Musical Acoustics
Homework 4

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A.Y. 2021/2022

1 Introduction

We aim at characterizing a loudspeaker with a vented cabinet.
The loudspeaker, not including the cabinet, can be thought of, in mechanical terms, as a mass-spring-resistance system attached to a force generator, while the vent is represented by a tube, with a circular cross section, that links the external air with the inner volume of the cabinet.
To characterize the system, we intend to use an equivalent electrical circuit.

2 Mechanical characterization

We are given the resonance frequency of the loudspeaker, $f_d = 49\,Hz$, the mass of the system, $m = 45.5\,g$, and the quality factor $Q = 50$.
Since the diaphragm is considered as a damped system, the relation between the given resonance frequency and the natural frequency of the system is:

$$\omega_d = 2\pi f_d = \sqrt{\omega_0^2 - \left(\frac{R}{2m}\right)^2}, \qquad \omega_0 = 2\pi f_0 = \sqrt{\frac{k}{m}}$$

The mechanical resistance coefficient of the system can be computed starting from the quality factor:

$$R = \frac{2\pi f_d m}{Q} = 0.2802\,\frac{kg}{s}$$

We can then derive the stiffness of the mechanical compliance of the loudspeaker, without considering the coupling with the cabinet:

$$k = m\left[(2\pi f_d)^2 + \left(\frac{R}{2m}\right)^2\right] = 4.3133 \times 10^3\,\frac{kg}{s^2}$$

2.1 Mechanical schematics of the system

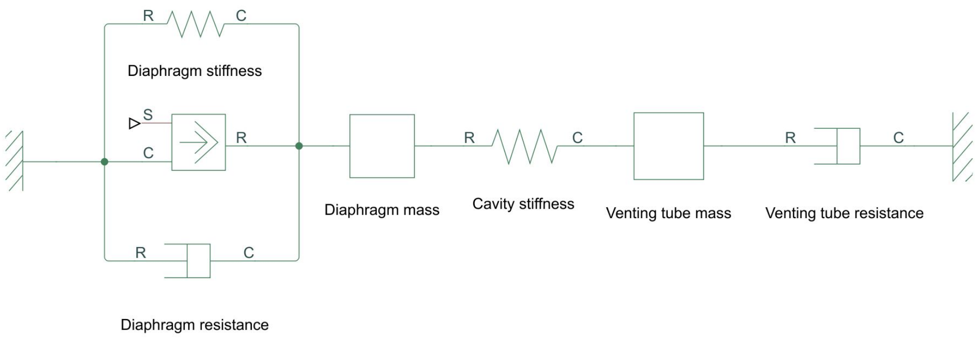
We want to represent the whole system with its mechanical schematics. To do so, we need to convert into mechanical elements (masses, springs, resistances) the air volume inside the cabinet and the air in the venting tube. The diaphragm, as previously mentioned, is considered as a mass-spring-damper system.
The venting tube, that we can consider as the short pipe of a Helmholtz resonator, can be viewed as a mass, but we need to remember to take into account the virtual elongation of the neck, which, at one end, is $\Delta L = 0.85r$, where r is the radius of the tube:

$$L_{tot} = L + 2 \times 0.85\frac{D}{2}$$

where $L = 24.7\,cm$ is the length of the pipe and $D = 7\,cm$ its diameter.
The pipe will also have some damping introduced by a resistance, which we can express as:

$$R = \frac{\rho c}{S}$$

where $\rho = 1.225\,kg/m^3$ is the air density, $c = 344\,m/s$ is the wave propagation speed in air, and S is the cross section of the tube.
The air volume, instead, can be considered equivalent to a spring.
With these equivalences we can derive the following mechanical schematics for the system:



3 Equivalent electric circuit

Starting from the mechanical system obtained before, we can derive the equivalent electric circuit for our system.
For the masses present in the mechanical system (the mass of the loudspeaker and the air mass of the tube), we can consider Newton's second law:

$$f(t) = m\frac{dv}{dt}$$

Moving to the frequency domain, the constitutive relation of the mass becomes:

$$f = j\omega mv$$

The impedance of a mass is thus:

$$Z_L(\omega) = f/v = j\omega m$$

This means that, in the electric circuit, we can represent each mass as an inductor.
For the springs (the one of the loudspeaker and the air volume), we can introduce the compliance $C = 1/k$, and we can write the following constitutive relation:

$$f(t) = \frac{1}{C}\int v(t)dt$$

In the frequency domain we get

$$f = \frac{v}{j\omega C}$$

The impedance of a spring is thus:

$$Z_C(\omega) = \frac{1}{j\omega C}$$

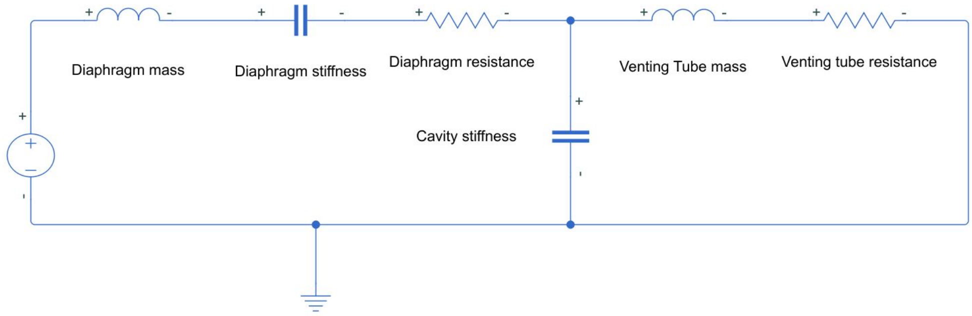
Which means we can represent springs as condensers in the equivalent electric circuit.
Finally, for the resistances, both in time and frequency we can write the following constitutive equation

$$f = Rv$$

Thus, the impedance of a resistance is:

$$Z_R(\omega) = R$$

Which means mechanical resistances can be represented as electrical resistances in the equivalent electric circuit.
We can therefore convert the mechanical system into an electrical system. We show here the electrical circuit we obtained inside Simulink:



The force generator from the mechanical circuit is here represented as a voltage generator.
We must give values to these impedances.
When dealing with the quantities relative to the air volume and the venting tube, we need to keep in mind that we are doing an analogy between an acoustical circuit and a mechanical circuit, so we are passing from an impedance computed as $Z = p/U$ to an impedance computed as $Z = f/v$. Since $f = pS$ and $v = U/S$, we have to multiply the impedances computed in the acoustic circuit by S^2.
Moving from the mechanical circuit to the electrical circuit, we have to substitute, therefore, the following quantities:

- Diaphragm:
  - impedance of the inductance L_1: $Z_L = j\omega m$;
  - impedance of the capacitor C_1: $Z_C = \frac{k}{j\omega}$;
  - impedance of the resistance R_1: $Z_R = R$.
- Cavity:
  - Impedance of the capacitor C_2: $Z_C = \frac{\rho c^2 S^2}{j\omega V}$, where S is the cross-sectional area of the venting tube, since it is the area of application of the pressure generated by the mass inside the tube.
- Venting tube:
  - Impedance of the inductance L_2: $Z_L = j\omega\rho L_{tot} S$
  - Impedance of the resistance R_2: $Z_R = \rho c S$

3.1 Transfer function

From the derived equivalent circuit, we want to study the transfer function of the system to observe what the resonance frequency of the loudspeaker is when coupled with the cabinet.
In particular, we chose to derive the admittance of the circuit, which is defined as:

$$Y(\omega) = \frac{1}{Z(\omega)}$$

We computed the admittance both analytically and with the results from the Simulink simulations. For the analytical computation, we first derived the impedance of the three elements (speaker, cavity, tube), then computed the overall impedance, and finally obtained the admittance.

$$Z_{speaker}(\omega) = j\omega L_{speaker} + R_{speaker} + \frac{1}{j\omega C_{speaker}}$$

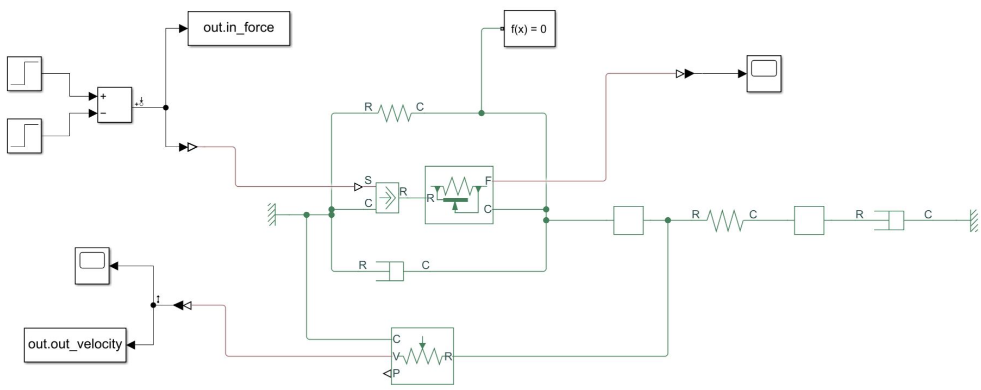
$$Z_{cavity}(\omega) = \frac{1}{j\omega C_{cavity}}$$

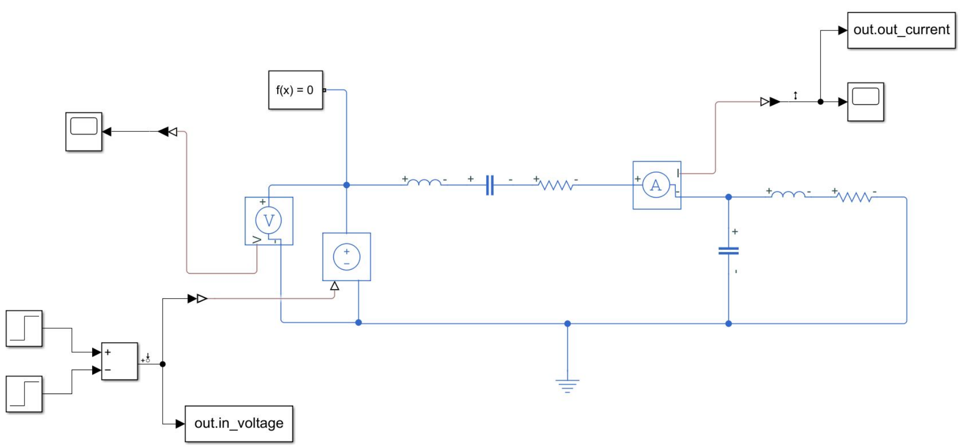
$$Z_{tube}(\omega) = j\omega L_{tube} + R_{tube}$$

$$Z_{system}(\omega) = Z_{speaker} + \frac{1}{\frac{1}{Z_{cavity}} + \frac{1}{Z_{tube}}}$$

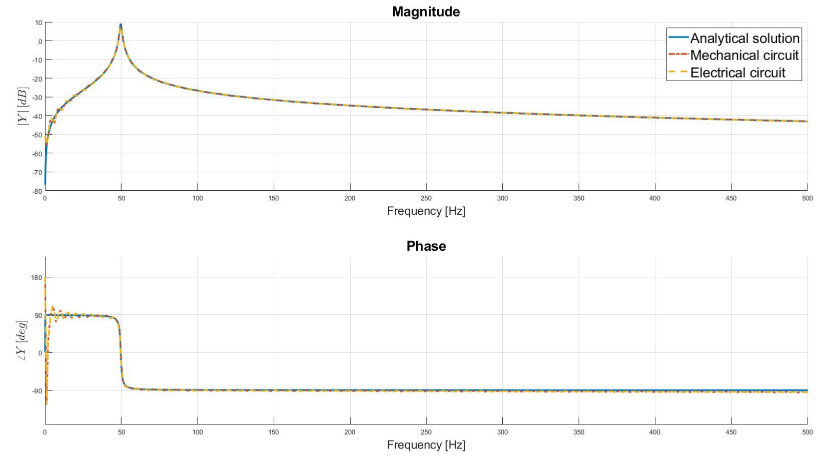
$$Y_{system}(\omega) = \frac{1}{Z_{system}(\omega)}$$

In order to derive the admittance from the Simulink simulation, we had to gather time response data from the simulation, send them to the Matlab workspace and compute the Fast Fourier Transform in order to pass from the time domain to the frequency domain.
To compute the time response we added an impulsive input signal as input (in order to excite every frequency) and sensors for both the input signal and the output signal. The input signals are the force applied on the diaphragm's mass inside the mechanical circuit, and the voltage applied inside the electrical one. The output signals are the velocity of motion of the diaphragm's mass in the mechanical circuit and the current passing through the diaphragm's elements inside the electrical one.
We decided to analyze the frequency response up to 20000 Hz, so we set a fixed time step for the evaluation of $T = 1/20000\,s$, then we implemented the impulsive input signal as the difference between two step functions both going from 0 to 1, but at different time instants, so that the total length of the part of the signal different from zero was $\Delta t = 1/20000\,s$. In this way we obtained a representation of the ideal Dirac's impulse.
The data computed by the simulation of the input signal and output signal were sent to Matlab workspace. Inside Matlab we processed the signals applying the fft on both of them. Then we computed the ratio between the fft of the output and the fft of the input in order to obtain the admittance of the system.
The complete circuits we used for the simulations are shown below:





The obtained results from all the described method are plotted in the following figure:



4 System characterization from the equivalent circuit

4.1 Resonance frequency

Once we computed the admittance, in order to find the resonance frequency of the whole system it is sufficient to find the peak in the magnitude of the transfer function. From the admittance computed analytically, we find a resonance frequency

$$f_{res} = 49.6Hz$$

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4.2 Considerations on the lumped element approximation

The lumped element approximation, which we are using when deriving the electric components equivalent to mechanical and acoustical ones, is valid as long as the dimensions of the element we are analyzing, in our case the size of the cabinet, do not exceed 1/10 of the wavelength.
In our case, at resonance:

$$\frac{\lambda}{10} = \frac{1}{10}\frac{c}{f_{res}} \simeq 0.69m$$

This means that the approximation holds as long as none of the dimensions of the cabinet exceeds 0.69m, which we can assume is true given the volume of our cabinet. Being an approximation, this method will still cause some errors compared to the transfer function of the system without approximations, especially looking at higher frequencies where the lumped element approximation does not hold anymore. This is why we need a strategy to assess the error introduced.
One way to do so could be to simulate the whole system using a finite element method, with a sufficiently fine mesh, and compute the transfer function of the obtained model. We could then compute the error between the FEM approximation and the lumped element method one, for example using an l^2 norm.
Since the approximation introduced with the finite element method has been thoroughly studied and can be found in literature, we can combine this with the result of the l^2 norm error to find an estimation of the error introduced with the lumped element method.