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Part A: Two Factor Hull-White Model

Introduction

According to Hull J (2015), the Hull–White model can be defined as a model of future interest rates. Generally, this model belongs to the family of no-arbitrage models that are tailored to fit the current term structure of interest rates. It is relatively simple to change the mathematical description of the developmental future interest rates to a matrix, so that the interest rate derivatives, such as Bermudan swaptions can be valued in the model.

In the Hull-White model, all parameters depend on time, implying that the drift and variable of interest rate ‘r’ becomes a function of time t, r and many other variables. With this idea, John Hull and Alan White published an article introducing the Hull-White model as follows: dr = [θ(t) + a(t)(b − r)]dt + σ(t)r β dz where θ(t) is a time-dependent on drift, a is the reversion rate, and σ is the volatility (John Hull and Alan White,1990). All the interest rate models that have been outlined are estimating the spot interest rate, thereby used to price the bond. However, in real life situation, it is not possible for us to predict the spot interest rate, this is because the data used in the yield curve fitting is just the historical interest rates, which easily favors the estimated risk neutral interest rate in the market.

In this write up, we are going to start with one-factor Hull-White Model which help us build on a two-factor Hull-White Model.

One-factor Hull-White Model

One-factor Hull-White model is a generalized model of the Vasicek model with time dependent parameters:

$$dr = (\theta(t) - a(t)r)dt + \sigma(t)dV(t)$$

. The parameters a and σ in the equation are calibrated against the volatility, where θ (t) is deterministic function which is also calibrated against the bond prices,

$\{p(0,T): T \geq 0\}$ to observed curve $\{p^*(0,T): T \geq 0$. Let us look at

$$\begin{cases} \frac{\partial F^T}{\partial t} + \{\mu(t,r) - \lambda(t,r)\sigma(t)\}\frac{\partial F^T}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 F^T}{\partial r^2} = r(t)F^T \\ F(r,T,T) = 1 \end{cases}$$

From the above function, we can see that the drift is given is μ(t,r) − λ(t,r)σ (t) . If these drift terms can be compared, the parameters in the Hull-White model can be clearly seen, which include the market price of risk and the volatility shown in the equation below:

$$\mu(t,r) - \lambda(t,r)\sigma(t) = \theta(t) - a(t)r(t)$$

This is why we say parameters a and σ are calibrated against the volatility. The model has an affine term structure.

$$p(t,T) = F(r(t),t,T) = e^{A(t,T)-B(t,T)r}$$

This model can be simplified further if the function is differentiated with respect to t.

$$\begin{cases} \frac{\partial B}{\partial t} - aB = -1 \\ B(T,T) = 1 \end{cases}$$

To obtain

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}$$

$$\begin{cases} \frac{\partial A}{\partial t} - \theta(t)B(t,T) + \frac{1}{2}\sigma^2 B^2(t,T) = 0 \\ A(T,T) = 0 \end{cases}$$

$$A(t,T) = \int_i^T \left\{\frac{\sigma^2}{2}B^2(s,T) - \theta(t)B(s,T)\right\}ds$$

The Forward Rate

$$f^*(0,T) = B_T(0,T)r(0) - A_T(0,T) = r(0)e^{-aT} + \int_t^T \theta(s)e^{-a(T-s)}ds - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2$$

$$f^*(0,T) = x(T) + g(T)$$

Where $\begin{cases} \dot{x}(t) = -ax(t) + \theta(t) \\ x(0) = r(0) \end{cases}$

And yield

$$x(t) = r(0)e^{-aT} + \int_t^T \theta(s)e^{-a(T-s)}ds$$

and

$$g(T) = \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2 = \frac{\sigma^2}{2a^2}B^2(0,T)$$

$$\theta = \dot{x}(T) + ax(T) = f_T^*(0,T) - \dot{g}(T) + ax(T)$$
$$= f_T^*(0,T) - \dot{g}(T) + a\left\{f^*(0,T) - g(T)\right\}$$

Martingale Measure

Using the function θ (T), the fixed values of a and σ

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)}\exp\left\{B(t,T)f^*(0,t) - \frac{\sigma^2}{4a^2}B^2(t,T)\left(1 - e^{-2at}\right) - B(t,T)r(t)\right\}$$

The spot rate volatility

$$\sigma_{\text{spot}} = \frac{\sigma}{a(T-t)}\left(1 - e^{-a(T-t)}\right)$$

The Two-Factor Hull-White Model

The risk-neutral process for the short rate, r,

This is clearly explained in Hull-White One factor model and it is given by:

$$df(r) = [\theta(t) + u - af(r)]dt + \sigma_1 dz_1$$

where u has an initial value of zero thereby follows the process

$$du = -\,bu\,dt + \sigma_2 dz_2$$

As this parameter θ(t) is already explained as a deterministic function of time, u is a stochastic variable component of the reversion level of r and itself degenerates to a level of zero at rate b. The parameters a, b, σ_1, and σ_2 are constants while dz_1 and dz_2 are Wiener processes with instantaneous correlation ρ. The model $df(r)$ provides a richer pattern of term structure movements as well as a richer pattern of volatility structures than the one-factor model. For example, when f (r) = r with all parameters known, the model will exhibit, at all times, a "humped" volatility structure similar to the common structures observed in most markets. When the model is analytically controllable. The price at time t of a zero-coupon bond that provides a payoff of say $1 at time T is

$$P(t,T) = A(t,T)\exp[-B(t,T)r - C(t,T)u]$$

$$where\, B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

$$C(t,T) = \frac{1}{a(a-b)}e^{-a(T-t)} - \frac{1}{b(a-b)}e^{-b(T-t)} + \frac{1}{ab}$$

Call and Put Price

At A(t, T), the prices of a call,'c' and put,'p', at time zero of European call and put options on a zero-coupon bond are given by

$$c = LP(0,s)N(h) - KP(0,T)N\left(h - \sigma_p\right)$$

$$p = KP(0,T)N\left(-h - \sigma_p\right) - LP(0,s)N(-h)$$

where T in A(t, T), is the maturity of the option, s is the maturity of the bond, K is the strike price, L is the bond's principal

$$h = \frac{1}{\sigma_p} Ln \frac{LP(0,s)}{P(0,T)K} + \frac{\sigma_p}{2}$$

The A(t, T) ,σ p , and θ (t) Functions

$$\text{In}A(t,T) = \text{In}\frac{P(0,T)}{P(0,t)} + B(t,T)F(0,t) - \eta$$

where

$$\eta = \frac{\sigma_1^2}{4a}\left(1 - e^{-2at}\right)B(t,T)^2 - \rho\sigma_1\sigma_2\left[B(0,t)C(0,t)B(t,T) + \gamma_4 - \gamma_2\right]$$
$$-\frac{1}{2}\sigma_2^2\left[C(0,t)^2B(t,T) + \gamma_6 - \gamma_5\right]$$

$$\gamma_1 = \frac{e^{-(a+b)T}\left(e^{(a+b)t} - 1\right)}{(a+b)(a-b)} - \frac{e^{-2aT}\left(e^{2at} - 1\right)}{2a(a-b)}$$

$$\gamma_2 = \frac{1}{ab}\left(\gamma_1 + C(t,T) - C(0,T) + \frac{1}{2}B(t,T)^2 - \frac{1}{2}B(0,T)^2 + \frac{t}{a} - \frac{e^{-a(T-t)} - e^{-aT}}{a^2}\right)$$

$$\gamma_3 = -\frac{e^{-(a+b)t} - 1}{(a-b)(a+b)} + \frac{e^{-2at} - 1}{2a(a-b)}$$

$$\gamma_4 = \frac{1}{ab}\left(\gamma_3 - C(0,t) - \frac{1}{2}B(0,t)^2 + \frac{t}{a} + \frac{e^{-at} - 1}{a^2}\right)$$

$$\gamma_5 = \frac{1}{b}\left[\frac{1}{2}C(t,T)^2 - \frac{1}{2}C(0,T)^2 + \gamma_2\right]$$

$$\gamma_6 = \frac{1}{b}\left[\gamma_4 - \frac{1}{2}C(0,t)^2\right]$$

The volatility function p

$$\sigma_p^2 = \int_0^t \left\{ \begin{array}{l} \{\sigma_1^2[B(\tau,T) - B(\tau,t)]^2 + \sigma_2^2[C(\tau,T) - C(\tau,t)]^2 \\ +2\rho\sigma_1\sigma_2[B(\tau,T) - B(\tau,t)][C(\tau,T) - C(\tau,t)]\} d\tau \end{array} \right.$$

σ_p^2 components

σ_p^2 component has three components and is defined as

$$U = \frac{1}{a(a-b)}\left(e^{-aT} - e^{-at}\right)$$
$$V = \frac{1}{b(a-b)}\left(e^{-bT} - e^{-bt}\right)$$

The first component of is σ_p^2 is given by

$$\frac{\sigma_1^2}{2a}B(t,T)^2\left(1-e^{-2at}\right)$$

The second component of is σ_p^2 is given by

$$\sigma_2^2\left(\frac{U^2}{2a}\left(e^{2at}-1\right)+\frac{V^2}{2b}\left(e^{2bt}-1\right)-2\frac{UV}{a+b}\left(e^{(a+b)t}-1\right)\right)$$

The third

$$\frac{2\rho\sigma_1\sigma_2}{a}\left(e^{-at}-e^{-aT}\right)\left(\frac{U}{2a}\left(e^{2at}-1\right)-\frac{V}{a+b}\left(e^{(a+b)t}-1\right)\right)$$

$$\theta(t)=F_t(0,t)+aF(0,t)+\phi_t(0,t)+a\phi(0,t)$$

$$\phi(t,T)=\frac{1}{2}\sigma_1^2B(t,T)^2+\frac{1}{2}\sigma_2^2C(t,T)^2+\rho\sigma_1\sigma_2B(t,T)C(t,T)$$

The Analogy with the Hull-White Two-Factor Model

The short rate

$$dr(t)=[\theta(t)+u(t)-\bar{a}r(t)]dt+\sigma_1dZ_1(t),r(0)=r_0$$

where the stochastic mean-reversion level satisfies

$$du(t)=-\bar{b}u(t)dt+\sigma_2dZ_2(t),u(0)=0$$

with ($Z1$, $Z2$) a two-dimensional Brownian motion with $dZ1(t)dZ2(t)$, σ_1 and σ_2 positive constants. The function θ is deterministic and properly chosen so as to exactly fit the current term structure of interest rates where simple integration leads to

$$r(t)=r(s)e^{-\bar{a}(t-s)}+\int_s^t\theta(v)e^{-\bar{a}(t-v)}dv+\int_s^t u(v)e^{-\bar{a}(t-v)}dv$$
$$+\sigma_1\int_s^t e^{-\bar{a}(t-v)}dZ_1(v)$$
$$u(t)=u(s)e^{-\bar{b}(t-s)}+\sigma_2\int_s^t e^{-\bar{b}(t-v)}dZ_2(v)$$

$$
\begin{aligned}
&\int_s^t u(v)e^{-\bar{a}(t-v)}dv \\
&= \int_s^t u(s)e^{-\bar{b}(v-s)-\bar{a}(t-v)}dv + \sigma_2\int_s^t e^{-\bar{a}(t-v)}\int_s^v e^{-\bar{b}(v-x)}dZ_2(x)dv \\
&= u(s)\frac{e^{-\bar{b}(t-s)} - e^{-\bar{a}(t-s)}}{\bar{a}-\bar{b}} + \sigma_2 e^{-\bar{a}t}\int_s^t e^{(\bar{a}-\bar{b})v}\int_s^v e^{\bar{b}x}dZ_2(x)dv
\end{aligned}
$$

Using by parts

$$
\begin{aligned}
&\int_s^t e^{(\bar{a}-\bar{b})v}\int_s^v e^{\bar{b}x}dZ_2(x)dv \\
&= \frac{1}{\bar{a}-\bar{b}}\int_s^t \left(\int_s^v e^{\bar{b}x}dZ_2(x)\right)d_v\left(e^{(\bar{a}-\bar{b})v}\right) \\
&= \frac{1}{\bar{a}-\bar{b}}\left[e^{(\bar{a}-\bar{b})t}\int_s^t e^{\bar{b}x}dZ_2(x) - \int_s^t e^{(\bar{a}-\bar{b})v}d_v\left(\int_s^v e^{\bar{b}x}dZ_2(x)\right)\right] \\
&= \frac{1}{\bar{a}-\bar{b}}\int_s^t \left[e^{(\bar{a}-\bar{b})t} - e^{(\bar{a}-\bar{b})v}\right]d_v\left(\int_s^v e^{\bar{b}x}dZ_2(x)\right) \\
&= \frac{1}{\bar{a}-\bar{b}}\int_s^t \left[e^{\bar{a}t-\bar{b}(t-v)} - e^{\bar{a}v}\right]dZ_2(v)
\end{aligned}
$$

$$
\begin{aligned}
r(t) = r(s)e^{-\bar{a}(t-s)} &+ \int_s^t \theta(v)e^{-\bar{a}(t-v)}dv + \sigma_1\int_s^t e^{-\bar{a}(t-v)}dZ_1(v) \\
&+ u(s)\frac{e^{-\bar{b}(t-s)} - e^{-\bar{a}(t-s)}}{\bar{a}-\bar{b}} + \frac{\sigma_2}{\bar{a}-\bar{b}}\int_s^t \left[e^{-\bar{b}(t-v)} - e^{-\bar{a}(t-v)}\right]dZ_2(v),
\end{aligned}
$$

and in particular,

$$
\begin{aligned}
r(t) = r_0 e^{-\bar{a}t} &+ \int_0^t \theta(v)e^{-\bar{a}(t-v)}dv + \sigma_1\int_0^t e^{-\bar{a}(t-v)}dZ_1(v) \\
&+ \frac{\sigma_2}{\bar{a}-\bar{b}}\int_0^t \left[e^{-\bar{b}(t-v)} - e^{-\bar{a}(t-v)}\right]dZ_2(v)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_3 &= \sqrt{\sigma_1^2 + \frac{\sigma_2^2}{(\bar{a}-\bar{b})^2} + 2\bar{\rho}\frac{\sigma_1\sigma_2}{\bar{b}-\bar{a}}} \\
dZ_3(t) &= \frac{\sigma_1 dZ_1(t) - \frac{\sigma_2}{\bar{a}-\bar{b}}dZ_2(t)}{\sigma_3} \\
\sigma_4 &= \frac{\sigma_2}{\bar{a}-\bar{b}},
\end{aligned}
$$

$$r(t) = r_0 e^{-\bar{a}t} + \int_0^t \theta(v) e^{-\bar{a}(t-v)} dv + \int_0^t e^{-\bar{a}(t-v)} \left[\sigma_1 dZ_1(v) + \frac{\sigma_2}{\bar{b} - \bar{a}} dZ_2(v) \right]$$
$$+ \frac{\sigma_2}{\bar{a} - \bar{b}} \int_0^t e^{-\bar{b}(t-v)} dZ_2(v)$$
$$= r_0 e^{-\bar{a}t} + \int_0^t \theta(v) e^{-\bar{a}(t-v)} dv$$
$$+ \sigma_3 \int_0^t e^{-\bar{a}(t-v)} dZ_3(v) + \sigma_4 \int_0^t e^{-\bar{b}(t-v)} dZ_2(v)$$

the G2++ model

$$a = \bar{a}$$
$$b = \bar{b}$$
$$\sigma = \sigma_3$$
$$\eta = \sigma_4$$
$$\rho = \frac{\sigma_1 \bar{\rho} - \sigma_4}{\sigma_3}$$
$$\varphi(t) = r_0 e^{-\bar{a}t} + \int_0^t \theta(v) e^{-\bar{a}(t-v)} dv$$

$$\bar{a} = a$$
$$\bar{b} = b$$
$$\sigma_1 = \sqrt{\sigma^2 + \eta^2 + 2\rho\sigma\eta}$$
$$\sigma_2 = \eta(a - b)$$
$$\bar{\rho} = \frac{\sigma\rho + \eta}{\sqrt{\sigma^2 + \eta^2 + 2\rho\sigma\eta}}$$
$$\theta(t) = \frac{d\varphi(t)}{dt} + a\varphi(t)\,.$$

A different way to prove this analogy is by defining the new stochastic process

$$\chi(t) = r(t) + \delta u(t),$$

Where $\delta = 1/(\bar{b} - \bar{a})$ in fact

$$\begin{aligned} d\chi(t) &= [\theta(t) + u(t) - \bar{a}r(t)]dt + \sigma_1 dZ_1(t) - \delta\bar{b}u(t)dt + \delta\sigma_2 dZ_2(t) \\ &= [\theta(t) + u(t) - \bar{a}\chi(t) + \bar{a}\delta u(t) - \bar{b}\delta u(t)]dt + \sigma_1 dZ_1(t) + \delta\sigma_2 dZ_2(t) \\ &= [\theta(t) - \bar{a}\chi(t)]dt + \sigma_3 dZ_3(t) \end{aligned}$$

with $\sigma 3$ and dZ_3 defined.

$$\begin{aligned} \psi(t) &= \frac{u(t)}{\bar{a} - \bar{b}} = -\delta u(t), \\ d\psi(t) &= -\frac{\bar{b}}{\bar{a} - \bar{b}} u(t)dt + \frac{\sigma_2}{\bar{a} - \bar{b}} dZ_2(t) \\ &= -\bar{b}\psi(t)dt + \sigma_4 dZ_2(t), \end{aligned}$$

with $\sigma 4$ defined. Therefore, we again obtain that $r(t)$ which can be written as

$$\begin{aligned} r(t) &= \tilde{\chi}(t) + \psi(t) + \varphi(t), \\ d\tilde{\chi}(t) &= -\bar{a}\tilde{\chi}(t)dt + \sigma_3 dZ_3(t) \\ d\psi(t) &= -\bar{b}\psi(t)dt + \sigma_4 dZ_2(t), \\ \varphi(t) &= r_0 e^{-\bar{a}t} + \int_0^t \theta(v) e^{-\bar{a}(t-v)} dv\,. \end{aligned}$$

Part B: Comparing Various Short Rate Models

Introduction

We now compare the Hull-White Model discussed above with two classical time-homogeneous short-rate models: the Vasicek (1977) model and the Cox, Ingersoll and Ross (1985) model.

These models are time-homogeneous, meaning that the short-rate dynamics depend only on constant coefficients. Further, the term structure in these models is endogenous. Typically, every model has a dynamics equation describing the instantaneous change in spot rates, and the structure of this equation has important implications for the model's theoretical and practical applications. Examples of such implications include

- negativity of interest rates,
- the distribution of the short rate,
- possibility of analytical computation of bond and bond option prices,
- mean reversion and volatility explosion,
- volatility structures and modelling of explicit short-rate dynamics under forward measures,
- suitability of Monte Carlo simulation and construction of recombining lattices,
- possibility to use historical estimation techniques for parameter estimation

That said, we start by introducing each of the aforementioned models and the implications of their dynamics. This is followed by a comparison of these models with the Hull-White model discussed earlier

The Vasicek (1977) Model

Dynamics Equation and Short Rate Distribution

Under this model, the short-rate follows Ornstein-Uhlenbeck process under the risk-neutral measure, as follows:

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t),\ r(0) = r_0$$

where r_0, k, θ and σ are positive constants. Integrating equation [XXX], we obtain for each $s \leq t$,

$$r(t) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma\int_s^t e^{-k(t-u)}dW(u)$$

so that $r(t)$ conditional on $\mathscr{F}_s$ is normally distributed with the following mean and variance

$$E\left\{r(t) \mid \mathscr{F}_s\right\} = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

$$\mathrm{Var}\left\{r(t) \mid \mathscr{F}_s\right\} = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right]$$

The dynamics equation shows that the short-rate is mean reverting to the level of θ in this model: it has a positive drift when it is below θ and negative drift when it is above θ.

A drawback of the Vasicek model is the possibility of negative interest rates with positive probability. However, its key advantage is the analytical tractability that is made possible by the short-rate's normal distribution, something which is difficult to achieve with other distributions.

Bond, Interest Rate Derivative Pricing and Dynamics Under a Forward Measure

The price of a pure-discount bond can be derived from the standard pricing equation which takes the expectation of the stochastic discount factor conditional on $\mathscr{F}_t$ under some pricing measure:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$A(t,T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4k}B(t,T)^2\right\}$$

$$B(t,T) = \frac{1}{k}\left[1 - e^{-k(T-t)}\right]$$

If we fix a maturity T,the dynamics equation under the T-forward measure Q^T is given by:

$$dr(t) = \left[k\theta - B(t,T)\sigma^2 - kr(t)\right]dt + \sigma dW^T(t)$$

where the Q^T-brownian motion W^T is defined by

$$dW^T(t) = dW(t) + \sigma B(t,T)dt$$

so that, for $s \leq t \leq T$,

$$r(t) = r(s)e^{-k(t-s)} + M^T(s,t) + \sigma\int_s^t e^{-k(t-u)}dW^T(u)$$

with

$$M^T(s,t) = \left(\theta - \frac{\sigma^2}{k^2}\right)\left(1 - e^{-k(t-s)}\right) + \frac{\sigma^2}{2k^2}\left[e^{-k(T-t)} - e^{-k(T+t-2s)}\right].$$

Under Q^T, the short rate distribution is still normal, with the mean and variance given by

$$E^T\left\{r(t) \mid \mathscr{F}_s\right\} = r(s)e^{-k(t-s)} + M^T(s,t)$$

$$\mathrm{Var}^T\left\{r(t) \mid \mathscr{F}_s\right\} = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right]$$

Under the Vasicek model, Jamshidian (1989) has used the distribution under Q^T to derive the price of a European option at time t with strike X and maturity T written on a zero-coupon bond maturing at time S. For the claim $H_T = (P(T,S) - X)^+$, the pricing equation is given by

$$\mathbf{ZBO}(t,T,S,X) = \omega\left[P(t,S)\Phi(\omega h) - XP(t,T)\Phi\left(\omega\left(h - \sigma_p\right)\right)\right]$$

where $\omega = 1$ for a call and $\omega = -1$ for a put, $\Phi(\cdot)$ denotes the standard normal cumulative distribution function, and

$$\sigma_p = \sigma\sqrt{\frac{1 - e^{-2k(T-t)}}{2k}}B(T,S),$$

$$h = \frac{1}{\sigma_p}\ln\frac{P(t,S)}{P(t,T)X} + \frac{\sigma_p}{2}$$

Objective Measure Dynamics and Historical Estimation

Under the objective measure Q_0, the dynamics equation takes the following form:

$$dr(t) = [k\theta - (k + \lambda\sigma)r(t)]dt + \sigma dW^0(t),\ r(0) = r_0$$

where λ captures the market price of risk. For $\lambda = 0$, the dynamics equation under the objective and risk-neutral measures become identical. With the above dynamics equation, we are tacitly assuming that the market price of risk process has the following functional form

$$\lambda(t) = \lambda\, r(t)$$

The above assumption is convenient since the short rate becomes tractable under both the risk-neutral and the objective measures. Tractability is desirable under the former since claims are priced under it and under the latter because it makes possible calibrating the dynamics equation to historical data and getting estimates of parameters $k,\ \sigma,\ \theta$ and λ.

The Cox, Ingersoll and Ross (1985) Model

Dynamics Equation and Short Rate Distribution

The advantage of this model over the Vasicek model is that the instantaneous short rate is always positive under its dynamics equation and the model is analytically tractable. This is achieved by introducing a square root term in the diffusion process, as shown below for the risk-neutral measure Q:

$$dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t),\ r(0) = r_0$$

with r_0, k, θ, σ positive constants. The condition $2k\theta > \sigma^2$ needs to be imposed to ensure positivity in the short rate.

The process follows a non-central chi-squared distribution with the following mean and variance.

$$E\{r(t) \mid \mathscr{F}_s\} = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

$$\text{Var}\{r(t) \mid \mathscr{F}_s\} = r(s)\frac{\sigma^2}{k}\left(e^{-k(t-s)} - e^{-2k(t-s)}\right) + \theta\frac{\sigma^2}{2k}\left(1 - e^{-k(t-s)}\right)^2$$

Bond, Interest Rate Derivative Pricing and Dynamics Under a Forward Measure

The price of a zero-coupon bond under this model is given by:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$A(t,T) = \left[\frac{2h\exp\{(k+h)(T-t)/2\}}{2h + (k+h)(\exp\{(T-t)h\} - 1)}\right]^{2k\theta/\sigma^2}$$

$$B(t,T) = \frac{2\left(\exp\{(T-t)h\} - 1\right)}{2h + \left(k+h\right)\left(\exp\{(T-t)h\} - 1\right)},$$

$$h = \sqrt{k^2 + 2\sigma^2}$$

The price of a European call option with maturity $T > t$, strike price X, written on a zero-coupon bond maturing at $S > T$, and with the instantaneous rate at time t is given by

$$\mathbf{ZBC}(t,T,S,X) = P(t,S)\chi^2\left(2\bar{r}\left[\rho + \psi + B(T,S)\right]; \frac{4k\theta}{\sigma^2}, \frac{2\rho^2 r(t)\exp\{h(T-t)\}}{\rho + \psi + B(T,S)}\right) - XP(t,T)\chi^2\left(2\bar{r}[\rho + \psi]; \frac{4k\theta}{\sigma^2}, \frac{2\rho^2 r(t)\exp\{h(T-t)\}}{\rho + \psi}\right)$$

where

$$\rho = \rho(T-t) := \frac{2h}{\sigma^2(\exp[h(T-t)] - 1)}$$

$$\psi = \frac{k+h}{\sigma^2}$$

$$\bar{r} = \bar{r}(S-T) := \frac{ln(A(T,S)/X)}{B(T,S)}$$

If we fix a maturity T, the dynamics equation under the T-forward measure Q^T is given by:

$$dr(t) = \left[k\theta - \left(k + B(t,T)\sigma^2\right)r(t)\right]dt + \sigma\sqrt{r(t)}dW^T(t)$$

where the Q^T-Brownian motion W^T is defined by

$$dW^T(t) = dW(t) + \sigma B(t,T)\sqrt{r(t)}dt$$

Objective Measure Dynamics

Under the objective measure Q_0, the dynamics equation takes the following form:

$$dr(t) = [k\theta - (k + \lambda\sigma)r(t)]dt + \sigma\sqrt{r(t)}dW^0(t),\ r(0) = r_0$$

The drift has been modified in exactly the same way as for the Vasicek case, with the aim to preserve the same structure under the objective and risk-neutral measures. However, here the change of measure has been designed to maintain the square-root process instead of the linear process as in the Vasicek model. So essentially, we are assuming that the market price of risk process has the following functional form

$$\lambda(t) = \lambda\sqrt{r(t)}$$

As in the Vasicek case, the above assumption is convenient since it ensure analytical tractability under the objective measure which can be helpful for historical estimation.

Comparing the Hull-White Model to the Vasicek and CIR Models

The Vasicek and CIR models are equilibrium models, i.e. they start with assumptions about economic variables and derive a process for the short-rate. Then, the implications of this process for bond prices and interest rate derivative prices are considered.

They are also endogeneous in nature. This means that if we have an initial zero-coupon bond curve which we wish to incorporate in our model, the model's parameters need to optimized such that the modelled curve is as close as possible to the market curve.

The problem with the above is that the three parameters in these models are not enough to satisfactorily reproduce a term structure observed in the market. Besides, some shapes of the zero coupon curve, such as the inverted curve, cannot be obtained by these models. Because of this inability to reproduce market-observed yield curves, speaking of volatility structures and other real-world aspects becomes meaningless.

This problem is rectified by exogeneous term structure models, and the model by Hull and White is one of them. Such models are built by transforming endogenous models via the inclusion of time-varying parameters. For instance, one can transform the dynamics equation in the Vasicek case as shown below:

$$dr(t) = k\left[\theta - r(t)\right]dt + \sigma dW(t) \longrightarrow dr(t) = k[\vartheta(t) - r(t)]dt + \sigma dW(t)$$

The advantage of doing the above is that the function $\vartheta(t)$ can be defined in terms of the initial yield curve observed in the market, so that the model reproduces exactly that very curve at time 0. This transformation yields the one-factor Hull-White model. Such models are also referred to as no-arbitrage models, since they are designed to be consistent with the term structure observed in the market.

The table below provides a comparative overview of the relevant properties of the instantaneous short rate for the models discussed above.

Model	**Dynamics**	$r > 0$	$r \sim$	**Analytical Bond Price**	**Analytical Option Price**
Vasciek	$dr_t = k[\theta - r_t]dt + \sigma dW_t$	N	$\mathcal{N}$	Y	Y
CIR	$dr_t = k[\theta - r_t]dt + \sigma\sqrt{r_t}dW_t$	Y	$\mathrm{NC}\chi^2$	Y	Y
Hull-White	$dr_t = k[\theta_t - r_t]dt + \sigma dW_t$	N	$\mathcal{N}$	Y	Y

Comparing the Hull-White Model to the Ho-Lee Model

Like the Hull-White model, the Ho-Lee model is also a no-arbitrage/ exogeneous term structure model. The dynamics equation under this model is given by:

$$dr_t = \theta(t)dt + \sigma dW_t$$

where W is a Brownian motion, $\theta(t)$ is deterministic and σ is a constant.

The above SDE can be solved to get the following:

$$r_t = r_0 + \int_0^t \theta(s)ds + \sigma W_t\,.$$

r_t is normally distributed with the following mean and variance

$$\mathbb{E}(r_t) = r_0 + \int_0^t \theta(s)ds$$

and

$$\mathrm{Var}(r_t) = \sigma^2 t$$

Since r_t is normally distributed, there is a positive probability of r_t being negative, no matter what the parameters are. This is not a desirable property of the model, assuming interest rates are typically positive.

The difference between the above dynamics and the HW dynamics is that the latter introduces mean reversion into the model. So, the Hull-White model can be thought of as the Ho-Lee Model with mean reversion. Equivalently, the Ho-Lee model can be thought of as the Hull-White model without mean reversion.

Concluding Critical Assessment

The equilibrium (endogenous) models discussed above are only useful when the time horizon under consideration is relatively long. So, for instance, if a pension fund or an insurance company wants to know the value of its portfolio in, say, 20 years, the failure of equilibrium models to accurately capture the exact shape of the present term structure of interest rates will not matter much.

However, in applications where derivative pricing is central, no-arbitrage (exogenous) models would be the preferred choice because in these situations, replicating the current term structure of interest rates is essential. This is reflected in the fact that a 1% error in the price of the underlying bond may lead to a 25% error in an option price.

Part C: Comparing Various Short Rate Models

Target Audience

Target Audience of the interest rate cap is in general public. This affects certain market cohorts like credit cards, payday loans, etc or cover credit ops in the industry or economy. So In effect, all banking community who have used or is planning to use some kind of credit based product is the target audience.

Inputs

$$N' = N\left(1 + X\left(T_2 - T_1\right)\right)$$

$$K = \frac{1}{\left(1 + X\left(T_2 - T_1\right)\right)}$$

$$\text{Cpl}\left(t, T_1, T_2, N, X\right) = N'\mathbb{E}\left[e^{-\int_t^{T_1} r_s ds}\left(K - P\left(T_1, T_2\right)\right)^+ \mid \mathscr{F}_t\right]$$

$$L(t,T) = \frac{1 - P(t,T)}{(T-t)P(t,T)}$$

X - Strike Rate

N - Notional

T1 and T2 - Start date and End Date

Excel file attached has rest data and simulation.

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