DEPARTMENT OF MATHEMATICS AND STATISTICS

The Malawi Polytechnic

Calculus II (Code: MTS-CAL-212)

BMEN2, BGEN2 & BMMP2.

Time: 1 hour 30 minutes June 18, 2019 *Test 1*

Question:	1	2	3	Total
Points:	26	16	8	50

Instruction: Attempt all questions.

Question 1

(a) Compute the trapezoidal approximation for the integral $\int_1^4 \sqrt{\ln x}\, dx$ using a regular partition of $n = 6$. [4]

Solution:

$$T_6 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5 + f(x_6))\right]$$

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{6} = \frac{3}{6} = 0.5$$

$$T_6 = \frac{\Delta x}{2}\left[f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)\right]$$

$$T_6 = \frac{0.5}{2}\left[\sqrt{\ln 1} + 2\sqrt{\ln 1.5} + 2\sqrt{\ln 2} + 2\sqrt{\ln 2.5} + 2\sqrt{3} + 2\sqrt{\ln 3.5} + \sqrt{4}\right]$$

$$= \frac{0.5}{2}\left[10.36533565\right]$$

$$= 2.591333912$$

$$\therefore \int_1^4 \sqrt{\ln x} \mathrm{d}x \approx T_6 = 2.591333912$$

(b) i. Use the Simpson's rule to approximate $\int_0^{\pi} \sin x \mathrm{d}x$ using a regular partition with $n = 6$. [5]

Solution:

$$S_6 = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right]$$

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$y = \sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

$$\begin{aligned} S_6 &= \frac{\pi}{6(3)}\left[0 + 4\left(\frac{1}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) + 4(1) + 2\left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{1}{2}\right) + 0\right] \\ &= \frac{\pi}{18}\left(0 + 2 + \sqrt{3} + 4 + \sqrt{3} + 2 + 0\right) \\ &= \frac{\pi}{18}\left(8 + 2\sqrt{3}\right) \\ &= \frac{\pi}{9}(4 + \sqrt{3}) \\ &= 2.00086319 \end{aligned}$$

$\therefore \int_0^{\pi} (\sin x)\mathrm{d}x \approx S_6 = 2.001$

ii. How large should n be to guarantee that the Simpson's rule approximation to $\int_0^{\pi} \sin x \mathrm{d}x$ is accurate to within 0.00001? [6]

Solution:

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \\ \left|f^{(4)}(x)\right| &= |\sin x| \leq 1, \quad 0 \leq x \leq \pi \\ \therefore k &= 1 \\ |E_s| &\leq \frac{k(b-a)^5}{180n^4} \\ \frac{1(\pi - 0)^5}{180n^4} &< 0.00001 \\ n^4 &> \frac{\pi^5}{180 \times 0.00001} \\ n &> 20.3057585 \\ \therefore n &\geq 22 \end{aligned}$$

$\therefore$ 22 subintervals will guarantee the required accuracy.

(c) Determine whether the following integrals are convergent or divergent:

i. $\int_1^{\infty} \frac{1}{(2x+1)^2}\mathrm{d}x.$ [5]

Solution:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{(2x+1)^2}\mathrm{d}x &= \lim_{t\to\infty} \int_1^t \frac{1}{(2x+1)^2}\mathrm{d}x \\
&= \lim_{t\to\infty} \left[-\frac{1}{2(2x+1)} \Big|_1^t \right] \\
&= \lim_{t\to\infty} \left[-\frac{1}{2(2t+1)} + \frac{1}{2(2(1)+1)} \right] \\
&= \lim_{t\to\infty} \left[-\frac{1}{2(2t+1)} + \frac{1}{6} \right] \\
&= \frac{1}{6}
\end{aligned}$$

ii. $\displaystyle\int_0^{\pi} \tan x \ \mathrm{d}x.$ [6]

Solution: The function $f(x) = \tan x$ is discontinuous at $x = \dfrac{\pi}{2}$.

$$\int_0^{\pi} \tan x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \tan x \mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} \tan x \mathrm{d}x$$

Computing the integrals separately:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \tan x \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x}\mathrm{d}x \\
&= \lim_{t\to\frac{\pi}{2}^-} \int_0^t \frac{\sin x}{\cos x}\mathrm{d}x \\
&= \lim_{t\to\frac{\pi}{2}^-} \left(-\ln|\cos x| \Big|_0^t \right) \\
&= \lim_{t\to\frac{\pi}{2}^-} \left(-\ln|\cot t| + \ln|\cos 0| \right) \\
&= \lim_{t\to\frac{\pi}{2}^-} \left(-\ln|\cos t| \right) \\
&= \infty
\end{aligned}$$

Now that we know one of the components diverges, we know that the entire integral diverges.

Question 2

(a) Suppose that 2 Joules of work is needed to stretch a spring from its natural length of 30cm to a length of 42cm. How much work is needed to stretch the spring from 35cm to 40cm? [5]

Solution: Using Hooke's Law, $f(x) = kx$.Let W represent work done then

$$
\begin{aligned}
W &= \int_a^b f(x)\mathrm{d}x \\
W &= \int_0^{0.12} kx\mathrm{d}x \\
2 &= k\frac{x^2}{2}\Big|_0^{0.12} \\
4 &= k\left(x^2\Big|_0^{0.12}\right) \\
4 &= k\big(0.12\big)^2 \\
4 &= 0.144k \\
k &= \frac{4}{0.0144} \\
\therefore k &= \frac{2500}{9}
\end{aligned}
$$

Work done after stretching the spring from 35 cm to 40 cm is calculated as follows:

$$
\begin{aligned}
W &= \int_{0.05}^{0.1} \frac{2500}{9} x\mathrm{d}x \\
&= \frac{2500}{9}\left[\frac{x^2}{2}\Big|_{0.05}^{0.1}\right] \\
&= \frac{2500}{9}\left(\frac{0.1^2}{2} - \frac{0.05^2}{2}\right) \\
&= \frac{2500}{9}\left(\frac{3}{800}\right) \\
&= \frac{25}{24} \\
&= 1.0416667 \\
&\approx 1.04\text{J}
\end{aligned}
$$

(b) Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$. [6]

Solution: The area of the region is

$$A = \int_0^1 (x - x^2)\mathrm{d}x = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

Therefore

$$\begin{aligned}
\bar{x} &= \frac{1}{A}\int_0^1 x\left[f(x)-g(x)\right]\mathrm{d}x = \frac{1}{\frac{1}{6}}\int_0^1 x(x-x^2)\mathrm{d}x \\
&= 6\int_0^1 (x^2-x^3)\mathrm{d}x = 6\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = \frac{1}{2} \\
\bar{y} &= \frac{1}{A}\int_0^1 \frac{1}{2}\left\{[f(x)]^2-[g(x)]^2\right\}\mathrm{d}x = \frac{1}{\frac{1}{6}}\int_0^1 \frac{1}{2}\left(x^2-x^4\right)\mathrm{d}x \\
&= 3\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1 = \frac{2}{5}
\end{aligned}$$

$\therefore$ The centroid is $\left(\frac{1}{2},\frac{2}{5}\right)$.

(c) The curve $y=\sqrt{9-x^2}$, $-2\le x\le 1$ is an arc of the circle $x^2+y^2=9$. Find the surface area obtained by rotating this arc about the x-axis. [5]

Solution:

$$\begin{aligned}
y &= \sqrt{9-x^2} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-x}{\sqrt{9-x^2}} \\
S &= \int_{-2}^1 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \\
&= 2\pi\int_{-2}^1 \sqrt{9-x^2}\sqrt{1+\frac{x^2}{9-x^2}}\mathrm{d}x \\
&= 2\pi\int_{-2}^1 \sqrt{9-x^2}\frac{3}{\sqrt{9-x^2}}\mathrm{d}x \\
&= 2\pi\int_{-2}^1 3\mathrm{d}x \\
&= 6\pi\int_{-2}^1 \mathrm{d}x \\
&= 6\pi\left[x\right]_{-2}^1 \\
&= 18\pi
\end{aligned}$$

Question 3

Newton's law of cooling(heating) states that the rate of change of temperature of a body is directly proportional to the difference between the temperature of the body and the temperature of the surrounding medium. If a thermometer is removed from a room where the temperature is $70^oF$and is taken outside, where the air temperature is 10^oF. After one-half minute the thermometer reads 50^oF. What is the reading of the thermometer at $t=1$min? [8]

Solution: Let T be the temperature of the body and T_s be the temperature of the surrounding.

$$\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}t} &\propto T - T_s \quad \text{but} \quad T_s = 10 \\
\frac{\mathrm{d}T}{\mathrm{d}t} &= k(T-10) \\
\frac{\mathrm{d}T}{T-10} &= k\mathrm{d}t \\
\int \frac{\mathrm{d}T}{T-10} &= k\int \mathrm{d}t \\
\ln|T-10| &= kt + c \\
T - 10 &= e^{kt+c} = e^{kt}e^{c} \\
T - 10 &= c_1 e^{kt}, \qquad c_1 = e^c \\
T &= 10 + c_1 e^k t
\end{aligned}$$

$T = 70$ when $t = 0$,

$$\begin{aligned}
70 &= 10 + c_1 e^{k(0)} \\
c_1 &= 60 \\
\therefore T &= 10 + 60e^{kt}
\end{aligned}$$

$T = 50$ when $t = \frac{1}{2}$,

$$\begin{aligned}
50 &= 10 + 60e^{\frac{1}{2}k} \\
k &= 2\ln\left(\frac{2}{3}\right) \\
k &= -0.810930216 \\
\therefore T &= 10 + 60e^{(0.810930216)t}
\end{aligned}$$

After 1 minute, $t = 1$,

$$\begin{aligned}
T &= 1060e^{(0.810930216)(1)} \\
T &= 36.666666667
\end{aligned}$$

$\therefore T \approx 37^o$ (to the nearest ^oF)