Problems: 6.15, 7.3, 7.6

Problem from homework 2021-2022

Find the curve $y(x)$ that minimises the functional

$$S[y] = \int_0^1 \sqrt{1 + x + (y')^2}\, dx$$

With $y(0) = 0$ and $y(1) = 1$.

Solution

We use the Euler equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

With

$$f = \sqrt{1 + x + (y')^2}$$

$$\frac{\partial f}{\partial y} = 0 \text{ and } \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + x + (y')^2}}$$

The Euler equation becomes

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + x + (y')^2}}\right) = 0$$

From which we conclude that

$$\frac{y'}{\sqrt{1 + x + (y')^2}} = C \Longrightarrow y' = C\sqrt{1 + x + (y')^2}$$

With C a constant of integration.

Squaring and rearranging the previous equation we get

$$\left(\frac{dy}{dx}\right)^2 = \frac{C^2}{1 - C^2}(1 + x) = k^2(1 + x)$$

With $k^2 = C^2/(1 - C^2)$

Taking the square root, we get

$$y' = k\sqrt{1 + x}$$

With k a constant, which can be positive or negative.

We can re-write the equation as

$$\frac{dy}{dx} = k\sqrt{1+x} \Longrightarrow \int dy = k \int \sqrt{1+x}\, dx$$

It follows that

$$y = \frac{2}{3}k(1+x)^{3/2} + B$$

Using the boundary conditions, we can calculate the constants C and B.

$$y(0) = \frac{2}{3}k + B = 0$$

$$y(1) = \frac{2}{3}k\; 2^{3/2} + B = 1$$

$$\Longrightarrow \frac{2}{3}k = -B \Longrightarrow \frac{2}{3}k\; 2^{3/2} - \frac{2}{3}k = 1$$

So, we have

$$k = \frac{3}{2}\frac{1}{2^{3/2}-1}$$

And

$$B = -\frac{1}{2^{3/2}-1}$$

So the final solution is

$$y = \frac{1}{2^{3/2}-1}(1+x)^{3/2} - \frac{1}{2^{3/2}-1}$$

$$y = \frac{(1+x)^{3/2}-1}{2^{3/2}-1}$$

6.15

$$I(y) = \int_0^1 \left[\left(\frac{dy}{dx}\right)^2 - y^2\right] dx = \int_0^1 (y'^2 - y^2)\, dx$$

a. Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = -2y - \frac{d}{dx}(2y') = 0$$

$$0 = -2y - 2\frac{d}{dx}\left(\frac{dy}{dx}\right) = -2y - 2\frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} + y = 0$$

b. Solution to equation (simple harmonic oscillator with $\omega_0^2 = 1$)

$$y = A\cos x + B\sin x$$

c. Boundary conditions:
$x = 0, y = 0$

$$0 = A\cos 0 + B\sin 0 \Longrightarrow A = 0$$

$x = 1, y = 1$

$$1 = B\sin 1 \Longrightarrow B = \frac{1}{\sin 1}$$

Therefore,

$$y = \frac{\sin x}{\sin 1}$$

d. Minimum value of integral:

$$I(y) = \int_0^1 \left[\left(\frac{dy}{dx}\right)^2 - y^2\right] dx = \frac{1}{(\sin 1)^2}\int_0^1 [(\cos x)^2 - (\sin x)^2] dx$$

Using

$$(\cos x)^2 = \frac{1}{2}(1 + \cos 2x);\ (\sin x)^2 = \frac{1}{2}(1 - \cos 2x)$$

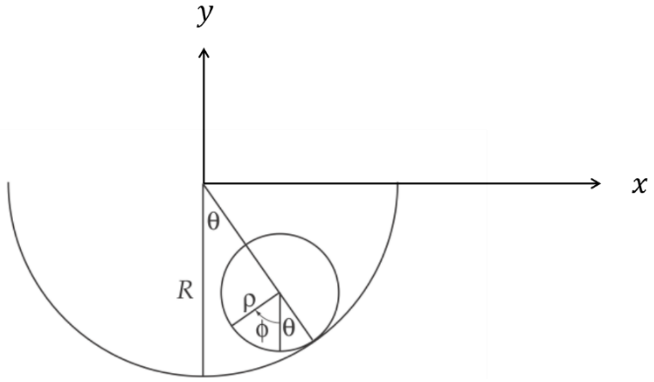
We obtain

$$I(y) = \frac{1}{(\sin 1)^2}\int_0^1 \cos 2x\ dx = \frac{1}{(\sin 1)^2}\left[\frac{\sin 2x}{2}\right]_0^1 = \frac{\sin 2}{2\,(\sin 1)^2} = 0.642$$

e. $x = y$

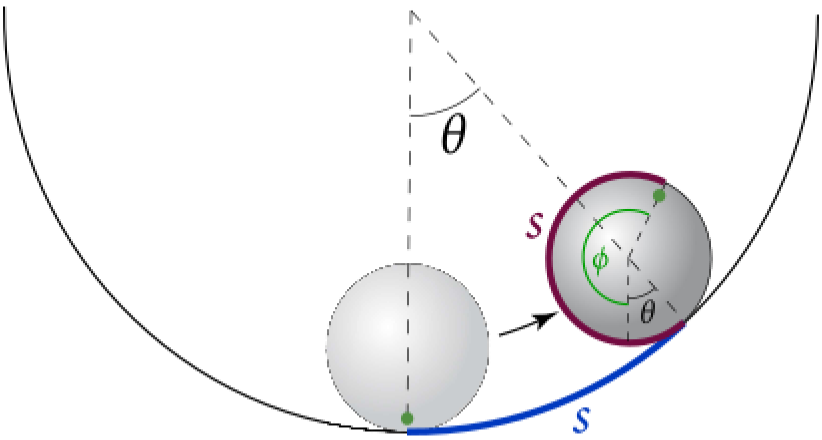
$$I = \int_0^1 (1 - y^2)\, dy = \left[y - \frac{y^3}{3}\right]_0^0 = 1 - \frac{1}{3} = \frac{2}{3}$$

7.3



We have 2 coordinates θ and ϕ. The no slip condition reduces the degree of freedom by 1, so we have one generalized coordinate θ.

The **equation of constraint**



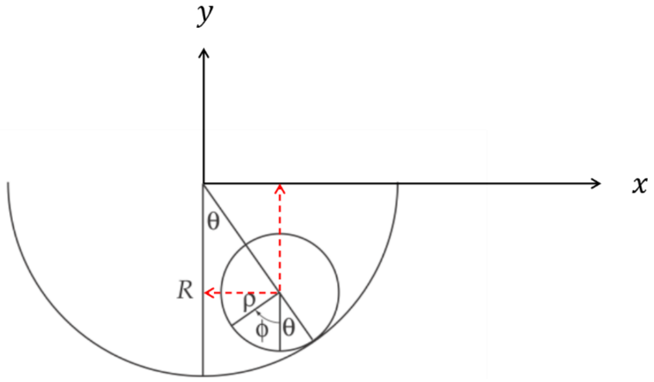
From the figure, we see that the distance S travelled by the sphere on the surface of the cylinder is $S = R\theta$. Since the sphere is rolling without slipping, the same distance S is also equal to $S = \rho\phi + \rho\theta$.
This leads to the equation of constraint:

$$R\theta = \rho\phi + \rho\theta$$

And finally

$$(R - \rho)\theta = \rho\phi$$

The **coordinates of the centre of the sphere** are



$$x = (R-\rho)\sin\theta$$
$$y = -(R-\rho)\cos\theta$$

The derivatives are

$$\dot{x} = (R-\rho)\dot{\theta}\cos\theta$$
$$\dot{y} = (R-\rho)\dot{\theta}\sin\theta$$

The **kinetic energy** is

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}I\dot{\phi}^2$$

Where I is the moment of inertia of sphere with respect to the diameter:

$$I = \frac{2}{5}m\rho^2$$

So, the kinetic energy becomes:

$$T = \frac{1}{2}m\left[\left((R-\rho)\dot{\theta}\cos\theta\right)^2+\left((R-\rho)\dot{\theta}\sin\theta\right)^2\right]+\frac{1}{2}\frac{2}{5}m\rho^2\dot{\phi}^2$$

$$T = \frac{1}{2}m(R-\rho)^2\dot{\theta}^2+\frac{1}{5}m\rho^2\dot{\phi}^2$$

From the equation of constraint $(R-\rho)\theta = \rho\phi$, we obtain a relationship between $\dot{\theta}$ and $\dot{\phi}$:

$$\dot{\phi} = \frac{R-\rho}{\rho}\dot{\theta}$$

Finally, we get for the kinetic energy

$$T = \frac{1}{2}m(R-\rho)^2\dot{\theta}^2+\frac{1}{5}m\rho^2\dot{\phi}^2 = \frac{1}{2}m(R-\rho)^2\dot{\theta}^2+\frac{1}{5}m\rho^2\left(\frac{R-\rho}{\rho}\dot{\theta}\right)^2$$

$$T = \frac{7}{10} m(R-\rho)^2\dot{\theta}^2$$

The **potential energy**, according to the system of coordinates, is

$$U = mg[R - (R-\rho)\cos\theta]$$

Finally, we can write the **Lagrangian** as:

$$L = T - U$$

$$L = \frac{7}{10} m(R-\rho)^2\dot{\theta}^2 - mg[R - (R-\rho)\cos\theta]$$

Applying the Lagrange equations:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$\frac{\partial L}{\partial \theta} = -mg(R-\rho)\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{7}{10} m(R-\rho)^2 2\dot{\theta} = \frac{7}{5} m(R-\rho)^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{7}{5} m(R-\rho)^2\ddot{\theta}$$

The **equation of motion** is then:

$$-mg(R-\rho)\sin\theta - \frac{7}{5} m(R-\rho)^2\ddot{\theta} = 0$$

Or rearranging the terms:

$$\frac{7}{5} m(R-\rho)^2\ddot{\theta} + mg(R-\rho)\sin\theta = 0$$

Applying the small angle approximation $\sin\theta \sim \theta$:

$$\frac{7}{5} m(R-\rho)^2\ddot{\theta} + mg(R-\rho)\theta = 0$$

$$\ddot{\theta} + \frac{5g}{7(R-\rho)}\theta = 0$$

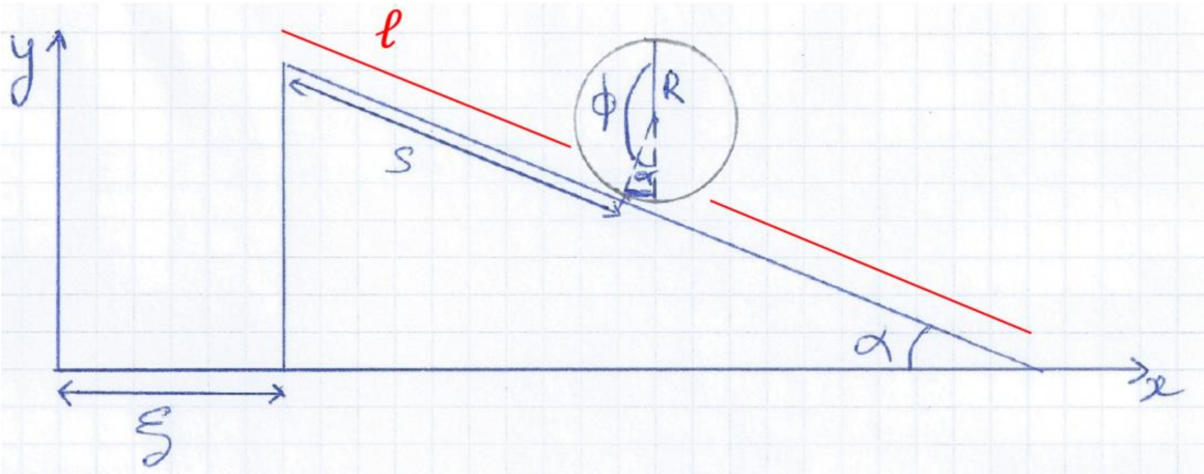
We have an equation of the form (harmonic oscillator):

$$\ddot{\theta} + \omega^2\theta = 0$$

Where ω is the frequency of small oscillations:

$$\omega = \sqrt{\frac{5g}{7(R-\rho)}}$$

7.6



Coordinates of the centre of the hoop:

$$x_h = \xi + S\cos\alpha + R\sin\alpha$$

$$y_h = R\cos\alpha + (\ell - S)\sin\alpha$$

Derivatives:

$$\dot{x}_h = \dot{\xi} + \dot{S}\cos\alpha$$

$$\dot{y}_h = -\dot{S}\sin\alpha$$

Kinetic energy of the hoop:

$$T_h = \frac{1}{2}m(\dot{x}_h^2 + \dot{y}_h^2) + \frac{1}{2}I\dot{\phi}^2$$

With $I = mR^2$ and $S = R\phi$.

$$T_h = \frac{1}{2}m\left(\dot{\xi}^2 + 2\dot{\xi}\dot{S}\cos\alpha + \dot{S}^2\cos^2\alpha + \dot{S}^2\sin^2\alpha\right) + \frac{1}{2}mR^2\frac{\dot{S}^2}{R^2}$$

$$T_h = \frac{1}{2}m\left(\dot{\xi}^2 + 2\dot{\xi}\dot{S}\cos\alpha + 2\dot{S}^2\right)$$

Kinetic energy of the wedge:

$$T_w = \frac{1}{2}M\dot{\xi}^2$$

Total kinetic energy:

$$T = T_h + T_w = \frac{1}{2}m\left(\dot{\xi}^2 + 2\dot{\xi}\dot{S}\cos\alpha + 2\dot{S}^2\right) + \frac{1}{2}M\dot{\xi}^2$$

Potential energy of the hoop:

$$U_h = mg(R\cos\alpha + (\ell - S)\sin\alpha)$$

Potential energy of the wedge:

$$U_w = 0$$

Lagrangian:

$$L = T - U = \frac{1}{2}(M+m)\dot{\xi}^2 + m\dot{S}^2 + m\,\dot{\xi}\dot{S}\cos\alpha - mg(R\cos\alpha + (\ell - S)\sin\alpha)$$

Equations of motion:

$$\frac{\partial L}{\partial \xi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}} = 0 - \frac{d}{dt}\left[(M+m)\dot{\xi} + m\dot{S}\cos\alpha\right] = -(M+m)\ddot{\xi} + m\ddot{S}\cos\alpha = 0 \qquad (1)$$

$$\frac{\partial L}{\partial S} - \frac{d}{dt}\frac{\partial L}{\partial \dot{S}} = mg\sin\alpha - \frac{d}{dt}\left[2m\dot{S} + m\dot{\xi}\cos\alpha\right] = mg\sin\alpha - 2m\ddot{S} - m\ddot{\xi}\cos\alpha = 0 \qquad (2)$$

We can write these equations in the form of decoupled equations:

From equation (1), we have

$$(M+m)\ddot{\xi} + m\ddot{S}\cos\alpha = 0 \Longrightarrow \ddot{\xi} = -\frac{m\cos\alpha}{M+m}\,\ddot{S}$$

Replacing $\ddot{\xi}$ in equation (2) gives:

$$mg\sin\alpha - 2m\ddot{S} - m\left(-\frac{m\cos\alpha}{M+m}\,\ddot{S}\right)\cos\alpha = 0$$

$$2\ddot{S} - \left(\frac{m\cos\alpha}{M+m}\,\ddot{S}\right)\cos\alpha = g\sin\alpha \Longrightarrow \ddot{S}\left[2 - \frac{m\cos^2\alpha}{M+m}\right] = g\sin\alpha$$

$$\ddot{S} = \frac{g\sin\alpha\,(M+m)}{2(M+m) - m\cos^2\alpha}$$

And replacing $\ddot{S}$ in the expression of $\ddot{\xi}$, we obtain

$$\ddot{\xi} = -\frac{m\cos\alpha}{M+m}\,\frac{g\sin\alpha\,(M+m)}{2(M+m) - m\cos^2\alpha} = \frac{mg\sin\alpha\cos\alpha}{2(M+m) - m\cos^2\alpha}$$

Looking at

$$\frac{d}{dt}\left[(M+m)\dot{\xi}+m\dot{S}\cos\alpha\right]=0$$

We can see that $(M+m)\dot{\xi}$ is the x-component of the linear momentum of the total (hoop + wedge) system and $m\dot{S}\cos\alpha$ the x-component of the linear momentum of the hoop with respect to the wedge.

This equation shows that the total linear momentum is a constant of motion, which makes sense since no external force is applied along the x-axis.