**Exercises**

# Symmetry in Physics

**Spring 2024**

**Exercises belonging to the subjects of Chapter 1 of the Lecture Notes**

**Exercises from Jones Chapter 1: 1.1-1.6**

**Exercise A1**
Let $G$ be a group and $K$ be a conjugacy class of $G$. Prove that the inverses of the elements of $K$ also form a conjugacy class.

**Exercise A2**
Show that the conjugacy classes of the symmetric group $S_n$ are characterized by the disjoint cycle structure of their elements, and that for $A_n$ this is not the case (e.g. by counterexample: show explicitly for $A_3$ that there are distinct conjugacy classes with the same disjoint cycle structure).

**Exercise from Jones Chapter 2: 2.2**

**Exercise A3**
Find the centers of $C_4$, $D_4$, $A_4$ and $S_4$.

**Exercise A4**
Show that $\mathsf{Z}_n \cong C_n$ by explicitly constructing the isomorphism.

**Exercises from Jones Chapter 1: 1.7**

**Exercise A5**
Consider the group $O(2)$ of orthogonal $2\times 2$ matrices and the group $U(1)$ of unitary $1\times 1$ matrices.
(a) Show that $O(2)$ is a non-Abelian group.
(b) Show that the group elements of $O(2)$ have either determinant 1 or $-1$. Show that the elements from $O(2)$ with determinant 1 form a group. This group is referred to as $SO(2)$, the special orthogonal group of $2 \times 2$ matrices with determinant 1. Show that the elements with determinant $-1$ do not form a group.
(c) Show that $SO(2) \cong U(1)$.
(d) Give an example of a physical system with an $SO(2)$ or $U(1)$ symmetry.

**Exercise A6**
The rotations around a fixed axis in three dimensions form an Abelian subgroup of $SO(3)$. Show that this subgroup is isomorphic to $SO(2)$.

**Exercises belonging to the subjects of Chapter 2 of the Lecture Notes**

**Exercise B1**
Three matrices $E, S$, and $T$ are given by

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \omega & 0 \\ 0 & 1/\omega \end{pmatrix},$$

where $\omega$ is one of the two roots of the equation $\omega^3 = 1$ and $\omega \neq 1$.
(a) Using these objects and ordinary matrix multiplication as the composition law construct the group $G(\omega)$ that consists of $E, S, T$, and their products.
(b) The matrix group $G(\omega)$ can be considered as a two-dimensional rep of an abstract group $G$. Show whether this rep is an irrep or not.
(c) Let $\omega_1$ and $\omega_2$ be the two cubic roots of unity not equal to one. Then $G(\omega_1)$ and $G(\omega_2)$ are two reps of $G$. Show whether these reps are equivalent or not.

**Exercise B2**
Let $G$ be a group and $\phi$ a representation of $G$. Consider the maps $\phi^*, (\phi^\mathsf{T})^{-1}$ and $(\phi^\dagger)^{-1}$ defined by

$$\begin{aligned} \phi^*(g) &= (\phi(g))^*, \\ (\phi^\mathsf{T})^{-1}(g) &= ((\phi(g))^\mathsf{T})^{-1}, \\ (\phi^\dagger)^{-1}(g) &= ((\phi(g))^\dagger)^{-1}, \end{aligned}$$

for all $g \in G$. Here $*, \mathsf{T}, \dagger, -1$ denote complex conjugation, transposition, Hermitian conjugation and the inverse, respectively.
(a) Prove that the three maps $\phi^*, (\phi^\mathsf{T})^{-1}$ and $(\phi^\dagger)^{-1}$ are also representations.
(b) Show that these maps are irreducible representations if $\phi$ is an irrep.

**Exercises from Jones Chapter 3: 3.3, 3.6**

**Exercise B3** Galilei group in 1+1 dimensions.
Consider the Galilei transformations in a $1 + 1$ dimensional space:

$$\begin{aligned} x' &= x + vt + a \\ t' &= t + b \end{aligned}$$

or in matrix form:

$$\begin{bmatrix} x' \\ t' \\ 1 \end{bmatrix} = T(a, b, v) \begin{bmatrix} x \\ t \\ 1 \end{bmatrix}, \quad T(a, b, v) \equiv \begin{bmatrix} 1 & v & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Verify the composition law

$$T(a_1, b_1, v_1) \circ T(a_2, b_2, v_2) = T(a_1 + a_2 + b_2 v_1, b_1 + b_2, v_1 + v_2),$$

(b) Show that the inverse is given by

$$T^{-1}(a, b, v) = T(-a + bv, -b, -v).$$

(c) Galilei transformations can be seen as non-relativistic symmetry transformations in a $1 + 1$ dimensional space. What is the natural definition of distance in this 1+1 dimensional space? How does this differ from $1 + 1$ dimensional Minkowski space and from a Euclidean two-dimensional space?

(d) Why is there no two-dimensional matrix representation for this group and do the three-dimensional matrices $T(a,b,v)$ form an irrep of the Galilei group?

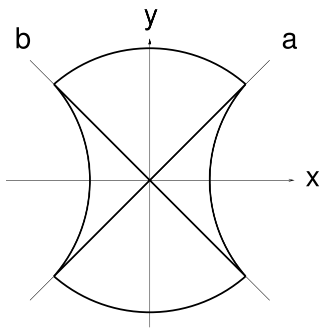
**Exercise B4**
Let $G$ be a group, $\phi$ an irreducible matrix representation of $G$, and $K$ a conjugacy class of $G$. Define

$$A = \sum_{T \in K} \phi(T).$$

Show that $A$ is proportional to the identity matrix.

**Exercises from Jones Chapter 4: 4.2, and Chapter 3: 3.1, 3.2**

**Exercise B5**




Denote the set of all symmetries of the figure shown above by $G$.
(a) Show that $G$ forms a group and determine the full group table of $G$.
(b) Determine the conjugacy classes of $G$.
(c) Determine the dimensions of the nonequivalent irreducible representations of $G$.

**Exercise B6**
Consider the so-called four-group $V$, consisting of four elements $e, a, b, c$, where $e$ is the identity, and the group multiplication is given by the following group table:

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $e$ | $c$ | $b$ |
| $b$ | $c$ | $e$ | $a$ |
| $c$ | $b$ | $a$ | $e$ |

(a) How many irreps does $V$ have?
(b) Write down all irreps of $V$.

**Exercise B7** The irreps of $S_3$
a) Write down the elements of $S_3$ in the standard $2 \times n$ matrix notation and in cycle notation.
b) Divide these elements in conjugacy classes. How many classes are there?
c) What are the dimensions of the irreps of $S_3$?
d) Write down the parity (also called sign) of the permutations of $S_3$. Does this furnish a one-dimensional irrep?

**Exercises from Jones Chapter 4: 4.5, 4.8, 4.4**

**Exercise B8**
Consider the cyclic group $C_5$: gp$\{c\}$ with $c^5 = e$.
(a) Construct the character table of $C_5$.
(b) Construct the three-dimensional vector representation $D^V$ for the generator $c$ of $C_5$.
(c) Decompose $D^V$ into irreps of $C_5$.

**Exercise from Jones Chapter 5: 5.1**

**Exercises belonging to the subjects of Chapter 3 of the Lecture Notes**

**Exercise C1**
Consider the cyclic group $C_4$: gp$\{c\}$ with $c^4 = e$.
(a) Construct the character table of $C_4$.
(b) Construct the three-dimensional vector representation $D^V$ of $C_4$ and demonstrate whether it is an irrep or not.
(c) Decompose $D^V$ into irreps of $C_4$ and use the result to conclude whether a crystal with $C_4$ symmetry can support a permanent electric dipole moment.
(d) Determine the Clebsch-Gordan series of the direct product representation $D^V \otimes D^V$ of $C_4$.

**Exercise from Jones Chapter 4: 4.9, and Chapter 5: 5.2**

**Exercise C2**
Consider the group $O(2)$ of orthogonal $2 \times 2$ matrices.
(a) Write down the two-dimensional representation of $O(2)$ obtained by its action on the vector $(x + iy, x - iy)$, corresponding to the spherical basis in two dimensions.
(b) Show whether this rep of $O(2)$ is an irrep or not.
(c) Explain why the restriction to the subgroup $SO(2)$ does not yield an irrep of $SO(2)$ over the complex numbers?

**Exercise from Jones Chapter 6: 6.1**

**Exercise C3**
(a) Explicitly give the matrix for a rotation around the $z$-axis in the defining representation of $SO(3)$.
(b) Explicitly give the matrix for a rotation around the $z$-axis in the $l = 1$ irrep of $SO(3)$.
(c) Show explicitly by a basis transformation that the reps $D^V$ and $D^{(1)}$ of $SO(3)$ are equivalent.

**Exercise C4**
(a) Explain the difference between the vector and axial-vector representations of $O(3)$.
(b) Show that the Kronecker delta $\delta_{ij}$ is invariant under $O(3)$ transformations.
(c) Determine the subgroup of $O(3)$ transformations that leave the tensor $\sigma_{ij} = \delta_{ij} + a\delta_{i3}\delta_{j3}$ invariant, for nonzero $a \in \mathsf{R}$.
(d) Explain what is a pseudoscalar and give an example of one.

**Exercises belonging to the subjects of Chapter 4 of the Lecture Notes**

**Exercises from Jones Chapter 5: 5.4, 5.5**

**Exercise D1**

Consider the angular momentum operators $L_i$ $(i = 1, 2, 3)$ acting on the states $|l, m\rangle$.

(a) Explain why the eigenvalues of the operator $\vec{L}^2 = \sum_i L_i^2$ can be used to label the irreps of $SO(3)$ (hint: use that $[\vec{L}^2, L_i] = 0$).

(b) Write down the explicit matrix for $L_z$ acting on the space of $|l, m\rangle$ states for non-negative integer $l$.

(c) Use the result of (b) to show that the characters of $SO(3)$ matrices are of the form:

$$\chi^{(l)}(\theta) = 1 + 2\left(\cos(\theta) + \ldots + \cos(l\theta)\right)$$

(d) Draw a picture of the parameter space of $SO(3)$ and indicate the conjugacy classes.