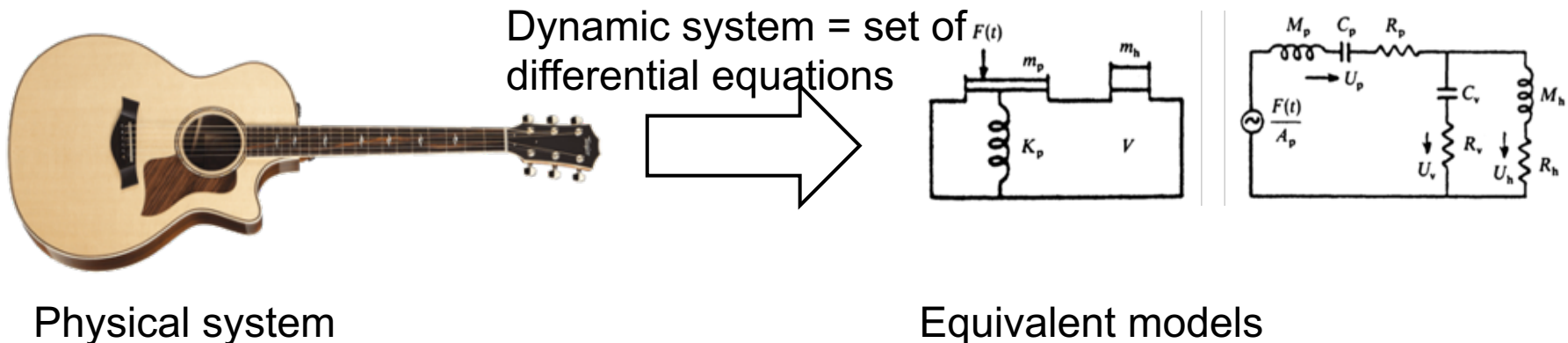




# Modeling of Musical Instruments

- Given a real system, we want to study it through a **mathematical approach**.
- For **dynamic systems**, this translates into building a **set of differential equations** which can describe **relationships among values** in systems.
- The **model complexity** determines the **accuracy** of the system representation. E.g: an ideal string model cannot represent real strings inharmonicities.
- One **same system** can often be described through **different models**, some of which can be **equivalent**, and therefore **interchangeable**.



- Models are built to study system behaviors.
- Sound models are built to reproduce sounds from real instruments.
- **Tradeoff** between model **accuracy** and **computational effort** (real-time issues).
- Sound models deal with:
  - **Timbre quality**: how much the timbre for a synthesized instruments equals the timbre complexity of musical instruments. Timbre mostly deals with the time-evolution of sound (envelope).
  - **Timbre dynamic range**: how the synthesized instrument can be able to adapt to the playing intensity, by reproducing related timbrical variations accordingly.
  - **Playability**: how much is easy to control the synthetic instrument sound by the player. A good synthetic instrument should change its sound according to same principles and parameters as real instruments.

- Very complex systems
- Their **sound behavior changes** according to the **playing style** and lots of other factors
- The sound highly depends on **the instrument system state**. The combination of a particular state and time/type of input can greatly modify the sound
- **Non linear interactions** (especially for stimulations) are ubiquitous (e.g: bow string interaction in violins)

- **Synthetic algorithms:**
  - Goal: try to **reproduce the sound** instead of the system
  - Not linked to the physics of real instruments
  - **Less computational complexity**
  - **Limited approach for real instruments: often they are not able to accurately reproduce the sound nuances and evolution in time**
  - Real common types of synthesis: **additive**, **subtractive**, FM, granular

**Real instrument sound synthesis is not a trivial task**

- One first approach: **Samplers**
  - They reproduce huge collections of accurately recorded real instruments
  - **Samplers sound as real instruments**
  - **Great samplers can reproduce different instruments playing styles by responding to input parameters (e.g. velocity)**
  - **Samplers require a lot of memory for storing all the samples**

- Goal: try to reproduce the **system behavior**, in terms of **physics**
- This approach automatically solves (or **attenuate**) problems such as **timbre quality**, **timbre dynamic range** and **playability**
- Physical modeling requires to start from the dynamic **differential equations** which describe the system
- Often, such equations are **not easy to be found**, and can become highly **complicated**
- This approach requires **great computational complexity**. That is why it has not been seriously taken into account until highly performing computers have become available.

$$\frac{dy}{dx} - x^2 + 3x - 2 = 0$$

$$\frac{dy}{dx} = x^2 - 3x + 2$$

$$dy = (x^2 - 3x + 2)dx$$

$$y = \int (x^2 - 3x + 2)dx$$

$$y = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + C$$

*Physical modeling starts from differential equations in order to represent musical instruments*

- Starting from constitutive equations, several techniques have been developed for the implementation of sound synthesis algorithms
- Of course, we deal with **discrete algorithms**. A discretization process is always concerned
- **Discretization schemes** can give rise to **problems** (aliasing, accuracy, stability, etc)
- We can basically divide between:
  - Methods which focus on the **system description** (i.e. differential equations) (e.g. **time-stepping methods**)
  - Methods which focus on the **system solution** (e.g. **modal synthesis, digital wave guides**)

- The PDE (partial differential equations) are **discretized in time and space**
- Partial differential operators are **approximated** through **finite differences**
- Finite differences equations describe the **next time instant value** for each discretized spatial point of the system
- For each temporal/spatial point, the next value depends only on **neighbouring points**

1) Being an approximation, finite differences equations can be obtained also from differential equations which cannot be solved analitically

2) Locality principles: separated systems can be connected by focusing only on their contact points.

Because of 1 and 2, non linearities can be 'easily' inserted in the systems description



- Discretization schemes
- Each scheme has different properties in terms of **accuracy**, **stability**, etc
- Basic idea: the derivative is **approximated** through **a finite difference**
- For time-dependent functions, this allows to separate the **current value from the next instant value**: no need for analytical solutions
- Start from **Taylor expansion** for functions

$$f(a + h) = f(a) + f'(a)h + R_1(x)$$



$$f'(a) \approx \frac{f(a + h) - f(a)}{h}$$

*Differential  
equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \kappa^2 \frac{\partial^4 y}{\partial x^4}$$



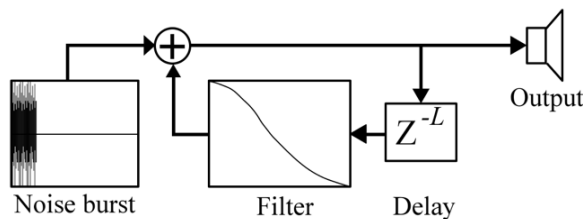
*Finite  
difference  
equation*

$$y_m^{n+1} = a_1(y_{m+2}^n + y_{m-2}^n) + a_2(y_{m+1}^n + y_{m-1}^n) \\ + a_3 y_m^n + a_4 y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1})$$

Boundary and initial conditions still need to be applied to complete the finite difference equation definition

$$\left. \frac{\partial^2 y}{\partial x^2} \right|_{x_{m_0}, t_n} = \left. \frac{\partial^2 y}{\partial x^2} \right|_{x_M, t_n} = 0$$

- Suitable for models characterized by the **propagation of waves**. Therefore often used for **pipes and string models**
- They start from a **discretization of the general analytical solution** in the time domain (travelling wave solution)
- They have been developed starting from a more ancient technique: the **Karplus-Strong algorithm**



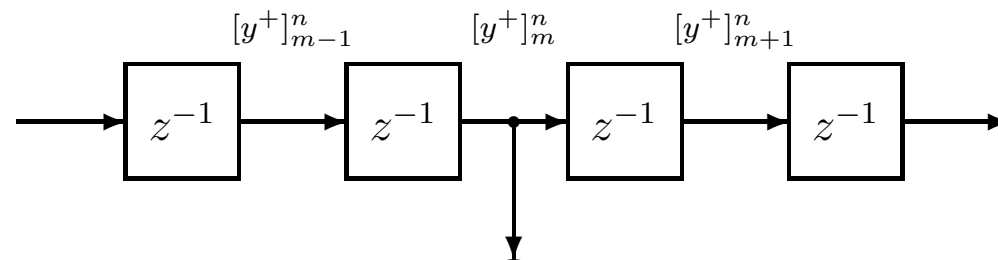
- Main idea: the signal is modeled according to the **D'Alembert solution**
- Therefore, we consider two waves travelling in opposite directions

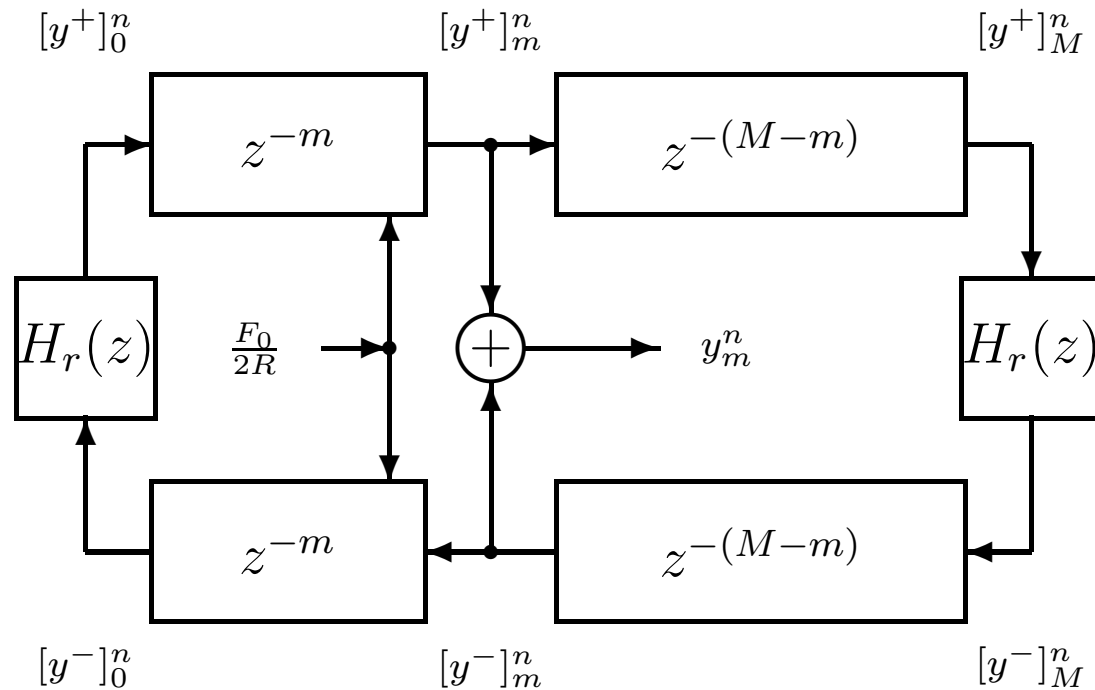
$$y(x, t) = f(t - x/c) + h(t + x/c)$$

$$[y^+]_m^n = f(t_n - x_m/c) = f[(n - m)T] = f(n - m)$$

$$[y^-]_m^n = h(t_n + x_m/c) = h[(n + m)T] = h(n + m)$$

- Core structure at the basis of digital waveguides
- It simulates a **M sample delay** for a **discrete signal** entering the delay line.
- The delay line can simulate the **propagation of a wave in a guide** (e.g a string, a tube).
- Hypothesis: the signal **does not undergo modifications while in the delay line**
- Dispersion, damping and other phenomena can be added through LTI filters **in one point of the delay line**
- The final delay guide becomes a generic **delay** connected to a **lumped system** which performs filtering

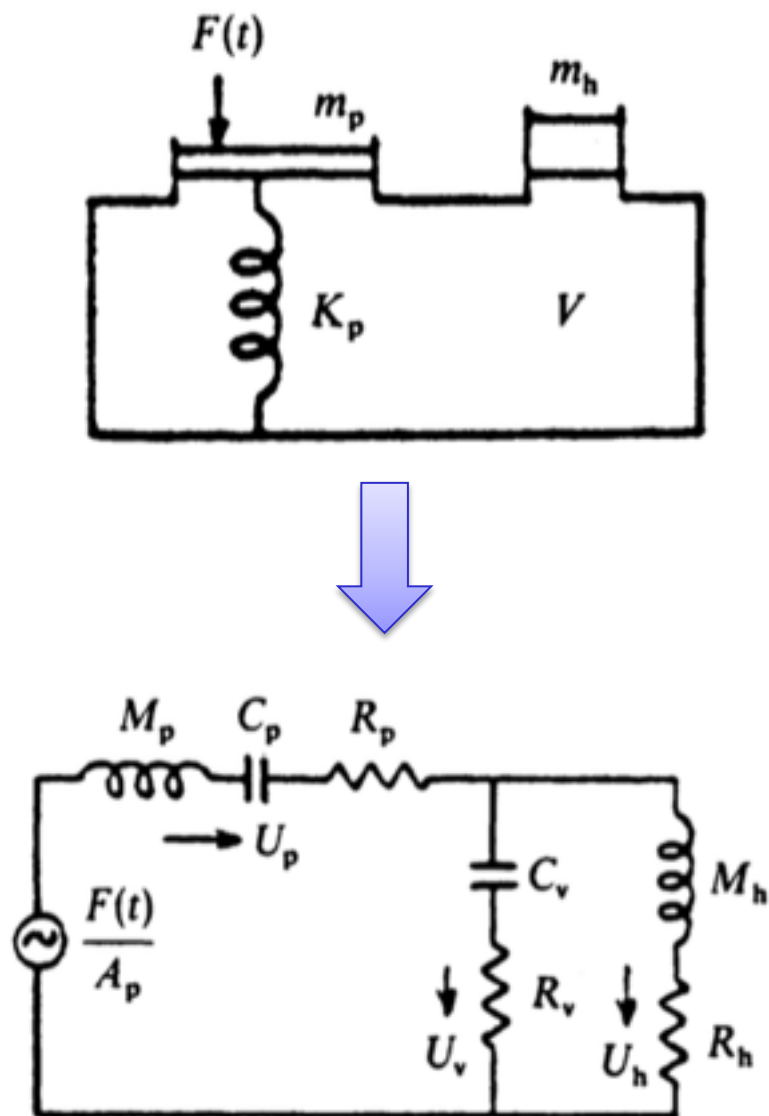




# Electric models for the guitar

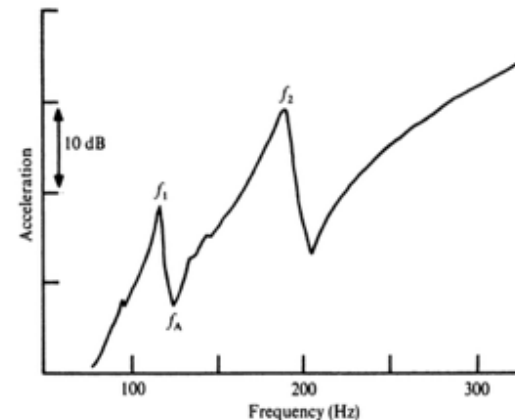
# One first simple model

- We model a guitar with **rigid (non vibrating) back plate and ribs**.
- This simple model will not be representative of the complete behavior of a guitar.
- The **vibrational elements** are the **top plate** and the **cavity**.
- We model those 2 elements as 2 **lumped oscillators**.
- We then compute **an electrical equivalent circuit**, which can be analyzed with classical electric circuit theory.



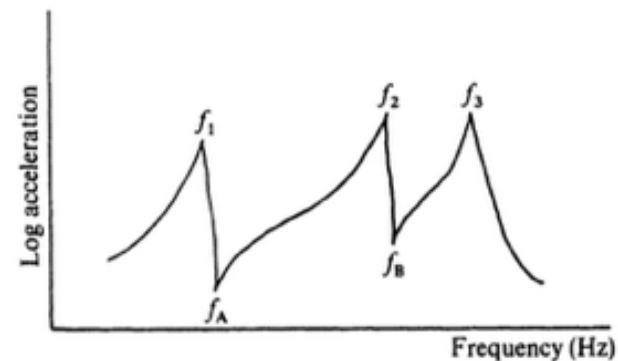
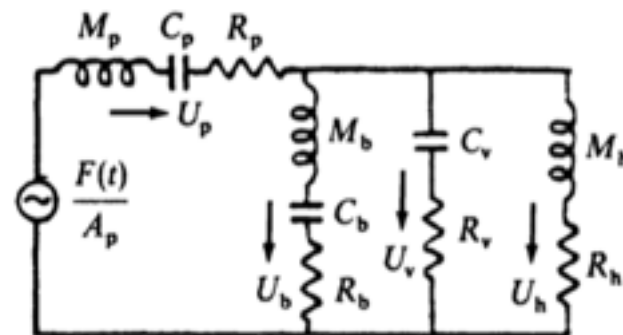
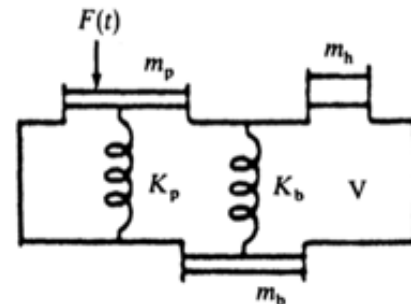
- This electrical system is described by a set of **two differential equations**. This means it is a **coupled system** with **two resonances**
- Because the excitation happens in the same place of the measure (the generator circuit branch, i.e the guitar bridge) we are dealing with a **driving point frf**
- Driving point admittances with more than one resonance are always characterized to have **one antiresonance between each couple of resonances**
- The **current** is associated with the **bridge velocity**, which is in turn associated with **sound** radiated from the instrument, thanks to the **Euler's equation**:

$$\rho_0 \frac{\partial v}{\partial t} = -\nabla p$$



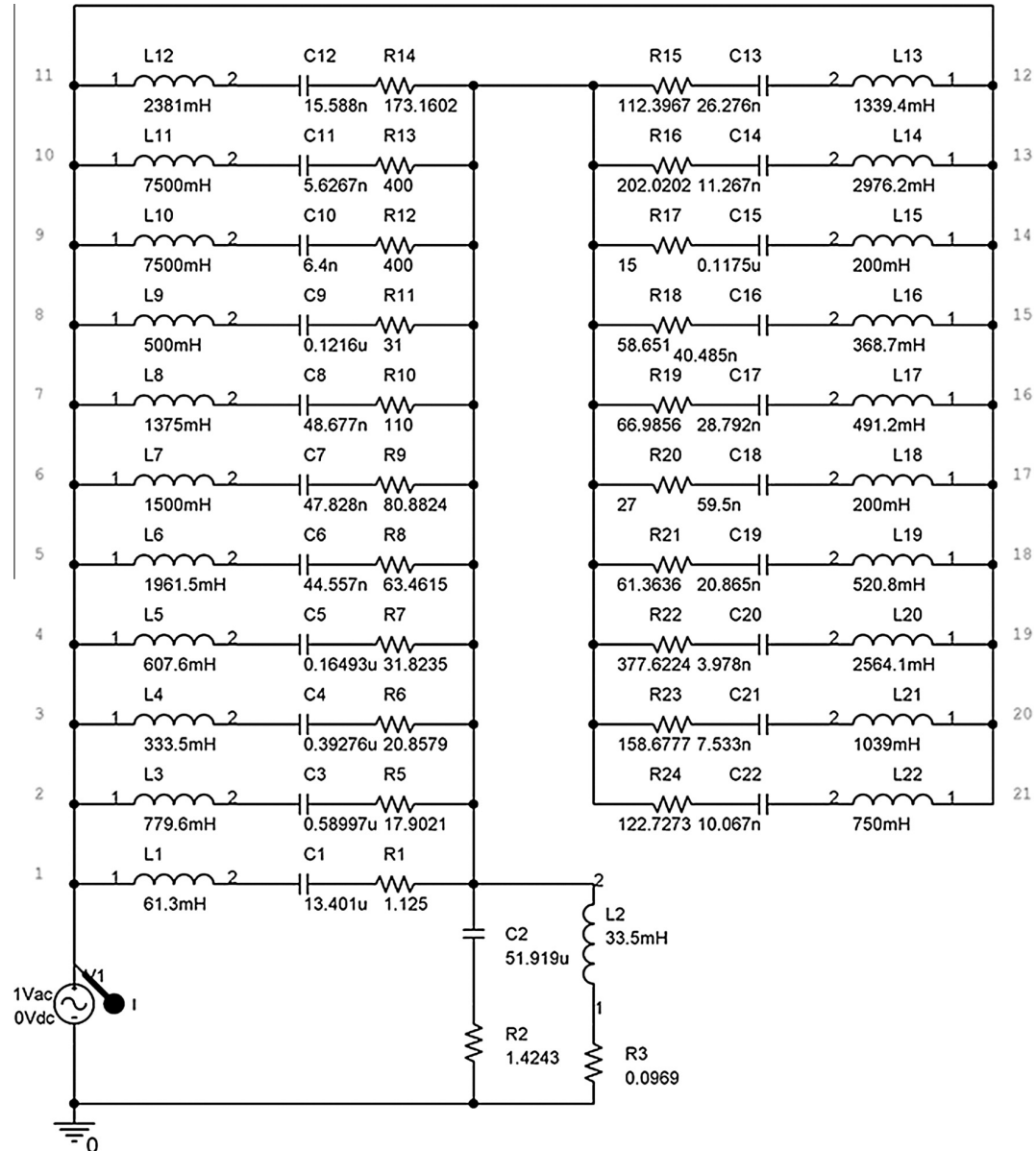


- We add the **back plate vibration**, while keeping the ribs fixed
- The back plate couples to the top plate through the enclosed air
- By adding a new spring and mass, the system now has **three coupled simple oscillators**
- We therefore predict **three resonances** in the corresponding frf

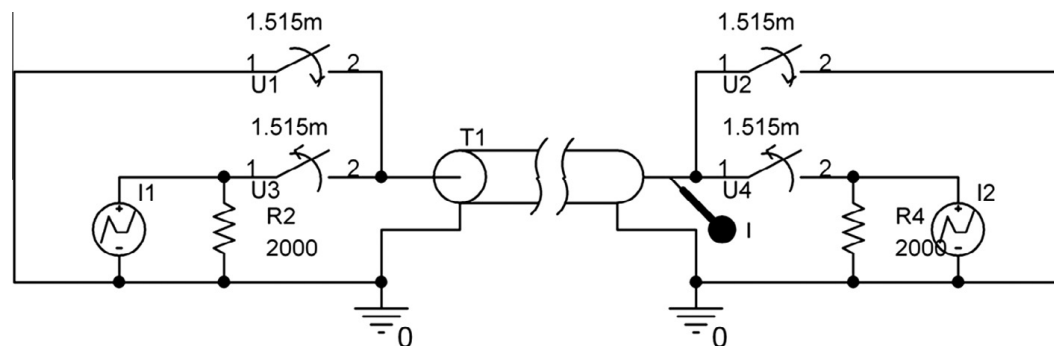


- A real guitar top plate has more than one **resonances**, which we need to add to the model
- **For this case we will consider again the system with fixed back plate.**
- Resonances are modeled by inserting a **filter bank** in the circuit. Each **RLC branch** will model a particular filter with one resonance.
- RLC branches values are empirically **tuned** to obtain back the resonances **measured from a real guitar**.
- We are interested in studying the **time response** of the guitar according to a **pluck** solicitation. We therefore add a particular **signal generator circuit** which can emulate the **plucked string vibration**.
- We consider an **ideal string** with no damping or stiffness. The real string closest to this representation is the **E1 string for guitars**. Therefore we will study the time response of the guitar when the E1 string is plucked.

- **20 top plate resonances** are added.
- The **current** probe is placed where the current represents the **bridge velocity** (i.e guitar sound).
- **Resistors** in the circuit are intended as **damping sources** attenuating the input signal.

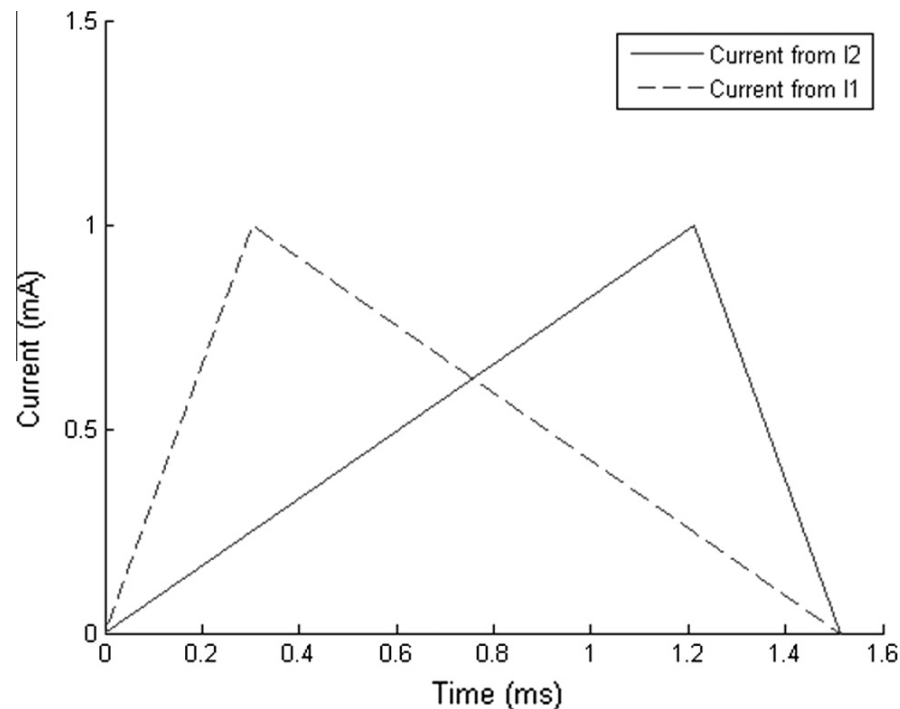


- **Standing** waves in the string are modeled through a transmission line, which allows to take into account **wave reflections** at the string boundaries
- The **string vertical displacement** at each spatial point is related to the corresponding **voltage** at the same place in the transmission line

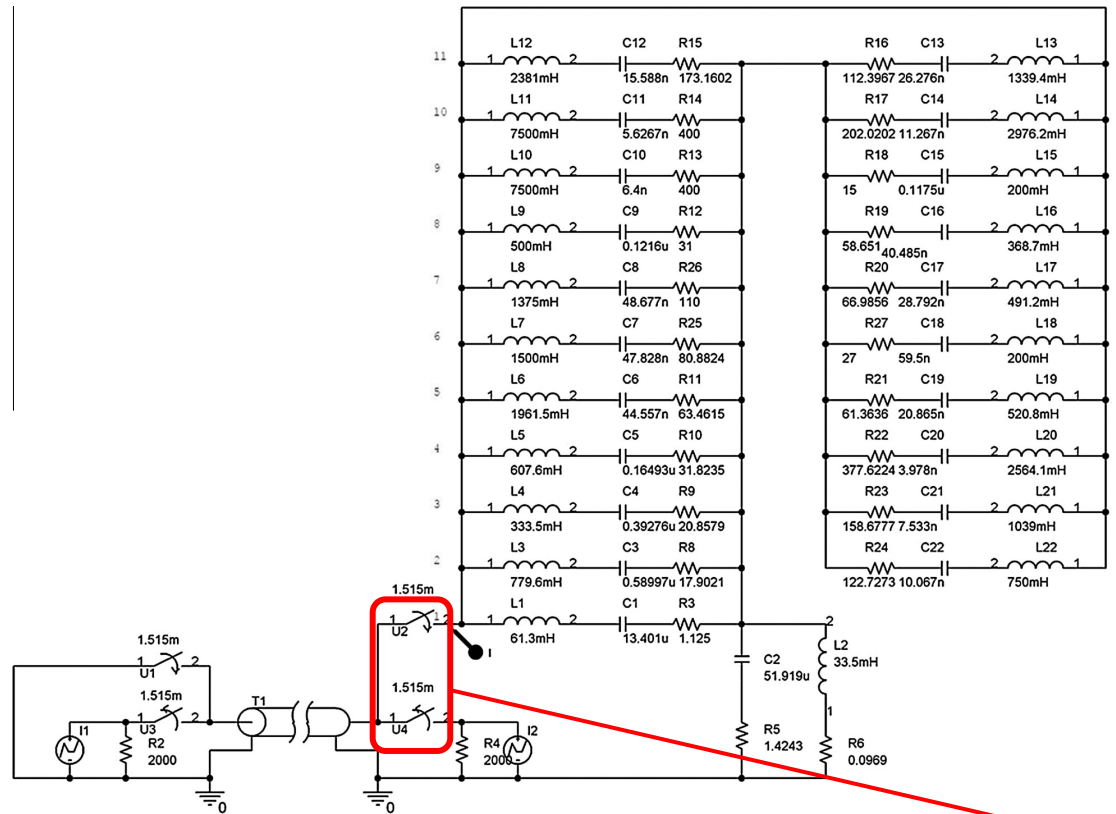


- The transmission line is fed on both sides with two **symmetrical triangular current pulses**. Their **duration** is linked to **half of the string fundamental frequency** of vibration
- The **shape of the pulses** determines the initial **shape of the string** plucked at a certain point on its length

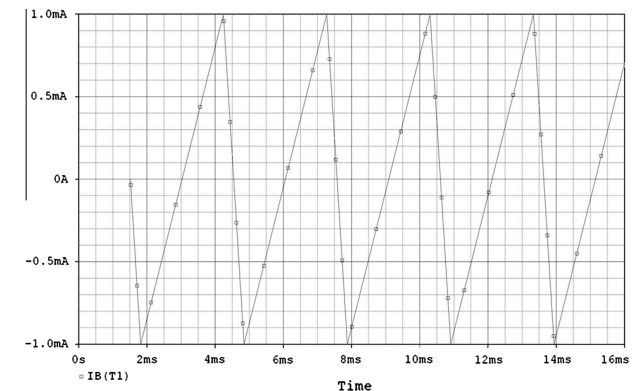
- These pulses correspond to two feasible propagating **D'Alembert solutions** for the transverse wave propagation on the string



*Pulse duration = 1.515 ms corresponds to  $f=329.5$  Hz for the string (E1 guitar string fundamental frequency)*

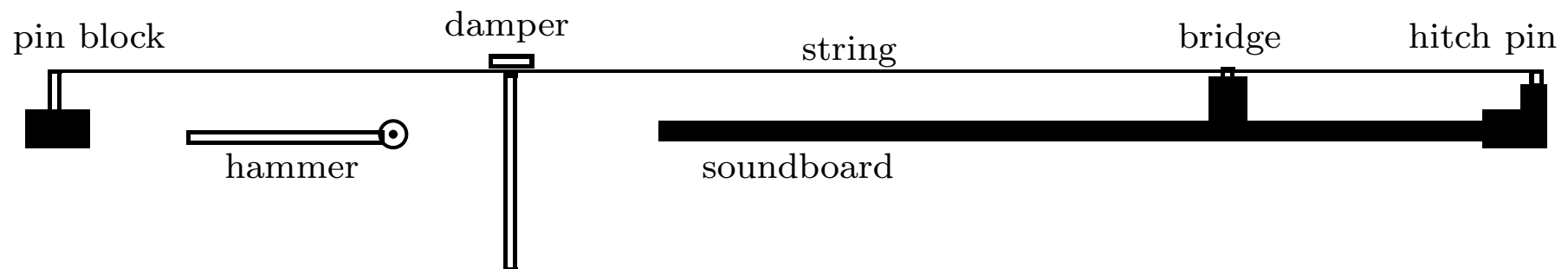


*After the current sources are switched off, the right end of the transmission line feeds the circuit with a triangular waveform that simulates the solicitation coming from a plucked string on a guitar body.*



## String-hammer interaction within a piano

- The **hammer motion mechanism** (capstan, whippen, jack, knuckle, repetition lever) is **omitted**
- Pianos have three string for each note (tricord). Here we consider **only one string**
- We omit the **damper** and the coupling with the **soundboard**
- Therefore we finally focus on a **real string** (damping+stiffness) left pinned and interacting with the **bridge** at the right, impacted through a **piano hammer**





- **Hard wood core** covered with two layers of **wool felt** of varying thickness that increases from treble to bass.
- Dynamic hardness properties: **the hammer hardness varies with hammer velocity** according to the material properties of the felt. **Higher velocity** causes the felt compression to **harden the hammer**, thus soliciting higher harmonics.
- **This process is highly non-linear (non-linear hardening)**
- The **dynamics of playing**, through the hammer, influences the **timbre** of the played note.

- The hammer felt can be considered as a **nonlinear spring**, whose stiffness increases with compression
- The complete hammer model is a **lumped mass attached to a nonlinear spring**
- We have finally a non-linear force interaction with the string, which is given by the following power law:

$$F_H = K \xi^p$$

*Stiffness exponent: it describes how the stiffness changes with the force*

*$\xi(t)$  is a time dependent function which describes the felt compression*

*Hammer stiffness*

- The interaction is modeled through a time dependent force exerted by the hammer **on a specific point**  $x_0$ , and is expressed by:

$$M_H \frac{d^2\eta}{dt^2} = -F_H(t) - b_H \frac{d\eta}{dt}$$

*Hammer mass*       *$\eta(t)$  is the hammer displacement wrt the string equilibrium position*      *Air damping*

- The hammer force can be further refined as:

$$F_H(t) = \phi[\xi(t)] = K\xi^p, \quad \xi(t) = |\eta(t) - y(x_0, t)|$$

*Hammer position*      *String vertical in  $x = x_0$  position at instant  $t$*

- The refined model takes into account a **stiff and lossy string** with **scalar impedance on both ends**, excited by the hammer, and implements the **hammer string interaction** previously introduced.
- The relative differential equation **has no analytical solution**.
- We therefore solve the model through an **approximation**, taking advantage of the **finite difference method**.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \kappa^2 \frac{\partial^4 y}{\partial x^4} - 2b_1 \frac{\partial y}{\partial t^2} + 2b_2 \frac{\partial^3 y}{\partial x^2 \partial t} + \rho^{-1} f(x, x_0, t)$$

*Hammer mass density*

$$f(x, x_0, t) = f_H(t)g(x, x_0)$$

$$f_H(t) = F_H(t) \left( \int_{x_0 - \delta x}^{x_0 + \delta x} g(x, x_0) dx \right)^{-1}$$

*g is a dimensionless spatial window of length  $2\delta$ . This window limits in space the force exerted by the hammer on the string*

- Boundary conditions at the string right end (at the bridge).  $x_M$  is the coordinate at the bridge end.

$$\frac{\partial^3 y}{\partial x^3} = -\frac{c^2}{\kappa^2} \frac{\partial y}{\partial x} + \frac{\zeta_b c}{\kappa^2} \frac{\partial y}{\partial t} \quad \zeta_b = \frac{R_b}{\rho c} \quad \left. \frac{\partial^2 y}{\partial x^2} \right|_{x_M, t_n} = 0$$

- At the bridge the string actually cannot move. It only acts with a force  $F_b$  on the bridge:

$$F_b = R_b \frac{\partial y}{\partial t} \quad F_b = -T_e \frac{\partial y}{\partial x} + \kappa^2 \rho \frac{\partial^3 y}{\partial x^3} \quad -T_e \frac{\partial y}{\partial x} + \kappa^2 \rho \frac{\partial^3 y}{\partial x^3} = R_b \frac{\partial y}{\partial t}$$

- Boundary conditions at the string left hinged end.  $x_M$  is the coordinate at the bridge end.  $x_0$  here stands for the string coordinate at its left end (not the interaction point with the hammer)

$$\frac{\partial^3 y}{\partial x^3} = -\frac{c^2}{\kappa^2} \frac{\partial y}{\partial x} + \frac{\zeta_l c}{\kappa^2} \frac{\partial y}{\partial t} \quad \zeta_l = \frac{R_l}{\rho c} \quad \left. \frac{\partial^2 y}{\partial x^2} \right|_{x_0, t_n} = 0$$

- Again, the string cannot move at the left hinged edge.

- The wave equation is approximated by the following finite difference equation:

$$y_m^{n+1} = a_1(y_{m+2}^n + y_{m-2}^n) + a_2(y_{m+1}^n + y_{m-1}^n) + a_3 y_m^n + a_4 y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + a_F F_m^n$$
$$F_m^n = F_H(n)g(m, m_0)$$

- m** is the **spatial index** (ranging from 0 to M), while **n** is the **time index** (ranging from 0 to an arbitrary value)
- This equation is **valid only in the interior of the string**. There is no m-2 for m=0 and m+2 for m=M
- Boundary conditions** need to be imposed as in the differential equation case for such points
- In this case we need to set **two conditions for the first and last two discrete points** at the string edges

$$a_1 = \frac{-\lambda^2 \mu}{1 + b_1 T}$$

$$a_2 = \frac{\lambda^2 + 4\lambda^2 \mu + \nu}{1 + b_1 T}$$

$$a_3 = \frac{2 - 2\lambda^2 - 6\lambda^2 \mu - 2\nu}{1 + b_1 T}$$

$$a_4 = \frac{-1 + b_1 T + 2\nu}{1 + b_1 T}$$

$$a_5 = \frac{-\nu}{1 + b_1 T}$$

$$\mu = \kappa^2 / c^2 X^2$$

$$\nu = 2b_2 T / X^2$$

$$a_1 = \frac{-\lambda^2 \mu}{1 + b_1 T}$$

$T$  = temporal resolution (time step)

$X$  = spatial resolution (spatial step)

$$a_2 = \frac{\lambda^2 + 4\lambda^2 \mu + \nu}{1 + b_1 T}$$

$\kappa$  = string stiffness coefficient

$$a_3 = \frac{2 - 2\lambda^2 - 6\lambda^2 \mu - 2\nu}{1 + b_1 T}$$

$\lambda = cT / X$  Courant number, for ensuring that the discrete scheme works

$$a_4 = \frac{-1 + b_1 T + 2\nu}{1 + b_1 T}$$

$$c = \sqrt{T_e / \rho} \quad T_e = \text{string tension}$$

$\rho$  = string linear density

$$a_5 = \frac{-\nu}{1 + b_1 T}$$

$b_1$  air damping coefficient

$b_2$  string internal friction coefficient

$b_H$  fluid damping coefficient

	C2	C4	C7	
<b>String</b>				
$f_1$	52.8221	262.1895	2112.1	Hz
$L$	1.92	0.62	0.09	m
$M_S$	$35 \times 10^{-3}$	$3.93 \times 10^{-3}$	$0.467 \times 10^{-3}$	Kg
$T_e$	750	670	750	N
$b_1$	0.25	1.1	9.17	$s^{-1}$
$b_2$	$7.5 \times 10^{-5}$	$2.7 \times 10^{-4}$	$2.1 \times 10^{-3}$	s
$\epsilon$	$7.5 \times 10^{-6}$	$3.82 \times 10^{-5}$	$8.67 \times 10^{-4}$	
<b>Hammer</b>				
$M_H$	$4.9 \times 10^{-3}$	$2.97 \times 10^{-3}$	$2.2 \times 10^{-3}$	Kg
$p$	2.3	2.5	3.0	
$b_H$	$1 \times 10^{-4}$	$1 \times 10^{-4}$	$1 \times 10^{-4}$	$s^{-1}$
$K$	$4 \times 10^8$	$4.5 \times 10^9$	$1 \times 10^{12}$	
$a$	0.12	0.12	0.0625	
<b>Boundary</b>				
$\zeta_l$	$1 \times 10^{20}$	$1 \times 10^{20}$	$1 \times 10^{20}$	$\Omega/\text{Kg.m}^{-2}.\text{s}^{-1}$
$\zeta_b$	1000	1000	1000	$\Omega/\text{Kg.m}^{-2}.\text{s}^{-1}$
<b>Sampling</b>				
$f_s$	$4 \times 44.1$	$4 \times 44.1$	$4 \times 44.1$	kHz
$M$	521	140	23	



- $m=1$

$$y_m^{n+1} = a_1(y_{m+2}^n - y_m^n + 2y_{m-2}^n) + a_2(y_{m+1}^n + y_{m-1}^n) + a_3y_m^n + a_4y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + a_F F_m^n$$

- $m=M-1$

$$y_m^{n+1} = a_1(2y_{m+1}^n - y_m^n + 2y_{m-2}^n) + a_2(y_{m+1}^n + y_{m-1}^n) + a_3y_m^n + a_4y_m^{n-1} + a_5(y_{m+1}^{n-1} + y_{m-1}^{n-1}) + a_F F_m^n$$

- $m=0$

$$y_m^{n+1} = b_{L1}y_m^n + b_{L2}y_{m+1}^n + b_{L3}y_{m+2}^n + b_{L4}y_m^{n-1} + b_{LF}F_m^n$$

- $m=M$

$$y_m^{n+1} = b_{R1}y_m^n + b_{R2}y_{m-1}^n + b_{R3}y_{m-2}^n + b_{R4}y_m^{n-1} + b_{RF}F_m^n$$

$$b_{R1} = \frac{2 - 2\lambda^2\mu - 2\lambda^2}{1 + b_1T + \zeta_b\lambda}$$

$$b_{R2} = \frac{4\lambda^2\mu + 2\lambda^2}{1 + b_1T + \zeta_b\lambda}$$

$$b_{R3} = \frac{-2\lambda^2\mu}{1 + b_1T + \zeta_b\lambda}$$

$$b_{R4} = \frac{-1 - b_1T + \zeta_b\lambda}{1 + b_1T + \zeta_b\lambda}$$

*for  $b_{Li}$  values, just substitute  $\zeta_b$  with  $\zeta_l$*

$$b_{R5} = \frac{T^2/\rho}{1 + b_1T + \zeta_b\lambda}$$

- $g(m, m_0)$  should be set as a **spatial window** with a **smooth shape** at its boundaries, to avoid for discretize infinite slope boundary problems.
- A possible choice for simulation purposes is a **Hanning window** of (discretized) length  $2\delta$ .

$$F_H(n) = K \left| \eta^n - y_{m_0}^n \right|^p \quad \eta^{n+1} = d_1 \eta^n + d_2 \eta^{n-1} + d_F F_H(n)$$

$$d_1 = \frac{2}{1 + b_H T / 2M_H}$$

$$d_2 = \frac{-1 + b_H T / 2M_H}{1 + b_H T / 2M_H}$$

$$d_F = \frac{-T^2 / M_H}{1 + b_H T / 2M_H}$$

- String speed at instant 0: the string is not moving, therefore the speed is zero.
- String displacement at instant 0: the string is in its equilibrium position.
- Hammer speed at instant 0: it is moving, therefore it has a non zero initial value equal to  $V_{H0}$ .
- Hammer displacement at instant 0: it is set to be in contact with the string. Its displacement  $\eta(0)$  wrt the string equilibrium position is equal to zero.
- The hammer displacement at  $n = 1$  depends on the initial velocity  $V_{H0}$  and on the temporal resolution between  $n$  and  $n + 1$  ( $\Delta T$ ), i.e  $\eta(1) = V_{H0}\Delta T$ . The displacement can be computed, according to the given formula, only from  $n = 2$ .
- Hammer force at instant 0 is equal to 0, being  $\eta(0)$  and  $y(m, 0) = 0$ .

- "Circuit based classical guitar model", J. Lee M. French, Applied Acoustics Volume 97, October 2015, Pages 96-103.
- "Numerical simulations of piano strings. I. A physical model for a struck string using finite difference methods", A. Chaigne, A. Askenfelt, The Journal of the Acoustical Society of America 95, 1112 (1994).
- "Physical modelling of the piano: An investigation into the effect of string stiffness on the hammer-string interaction", C. Saitis, dissertation, Master of Arts in Sonic Arts, Queen's University Belfast.