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** Note, you may be required to provide proof of your outreach to non-contributing members upon request.*

Part a

Introduction

Louis Bachelier and Kiyosi Itô were two of the most influential 20th century personalities in the areas of probability theory and stochastic processes.

In the area of Mathematics, Bachelier's work is considered to form the bedrock of the works of mathematicians such as Wiener, Kolmogorov, Feller, Doob and Meyer who used Bachelier's ideas to advance the field of stochastic calculus. In Economics, Paul Samuelson adapted Bachelier's Brownian motion for his seminal paper on option pricing.

Kiyosi Itô, on the other hand, is considered to be the father of stochastic integration. His seminal papers in 1940 and 1942 led to the invention of the stochastic integral and Itô's lemma, and was used as a basis for the advancement of the field by mathematicians in Japan and France.

Bachelier's Contributions

In 1900, Bachelier's PhD thesis Théorie de la Spéculation introduced several mathematical ideas which are key to the modern theory of Brownian motion. He obtained Brownian motion as the diffusion limit of a random walk, by arguing that small price fluctuations over a short time interval are independent of the current and past prices and, combined with the Central Limit Theorem, concluded that these increments are normally distributed [2]. Further, he used the memoryless property (formalised by A.A. Markov in 1906 in his work on Markov chains and now known as the Markov property) to write the Chapman-Kolmogorov equation but did not treat it rigorously.

Perhaps, one of Bachelier's most important ideas emerges in his work relating the Brownian motion to heat equations, wherein he views random processes not as vibrations but as trajectories. Based on this, he published an article in 1901 after defending his thesis revising all of the classical results on games (the hyper asymptotic view) with his technique of approximation by diffusion [2].

Just before the Second World War, Bachelier continued his work on Brownian motion via a new book and then a paper in 1941. Fundamental to Bachelier's calculations is what he calls the true price of a security. All his calculations are in terms of the true price for which 'the mathematical expectation of the speculator is null' [2]. Combined with the implicit lack of memory property of the price process, Bachelier is saying that the true price is a martingale [2].

Brief Account of the Works of Mathematicians Built on Bachelier's Ideas

Although Bachelier is credited with providing the mathematical ideas and approaches central to modern Brownian motion theory, it is Wiener, Kolmogorov, Feller, Doob and Meyer who treated those ideas with the mathematical rigour required for probability theory to be worthy enough to be considered a branch of mathematics. What follows is a brief account of these contributions, and their relation to Bachelier's work.

Through his seminal paper 'Differential Space', Wiener converted his previous work on integration to construct the probability measure and the mathematical model of Brownian motion as a continuous process, often called the Wiener's process.

Kolmogorov is credited with two key papers which give Bachelier's ideas the mathematical structure to make them central to modern mathematical finance. His 1931 paper made Bachelier's approach of the passage from discrete to continuous schemes rigorous by extending Lindeberg's method for proving

Central Limit Theorem to this setting. The Kolmogorov's partial differential equations (continuous) then followed from the corresponding difference equations (discrete). Secondly, his 1933 monograph gave a mathematical definition to conditional probabilities and expectations as random variables whose existence is guaranteed by the Randon-Nikodym Theorem.

Finally, in 1953, Doob authored Stochastic Processes, one of the most influential books on probability theory, for which he credits André for his reflection principle and Bachelier for his extensive use of it in his research, another evidence demonstrating Bachelier's unique view of stochastic processes as sample paths/ trajectories. Doob is also one of the first people to study stochastic differential equations and extended Itô's stochastic calculus to martingales.

As the above account clearly demonstrates, Bachelier's ideas were used, built upon and improved time and again throughout the 20th century for the advancement of stochastic calculus. But it was only in the mid-1950s that awareness about his works gained momentum in the Economics community. Bachelier's theory of Brownian motion was then used as the foundation for option valuation, and one figure central to this development was Paul Samuelson.

Samuelson, Bachelier and Option Valuation

Samuelson credits Bachelier's work for providing him with the mathematical tools on Brownian motion, the heat equation, Markov processes and martingales, which would aid him in his work on option and warrant valuation. However, a deficiency of Bachelier's Brownian motion as a model of asset prices was that it allowed for them to be negative at any given point in time. Samuelson, therefore, remedied this by using an exponential form which did not allow for negative values. This came to be known as the geometric Brownian motion wherein the asset price $S(t)$ is given by

$$S(t) = S(0)\,.\,exp(at + \sigma W(t))$$

where $W(t)$ is the Brownian motion and a, σ are constants. Since $W(t) \sim N(0,t)$ we have $\mathbb{E}\left[e^{\sigma W(t)}\right] = \exp\left(\frac{1}{2}\sigma^2 t\right)$, so if

$$a = \alpha - \frac{1}{2}\sigma^2$$

then $\mathbb{E}[S(t)] = S(0)e^{\alpha t}$. Thus α is the expected growth rate. The parameter σ is known as the volatility and measures the standard deviation of log-returns: the standard deviation of $\log(S(t+h)/S(t))$ which is $\sigma\sqrt{h}$.

In addition to avoiding negative prices, the use of the above exponential form had another profound implication on the mathematics of option pricing. The analysis of non-linear (exponential) transformation of the Brownian motion involved the use of Itô calculus. In fact, applying the Itô formula to the geometric Brownian motion equation shows that the asset price $S(t)$ satisfies the following stochastic differential equation:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

Samuelson then used the above model in his 1965 paper *Rational Theory of Warrant Pricing* wherein he provides the first formal analysis of American options. In brief, Samuelson argues that if the warrant price has an expected growth rate of α, it would never be optimal to exercise the warrant before its expiry. Therefore, any option pricing theory would require such a growth rate β to be greater than α. McKean gives a precise formulation of this problem and a complete solution in terms of a free boundary problem in partial differential equations (PDE) [2].

In hindsight, this was an important insight since McKean's solution is identical to Black-Scholes if β is interpreted as the risk-free rate and $\alpha = \beta - q$ where q is the underlying stock's dividend yield.

Part b

Kiyosi Itô and Stochastic Calculus

The Japanese mathematician Kiyosi Itô is widely considered to be the central figure behind the development of differential and integral calculus for a class of random functions called Itô processes.

Notably, two of Itô's contributions constitute the origins of the stochastic integral. Itô undertook a rigorous treatment of Lévy's intuitive ideas in his Theory of the Sum of Independent Random Variables based on works of Kolmogorov and Doob. More precisely, he gave a rigorous proof of what is now called the Lévy-Itô theorem for the structure of sample functions of Lévy processes [5].

The motivation for Itô's work stemmed from his desire to establish a stochastic differential equation to study Markov processes. In his own words, "Observing the intuitive background in which Kolmogorov derived his equation (explained in the introduction of the paper), I noticed that a Markovian particle would perform a time homogeneous differential process for infinitesimal future at every instant, and arrived at the notion of a stochastic differential equation governing the paths of a Markov process that could be formulated in terms of the differentials of a single differential process" [6].

So, in another work, he developed a complete theory of stochastic differential equations determining sample functions of diffusion processes whose laws are described by Kolmogorov's differential equations [5]. Itô created the following stochastic differential equation, in an attempt to model Markov processes:

$$dX_t = \sigma(X_t)dW_t + \mu(X_t)dt$$

where W represents a standard Wiener process.

In an attempt to link Kolmogorov's work on Markov processes to his, Itô then wanted to relate the paths of X to the transition function of the diffusion. This amounted to showing that the distribution of X solves Kolmogorov's forward equation [1]. It is in this work that a rigorous definition of the stochastic integral was provided and what is now known as Itô's lemma came into being. These have been stated below in its modern form.

Itô Integral

If φ is a simple process, we will define the Ito integral of φ with respect to a Brownian motion W as the stochastic process $\{(\varphi \bullet W)_t : 0 \le t \le T\}$ defined by

$$(\varphi \bullet W)_t := \int_0^t \varphi_s dW_s := \sum_{i=1}^{n} H_i \left(W_{t_i \wedge t} - W_{t_{i-1} \wedge t}\right),\ 0 \le t \le T$$

Note that what differentiates the Itô integral from the Wiener integral is that in the former time is crucial and the integrand can itself be a random function.

Itô's Lemma

Let $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a function of two variables $f = f(t, x)$ such that the partial derivatives

$$\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x} \text{ and } \frac{\partial^2 f}{\partial x^2}$$

all exist and are continuous. Suppose that X is an Ito process with

$$dX_t = \mu_t dt + \sigma_t dW_t$$

Then $Y_t = f\left(t, X_t\right)$ is also an Ito process, and

$$dY_t = df\left(t, X_t\right) = \left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2} + \mu_t \frac{\partial f}{\partial x}\right] dt + \sigma_t \frac{\partial f}{\partial x} dW_t .$$

In integral notation, we have

$$Y_t = Y_0 + \int_0^t \left[\frac{\partial f}{\partial s} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 f}{\partial x^2} + \mu_s \frac{\partial f}{\partial x}\right] ds + \int_0^t \sigma_s \frac{\partial f}{\partial x} dW_s .$$

As a consequence, just like Bachelier, Itô's hallmark contributions were momentous in the advancement of stochastic processes and stochastic calculus. As a side note, one may observe that Itô's training as a mathematician enabled him to provide a more rigorous treatment of his ideas, when compared to Bachelier. As previously mentioned, Itô's calculus went on to be extended to processes with orthogonal increments and conditionally orthogonal increments (i.e. martingales) by Doob in his 1953 book. In order to be able to define the stochastic integral with respect to the martingale M, he required the existence of a non-random increasing function $F(t)$ such that $M_t^2 - F(t)$ is also a martingale. This eventually led him to derive the Doob and the Doob-Meyer Decompositions.

The Contributions of Other Japanese Mathematicians (Motoo, Kunita and Watanbe)

Following Itô's remarkable papers, the development of stochastic integration continued primarily in Japan and France. In 1963, Courrège built upon the work of Meyer in this area to provide a more systematic development of the stochastic integral. The key contribution of Courrège to the Doob-Itô framework was the assumption that if X is a submartingale in the Doob-Meyer decomposition $X = M + A$, then A has continuous sample paths [1]. This implies that in the L^2 isometry

$$E\left(\left(\int_0^t H_s dM_s\right)^2\right) = E\left(\int_0^t H_s^2 dA_s\right)$$

where A is the increasing process corresponding to the submartingale $X = M^2$ [1].

Two years later, Motoo and Watanabe contrasted Courrège's work by treating the stochastic integral as an operator on martingales having specific properties, utilizing the Hilbert space structure of L^2 by using the Doob-Meyer increasing process to inspire an inner product through the quadratic variation of martingales [1]. In the same paper, they also showed that all L^2 martingales defined on a probability space obtained via the construction of a Hunt process were generated by a collection of additive functionals which were also L^2 martingales, and which were obtained in a way now associated with Dynkin's formula [1].

Thereafter, in 1967, Kunita and Watanabe developed the ideas on orthogonality of martingales pioneered by P. A. Meyer, and Motoo and Watanabe, and used it to develop a theory of stable spaces of martingales. This was fundamental to the theory of market completeness in Finance. They also clarified the idea of quadratic variation as a pseudo inner product, and used it to prove a general change of variables formula, profoundly extending Itô's formula for Brownian martingales [1].

The 1967 paper by Kunita and Watanabe inspired a series key papers in Springer's famed *Lecture Notes in Mathematics*. Firstly, these led to the development of the square bracket pseudo inner product $[X, Y]$ which turned out to be important in the development of semimartingales and the extension of the stochastic integral to all local martingales. Secondly, they also led Meyer to show that a stochastic integral should agree with a path-by-path construction using Lebesgue-Stieltjes integration when a martingale has paths of finite variation [1]. Finally, Meyer also discarded the Markov process framework used by Kunita and Watanabe and established the general change of variables formula used today without using Lévy systems [1]. Meyer then applied his more general results to Markov processes.

Concluding Remarks for part a and part b

It is evident from the above account that the works of Bachelier and Itô formed the basis of modern stochastic calculus. Bachelier was the first to conceive the idea of a Brownian motion, which was then used by mathematicians and economists in France and the USA for the advancement of the theory of stochastic processes and option valuation. Itô's work was a direct result of his motivation to derive stochastic differential equations to study Markov processes. While the contributions of these two personalities may not appear monumental when look at in hindsight, one must appreciate that all of the work in the field of probability theory and stochastic processes was essentially derived from the works of these two figures.

Part C

Starting from the brownian motion, Ito's lemma and European call option price C as a function of time t and price of the asset S, C = C(S,t). Assuming that the underlying asset price S follows brownian motion (geometric) as -

$$dS(t) = \mu S(t)dt + \sigma S(t)dX(t)$$

Where μ and σ are constants.
Using Ito's lemma on call price option we get -

$$dE = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}(S,t)dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S,t)dS^2$$

Substituting brownian motion assumption into Ito's lemma -

$$dC = \left(\frac{\partial C}{\partial t}(S,t) + \mu S\frac{\partial C}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t)\right)dt + \sigma S\frac{\partial C}{\partial S}(S,t)dX$$

Now, we will see the effect of infinitesimal change in call price and change in quantity of assets-

$$d(C+\Delta S) = \left(\frac{\partial C}{\partial t}(S,t) + \mu S\frac{\partial C}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t) + \Delta\mu S\right)dt + \Delta S\left(\frac{\partial C}{\partial S} + \Delta\right)dX$$

choosing a pertinent value of Δ, $\Delta = -\frac{\partial C}{\partial S}(S,t)$, we get -

$$d(C+\Delta S) = \left(\frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t)\right)dt$$

Value of Δ is derived/worked out from delta-hedging method. Delta hedged portfolio's growth rate should be tantamount to compounding risk free rate r to avoid the arbitrage. Hence we can get this -

$$\frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t) = r\left(C - S\frac{\partial C}{\partial S}\right)$$

Re-arranging and simplifying the above equation gives us famous Black Scholes equation -

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} - rC = 0$$

Part D

Mathematically, the Delta is the first derivative and Gamma is the second derivative with respect to the underlying security's price. On the other hand, Theta is the first order partial derivatives of the portfolio value with respect to the underlying parameters.

Delta, Δ, Delta is referred as hedge ration of the amount of a position hedged to the entire position. Hence, it is used as primary indicator of the portfolio.

Gamma, Γ, is the rate of change in the Delta with regards to time in the underlying price.

Theta, **Θ**, is the factor which determines the value of an option. It measures the sensitivity of the derivative value to the passage of time.

Part E

The derivatives of the Black-Scholes-Merton function, c(t, x) with respect to various Greek variables(delta, theta, gamma, Vega) are called the Greeks.
The Black-Scholes-Merton partial differential equation is given by:

rc(t, x) = c_t(t,x) + rxc_x(t,x) + ½$\sigma^2 x^2$$c_{xx}$(t, x) for €[0, T] x ≥0, with a terminal condition c(T, x) = (x - K)+. If c(t, x) is differentiated

(i) With respect to delta:
cx (t, x) = N (d+ (T - t, x))

(ii) With respect to Theta:

$$C_t(t, x) = -rKe^{-r(T-t)}N(d-(T-t, x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x))$$

(iii) With respect to Gamma:

$$C_{xx}(t, x) = N'(d_+(T-t, x)) \frac{\vartheta}{\vartheta x} d_+(T-t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t, x))$$

(iv) If c(t, s) is differentiated with respect to σ to get a Greek called Vega, is given by:

$$\frac{\vartheta u}{\vartheta \sigma} = \frac{1}{P}\left(\frac{2c}{\sigma^3}s(1-\mathrm{Lns}) + ts(\frac{\sigma}{8} + \frac{2c^2}{\sigma^3}) - \frac{\sigma}{8c}e^{ct+s_0}\right) \text{ for P,s,}\sigma\text{, c>0}$$

Part F

$$U_{SS} = \frac{1}{Ps}\left(\frac{1}{4\sqrt{s}}e^{\frac{ct+S_0}{2}} + \frac{c}{\sigma^2} - \frac{1}{4}\right) \quad \text{for P,s,}\sigma\text{, c>0}$$

An investor is one type of delta hedging involving either buying or selling of options with the aim of offsetting the delta risk through buying or selling equivalent amount of stock. In order to offset the delta, one has to borrow more on the part of a short position and the opposite will be appropriate for a long position in the same underlying.

Part G

(i) Theta, $\Theta = \frac{\vartheta V}{\vartheta t} = \frac{\vartheta V}{\vartheta T}\frac{\vartheta T}{\vartheta t} = \frac{\vartheta V}{\vartheta T}(-1) = -\frac{\vartheta V}{\vartheta T}$

Which implies that $C_t(t, x) = -rKe^{-r(T-t)}N(d-(T-t, x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x))$

Theta is expressed as negative because its option loses the value as the expiry date approaches.

(ii) Gamma, $C_{xx}(t, x) = N'(d_+(T-t, x))\frac{\vartheta}{\vartheta x}d_+(T-t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t, x))$

Is expressed as positive because its prices increase which causes delta to approach 1 from 0. This is also clearly seen from a call of an option, given by:

$\Gamma = \frac{\vartheta^2 C}{\vartheta S^2} = \frac{\vartheta\left(\frac{\vartheta C}{\vartheta S}\right)}{\vartheta S} = \frac{\vartheta N(d_1)}{\vartheta S} = \frac{\vartheta N(d_1)}{\vartheta d_1}\frac{\vartheta(d_1)}{\vartheta S} = N'(d_1)\frac{\frac{1}{S}}{\sigma\sqrt{t}} = \frac{N'(d_1)}{S\sigma\sqrt{t}}$, which is also positive.

Despite N and N' being positive in both theta and gamma, theta is always negative and gamma positive.

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