Relativistic Quantum Mechanics
Tutorial 1

Exercise one and two are for the tutorial, exercise 3 is to be handed in at the start of tutorial 5 as part of the (optional) graded assignment.

1. We define $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $x^2 = x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu$ and $x^4 = (x^2)(x^2)$. Evaluate:

$$\partial_\mu(x^2), \qquad \partial^\lambda \partial_\lambda (x^\mu x^\rho), \qquad \partial_\mu \partial_\nu (x^4).$$

2. The motion of a complex scalar field ψ is governed by the Lagrangian density

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2}(\psi^* \psi)^2.$$

   a) Derive this system's equations of motion using the Euler-Lagrange equations. (We treat ψ and ψ^* as separate fields, for which we use the Euler-Lagrange equation separately.)

   b) Verify that the Lagrangian is invariant under the infinitesimal transformation

$$\psi \to (1 + i\alpha)\psi, \qquad \psi^* \to (1 - i\alpha)\psi^*,$$

   where α is an infinitesimal parameter.

   c) Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations derived in a).

3. A Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is such that it preserves the Minkowski metric $\eta_{\mu\nu}$, meaning that $\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu$ for all x.

   a) Show that this is equivalent to

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}.$$

b) Use this result to show that an infinitesimal transformation of the form

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$$

is in fact a Lorentz transformation if $\omega^{\mu\nu}$ is anti-symmetric, i.e. $\omega^{\mu\nu} = -\omega^{\nu\mu}$.

c) Derive the Noether current associated with the infinitesimal Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu},$$

where

$$\omega^{\mu}{}_{\nu} \equiv \theta \Omega^{\mu}{}_{\nu},$$

θ being infinitesimal and $\Omega^{\mu}{}_{\nu}$ not, for Lagrangian density

$$\mathcal{L} = \partial^{\mu}\psi^{*}\,\partial_{\mu}\psi - \psi^{*}\psi.$$

The answer should be:

$$j^{\mu} = \Omega^{\rho}{}_{\nu} x^{\nu} \left[\delta^{\mu}_{\nu}\mathcal{L} - \partial^{\mu}\psi\,\partial_{\rho}\psi^{*} - \partial^{\mu}\psi^{*}\,\partial_{\rho}\psi\right].$$

Hint: This is a transformation that changes the fields and the Lagrangian by changing their argument, because ψ, ψ^{*} and $\mathcal{L}$ are scalar fields. Determine $\delta\mathcal{L}$, $\delta\psi$ and $\delta\psi^{*}$ by doing a first order Taylor expansion for the fields and the Lagrangian.

d) Show that

$$j^{\mu} = -\Omega^{\rho}{}_{\nu} x^{\nu} T^{\mu}{}_{\rho}.$$