AE353 (Spring 2021)

Day 18

T. Bretl

Acker

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Controllability

Controllable Canonical Form (CCF)

$$A = \begin{bmatrix} \begin{bmatrix} -a_1 & \cdots & -a_n \end{bmatrix} \\ \begin{bmatrix} I_{(n-1)\times(n-1)} \end{bmatrix} \begin{bmatrix} 0_{(n-1)\times 1} \end{bmatrix} \end{bmatrix} \qquad B = \begin{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \\ \begin{bmatrix} 0_{(n-1)\times 1} \end{bmatrix} \end{bmatrix}$$

Facts

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

$$A - BK = \begin{bmatrix} \begin{bmatrix} -a_1 - k_1 & \cdots & -a_n - k_n \end{bmatrix} \\ \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \end{bmatrix}$$

$$\det(sI - (A - BK)) = s^n + (a_1 + k_1) s^{n-1} + \cdots + (a_{n-1} + k_{n-1}) s + (a_n + k_n)$$

Consequence

if you want $\quad s^n + r_1 s^{n-1} + \cdots + r_{n-1} s + r_n$

no symbolic computation!

then $\quad k_1 = r_1 - a_1 \quad \cdots \quad k_n = r_n - a_n$

If we could put a system in CCF...

$$\dot{x} = Ax + Bu$$

$$\boxed{x = Vz} \leftarrow z = V^{-1}x$$

$$\dot{z} = A_{CCF}\, z + B_{CCF}\, u$$

$$\dot{x} = Ax + Bu$$

$$V\dot{z} = AVz + Bu$$

$$\dot{z} = V^{-1}AVz + V^{-1}Bu$$

Then ...

$$u = -\overbrace{K_{CCF}}^{\text{easy to find}}\, z$$

$$= -\underbrace{\qquad\qquad}_{K \text{ (what we want)}}\, x$$

How?

$$\underbrace{A_{CCF} = V^{-1} A V \qquad B_{CCF} = V^{-1} B}_{\text{solve for } V^{-1}}$$

(that's what we need to find K given K_{CCF})

$B_{CCF} =$

$A_{CCF} B_{CCF} =$

$A_{CCF}^2 B_{CCF} =$

$\vdots$

$A_{CCF}^{n-1} B_{CCF} =$

$$\underbrace{\begin{bmatrix} B_{CCF} & A_{CCF} B_{CCF} & A_{CCF}^2 B_{CCF} & \cdots & A_{CCF}^{n-1} B_{CCF} \end{bmatrix}}_{W_{CCF}} = V^{-1} \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{W}$$

$$\boxed{V^{-1} = \qquad\qquad}$$

works as long as is invertible

ACKERMANN'S METHOD

- Compute the characteristic equation that we want:

$$(s-p_1)\cdots(s-p_n) = s^n + r_1 s^{n-1} + \cdots + r_{n-1}s + r_n$$

- Compute the characteristic equation that we have:

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

- Compute the controllability matrix of the original system (and check that $\det(W) \neq 0$):

$$W = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

- Compute the controllability matrix of the transformed system:

$$W_{\text{ccf}} = \begin{bmatrix} B_{\text{ccf}} & A_{\text{ccf}}B_{\text{ccf}} & \cdots & A_{\text{ccf}}^{n-1}B_{\text{ccf}} \end{bmatrix}$$

where

$$A_{\text{ccf}} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_{\text{ccf}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Compute the gains for the transformed system:

$$K_{\text{ccf}} = \begin{bmatrix} r_1 - a_1 & \cdots & r_n - a_n \end{bmatrix}$$

- Compute the gains for the original system:

$$K = K_{\text{ccf}} W_{\text{ccf}} W^{-1}$$

The system

$$\dot{x} = Ax + Bu$$

is controllable if

$$W = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

"n" is the number of states (i.e., the size of x)

has $\underbrace{\text{full rank}}$.

in python this means "np.linalg.matrix_rank(W)" is the same as "n"

if there is only one input, W is square, and so "full rank" and "invertible" mean the same thing — so, for a system with only one input, you can simply check if $\det(W) \neq 0$