Units and Scales

$c, \quad \hbar = \frac{h}{2\pi}, \quad G$

$[c] = LT^{-1}, \quad [\hbar] = L^2 M T^{-1},$

$[G] = L^3 M^{-1} T^{-2}$

$c = \hbar = 1.$ (Natural Units), $k_B = 1$

$[\text{Energy}] = [\text{Mass}] = [\text{Temperature}]$

$= [\text{Length}]^{-1} = [\text{Time}]^{-1}$

$1\ \text{GeV} = 10^3\ \text{MeV} = 10^6\ \text{keV} = 10^9\ \text{eV}$

- $1\ \text{GeV} \simeq 1.602 \times 10^{-3}\ \text{erg}$
- $1\ \text{GeV} \simeq 1.1605 \times 10^{13}\ \text{K}$
- $1\ \text{GeV} \simeq 1.7827 \times 10^{-24}\ \text{g}$
- $1\ \text{GeV}^{-1} = 1.9733 \times 10^{-14}\ \text{cm}$
- $1\ \text{GeV}^{-1} = 6.5822 \times 10^{-25}\ \text{sec}$
- $1\ \text{GeV}^2 = 2.431 \times 10^{24}\ \text{erg sec}^{-1}$

• Compton wavelength : $\lambda \sim \frac{\hbar}{mc}$

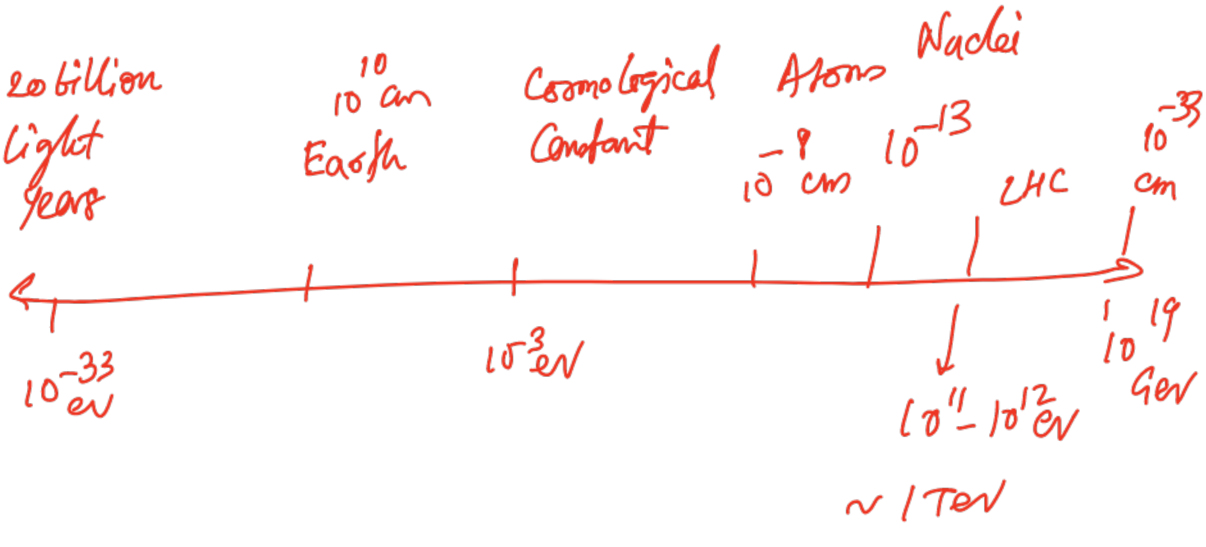
$m_e \sim 10^6 eV, \quad \lambda_e \sim 10^{-12} m$

Sometime Compton wavelength $\Rightarrow \lambda \sim \frac{2\pi\hbar}{mc}$

Similarly $G = \frac{\hbar c}{M_p^2} = \frac{1}{M_p^2}$

$M_p \approx 10^{19} GeV$ Planck Scale

$l_p \approx 10^{-33} cm, \quad t_p \approx 10^{-44} sec$



- Neutrinos $\sim 10^{-2}$ eV
- electrons $\sim$ 0.5 MeV
- Muon $\sim 10^{2}$ MeV
- Pion $\sim$ 140 MeV
- Proton, Neutron $\sim$ 1 GeV
- Tau $\sim$ 2 GeV
- W, Z Boson $\sim$ 80–90 GeV
- Higgs $\sim$ 125 GeV

$\Rightarrow$ All belong to Standard Model Physics

there are plethora of Beyond the Standard Model particles (hypothetical) as well, whose properties are not so well (directly) constrained.

Classical Field Theory

$(\vec{x}, t)$

Field.

$\phi_a(\vec{x}, t)$

label

e.g. $\vec{E}(\vec{x}, t)$, $\vec{B}(\vec{x}, t)$

$A^\mu(\vec{x}, t) = (\phi, \vec{A})$

3 Vector

$\mu = 0, 1, 2, 3$

0^{th} component.

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \frac{d\vec{B}}{dt} = -\vec{\nabla} \times \vec{E}$$

Lagrangian

- $L(t) = \int d^3x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a)$

Lagrange density (Lagrangian)

$$S = \text{action} = \int_{t_1}^{t_2} dt \int d^3x \, \mathcal{L}$$

$$= \int d^4x \, \mathcal{L}$$

in general, $\boxed{S_d = \int d^dx \, \mathcal{L}}$

in 'd' Spacetime dimensions.

In particle theory

$\mathcal{L}(q, \dot{q})$ but no $\ddot{q}$

In field theory

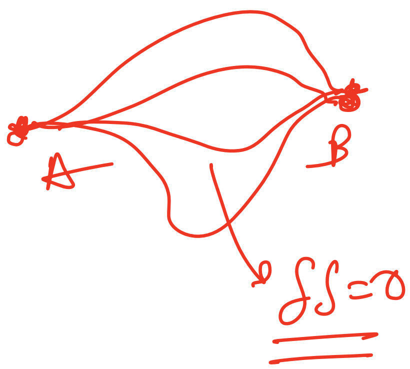
$$\mathcal{L}(\nabla\phi, \nabla^2\phi, \nabla^3\phi, \ldots\ldots)$$

* <u>NOTE</u> : Make sure the Lagrangian is Lorentz invariant.

- we will make sure, we study only 2 derivative theories.
- higher derivative theories bring issues such as "Ghosts", we will discuss this later.
- In principle you can study infinite derivative theories as well.

$$S \longrightarrow \delta S = 0$$

↳ <u>Variation</u>



Least Action Principle.

By keeping A & B fixed we vary the paths, and make sure. we consider the least action path. i.e. path which minimises the action.

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \underline{\delta (\partial_\mu \phi_a)} \right]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \right] \delta \phi_a$$

$$+ \underline{\partial_\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a \right)$$

Note $[\partial (AB) = (\partial A) B + A (\partial B)]$

Last term is a total derivative

also the last term obeys: $\delta\phi_a(\vec{x}, t_1) = 0$

$$\delta\phi_a(\vec{x}, t_2) = 0$$

Now, requiring $\delta S = 0$

$\Rightarrow$ Euler-Lagrange Equations of Motion

$$\boxed{\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right] - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0}$$

Equations of motion.

E.g. $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2$

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Mostly $-$ve, you can select mostly $+$ve as well.

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2$$

using $(+,-,-,-)$ signature

$$= -\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2$$

using $(-,+,+,+)$ signature.

- Let's use $(+,-,-,-)$ signature and you can repeat the exercise for $(-,+,+,+)$ signature

T = Kinetic Energy $= \int d^3x \, \frac{1}{2}\dot{\phi}^2$

V = Potential Energy $= \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]$

$L = T - V$,

$\nabla\phi$ = energy gradient.

$\frac{m^2\phi^2}{2}$ = Potential Energy

Equations of motion

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \quad , \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$= \partial^0\phi - \vec{\nabla}\phi$$

$$= (\dot{\phi} - \vec{\nabla}\phi)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)$$

$$\Rightarrow -m^2\phi = \partial_\mu \left(\partial^0\phi , -\vec{\nabla}\phi \right)$$

$$-m^2\phi = \partial_0 \partial^0 \phi - \vec{\nabla}\vec{\nabla}\phi$$

$$-m^2\phi = \ddot{\phi} - \vec{\nabla}^2\phi$$

$$\Rightarrow \quad \ddot{\phi} - \vec{\nabla}^2\phi + m^2\phi = 0$$

or, $\quad \partial_\mu \partial^\mu \phi + m^2\phi = 0$

$$\Box\phi + m^2\phi = 0$$

$$(\partial_0\partial^0\phi - \partial_i\partial^i\phi) + m^2\phi = 0$$

$$\Rightarrow \quad \ddot{\phi} - \partial_i^2\phi + m^2\phi = 0$$

$$\Rightarrow \quad \ddot{\phi} - \vec{\nabla}^2\phi + m^2\phi = 0$$

[See the Steps] $\Rightarrow$ Now do it for $(-,+,+,+)$ Signature you will find the Same equations of motion [So, physics doesn't depend on the choice of metric signature].

For a generic Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\partial^\mu\phi - V(\phi)$$

$$\Rightarrow \boxed{\partial_\mu\partial^\mu\phi + \frac{\partial V}{\partial\phi} = 0}$$

- Home work $\Rightarrow$ do the First order Lagrangian

$$\mathcal{L} = \frac{i}{2}\left(\psi^* \dot{\psi} - \dot{\psi}^* \psi\right) - \vec{\nabla}\psi^* \cdot \vec{\nabla}\psi - m\psi^*\psi$$

Find equations of motion ?

"Let's Play with Indices"

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2$$

Lagrangian doesn't depend on $(\mu, \nu, \ldots)$ all indices are contracted

- Note: $-$ve sign $-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)$ expand it $(0, 1, 2, 3)$ all the components, you will see that the kinetic term becomes $\mathcal{L} \sim \frac{1}{2}\dot{A}_i^2$ and has no $\dot{A}_0^2$ term.

[We will see it's consequence later on].

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + (\partial_\rho A^\rho)\eta^{\mu\nu}$$

Now, $\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\partial^2 A^\nu + \partial^\nu(\partial_\rho A^\rho)$

$$= -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\partial_\mu F^{\mu\nu}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ or, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
is the field strength.

Now, we $A^\mu(\vec{x}, t) = (\phi, \vec{A})$

$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$ & $\vec{B} = \vec{\nabla} \times \vec{A}$

and show that all these equations of motion arise from $\partial_\mu F^{\mu\nu} = 0$

We can write down Maxwell Lagrangian up to total derivative (or up to an integration by parts)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

(Birth place for QED if we add electron term in the Lagrangian)

Very Important [Locality]

- Lagrangian is Local

Locality is one of the pillars of relativistic quantum mechanics / (Field theory)

↳ Quantum

- We can describe non-local field theory but as an advanced course. Not now!!

- What does it mean?

$$\phi(\vec{x}, t), \quad \phi(\vec{y}, t) \text{ with } x \neq y$$

$$L = \int d^3x \, d^3y \, \underbrace{\phi(x)\phi(y)}_{}$$

Non-local if $x \neq y$.

The closest non-locality we will approach will be by taking

$\vec{y} = \vec{x} + \delta\vec{x}$ → Expanding it we get $(\nabla\phi)^2$ gradient term

* Very Important [Lorentz Invariance]

* Laws of nature are relativistic

* Physics is also frame independent

$$x^\mu \longrightarrow (x')^\mu = \Lambda^\mu{}_\nu \, x^\nu$$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-v^2}}$$

(a boost in $\vec{x}$ axis).

a metric $ds^2 = c^2dt^2 - (dx^2+dy^2+dz^2)$

[$c=1$]

$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$

$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$

Note $c=1$ (we take)

$ds^2 = dx^\mu dx_\mu = dt^2 - dx^2 - dy^2 - dz^2$

Similarly $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \partial_1, \partial_2, \partial_3)$

$$= \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$\partial^\mu = \eta^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right).$$

$$\Box = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

- $$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Lorentz transformation have a representation on the fields

$$x \to \Lambda x$$

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$$

$$(\partial_\mu \phi)(x) \to (\Lambda^{-1})^{\nu}{}_{\mu} (\partial_\nu \phi)(\Lambda^{-1} x)$$

$$\to (\Lambda^{-1})^{\nu}{}_{\mu} (\partial_\nu \phi)(y)$$

- $\mathcal{L}_{derivative}(x) = \partial_\mu \phi(x) \partial_\nu \phi(x) \eta^{\mu\nu}$

$$\to \left[(\Lambda^{-1})^{\rho}{}_{\mu} (\partial_\rho \phi)(y)\right]\left[(\Lambda^{-1})^{\sigma}{}_{\nu} (\partial_\sigma \phi)(y)\right] \eta^{\mu\nu}$$

$$= (\partial_\rho \phi)(y) (\partial_\sigma \phi)(y) (\Lambda^{-1})^{\rho}{}_{\mu} (\Lambda^{-1})^{\sigma}{}_{\nu} \eta^{\mu\nu}$$

$$= (\partial_\rho \phi)(y) (\partial_\sigma \phi)(y) \left[\eta^{\rho\sigma} \right]$$

$$= \mathcal{L}_{derivative}(y).$$

where, $\boxed{y = \Lambda^{-1} x}$

- Potential term $\phi^2(x) \to \phi^2(y)$

- Adding both

$$S = \int d^4x \, \mathcal{L}(x) \to \int d^4x \, \mathcal{L}(y)$$

$$= \int d^4y \, \mathcal{L}(y) = S.$$

$$\int d^4x = \int d^4y$$ [No Jacobian factor]

This is because $\det \Lambda = 1$.

$$d^4x = J\left(\frac{x}{y}\right) d^4y.$$

↳ Jacobian of the transformation $x \to y$

let's say $y \to x + \delta x$

then, $$\frac{\partial y^\mu}{\partial x^\lambda} = \delta^\mu_\lambda + \partial_\lambda \delta x^\mu$$

So, $J\left(\frac{y}{x}\right) = \det\left(\frac{\partial y^\mu}{\partial x^\nu}\right) = 1 + \partial_\mu(\partial x^\mu)$

$\underbrace{}_{\text{leading order term}}$

See discussion in Ryder's book
Section 3.2

- For any Lagrangian you should be able to check that the action S is Lorentz invariant. (spin 0, spin $-\frac{1}{2}$, spin-1, ... spin-2, ...)

Very Important [Noether's theorem].

- Key to understand symmetries in the action. e.g. Lorentz symmetry, internal symmetries, gauge symmetries, etc

$\Rightarrow$ For every continuous symmetry of the Lagrangian there exists a Conserved Current $J^{\mu}(x)$

$\Rightarrow$ Such that $\partial_\mu J^\mu = 0$

$$\Rightarrow \frac{\partial J^0}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

Comment: A conserved current $\Rightarrow$ Conserved Q (Charge)

$$Q = \int_{R^3} d^3x \, J^0$$

$$\frac{dQ}{dt} = \int_{R^3} d^3x \frac{\partial J^0}{\partial t} = - \int_{R^3} d^3x \, \vec{\nabla}\cdot\vec{J} = 0$$

Assumption $\vec{J} \to 0$ as $|\vec{x}| \to \infty$

"Charge is conserved locally"

How to show?

$$\Rightarrow \quad Q_V = \int_V d^3x\, J^0$$

$$\frac{dQ_V}{dt} = -\int_V d^3x\, \vec{\nabla}\cdot\vec{J} = -\int_A \vec{J}\cdot d\vec{S}$$

(Stokes' theorem)

Local conservation of charge holds in any local quantum field theory.

Proof of Noether's theorem:

$\delta\phi_a(x) = X_a(\phi)$ is a symmetry if the Lagrangian changes by a total derivative.

$$\delta\mathcal{L} = \partial_\mu F^\mu \quad \text{for some set of function } F^\mu(\phi)$$

Let's consider making 'arbitrary' transformation of the field $\delta\phi_a$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a)$$

$$= \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)$$

When the equations of motion are satisfied, we have $[\ldots\ldots] = 0$, so,

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a \right) \quad \text{— (A)}$$

But for the symmetry transformation $\delta\phi_a = X_a(\phi)$, we have $\delta \mathcal{L} = \partial_\mu F^\mu$. — (B)

By equating the above two equations (A) & (B), we get:

$$\partial_\mu J^\mu = 0, \quad \text{with } j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X_a(\phi) - F^\mu(\phi)$$

Take some Examples

- Translation & the Energy-momentum Tensor.

$$x^\mu \to x^\mu - \epsilon^\mu \Rightarrow \phi_a(x) \to \phi_a(x) + \epsilon^\mu \partial_\mu \phi_a(x)$$

infinitesimal transformation

$$\mathcal{L}(x) \to \mathcal{L}(x) + \epsilon^{\mu}\partial_{\mu}\mathcal{L}(x) \quad , \quad \mu, \nu = 0,1,2,3.$$

$$(j^{\mu})_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi_a)}\partial_{\nu}\phi_a - \delta^{\mu}_{\nu}\mathcal{L} \underset{\text{define}}{\equiv} T^{\mu}_{\nu}$$

where, T^{μ}_{ν} = Energy-momentum Tensor.

It satisfies $\partial_{\mu}T^{\mu}_{\nu} = 0$

$E = \int d^3x \, T^0_0$ & $P^i = \int d^3x \, T^{0i}$

E → Energy

P^i → momentum

e.g. of Energy-momentum tensor

$$T^{\mu\nu} = \partial^{\mu}\phi\,\partial^{\nu}\phi - \eta^{\mu\nu}\mathcal{L}$$

Take $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}m^2\phi^2$

By using equations of motion, you can verify $\partial_{\mu}T^{\mu\nu} = 0$, you can show

$$E = \int d^3x \underbrace{\left[\frac{\dot{\phi}^2}{2} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]}_{\text{+ve definite.}}$$

$$p^i = \int d^3x \, \dot{\phi}\, \partial^i \phi$$

(For this example $T^{\mu\nu} = T^{\nu\mu}$ = Symmetric in μ & ν.)

- Example-2.

Lorentz Transformation & Angular Momentum

$\Rightarrow$ Rotational invariance $\Rightarrow$ Conservation of angular momentum?

$\Rightarrow$ What is the analogy in field theory?

$$\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\ \nu}$$

$\omega^{\mu}_{\ \nu} \longrightarrow$ Infinitesimal !!

Since $\Lambda^{\mu}_{\ \sigma} \Lambda^{\nu}_{\ \tau} \eta^{\sigma\tau} = \eta^{\mu\nu}$

$$\Rightarrow \left(\delta^{\mu}_{\sigma} + \omega^{\mu}_{\ \sigma}\right)\left(\delta^{\nu}_{\tau} + \omega^{\nu}_{\ \tau}\right)\eta^{\sigma\tau} = \eta^{\mu\nu}$$

$$\Rightarrow \omega^{\mu\nu} + \omega^{\nu\mu} = 0$$

$$\therefore \omega^{\mu\nu} = -\omega^{\nu\mu} \quad \text{[Anti-Symmetric]}$$

Note: number of terms: $\begin{pmatrix} & 3 \\ 3 & \end{pmatrix} = 6 = 3$ rotations $+ 3$ boosts

Now, $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$

$$= \phi(x^\mu - \omega^\mu{}_\nu x^\nu)$$

$$= \phi(x^\mu) - \underbrace{\omega^\mu{}_\nu x^\nu \partial_\mu \phi(x)}_{\delta\phi}$$

$$\delta\phi = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi(x)$$

$$\delta\mathcal{L} = -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = \boxed{-\partial_\mu(\underbrace{\omega^\mu{}_\nu x^\nu \mathcal{L}}).}$$

$\underline{\omega^\mu{}_\mu = 0}$ (anti-symmetric)

Since Lagrangian changes be total derivative we can apply Noether's theorem:

Now, $F^\mu = -\omega^\mu{}_\nu x^\nu \mathcal{L}$

$\therefore$ Conserved Current

$$j^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \omega^\rho{}_\nu x^\nu \partial_\rho \phi + \omega^\mu{}_\nu x^\nu \mathcal{L}$$

we have used:

$\left[J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} X_a(\phi) - F^\mu(\phi) \right.$, where $\delta\phi_a = X_a(\phi)$ & $\left. \delta\mathcal{L} = \partial_\mu F^\mu. \right]$

For each choice of $\omega_{\rho\sigma}$

$$(J^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

which satisfies $\partial_\mu (J^\mu)^{\rho\sigma} = 0 \Rightarrow$ 6 Conserved currents.

for $\rho, \sigma = 1, 2, 3.$

$$\left.\begin{aligned} Q^{ij} &= \int d^3x \left(x^i T^{0j} - x^j T^{0i}\right) \\ Q^{0i} &= \int d^3x \left(x^0 T^{0i} - x^i T^{i0}\right) \end{aligned}\right\}$$

they are conserved $\Rightarrow 0 = \frac{dQ^{0i}}{dt} = \int d^3x\, T^{0i}$

$$+ t \int d^3x \frac{\partial T^{0i}}{\partial t}$$

$$- \frac{d}{dt} \int d^3x\, x^i T^{00}$$

$$\Rightarrow 0 = p^i + t \frac{dp^i}{dt} - \frac{d}{dt} \int d^3x\, x^i T^{00}$$

$\Rightarrow$ We know that p^i is conserved, so $\frac{dp^i}{dt} = 0$

$$\Rightarrow \frac{d}{dt} \int d^3x\, x^i T^{00} = \text{Constant}$$

$\Rightarrow$ Centre of energy of the field travels with a

Constant velocity. (Analogue of Newton's first Law)

- Internal Symmetries.

def: An internal symmetry is one that involves a transformation of the fields and acts the same at every point in spacetime.

e.g. $\psi(x) = \dfrac{\phi_1(x) + i\phi_2(x)}{\sqrt{2}}$ Complex Scalar field

$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - V(|\psi|^2)$

$|\psi|^2 = \psi^* \psi$.

Let's vary w.r.t. $\psi^* \Rightarrow \partial_\mu \partial^\mu \psi + \dfrac{\partial V(\psi^* \psi)}{\partial \psi^*} = 0$

Has a Continuous symmetry

$\psi = e^{i\alpha}\psi$ or $\delta\psi = i\alpha\psi$

Infinitesimal.

Lagrangian $\delta L = 0$.

$j^\mu = i(\partial^\mu \psi^*)\psi - i\psi^*(\partial^\mu \psi)$

Very Important: Hamiltonian Formalism.

momentum $\quad \pi^a(x) = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}_a}$

$$\mathcal{H} = \pi^a(x)\,\dot{\phi}_a(x) - \mathcal{L}(x)$$

$$H = \int d^3x\, \mathcal{H}$$

e.g. $\quad \mathcal{L} = \dfrac{\dot{\phi}^2}{2} - \dfrac{(\nabla\phi)^2}{2} - V(\phi), \quad \pi = \dot{\phi}$

$$H = \int d^3x \underbrace{\left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]}_{+\text{ve definite}}$$

$$\dot{\phi}(\vec{x},t) = \frac{\partial H}{\partial \pi(\vec{x},t)}, \quad \dot{\pi}(\vec{x},t) = -\frac{\partial H}{\partial \phi(\vec{x},t)}$$