Musical Acoustics
Homework 1

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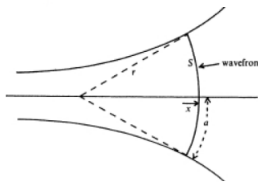
Introduction

We aim at designing an exponential horn.



Firstly, we are interested in approximating it with a series of conical horns considering the open end at the mouth as ideal. Secondly, we will introduce a numerical model to describe the impedance load at the mouth of the horn. Lastly, we will attach this approximated model to a cylinder to obtain a compound horn.
The study is made upon the input impedance of the horn, with increasingly precision of the approximation, inside a fixed range of frequencies $[0, 2000]\, Hz$.

1 Part 1: design of the exponential section

The horn we want to design is a Salmon horn. The cross-section's radius of these kind of horns follows the following formula:

$$a = a_0[\cosh mx + T \sinh mx] \tag{1}$$

The one we are considering is called exponential horn, which means it has a value of the coefficient T which is $T = 1$, so that the previous formula can be rewritten as:

$$a = a_0 e^{mx} \tag{2}$$

where $a_0 = 0.01\, m$ and $m = 4\, m^{-1}$. The length of the horn is $L = 0.4\, m$

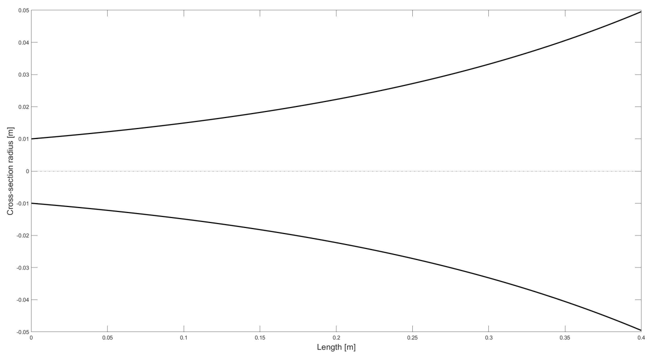


Figure 1: Axial section of the exponential horn.

1.1 Analytical formulation of the input impedance

The input impedance of an exponential horn is given by:

$$Z_{IN} = \frac{\rho c}{S_1} \left[\frac{Z_L \cos(bL + \theta) + j(\rho c/S_2) \sin bL}{j Z_L \sin bL + (\rho c/S_2) \cos(bL - \theta)} \right] \tag{3}$$

where k is the wavelength, $b^2 = k^2 - m^2$, $\theta = \tan^{-1}(m/b)$, $S_{1,2}$ are, respectively the areas of the throat and mouth section and ρ and c are here (and from now on)

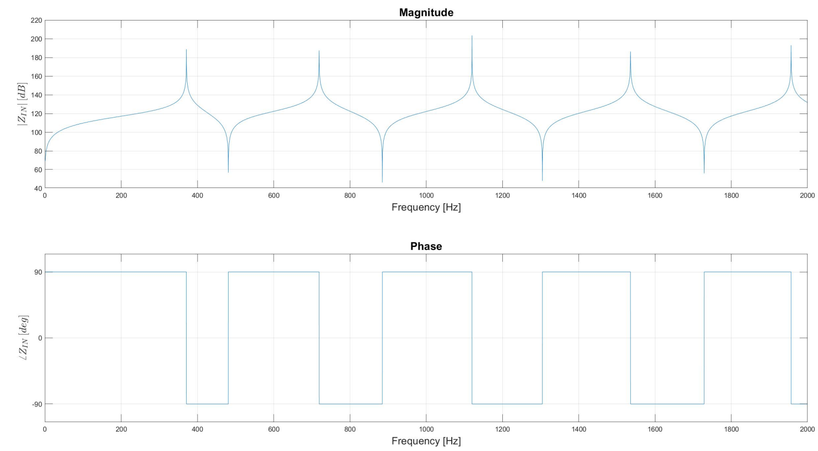


Figure 2: Input impedance of the exponential horn through analytical model.

the density of the air and the speed of sound in air. Considering the mouth of the horn as an ideal open end, we have $Z_L = 0$, which leads to the results in figure 2. Inside the desired range of frequencies, the diagram shows 5 peaks and 4 valleys, which corresponds to resonances and anti-resonances. Since we considered the ideal case of zero losses and ideal open end, values at resonances tend to infinity, values at anti-resonances are zero and the pace is not damped we the increase in frequency.

1.2 Approximation with one conical horn

The most basic approximation we are able to implement is done through the use of just 1 conical horn.
We have to retrieve the radius of the throat (a_1) and of the mouth (a_2):

$$a_1 = a_0 \qquad\qquad a_2 = a_0 e^{mL} \tag{4}$$

Then, knowing that the length of the conical horn is the same of the exponential horn, we can obtain the inclination of the edge with respect to the symmetry axis:

$$\alpha = \frac{a_2 - a_1}{L} \tag{5}$$

The geometry of the conical horn is plotted in figure 3.

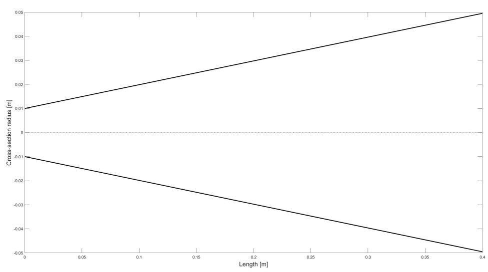


Figure 3: Axial section of the conical horn.

The input impedance of a conical horn is given, in general, by the following formula:

$$Z_{IN} = \frac{\rho c}{S_1}\left\{ \frac{jZ_L[\sin(kL-\theta_2)/\sin\theta_2] + (\rho c/S_2)\sin(kL)}{Z_L[\sin(kL+\theta_1-\theta_2)/\sin\theta_1\sin\theta_2] - (j\rho c/S_2)\sin(kL+\theta_1)/\sin\theta_1} \right\} \tag{6}$$

where $S_{1/2}$ are the section areas of the throat and mouth respectively, $\theta_{1/2} = \tan^{-1}(kx_{1/2})$ while x_1 and x_2 can be found geometrically considering the apex of the cone as in figure 4.

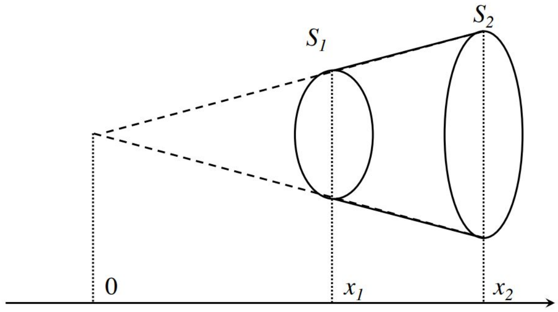


Figure 4: Geometry of a cone.

So that we have, considering a proportion between similar triangles:

$$x_1 = \frac{a_1 L}{a_2 - a_1} \qquad\qquad x_2 = \frac{a_2 L}{a_2 - a_1} \tag{7}$$

Considering again the mouth section as an open end ($Z_L = 0$) and substituting everything into equation (6) we obtain the input impedance in figure 5.

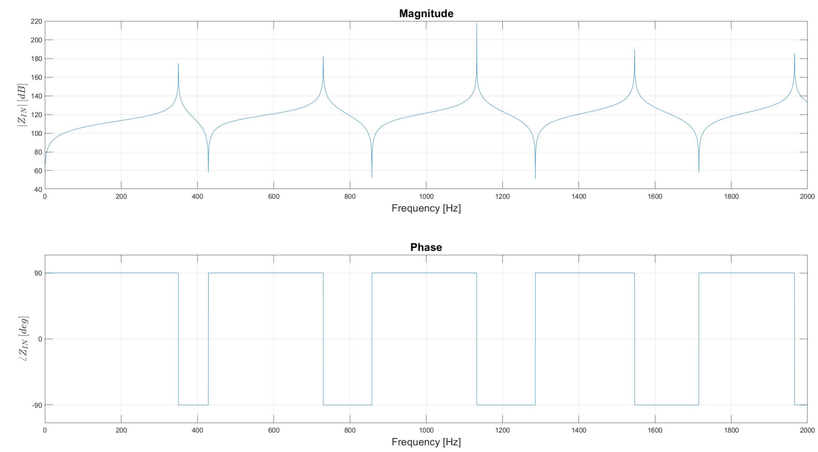


Figure 5: Input impedance of the horn approximated with one conical horn.

1.3 Approximation with a series of conical horns

Let's divide now the length of the exponential horn into n identical segments, for each of them we are able to approximate the correspondent part of the horn with a conical horn with the following geometrical considerations:

- let's define $\delta = L/n$ as the length of the segments into which we split the horn;
- the *i-th* cone (with i that goes from 1, in correspondence of the throat of the exponential horn, to n, in correspondence of the mouth of the exponential horn) is characterized by the two radius $a_1 = a_0 e^{(i-1)\delta}$ (throat of the cone) and $a_2 = a_0 e^{i\delta}$ (mouth of the cone);

- as we did for the single cone approximation, we can find the values for $S_{1/2}$, $x_{1/2}$ and $\theta_{1/2}$ for the *i-th* cone knowing the values of the two radius and the length δ.

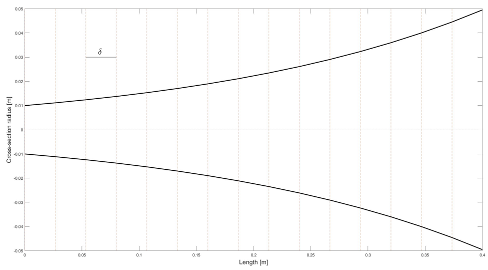


Figure 6: Axial section of the exponential horn approximated with a series of 15 conical horns.

The obtained geometry is shown in figure 6. In a series of cones, the load impedance of one of them is equal to the input impedance of the following one moving along the x-axis. Inside Matlab, we use an iterative approach where we start from the *n-th* cone, imposing $Z_L = 0$, and we go towards the first one always updating the load impedance of the *i-th* cone to be equal to the input impedance of the *(i+1)-th* one. By doing so we obtain the impedance diagram shown in figure 7.

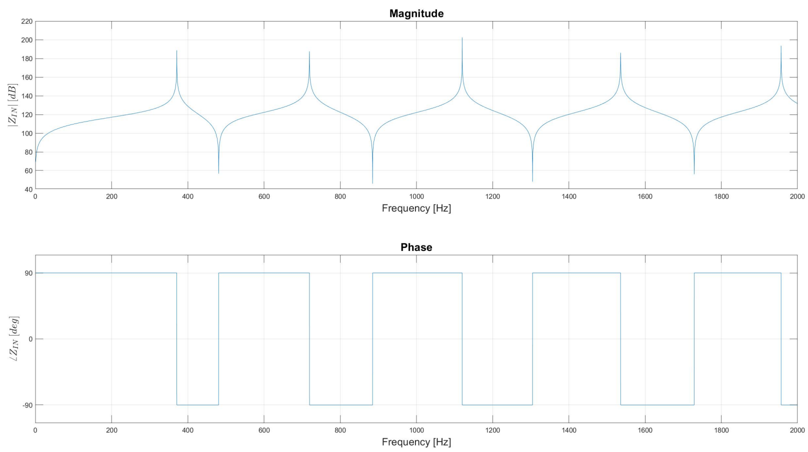


Figure 7: Input impedance of the series of 15 conical horns.

1.4 Comparison between the analytical model and the approximation model

We want to evaluate the similarity between the impedance of the exponential horn $Z_1(\omega)$ and the impedance of the series of cones $Z_2(\omega)$ through the computation of two metrics:

$$e_1 = \frac{1}{(\omega_{max} - \omega_{min})} \int_{\omega_{min}}^{\omega_{max}} |Z_1(\omega) - Z_2(\omega)|^2 d\omega \tag{8}$$

which evaluates the mean squared error between the two impedances, and:

$$e_2 = \sum_{1}^{5} \min |\arg max_{\omega} \Re(Z_1(\omega)) - \arg max_{\omega} \Re(Z_2(\omega))| \tag{9}$$

which evaluated the difference in frequency between the first five maxima of the two impedances.
Using Matlab, we can plot the errors of the approximation as functions of the length δ of the cones. This is shown in figure 8. From the plot we can see that the first

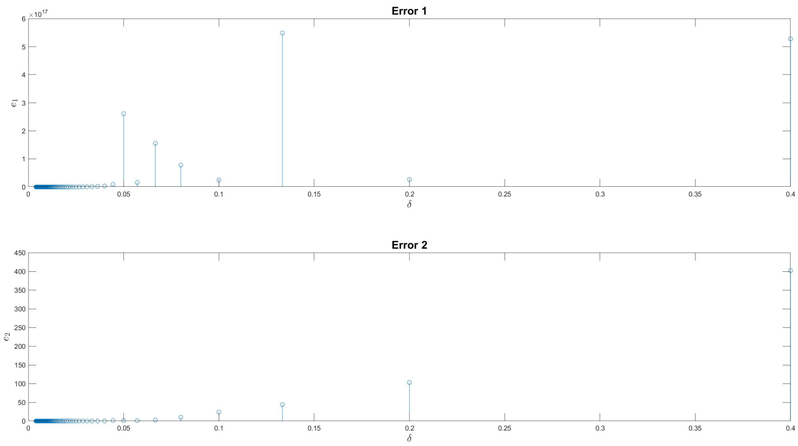
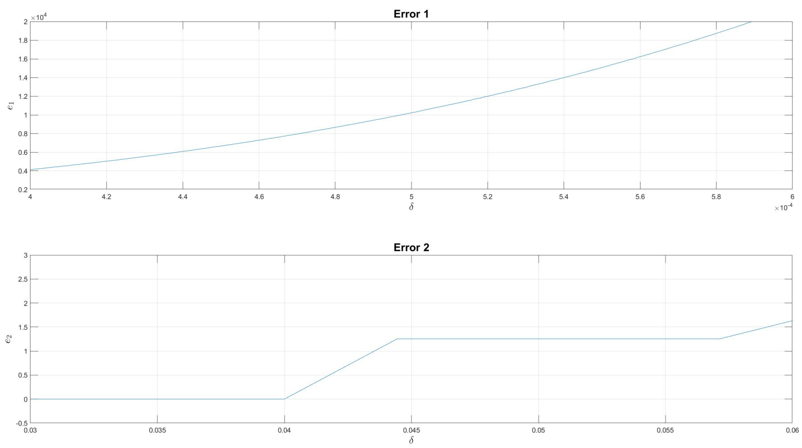


Figure 8: Errors due to the approximation of the exponential horn as a concatenation of equal length conical horns with respect to the length of them.

metric shows really high error values, which, after a minimum number of 8 cones, tend to maintain a decreasing pace. The second metric, instead, seems more reliable since from the start it decreases in value with decreasing length δ of the sections.
By evaluating the metrics up to a maximum of 1000 cones, and by plotting a zoomed



version of the same diagram, we can see that the first metric keeps decreasing by still doesn't reach zero, also with a length δ of the sections of $1/1000$ of the total length of the horn, the values of the error are really high, up to an order of 10^3. The second metric is shown to reach a value of zero for the exact minimum number of 10 cones ($\delta = 0.04 = L/10$).

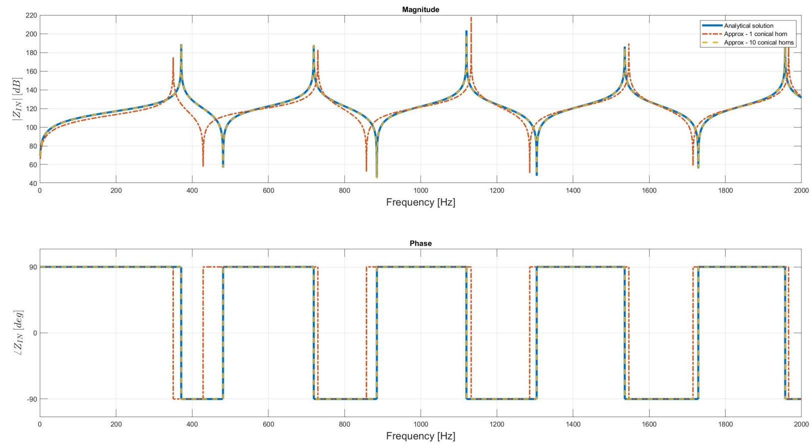


Figure 9: Comparison between the impedances of the analytical model, of the approximation with a single cone and of the approximation with the concatenation of 10 cones.

1.5 Different sampling strategies

Since the horn has an exponential pace, we decided to try a different approach for the approximation of the horn. Instead of using a uniform sampling along the x-axis, we tried a uniform sampling on the y-axis so that the closer we get to the mouth of the exponential horn, the smaller the length δ of the sections become. In order to do so, we had to apply a few preliminary steps with respect to the previous procedure:

- δ is not a constant anymore, we divide instead the height of the horn (considered as the difference between the radius of the mouth and the radius of the throat $H = a_2 - a_1$) into segments of equal length, let's call this length h;
- we use this new parameter to define the mouth and throat radius of the *i-th* cone as $a_2 = ih + a_0$ and $a_1 = (i-1)h + a_0$;
- now we can find the length along the x-axis of the *i-th* cone as $\delta = d_2 - d_1$, where $d_{1/2} = (1/m) * \ln(a_{1/2}/a_0)$;
- from now on the procedure is the same that we used before.

The obtained geometry is shown in figure 10 and it gives the input impedance diagram shown in figure .

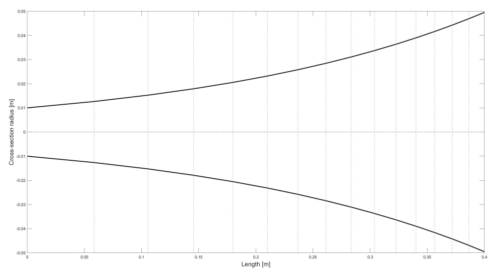


Figure 10: Axial section of a concatenation of 15 conical horns built with a uniform sampling along the y-axis of the exponential horn.

With this particular geometry, we compute again the same two metrics used before.

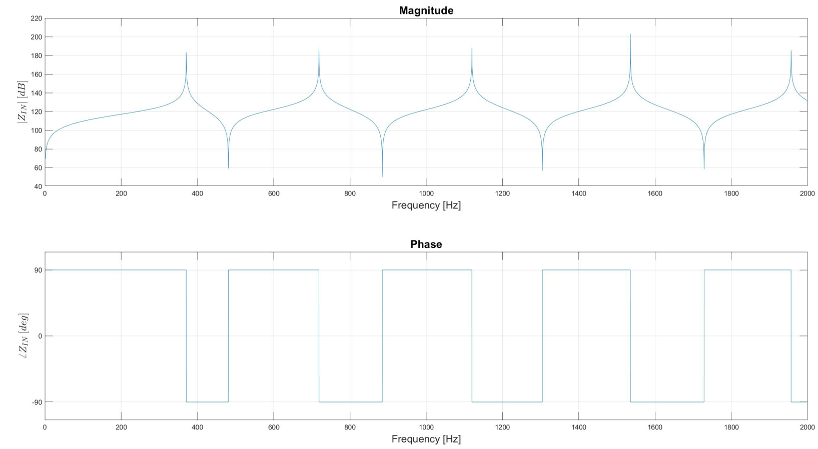


Figure 11: Input impedance obtained with a concatenation of 15 conical horns built with a uniform sampling along the y-axis of the exponential horn.

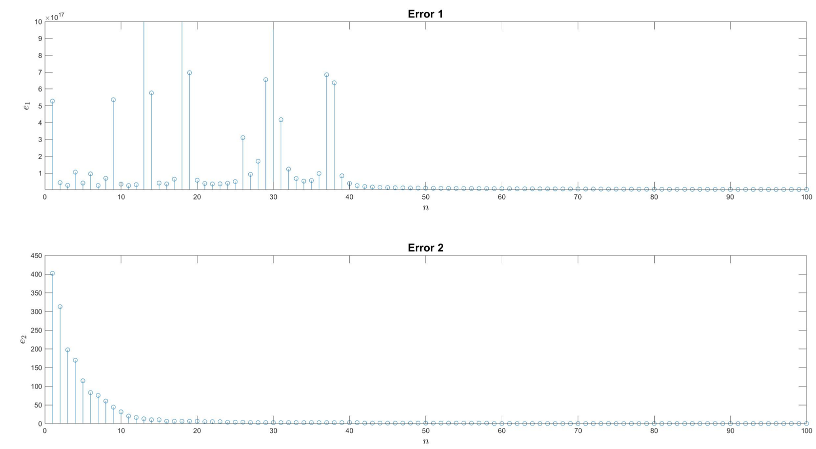
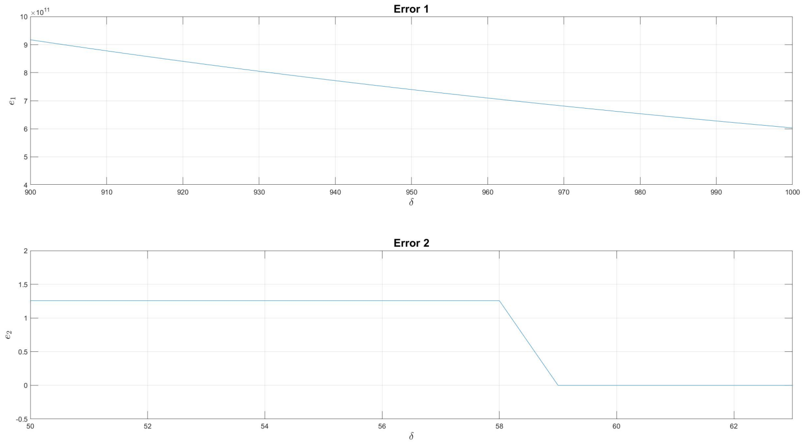


Figure 12: Errors due to the approximation of the exponential horn as a concatenation of conical horns of decreasing length as a function of the number of cones itself.

The results are plotted in figure 12. We can see that for low numbers of cones the results are quite similar to the one obtained before: the first metric has a fluctuating behaviour while the second metric is always decreasing.

Again, we compute the to metrics for up to 1000 cones and show a zoomed version of the plots. We can see that the first metric gives us a much higher value even for high numbers of cones. The second metrics reaches zero as well as the first one, but the minimum number for this condition has become 69 against the 10 cones from before. The comparison between the impedances obtained with different sampling strategies is shown in figure 13.

Our idea was, since the radius of the cross-section of the horn follows an exponential behaviour, to apply an increasing precision in describing the first derivative of the radius going towards the mouth of the horn. But, since the input impedance is calculated with an iterative approach from the cone at the mouth to the cone at the throat of the exponential horn, it seams that high errors occurs when we get to cones near the throat, which means that we need a high resolution also toward the throat.



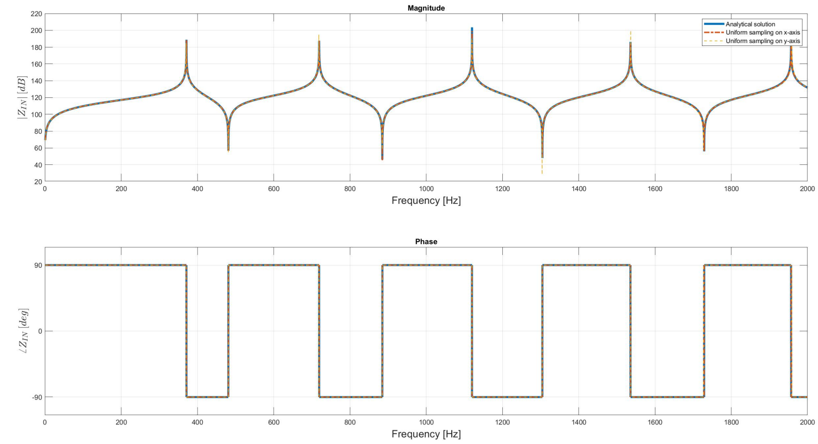


Figure 13: Impedance comparison between the analytical model and the approximation model made with a concatenation of 10 conical horns with both the uniform sampling along x-axis and the uniform sampling along y-axis.

1.6 Introduction of the impedance load

From now on we use the approximation model with the use of 10 conical horns, which is the minimum number of horns forming a concatenation characterized by a value of $e_2 = 0$. The cones are found with a uniform sampling of the exponential horn along the x-axis.
The adaptation to the external air is modeled through a further section whose impedance is: $Z_L(\omega) = Z_{L0}(\omega)S_p/S_s$, where $Z_{L0}(\omega) = 0.25(\omega^2\rho/c\pi) + j0.61(\omega\rho/a\pi)$ is the impedance of an unflanged cylindrical pipe of radius a. S_p is the cross-sectional area of the cylinder and $S_s = 2S_p/(1+\cos\theta)$ is the spherical wave front area at the open end of the cone, where θ is the flaring angle of the last conical section. Given $n = 10$ as number of cones and $\delta = L/10$ as length of each of them, we can compute the flaring angle as:

$$\theta = \arctan(\frac{a_2 - a_1}{\delta}) \tag{10}$$

where $a_1 = a_0 e^{m(9\delta)}$ and $a_2 = a_0 e^{m(10\delta)}$.
The impedance load of the horn is therefore plotted in figure 14.
The introduction of the impedance load has brought damping to the system. We

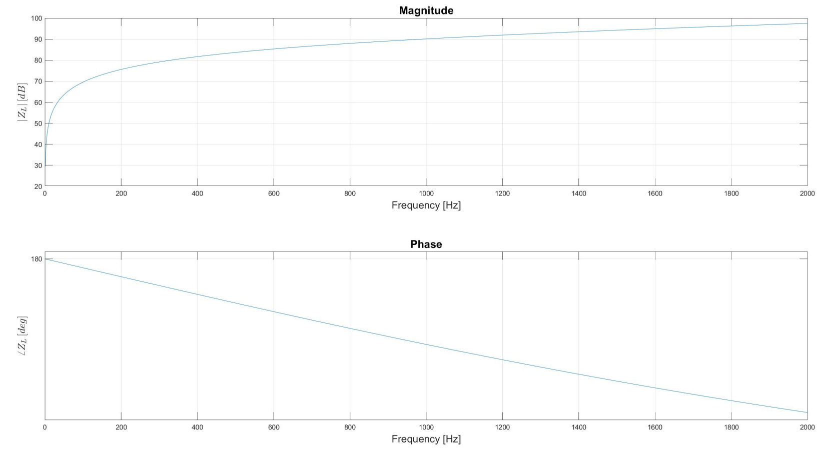


Figure 14: Impedance load diagram of the adaptation to the external air at the mouth of the horn.

can see, in fact, that peaks tend to have lower values going up with the frequency and valleys tend, instead, to have higher values. The phase as well shows as the introduction of damping since its slope levels down to finite, smaller values.

2 Part 2: design of the compound horn

Consider now a compound horn, composed by a cylindrical pipe, whose length is $0.6\,m$, followed by the approximated model of the exponential horn, which is the concatenation of 10 conical horns. We want to compute the input impedance of the compound horn considering the radiation load as previously modeled in section 1.6.
The input impedance of a cylindrical pipe is defined as:

$$Z_{IN} = Z_0 \left[\frac{Z_L \cos(kL) + jZ_0 \sin(kL)}{jZ_L \sin(kL) + Z_0 \cos(kL)} \right] \tag{11}$$

where L is the length of the pipe, $Z_0 = \rho c/S$ is the acoustic impedance of the pipe ($S = \pi a_0^2$ is the cross-sectional area of this element) and Z_L is the load impedance. Since in a compound horn the pipe and the horn are attached in series, the load impedance of the cylindrical pipe must be imposed equal to the input impedance of the horn, which we computed in section 1.6. The obtained input impedance of the compound horn is plotted in figure 15. We can see from the diagram that the damping remains, since it is given by the radiation load, but the combination of the two elements into the compound horn gives us an impendance that has a lot more resonances and anti-resonances into the chosen range of frequencies.
The first ten maxima of the compound horn's impedance correspond to the following frequencies:

Frequency [Hz]				
118.15	323.07	440.42	614.35	765.49
931.23	1097.96	1256.50	1430.03	1587.96

Table 1: First ten resonance frequencies of the compound horn.

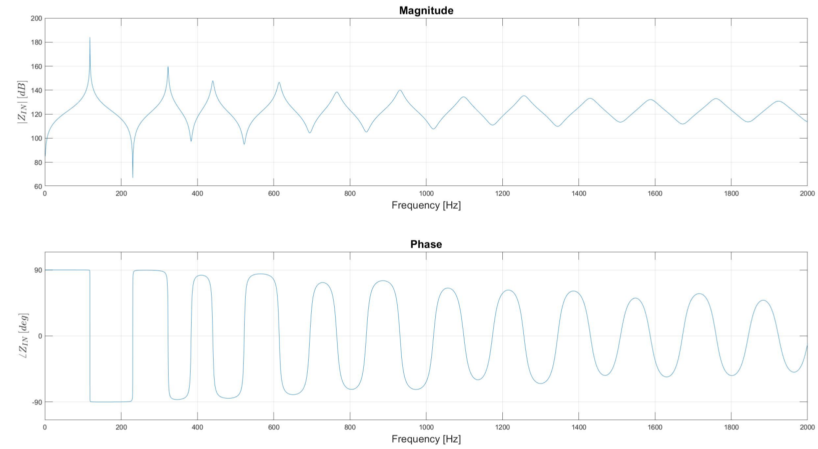


Figure 15: Input impedance of a compound horn composed by a cylindrical pipe followed by an exponential horn.

Conclusions

Inside Matlab we defined a lot of parameters that influences the computations of the acoustic properties of the models and their resolution. In particular we've set a value for the resolution of the frequency interval and we've seen that the higher the resolution the higher the number of cones needed to obtain a condition of $e_2 = 0$. This is due to the fact that having higher resolution should give us higher chance to exactly approximate the impedance of the exponential horn through the series of conical horns, but since it is a geometrical approximation, a zero error cannot be reached with a finite number of cones. So having a higher frequency resolution gives instead the need of a higher number of cones in order to have the impedance peaks at exactly the same frequency values. With the same consideration, we obtain for higher resolutions higher values of the first metric. This is due to the fact that the addends of the Riemann's sum inside the formula decrease in value (still not reaching zero though), but increase in number with increasing resolution.