Submission Number: 3

Group Number: 30

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Statement of integrity: By typing the names of all group members in the text box below, you confirm that the assignment submitted is original work produced by the group (*excluding any non-contributing members identified with an "X" above*).

Anubhav Mishra, Ishaan Narula

Use the box below to explain any attempts to reach out to a non-contributing member. Type (N/A) if all members contributed.

Aishat – had stopped contributing to the Group Work Activity since assignment 1. Click here for details

** Note, you may be required to provide proof of your outreach to non-contributing members upon request.*

Review Note: Please review all answers in conjunction with WQU_MScFE 620_Project3_Group30_Code_vF.ipynb file.

Inputs/ Assumptions

The following inputs/ assumptions have been used in the analysis.

Inputs			
$\boldsymbol{X_o}$	\$95*	T	1 year
$\boldsymbol{K}$	\$90	n	5 steps
$\boldsymbol{u}$	1.40	r	0%

**X_o is taken to be \$100 for pricing the European Up-and-Out Call Option in Answer 3*

Answer 1

In contrast to a European Call Option, an American Call Option can be exercised at any time t, where $0 \leq t \leq T$. So, its payoff function at time t is given by $H_t = (X_t - K)^+$. The early exercise feature makes American derivatives a sequence of non-negative random variables $H = \{H_t : t = 0, 1, \dots, T\}$, where H_t is $\mathcal{F}_t$-measurable in contrast to European derivatives which are represented by only one random variable.

Part (a)

To price the American call option H, we find the Snell envelope of H which is essentially a stochastic process $U = \{U_t : t \in \mathbb{I}\}$ defined by backward induction as follows:

$$U_T := H_T, \quad U_T := \max\{H_t, \mathbb{E}^*(U_{t+1}|\mathcal{F}_t)\}, \quad t = T-1, \dots, 0$$

The Snell envelope is the smallest supermartingale that dominates H and is essentially the minimal price required to hedge the call option. This is because at any time before maturity $t = T - 1$, the price of H given by the Snell envelope is enough to pay for any exercise at the time (i.e., $U_{T-1} \geq H_{T-1}$) and also enough to cover subsequent values of H if the derivative is not exercised (i.e., $U_{T-1} \geq \mathbb{E}^*(H_T \mid \mathcal{F}_{T-1})$). For this reason, U_0 is the minimal price of the American call option H.

With the above procedure, we get the following outputs:

<u>Stock Price Evolution</u>

0	1	2	3	4	5
					133.00
				124.34	116.25
			116.25	108.69	101.61
		108.69	101.61	95.00	88.82
	101.61	95.00	88.82	83.04	77.63
95.0	88.82	83.04	77.63	72.58	67.86

<u>Call Payoffs at Each Node</u>

0	1	2	3	4	5
					43.00
				34.34	26.25
			26.25	18.69	11.61
		18.69	11.61	5.00	0.00
	11.61	5.00	0.00	0.00	0.00
5.0	0.00	0.00	0.00	0.00	0.00

<u>Call Value H at each Node</u>

0	1	2	3	4	5
					43.00
				34.34	26.25
			26.25	18.69	11.61
		18.85	11.93	5.61	0.00
	12.81	7.17	2.71	0.00	0.00
8.33	4.14	1.31	0.00	0.00	0.00

Part (b)

The buyer of an American call option would benefit from an early exercise at a given node when the payoff from exercising the option $H_t = (X_t - K)^+$ exceeds the expected value of future cash flows at the subsequent up- and down-states of that node $\mathbb{E}^*(U_{t+1}|\mathcal{F}_t)$.

0	1	2	3	4	5
					0.0
				1.0	0.0
			1.0	1.0	0.0
		0.0	0.0	0.0	0.0
	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0

As shown in the lattices in Answer 1 Part (a) and the one alongside, early exercise occurs at the 3rd and 4th price steps where the option value at the relevant nodes (marked alongside) is equal to the exercise price.

Part(c)

It is not always the case that an early exercise is beneficial. There are several states where the expected value of future cash flows $\mathbb{E}^*(U_{t+1}|\mathcal{F}_t)$ exceeds the option's payoff, making it suboptimal to exercise. These are shown in the lattice in Answer 1 Part (b) with where cells take a 0.0 value.

<u>Answer 2</u>

An American Put option has the payoff function at time t if given by $H_t = (K - X_t)^+$ since it offers the option holder the right but not the obligation to exercise at any time before maturity.

Part (a)

Following the same procedure as in Part (a) of Answer 1, we get the following outputs:

<u>Stock Price Evolution</u>

0	1	2	3	4	5
					133.00
				124.34	116.25
			116.25	108.69	101.61
		108.69	101.61	95.00	88.82
	101.61	95.00	88.82	83.04	77.63
95.0	88.82	83.04	77.63	72.58	67.86

<u>American Put Payoffs at Each Node</u>

0	1	2	3	4	5
					0.00
				0.00	0.00
			0.00	0.00	0.00
		0.00	0.00	0.00	1.18
	0.00	0.00	1.18	6.96	12.37
0.0	1.18	6.96	12.37	17.42	22.14

<u>American Put Option Value H at each Node</u>

0	1	2	3	4	5
					0.00
				0.00	0.00
			0.00	0.00	0.00
		0.16	0.32	0.61	1.18
	1.20	2.16	3.89	6.96	12.37
3.33	5.32	8.27	12.37	17.42	22.14

Part (b)

The buyer of an American put option would benefit from an early exercise at a given node when the payoff from exercising the option $H_t = (K - X_t)^+$ exceeds the expected value of future cash flows at the subsequent up- and down-states of that node $\mathbb{E}^*(U_{t+1}|\mathcal{F}_t)$.

0	1	2	3	4	5
					0.0
				0.0	0.0
			0.0	0.0	0.0
		0.0	0.0	0.0	0.0
	0.0	0.0	0.0	1.0	0.0
0.0	0.0	0.0	1.0	1.0	0.0

As shown in the lattices in Answer 2 Part (a) and the one alongside, early exercise occurs at the 3rd and 4th price steps where the option value at the relevant nodes (marked alongside) is equal to the exercise price.

Part (c)

As argued in Answer 1 Part(c), it is not always the case that an early exercise is beneficial. There are several states where the expected value of future cash flows $\mathbb{E}^*(U_{t+1}|\mathcal{F}_t)$ exceeds the option's payoff, making it suboptimal to exercise. These are shown in the lattice in Answer 2 Part (b) with where cells take a 0.0 value.

<u>Answer 3</u>

Part (a)

To price a European Up-and-Out (UAO) Call option, we start by constructing the binomial tree with $\boldsymbol{X_o} = 100$. We then compare the payoff of the option at the final price step $v(S_T) = (S_T - K)^+$ with the barrier L, which in this case is \$130. When this exceeds, the barrier, the payoff is taken to be zero.

Once the payoffs for the final price step have been calculated, we use backward induction with the risk-neutral probabilities to calculate the value of the option at each intermediate price step, or in accordance with the following European call option pricing formula:

$$\pi_C := \pi(H) = \mathbb{E}^*(H) = \sum_{y=0}^{T} (X_0 u^y d^{T-y} - K)^+ \binom{T}{y} p^{*y} (1-p^*)^{T-y}$$

As a result, we get the following outputs.

Stock Price Evolution

0	1	2	3	4	5
					140.00
				130.89	122.37
			122.37	114.41	106.96
		114.41	106.96	100.00	93.49
	106.96	100.00	93.49	87.41	81.72
100.0	93.49	87.41	81.72	76.40	71.43

UAO Call Value at Each Node

0	1	2	3	4	5
					0.00
				16.73	32.37
			20.70	24.41	16.96
		18.77	16.96	10.00	3.49
	14.83	11.14	5.70	1.69	0.00
10.79	7.03	3.18	0.81	0.00	0.00

Part (b)

A procedure on the same lines as above is employed to then estimate the value of a vanilla European call with the same features. This results in the following output.

European UAO Call Value at Each Node

0	1	2	3	4	5
					50.00
				40.89	32.37
			32.37	24.41	16.96
		24.41	16.96	10.00	3.49
	17.55	11.14	5.70	1.69	0.00
12.11	7.03	3.18	0.81	0.00	0.00

Clearly, the European call option is more expensive. This is because when a barrier is introduced to a vanilla European call, its pay-off reduces to zero when the stock price crosses this barrier. For a vanilla call whose value increases with an increase in the underlying's price, the introduction of a barrier prevents it from realising this value once it is breached. For this reason, a European Up-and-Out (UAO) Call option will typically trade at a discount to the value of a plain Vanilla European Call option.

Part (c)

The primary advantage of the European Up-and-Out Call option is that it entails a lower investment outlay relative to a Plain Vanilla European option. So, the loss is smaller if the option trade does not work out.

Such an option may be suitable for speculators who, based on their research, have confidence that the underlying's price will not cross a certain level. They can then employ a smaller outlay in an Up-and-Out call to capitalise on their view.

Lastly, Knock-out options are typically OTC instruments and, therefore, afford the investor much more customisability relative to vanilla options which are exchange-traded.

Part (d)

Being long a Knock-out option and a Knock-in option with the same features is equivalent to owning a comparable vanilla option independently from the behaviour of the spot with respect to the barrier level.

$$Knock - In\ (K, T, B)\ +\ Knock - Out\ (K, T, B)\ =\ Vanilla\ (K, T)$$

It is very easy to see that, for any given scenario of the underlying asset path before maturity, the portfolio (Knock-In + Knock-Out) will always have the same payoff as the corresponding vanilla option. This relationship holds for both the put and call options in the absence of rebates.[1]

0	1	2	3	4	5
					50.0
				24.16	0.0
			11.67	0.00	0.0
		5.64	0.00	0.00	0.0
	2.72	0.00	0.00	0.00	0.0
1.32	0.00	0.00	0.00	0.00	0.0

The price of the nodes of a UAI Call Option equals price of nodes for a European call option minus the price corresponding nodes of a UAO Call Option

Answer 4

Part (a)

If we can price an American call or put option using a binomial tree, we must have assumed that the market for the underlying is complete. It is necessary to assume market completeness in order to be able to price and American call or put option. This is because to price such an option (i.e. to obtain U_0), the Snell envelope U needs to be expressed as

$$U = U_0 + M - A \qquad (Doob\ Decomposition)$$

where M is a martingale and A is an increasing process, both null at zero, and we need to use the Predictable Representation Property (PRP) of the underlying's complete market to find the hedging strategy φ with initial capital U_0 such that

$$M_t = \sum_{k=1}^{t} \varphi_k (X_k - X_{k-1}) \qquad (PRP)$$

and the strategy (U_0, φ) dominates H. In other words, $V_\tau \geq H_\tau$ for any exercise strategy τ.

Part (b)

As a reminder, a market is said to be complete *iff* there exist no arbitrage opportunities and every contingent claim is attainable i.e. its payoffs can be replicated by a strategy φ so that $V_T(\varphi) = H$.

Mathematically, completeness implies that the set of Equivalent Martingale Measures P has only one element, and that every contingent claim has a unique price. That said, we can only derive a unique price for a European Up-and-Out (UAO) Option *iff* the market for the underlying is complete.

Part (c)

For a market to be complete, every contingent claim is attainable. This can be achieved through a **unique** replication strategy due to the no-arbitrage rule. In continuation to 3(d) of Submission 2 we know that:

$$\mathbf{Ax = b.}$$
$$\mathbf{\Rightarrow A^{-1}Ax = A^{-1}b.}$$
$$\mathbf{\Rightarrow Ix = A^{-1}b.}$$
$$\mathbf{\Rightarrow x = A^{-1}b.}$$

Matrix A with rows indicating the state of the world (up and down). Each column contains a security (stock and bond). Column **vector b** contains the option value if the stock price went down.

Matrix x represents the number of units of Bond and Stock that needs to be purchased at time t = 0 so that it replicates the call option at time t=0.

In the above equation **x has exactly one solution only if Matrix A has full rank**. In this case, the null space of A contains just the zero vector. Thus, the market can only be complete if there is one unique solution to **x** i.e. Matrix A is a full rank matrix.

Part (d)

The market for an asset is complete when it is not possible to make riskless profits because every conceivable derivative's cash flows can be replicated with some combination of that asset and a riskless borrowing.

Reference

[1] Chapter 12, Barrier Options: Derivatives Academy, https://bookdown.org/maxime_debellefroid/MyBook/barrier-options.html
[2] Shreve, S. (2005). Stochastic Calculus for Finance I: The Binomial Asset Pricing Model. New York: Springer