- Canonical Quantisation

From $\underline{\underline{QM}}$ $\quad [q_a, q_b] = [p^a, p^b] = 0$
$[q_a, p^b] = i\delta_a^b$ $\Bigg\}$ Poisson Bracket

In Field theory $\quad [\phi_a(\vec{x}), \phi_b(\vec{y})] = [\pi^a(x), \pi^b(y)] = 0$

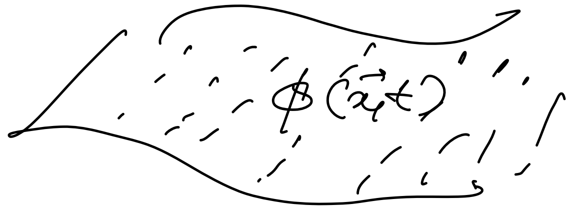
$$[\phi_a(\vec{x}), \pi^b(\vec{y})] = i\delta^{(3)}(x-y)\,\delta_a^b$$

Note: These commutating relations are not Lorentz Invariant. (We have separated $(\vec{x})$ & (t))

Also, we are working in Schröedinger's picture, where $\phi_a(\vec{x})$ and $\pi^a(\vec{x})$ dont depend on time at all, - only on space.

All time dependences are in $\Psi(t)$ or $|\Psi\rangle$ which evolves by the usual Schrödinger's equation
$$i\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle$$
Now, we are applying the same to the fields.

Example: $\phi(\vec{x},t)$



Real Klein-Gordon field.

Equations of motion: $\partial_\mu \partial^\mu \phi + m^2 \phi = 0$

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi(\vec{p},t) \quad [\text{F.T.}]$$

Fourier transform

$\phi(\vec{p},t)$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} + \left(\vec{p}^{\,2} + m^2\right)\right) \phi(\vec{p},t) = 0$$

For each $\vec{p}$, we have a Simple Harmonic Oscillator vibrating at frequency.

$$\omega_{\vec{p}} = \pm\sqrt{\vec{p}^{\,2} + m^2}$$

Most general solution to the KG is
Linear Superposition of harmonic Oscillators each vibrating at different frequency/Amplitude

We must quantise $\phi(\vec{x}, t)$, & there are ∞ number of harmonic Oscillators.

- Revision of Simple Harmonic Oscillator

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 \quad \text{(particle picture)}$$

$$[q, p] = i\hbar \quad \text{(I suppress } \hbar\text{)}.$$

$$[q, p] = i$$

we know $\quad a = \sqrt{\frac{\omega}{2}}\, q + \frac{i}{\sqrt{2\omega}}\, p$

$$a^{\dagger} = \sqrt{\frac{\omega}{2}}\, q - \frac{i}{\sqrt{2\omega}}\, p$$

$$\Rightarrow q = \frac{1}{\sqrt{2\omega}}(a + a^{\dagger}) \quad \& \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^{\dagger})$$

Since, $[q, p] = i$

$$\Rightarrow \left[\frac{1}{\sqrt{2\omega}}(a + a^{\dagger}),\ -i\sqrt{\frac{\omega}{2}}(a, -a^{\dagger})\right] = i$$

$$\Rightarrow -\frac{\cancel{i}}{2}(a + a^{\dagger})(a - a^{\dagger}) + \frac{\cancel{i}}{2}(a - a^{\dagger})(a + a^{\dagger}) = \cancel{i}$$

$$\Rightarrow -\frac{1}{2}(\cancel{aa} - aa^{\dagger} + a^{\dagger}a - \cancel{a^{\dagger}a^{\dagger}}) + \frac{1}{2}(\cancel{aa} + aa^{\dagger} - a^{\dagger}a - \cancel{a^{\dagger}a^{\dagger}}) = 1$$

$\Rightarrow \quad aa^\dagger - a^\dagger a = 1 \quad \Rightarrow \quad [a, a^\dagger] = \mathbb{1}$

$$\Rightarrow \quad H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 = \frac{\omega}{4}(a-a^\dagger)^*(a-a^\dagger) + \frac{\omega}{4}(a-a^\dagger)^*(a-a^\dagger)$$

$$H = \frac{1}{2}\omega\left(aa^\dagger + a^\dagger a\right)$$

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) \longrightarrow \text{Use } [a, a^\dagger] = 1$$

$\Rightarrow$ We can also show: $[H, a^\dagger] = \omega a^\dagger$

$[H, a] = -\omega a$

Let E be the energy eigenvalue of $H|E\rangle = E|E\rangle$

$$H a^\dagger|E\rangle = (E+\omega)\, a^\dagger|E\rangle$$

$$H a|E\rangle = (E-\omega)\, a|E\rangle$$

We have a ladder of states with energies

$\ldots, E-\omega, \quad E, \quad E+\omega, \quad E+2\omega, \quad \ldots$

If the Energy is bounded from below.

then, $H|0\rangle = \frac{1}{2}\omega|0\rangle$, $a|0\rangle = 0$

Excited States $|n\rangle = (a^\dagger)^n|0\rangle$

with $H|n\rangle = (n+\frac{1}{2})\omega\,|n\rangle$

$\langle n|n\rangle = 1$ (Normalization)

Zero-point Energy

- Free Scalar Field, we follow a similar recepie

$\phi(\vec{x})$, $\pi(\vec{x})$ (Lorentz invariance will be checked later on) just hold on!!

we expand

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right] \quad (A)$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right] \quad (B)$$

Claim $[\phi(\vec{x}), \phi(\vec{y})] = 0$, $[\pi(\vec{x}), \pi(\vec{y})] = 0$

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$$

$\Updownarrow$ Equivalent to

$$[a_{\vec{p}}, a_{\vec{q}}] = 0 = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}]$$

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^3(\vec{p}-\vec{q})$$

Show this (part of your Home-work)

We will illustrate only one of them here

assume $[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^3(\vec{p}-\vec{q})$ &

Show $[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$.

Proof $[\phi(\vec{x}), \pi(\vec{y})] = \displaystyle\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{(-i)}{2}\sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}}$

$$\times\left(-[a_{\vec{p}}, a^\dagger_{\vec{q}}]e^{i\vec{p}\cdot\vec{x}-i\vec{q}\cdot\vec{y}} + [a^\dagger_{\vec{p}}, a_{\vec{q}}]e^{-i\vec{p}\cdot\vec{x}+i\vec{q}\cdot\vec{y}}\right]$$

$$[\phi(\vec{x}), \pi(\vec{y})] = \int \frac{d^3p}{(2\pi)^3} \frac{(-i)}{2} \left(-e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{i\vec{p}(\vec{y}-\vec{x})} \right)$$

we have used commutating relationship

$$[a_{\vec{p}}, a^{\dagger}_{\vec{q}}] = (2\pi)^3 \delta^3(\vec{p}-\vec{q})$$ and

then we integrated d^3q, mostly "Dirac Delta" function.

$$[\phi(\vec{x}), \pi(\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y}).$$ ∎

we used the definition

$$-\frac{(-i)}{2} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \frac{i}{2}\delta^3(\vec{x}-\vec{y})$$

—— ~ —— ~ ——

Hamiltonian

* Let's compute the Hamiltonian in terms of $a_{\vec{p}}, a^{\dagger}_{\vec{p}}$

$$H = \frac{1}{2}\int d^3x\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right]$$

Use $\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left[a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right]$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{p}}}{2}}\left[a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right]$$

& $[a_{\vec{p}}, a_{\vec{q}}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0$

$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^3(\vec{p}-\vec{q})$.

$$H = \frac{1}{2}\int \frac{d^3x\, d^3p\, d^3q}{(2\pi)^6}\left[-\frac{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}{2}\left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right)\left(a_{\vec{q}}\, e^{i\vec{q}\cdot\vec{x}} - a^\dagger_{\vec{q}}\, e^{-i\vec{q}\cdot\vec{x}}\right)\right.$$

$$+ \frac{1}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}\left(i\vec{p}\, a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - i\vec{p}\, a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right)\cdot\left(i\vec{q}\, a_{\vec{q}}\, e^{i\vec{q}\cdot\vec{x}} - i\vec{q}\, a^\dagger_{\vec{q}}\, e^{-i\vec{q}\cdot\vec{x}}\right)$$

$$\left.+ \frac{m^2}{2\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}}\left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right)\cdot\left(a_{\vec{q}}\, e^{i\vec{q}\cdot\vec{x}} + a^\dagger_{\vec{q}}\, e^{-i\vec{q}\cdot\vec{x}}\right)\right]$$

$= 0$

$$= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\vec{p}}}\left[\cancel{(-\omega^2_{\vec{p}} + \vec{p}^2 + m^2)}\left(a_{\vec{p}}\, a_{-\vec{p}} + a^\dagger_{\vec{p}}\, a^\dagger_{-\vec{p}}\right) + (\omega^2_{\vec{p}} + \vec{p}^2 + m^2)\left(a_{\vec{p}}\, a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}}\, a_{\vec{p}}\right)\right]$$

$$H = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \omega_{\vec{p}} \left[a_{\vec{p}}\, a^{\dagger}_{\vec{p}} + a^{\dagger}_{\vec{p}}\, a_{\vec{p}} \right]$$

Use $\omega_{\vec{p}}^2 = \vec{p}^{\,2} + m^2$ (only 2nd term survives)

$$H = \int \frac{d^3p}{(2\pi)^3}\, \omega_{\vec{p}} \left[a^{\dagger}_{\vec{p}}\, a_{\vec{p}} + \frac{1}{2}(2\pi)^3 \delta^3(0) \right]$$

H blows up

- $\delta^3(0)$ gives a spike which is ∞
- $\omega_{\vec{p}}$ diverges at large $\vec{p}$.

- The Vacuum

Remind ourselves of Harmonic Oscillator problem, where the vacuum is defined by

- $a_{\vec{p}}|0\rangle = 0|0\rangle = 0$ for all $\vec{p}$

Now, Evaluate E_0 of the ground state from $\delta^3(0)$ contribution from $\underline{\underline{H}}$

$$H|0\rangle \equiv E_0|0\rangle = \left[\int \underline{d^3p}\ \frac{1}{2}\omega_{\vec{p}}\ \delta^3(0)\right]|0\rangle$$

$$= \infty|0\rangle$$

$\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

for $\vec{p} \to \infty$

$\omega_p \to \infty$

$4\pi p^2 dp$

$= p \to \infty$

Note: There are 2 kinds of infinities

(a) Space is ∞ (Infrared divergence)

$\downarrow$

IR

(b) $\vec{p} \to \infty$ (Ultraviolet (UV) divergence)

"Arbitrary short distances"

may be local formulation of QFT may not be valid, or there are new dynamical degrees of freedom !!

Dealing with infinities

* We are keen on Energy difference & not absolute Energy. So, we can redefine our Hamiltonian by subtracting off this infinity.

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} a^\dagger_{\vec{p}} a_{\vec{p}}}$$

With this new definition $H|0\rangle = 0$

$$a_{\vec{p}}|0\rangle = 0$$

Quantum Mechanically, we will define always the Hamiltonian by subtracting off the infinities.

So $a_{\vec{p}}$ will come towards the rightmost.

The above procedure is known as

NORMAL ORDERING

(VERY IMPORTANT)

If we have string of operators

$$\phi_1(\vec{x}_1)\,\phi_2(\vec{x}_2)\,\cdots\,\phi_n(\vec{x}_n)$$

we will NORMAL ORDER It by

$$:\phi_1(\vec{x}_1)\,\phi_2(\vec{x}_2)\,\cdots\,\phi_n(\vec{x}_n):$$

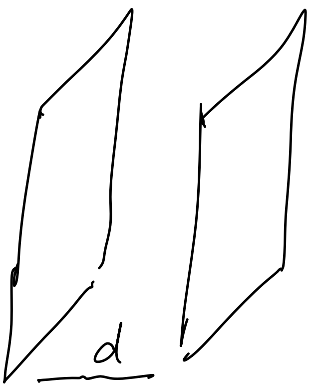
So, the Normal Ordered Hamiltonian is

$$:H: = \int \frac{d^3p}{(2\pi)^3}\,\omega_{\vec{p}}\,a_p^\dagger a_p$$

Beautiful Example: Cassimir Potential

Home-work Exercise

Differences in the energy of vacuum fluctuations can be measured !!



$$\frac{\partial E_{(+)}}{\partial d} = \underline{\underline{\frac{\pi}{24 d^2}}}$$

plays an important role in the development of String theory.

there we get $26 = \underbrace{24}_{\text{Space}} + \underbrace{(1+1)}_{\text{time}}$

Strings are extended objects (1 space & 1 time dimensional).

Excitations (1-particle State)

- What does excitation of the field means?

$$[H, a^\dagger_{\vec{p}}] = \omega_{\vec{p}}\, a^\dagger_{\vec{p}}, \quad [H, a_{\vec{p}}] = -\omega_{\vec{p}}\, a_{\vec{p}}$$

$\therefore$ we can construct energy eigenstates by acting on the vacuum $|0\rangle$ with $a^\dagger_{\vec{p}}$.

$$|\vec{p}\rangle = a^\dagger_{\vec{p}} |0\rangle$$

This state has energy $H|\vec{p}\rangle = \omega_{\vec{p}} |\vec{p}\rangle$

with $\omega_{\vec{p}}^2 = \vec{p}^{\,2} + m^2$.

This is equivalent to energy $E_{\vec{p}}^2 = \vec{p}^{\,2} + m^2$

$$\boxed{\omega_{\vec{p}}^2 = E_{\vec{p}}^2.}$$

- We can define $\vec{P}$ (momentum) $= -\int d^3x\, \pi\, \vec{\nabla}\phi$

$$\vec{P} = -\int d^3x \, \pi \vec{\nabla}\phi = \int \frac{d^3p}{(2\pi)^3} \vec{p} \, a^\dagger_{\vec{p}} a_{\vec{p}}$$

$$\boxed{\vec{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle}$$

State $|\vec{p}\rangle$ has a momentum $\vec{p}$

- We can define $J^i = \epsilon^{ijk} \int d^3x (J^0)^{jk}$

one can show, one-particle state has

$J^i|\vec{p}=0\rangle = 0$, means a particle carries no internal angular momentum.

Quantising a scalar field will give rise to a zero spin particle

Multi-particle States (Bosonic Statistics) and Fock State

$\Rightarrow$ Very simple multi particle states are given by multi-particle momentum states

$$|\vec{p}_1, \vec{p}_2, \ldots \vec{p}_n\rangle = a^\dagger_{\vec{p}_1} \ldots a^\dagger_{\vec{p}_n} |0\rangle$$

all $a^\dagger$ commute among themselves
the $|\vec{p}_1 \ldots \vec{p}_n\rangle$ is Symmetric under exchange of two particles.

$$|\vec{p}, \vec{q}\rangle = |\vec{q}, \vec{p}\rangle. \Rightarrow \text{Bosons}$$

Full Hilbert space will be:

$$|0\rangle, \; a^\dagger_{\vec{p}}|0\rangle, \; a^\dagger_{\vec{p}} a^\dagger_{\vec{q}}|0\rangle, \; a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} a^\dagger_{\vec{r}}|0\rangle$$

$\ldots$

Such a Hilbert Space is known as

Fock State

Fock State = Simply the sum of n-particle Hilbert space for all $n \geq 0$

Useful Operator ⇒ Number Operator

$$N = \int \frac{d^3p}{(2\pi)^3} \, a^\dagger_{\vec{p}} \, a_{\vec{p}}$$

$$N \, |\vec{p}_1, \vec{p}_2, \ldots \vec{p}_n\rangle = n \, |\vec{p}_1, \vec{p}_2, \ldots \vec{p}_n\rangle$$

$$[N, H] = 0$$

Note: Property of a Free theory & not of an interacting theory.

What does it mean by $|\vec{p}\rangle$

- Are they localised?
  No they are momentum eigenstate.

- Recall QM (position & momentum eigenstates are not good elements of Hilbert space)
  They are not Normalised States

Means, $\langle 0| a_p \, a^\dagger_{\vec{p}} |0\rangle = \langle \vec{p} | \vec{p} \rangle = (2\pi)^3 \delta(0)$

and $\langle 0 | \phi(\vec{x}) \, \phi(\vec{x}) | 0 \rangle = \langle \vec{x} | \vec{x} \rangle = \delta(0)$.

These are operator valued distributions rather than functions.

$\Rightarrow$ Although $\phi(x)$ has a well defined vacuum expectation value

$\langle 0|\phi(\vec{x})|0\rangle = 0$, the fluctuations of the operator at a fixed point are $\langle 0|\phi(\vec{x})\phi(\vec{x})|0\rangle = \infty$

We can construct a wavepacket

$$|\varphi\rangle = \int \frac{d^3p}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{x}}\, \varphi(\vec{p})\, |\vec{p}\rangle$$

A typical state might be

$$\varphi(\vec{p}) = e^{-\vec{p}^2/2m^2}$$

Gaussian State

Relativistic Normalisation

Vacuum $|0\rangle \Rightarrow \langle 0|0\rangle = 1$ (Normalised)

$|\vec{p}\rangle = a^\dagger_{\vec{p}}|0\rangle \Rightarrow \langle \vec{p}|\vec{q}\rangle = (2\pi)^3\delta^3(\vec{p}-\vec{q})$.

But we have only 3 vectors $\vec{p}$!!

is it Lorentz Invariant?

Answer is NO

How to make it Lorentz Invariant?

Note: $p^\mu \to p'^\mu = \Lambda^\mu{}_\nu p^\nu$

$$\vec{p} \longrightarrow \vec{p}\,'$$

In Quantum theory $\quad |\vec{p}\rangle \to |\vec{p}\,'\rangle = \underline{\underline{U(\Lambda)}}\,|\vec{p}\rangle$

Unitary Operator / Unitary transformation

In general, $\quad |\vec{p}\rangle \Longrightarrow \underbrace{\lambda(\vec{p},\vec{p}\,')}_{\text{Some Unknown function}}\,|\vec{p}\,'\rangle$

The trick is to find an object which Lorentz invariant, One such object is

$$\boxed{1 = \int \frac{d^3p}{(2\pi)^3}\,|\vec{p}\rangle\langle\vec{p}|} \Rightarrow \text{Lorentz Invariant}$$

But, Individually $\int \frac{d^3p}{(2\pi)^3}$ is not Lorentz invariant and nor $|\vec{p}\rangle\langle\vec{p}|$

Claim $\int d^4p \rightarrow$ Lorentz Invariant

and $\displaystyle\int \frac{d^3p}{2E_{\vec{p}}}$ is Lorentz Invariant.

$\Rightarrow$ Note $\underline{p_\mu p^\mu = m^2} \Rightarrow p_0^2 = E_{\vec{p}}^2 = \vec{p}^2 + m^2$

Lorentz Invariant.

Solving $p_0 = \pm\sqrt{\vec{p}^2 + m^2}$ or $\pm E_{\vec{p}}$

+ or −ve branch is another Lorentz invariant

Concept, So

$$\boxed{\int d^4p \, \delta(p_0^2 - \vec{p}^2 - m^2)\Big|_{p_0>0} = \int \frac{d^3p}{2p_0}\Big|_{p_0 = E_{\vec{p}}}}$$

is a Lorentz Invariant quantity.

$\Rightarrow$ Lorentz invariant δ function for 3 vectors is

$$\boxed{2E_{\vec{p}}\, \delta^{(3)}(\vec{p}-\vec{q}).}$$ Lorentz Invariant

$$\Rightarrow \int \frac{d^3p}{2E_{\vec{p}}}\, 2E_{\vec{p}}\, \delta^{(3)}(\vec{p}-\vec{q}) = 1$$

$\therefore$ Relativistically Normalised States are

$$|p\rangle = \sqrt{2E_{\vec{p}}}\, |\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a^{\dagger}_{\vec{p}} |0\rangle$$

Relativistically normalised state $|p\rangle$ differs by $\sqrt{2E_{\vec{p}}}$ factor compared to $|\vec{p}\rangle$

$$\underline{\langle p|q\rangle} = (2\pi)^3\, 2E_{\vec{p}}\, \delta^{(3)}(\vec{p}-\vec{q})$$

$$1 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} |p\rangle\langle p|$$

Note, Some texts take $\boxed{a^{\dagger}(p) = \sqrt{2E_{\vec{p}}}\, a^{\dagger}_{\vec{p}}}$

Complex Scalar Field

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi^* - M^2 \psi^* \psi.$$

$$\psi = (\phi_1 + i\phi_2)/\sqrt{2}$$ (You get the right K.E. for ϕ_1 & ϕ_2)

Equations of motion:

$$\partial_\mu \partial^\mu \psi + M^2 \psi = 0, \quad \partial_\mu \partial^\mu \psi^* + M^2 \psi^* = 0$$

$$\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(b_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\psi^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{(2E_{\vec{p}})^{1/2}} \left(b_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}} + c_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} \right).$$

Since ψ is not real, the corresponding quantum field ψ is not hermitian. This is the reason we have $b_{\vec{p}}$ & $c_{\vec{p}}$ are two different operators.

$$\pi = \partial\mathcal{L}/\partial\dot{\psi} = \dot{\psi}^*$$

$$\pi = \int \frac{d^3p}{(2\pi)^3}\, i \sqrt{\frac{E_{\vec{p}}}{2}} \left(b^{\dagger}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} - c_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} \right)$$

$$\pi^{\dagger} = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(b_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - c^{\dagger}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right).$$

$$[\psi(\vec{x}), \pi(\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y}).$$

$$[\psi(x), \pi^{\dagger}(y)] = 0. = [\psi(\vec{x}), \psi(\vec{y})] = 0$$

$$0 = [\psi(\vec{x}), \psi^{\dagger}(\vec{y})].$$

One can show,

$[b_{\vec{p}}, b^{\dagger}_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$, $[c_{\vec{p}}, c^{\dagger}_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$

Rest $[b_{\vec{p}}, b_{\vec{q}}] = [c_{\vec{p}}, c_{\vec{q}}] = [b_{\vec{p}}, c_{\vec{q}}] = [b_{\vec{p}}, c^{\dagger}_{\vec{q}}] = 0$

$\Rightarrow$ Complex Scalar field = 2 particle states with $b^{\dagger}_{p}, c^{\dagger}_{p}$ and both are spin-0 states.

$$Q = i\int d^3x \left(\dot{\psi}^{*}\psi - \psi^{*}\dot{\psi} \right) = i\int d^3x \left(\pi\psi - \psi^{*}\pi^{*} \right)$$

After Normal Ordering $Q = N_c - N_b$.

$[H, Q] = 0$

Heisenberg Picture

- Why time is so special?
- What happens to Lorentz Invariance?
- $\phi(\vec{x})$ depends on $\vec{x}$ and not on time, why?
- One-particle Schrödinger equation evolves with respect to time only.

i.e. $i\dfrac{d|\vec{p}(t)\rangle}{dt} = H|\vec{p}(t)\rangle \Rightarrow |\vec{p}(t)\rangle = e^{-iE_{\vec{p}}t}|\vec{p}\rangle$

Recall Quantum Physics-II, where we studied Heisenberg & Interaction pictures.

- In Heisenberg picture, time dependence is assigned to the operator $\mathcal{O}$,

$$\mathcal{O}_H = e^{iHt}\,\underbrace{\mathcal{O}_S}_{\text{Schrödinger picture operator.}}\,e^{-iHt} \quad \text{— (1)}$$

Differentiating above eqn (i), we get

$$\frac{dO_H}{dt} = i[H, O_H]$$

Subscripts. S & H correspond to Schrödingers' and Heisenberg picture.

In field theory $\phi(\vec{x})$ corresponds to Schrödingers' picture

$\phi(\vec{x}, t)$ corresponds to Heisenberg's picture !!

Two pictures agree at $t=0$

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = [\pi(\vec{x}, t), \pi(\vec{y}, t)] = 0$$

$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y})$$

Eventually $\phi(x)$ will evolve in time, and become $\phi(\vec{x}, t)$!!

$$\dot{\phi} = i[H, \phi] = \frac{i}{2}\left[\int d^3y \, \pi(y)^2 + \nabla\phi(y)^2 + m^2\phi(y)^2, \phi(x)\right]$$

$$= i\int d^3y \, \pi(y)(-i)\delta^3(\vec{y} - \vec{x}) = \pi(x)$$

— (A)

Similarly $\dot{\pi} = i[H, \pi]$

$$\dot{\Pi} = \frac{i}{2}\left[\int d^3y\, \pi(y)^2 + \nabla\phi(y)^2 + m^2\phi(y)^2,\ \pi(x)\right]$$

$$= \frac{i}{2}\int d^3y\, \Big(\nabla_y [\phi(y), \pi(x)]\nabla\phi(y) + \nabla\phi(y)[\phi(y), \pi(x)]$$

$$+ 2im^2\phi(y)\, \delta^{(3)}(\vec{y}-\vec{x})\Big]$$

$$= -\left(\int d^3y \left(\nabla_y \delta^{(3)}(\vec{x}-\vec{y})\nabla_y \phi(y)\right) - m^2\phi(x)\right)$$

$$= \nabla^2\phi - m^2\phi \qquad - (B)$$

Now taking both (A) & (B) into account, we get

$$\partial_\mu \partial^\mu \phi + m^2\phi = 0 \qquad \text{(Klein-Gordan equation)}$$

We can go from Schrödinger to Heisenberg picture

$$e^{iHt} a_{\vec{p}}\, e^{-iHt} = e^{-iE_{\vec{p}}t} a_{\vec{p}}$$

$$e^{iHt} a^\dagger_{\vec{p}}\, e^{-iHt} = e^{iE_{\vec{p}}t} a^\dagger_{\vec{p}}.$$

$$\Rightarrow [H, a_{\vec{p}}] = -E_{\vec{p}} a_{\vec{p}} \ \& \ [H, a^\dagger_{\vec{p}}] = +E_{\vec{p}} a^\dagger_{\vec{p}}.$$

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip.x} + a^\dagger_{\vec{p}}\, e^{+ip.x} \right)$$

Lorentz Invariant.

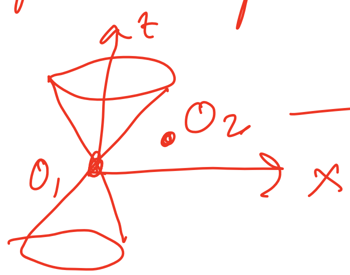
$$\boxed{\begin{aligned} p\cdot x &= 4 \text{ Momenta } p \ \& \ x \\ &= E_{\vec{p}}\, t - \vec{p}\cdot\vec{x} \end{aligned}}$$

Important: Causality

What Causality means?

$$[O_1(x), O_2(y)] = 0 \quad \forall \ (x-y)^2 < 0$$

Space like operators must commute



$\longrightarrow$ Must Commute

O_1 doesn't know about O_2 (Not inside the light cone)

Measurement in X can't influence measurement in Y? and vice-verse.

$$\Delta(x-y) = [\phi(x), \phi(y)].$$

• Note $\Delta(x-y)$ is a Lorentz invariant

However $[\phi(\vec{x},t), \pi(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y})$ is not

"equal time Commutation relationship"

$$\boxed{\begin{aligned} &\text{Now, } \Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip(x-y)}\right) \\ &\Delta(x-y) = [\phi(x), \phi(y)] \end{aligned}}$$

• Above expression is Lorentz invariant. $\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}$

• $[\phi(\vec{x},0), \phi(\vec{x},t)] \sim e^{-imt} - e^{imt}$

$x-y = (t, 0, 0, 0)$

• $\Delta(x-y) = 0$ at equal times for all

$(x-y)^2 = -(\vec{x}-\vec{y})^2 < 0$

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\vec{p}^2+m^2}} \left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right)$$

Note: $\vec{p} \longrightarrow -\vec{p}$,

Since $\Delta(x-y)$ is Lorentz Invariant, it can only depend on $(x-y)^2$ $\therefore [\phi(\vec{x},t), \phi(\vec{y},t)] = 0$

for $(x-y)^2 < 0$

Since $\displaystyle\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\vec{p}^2+m^2}} \left[e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right]$

$$\rightarrow \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\vec{p}^2+m^2}} e^{-i\vec{p}\cdot(x-y)}$$

$$= \int \frac{d^3p}{(2\pi)} \frac{e^{i\vec{p}\cdot(x-y)}}{\sqrt{\vec{p}^2+m^2}}$$

- However $\Delta(x-y)$ must and only depends on $(x-y)^2$ and must therefore vanish for all $(x-y)^2 < 0$!!

Propagator(s)

- Very Important.
- A powerful way to probe the structure of Causality.

prepare your State

• y

what is the probability of finding the state here?

• x

Mathematically: $\langle 0 | \phi(x)\, \phi(y) | 0 \rangle = ?$

$$\langle 0 | \phi(x)\, \phi(y) | 0 \rangle = \left[\int \frac{d^3p \, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{4 E_{\vec{p}} E_{\vec{p}}}} \langle 0 | a_{\vec{p}}\, a^\dagger_{\vec{p}} | 0 \rangle \, e^{-i\vec{p}\cdot\vec{x} + i\vec{p}\cdot\vec{y}} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-i\vec{p}\,(\vec{x}-\vec{y})} = D(x-y)$$

Function $D(x-y)$ is called the <u>propagator</u>

For $(x-y)^2 < 0$, $D(x-y) \sim e^{-m|x-y|}$

- So, it decays exponentially outside the <u>light cone</u>
- Inside the lightcone it is non-vanishing
- Quantum fields do leak out of <u>lightcone</u>

However, we have seen that space-like measurements must commute and the theory is Causal. How shall we reconcile this?

- Recall $[\phi(\vec{x},t), \phi(\vec{y},t)]$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\vec{p}^2+m^2}} \left(e^{+i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}(\vec{x}-\vec{y})} \right)$$

$\Rightarrow$ We can re-write the above equation as

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x)$$

where

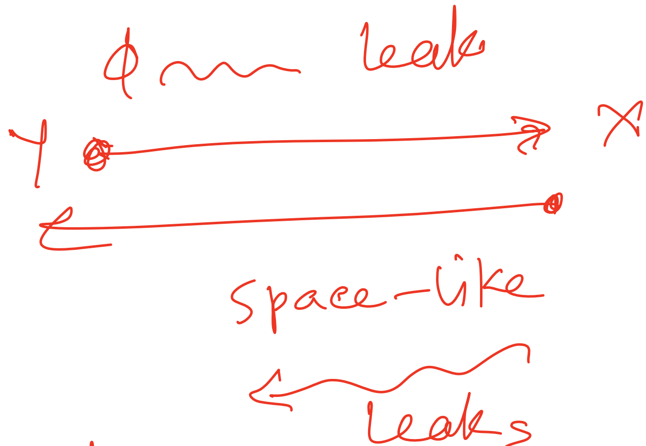
$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$$E_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m^2}.$$

Now $[\phi(x), \phi(y)] = 0$ if $(x-y)^2 < 0$

When $(x-y)^2<0$, there is no Lorentz-invariant way to order events.

if



In any measurement, the amplitudes must cancel!!

- For Complex field $[\phi(x), \phi^\dagger(y)] = 0$ outside lightcone.

- Interpretation is particle to propagat $x \longrightarrow y$ Cancels the amplitude for the anti-particle to travel from $y \rightarrow x$.

- Such an interpretation is there for a real field as well, because particle is its own anti-particle!!

Feynman Propagator

Most important quantities in interacting field theory is (F-P)

- Defined as

$$\Delta_F(x-y) = \langle 0 | T\phi(x)\phi(y) | 0 \rangle$$

$$= \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & y^0 < x^0 \end{cases}$$

T = Time Ordered / Time Ordering

Place all operators evaluated at later times to the left, so

$$T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y) & x^0 > y^0 \\ \phi(y)\phi(x) & y^0 > x^0 \end{cases}$$

- $$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip.(x-y)}$$

4-momentum

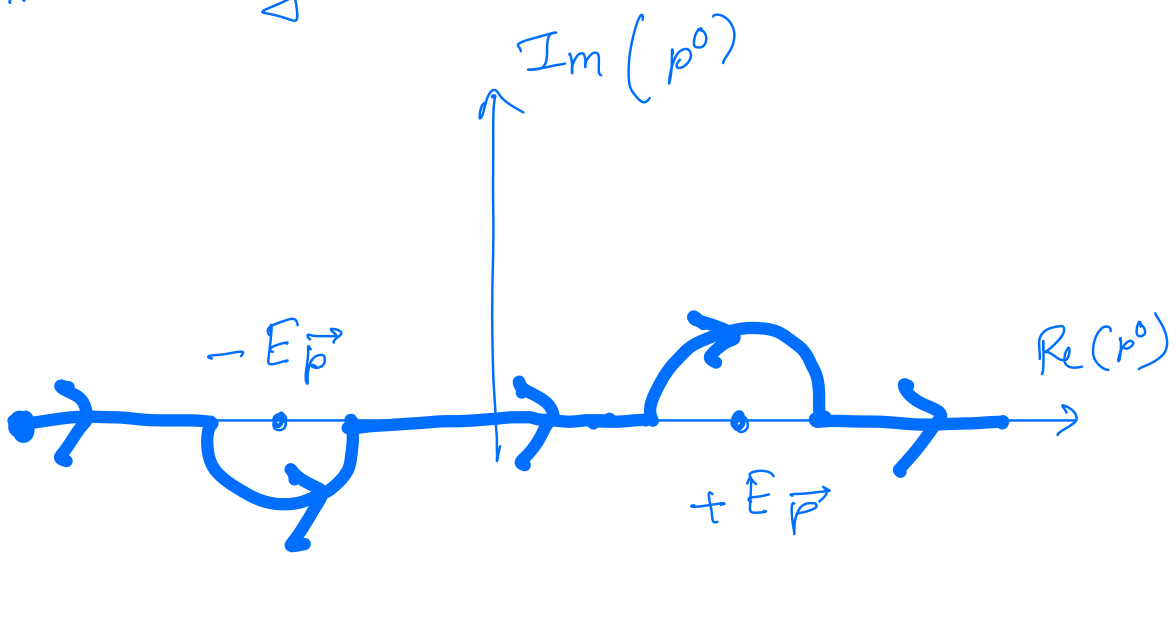
p^0 is fixed by mass-shell condition

$$\boxed{p^0 = E_{\vec{p}}}$$

Observe: for each value of $\vec{p}$, the denominator: $p^2 - m^2 = (p^0)^2 - \vec{p}^2 - m^2$

We have a pole: $p^0 = \pm E_{\vec{p}} = \pm\sqrt{\vec{p}^2 + m^2}$

- $\Delta_F(x-y)$ blows up at these poles.
- We need to define a prescription to evaluate the integration.



$$\frac{1}{p^2-m^2} = \frac{1}{(p^0)^2 - E_{\vec{p}}^2} = \frac{1}{(p^0 - E_{\vec{p}})(p^0 + E_{\vec{p}})}$$

Residue of the pole at $p^0 = \pm E_{\vec{p}}$ is

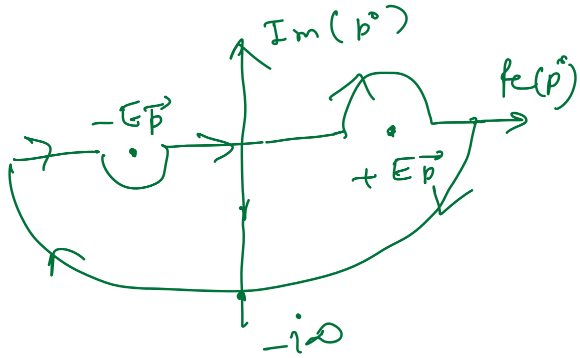
$$\pm \frac{1}{2 E_{\vec{p}}}$$

[Simple pole

$$\text{Res}\,[f, c] = \lim_{z \to c} (z-c) f(z) \quad]$$

When $x^0 > y^0$ we close the contour in the lower half plane, ie

$$T\phi(x)\phi(y) = \phi(x)\phi(y) \qquad x^0 > y^0$$



where $p^0 \to -i\infty$

ensures $e^{-ip_0(x^0-y^0)} \to 0$.

The integral over p^0 picks up the residue at $p^0 = +E_{\vec{p}}$

The residue is $\frac{-2\pi i}{2E_{\vec{p}}}$ $\longrightarrow$ clockwise contour

when $x^0 > y^0$

$$\therefore \quad \Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^4}\left[\frac{-2\pi i}{2E_{\vec{p}}}\; i\; e^{-iE_{\vec{p}}(x_0-y_0)+i\vec{p}\cdot(\vec{x}-\vec{y})}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\, e^{-ip(x-y)} = D(x-y)$$

actually $\vec{p}\cdot(\vec{x}-\vec{y})$
but we can write as $p(x-y)$
because 0^{th} component vanishes exponentially.

- Repeat the same computation when $y^0 > x^0$, the contour will be upper half plane and the arrow will be anti-clockwise

Steps: $\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^4}\frac{+2\pi i}{(-2E_{\vec{p}})}\; i\; e^{+iE_{\vec{p}}(x^0-y^0)+i\vec{p}(\vec{x}-\vec{y})}$

$$\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-iE_{\vec{p}}(y^0-x^0) - i\vec{p}(\vec{y}-\vec{x})}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip.(y-x)} = D(y-x)$$

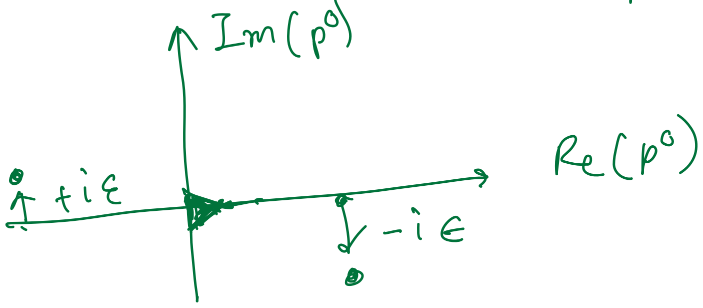
actually $\vec{p}(\vec{y}-\vec{x})$ only contributes.

$$\Delta_F(x-y) = D(x-y) \quad \text{when } x^0 > y^0$$
$$= D(y-x) \quad \text{when } y^0 > x^0$$

Succinctly, we can write Feynman Propagator as

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip.(x-y)}}{p^2-m^2+i\varepsilon}, \quad \epsilon > 0$$

small



This way of writing the propagator is known as $i\varepsilon$ prescription.

Green Functions

- Another way to write a propagator.

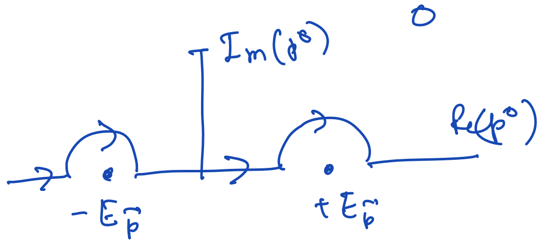
Green's function for the Klein-Gordon operator

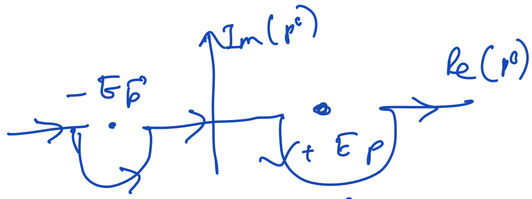
$$\left(\partial_t^2 - \vec{\nabla}^2 + m^2\right)\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2}\left(-p^2+m^2\right)e^{-ip\cdot(x-y)}$$

$$= -i\int \frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot(x-y)} = -i\,\delta^{(4)}(x-y)$$

Note: we avoided contour here, (no contours at all)

$$\Delta_R(x-y) = D(x-y) - D(y-x) \quad \text{for } x^0 > y^0$$
$$0 \qquad\qquad\qquad\qquad\qquad\qquad y^0 > x^0$$





Retarded

↓

Useful for Classical field theory (initial Conditions known)

$$\partial_\mu\partial^\mu\phi + m^2\phi = J(x).$$

Advanced

Useful for QFT.

(final Conditions are known.

$$\Delta_A(x-y) = 0 \text{ for } y^0 < x^0$$

Non-Relativistic Limit

How to go from QFT $\longrightarrow$ Schrödinger's Interpretation

- Let's take complex scalar field obeying Klein-Gordon equation

$$\psi(\vec{x},t) = e^{-imt}\tilde{\psi}(\vec{x},t)$$

$$\Rightarrow \text{K-G equation: } \partial_t^2\psi - \vec{\nabla}^2\psi + m^2\psi = e^{-imt}\left[\ddot{\tilde{\psi}} - 2im\dot{\tilde{\psi}} - \vec{\nabla}^2\tilde{\psi}\right] = 0$$

Non-Relativistic limit $|\vec{p}| \ll m$

$$\Rightarrow |\ddot{\tilde{\psi}}| \ll m|\dot{\tilde{\psi}}|, \text{ in this limit}$$

$$i\frac{\partial\tilde{\psi}}{\partial t} = -\frac{1}{2m}\vec{\nabla}^2\tilde{\psi} \Longrightarrow$$ Schrödinger's equation for non-relativistic free particle

Hamiltonian Formulation

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^*$$ [Momentum Conjugate of ψ is $i\psi^*$]

$$\mathcal{H} = \frac{1}{2m}\nabla\psi^*\nabla\psi$$

Impose $[\psi(\vec{x}), \psi(\vec{y})] = 0$

$$[\psi^\dagger(\vec{x}), \psi^\dagger(y)] = 0$$

$$[\psi(x), \psi^\dagger(y)] = i\delta^3(\vec{x}-\vec{y})$$

$$\psi(\vec{x}) = \int\frac{d^3p}{(2\pi)^3}a_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}}$$

$$\longrightarrow [a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^3(\vec{p}-\vec{q})$$

Vacuum satisfies $a_{\vec{p}}|0\rangle = 0$ & 1-particle state has $H|\vec{p}\rangle = \frac{\vec{p}^2}{2m}|\vec{p}\rangle$

- Only one particle state of a Complex field & not 2.
  No anti-particle. "It is a Concept of relativity"
  or Consequence of relativity

- Conserved charge $Q = \int d^3x : \psi^\dagger \psi :$ is the particle number.

- It's only with relativity, and the presence of anti-particle that particle number can change.

- There is no non-relativistic limit of a real scalar field.
  In relativistic theory particle = anti-particle in the Case of a real scalar field theory.

Recovering QM

In QM we have $\vec{X}$ & $\vec{P}$ operators. In QFT, positions $(\vec{X}, \vec{P}) \longrightarrow$ are relegated to a label.

How do we get back to QM?

- $\vec{P} = \int \frac{d^3p}{(2\pi)^3} \, \vec{p} \, a^\dagger_{\vec{p}} a_{\vec{p}} \Rightarrow$ 1 particle State
  $\vec{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$

• In N-R Limit $\psi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}$

↳ Creates a particle with δ-function localisation at $\vec{x}$.

$|\vec{x}\rangle = \psi^\dagger(\vec{x})|0\rangle$. A natural position operator will be: $\vec{X} = \int d^3x \; \vec{x} \; \psi^\dagger(\vec{x})\psi(\vec{x})$ so that

$\vec{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$.

Now, let's take $|\psi\rangle = \int d^3x \; \psi(\vec{x})|\vec{x}\rangle$.

↳ Schrödinger's wave function.

$$X^i|\psi\rangle = \int d^3x \; x^i \psi(\vec{x})|\vec{x}\rangle$$

$$p^i|\psi\rangle = \int d^3x \left(-i\frac{\partial\psi}{\partial x^i}\right)|\vec{x}\rangle \longrightarrow$$ You can show this little algebra.

You can show $[X^i, p^j] = i\delta^{ij}$ [H-W exercise]

$$H = \int d^3x \frac{1}{2m}\nabla\psi^*\nabla\psi = \int \frac{d^3p}{(2\pi)^3}\frac{\vec{p}^2}{2m} a^\dagger_{\vec{p}} a_{\vec{p}}$$

& $$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\vec{\nabla}^2\psi.$$

— x — x —