1)(x-1)}\\&=\lim_{x\to 1}\frac{1}{(x+1)}\\&=\frac{1}{2}\end{aligned}$$

Find the limit $\displaystyle\lim_{x\to\infty}\frac{x^{2/3}+x}{1+x^{3/4}}$

$$\begin{aligned}\lim_{x\to\infty}\frac{x^{2/3}+x}{1+x^{3/4}}&=\lim_{x\to\infty}\frac{x^{2/3-3/4}+x^{1-3/4}}{\frac{1}{x^{3/4}}+1}\\&=\lim_{x\to\infty}\frac{\frac{1}{x^{1/12}}+x^{1/4}}{\frac{1}{x^{3/4}}+1}\\&=\infty\end{aligned}$$

Find the limit $\lim_{x\to\infty} \frac{\sqrt{x}+1}{x^2+x^3}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x}+1}{x^2+x^3} &= \lim_{x\to\infty} \frac{x^{\frac{1}{2}-2}+\frac{1}{x^2}}{1+x} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x^{3/2}}+\frac{1}{x^2}}{1+x} \\
&= \infty
\end{aligned}$$

Evaluate the following Limits.

(a). $\lim_{x\to 4} \frac{x^2-16}{x-4}$

(b). $\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{4}{x^2+2}\right)$

(c). $\lim_{x\to 4} \frac{x^2-x-12}{x-4}$

(d). $\lim_{x\to 2} \frac{2-x}{x^2-4}$

(e). $\lim_{x\to 1} \frac{x^2+x-2}{x^2-1}$

(f). $\lim_{x\to 2} \frac{x^3-5x^2+2x-4}{x^2-3x+3}$

(g). $\lim_{x\to 3} \frac{x^2-5x+6}{x-3}$

(h). $\lim_{x\to 5} \frac{x-5}{x^2-25}$

(i). $\lim_{x\to 9} \frac{x^2-81}{x-9}$

(j). $\lim_{x\to 2} \frac{x^2-4}{x-2}$

(k). $\lim_{x\to 3} \frac{x^2-5x+6}{x-3}$

(l). $\lim_{x\to 0} \frac{x^3-4x+4}{x^3}$

(m). $\lim_{x\to 1} \left(1+\frac{1}{x}\right)\left(\frac{1}{x}-1\right)$

(n). $\lim_{x\to 1} \frac{x^3+x^2-x-1}{x-1}$

(o). $\lim_{x\to 2} \frac{x^5-32}{x-2}$

(p). $\lim_{x\to 3} \left(\frac{x+4}{x^2+1} + \frac{x^3-27}{x-3}\right)$

(q). $\lim_{x\to 1} \frac{x^4+3x^3-13x^2-27x+36}{x^2+3x-4}$

(r). $\lim_{x\to 4} \frac{x^3-64}{x-4}$

1.4.4 Rationalization method

Calculate

$$\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$$

Rationalizing the function

$$\begin{aligned}\frac{\sqrt{4+x}-2}{x} &= \frac{(\sqrt{4+x}-2)(\sqrt{4+x}+2)}{x(\sqrt{4+x}+2)}\\ &= \frac{4+x-4}{x(\sqrt{4+x}+2)}\\ &= \frac{1}{\sqrt{4+x}+2}\end{aligned}$$

$$\therefore \lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0}\frac{1}{\sqrt{4+x}+2} = \frac{1}{4}$$

Evaluate

$$\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}$$

Rationalizing the function

$$\begin{aligned}\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x} &= \lim_{x\to 1}\frac{(1-\sqrt{x})(1+\sqrt{x})}{(1-x)(1+\sqrt{x})}\\ &= \lim_{x\to 1}\frac{(1-x)}{(1-x)(4\sqrt{x})} = \frac{1}{2}\end{aligned}$$

Find the limit: $\lim_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}$

By direct substitution, you obtain the indeterminate form 0/0.

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} \qquad \text{Direct substitution fails.}$$

In this case, you can rewrite the fraction by rationalizing the numerator[2]

$$\begin{aligned}\frac{\sqrt{x+1}-1}{x} &= \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)\\ &= \frac{(x+1)-1}{x(\sqrt{x+1}+1)}\\ &= \frac{x}{x(\sqrt{x+1}+1)}\\ &= \frac{1}{\sqrt{x+1}+1}, \qquad x\neq 0\end{aligned}$$

[2]The rationalization technique for evaluating limits is based on multiplication by a convenient form of 1. In the above example, the convenient form is $1 = \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$.

Now, using direct substitution, you can evaluate the limit as follows.

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}\\ &= \frac{1}{1+1}\\ &= \frac{1}{2}\end{aligned}$$

Evaluate the following limits Find the limit $\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}$ Rationalizing the numerator

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x} &= \lim_{x\to 0}\left(\frac{\sqrt{x+3}-\sqrt{3}}{x}\right)\left(\frac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}}\right)\\ &= \lim_{x\to 0}\frac{(x+3)-3}{x(\sqrt{x+3}+\sqrt{3})}\\ &= \lim_{x\to 0}\frac{3}{x(\sqrt{x+3}+\sqrt{3})}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{x+3}+\sqrt{3}}\\ &= \frac{1}{2\sqrt{3}}\end{aligned}$$

Find the limit $\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-2x}$ Rationalizing the numerator

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-2x} &= \lim_{x\to 2}\left(\frac{\sqrt{x^2+5}-3}{x^2-2x}\right)\left(\frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}\right)\\ &= \lim_{x\to 2}\frac{(x^2+5)-9}{(x^2-2x)(\sqrt{x^2+5}+3)}\\ &= \lim_{x\to 2}\frac{x^2-4}{(x^2-2x)(\sqrt{x^2+5}+3)}\\ &= \lim_{x\to 2}\frac{(x-2)(x+2)}{x(x-2)(\sqrt{x^2+5}+3)}\\ &= \lim_{x\to 2}\frac{(x+2)}{x(\sqrt{x^2+5}+3)}\\ &= \frac{4}{2(\sqrt{9}+3)} = \frac{1}{3}\end{aligned}$$

Evaluate the following Limits.

(a). $\lim_{x\to 0} \frac{x(1+\sqrt{1-x})}{1-(1-x)}$ (b). $\lim_{x\to 1} \frac{2}{2-\sqrt{3+x}}$

(c). $\lim_{x\to 0} \frac{\sqrt{x+1}-2}{x+1}$ (d). $\lim_{x\to 0} \frac{x}{2-\sqrt{x-2}}$

(e). $\lim_{x\to 16} \frac{4-\sqrt{x}}{x-16}$ (f). $\lim_{x\to 0} \frac{[1/(2+x)]-(1/2)}{x}$

(g). $\lim_{x\to 2} \frac{x^5-32}{x-2}$ (h). $\lim_{x\to 0} \frac{2\left(2+\sqrt{4+x}\right)}{4-(4+x)}$

(i). $\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-2x}$ (j). $\lim_{x\to 0} \frac{1-\sqrt{1-x}}{x}$

(k). $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$ (l). $\lim_{x\to \infty} \frac{5+x}{x^2+x+1}$

(m). $\lim_{x\to 1} \frac{x^3-1}{x-1}$ (n). $\lim_{x\to 0} \frac{5x^2-4}{x+1}$

(o). $\lim_{x\to 0} \frac{x}{2-\sqrt{2-x}}$ (p). $\lim_{x\to 0} \frac{\sqrt{x}+1}{x^2+x^3}$

A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x\to 0} \frac{\tan x}{x}$. Direct substitution yields the intermediate form $0/0$. To solve this problem, you can write $\tan x$ as $(\sin x)/(\cos x)$ and obtain

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right) = \lim_{x\to 0} \left(\frac{\sin x}{x}\right) \lim_{x\to 0} \left(\frac{1}{\cos x}\right)$$

Now, because

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \therefore, \qquad \lim_{x\to 0} \left(\frac{\sin x}{x}\right) \lim_{x\to 0} \left(\frac{1}{\cos x}\right) = \frac{1}{1} = 1.$$

1.5 One sided Limits

Let L be a real number

(i). Suppose that $f(x)$ is defined near x_0 for $x > x_0$ and that the x gets close to x_0 (with $x > x_0$), $f(x)$ gets close to L. Then we say that L is the right hand limit of $f(x)$ as x approaches x_0 and we write

$$\lim_{x\to x_0^+} f(x) = L$$

(ii). Suppose that $f9x)$ is defined near x_0 for $x < x_0$ and that as x gets close to L. Then we say that L is the left hand limit of $f(x)$ as x

$$\lim_{x \to x_0^-} f(x) = L$$

For function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

It follows that

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1$$

and

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1$$

$\lim_{x \to x_0} f(x) = L$ exists if and only if the following hold:

(i). $\lim_{x \to x_0^+} f(x)$ exists

(ii). $\lim_{x \to x_0^-} f(x)$ exists

(iii). $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$

i.e., the limits if and only if the right hand side and left hand side exist and are equal. Let

$$f(x) = \begin{cases} x + 1 & x > 0 \\ x - 1 & x < 0 \end{cases}$$

$\lim_{x \to 0} f(x)$ does not exist.

1.6 Infinite limits and limits at infinity

Consider

$$\lim_{x \to 0} \frac{1}{x^2}$$

x	$1/x^2$	x	$1/x^2$
1	1	-1	1
0.5	4	-0.5	4
0.1	100	-0.1	100
0.01	10,000	-0.01	10,000
0.001	1,000,000	-0.001	1,000,000
0.0001	100,000,000	-0.0001	100,000,000

It appears $f(x)$ tends to infinity as x approaches zero, and we write

$$\lim_{x\to 0}\frac{1}{x^2}=\infty$$

1.7 Infinite Limits

(i). If $f(x)$ grows without bound in the positive direction as x gets close to the number x_0 from either side, then we say that $f(x)$ tends to infinity as x approaches x_0, and we write

$$\lim_{x\to x_0} f(x)=\infty$$

(ii). If $f(x)$ grows without bound in the negative direction as x gets close to the number x_0 from either side, then we say that $f(x)$ tends to minus infinity as x approaches x_0, and we write

$$\lim_{x\to x_0} f(x)=-\infty$$

Since $1/x^2$ grows without bound as $x\to 0$, $-1/x^2$ grows without bound in the negative direction as $x\to 0$.

$$\lim_{x\to 0}-\frac{1}{x^2}=-\infty$$

Calculate

$$\lim_{x\to 0}\frac{1}{x^2+x^3}$$

Using the table method

x	0.5	0.1	0.01	0.001	0	-0.001	-0.01	-0.1	-0.5
$f(x)$	2.66667	90.90909	9,900.99	999,000.999		1,001,001	10,101.01	111.1111	8

$$\therefore \lim_{x\to 0}\frac{1}{x^2+x^3}=\infty$$

Calculate

$$\lim_{x\to 0}\frac{1}{x}$$

$$\lim_{x\to 0^+}\frac{1}{x}=\infty,\qquad \lim_{x\to 0^-}\frac{1}{x}=-\infty$$

To this point we have considered limits as $x\to x_0$, where x_0 is a real number. But in many important applications it is necessary to determine what happens to $f(x)$ as x becomes very large. For example, what happens to function $f(x)=\frac{1}{x}$ as x becomes large

These examples suggest the following definition

1.7.1 Limits at infinity

If you have a limit of a rational function (i.e. a quotient of polynomials), then there are three cases.

(1) **Degree of numerator $<$ degree of denominator:** In this case, the denominator dominates the quotient, so you multiply the numerator and denominator of the function by 1 over the highest degree term. As an example, if we have

$$\lim_{x\to\infty} \frac{x^2+3x+1}{x^3-2},$$

then the degree of the bottom is 3 and the top has degree 2. In this case, we multiply the top and bottom by $1/x^3$ to get

$$\lim_{x\to\infty} \frac{(x^2+3x+1)\frac{1}{x^3}}{(x^3-2)\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{3}{x^2}+\frac{1}{x^3}}{1-\frac{2}{x^3}}.$$

At this point, the usual limit laws work, and since the top goes to 0 and the bottom goes to 1, the limit is $0/1 = 0$.

(2) **Degree of numerator $=$ degree of denominator:** In this case, you do the same procedure as above. For example, if the limit is

$$\lim_{x\to\infty} \frac{-3x^2+x+1}{x^2-2},$$

then we can multiply the top and bottom by $1/x^2$, and we find

1-2$\frac{}{x^2,}$

and now we use the usual limit laws to get $-3/1 = -3$.

- **Degree of numerator $>$ degree of denominator:** In this case, the limit will be ∞ or $-\infty$. While the method I showed

(i). The limit as x approaches infinity of $f(x)$ is L, written

$$\lim_{x\to\infty} f(x) = L$$

if $f(x)$ is defined for all large values of x and if $f(x)$ gets close to L as as x increases without bound.

(ii). The limit as x approaches minus infinity of $f(x)$ is L, written

$$\lim_{x\to -\infty} f(x) = L$$

if $f(x)$ is defined for all values of x that are large in the negative direction and $f(x)$ gets close to L as x increases without bound in the negative direction.

Let f be the function given by

$$f(x) = \frac{3}{x-2}.$$

From the table below, you can see that $f(x)$ *decreases without bound* as x approaches 2 from the left, and $f(x)$ *increases without bound* as x approaches 2 from the right. This behaviour is denoted as

$$\lim_{x\to 2^-} \frac{3}{x-2} = -\infty \qquad f(x) \text{ decreases without bound as x approaches 2 from the left}$$

and

$$\lim_{x\to 2^+} \frac{3}{x-2} = \infty \qquad f(x) \text{ increases without bound as x approaches 2 from the right}$$

x	1.2	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	-6	-30	-300	-3000	?	3000	300	30	6

A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an **infinite limit.** Evaluate $\lim\limits_{x\to\infty} \dfrac{5+x}{x^2+x+1}$

$$\begin{aligned}\lim_{x\to\infty} \frac{5+x}{x^2+x+1} &= \lim_{x\to\infty} \frac{5/x^2+1/x}{1+1/x+1/x^2} \\ &= \frac{0}{1} = 0\end{aligned}$$

Evaluate the following Limits.

$$\text{(a). } \lim_{x\to\infty} 1-\frac{1}{x} \quad \text{(b). } \lim_{x\to\infty} \frac{x^{\frac{2}{3}}+x}{1+x^{\frac{3}{4}}} \quad \text{(c).}$$

Use the position function $s(t) = -16t^2 + 1000$, which gives the height (in feet) of an object that has fallen for t seconds from a height of 1000 ft. The velocity at time $t = a$ seconds is given by

$$\lim_{t\to a} \frac{s(a)-s(t)}{a-t}$$

(a). If a construction worker drops a wrench from a height of 1000 feet, how fast will the wrench be falling after 5 seconds?

(b). If a construction worker drops a wrench from a height of 1000 feet, when will the wrench hit the ground? At what velocity will the wrench impact the ground.

Evaluate the following Limits.

(a). $\lim_{x\to 4} x^2$

(b). $\lim_{x\to -3}(3x+2)$

(c). $\lim_{x\to 0}(2x-1)$

(d). $\lim_{x\to 1}(-x^2+1)$

(e). $\lim_{x\to 2}(-x^2+x-2)$

(f). $\lim_{x\to 1}(3x^3-2x^2+4)$

(g). $\lim_{x\to 3}\dfrac{x^2+x}{x+5}$

(h). $\lim_{x\to 0}\dfrac{1}{x-3}$

(i). $\lim_{x\to -1}\dfrac{1}{(1+x)^2}$

(j). $\lim_{x\to 2}\dfrac{x^3-x^2-x-2}{x-2}$

(k). $\lim_{x\to 3}\dfrac{x^2+x}{x+5}$

(l). $\lim_{x\to 3}\sqrt{x+1}$

(m). $\lim_{x\to -4}(x+3)^2$

(n). $\lim_{x\to 2}\dfrac{1}{x}$

(0). $\lim_{x\to -1}\dfrac{x^2+1}{x}$

(p). $\lim_{x\to 4}\sqrt[3]{x+4}$

(q). $\lim_{x\to 0}(2x-1)^3$

(r). $\lim_{x\to -3}\dfrac{2}{x+2}$

(s). $\lim_{x\to 3}\sqrt{\dfrac{x+1}{x-4}}$

(t). $\lim_{x\to 5}(3x^4-275)$

(u). $\lim_{x\to 4}\dfrac{x^3}{x+5}$

(v). $\lim_{x\to 1}\dfrac{6x^3+2x^2-12x}{3x-2}$