Musical Acoustics
Homework 5: Design of a piano

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1 Introduction

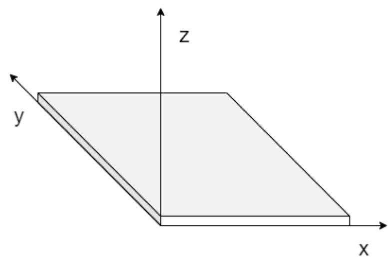
We are asked to design some components of a piano. In particular we will study the soundboard, the bridge and the strings:

- soundboard: we will compute the driving point input impedance as a function of the frequency and of the coordinates of the considered point;
- bridge: with the results obtained at the previous point, we will design the position of the bridge;
- strings: we will compute the eigenfrequencies of some note's pair of strings. After that we will calculate the decay time for both the eigenfrequencies.

2 Soundboard characterization

The soundboard has dimensions $1\,m \times 1.4\,m \times 1\,cm$ and it is composed of Sitka spruce. The study of the soundboard is characterized by the following approximations:

- it can be considered, by an acoustic point of view, as a rectangular thin plate;
- the material is homogeneous and isotropic.



2.1 Input impedance

To compute the driving point impedance, we will apply the modal approach and in order to do so we have to consider the mode shapes of the plate. We start from the wave equation for the transverse displacement of a thin plate:

$$\nabla^4 w + \frac{\rho s}{B}\frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

By applying Cartesian coordinates we can rewrite the previous equation:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{\rho s}{B}\frac{\partial^2 w}{\partial t^2} \tag{2}$$

Now we substitute a standing wave solution:

$$w(x, y, t) = \Phi(x, y)G(t) \tag{3}$$

where the dependency on time is given by:

$$G(t) = \overline{G}e^{j\omega t} \tag{4}$$

$\overline{G}$ being a constant of amplitude. The dependency on the position is, instead, given by the mode shape $\Phi(x, y)$. In a system we have as much mode shapes as the number of resonance frequencies. Considering a rectangular thin plate each one of them is given by:

$$\Phi_{n,m}(x, y) = A\sin(k_{x,n}x)\sin(k_{y,m}y) \tag{5}$$

Where A is a another constant of amplitude, while n and m, $n, m = 0, 1, 2, \ldots$ define the number of nodal lines on the plate corresponding to the eigenfrequency $\omega_{n,m}$. Inside the previous formula we have the wavenumbers $k_{x,n}$ on the x direction and $k_{y,m}$ on the y direction, together they define the wavenumber of the wave propagating on the plate:

$$k_{n,m}^2 = k_{x,n}^2 + k_{y,m}^2 \tag{6}$$

So the wavenumbers $K_{x,n}$ and $k_{y,m}$ are the components of the wavenumber $k_{n,m}$ projected on the two axis. They are given by:

$$k_{x,n} = \frac{(n+1)\pi}{L_x} \qquad k_{y,m} = \frac{(m+1)\pi}{L_y} \tag{7}$$

where $L_x = 1\,m$ and $L_y = 1.5\,m$ are the dimensions of the soundboard. From $k_{n,m}$ we can obtain the corresponding resonance frequency:

$$\omega_{n,m} = k_{n,m}^2 \sqrt{\frac{B}{\rho h}} \tag{8}$$

where $s = 0.01\,m$ is the thickness of the plate.
So now we have are able to compute the mode shapes of the plate. The receptance of the system is given, following the modal approach, by:

$$X = \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{\Phi_{n,m}(x_i, y_i)\Phi_{n,m}(x_j, y_j)}{-\omega^2 m_{mod} + (1 + j\eta)k_{mod}} \tag{9}$$

(x_i, y_i) and (x_j, y_j) are, respectively, the point of the measurement of the displacement and the point of application of the force. Since we are looking for the driving point input impedance they coincide. m_{mod} and k_{mod} are the modal mass and modal stiffness of the (n, m) one degree of freedom system into which the whole system has being divided by the modal approach, the term η is, instead, the isotropic loss factor of the material. For a rectangular thin plate they are given by:

$$m_{mod} = \rho s \frac{ab}{4} \quad k_{mod} = m_{mod}\omega_{n,m}^2 \tag{10}$$

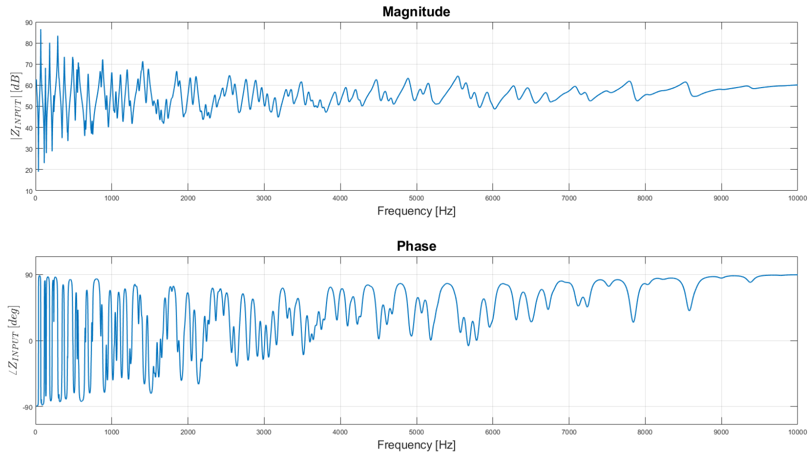
In order to obtain the input impedance we have to pass from the receptance to the admittance:

$$X = \frac{displacement}{force} \quad Y = \frac{velocity}{force} \implies Y = j\omega X \tag{11}$$

finally, after having compute the admittance of the system, we obtain the impedance as the inverse of the admittance:

$$Z = \frac{1}{Y} \tag{12}$$

We did the implementation of the algorithm inside the Matlab software. In the script we decided to approximate the impedance with the use of the coefficients n and m ranging from 0 to 15 and obtain, therefore, all the natural frequencies of the soundboard up to $f_{15,15} = 10335.58\,Hz$. In the following figure we show the obtained driving point input impedance measured on a point at the very center of the soundboard:



We notice that the diagram shows a lot of resonances in the low register, while, as the frequency increase, we see less peaks with a smaller difference of amplitude between peaks and valleys. This is expected as an effect of the isotropic loss factor of the material. We also notice that this diagram does not show the decreasing in magnitude above the limit of 1000 Hz that we would expect from the soundboard of a piano; we suppose that this fact is due to the strong approximations that we applied in this study.

3 String pairing

As an approximation with respect to a real piano, where the number of strings dedicated to producing a single note is variable, from 1 to 3, we are given pairs of strings deputed to producing the notes F4 (349.23 Hz), A4 (440.00 Hz), C5 (524.25 Hz), E5 (659.25 Hz), G5 (783.99 Hz).

3.1 Shape of the bridge

To design the bridge of the piano, we assume that the bridge transfers energy to the soundboard at the point where each string is mounted.
Observing the patterns of the impedance of the board for each of the frequencies of the strings, we need to choose a point, in which to attach the string, that is not in correspondence of a maximum of impedance, so that the board will vibrate well.
In making these choices, we decided to start from a value of tension equal for the strings of 900N, by looking at the plot below, knowing our notes are all in the "trichords" portion.
PLOT
Once we fixe the tension, we can use it to derive the wave speed in the strings, assuming they all have the same mass per unit length $\mu = 10.8g/m$:

$$c = \sqrt{\frac{T}{\mu}} \tag{13}$$

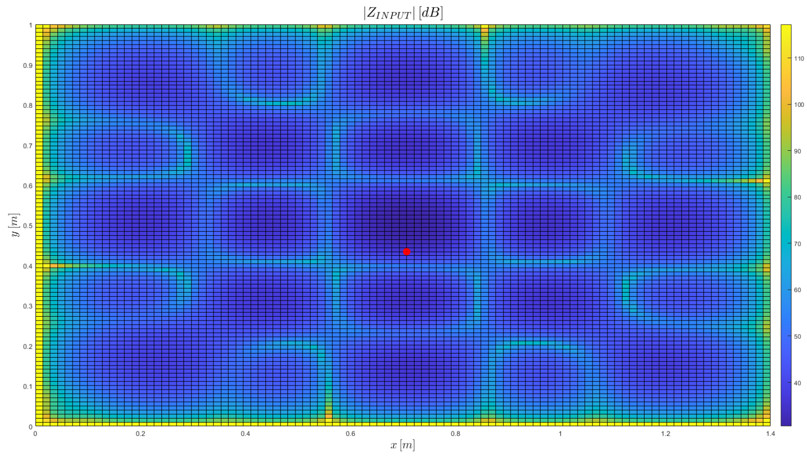
We then derive the length of each string, according to the node they should produce:

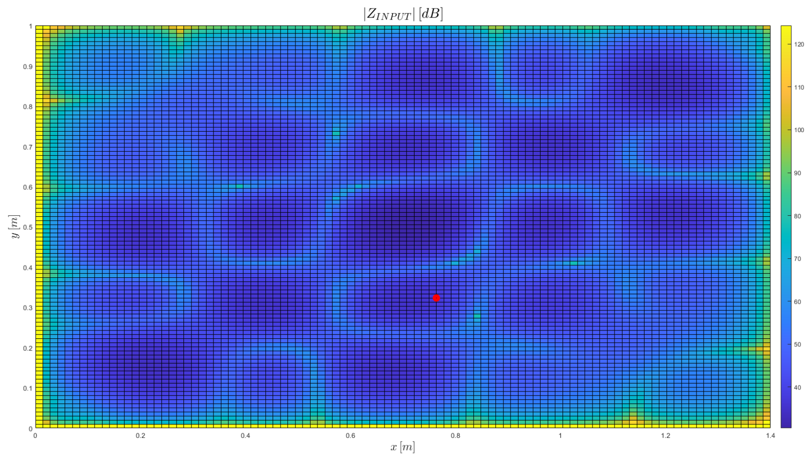
$$L = \frac{c}{2f} \tag{14}$$

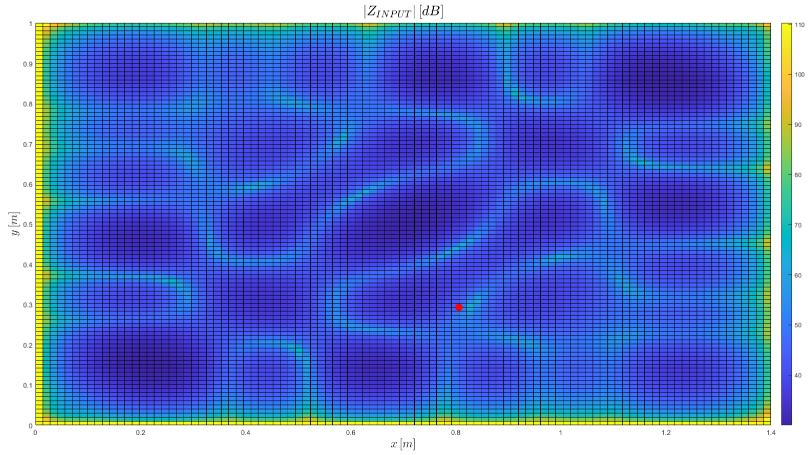
We then checked that the length of the string, dedicated to a note, calculated this way did not place the bridge too close to a maximum for the impedance at that frequency. In case it did, we moved the position in a better place.
The values we settled on are shown in the table below.

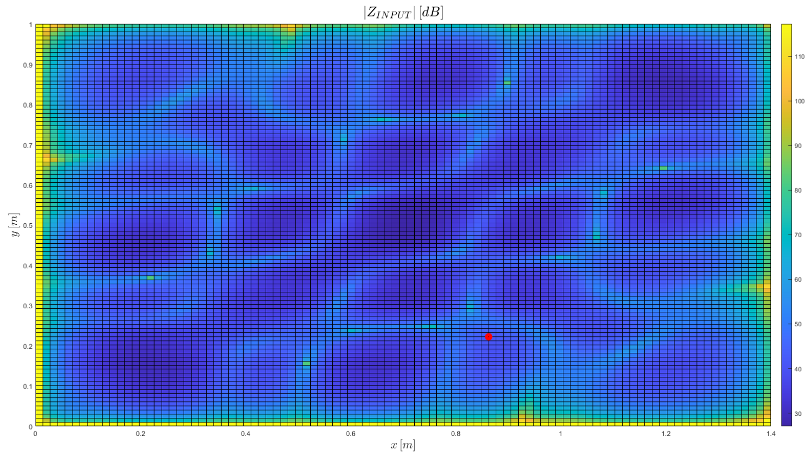
Note	Length [m]	Tension [N]
F4	0.434	992.4
A4	0.328	900
C5	0.293	1014.7
E5	0.219	900
G5	0.184	900

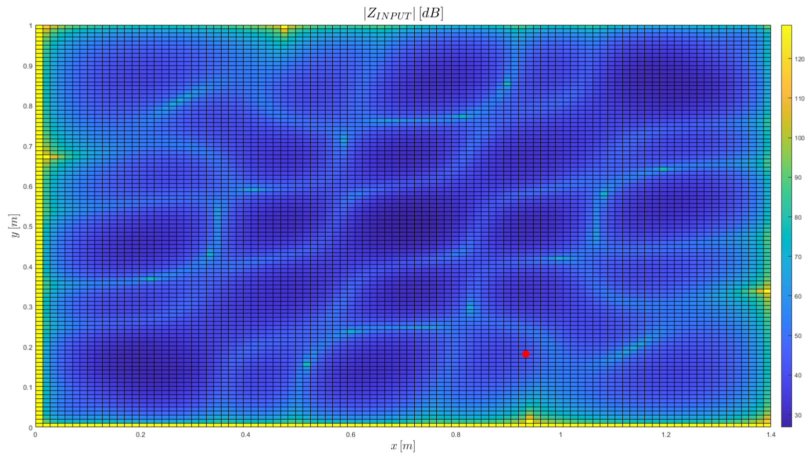
Below, we show the impedance pattern of the plate for each of the frequencies of the strings. Highlighted in red are the points of application of the strings.



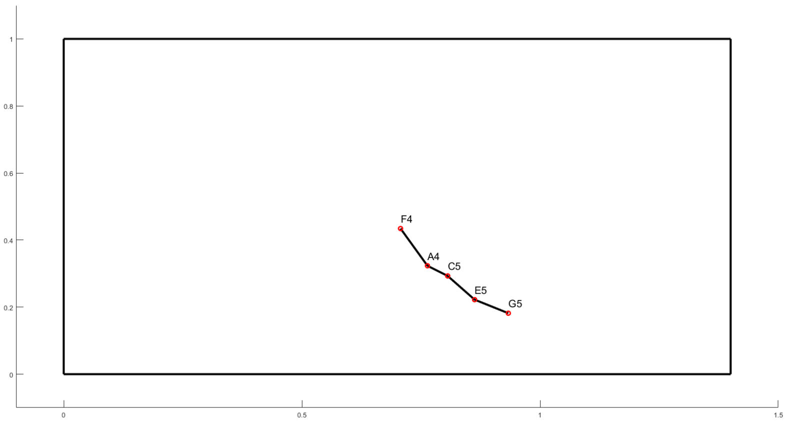








Finally, the following picture shows the design of the portion of the bridge where the five string pairs are attached, and its placement on the soundboard. We can imagine that, given the result, if we were to continue this procedure for other notes, the bridge would probably end up looking decently similar to the one of a real upright piano



3.2 Eigenfrequencies of the strings

The two strings in each couple are tuned so that one produces $f_0 = 2\pi\omega_0$, while the other produces $f_0(1+2\epsilon)$. We are interested in computing the eigenfrequencies of the two strings in each couple, considering this detuning and the coupling at the bridge. From the previous section, we compute the characteristic impedance of the string:

$$Z_0 = \sqrt{T\mu} \tag{15}$$

If we approximate the admittance of the bridge to be the same of the soundboard, at the point in which a string is attached, we can normalize its expression with respect to the characteristic impedance of the string, obtaining the following expression:

$$Y_B = \frac{\pi}{j\omega_0 Z_0}\zeta = \frac{\pi}{jZ_0}\chi \tag{16}$$

where χ is, in general, a complex value.
Following the procedure described by Chaigne and Kergomard in the chapter on the coupling of piano strings in their book "Acoustics of Musical Instruments", we arrive at the expression:

$$a_{\pm} = \chi + \epsilon \pm \sqrt{\chi^2 + \epsilon^2} \tag{17}$$

Taking the real part of this expression, we can derive the eigenfrequencies of the coupled strings as:

$$\omega_{\pm} = \omega_0 a_{\pm} + \omega_0 \tag{18}$$

To do so, we first need to fix a value for the detuning, ϵ. Since we want one string to have a much higher damping than the other, we want to choose the value in the range $-0.5 < \omega_0\epsilon < 0.5$.
In reality, this value would not be the same for every note, but it would be determined by an experienced piano tuner in order to achieve the best sound and decay possible for every note. In our case, to simplify the procedure, we can chosse for example $\epsilon = 10^{-4}$, so that we are inside the interval for each of the considered frequencies.
With this value, we find the following eigenfrequencies:

Note	**Frequency 1 [Hz]**	**Frequency 2 [Hz]**
F4	349.26	355.12
A4	432.44	440.04
C5	523.30	523.60
E5	659.13	659.31
G5	782.04	784.07

We observe that, while most of the frequencies as just slightly detuned, some deviate quite a lot from the initial value, especially in the two lower frequencies for which the coupling generates a not welcome beating effect.

3.3 Decay time

We finally want to compute the decay time for the two eigenfrequencies of each of the couples of strings.
To do so, we first need to write the expression of the bridge velocity in time, which we can do starting from the expression of the forces exerted by the two strings:

$$\begin{cases} F_1 = \frac{F_0}{2\mu}[(\chi - \epsilon + \mu)e^{ja_+t} + (\epsilon - \chi + \mu)e^{ja_-t}], \\ F_2 = \frac{F_0}{2\mu\chi}[(\epsilon + \mu)(\chi - \epsilon + \mu)e^{ja_+t} + (\epsilon - \mu)(\epsilon - \chi + \mu)e^{ja_-t}] \end{cases} \tag{19}$$

Where we defined $\mu = \sqrt{\chi^2 + \epsilon^2}$ and we imposed as initial condition $f_1(0) = f_2(0) = F_0 = 1N$
We can then obtain the velocity of the bridge as

$$V_B = Y_B(F_1 + F_2) = \frac{2\pi F_0 \chi}{\mu Z_0} e^{j(\epsilon+\chi)\omega_0 t}[\mu \cos \mu\omega_0 t + j\chi \sin \mu\omega_0 t]e^{\omega_0 t} \tag{20}$$

Taking the real part of this expression yields the time envelope of the bridge velocity.
We can finally compute the decay time computing the T_{60}, observing how long it takes for the envelope of the bridge velocity to decrease by 60 dB.
Unfortunately, we were not able to obtain results that were in accordance with what we expected theoretically, since the resulting envelopes for the motion of the bridge clearly showed unreliable results.
We show the most convincing looking plot, corresponding to the lowest frequency, which still gives a decay time T_{60} of more than two minutes.