Musical Acoustics
Homework 6: Design of a recorder flute

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Introduction

The aim of this study is to design the bore, the last two finger holes, the flue channel and the instrument mouth of a recorder flute.

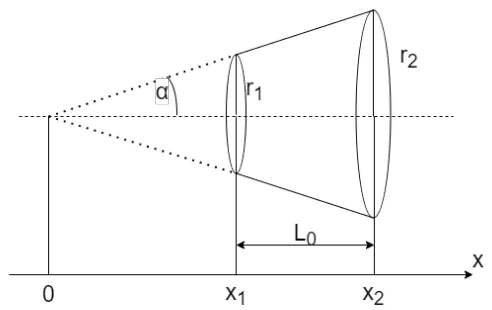
1 Resonator

The resonator is shaped as a cone whose conical semiangle is $\alpha = 0.75°$. The instrument is aimed at being a treble recorder, with a length of $L_0 = 0.45\,m$.

1.1 Diameters at bore's foot and head

The flute with all the holes closed must produce the note F4 with frequency $f_{F4} = 349.23\,Hz$.
The bore of a recorder is a truncated tapering cone, we decided to adopt the following conventions:

- the symmetry axis of the cone is parallel to the x-axis;
- the virtual vertex of the cone is placed at $x = 0$;
- the foot section is placed at coordinate $x_1 > 0$;
- the head section is placed at coordinate $x_2 > x_1$.



Since the input is the wider end in correspondence of x_2, the formula for the input impedance is:

$$Z_{IN} = \frac{j\omega\rho}{S_2} \times \frac{\sin(kL)\sin(k\theta_1)}{sin(k(L+\theta_1))} \tag{1}$$

where S_2 is the cross sectional area of the resonator in correspondence of x_2, $\rho = 1.225\ kg/m^3$ is the air density, $\theta_1 = \frac{\arctan(kx_1)}{k}$ and L is the length of the resonator comprising the end correction at the open end, which is $dL = 0.6 \times r_2$.
Considering that we are studying a recorder, we need to insert into the previous equation the impedance of the mouth window, which is given by:

$$Z_m = j\omega M = j\omega\frac{\rho\Delta l}{S_2} \tag{2}$$

for a treble recorder, a typical value is $\Delta l = 40\ mm$.
So the total impedance is given by:

$$Z_{IN_{tot}} = Z_{IN} + Z_m \tag{3}$$

The instrument works at minima of impedance, so we fix the frequency ω at $\omega_{F4} = 2\pi f_{F4}$ and we search for a minimum. Inside Matlab we plot the input impedance as a function of the radius r_1. The value of the radius r_1 that minimizes the input impedance is shown in figure 1, the radius at the input section is $r_2 = r_1 + L_0\tan(\alpha)$, the obtained diameters are therefore:

$$d_1 = 1.46\ cm \qquad d_2 = 2.64\ cm \tag{4}$$

With these dimensions we obtain the input impedance shown in figure 2

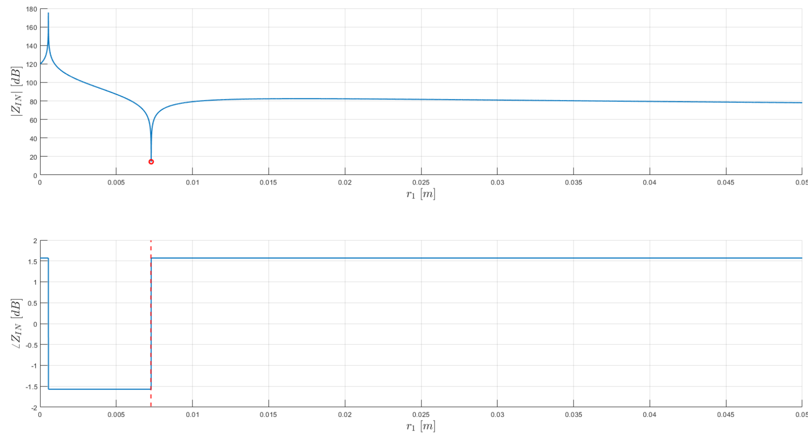


Figure 1: Input impedance of the cone of the treble recorder as a function of the radius of the smallest section.

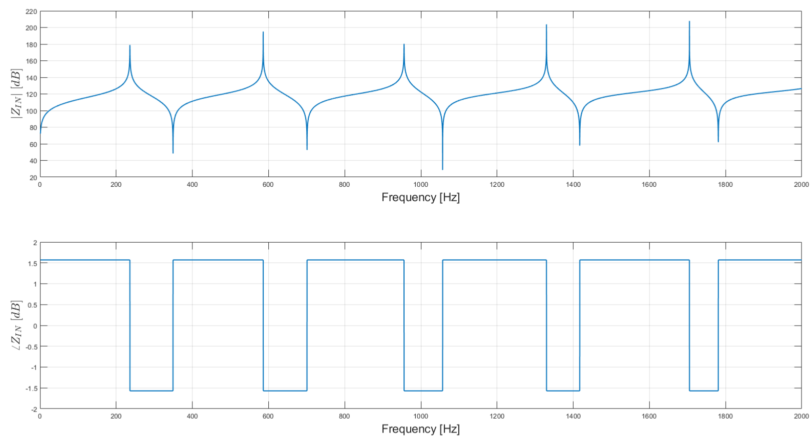


Figure 2: Input impedance of the resonator bore.

1.2 Finger holes

For simplicity, we assume the resonator has only two finger holes. We want to find their position in order for the instrument to produce a G4, with $f_{G4} = 392$, when the last finger hole (i.e. the one closest to the resonator foot) is open, and an A4, with $f_{A4} = 440$, when both finger holes are open.
We take into account the following semplifications:

- we consider the diameter of the two holes to be equal to the resonator section at its foot;
- the resonator is approximated as cylindrical.

In figure 3 we can see the resonator with one hole, in correspondence of Z_{in1}, and the equivalent resonator we gather when leaving the hole open, in correspondence of Z_{in2}. Δ is the end correction at the foot of the resonator, $D1$ is the distance from the foot to the center of the finger hole, and, because of the absence of a hole's chimney, the end correction applied at the finger hole has length $l = \Delta$. $L0$ is the length of the resonator and L is the length with the end correction.
The equivalent resonator will produce the desired note without any finger hole, just as

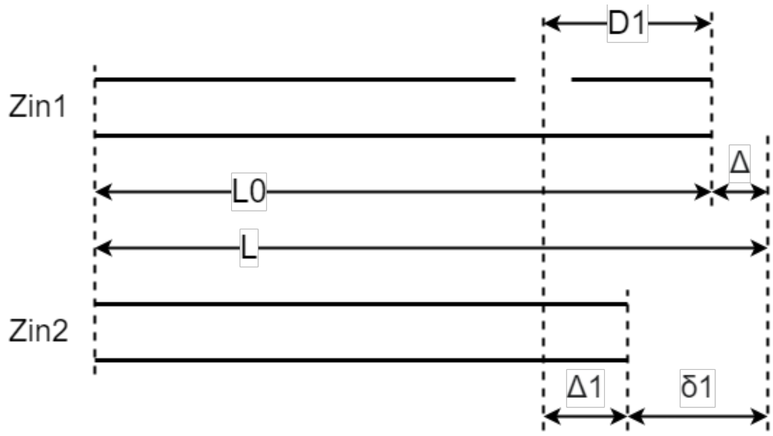


Figure 3: Schematics of the last hole and the equivalent resonator when the hole is open.

the original one would with the hole opened. This means that the impedance at the finger hole seen from the input section must be equal in the two cases. Since the hole and the portion of resonator corresponding to $D1$ are in parallel, it is more convenient to consider the admittance instead of the impedance, let's call Y_h the admittance of the finger hole and Y_p the admittance corresponding to the pipe of length $D1$:

$$Y_{in1} = Y_h + Y_p = -\frac{j}{\rho\omega}\left(\frac{S_h}{l} + \frac{S_p}{D1+\Delta}\right) \tag{5}$$

$$Z_{in1} = j\rho\omega\frac{l(D1+\Delta)}{S_h(D1+\Delta)+S_p l} \tag{6}$$

$$Z_{in2} = i\rho\omega\frac{\Delta 1}{S_p} \tag{7}$$

It must be $Z_{in1} = Z_{in2}$, so:

$$\Delta 1 = \frac{S_p l(D1+\Delta)}{S_h(D1+\Delta)+S_p l} \tag{8}$$

We can then define the difference between L and the length of the new resonator as δ:

$$\delta 1 = D1 + \Delta - \Delta 1 = \frac{S_h(D1+\Delta)^2}{S_h(D1+\Delta)+S_p l} \tag{9}$$

Applying our simplifications, δ can be rewritten as

$$\delta 1 \approx D1 + \frac{\Delta^2}{D1+2\Delta} \tag{10}$$

We can now express the length of the equivalent resonator as $L_{G4} = L - \delta 1$, which is a function of $D1$, the value we want to find. We can use L_{G4} in the computation of the input impedance of the series mouth-resonator, as in equation (1), working this time at the frequency f_{G4}. The resulting impedance as function of $D1$ is shown in figure 3.
Positioning ourselves at the minimum of the impedance, we find a distance for the last finger hole of $D_{G4} = 5.45\ cm$ from the foot of the resonator. For the second to last finger hole we adopt a similar strategy: Since we already know the position of the last hole, we can compute the same reasoning as before starting from the equivalent resonator shown in figure 5 and then going back to the original bore.

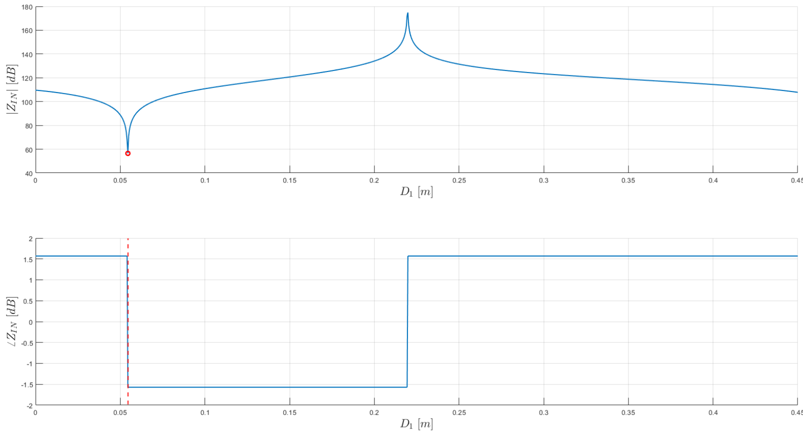


Figure 4: Input impedance of the resonator with the last finger hole open.

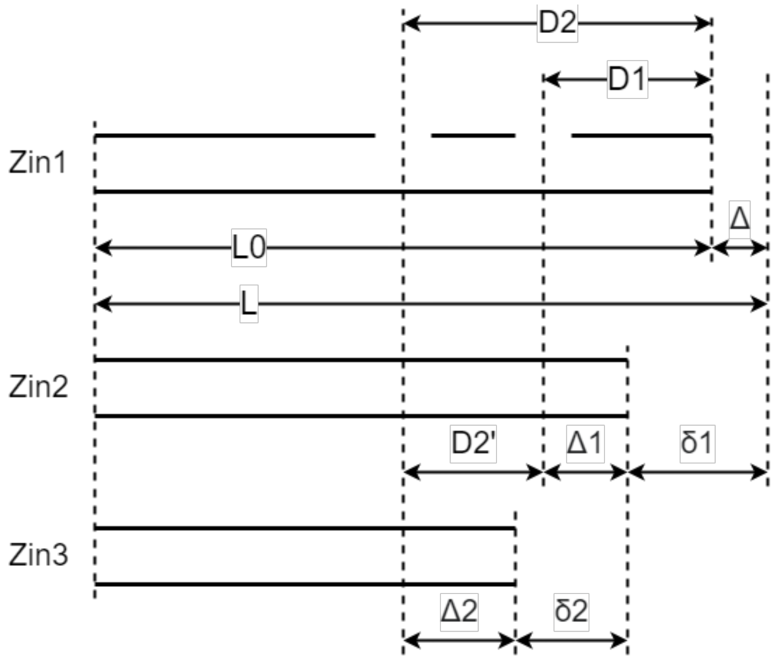


Figure 5: Schematics of the two holes and the equivalent resonators when the second to last hole is closed and the last hole is open and when both the holes are open.

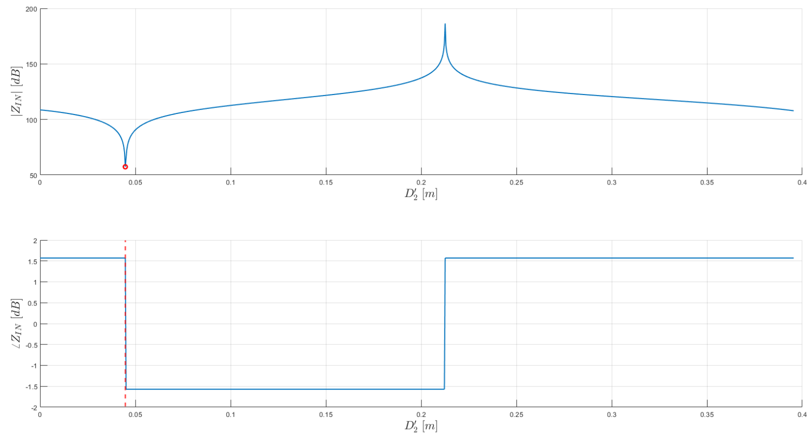


Figure 6: Input impedance of the resonator with the second to last finger hole open.

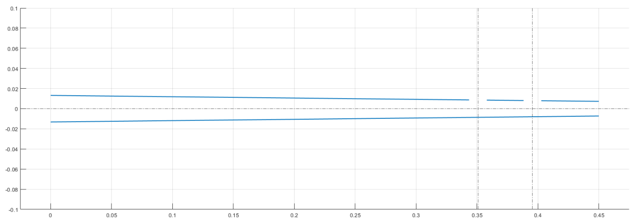


Figure 7: Schematics of the resonator

We have to consider the resonator corresponding to Z_{in3} and find the minimum. We start from $D2'$ and we define the geometry as a function of this parameter:

$$\delta 2 = D2' + \frac{\Delta 1^2}{D2' + 2\Delta 1} \tag{11}$$

The impedance is calculated based on the frequency f_{A4} and the length $L - \delta 1 - \delta 2$: From the figure 6 we can find the exact value of $D2'$ that minimize the input impedance at the frequency f_{A4} and so we obtain $D2$ $D_{A4} = D1 + D2' = 9.92\ cm$ Figure 5 shows the obtained geometry of the conical resonator with the two finger holes in their computed positions.

2 Flue channel and mouth

The instrument is aimed at producing a spectrum whose centroid is at $f_c = 2.0kHz$ when the pressure difference between the player's mouth and the flue channel entrance is $\Delta p = 62Pa$.

2.1 Flue channel thickness and Reynolds number

From the given specification, we can compute the central velocity U_j of the jet in the channel using the Bernoulli equation:

$$U_j = \sqrt{\frac{2\Delta p}{\rho}} = 10.061\ \frac{m}{s} \tag{12}$$

where ρ is the air density. At the frequency f_c, centroid of the spectrum of the instrument, the amplification is maximum. We can thus derive the thickness h of the flue channel as:

$$h = \frac{0.3U_j}{f_c} = 1.5\ mm \tag{13}$$

From the previous results, we can compute the Reynolds number as

$$Re = \frac{U_j h}{\nu} = 1012.2 \tag{14}$$

where $\nu = 1.5 \times 10^{-5}\ m^2/s$ is the kinematic viscosity.
Since the Reynolds number is lower than 2000, the jet remains laminar for a short distance after the flue channel exit.

2.2 Boundary layer thickness

Near the walls of the channel, the viscosity determines the presence of a boundary layer, causing a reduction of the velocity of the jet near the walls.
We are interested in finding the thickness of the boundary layer near the end of the channel, which is $L_{channel} = 20\ mm$ long.
The thickness is given by

$$\delta(x) = \sqrt{\frac{\nu x}{U_j}}, \tag{15}$$

which for our case gives a result of $\delta = 0.172mm$ at the channel exit.

2.3 Magnitude of oscillation of the jet

Finally we are interested in predicting the magnitude of oscillation of the jet at the labium, knowing that the sound pressure level at the flue channel exit is 50 dB and the distance between the exit of the channel and the labium is $W = 4\ mm$.
Following the jet receptivity model, the transverse displacement η(x,t) of the jet in the y direction can be calculated by integration of the transverse velocity from the flue exit to the current position x in the mouth.
A good experimental model of the receptivity is given by

$$\eta(x,t) \approx \eta_0 e^{\alpha_i x} e^{j\omega(t - x/c_p)}, \tag{16}$$

where c_p is the velocity of convection of the perturbation.
Since we're interested in the magnitude, we can neglect the complex exponential, arriving at

$$\eta_{max}(x) = \eta_0 e^{\alpha_i x}, \tag{17}$$

where α_i is the amplification factor and will be approximated through inspection, while η_0 is:

$$\eta_0 = \frac{V_{ac} h \delta_j}{U_j \delta_{ac}} \tag{18}$$

In the previous equation $L = 20\ mm$ the length of the flue channel, δ_j is the relative thickness of the shear layers in the jet and δ_{ac} is the thickness of the acoustic boundary layers:

$$\delta_j = \frac{L}{\sqrt{Re}} = 6.29 \times 10^{-4} \qquad \delta_{ac} = \sqrt{\frac{2\nu}{\omega}} = 1.17 \times 10^{-4} \tag{19}$$

while V_{ac} is the acoustic velocity, and can be derived knowing the input impedance of the system and the pressure of the source:

$$Z_{in} = \frac{\Delta P}{V_{ac}} \tag{20}$$

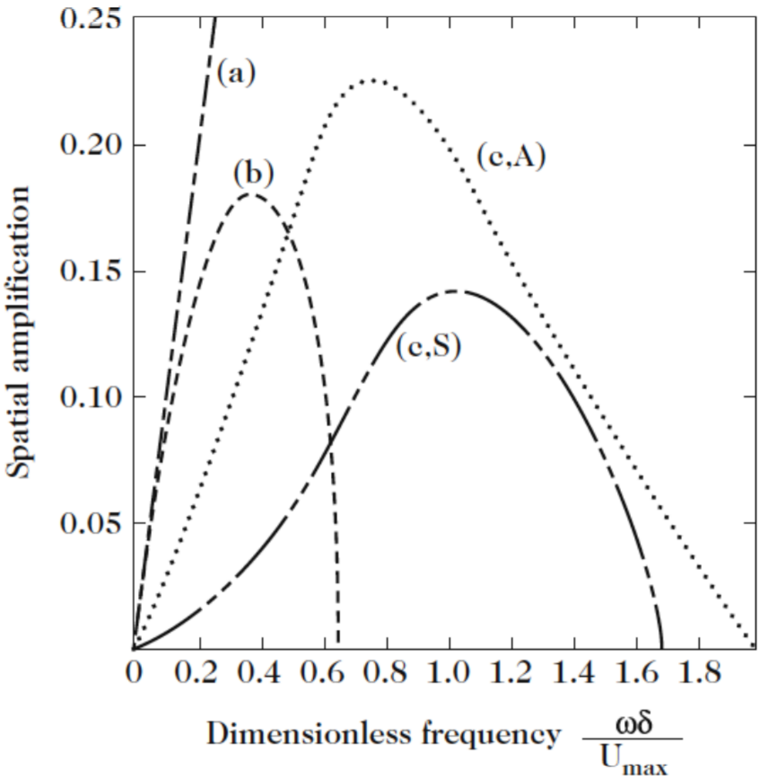


Figure 8: Normalized amplification factor as a function of the dimensionless frequency.

The pressure ΔP can be obtained converting the sound pressure level at the flue channel exit into pascal, which gives a value of:

$$\Delta p = 20 \times 10^{-6} \times 10^{\frac{SPL}{20}} = 0.0063\ Pa \tag{21}$$

Like in the first section of this assignment, the input impedance is made of the concatenation of the input impedance of the resonator with the one of the player's mouth. Computing this impedance for the frequency of a note F4 (349.23 Hz), leads to a value of:

$$V_{ac} = \frac{\Delta P}{Z_{in}} = 0.0012 \frac{m}{s} \tag{22}$$

This brings to:

$$\eta_0 = 9.86 \times 10^{-7} \tag{23}$$

From the value of δ_j we can obtain the dimensionless frequency:

$$\frac{\omega_{F4}\delta_j}{U_j} = 0.1371 \tag{24}$$

Applying the value just found to the diagram of the normalized amplification factor we can obtain the value of α_i: From the diagram we obtain the value:

$$\alpha_i \delta_j = 0.045 \tag{25}$$

which leads to:

$$\alpha_i = 715.85 \tag{26}$$

And the resulting oscillation amplitude is:

$$\eta_{max}(W) = \eta_0 e^{\alpha_i W} = 1.31 \times 10^{-3}\ mm \tag{27}$$