Numerical simulations of piano strings. I. A physical model for a struck string using finite difference methods

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The first attempt to generate musical sounds by solving the equations of vibrating strings by means of finite difference methods (FDM) was made by Hiller and Ruiz [J. Audio Eng. Soc. **19**, 462–472 (1971)]. It is shown here how this numerical approach and the underlying physical model can be improved in order to simulate the motion of the piano string with a high degree of realism. Starting from the fundamental equations of a damped, stiff string interacting with a nonlinear hammer, a numerical finite difference scheme is derived, from which the time histories of string displacement and velocity for each point of the string are computed in the time domain. The interacting force between hammer and string, as well as the force acting on the bridge, are given by the same scheme. The performance of the model is illustrated by a few examples of simulated string waveforms. A brief discussion of the aspects of numerical stability and dispersion with reference to the proper choice of sampling parameters is also included.

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LIST OF SYMBOLS

$a_1 - a_5$	coefficients in the discrete wave equation
$a_H(t)$	hammer acceleration
b_1, b_3	damping coefficients
$c = \sqrt{T/\mu}$	transverse wave velocity of string
E	Young's modulus of string
$f(x, x_0, t)$	force density
f_1	fundamental frequency
f_e	sampling frequency
$F_B(t)$	bridge force
$F_H(t)$	hammer force
$g(x, x_0)$	spatial window
i	spatial index
K	coefficient of hammer stiffness
L	string length
$M_S = \mu L$	string mass
M_H	hammer mass
M_H / M_S	hammer–string mass ratio (HSMR)
n	time index
N	number of string segments
p	stiffness nonlinear exponent
S	cross-sectional area of the core
T	string tension
$v(x, t)$	transverse string velocity
$v_H(t)$	hammer velocity
V_{H0}	initial hammer velocity ($t = 0$)
x_0	distance of hammer from agraffe
$y(x, t)$	transverse displacement of string
$\alpha = x_0 / L$	relative hammer striking position (RHSP)
$\Delta t = 1/f_e$	time step
$\Delta x = L/N$	spatial step
ϵ	string stiffness parameter
$\eta(t)$	hammer displacement
κ	radius of gyration of string
μ	linear mass density of string
σ	decay rate
$\tau = 1/\sigma$	decay time
ω	angular frequency

INTRODUCTION

The vibrational properties of a musical instrument—like any other vibrating structure—can be described by a set of differential and partial differential equations derived from the general laws of physics. Such a set of equations, which define the instrument with a higher or lesser degree of perfection, is often referred to as a physical model. Due to the complex design of the traditional instruments, which in most cases also include a nonlinear excitation mechanism, no analytical solutions can, however, be expected from such a set of equations. Consequently, it is necessary to use numerical methods when testing the validity of a physical model of a musical instrument.

Once the numerical difficulties have been mastered, a simulation of a traditional instrument by a physical model means that the influence of step-by-step variations of significant design parameters like string properties, plate resonances, and others, can be evaluated. Such a systematic research method could hardly be achieved when working with real instruments, not even with the assistance of skilled instrument makers. In the future, it is hoped that

advanced physical models, which reproduce the performance of traditional instruments with high fidelity, can be used as a tool for computer-aided-lutherie (CAL).

Various numerical methods have been used extensively for many years in other branches of acoustics, for example in underwater acoustics where the goal is to solve the elastic wave equation in a fluid.[1] In musical acoustics, it is of great value to obtain a solution directly in the time domain, since it allows us to listen to the computed waveform directly, and judge the realism of the simulation. Among the large number of numerical techniques available, finite difference methods (FDM) are particularly well suited for solving hyperbolic equations in the time domain.[2] For systems in one dimension, like the transverse motion of a vibrating string, the use of FDM leads to a recurrence equation that simulates the propagation along the string.[3] The generality of FDM makes it possible to also use them for solving problems in two and three dimensions. The main practical limit then is set by the rapidly increasing computing time.

Historically, Hiller and Ruiz were the first to solve the equations of the vibrating string numerically in order to simulate musical sounds.[4] The model of the piano string and hammer used by these pioneers was, however, rather crude in view of the improvements in piano modeling over the last two decades.[5] For example, the crucial value of the contact duration between hammer and string, in reality being a result of the complex hammer–string interaction, was set beforehand as a known parameter.

Some years later, Bacon and Bowsher developed a discrete model for the struck string where the hammer was defined by its mass and its initial velocity.[6] Displacement waveforms were computed for both hammer and string at the contact point. Their model can be regarded as the first serious attempt to achieve a realistic description of the hammer–string interaction in the time domain. However, several effects were not modeled in detail. The damping was included as a single fluid (dashpot) term, and the stiffness of the string was neglected. The model assumed further a linear compression law of the felt. From a numerical point of view, no attempts were made to investigate stability, dispersion, and accuracy problems.

More recently, Boutillon made use of finite differences for modeling a piano string without stiffness, assuming a nonlinear compression law and the presence of a hysteresis in the felt. He investigated, in particular, the hammer–string interaction for two notes, in the bass and mid range, respectively.[7]

In all three papers mentioned, the numerical velocity, i.e., the ratio between the discrete spatial and time steps, was set equal to the physical transverse velocity of the string. It has been shown that this particular choice is possible for an ideal string only, and that the numerical scheme becomes unstable if stiffness, or nonlinear effects due to large vibration amplitudes, are taken into account in the model.[3]

At about the same time, Suzuki presented an alternative for simulating the motion of hammer and string, using a string model with lumped elements struck by a hammer

with a nonlinear compression characteristic. He investigated, in particular, some details of the hammer–string interaction, and the efficiency in the energy transmission from hammer to string. The effect of string inharmonicity was taken into account in a simplified manner by slightly modifying the values of the lumped string compliances.[8]

In a recent paper, Hall made use of another approach for simulating a stiff string excited by a nonlinear hammer, which he named a standing-wave model. His method can be regarded as a seminumerical approach, since it partially makes use of analytical results. By this method, he investigated systematically the effects of step by step variations of hammer nonlinearity and stiffness parameters, among other things.[9]

In comparison with the earlier studies mentioned above, the present model has the feature of a detailed modeling of the piano string and hammer as closely as possible to the basic physical relations: Our model is entirely based on finite difference approximations of the continuous equations for the transverse vibrations of a damped stiff string struck by a nonlinear hammer. The blow of the hammer is represented by a force density term in the wave equation, distributed in time and space, and the damping is frequency dependent.

The presentation is organized as follows. In Sec. I, the continuous model for the damped stiff string is briefly reviewed, with regard to the wave equation, and to the equations governing the hammer–string interaction. In Sec. II, it is shown how this theoretical background can be put into a discrete form for time-domain simulations. Some important aspects of numerical stability, dispersion, and accuracy are briefly discussed here, in particular the selection of the appropriate number N of spatial steps as a function of the fundamental frequency f_1 of the string, for a given sampling frequency f_e. A detailed treatment of the numerical aspects can be found in a previous paper by the first author.[3] In Sec. III, the structure of the computer program is presented, and a few examples of the capabilities of the model for representing the wave propagation on the string are given.

A thorough evaluation of the model by systematic comparisons between simulated and measured waveforms and spectra was left as a separate study. That work will also include a systematic exploration of the influence of the hammer–string parameters on the piano tone.

I. THEORETICAL BACKGROUND

A. Wave propagation on a damped stiff string

The present model describes the transverse motion of a piano string in a plane perpendicular to the soundboard. The vibrations are governed by the following equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \epsilon c^2 L^2 \frac{\partial^4 y}{\partial x^4} - 2b_1 \frac{\partial y}{\partial t} + 2b_3 \frac{\partial^3 y}{\partial t^3} + f(x, x_0, t), \quad (1)$$

in which stiffness and damping terms are included. The stiffness parameter is given by

$$\epsilon = \kappa^2 (ES/TL^2). \quad (2)$$

It has been shown that this stiffness term, which is the main cause of dispersion in piano strings, especially in the lowest range of the instrument, gives rise to a "precursor" which precedes the main pulses in the string waveform. Possibly it could also affect the perceived attack transient.[10]

The two partial derivatives of odd order with respect to time in Eq. (1) simulate a frequency-dependent decay rate of the form,

$$\sigma = 1/\tau = b_1 + b_3 \omega^2. \tag{3}$$

As a consequence, the decay times of the partials in the simulated tones will decrease with frequency, as can be observed in real pianos.[11] It must be pointed out that this simplified formula yields a smooth law of damping which is only a fair approximation of the reality. The constants b_1 and b_3 in Eq. (3) were derived from experimental values through standard fitting procedures, and it is assumed that these empirical laws account globally for the losses in the air and in the string material, as well as for those due to the coupling to the soundboard. No attempts were made toward an accurate modeling of each individual physical process that causes energy dissipation in the strings. The form of Eq. (3) is particularly attractive as it has been shown in previous studies that the time response of mechanical systems is stable and causal for laws of damping involving even order polynomials in frequency.[12]

The model does not include the mechanisms which give rise to two different decay times in the piano tone, "prompt sound" and "aftersound."[13] This effect is mainly due to string polarization, differences in horizontal and vertical soundboard admittance, and "mistuning" within a string triplet.

The force density term $f(x,x_0,t)$ in Eq. (1) represents the excitation by the hammer. This excitation is limited in time and distributed over a certain width. It is assumed that the force density term does not propagate along the string, so that the time and space dependence can be separated,

$$f(x,x_0,t) = f_H(t)g(x,x_0). \tag{4}$$

From a physical point of view, it is clear that the dimensionless spatial window $g(x,x_0)$ accounts for the width of the hammer. Within the context of numerical analysis, it is interesting to notice that the use of such a smoothing window eliminates the artifacts that occur in the solution (in the form of strong discontinuities), when the excitation is concentrated in a single point.

The density term $f_H(t)$ is related to the time history of the force $F_H(t)$ exerted by the hammer on the string by the following expression:

$$f_H(t) = F_H(t) \bigg/ \left(\mu \int_{x_0-\delta x}^{x_0+\delta x} g(x,x_0)dx \right), \tag{5}$$

where the length of the string segment interacting with the hammer is equal to $2\delta x$.

B. Initial and boundary conditions

For the struck string, it is now well known that the force $F_H(t)$ is a result of a nonlinear interaction process between hammer and string.[5] In our model, the motion of the string starts at $t=0$ as the hammer with velocity V_{H0} makes contact with the string at the striking position x_0. It is assumed that $F_H(t)$ is given by a power law,[9]

$$F_H(t) = K|\eta(t) - y(x_0,t)|^p, \tag{6}$$

where the displacement $\eta(t)$ of the hammer head is given by

$$M_H \frac{d^2\eta}{dt^2} = -F_H(t), \tag{7}$$

and where the stiffness parameters K and p of the felt are derived from experimental data on real piano hammers. The losses in the felt are neglected.

In the computer program, the interaction process ends when the displacement of the hammer head becomes less than the displacement of the string at the center of the contact segment (x_0). This yields, among other things, the contact duration between hammer and string.

The string is assumed to be hinged at both ends, which corresponds to the following four boundary conditions:[14]

$$y(0,t) = y(L,t) = 0$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (8)

$$\frac{\partial^2 y}{\partial x^2}(0,t) = \frac{\partial^2 y}{\partial x^2}(L,t) = 0.$$

These boundary conditions do not correspond strictly to the string terminations in real pianos, and will be reconsidered in a future work.

The continuous model of piano strings developed in this section forms the basis of our numerical model. Emphasis will now be put on the computational methods used for solving the equations, and the obtained algorithms will be discussed.

II. TIME-DOMAIN SIMULATIONS

A. String model

The equations of motion for the string and hammer presented in Sec. I are formulated in discrete form using standard explicit differences schemes centered in space and time.[2] The main variable is the transverse string displacement $y(x,t)$ which is computed for the discrete positions $x_i = i\,\Delta x$, and at discrete time steps $t_n = n\,\Delta t$. Values of the hammer position $\eta(t)$ are computed, using the same time grid and the same increment Δt. In the following, the simplified notation,

$$y(x,t) \rightarrow y(x_i,t_n) \rightarrow y(i,n), \tag{9}$$

will be used for convenience

In a second stage, the velocity and acceleration of the hammer, and of each discrete point of the string, are derived from the corresponding displacement values by means of finite differences centered in time. Finite differ-

TABLE I. Coefficients of the recurrence equation for the damped stiff string.

$a_1 = [2 - 2r^2 + b_3/\Delta t - 6\epsilon N^2 r^2]/D$	$a_2 = [-1 + b_1\Delta t + 2b_3/\Delta t]/D$
$a_3 = [r^2(1 + 4\epsilon N^2)]/D$	$a_4 = [b_3/\Delta t - \epsilon N^2 r^2]/D$
$a_5 = [-b_3/\Delta t]/D$	
where	
$D = 1 + b_1\Delta t + 2b_3/\Delta t$	and $r = c\Delta t/\Delta x$

FIG. 1. Maximum number of spatial steps N_{max} as a function of the fundamental frequency f_1 of the string for different values of the stiffness parameter. (a) $\epsilon = 10^{-8}$; (b) $\epsilon = 10^{-6}$; (c) $\epsilon = 10^{-4}$. The sampling frequency is $f_e = 48$ kHz.

ences centered in space are used for computing the force transferred from one segment of the string to its adjacent segments. This gives, in particular, the force $F_B(t)$ exerted by the string on the bridge. The interaction force between the hammer and the string is obtained in a straightforward way by putting Eq. (6) into a discrete form.

These numerical schemes lead to convenient recurrence equations where, for each point i, the variable under examination at a future time step $(n+1)$ is a function of the same variable at the same position i and at adjacent positions $(i-2, i-1, i+1, i+2)$ at present and past time steps $(n, n-1,$ and $n-2)$. The recurrence equation for the transverse displacement of a damped stiff string corresponding to Eq. (1), is given by

$$y(i,n+1) = a_1 y(i,n) + a_2 y(i,n-1)$$
$$+ a_3[y(i+1,n) + y(i-1,n)] + a_4[y(i+2,n)$$
$$+ y(i-2,n)] + a_5[y(i+1,n-1)$$
$$+ y(i-1,n-1) + y(i,n-2)]$$
$$+ [\Delta t^2 N F_H(n) g(i,i_0)]/M_S, \qquad (10)$$

where the coefficients a_1 to a_5 are given in Table I.

Before starting the computation, an appropriate number (N) of spatial steps must be selected. For a standard explicit finite difference scheme, it has been shown theoretically that this selection is critical for stability and numerical dispersion.[3] In practice, the stability condition provides us with a maximum number (N_{max}) of discrete string segments, (i.e., with a minimum segment length Δx_{min}), assuming that the (dimensionless) stiffness parameter (ϵ), the fundamental frequency (f_1) of the string, and the sampling frequency (f_e) are given. The stability condition can be written as

$$N_{max} = \{[-1 + (1 + 16\epsilon\gamma^2)^{1/2}]/8\epsilon\}^{1/2}, \qquad (11)$$

where

$$\gamma = f_e/2 f_1. \qquad (12)$$

If the stiffness is neglected ($\epsilon = 0$), then Eq. (11) reduces to

$$N_{max} = \gamma. \qquad (13)$$

In addition to these stability requirements, the problem of numerical dispersion must also be taken into account. It has been shown that some unwanted dispersive effects (grid dispersion) may be present in the solution if an explicit finite difference scheme is used for solving the stiff string equa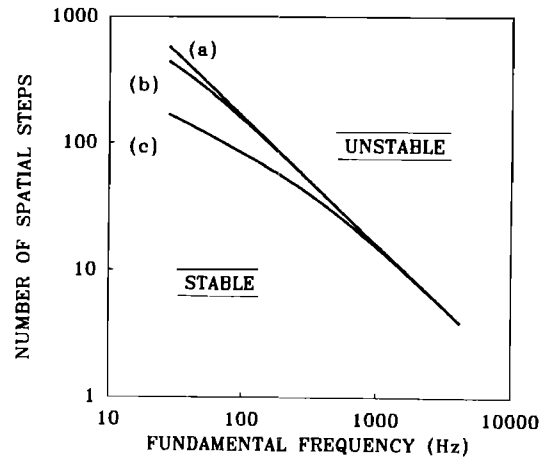tion.[3] This numerical dispersion should not be confused with the intrinsic physical dispersion due to the stiffness term in Eq. (1). As a result of the grid dispersion, the eigenfrequencies of the string and the inharmonicity are slightly underestimated for a given stiffness parameter. Fortunately, this applies primarily to the frequency range just below the Nyquist frequency ($f_e/2$). By using a sufficiently high sampling rate so that the string partials near the Nyquist frequency contain no significant energy, the effects of this underestimation can be made inaudible. Further, in order to limit the dispersion as much as possible, N should be equal to the highest possible integer value which is immediately lower than N_{max}.

Usually, the actual sampling frequency f_e, is determined by the audio equipment. Therefore, it was decided to select, in this particular experiment, one of the standard values (32, 44.1, and 48 kHz) for the output sampling rate. Figure 1 shows N_{max} as a function of the fundamental frequency (f_1) of the string, at a sampling rate of $f_e = 48$ kHz for three different values of the stiffness parameter. The three decades for f_1 shown in the figure cover the range of a grand piano. Notice that N_{max} is not directly dependent on the string length, but rather on the ratio between this parameter and the transverse wave velocity.

In practice, the computation will be made at a lower sampling rate (say $f_e = 16$ kHz) for notes with fundamental frequency below 100 Hz, in order to limit N to an acceptable value. The synthesized signals will be then interpolated by a factor 2 or 3 and played back at a standard sampling rate. At the other end, oversampling will be necessary for the highest notes of the instrument (typically for f_1 greater than 1 kHz, i.e., for note C6 and above), since truncation errors may appear in the solution for too small values of N. In this range, the computations were made with a sampling rate of 64 kHz, or even 96 kHz for note C7, and the signals were played back after low-pass filtering and decimation.

B. Modeling the initial and boundary conditions

At time $t = 0$ ($n = 0$), the hammer velocity is assumed to be equal to V_{H0}, and its displacement and the force

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exerted on the string are taken equal to zero. For the sake of simplicity, only the simplest case, where the string is assumed to be at rest at the origin of time, will be presented below. Note, however, that the model can handle any initial condition. With the string at rest at $t=0$,

$$y(i,0)=0. \tag{14}$$

At time $t=\Delta t$ ($n=1$), the hammer displacement is given by

$$\eta(1)=V_{H0}\,\Delta t. \tag{15}$$

At that time, Eq. (10) generally cannot be used for computing the string displacement, since four time steps are involved in the general recurrence equation. One solution, however, consists in assuming that the string is at rest for the first three time steps. Another technique used here is to estimate $y(i,1)$ by the approximated Taylor series:[2]

$$y(i,1)=[y(i+1,0)+y(i-1,0)]/2. \tag{16}$$

Thus the force exerted by the hammer on the string becomes

$$F_H(1)=K\,|\eta(1)-y(i_0,1)|^p. \tag{17}$$

This enables us to compute a first estimate of the displacement $y(i,2)$. In order to limit the time and space dependence for $n=2$, a simplified version of Eq. (10) is used, where the stiffness and damping terms are neglected. This yields

$$y(i,2)=y(i-1,1)+y(i+1,1)-y(i,0)$$
$$+[\Delta t^2 N F_H(1)g(i,i_0)]/M_S. \tag{18}$$

Similarly, the hammer displacement $\eta(2)$ is given by

$$\eta(2)=2\eta(1)-\eta(0)-[\Delta t^2 F_H(1)]/M_H, \tag{19}$$

and the hammer force is now written as

$$F_H(2)=K\,|\eta(2)-y(i_0,2)|^p. \tag{20}$$

At this stage, one may ask if it is fully justified to compute the displacements in Eqs. (18) and (19) at time $n=2$ using the value of the force at time $n=1$, i.e., with a time delay equal to Δt. This follows from the implicit form of Eq. (20), which requires the values of the displacements in order to compute the hammer force.

Normally, the effects of this approximation can be neglected, provided that the sampling frequency is sufficiently high. In that case, only the high-frequency content of the synthesized signal will be affected by the delay, and the influence on the computations will be small. An accurate estimation of the effect can be obtained by iterating the procedure described above, and calculating a second estimate of the displacements using Eq. (20), which in turn leads to a more accurate estimate of the hammer force. This procedure can be repeated until no significant differences between successive results are observed. In our simulations, the algorithm converged rapidly, and the differences between the first and second estimates for displacements and forces were never greater than 1% in
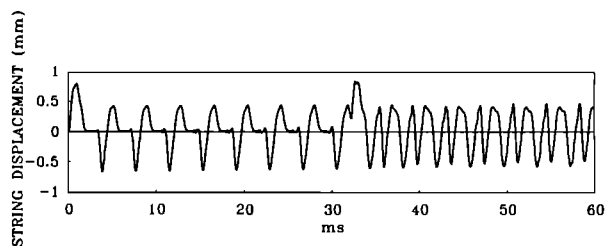

FIG. 2. Illustration of a repetition of a note showing computed string displacement (at 40 mm from the hammer, bridge side). First blow of hammer at $t=0$ with string initially at rest, followed by a repeated blow at $t=32$ ms with string in motion.

the worst cases. It was therefore decided to calculate only the first estimate of the variables, in order to limit the computational time.

Once the values of the displacements are known for the first three time steps, it is possible to start using the general recurrence formula given in Eq. (10), where the future displacement $y(i,n+1)$ is computed assuming that the present force $F_H(n)$ is known. The hammer leaves the string when

$$\eta(n+1)<y(i_0,n+1), \tag{21}$$

after which time the string is left to free vibrations. In this case, Eq. (10) still applies, but the force term is temporarily removed. By further comparisons of string and hammer displacements, the possibility of hammer recontact can be taken into account. This latter feature has been observed however only for the low bass strings.

An attractive feature of the method is that there is no need to assume that the string initially is at rest. The force density term $f(x,x_0,t)$ can be introduced at any time in the wave equation, whatever the vibrational state of the string. Thus the model makes it possible to simulate not only isolated tones, but also a musical fragment with realistic transitions between notes (see Fig. 2). In this case, repeated notes are obtained by re-initializing the hammer position to zero before striking the moving string. This feature is not available in today's commercial synthesizers.

As for the boundary conditions, the numerical expressions corresponding to hinged ends case in Eq. (8) are straightforward and yield:

$$y(0,n)=0 \quad\text{and}\quad y(N,n)=0, \tag{22}$$

$$y(-1,n)=-y(1,n) \quad\text{and}$$
$$y(N+1,n)=-y(N-1,n). \tag{23}$$

If the load of the soundboard at $i=N$ is modeled by a frequency-dependent admittance, then the second condition in Eq. (23) can conveniently be replaced by the discrete form of the appropriate differential equation. This refinement has already been successfully applied to the guitar.[3]

The conditions given in Eq. (23) are important for deriving specific recurrence equations for the points $i=1$ and $i=N-1$ which are close to the string terminations. Due to the stiffness term, Eq. (1) is of the fourth-order in
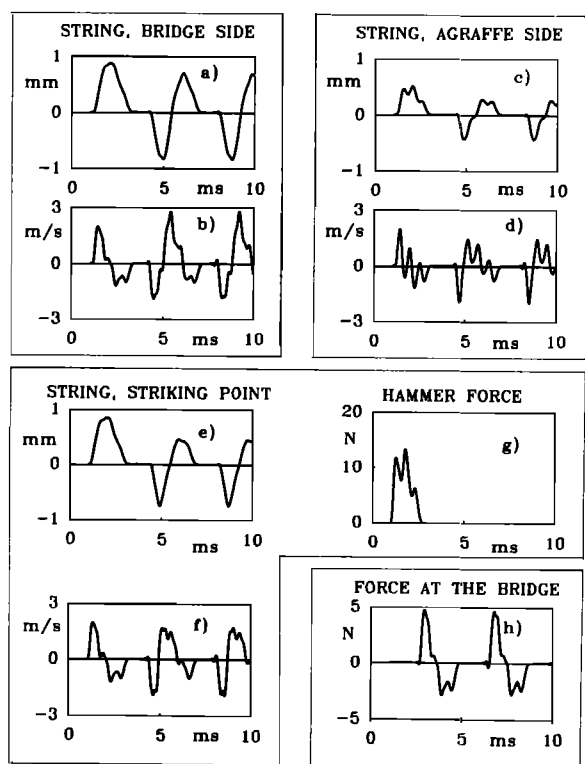
FIG. 3. Computed waveforms for piano string C4 at four positions. **String, bridge side** (a) string displacement (at 40 mm from the hammer), (b) string velocity. **String, agraffe side** (c) string displacement (at 40 mm from the hammer); (d) string velocity. **Striking point** (e) string displacement, (f) string velocity. (g) hammer force. **Bridge** (h) force transmitted to the bridge.

space, and thus the recurrence equation for the point i will depend on the vibrational state of points $i-2$ to $i+2$. Therefore, it is necessary to know the values of the displacements $y(-1,n)$ and $y(N+1,n)$ in order to compute the solution at $i=1$ and $i=N-1$, respectively. Because $i=-1$ and $i=N+1$ do not belong to the "physical" string, Eq. (23) must be used for replacing $y(-1,n)$ and $y(N+1,n)$ by expressions involving only the values of the displacements for i within the interval $[0,N]$.

III. STRUCTURE AND PERFORMANCE OF THE COMPUTER PROGRAM

The simulation program is written in Turbo-Pascal, and runs on a 80486 based Personal Computer DEC station 425 PC677-A3. At a clock speed of 25 MHz, it takes about 100 s to obtain 1 s of sound at a sampling frequency of 32 kHz, with the string divided into $N=100$ spatial steps. This value is a typical order of magnitude for the computing time, although it may vary slightly from one string to the other.

The main part of the program holds the model of the string motion, described in the previous section. This part is linked with data files which contain the values of the hammer and string parameters actually used in the simulations. Some of the parameters were measured by the authors, while others were extracted from the literature.[9,15,17] The stiffness parameters K and p of the hammer felt, and
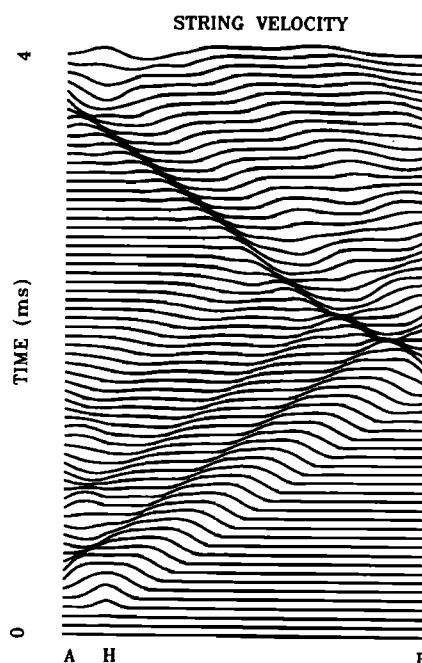

FIG. 4. Simulated velocity profile of a piano string (C4) during the first 4 ms after the blow. The time step between successive plots is 62.5 μs. The string terminations are indicated by A (agraffe) and B (bridge), and the striking point by H (hammer).

the damping coefficients b_1 and b_3 of the string, were derived from experimental data by means of standard curve-fitting procedures.

In its standard executable version, the program starts with an interaction with the user, requesting the sampling frequency (in kHz), the fundamental frequency (in Hz), and the duration of the computed note (in s). This enables the program to compute the number (N) of spatial steps, using Eqs. (11)–(13). This procedure is followed by the computation of the first three time steps ($n=0$ to $n=2$), as described in the previous section, using Eqs. (14)–(20). Then, the recurrence parameters of the damped stiff string given in Table I are calculated once for all. For $n \geqslant 3$, the program computes the hammer force $F_H(n)$, string displacement $y(i,n)$, and hammer displacement, $\eta(n)$, in parallel. If the condition in Eq. (21) is met, the force term is removed from the recurrence scheme in Eq. (10) before the computations proceed.

At each time step, the program can provide a complete set of signals, adding four variables—$v(i,n)$, the string velocity at each point of the string, $F_B(n)$, the force exerted by the string on the bridge, $v_H(n)$, hammer velocity, and $a_H(n)$ hammer acceleration—to $y(i,n)$, $\eta(n)$, and $F_H(n)$, which are the three principal variables in the computations. Examples of waveforms generated by the model for note C4 are shown in Fig. 3.

A great advantage of using a finite difference method is that each physical quantity (displacement, velocity, force) is directly available for all discrete points at each time step. In this way, it becomes straightforward to plot the state of the string at successive instants, in order to obtain a view of the wave propagation along the string. This feature is illustrated in Fig. 4, which shows the velocity profile of a C4 string during the first 4 ms after the blow of the hammer.
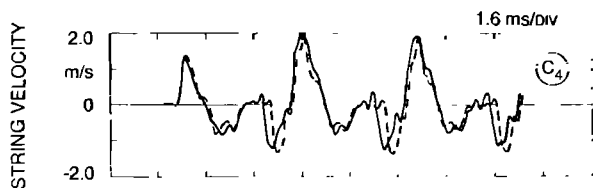
FIG. 5. Comparison of string velocities at the bridge side of the striking point (40 mm from the hammer), for a mid range note (C4) played *mezzo forte*, simulated (dashed) and measured (full line).[16]

In particular, the propagating wave front and its reflection at the bridge can be clearly seen. Similar plots of the wave propagation on a piano string have been presented by Suzuki, however, using a string model with lumped elements.[8]

A detailed test of the model by comparisons between simulated and measured waveforms will be the topic of a separate study. An example of the strength of the model is given in Fig. 5, which compares string waveforms for the note C4. It can be seen that our model reproduces the characteristics of the measured waveform convincingly, using measured values of string and hammer parameters. The small discrepancies which can be observed in the actual timing relations between the pulses are mostly due to slight differences in observation points.

IV. CONCLUSION

The numerical model presented in this paper has interesting features, which allow a simulation of a piano string very closely to the basic physical relations. The method is time efficient, and the numerical advantages and limitations have been thoroughly investigated and are well documented. The first examples and comparisons with measurements indicate that the model generates waveforms and spectra which closely resemble the signals observed in real pianos. Although all details in the design of the piano still are not modeled as realistically as desired (in particular the boundary conditions), we consider the model to be a promising tool for exploring the space of string–hammer parameters and their influence on piano tone.

ACKNOWLEDGMENTS

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