Introduction to Tensor Notation

- A tensor is denoted by a symbol follow by a collection of subscripts, superscritps, e.g

$$X_i \;,\; X^j ,\; \beta_{jkl} ,\; A^{\alpha\beta} , \ldots$$

- Tensor of rank-0 : No super/Subsript
  a scalar ϕ

- Tensor for rank-1 : Vector

$U_j \quad (j = 1, 2, 3) \qquad \vec{U} = u_1 \hat{e}_x + u_2 \hat{e}_y + u_3 \hat{e}_z$

$U_1 = u_1$

$U_2 = u_2$

$U_3 = u_3$

- Tensor of rank-2 , Tensor σ_{ij}

$i = 1, 2, 3, \qquad j = 1, 2, 3$

$$\sigma_{ij} = \begin{Bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{Bmatrix}$$

- Operations on Tensors :

$$\sum_{i=1}^{3} A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$

Contraction

$$\sum_{i=1}^{3} A_i B^i = A_1 B^1 + A_2 B^2 + A_3 B^3$$

$$\sum_{i=1}^{3} A^i B_i = A^1 B_1 + A^2 B_2 + A^3 B_3$$

$$\vec{A} \cdot \vec{B} = A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$

Similarly : $\eta_{ij} x^i x^j$

$i, j = 1, 2, 3.$

$$\Rightarrow \eta_{11} x^1 x^1 + \eta_{12} x^1 x^2 + \eta_{13} x^1 x^3$$
$$+ \eta_{21} x^2 x^1 + \eta_{22} x^2 x^2 + \eta_{23} x^2 x^3$$
$$+ \eta_{31} x^3 x^1 + \eta_{32} x^3 x^2 + \eta_{33} x^3 x^3.$$

$$\eta^{ij} x_i x_j = \eta^{11} x_1 x_1 + \eta^{12} x_1 x_2 + \eta^{13} x_1 x_3$$
$$+ \eta^{21} x_2 x_1 + \eta^{22} x_2 x_2 + \eta^{23} x_2 x_3$$
$$+ \eta^{31} x_3 x_1 + \eta^{32} x_3 x_2 + \eta^{33} x_3 x_3$$

Similarly

$$w_i = u_j \sigma_{ij}$$

$$w_1 = u_1 \sigma_{11} + u_2 \sigma_{12} + u_3 \sigma_{13}$$

$$w_2 = u_2 \sigma_{21} + u_2 \sigma_{22} + u_3 \sigma_{23}$$

$$w_3 = u_1 \sigma_{31} + u_2 \sigma_{32} + u_3 \sigma_{33}$$

- Divergence of a Tensor

$$\underbrace{\vec{\nabla} \cdot \vec{u}}_{\text{Scalar.}} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$$

$$\vec{\nabla}_i \cdot \sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial x_i} = r_i \text{ (vector)}$$

$$r_1 = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}$$

$$r_2 = \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3}$$

$$r_3 = \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3}$$

Similarly.

$$\nabla \cdot (u \cdot \sigma) = \frac{\partial u_j \sigma_{ij}}{\partial x_i}$$

$$= \frac{\partial}{\partial x_1}\left(u_1 \sigma_{11} + u_2 \sigma_{12} + u_3 \sigma_{13}\right)$$

$$+ \frac{\partial}{\partial x_2}\left(u_1 \sigma_{21} + u_2 \sigma_{22} + u_3 \sigma_{23}\right)$$

$$+ \frac{\partial}{\partial x_3}\left(u_1 \sigma_{31} + u_2 \sigma_{32} + u_3 \sigma_{33}\right)$$

Gradient of a Tensor:

$$\nabla \phi = \frac{\partial \phi}{\partial x_i} = A_i \text{ (Vector)}$$

$$A_1 = \frac{\partial \phi}{\partial x_1}, \quad A_2 = \frac{\partial \phi}{\partial x_2}, \quad A_3 = \frac{\partial \phi}{\partial x_3}$$

$$\vec{\nabla} \vec{u} = \frac{\partial u_i}{\partial x_j} = \beta_{ij} \quad (\text{Tensor})$$

$$\beta_{ij} = \begin{Bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{Bmatrix}$$

$$\vec{u} \cdot \vec{\nabla} \vec{u} = u_j \frac{\partial u_i}{\partial x_j} = S_i$$

$$S_1 = u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial x_3}$$

$$S_2 = u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_2}{\partial x_3}$$

$$S_3 = u_1 \frac{\partial u_3}{\partial x_1} + u_2 \frac{\partial u_3}{\partial x_2} + u_3 \frac{\partial u_3}{\partial x_3}$$

The Kronecker Delta

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0 & i \neq j \end{cases}$$

$$\nabla \cdot \delta = \frac{\partial \delta_{ij}}{\partial x_j} = \frac{\vec{\partial}}{\partial x_j} = \nabla$$

Levi-Civita

$$c_i = \epsilon_{ijk} b_k \Rightarrow \vec{c} = \vec{a} \times \vec{b}$$

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$$

$$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1.$$

repeated indices $= 0$

e.g. $\epsilon_{112} = 0$

$$c_i = \epsilon_{ijk} a_j b_k = \epsilon_{i11} a_1 b_1 + \epsilon_{i12} a_1 b_2 + \epsilon_{i13} a_1 b_3$$
$$+ \epsilon_{i21} a_2 b_1 + \epsilon_{i22} b_2 a_2 + \epsilon_{i23} a_2 b_3$$
$$+ \epsilon_{i31} a_3 b_1 + \epsilon_{i32} a_3 b_2 + \epsilon_{i33} a_3 b_3$$

Set $i=3$, $c_3 = a_1 b_2 - a_2 b_1$ (all other terms are 0)

Minkowski Space-time.

$$\eta_{\mu\nu} = \begin{cases} -1 & \mu=0,\ \nu=0 \\ +1 & \mu = 1,2,3 \\ & \nu = 1,2,3 \\ 0 & \mu \neq \nu \end{cases}$$

$\eta_{00} = -1$, $\eta_{11} = +1$, $\eta_{22} = +1$, $\eta_{33} = +1$

$(- + + +)$ Signature.

$$\eta_{\mu\nu} = \begin{cases} +1 & \mu=0,\ \nu=0 \\ -1 & \mu=1,2,3,\ \nu=1,2,3 \\ 0 & \mu \neq \nu \end{cases}$$

$\eta_{00} = +1$, $\eta_{ii} = -1$, $i = 1,2,3$.

$(+ - - -)$ Signature

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = \eta_{00}\, dx^0 dx^0 + \eta_{11}\, dx^1 dx^1 + \eta_{22}\, dx^2 dx^2 + \eta_{33}\, dx^3 dx^3$$

For $(-+++)$ Signature

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2$$

$$-c^2 dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$ds^2 \begin{cases} < 0 & \text{time like} \\ > 0 & \text{Space like} \\ = 0 & \text{light like} \end{cases}$$