1 Aim and Approach

Tournament is a group of robots in a perfect backet with no byes

$$R_T = 2^n, n \in Z^+,$$

Rearranging above equation

$$n = \log_2(R_T).$$

$$\begin{aligned} n &= \log_2(4) \\ &= 2, \end{aligned}$$

Tournament winning general equation

$$P(W_n) = P(W_1) \cdot P(W_2|W_1) \cdot P(W_3|W_2) \cdot \ldots P(W_{n-1}|W_{n-2}) \cdot P(W_n|W_{n-1}),$$

2 Robot game winning probability

Setting variables and initial info

$$\log_2(4) = 2,$$

$$0 \leq p \leq 1000,$$

$$0 \leq s \leq 1.$$

$$0.25 \leq s \leq 0.75,$$

(Partition Picture)
General equation for partitioned sample space

$$P(W_H) = P(B_1) \cdot P(W_H|B_1) + P(B_2) \cdot P(W_H|B_2) + P(B_3) \cdot P(W_H|B_3) + P(B_4) \cdot P(W_H|B_4).$$

$$P(B_1) = P(s'_A \cap s'_B)$$

sA and sB are independenet, so:

$$\begin{aligned} P(s'_A \cap s'_B) &= P(s'_A) \cdot P(s'_B) \\ &= (1 - P(s_A)) \cdot (1 - P(s_B)). \end{aligned}$$

$$P(B_1) = (1 - P(s_A)) \cdot (1 - P(s_B)).$$

B2 and B3 are easy because

$$p_A > p_B.$$

Flip a coin method

$$\lim_{x \to \infty} (s^x) = 0, 0 \leq s < 1.$$

Edge case for infinite game.

$$p_H = p_E$$

$$s_H = s_E = 1$$

Calculations for flip a coin method for Robot A

$$\begin{aligned}
P(W_A|B_1) &= 0.5 \\
P(W_A|B_2) &= 0 \\
P(W_A|B_3) &= 1 \\
P(W_A|B_4) &= 1.
\end{aligned}$$

Plug everything in for $P(W_A)$:

$$\begin{aligned}
P(W_A) &= P(B_1) \cdot P(W_A|B_1) + P(B_2) \cdot P(W_A|B_2) + P(B_3) \cdot P(W_A|B_3) + P(B_4) \cdot P(W_A|B_4) \\
&= 0.375 \cdot 0.5 + 0.375 \cdot 0 + 0.125 \cdot 1 + 0.125 \cdot 1 \\
&= 0.1875 + 0 + 0.125 + 0.125 \\
&= 0.4375
\end{aligned}$$

Redo method

$$\begin{aligned}
P(W_H) &= (1 - P(s_1))(1 - P(s_2))P(W_H|B_1) + (1 - P(s_1))P(s_2)P(W_H|B_2) \\
&\quad + P(s_1)(1 - P(s_2))P(W_H|B_3) + P(s_1)P(s_2)P(W_H|B_4) \\
\\
&= P(B_1) \cdot P(W_H|B_1) + P(B_2) \cdot P(W_H|B_2) \\
&\quad + P(B_3) \cdot P(W_H|B_3) + P(B_4) \cdot P(W_H|B_4).
\end{aligned}$$

Since we redo the game when both robots fail,

$$P(W_H|B_1) = P(W_H).$$

$$
\begin{aligned}
P(W_H) &= P(B_1) \cdot P(W_H) + P(B_2) \cdot P(W_H|B_2) \\
&\quad + P(B_3) \cdot P(W_H|B_3) + P(B_4) \cdot P(W_H|B_4).
\end{aligned}
$$

Let

$$
\begin{aligned}
r &= P(B_1) \\
u_1 &= P(B_2)P(W_H|B_2) + P(B_3)P(W_H|B_3) + P(B_4)P(W_H|B_4).
\end{aligned}
$$

$$
\begin{aligned}
P(W_H) &= P(B_1) \cdot P(W_H) + P(B_2) \cdot P(W_H|B_2) \\
&\quad + P(B_3) \cdot P(W_H|B_3) + P(B_4) \cdot P(W_H|B_4) \\
&= r(P(W_H)) + u_1 \\
&= u_1 + r(u_1 + r(P(W_H))) \\
&= u_1 + r(u_1 + r(u_1 + r(u_1 + r \ldots))) \\
&= u_1 + ru_1 + r^2(u_1 + r(u_1 + r \ldots)) \\
&= u_1 + ru_1 + r^2u_1 + r^3u_1 + r^4u_1 \ldots.
\end{aligned}
$$

The sum of a geometric series $y = u_1(r)^x$ is

$s_\infty = \frac{u_1}{1-r}$

which I got from the IB Math formula booklet. Since $r < 1$, we know that this series converges and we can use this formula to find a definite answer to this infinite series. Plugging in actual values in place of stand in variables:

$$
\begin{aligned}
&\sum_{n=1}^{\infty} u_1(r)^n, |r| < 1, \\
S_\infty &= \frac{u_1}{1-r}, \\
P(W_H) &= \frac{P(B_2)P(W_H|B_2) + P(B_3)P(W_H|B_3) + P(B_4)P(W_H|B_4)}{1 - P(B_1)}.
\end{aligned}
$$

Therefore,

$$
P(W_H) = \frac{P(B_2)P(W_H|B_2) + P(B_3)P(W_H|B_3) + P(B_4)P(W_H|B_4)}{1 - P(B_1)}.
$$

Simplified infinite geometric series equation:

$$
P(W_H) = \frac{P(B_3) + P(B_4)P(W_H|B_4)}{1 - P(B_1)}.
$$

Plugging in values for Robot A against Robot B:

$$\begin{aligned} P(W_{A,1}) &= \frac{0.125 + 0.125 \cdot 1}{1 - 0.375} \\ &= \frac{0.25}{0.625} \\ &= 0.40. \end{aligned}$$

3 Bayes' Theorem

The question What is the probability a robot wins given that we redo all double fails?" can be restated as "What is the probability a robot wins given that at least one robot succeeds?" So, the "redo" method can be described as

$$\begin{aligned} P(W_H) &= P(W_H | s_1 \cup s_2) \\ &= P(W_H | s_H \cup s_E). \end{aligned}$$

We can relate this conditional probability with $P(s_H \cup s_E | W_H)$ using Bayes' Theroem with the following equation:

$$P(W_H | s_H \cup s_E) = \frac{P(W_H) P(s_H \cup s_E | W_H)}{P(W_H) P(s_H \cup s_E | W_H) + P(W'_H) P(s_H \cup s_E | W'_H)}.$$

This equation is in the Math HL formula booklet. This equation modifies an existing method of finding a robot's game winning probability since it needs $P(W_H)$. For simplicity's sake, I will use the flip a coin method to find the initial $P(W_H)$ for calculations. For our situation, the equation can be simplified to the following:

$$P(W_H | s_H \cup s_E) = \frac{P(W_H) P(s_H \cup s_E | W_H)}{P(s_H \cup s_E)}.$$

This makes Bayes' Theorem more practical for us given the variables known in this situation. Using past equations, we know that

$$\begin{aligned} P(W_H) = {} & P(B_1) \cdot P(W_H | B_1) + P(B_2) \cdot P(W_H | B_2) \\ & + P(B_3) \cdot P(W_H | B_3) + P(B_4) \cdot P(W_H | B_4) \end{aligned}$$

and that

$$P(s_H \cup s_E) = P(B_2) + P(B_3) + P(B_3).$$

since the partitions B_1 along with B_2, B_3, and B_4 all add up to 1, so the latter makes up the probability that at least one robot functions. If we can find $P(s_H \cup s_E|W_H)$, we can find $P(W_H|s_H \cup s_E)$. ???:

$$\begin{aligned}
P(s_H \cup s_E|W_H) &= P((B_2 \cup B_3 \cup B_4)|W_H) \\
&= P(B_2|W_H \cup B_3|W_H \cup B_4|W_H) \\
&= P(B_2|W_H) + P(B_3|W_H) + P(B_4|W_H) \\
P(s_H \cup s_E|W_H) &= \frac{P(B_2)P(W_H|B_2)}{P(W_H)} \\
&+ \frac{P(B_3)P(W_H|B_3)}{P(W_H)} \\
&+ \frac{P(B_4)P(W_H|B_4)}{P(W_H)} \\
&= \frac{P(B_2)P(W_H|B_2) + P(B_3)P(W_H|B_3) + P(B_4)P(W_H|B_4)}{P(W_H)}
\end{aligned}$$

This is the portion of wins that come from one of the robots working. Plugging this into the above equation:

$$\begin{aligned}
P(W_H|s_H \cup s_E) &= \frac{P(W_H)\frac{P(B_2)P(W_H|B_2)+P(B_3)P(W_H|B_3)+P(B_4)P(W_H|B_4)}{P(W_H)}}{P(s_H \cup s_E)} \\
&= \frac{P(B_2)P(W_H|B_2) + P(B_3)P(W_H|B_3) + P(B_4)P(W_H|B_4)}{P(s_H \cup s_E)}
\end{aligned}$$

Simplified Bayes' Theorem equation:

$$P(W_H|s_H \cup s_E) = \frac{P(B_3) + P(B_4)P(W_H|B_4)}{P(s_H \cup s_E)}$$

This is the wins that come from at least one robot working divided by the portion of times that at least one robot workks. This equation is identical to the sum of an infinite geometric series equation found earlier. Thus, with two methods arriving at the same result, both are validated.

4 Winning Consecutive Rounds

What we know immediately

$$P(W_2) = P(W_1 \cap W_2),$$

Going further,

$$
\begin{aligned}
P(W_2) &= \frac{P(W_1) \cdot P(W_1 \cap W_2)}{P(W_1)} \\
&= P(W_1) \cdot P(W_2|W_1),
\end{aligned}
$$

This can be shown as

$$P(W_{2,A}|W_{1,A}) = P(B_C) \cdot P(W_{2,A}|B_C) + P(B_D) \cdot P(W_{2,A}|B_D).$$

So,

$$P(W_{2,A}|W_{1,A}) = P(W_{1,C}) \cdot P(W_{2,A}|W_{1,C}) + P(W_{1,D}) \cdot P(W_{2,A}|W_{1,D}).$$

By hand:
Defining variables

$$
\begin{aligned}
P(W_{1,C}) &= \frac{P(B_3) + P(B_4)P(W_{1,C}|B_4)}{1 - P(B_1)} \\
P(W_{1,D}) &= \frac{P(B_3) + P(B_4)P(W_{1,D}|B_4)}{1 - P(B_1)} \\
P(W_{2,A}|W_{1,C}) &= \frac{P(B_3) + P(B_4)P((W_{2,A}|W_{1,C})|B_4)}{1 - P(B_1)} \\
P(W_{1,A}|W_{1,D}) &= \frac{P(B_3) + P(B_4)P((W_{1,A}|W_{1,D})|B_4)}{1 - P(B_1)},
\end{aligned}
$$

Setting up the equation

$$
\begin{aligned}
P(W_{2,A}|W_{1,A}) = &\frac{P(B_3) + P(B_4)P(W_{1,C}|B_4)}{1 - P(B_1)} \\
&* \frac{P(B_3) + P(B_4)P((W_{2,A}|W_{1,C})|B_4)}{1 - P(B_1)} \\
&+ \frac{P(B_3) + P(B_4)P(W_{1,D}|B_4)}{1 - P(B_1)} \\
&* \frac{P(B_3) + P(B_4)P((W_{1,A}|W_{1,D})|B_4)}{1 - P(B_1)}.
\end{aligned}
$$

Plugging it all in

$$\begin{aligned}
P(W_{2,A}|W_{1,A}) &= \frac{0.75 \cdot (1 - 0.125) + 0}{(1 - (1 - 0.75)(1 - 0.125))} \\
&* \frac{0.25 \cdot (1 - 0.75) + 0.25 \cdot 0.75 \cdot 1}{1 - (1 - 0.25)(1 - 0.75)} \\
&+ \frac{0.125 \cdot (1 - 0.75) + 0.125 \cdot 0.75 \cdot 1}{(1 - (1 - 0.125)(1 - 0.75))} \\
&* \frac{0.25 \cdot (1 - 0.125) + 0}{1 - (1 - 0.25)(1 - 0.125)} \\
&= 0.36028.
\end{aligned}$$

Solving for P(WA)

$$\begin{aligned}
P(W_A) &= P(W_{1,A}) \cdot P(W_{2,A}|W_{1,A}) \\
&= 0.4 \cdot 0.36028 \\
&= 0.144111.
\end{aligned}$$