Politecnico di Milano

INGEGNERIA INDUSTRIALE E DELL'INFORMAZIONE
Music and Acoustic Engineering

Musical Acoustics

Module 1 - Modeling of musical instruments:
Assignment 2: 2d systems

Candidate:
De Bortoli Gian Marco 10805035
Marin Pasin Davide 10799610

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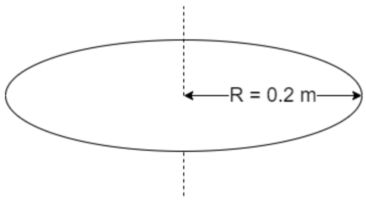
Introduction

We are given three different systems: the first two are composed by one structure each, the last one is made up of two coupled components.
We are going to study some acoustic properties of these systems using the Matlab software for the computations.

Circular membrane characterization



The first system is a circular membrane with radius $R = 0.2\,m$ and tension $T = 10\,N/m$. the unit surface weight is $\sigma = 0.1\,kg/m^2$.

a) Compute the propagation speed in the membrane.

It is given by the following equation:

$$c = \sqrt{\frac{T}{\sigma}} = 10 \quad \frac{m}{s} \tag{1}$$

b) Compute the frequency of the first five modes for this membrane and draw in Matlab the corresponding modeshapes.

In order to obtain the mode shapes of a circular membrane we need to start from the equation of motion in cylindrical coordinates:

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \phi^2} \right) \tag{2}$$

Assuming then $\tilde{z}(r, \phi, t) = R(r)\Phi(\phi)e^{j\omega t}$ we obtain:

$$\Phi(\phi) = Ae^{\pm jm\phi} \tag{3}$$

and:

$$R(r) = J_m(kr) \tag{4}$$

Considering the modal approach, we can describe the general displacement of a point of the membrane as a linear combination of the eigenmodes of the system:

$$z(r, \phi) = \Re(\tilde{z}(r, \phi)) = A\cos(m\phi)J_m(k_n r) \tag{5}$$

Where k_n is the wavelength of the *n-th* eigenmode considering, as boundary condition, a zero displacement of the membrane in correspondence of the maximum radius $R = 0.2\,m$.
This means that we can compute the natural frequencies knowing that $k_n\,R$

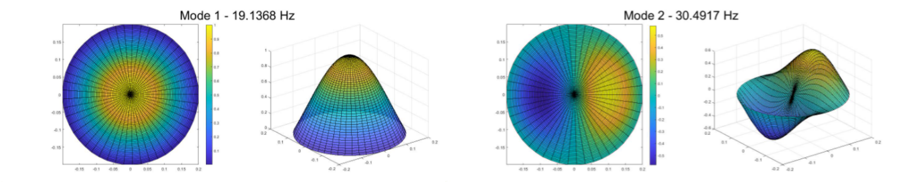
must corresponde to a zero of the bessel function of first kind. Therefore, we can compute them with the following:

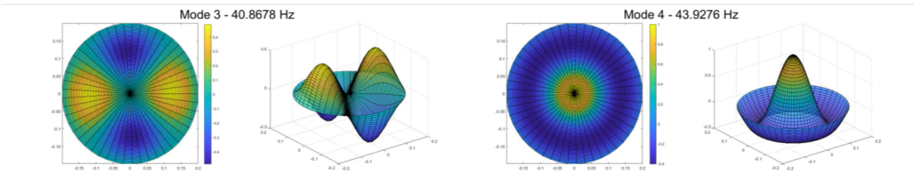
$$f_{mn} = \frac{Z_n(J_m(kr))}{2\pi R}\sqrt{\frac{T}{\sigma}} \tag{6}$$

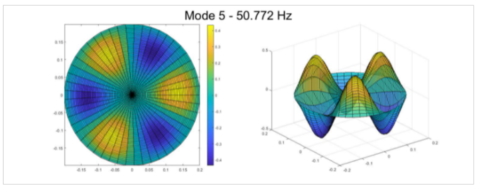
Where $Z_n(J_m(kr))$ is the *n-th* zero of the bessel function J_m.
With the above equation we obtain the natural frequencies of the first five mode shapes:

$$f_{01} = 19.14\,Hz \qquad f_{11} = 30.49\,Hz \qquad f_{21} = 40.87\,Hz \tag{7}$$
$$f_{02} = 43.93\,Hz \qquad f_{31} = 50.77\,Hz$$

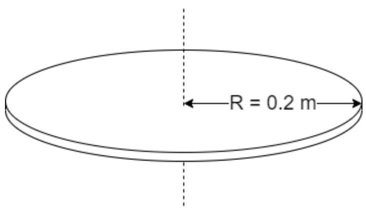
Finally, using (5), we can plot the mode shapes of the membrane corresponding to the first five natural frequencies:







Circular plate characterization



We consider now a thin plate with clamped edges and with the same size of the membrane ($R = 0.2\,m$). The plate has a thickness of $h = 1\,mm$, and it is made with aluminum ($E = 69\,GPa$, $\rho = 2700\,kg/m^3$, $\nu = 0.334$).

c) Compute the propagation speed of quasi-longitudinal and longitudinal waves.

In order to compute these two velocities we use two relations valid indipendently on the shape of the plate.
Quasi-longitudinal waves:

$$c_{QL} = \sqrt{\frac{E}{\rho(1-\nu^2)}} = 5363.2\,\frac{m}{s} \tag{8}$$

Longitudinal waves:

$$c_L = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} = 6199.2\,\frac{m}{s} \tag{9}$$

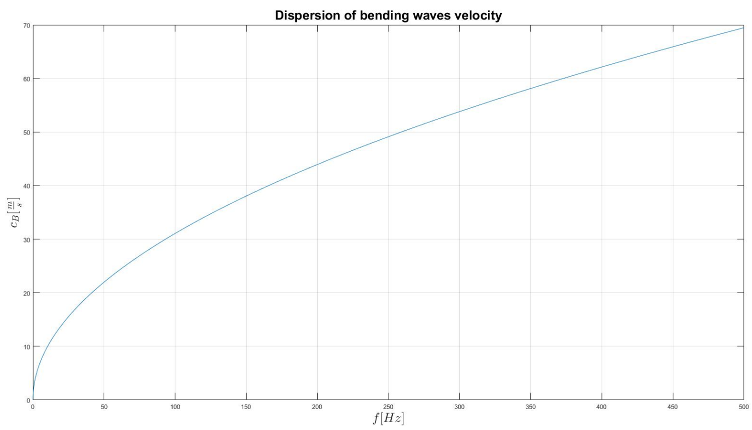
As we can see, both the quasi-longitudinal and the longitudinal waves are constants depending only on the material properties.

d) Plot the propagation speed of the bending waves as a function of the frequency for the considered plate.

We can compute the bending waves velocity with a this third equation:

$$c_B = \sqrt{1.8 f h c_L} = c_B(f) \tag{10}$$

This time the speed is not dependent only on material properties, it is also proportional to the square root of the frequency of the wave itself: $c_B = c_B(f)\ \alpha\ \sqrt{f}$.
This behaviour is called dispersion, we can plot it as follows:



e) Find the modal frequencies for the first five bending modes of the plate.

To compute the first five natural frequencies of the plate use the following empirical coefficients:

$$f_{01} = 0.4694\frac{c_L h}{R^2} = 62.94\,Hz \qquad f_{11} = 2.08 f_{01} = 130.91\,Hz$$

$$f_{21} = 3.41 f_{01} = 214.62\, Hz \qquad f_{02} = 3.89 f_{01} = 244.83\, Hz \tag{11}$$
$$f_{31} = 5.00 f_{01} = 314.69\, Hz$$

f) Plot the modeshape displacement of the first five bending modes.

We need to find the expression of the mode shapes. In order to do so, we start from the equation of motion:

$$\nabla^4 z + \frac{12\rho(1-\nu^2)}{Eh^2}\frac{\partial^2 z}{\partial t^2} = 0 \tag{12}$$

and we impose the general solution:

$$z(r,\theta,t) = Z(r,\theta)e^{j\omega t} \tag{13}$$

By substituting this solution into equation (12) we can resolve the dependency on time and we obtain:

$$(\Delta^2 + k^2)(\Delta^2 - k^2)Z = 0 \tag{14}$$

where we defined $k^2 = \frac{\omega\sqrt{12\rho(1-\nu^2)}}{h\sqrt{E}}$.
So the general component $Z(r,\theta)$ can be a solution of any of the following two:

$$\begin{cases} (\Delta^2 + k^2)Z = 0 \\ (\Delta^2 - k^2)Z = 0 \end{cases}$$

The first one is solved by a bessel function of first kind, the second one is solved by a hyperbolic bessel function of first kind, so the general solution must be a linear combination of the two multiplied by a term that presents the dependency on the variable θ:

$$Z(r,\theta) = [AJ_m(kr) + BI_m(kr)]\cos(m\theta + \alpha) \tag{15}$$

Since the plate is circular, the constant α is totally arbitrary and we can impose $\alpha = 0$, but we need to find the constants A and B. Hence, we add the boundary conditions of a clamped plate:

$$\begin{cases} Z|_{r=R} = 0 \\ \frac{\partial Z}{\partial r}|_{r=R} = 0 \end{cases}$$

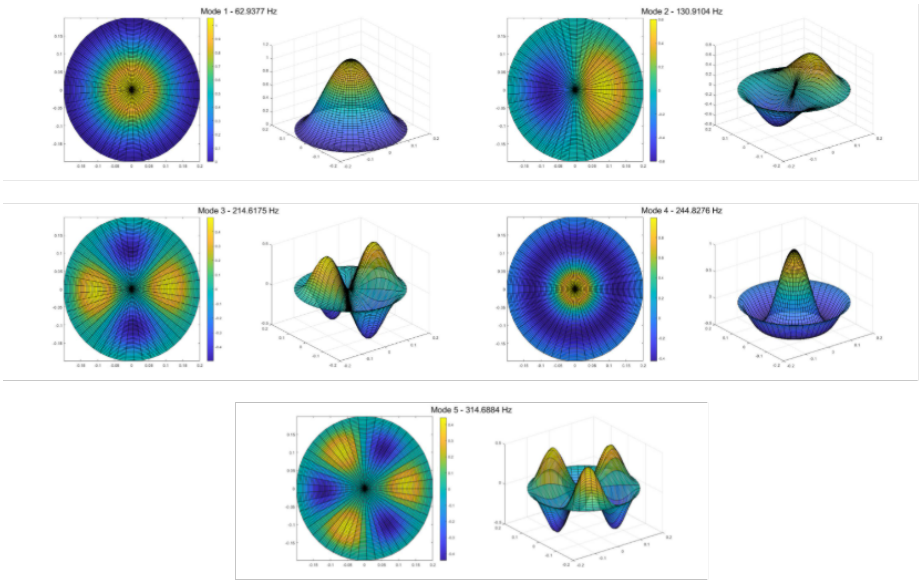
from which we obtain:

$$\begin{cases} AJ_m(kR) + BI_m(kR) = 0 \\ A\frac{\partial J_m}{\partial r}(kR) + B\frac{\partial I_m}{\partial r}(kR) = 0 \end{cases}$$

We can now derive the constant B as a function of the constant A and rewrite equation (12) as follows:

$$Z(r,\theta) = A(J_m(kr) - \frac{J'_m(ka)}{I'_m(ka)}I_m(kr))\cos(m\theta + \alpha) \tag{16}$$

By imposing an arbitrary normalization, like $A = 1$, we can plot the mode shapes:



We can notice that the mode shapes of the plate are very similar to the ones of the membrane, but given the fact that the edge is clamped and given, as a consequence, the two boundary conditions, we can see that the displacement is more concentrated towards the center and near the edge the slope goes to zero.

Interaction between coupled systems

Finally we consider that a string is attached to the considered plate, and its fundamental mode is tuned to the frequency of the first mode of the plate. The string is made with iron($\rho = 5000 kg/m^3$), its cross section is circular with a radius of $a = 0.001\, m$, and its length is $L = 0.4\, m$.
Due to internal losses and sound radiation, the plate at the frequency of the considered mode dissipates energy, and the merit factor is $Q = 50$.

g) Compute the tension of the string so that its fundamental mode is tuned with the first mode of the soundboard.

We know that the wave propagation speed on a string is given by:

$$c = \sqrt{\frac{T}{\mu}} \tag{17}$$

From which we can obtain the tension T on the string, but first we need to compute the linear density on the string and the wave velocity.
We know that the frequency of the n-th mode of vibration of the string is:

$$f_n = \frac{nc}{2L} \tag{18}$$

But since the string is tuned so that its fundamental is equal to the fundamental of the plate:

$$\mu = \rho \pi a^2 = 0.0157\, \frac{kg}{m} \tag{19}$$

$$c = 2L * 72.75\,Hz = 50.35\,\frac{m}{s} \tag{20}$$

So that the tension on the string is:

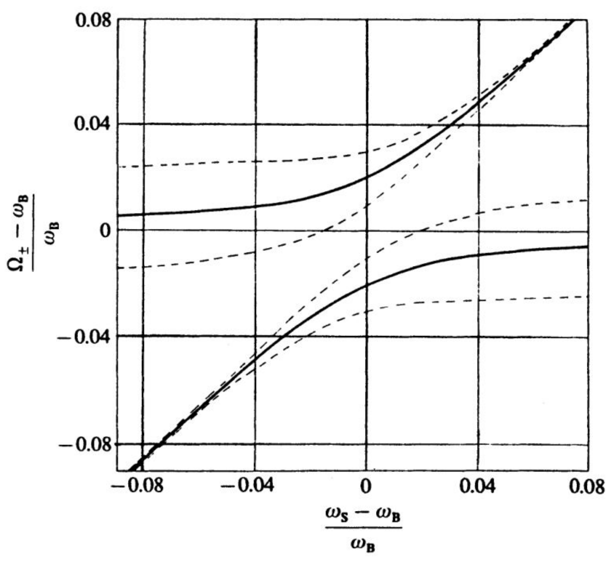
$$T = c^2\mu = 39.82\,N \tag{21}$$

h) Compute the frequencies of the modes of the string-soundboard system resulting from their coupling.

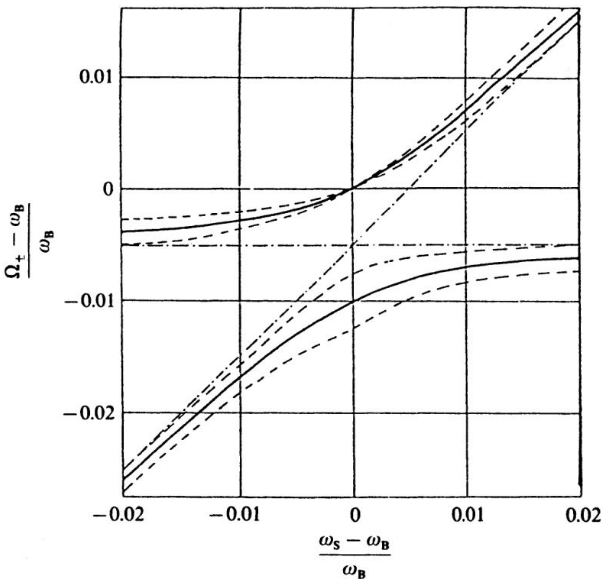
We know that in a coupling situation we can have weak or strong coupling, so the first thing we do is to study if the first five modes of the system are characterized by weak coupling or by strong coupling. Since we are considering the interaction between a string and a soundboard, the two situations are present based on the following relation:

- Weak coupling: $\frac{m}{n^2M} < \frac{\pi^2}{4Q^2}$
- Strong coupling: $\frac{m}{n^2M} > \frac{\pi^2}{4Q^2}$

Where m is the mass of the string, n indicates the *n-th* mode of vibration and M is the mass of the plate.
By computing the relation for each of the first five modes, we discover that modes 1 to 4 are characterized by strong coupling, while the fifth is characterized by weak coupling.
The second thing we do is to measure, for each of the five modes, the ratio $\frac{\omega_S - \omega_B}{\omega_B}$ where ω_S is the angular frequency of the string and ω_B is the angular frequency of the soundboard. We do this because we are going to measure the frequency separation due to coupling through inspection of two diagrams.
For strong coupling, so for the first four modes, we will use the following diagram:



While for the weak coupling, so for the fifth mode, we will use:



The values of the ratios we previously calculated are:

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Mode 1 ratio: 0
Mode 2 ratio: -0.038462
Mode 3 ratio: -0.12023
Mode 4 ratio: 0.028278
Mode 5 ratio: 0
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So, through inspection, we find:

$f_{1+} = 64.20\,Hz$ $\quad f_{2+} = 132.09\,Hz$ $\quad f_{3+} = 215.05\,Hz$
$f_{4+} = 254.62\,Hz$ $\quad f_{5+} = 314.69\,Hz$

$f_{1-} = 61.68\,Hz$ $\quad f_{2-} = 125.02\,Hz$ $\quad f_{3-} = 188.86\,Hz$
$f_{4-} = 242.13\,Hz$ $\quad f_{5-} = 311.54\,Hz$