Relativistic Quantum Mechanics
Tutorial 1 Partial Solutions

1. We define $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $x^2 = x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu$ and $x^4 = (x^2)(x^2)$. Evaluate:

$$\partial_\mu(x^2), \qquad \partial^\lambda \partial_\lambda (x^\mu x^\rho), \qquad \partial_\mu \partial_\nu (x^4).$$

$$\begin{aligned}
\partial_\mu(x^2) &= \partial_\mu(x^\nu x_\nu), \\
&= (\partial_\mu x^\nu) x_\nu + x^\nu(\partial_\mu x_\nu), \\
&= \delta^\nu_\mu x_\nu + x^\nu \eta_{\mu\nu}, \\
&= 2x_\mu.
\end{aligned}$$

Remember: you cannot use the same index three times in the same term. Always remember what you want to calculate. You want to calculate the derivative with respect to x^μ of a sum of $x_0 x^0 + x_1 x^1 + \ldots$

$$\begin{aligned}
\partial^\lambda \partial_\lambda (x^\mu x^\rho) &= \partial^\lambda ((\partial_\lambda x^\mu) x^\rho + x^\mu (\partial_\lambda x^\rho)), \\
&= \partial^\lambda (\delta^\mu_\lambda x^\rho + x^\mu \delta^\rho_\lambda), \\
&= \delta^\mu_\lambda \eta^{\lambda\rho} + \eta^{\lambda\mu} \delta^\rho_\lambda, \\
&= 2\eta^{\mu\rho}.
\end{aligned}$$

Since in the example above, we have two free upper indices, the result should also contain the same two free upper indices.

$$\begin{aligned}
\partial_\mu \partial_\nu (x^4) &= 2\partial_\mu \left(x^2 \partial_\nu (x^2) \right), \\
&= 2\partial_\mu (2 x_\nu x^2), \\
&= 4(\eta_{\mu\nu} x^2 + 2 x_\mu x_\nu).
\end{aligned}$$

2. The motion of a complex field ψ is governed by the Lagrangian density

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2}(\psi^* \psi)^2.$$

a) Derive this system's equations of motion using the Euler-Lagrange equations. (We treat ψ and ψ^* as separate fields, for which we use the Euler-Lagrange equation separately.)

We have

$$\begin{aligned} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) &= \partial^2 \psi^* \\ = \frac{\partial \mathcal{L}}{\partial \psi} &= -m^2 \psi^* - \lambda (\psi^*)^2 \psi \end{aligned}$$

and

$$\begin{aligned} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \right) &= \partial^2 \psi \\ = \frac{\partial \mathcal{L}}{\partial \psi^*} &= -m^2 \psi - \lambda \psi^2 \psi^*. \end{aligned}$$

b) Verify that the Lagrangian is invariant under the infinitesimal transformation

$$\psi \to (1 + i\alpha)\psi, \qquad \psi^* \to (1 - i\alpha)\psi^*,$$

where α is an infinitesimal parameter.

$$\begin{aligned} \mathcal{L} \to \mathcal{L}' &= \partial_\mu (1 - i\alpha)\psi^* \partial^\mu (1 + i\alpha)\psi - m^2 (1 - i\alpha)\psi^* (1 + i\alpha)\psi - \frac{\lambda}{2}((1 - i\alpha)\psi^*(1 + i\alpha)\psi)^2, \\ &= (1 + \alpha^2)\partial_\mu \psi^* \partial^\mu \psi - (1 + \alpha^2) m^2 \psi^* \psi - \frac{\lambda}{2}((1 + \alpha^2)\psi^* \psi)^2, \\ &= \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2}(\psi^* \psi)^2, \\ &= \mathcal{L}. \end{aligned}$$

Since α is infinitesimal, α^2 drops away.

c) Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations derived in a). Since we have two fields, the Noether current is given by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} X_\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} X_{\psi^*} - F^\mu.$$

We know that $\delta \mathcal{L} = 0$, since the Lagrangian is invariant. Hence, F^μ is constant. Adding a constant to any component of a conserved current also results

in a conserved current. Hence, we do not care about the value of F^μ and pick $F^\mu = 0$.
X_ψ is $i\psi$ and X_{ψ^*} is $-i\psi^*$. (In Tong's notation convention, $\delta\psi$ does not contain the infinitesimal parameter.)
Hence,

$$j^\mu = i(\partial^\mu \psi^*)\psi - i(\partial^\mu \psi)\psi^*.$$

To check if this is indeed a conserved current, we simply calculate the divergence:

$$\begin{aligned}
\partial_\mu j^\mu &= i\left[(\partial^2\psi^*)\psi - (\partial^2\psi)\psi^*\right], \\
&= i\left[(-m^2\psi^* - \lambda(\psi^*)^2\psi)\psi - (-m^2\psi - \lambda\psi^2\psi^*)\psi^*\right], \\
&= i\left[-m^2|\psi|^2 - \lambda|\psi|^4 + m^2|\psi|^2 + \lambda|\psi|^4\right], \\
&= 0.
\end{aligned}$$

Therefore, this is indeed a conserved current.

3. A Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is such that it preserves the Minkowski metric $\eta_{\mu\nu}$, meaning that $\eta_{\mu\nu}x'^\mu x'^\nu = \eta_{\mu\nu}x^\mu x^\nu$ for all x.
   a) Show that this is equivalent to

   $$\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}.$$

   b) Use this result to show that an infinitesimal transformation of the form

   $$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

   is in fact a Lorentz transformation if $\omega^{\mu\nu}$ is anti-symmetric, i.e. $\omega^{\mu\nu} = -\omega^{\nu\mu}$.

   c) Derive the Noether current associated with the infinitesimal Lorentz transformation

   $$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu,$$

   where

   $$\omega^\mu{}_\nu \equiv \theta\Omega^\mu{}_\nu,$$

   θ being infinitesimal and $\Omega^\mu{}_\nu$ not, for Lagrangian density

   $$\mathcal{L} = \partial^\mu\psi^*\,\partial_\mu\psi - \psi^*\psi.$$

   The answer should be:

   $$j^\mu = \Omega^\rho{}_\nu x^\nu\left[\delta^\mu_\nu \mathcal{L} - \partial^\mu\psi\,\partial_\rho\psi^* - \partial^\mu\psi^*\,\partial_\rho\psi\right].$$

   Hint: This is a transformation that changes the fields and the Lagrangian by changing their argument, because ψ, ψ^* and $\mathcal{L}$ are scalar fields. Determine $\delta\mathcal{L}$, $\delta\psi$ and $\delta\psi^*$ by doing a first order Taylor expansion for the fields and the Lagrangian.

   d) Show that

   $$j^\mu = -\Omega^\rho{}_\nu x^\nu T^\mu{}_\rho.$$