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** Note, you may be required to provide proof of your outreach to non-contributing members upon request.*

Part A

Lookback Option

Definition

A lookback option is a form of option that allows the owner of the option the opportunity of having prior knowledge about the option when deciding on the suitable time to exercise it. A lookback option has an advantage of lowering the uncertainties arising from timing the market entry and the option will expire worthlessly. This option is part of exotic option possessing path dependency. Exotic options are expensive to execute. (Boyle, P.P, Tian, Y & Iman, J, 1999)

Variations

Despite lookbacks option having different varieties, it is classified into two major categories: Floating and Fixed.

Floating Option

The strike price for floating option is determined at maturity automatically. This strike price is as a result of the optimal value of the price of the underlying asset during the lifetime of the option. Subsequently, the floating option consents with the holder of the option to assess the historical prices of the underlying asset over its lifespan. The main purpose of the floating strike option is to address the issue of market entry (Babbs,S,2000).

Call Option Value

$$Se^{-q(T-t)}N(d_1) - Me^{-r(T-t)}N(d_2) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-q)} \times \left(\left(\frac{S}{M}\right)^{-2(r-q)/\sigma^2} N\left(-d_1 + \frac{2(r-q)\sqrt{T-t}}{\sigma}\right) - e^{(r-q)(T-t)}N(-d_1)\right)$$

where

$$d_1 = \frac{log(S/M) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and $d_2 = d_1 - \sigma\sqrt{T-t}$

Here,
$S =$ Underlying's spot price
$M =$ Strike price
$r =$ Risk-free interest rate
$T - t =$ Time to expiration of the option
$q =$ Dividend yield on the underlying stock and
$N(.) =$ Cumulative distribution function of the standard normal curve

Put Option Value

$$Me^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-q)}\left(-\left(\frac{S}{M}\right)^{-2(r-q)/\sigma^2} N\left(d_1 - \frac{2(r-q)\sqrt{T-t}}{\sigma}\right) + e^{(r-q)(T-t)}N(-d_1)\right)$$

where all parameters have been introduced above.

Fixed Option

The strike price of the fixed option is set right at the time of purchase, as it is done with other normal options. However, at the time of executing the option, the most favorable underlying asset price is taken over the option term, (Babbs,S,2000). Thereby, this stands to be the key feature that distinguish from other options as the current market price is considered at the time of the execution.

Call Option Value

If $Underlying\ Price > Strike$,

$$Se^{-q(T-t)}N(d_1) - Xe^{-r(T-t)}N(d_2) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-q)}\left(-\left(\frac{S}{X}\right)^{-2(r-q)/\sigma^2} N\left(d_1 - \frac{2(r-q)\sqrt{T-t}}{\sigma}\right) + e^{(r-q)(T-t)}N(d_1)\right)$$

Otherwise,

$$(M-S)e^{-r(T-t)} + Se^{-q(T-t)}N(d_1) - Me^{-r(T-t)}N(d_2) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-q)}\left(-\left(\frac{S}{M}\right)^{-2(r-q)/\sigma^2} N\left(d_1 - \frac{2(r-q)\sqrt{T-t}}{\sigma}\right) + e^{(r-q)(T-t)}N(d_1)\right)$$

Put Option Value

If $Underlying\ Price > Strike$,

$$Xe^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-q)}\left(\left(\frac{S}{X}\right)^{-2(r-q)/\sigma^2} N\left(-d_1 - \frac{2(r-q)\sqrt{T-t}}{\sigma}\right) - e^{(r-q)(T-t)}N(-d_1)\right)$$

Otherwise,

$$(X-M)e^{-r(T-t)} - Se^{-q(T-t)}N(-d_1) + Me^{-r(T-t)}N(-d_2) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-q)}\left(\left(\frac{S}{M}\right)^{-2(r-q)/\sigma^2} N\left(-d_1 + \frac{2(r-q)\sqrt{T-t}}{\sigma}\right) - e^{(r-q)(T-t)}N(-d_1)\right)$$

Strategies

A lookback option can be designed as a put or call option, which is applicable to either fixed or floating. In case of a call option, the holder of the option would review the price history and then choose the appropriate time to execute an option price which would yield the maximum profit. On the other hand, in the case of a put, the holder of the option would execute the contract at the lowest price differential so as to ensure maximum profit (Bermin.H, 2000).

Conclusion

Lookback option is a useful option which helps the holder to maximize their profit and minimize the risk. This is done in order to minimize the uncertainty. However, this option is relatively expensive because the seller assumes the higher risk. Thus, holders must consider the cost of this option when taking up this option. These types of options are unlisted and therefore do not trade on the formal exchanges. Upon their execution, the holder is awarded a cash settlement equivalent to the profits that would have been realized in the buying or selling of the underlying asset.

Barrier Option

Definition

A barrier option is a kind of derivative whose payoff depends on whether or not the underlying asset has reached or surpassed a predetermined price, (Ritchken, P,1995). Subsequently, barrier options are best described options that offer cheaper premiums than standard options and can also be used to hedge positions. A barrier option can expire worthless if the underlying asset exceeds either a certain price or limiting profits in case of the for the holder and limiting losses, in case of the writer. A barrier option can lose value until when the underlying asset reaches an appropriate price, (Boyle, P.P.& Lau, S.H,1994). Therefore, barrier options are typically classified as either knock-in or knock-out. Barrier Options cannot easily be publicly accessible through the various stock & options exchanges but only traded in OTC markets.

Variations

Barrier option is categorized into two major variations: The knock-in and knock-out

Knock-in Barrier Option

A knock-in barrier option is an option whose associated rights commence once an underlying asset reaches a certain price. This means that the holder can exercise the option only at and right after the price reaches a particular level in the open market. However, If the knock-in price level is never reached, then the knock-in barrier option expires worthless, (Boyle, P.P.& Lau, S.H,1994).

Knock-in barrier option is further classified into up-and-in or down-and-in options. In an up-and-in barrier option, the option contract starts only when the price of the underlying asset exceeds the predetermined price barrier. On the contrary, if it is a down-and-in barrier option, it turns valid as the underlying asset value drops below the initially set barrier price, (Ritchken, P,1995).

Valuation

$$V_{DI\,Call} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} \left(C_{BS}\left[\frac{H^2}{S}, max(H,K)\right] + [max(H,K) - K]e^{-r\tau} N\left\{ d_{BS}\left[\frac{H^2}{S}, max(H,K)\right]\right\}\right)$$
$$+ \{P_{BS}(S,K) - P_{BS}(S,H) + [H-K]e^{-r\tau}N[-d_{bs}[S,H]\}B_{H>K}$$

$$V_{UI\,Call} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} \left\{P_{BS}\left(\frac{H^2}{S}, K\right) - P_{BS}\left(\frac{H^2}{S}, H\right) + (H-K)e^{-r\tau}N[-d_{BS}(H,S)]\right\} B_{H>K} + [C_{BS}[S, max(H,K)] + [max(H,K) - K]e^{-r\tau}N\{d_{BS}[S, ma\,x(H,K)]\}]$$

$$V_{DI\,Put} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} \left\{C_{BS}\left(\frac{H^2}{S}, K\right) - C_{BS}\left(\frac{H^2}{S}, H\right) - (H-K)e^{-r\tau}N[d_{BS}(H,S)]\right\} B_{H>K} + [P_{BS}[S, min(H,K)] - [min(H,K) - K]e^{-r\tau}N\{-d_{BS}[S, mi\,n(H,K)]\}]$$

$$V_{UI\,Put} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} \left(P_{BS}\left[\frac{H^2}{S}, min(H,K)\right] - [min(H,K) - K]e^{-r\tau}N\left\{-d_{BS}\left[\frac{H^2}{S}, min(H,K)\right]\right\}\right) + \{C_{BS}(S,K) - C_{BS}(S,H) - [H-K]e^{-r\tau}N[d_{bs}[S,H]\}B_{K>H}$$

where $C_{BS}(m,n)$ and $P_{BS}(m,n)$ denote the values of standard call and put options with underlying price m and strike n and

$$C_{BS} = \omega X_t.N(\omega d_{1BS}) - \omega K e^{-\tau r}.N(\omega d_{BS})$$

$$d_{1BS} = \frac{ln(X_t/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_{BS} + \sigma\sqrt{\tau}$$

$$d_{BS} = \frac{ln(X_t/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

Here $X = S/B$ is the discounted asset price, K is the strike, r is the risk-free rate, σ is the volatility of the underlying's return, τ is the life of the option contract, $N(.)$ is the cumulative standard normal random variable and ω is a binary operator which takes values 1 for a call option and -1 for a put option.

Knock-out Barrier Option

A knock-out barrier option's validity ceases when the underlying asset hits a barrier. This happens during the time horizon of the contract. Knock-out barrier option equally divided into categories: The up-and-out or down-and-out options, (Ritchken, P,1995). An up-and-out option stops existing when the underlying security moves above the barrier that was set above the initial price of the underlying security. On the other hand, a down-and-out option stops existing when the underlying security moves below the barrier that was set below the initial price of the underlying security. If an asset underlying the barrier option strikes the barrier anytime during the option's life, the option is terminated or knocked out.

Valuation

$$V_{DO\ Call} = C_{BS}[S, max(H,K)] - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} C_{BS}\left[\frac{H^2}{S}, max(H,K)\right] + [max(H,K) - K]e^{-r\tau}\left(N\{d_{BS}[S, max(H,K)]\} - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} N\left\{d_{BS}\left[\frac{H^2}{S}, max(H,K)\right]\right\}\right)$$

$$V_{UO\ Call} = B_{H>K}\left\{C_{BS}(S,K) - C_{BS}(S,H) - (H-K)\mathrm{e}^{-r\tau}N[d_{BS}(S,H)] - \left(\frac{H}{S}\right)^{2v/\sigma^2}\left[C_{BS}\left(\frac{H^2}{S}, K\right) - C_{BS}\left(\frac{H^2}{S}, H\right) - (H-K)\mathrm{e}^{-r\tau}N[d_{BS}(H,S)]\right]\right\}$$

$$V_{DO\ Put} = B_{K>H}\left\{P_{BS}(S,K) - P_{BS}(S,H) + (H-K)\mathrm{e}^{-r\tau}N[-d_{BS}(S,H)] - \left(\frac{H}{S}\right)^{2v/\sigma^2}\left[P_{BS}\left(\frac{H^2}{S}, K\right) - P_{BS}\left(\frac{H^2}{S}, H\right) + (H-K)\mathrm{e}^{-r\tau}N[-d_{BS}(H,S)]\right]\right\}$$

$$V_{UO\ Put} = P_{BS}[S, \min(H,K)] - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} P_{BS}\left[\frac{H^2}{S}, min(H,K)\right] - [min(H,K) - K]e^{-r\tau}\left(N\{-d_{BS}[S, min(H,K)]\} - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} N\left\{-d_{BS}\left[\frac{H^2}{S}, min(H,K)\right]\right\}\right)$$

Strategies

Smith, C. (2008) outlines the following strategies in order to maximize the profit and minimize risks in barrier option.

Long Combo Strategy

Formation

This is a Bullish strategy. If an investor is expecting the price of a stock to move up he can do a Long Combo strategy. It involves selling a lower strike (OTM) Put and buying a higher strike that’s OTM Call. A call option is OTM if the underlying price is trading below the strike price of the call. A put option is OTM if the underlying's price is above the put's strike price. On the other hand, it can also help in comparison with other trading strategies and in construction of hedging option strategies.

Hedging Portfolio

Hedging using long combo with barrier options which expands hedging opportunities in different directions, thereby offers more alternatives for price hedging which allows adaptation of hedger's specific individual requirements.

Hedging by Long Combo strategy using barrier options enables us to minimize risk from the price drop and also to profit from the opposite underlying price movement.

Hedging using Long Combo strategy with vanilla options. This can minimize and maximize selling price. The advantage of hedging against a price drop using the Long Combo strategy with the barrier and vanilla options is the possibility of its formation at zero initial cost.

Compound Option

Definition

A compound option is a type of option that receives another option as underlying security. This underlying option is termed as second option while the initial option is called the underlying. In this case, the compound options contain two strike prices and two expirations dates. This implies that when a compound option is exercised, two premiums are involved. On the other hand, if the holder of the option exercises overlying option, a back fee inform premium based on the strike price of the compound option must be paid to the seller of the underlying option. (Smith, C,2008).

Variations

Amaitiek, O.F.S., Bálint, T and Rešovský, M (2010) outlined the following variations on compound option:

Call on a Put

Is a call option on an underlying put option. This implies that if the holder of the option decides to exercise the call option will receive a put option in return. In view of the foregoing statement, this option can be used by an investor to extend their exposure to an underlying asset at a low cost, and can also be used in real estate development to secure property rights without being obliged to commit. The call on a put is given by the equation below:

Payoff

$$Payoff_T = Max\{0, P_{std}(S_T, K^*, T^* - T) - K\}$$

Valuation

$$c_{put} = X_2 e^{-rT_2} N_2\left(-a_-, -b_-; \sqrt{T_1/T_2}\right) - Se^{-qT_2} N_2\left(-a_+, -b_+; \sqrt{T_1/T_2}\right) - X_1 e^{-rT_1} N(-a_-)$$

where a_+ and a_- (for the first option) and b_+ and b_- (for the second option) have the same meaning as the notation d_{1BS} and d_{BS} respectively introduced previously. The rest of the notation is self-explanatory.

Call on a Call

In this option, the investor is required to buy another call option with modified provisions. The underlying provisions include the right to buy a plain vanilla call option on an underlying security. This option is mostly used by companies during a bidding process for a potential work contract. The call on a call option is defined by the equation below:

Payoff

$Payoff_T = Max\{0, C_{std}(S_T, K^*, T^* - T) - K\}$

Valuation

$c_{call} = Se^{-qT_2}N_2\left(a_+, b_+; \sqrt{T_1/T_2}\right) - X_2 e^{-rT_2}N_2\left(a_-, b_-; \sqrt{T_1/T_2}\right) - X_1 e^{-rT_1}N(a_-)$

Put on a Call

In this option, if an investor delivers the underlying call option to the seller, a premium based on the strike price of the overlying put option must be collected. Investors can use a put to extend their hedge on an underlying asset at a low cost, and can also be used in real estate development to get out a property rights without being obligated to the agreements. A put on a call equation is given below:

Payoff

$Payoff_T = Max\{0, K - C_{std}(S_T, K^*, T^* - T)\}$

Valuation

$p_{call} = X_2 e^{-rT_2}N_2\left(-a_-, b_-; -\sqrt{T_1/T_2}\right) - Se^{-qT_2}N_2\left(-a_+, b_+; -\sqrt{T_1/T_2}\right) + X_1 e^{-rT_1}N(-a_-)$

Put on a Put

This is an option in which a put is purchased on a put contract and wait for it to gain in value when its contract falls in value. This option is used when a bullish trader wants to employ leverage. The equation below shows a put on a put:

Payoff

$Payoff_T = Max\{0, K - P_{std}(S_T, K^*, T^* - T)\}$

Valuation

$p_{put} = Se^{-qT_2}N_2\left(a_+, -b_+; -\sqrt{T_1/T_2}\right) - X_2 e^{-rT_2}N_2\left(a_-, -b_-; -\sqrt{T_1/T_2}\right) + X_1 e^{-rT_1}N(a_-)$

Note that C_{std} and P_{std} can be found by Black-Scholes:

$C_{std}(S, K, \tau) = Se^{-q\tau}N(d_1) - Ke^{-r\tau}N(d_2)$

$P_{std}(S, K, \tau) = Ke^{-r\tau}N(-d_2) - Se^{-q\tau}N(-d_1)$

Real World Applications

Compound options are mostly used in forex or fixed income markets where the risk protection abilities of a standard option may be uncertain and called into question. In forex markets, this is done in order to eliminate high risks that the traders might encounter during forex trading. In fixed income markets such as the mortgage market, one may use a Call on a Put to hedge the risk arising from interest rate fluctuations between the time a mortgage contract has been entered into and the scheduled date from which the contract takes effect.

Such instruments may also be used in situations requiring high leverage levels, since compound options are, at least initially cheaper relative to standard options. However, if the holder ends up exercising both options, the total premium payable is larger than what one would pay for a single standard option.

Furthermore, traders typically extend the life of a pre-existing options position using a compound option. Besides, this also helps them participate in the gains of the underlying option without putting up the full amount to buy it at the onset. However, as mentioned above, the caveat of the possibility to pay two premiums is always present if the second option is exercised.

Besides hedging and speculation in financial markets, compound options also find applications in business, particularly when companies are making bids for large projects. Such companies typically need to secure resources and financing before they win the bid and start the project. In the event they lose, companies could be left with spare resources and financing. That said, compound options can be used to avoid such a situation and can, therefore, act as an insurance policy.

Part B (Choice 2)

Black-Scholes Pricing Equation

We are given that the risky asset class S_t and the risk-free bank account B_t satisfy the following SDEs

$$dS_t = S_t(\mu dt + \sigma dW_t) \quad \text{...[B.1]}$$

$$dB_t = rB_t dt \quad \text{...[B.2]}$$

where μ and σ are the instantaneous mean and standard deviation of the asset price respectively and W_t is the standard Brownian motion.

When the above holds along with other standard Black-Scholes assumptions, the Black-Scholes option pricing equation for a vanilla European call/ put option is given by

$$C_{BS} = \omega X_t . N(\omega d_{1BS}) - \omega K e^{-\tau r} . N(\omega d_{BS}) \quad \text{...[B.3]}$$

where

$$d_{1BS} = \frac{\ln(X_t/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_{BS} + \sigma\sqrt{\tau} \quad \text{...[B.4]}$$

$$d_{BS} = \frac{\ln(X_t/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad \text{...[B.5]}$$

Here $X = S/B$ is the discounted asset price, K is the strike, r is the risk-free rate, σ is the volatility of the underlying's return, τ is the life of the option contract, $N(.)$ is the cumulative standard normal random variable and ω is a binary operator which takes values 1 for a call option and -1 for a put option.

Barrier Options Overview

Barrier options, in brief, are derivatives the right to exercise of which depends on whether the underlying's price touches a pre-defined fixed barrier throughout the life of the contract. Barrier options may be of the *knock-in* or *knock-out* variety, whereby the right to exercise either appears or disappears respectively, upon breach of the barrier.

With that context, we now write the payoffs and derive the pricing equations of four Barrier options: Up-and-In Calls, Up-and-In Puts, Down-and-Out Calls and Down-and-Out Puts. For simplicity, we assume all of these are European style and none of the options offer rebates.

Down-and-In Calls and Puts

Down-and-in options are ones where the barrier H is set *below (down)* the spot price of the underlying. In case this barrier is touched at any point throughout the life of the contract, the underlying derivative (call or put) gets activated. If it is not, the holder gets nothing or, sometimes, a rebate upon maturity of the contract.

Up-and-In Calls and Puts

In case of an up-and-in option, the barrier H is set *above (up)* the spot price of the underlying upon the derivative's inception. If the barrier is touched/ breached throughout the life of the contract, the holder *earns the right (in)* to receive a European call or put payoff upon maturity. If it is not, the holder gets nothing or, sometimes, a rebate upon maturity of the contract.

Down-and-Out Calls and Puts

The only differences here relative to up-and-in options are that the barrier H is set *below (down)* the spot price of the underlying upon the derivative's inception. If the barrier is touched/ breached throughout the life of the contract, the holder *loses the right (out)* to receive a European call or put's payoff upon maturity. He may get nothing or, sometimes, a rebate in this case. If the barrier is not touched, the holder gets the standard call/ put payoff.

Up-and-Out Calls and Puts

Relative to down-and-out options, up-and-out options are different in that the barrier for these contracts is set above the underlying's spot price. The other features remain the same as mentioned above.

Payoff Functions

Based on the above definitions, the payoff of the down-and-in option (call or put) can be formalised as follows:

$$\boxed{Payoff\ DI := \begin{cases} max\{[\omega S(t^*) - \omega K, 0] \mid S(t) > H \ and \ S(T) \leq H\}, for\ some\ t < T \leq t^* \\ \\ Rm(\tau) \mid S(t) > H \ and \ S(T) > H, for\ all\ t < T \leq t^* \end{cases}} \quad \text{...[B.6]}$$

where t and t^* stand for the current and expiration time of the contract respectively and $Rm(\tau)$ is the rebate paid on maturity in case the barrier is not touched. The rest of the notation is standard or has been introduced before.

The payoff of the up-and-in option (call or put) is given by:

$$Payoff\ UI := \begin{cases} max\{[\omega S(t^*) - \omega K, 0] \mid S(t) < H\ and\ S(T) \geq H\}, for\ some\ t < T \leq t^* \\ Rm(\tau) \mid S(t) < H\ and\ S(T) < H, for\ all\ t < T \leq t^* \end{cases}$$ ...[B.7]

The payoff of the down-and-out option (call or put) is formalised below:

$$Payoff\ DO := \begin{cases} max\{[\omega S(t^*) - \omega K, 0] \mid S(t) > H\ and\ S(T) > H\}, for\ all\ t < T \leq t^* \\ Rd(\tau) \mid S(t) > H\ and\ S(T) \leq H, for\ some\ t < T \leq t^* \end{cases}$$ ...[B.8]

where all parameters are the same as before except for $Rd(\tau)$ which represents the rebate deferred to maturity.

Finally, the payoff of the up-and-out option (call or put) can be written as:

$$Payoff\ UO := \begin{cases} max\{[\omega S(t^*) - \omega K, 0] \mid S(t) < H\ and\ S(T) < H\}, for\ all\ t < T \leq t^* \\ Rd(\tau) \mid S(t) < H\ and\ S(T) \geq H, for\ some\ t < T \leq t^* \end{cases}$$ ...[B.9]

Pricing Equations

Derivation of Restricted Distributions

We know that under Black-Scholes, the log return of the underlying's price is normally distributed with mean $v\tau$ and variance $\sigma^2\tau$. The resulting density function is an unrestricted distribution due to absence of any explicit constraints imposed on it.

To price barrier options, we need a probability distribution conditioned on whether the barrier is touched. This is derived below, before writing the pricing equations.

Consider the following two variables

$$M_t^{t^*} = max\{S(s) \mid s \in [t, t^*]\}$$ ...[B.10]

$$m_t^{t^*} = min\{S(s) \mid s \in [t, t^*]\}$$ ...[B.11]

which simply represent the maximum and minimum of all the prices of the underlying over the life of the option contract.

Transforming these into log-returns, we get

$$Y_\tau = ln\left(\frac{M_t^{t^*}}{S}\right) \text{ and } y_\tau = ln\left(\frac{m_t^{t^*}}{S}\right)$$ ...[B.12]

<u>Case 1: Up Barrier</u>

Let T_a stand for the time the underlying asset price first reached the up barrier U. Then, the following probability equations always hold:

$$P(T_a > \tau) = P\left(M_t^{t^*} < U\right) = P(Y\tau < a) \quad \text{...[B.13]}$$

$$P(T_a \leq \tau) = P\left(M_t^{t^*} \geq U\right) = P(Y\tau \geq a) \quad \text{...[B.14]}$$

where $a = \ ln\left(\frac{U}{S}\right)$.

In simple terms, the first equation follows from the fact that if the asset price reached U for the first time after the option's expiry, then its maximum $M_t^{t^*}$during the option's life must have been less than the barrier. Similarly, in case of the second equation, if the asset price reached U for the first time before the option's expiry, then its maximum $M_t^{t^*}$during the option's life must have exceeded the barrier.

<u>Case 1(a): The Up Barrier is Not Touched</u>

Using the above and some well-known results of Brownian motion, we can get the following joint cumulative density of the underlying's return and the transformed maximum log return.

$$F(X_\tau \leq x, Y_\tau < a) = N\left(\frac{x-v\tau}{\sigma\sqrt{\tau}}\right) - \mathrm{e}^{2av/\sigma^2} N\left(\frac{x-2a-v\tau}{\sigma\sqrt{\tau}}\right) \quad \text{...[B.15]}$$

The above equation can be interpreted as the cumulative density of the underlying's log return X_τ conditional on the fact that the up barrier is never touched during the option's life. Here $v = r - \sigma^2/2$.

When differentiated w.r.t x, we obtain the restricted density function of the underlying's log return subject to the barrier never being reached. We get the following.

$$\phi(x \mid Y_\tau < a) = f(x) - \mathrm{e}^{2av\tau/\sigma^2} f(x - 2a) \quad \text{...[B.16]}$$

or

$$\boxed{\phi(x \mid Y_\tau < a) = \begin{cases} f(x) - \left(\frac{U}{S}\right)^{\frac{2v}{\sigma^2}} f(x-2a) \ for \ x < a \\ 0 \ for \ x \geq a \end{cases}} \quad \text{...[B.17]}$$

where $f(x)$ is the unrestricted density function (i.e. the normal density) of the log-return of the underlying.

<u>Case 1(b): The Up Barrier is Touched</u>

The density for this case can be obtained with the following identity which is quite intuitive.

$$\phi(x \mid Y_\tau \geq a) + \phi(x \mid Y_\tau < a) = f(x) \quad \text{...[B.18]}$$

Thus, the density function of the underlying's log return subject to the up barrier being touched is given by

$$\boxed{\phi(x \mid Y_\tau \geq a) = \begin{cases} \left(\frac{U}{S}\right)^{\frac{2v}{\sigma^2}} f(x-2a) \; for \; x < a \\ f(x) \; for \; x \geq a \end{cases}} \quad \text{...[B.19]}$$

Case 2: Down Barrier

In case of a down barrier L, the probability equations [B.13] and [B.14] change to

$$P(T_a > \tau) = P\left(m_t^{t^*} > L\right) = P(y_\tau > b) \quad \text{...[B.20]}$$

$$P(T_a \leq \tau) = P\left(m_t^{t^*} \leq L\right) = P(y_\tau \leq b) \quad \text{...[B.21]}$$

where $b = \ ln\left(\frac{L}{S}\right)$.

Case 2(a): The Down Barrier is Not Touched

What follows is the cumulative density of the underlying's log return X_τ conditional on the fact that the down barrier is never touched during the option's life.

$$F(X_\tau \geq x, y_\tau > b) = N\left(\frac{-x+v\tau}{\sigma\sqrt{\tau}}\right) - \mathrm{e}^{2bv/\sigma^2} N\left(\frac{-x+2b+v\tau}{\sigma\sqrt{\tau}}\right) \quad \text{...[B.22]}$$

Using the above, we can obtain the density function of the underlying's log return subject to the barrier never being touched.

$$\boxed{\phi(x \mid y_\tau > b) = \begin{cases} f(x) - \left(\frac{L}{S}\right)^{\frac{2v}{\sigma^2}} f(x-2b) \; for \; x > b \\ 0 \; for \; x \leq b \end{cases}} \quad \text{...[B.23]}$$

Case 2(b): The Down Barrier is Touched

Following a similar approach to the one used in Case 1(b), the density function of the underlying's log return subject to the down barrier being touched is given by

$$\boxed{\phi(x \mid y_\tau \leq b) = \begin{cases} \left(\frac{L}{S}\right)^{\frac{2v}{\sigma^2}} f(x-2b) \; for \; x > b \\ f(x) \; for \; x \leq b \end{cases}} \quad \text{...[B.24]}$$

Pricing Equations for Knock-In Options

Down-and-In Call

Case 1: Assume Barrier < Strike

The expected payoff of a down-and-in call assuming zero rebate is as follows.

$$E(Payoff\ DI\ Call \mid S(t) > H\ and\ S(T) \leq H\ for\ some\ t < T \leq t^*)$$
$$= \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} . \left\{ S\left(\frac{H}{S}\right)^2 . e^{r\tau} . N\left(d_{1BS}\left(\frac{H^2}{S}, K\right)\right) - K.N\left(d_{BS}\left(\frac{H^2}{S}, K\right)\right)\right\}$$

Note that the above expected payoff uses the restricted density function [B.24] instead of the unrestricted density (normal distribution).

The value of the down-and-in call with zero rebate, if the barrier is reached and if $H < K$ is simply the expected value of the above payoff discounted at the risk-free rate

$$V_{DI\ Call} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} . \left\{\frac{H^2}{S} . N\left(d_{1BS}\left(\frac{H^2}{S}, K\right)\right) - K.e^{-r\tau}.N\left(d_{BS}\left(\frac{H^2}{S}, K\right)\right)\right\} \quad ...[B.25]$$

or

$$V_{DI\ Call} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} . C_{BS}\left(\frac{H^2}{S}, K\right) \quad ...[B.26]$$

<u>Case 2: Assume Barrier > Strike</u>

In this case, the entire range (K, ∞) needs to be divided into (K, H) and (H, ∞) because the density functions applicable to the two ranges are different.

Over (H, ∞)

$$V_{DI\ Call}\ over\ (H, \infty) = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} . \left\{C_{BS}\left(\frac{H^2}{S}, K\right) + (H - K).e^{-r\tau}.N\big(d_2(H, S)\big)\right\} \quad ...[B.27]$$

For the range (K, H), we obtain the value of the down-and-in call as a difference of its value over the ranges $(-\infty, H)$ and $(-\infty, K)$.

$$V_{DI\ Call}\ over\ (K, H) = P_{BS}(S, K) - P_{BS}(S, H) + (H - K)\mathrm{e}^{-r\tau}N[-d_{BS}(S, H)] \quad ...[B.28]$$

where $P_{BS}(m, n)$ denotes the value of a put option with underlying price m and strike n. Under this case, therefore, the value of the down-and-in call option is the sum of equations [B.27] and [B.28].

<u>Combining Cases 1 and 2</u>

The above two cases can be combined as shown below.

$$\boxed{\begin{aligned} V_{DI\ Call} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} \left(C_{BS}\left[\frac{H^2}{S}, max(H, K)\right] + [max(H, K) - K]e^{-r\tau}N\left\{d_{BS}\left[\frac{H^2}{S}, max(H, K)\right]\right\}\right) \\ + \{P_{BS}(S, K) - P_{BS}(S, H) + [H - K]e^{-r\tau}N[-d_{bs}[S, H]\}B_{H>K} \end{aligned}}$$

...[B.29]

where $B_{H>K}$ is a digital number which takes a value of 1 when $H > K$ and 0 otherwise.

<u>Up-and-In Call</u>

Using a procedure on the same lines as above while switching the restricted density function for the up-barrier, we can write the value of the up-and-in call as follows.

$$V_{UI\ Call} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} \left\{P_{BS}\left(\frac{H^2}{S}, K\right) - P_{BS}\left(\frac{H^2}{S}, H\right) + (H-K)e^{-r\tau}N[-d_{BS}(H,S)]\right\} B_{H>K} + [C_{BS}[S, max(H,K)] + [max(H,K) - K]e^{-r\tau}N\{d_{BS}[S, ma\,x(H,K)]\}]$$

...[B.30]

Down-and-In Put

The pricing equation for a down-and-in put can be derived using its pricing equation's symmetry with the up-and-in call pricing equation.

This is done by replacing in equation [B.30] the prices of all calls with corresponding puts, $max(H, K)$ with $min(H, K)$, the digital number $B_{H>K}$ with $B_{K>H}$ and d_{BS} with $-d_{BS}$. Therefore, we get

$$V_{DI\ Put} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} \left\{C_{BS}\left(\frac{H^2}{S}, K\right) - C_{BS}\left(\frac{H^2}{S}, H\right) - (H-K)e^{-r\tau}N[d_{BS}(H,S)]\right\} B_{H>K} + [P_{BS}[S, min(H,K)] - [min(H,K) - K]e^{-r\tau}N\{-d_{BS}[S, min(H,K)]\}]$$

...[B.31]

Up-and-In Put

As done above, we use the up-and-in put pricing equation's symmetry with that of a down-and-in call to obtain the following.

$$V_{UI\ Put} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} \left(P_{BS}\left[\frac{H^2}{S}, min(H,K)\right] - [min(H,K) - K]e^{-r\tau}N\left\{-d_{BS}\left[\frac{H^2}{S}, min(H,K)\right]\right\}\right) + \{C_{BS}(S,K) - C_{BS}(S,H) - [H-K]e^{-r\tau}N[d_{bs}[S,H]\}B_{K>H}$$

...[B.32]

Pricing Equations for Knock-Out Options

Down-and-Out Call

Such an option yields a European call's payoff if the barrier is not touched and a rebate if it is.

In the absence of a rebate, we simply discount the expected payoff calculated based on equation [B.8] using the restricted density functions in equations [B.23] and [B.24] for a down barrier. We get

$$V_{DO\ Call} = C_{BS}[S, max(H,K)] - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} C_{BS}\left[\frac{H^2}{S}, max(H,K)\right] + [max(H,K) - K]e^{-r\tau}\left(N\{d_{BS}[S, max(H,K)]\} - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} N\left\{d_{BS}\left[\frac{H^2}{S}, max(H,K)\right]\right\}\right)$$

...[B.33]

Up-and-Out Call

By discounting the expected payoff using the relevant restricted density, the pricing equation for an up-and-out call is

$$V_{UO\ Call} = B_{H>K}\left\{C_{BS}(S,K) - C_{BS}(S,H) - (H-K)\mathrm{e}^{-r\tau}N[d_{BS}(S,H)] - \left(\frac{H}{S}\right)^{2v/\sigma^2}\left[C_{BS}\left(\frac{H^2}{S},K\right) - C_{BS}\left(\frac{H^2}{S},H\right) - (H-K)\mathrm{e}^{-r\tau}N[d_{BS}(H,S)]\right]\right\}$$

…[B.34]

Down-and-Out Put

We use the down-and-out put pricing equation's symmetry with that of an up-and-out call to obtain the following.

$$V_{DO\ Put} = B_{K>H}\left\{P_{BS}(S,K) - P_{BS}(S,H) + (H-K)\mathrm{e}^{-r\tau}N[-d_{BS}(S,H)] - \left(\frac{H}{S}\right)^{2v/\sigma^2}\left[P_{BS}\left(\frac{H^2}{S},K\right) - P_{BS}\left(\frac{H^2}{S},H\right) + (H-K)\mathrm{e}^{-r\tau}N[-d_{BS}(H,S)]\right]\right\}$$

…[B.35]

Up-and-Out Put

Once again, we use the up-and-out put pricing equation's symmetry with that of an down-and-out call to obtain the following.

$$V_{UO\ Put} = P_{BS}[S,\min(H,K)] - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} P_{BS}\left[\frac{H^2}{S}, min(H,K)\right] - [min(H,K) - K]e^{-r\tau}\left(N\{-d_{BS}[S,min(H,K)]\} - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} N\left\{-d_{BS}\left[\frac{H^2}{S}, min(H,K)\right]\right\}\right)$$

…[B.36]

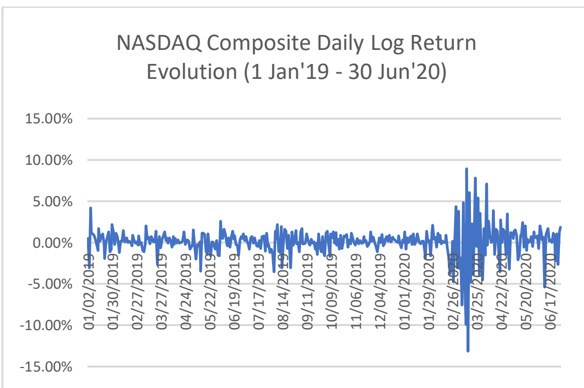
Pricing for an Up-and-In (Call/ Put) Option at 10,000 and Down-and-Out (Call/ Put) Option at 9000

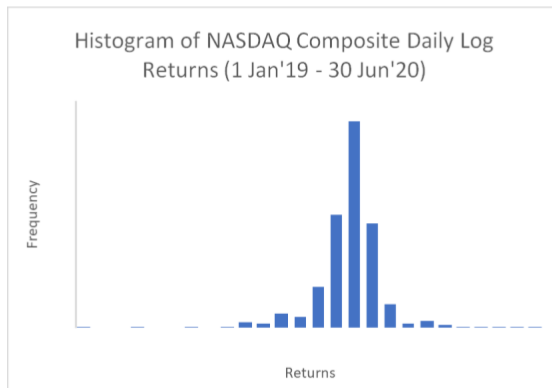
Assumptions

- We are pricing Barrier options where the payoff is that of a vanilla European call/ put
- We assume zero rebate, in case the holder loses his/ her right to the option payoff
- All standard assumptions of Black-Scholes theory hold:
  - Lognormality of asset prices/ normality of log returns (ref. chart below)

- ○ Constant volatility in the underlying's returns
- ○ Absence of dividends
- ○ Known and constant risk-free rate
- ○ No transaction costs (perfect markets)
- ○ Fractional borrowing of the price of a security
- ○ Trading in continuous time

- Since the hedge fund wants to buy a barrier option in the midst of market volatility due to COVID-19, we assume that the option contract starts at 1 Jan'20 and expires after 6 months, since return volatility was quite high during this period, as shown in the chart below





*Inputs**

$S = 9092.19$ (index value on 2 Jan'20)**
$K = 9092.19$ (assumed)
$H = 10{,}000$ (up-and-in) or 9,000 (down-and-out)
$\sigma = 44.87\%\ p.a.$ (H1 2020 annualised volatility)
$\tau = 0.5$ (6 months time to expiration)
$r = 1.57\%\ p.a.$ (6-month UST as on 2 Jan'20 source)

**Refer to attached Excel file for calculations of some of the above inputs*
***We did not take the index value of 8972.61 on 1 Jan'19, the date of contract inception, since that would have led to negative valuations for down-and-out calls and puts.*

Up-and-In Call Pricing at H = 10,000

Pricing Equation

$$V_{UI\ call} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} \left\{ P_{BS}\left(\frac{H^2}{S}, K\right) - P_{BS}\left(\frac{H^2}{S}, H\right) + (H-K)e^{-r\tau} N[-d_{BS}(H,S)] \right\} B_{H>K} + [C_{BS}[S, max(H,K)] + [max(H,K) - K]e^{-r\tau} N\{d_{BS}[S, ma\,x(H,K)]\}]$$

Calculation *(refer to attached Excel file for details)*

$$V_{UI\ call} = \left(\frac{10000}{9092.19}\right)^{\frac{2(-0.085)}{0.4487^2}} \{508.52 - 848.39 + (10000 - 9092.19)e^{-0.0157\times 0.5} \times 0.4341\}(B_{H>K} = 1) + [828.21 + [10000 - 9092.19]e^{-0.0157\times 0.5} \times 0.3322]$$

$V_{UI\ Call} = 1174.56$

Up-and-In Put Pricing at H = 10,000

Pricing Equation

$$V_{UI\ Put} = \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}}\left(P_{BS}\left[\frac{H^2}{S}, min(H,K)\right] - [min(H,K) - K]e^{-r\tau}N\left\{-d_{BS}\left[\frac{H^2}{S}, min(H,K)\right]\right\}\right) + \{C_{BS}(S,K) - C_{BS}(S,H) - [H-K]e^{-r\tau}N[d_{bs}[S,H]\}B_{K>H}$$

Calculation *(refer to attached Excel file for details)*

$V_{UI\ Put} = \left(\frac{10000}{9092.19}\right)^{\frac{2(-0.085)}{0.4487^2}}(508.52 - [9092.19 - 9092.19]e^{-0.0157\times0.5} \times 0.3206) + 0$ [since $B_{K>H} = 0$]

$V_{UI\ Put} = 469.27$

Down-and-Out Call Pricing at H = 9,000

Pricing Equation

$$V_{DO\ Call} = C_{BS}[S, max(H,K)] - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} C_{BS}\left[\frac{H^2}{S}, max(H,K)\right] + [max(H,K) - K]e^{-r\tau}\left(N\{d_{BS}[S, max(H,K)]\} - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} N\left\{d_{BS}\left[\frac{H^2}{S}, max(H,K)\right]\right\}\right)$$

Calculation *(refer to attached Excel file for details)*

$$V_{DO\ Call} = 1177.46 - \left(\frac{9000}{9092.19}\right)^{\frac{2(-0.085)}{0.4487^2}} \times 1074.70 + [9092.19 - 9092.19]e^{-0.0157\times0.5}\left(N\{d_{BS}[S, max(H,K)]\} - \left(\frac{H}{S}\right)^{\frac{2v}{\sigma^2}} N\left\{d_{BS}\left[\frac{H^2}{S}, max(H,K)\right]\right\}\right)$$

$V_{DO\ Call} = 93.47$

Down-and-Out Put Pricing at H = 9,000

Pricing Equation

$$V_{DO\ Put} = B_{K>H}\left\{P_{BS}(S,K) - P_{BS}(S,H) + (H-K)\mathrm{e}^{-r\tau}N[-d_{BS}(S,H)] - \left(\frac{H}{S}\right)^{2v/\sigma^2}\left[P_{BS}\left(\frac{H^2}{S},K\right) - P_{BS}\left(\frac{H^2}{S},H\right) + (H-K)\mathrm{e}^{-r\tau}N[-d_{BS}(H,S)]\right]\right\}$$

Calculation *(refer to attached Excel file for details)*

$$V_{DO\ Put} = (B_{K>H} = 1)\left\{1106.36 - 1056.34 + (9000 - 9092.19)e^{-0.0157\times 0.5} \times 0.5405 - \left(\frac{9000}{9092.19}\right)^{\frac{2(-0.085)}{0.4487^2}}[1187.05 - 1134.71 + (9000 - 9092.19)e^{-0.0157\times 0.5} \times 0.5659]\right\}$$

$$V_{DO\ Put} = 0.0004$$

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