# BP + OSD Algorithm for Quantum Error Correction

Lecture Note based on arXiv:2005.07016

*Roffe, White, Burton, and Campbell (2020)*

## Contents

# 1 Introduction

## 1.1 Overview

This lecture note introduces the **BP+OSD decoder** for quantum error correction:

- **BP** = Belief Propagation (a classical decoding algorithm)
- **OSD** = Ordered Statistics Decoding (a post-processing technique)

Together, BP+OSD provides a general-purpose decoder for **quantum LDPC codes** (Low-Density Parity Check codes).

## 1.2 Learning Objectives

By the end of this note, you will understand:

1. How classical error correction codes work
2. The Belief Propagation algorithm for decoding
3. Why BP fails for quantum codes (the degeneracy problem)
4. How OSD fixes the degeneracy problem
5. The complete BP+OSD decoding algorithm

# 2 Classical Error Correction

## 2.1 Linear Codes

All arithmetic in this note is performed in **binary** (modulo 2):

- $0 + 0 = 0, \quad 1 + 0 = 0 + 1 = 1, \quad 1 + 1 = 0$
- This is also written as XOR: $a \oplus b = (a + b) \bmod 2$
- Vectors and matrices use element-wise mod-2 arithmetic

## 2.2 Hamming Weight and Distance

> **Definition.** The **Hamming weight** of a binary vector $\boldsymbol{v}$ is the number of 1s it contains $|\boldsymbol{v}| = \sum_i v_i$. The **Hamming distance** between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is the number of positions where they differ:$d(\boldsymbol{u}, \boldsymbol{v}) = |\boldsymbol{u} + \boldsymbol{v}|$.

For example, for $\boldsymbol{v} = (1, 0, 1, 1, 0)$: Hamming weight $|\boldsymbol{v}| = 3$ and for $\boldsymbol{u} = (1, 1, 0, 1, 0)$ and $\boldsymbol{v} = (1, 0, 1, 1, 0)$: $\boldsymbol{u} + \boldsymbol{v} = (0, 1, 1, 0, 0)$ and $d(\boldsymbol{u}, \boldsymbol{v}) = 2$.

> **Definition.** An $[\boldsymbol{n}, \boldsymbol{k}, \boldsymbol{d}]$ **linear code** $\mathcal{C}$ is a set of binary vectors (called **codewords**) where $n$ is the **block length** (number of bits in each codeword), $k$ is the **dimension** (number of information bits encoded), and $d$ is the **minimum distance** (minimum Hamming weight among non-zero codewords). The **rate** of the code is $R = k/n$.

A linear code can be defined by an $m \times n$ **parity check matrix** $H$. $H_{ij}$ denotes the entry in row $i$, column $j$ of matrix $H$. $m$ is the number of rows in $H$ (number of parity checks), $n$ is the number of columns in $H$ (number of bits), and $\operatorname{rank}(H)$ is the number of linearly independent rows. By the rank-nullity theorem: $k = n - \operatorname{rank}(H)$.

For example, The $[\mathbf{3}, \mathbf{1}, \mathbf{3}]$ **repetition code** encodes 1 bit into 3 bits by triplication.

Parity check matrix:

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{1}$$

Verification: $H \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓ and $H \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓

So the codewords are: $\mathcal{C}_H = \{(0, 0, 0), (1, 1, 1)\}$

Parameters: $n = 3$ bits, $k = 3 - 2 = 1$ info bit, $d = 3$ (weight of $(1, 1, 1)$)

## 2.3 Error Model and Syndrome

**Definition.** In the **binary symmetric channel** (BSC) with error probability $p$:

- Each bit is independently flipped with probability $p$
- Original codeword: $\boldsymbol{c}$
- Error pattern: $\boldsymbol{e}$ (a binary vector, $e_i = 1$ means bit $i$ flipped)
- Received word: $\boldsymbol{r} = \boldsymbol{c} + \boldsymbol{e}$

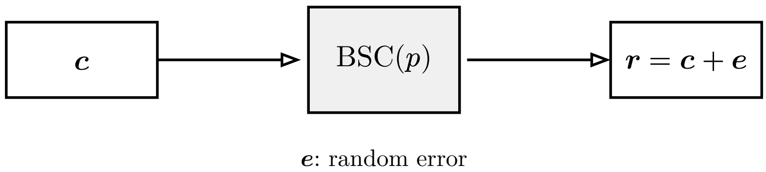



Figure 1: Binary symmetric channel model

**Definition.** The **syndrome** of a received word $\boldsymbol{r}$ is:

$$\boldsymbol{s} = H \cdot \boldsymbol{r} \tag{2}$$

Since $H \cdot \boldsymbol{c} = \boldsymbol{0}$ for any codeword, we have:

$$\boldsymbol{s} = H \cdot \boldsymbol{r} = H \cdot (\boldsymbol{c} + \boldsymbol{e}) = H \cdot \boldsymbol{c} + H \cdot \boldsymbol{e} = \boldsymbol{0} + H \cdot \boldsymbol{e} = H \cdot \boldsymbol{e} \tag{3}$$

**Key Point.** The syndrome depends **only on the error**, not on which codeword was sent! This is what makes syndrome-based decoding possible.

## 2.4 The Decoding Problem

Given: Parity check matrix $H$ and syndrome $\boldsymbol{s} = H \cdot \boldsymbol{e}$

Find: The most likely error $\boldsymbol{e}^*$ that could have produced $\boldsymbol{s}$

**Definition. Maximum likelihood decoding** finds:

$$\boldsymbol{e}^* = \arg \min_{\boldsymbol{e}: H \cdot \boldsymbol{e} = \boldsymbol{s}} |\boldsymbol{e}| \tag{4}$$

That is, the minimum Hamming weight error consistent with the syndrome.

## 2.5 Probabilistic Graphical Models

Before introducing the Belief Propagation algorithm, we need to understand how probabilistic inference problems can be represented as graphs.

**Definition.** A **probabilistic graphical model** (PGM) is a graph-based representation of a probability distribution. Nodes represent random variables, and edges encode conditional independence relationships. PGMs enable efficient inference algorithms by exploiting the structure of the distribution.

There are two main families of PGMs:

- **Directed graphical models** (Bayesian networks): edges have direction, representing causal relationships
- **Undirected graphical models** (Markov networks): edges are undirected, representing symmetric dependencies

For error correction, we use undirected models because parity constraints are symmetric — no variable "causes" another.

## 2.6 Undirected Probabilistic Graphical Models

**Definition.** An **undirected probabilistic graphical model** (also called a **Markov network** or **Markov random field**) represents a joint probability distribution as:

$$P(\boldsymbol{x}) = \frac{1}{Z} \prod_{c \in \mathcal{C}} \psi_c(\boldsymbol{x}_c) \tag{5}$$

where:

- $\boldsymbol{x} = (x_1, ..., x_n)$ are random variables
- $\mathcal{C}$ is a set of **cliques** (fully connected subgraphs)
- $\psi_c(\boldsymbol{x}_c)$ is a **potential function** over variables in clique $c$
- $Z = \sum_{\boldsymbol{x}} \prod_c \psi_c(\boldsymbol{x}_c)$ is the **partition function** (normalization constant)

**Key Point.** The UAI format mentioned in the getting started guide represents exactly this structure: variables (detectors), cliques (error mechanisms), and potential functions (error probabilities).

For binary error correction with syndrome $\boldsymbol{s}$, we want to compute:

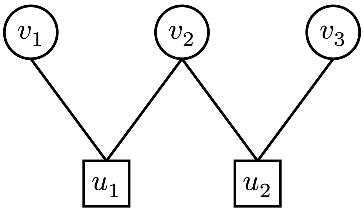
$$P(\boldsymbol{e} \mid \boldsymbol{s}) \propto \prod_c \psi_c(\boldsymbol{e}_c) \tag{6}$$

where each potential $\psi_c$ encodes a parity constraint.

## 2.7 Factor Graphs

To understand the Belief Propagation algorithm, we need the concept of **factor graphs**.

**Definition.** A **factor graph** is a bipartite graph $G = (V, U, E)$ representing the parity check matrix $H$. The **data nodes** are set $V = \{v_1, v_2, ..., v_n\}$ such that each node $v_j$ corresponds to each column of $H$. A **parity nodes** are set $U = \{u_1, u_2, ..., u_m\}$ such that each node $u_i$ corresponds to each row of $H$. An **edges** $E = \{(v_j, u_i) : H_{ij} = 1\}$ connects $v_j$ to $u_i$ exists if $H_{ij} = 1$.




Factor graph for $H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

Figure 2: Factor graph for the $[3, 1, 3]$ repetition code with node conventions

The **neighborhoods** of nodes are defined as: $V(u_i) = \{v_j : H_{ij} = 1\}$.

## 2.8 Comparing Graph Representations

Three related graph representations appear in coding theory and probabilistic inference:

**Definition. Comparison of graph representations:**

1. **Undirected Probabilistic Graphical Model (Markov Network)**:
   - Nodes = random variables
   - Edges = direct dependencies between variables
   - Cliques = groups of mutually dependent variables
   - Represents: $P(\boldsymbol{x}) = \frac{1}{Z} \prod_c \psi_c(\boldsymbol{x}_c)$
2. **Factor Graph**:
   - Two types of nodes: variable nodes AND factor nodes
   - Bipartite structure: edges only between variables and factors
   - Explicitly represents factorization of the distribution
   - Represents: $P(\boldsymbol{x}) = \frac{1}{Z} \prod_i f_i\left(\boldsymbol{x}_{N(i)}\right)$
3. **Tanner Graph**:
   - A special case of factor graph for error correction codes
   - Variable nodes = bits (columns of $H$)
   - Factor nodes = parity checks (rows of $H$)
   - Represents: parity check matrix $H$ structure

**Key Point. Key relationships:**

- Factor graphs are a **bipartite refinement** of Markov networks that make the factorization explicit
- Tanner graphs are factor graphs **specialized for linear codes** where factors represent parity constraints
- All three represent the same probability distribution, but factor graphs enable more efficient message-passing algorithms
- The UAI format represents Markov networks (cliques and potentials), while BP operates on the factor graph representation

| Property | Markov Network | Factor Graph | Tanner Graph |
|---|---|---|---|
| Node types | Variables only | Variables + Factors | Bits + Checks |
| Graph structure | General | Bipartite | Bipartite |
| Edges represent | Dependencies | Factor membership | Parity constraints |
| Used for | General inference | Message passing | Code decoding |
| BP efficiency | Less efficient | Efficient | Efficient |

Table 1: Comparison of graph representations

**Why use factor graphs for BP?** The bipartite structure of factor graphs makes message passing natural:

- Messages flow between variables and factors
- Each factor collects evidence from its variables
- Each variable aggregates information from its factors
- No need to handle complex clique structures

For error correction, the Tanner graph (factor graph) representation is ideal because:

- Parity checks are naturally factors (XOR constraints)
- Bits are naturally variables (error indicators)
- The sparse structure ($H$ has few 1s) gives efficient $O(n)$ BP iterations

**Definition.** An **$(l, q)$-LDPC code** is a linear code whose parity check matrix $H$ satisfies:

- Each column has at most $l$ ones (each bit is in at most $l$ checks)
- Each row has at most $q$ ones (each check involves at most $q$ bits)

The matrix $H$ is called **sparse** because $l$ and $q$ are small constants independent of $n$.

**Key Point.** LDPC codes are important because their sparse structure enables efficient decoding via Belief Propagation with complexity $O(n)$ per iteration.

# 3 Belief Propagation Decoder

## 3.1 Introduction and Motivation

The rediscovery of Low-Density Parity-Check (LDPC) codes in the late 1990s marked a paradigm shift in coding theory, transitioning from algebraic decoding algorithms to probabilistic iterative decoding that approaches the Shannon limit [1]. Central to this revolution is the **Belief Propagation** (BP) algorithm [2], a message-passing protocol that operates on the graphical representation of codes.

**Key Point. BP in Modern Communications:** BP decoding powers critical communication standards:

- Wi-Fi (IEEE 802.11n/ac/ax)
- Satellite communication (DVB-S2)
- 5G New Radio

While its practical efficacy is undisputed, the mathematical rigor underlying its convergence behavior involves multiple theoretical frameworks: Density Evolution for asymptotic analysis, Bethe Free Energy for variational optimization, and trapping set theory for failure mechanisms.

The convergence of BP is understood through different lenses depending on the regime. In the asymptotic limit of infinite block length, convergence is probabilistic and governed by Density Evolution [3]. In finite-length regimes, convergence is variational, linked to minimization of the Bethe Free Energy [4]. However, combinatorial substructures known as trapping sets can arrest decoding, creating error floors [5].

## 3.2 The Message Passing Mechanism

**Belief Propagation** (BP), also called the **sum-product algorithm**, is an iterative message-passing algorithm on the factor graph.

**Definition.** The goal of BP is to compute, for each bit $j$, the **marginal probability**:

$$P_1(e_j) = P(e_j = 1 \mid \boldsymbol{s}), \tag{7}$$

given $\boldsymbol{s} = H \cdot \boldsymbol{e}$ is the syndrome of the error $\boldsymbol{e}$. This is called a **soft decision** – it tells us how likely each bit is to be flipped.

We use the following notation throughout:

- $p$ as the channel error probability (probability each bit flips)
- $m_{v_j \to u_i}$ as the message from data node $v_j$ to parity node $u_i$
- $m_{u_i \to v_j}$ as the message from parity node $u_i$ to data node $v_j$
- Messages represent beliefs about whether $e_j = 1$

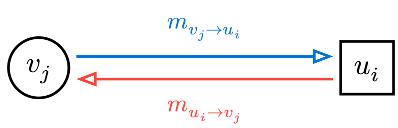



Figure 3: Messages passed between data and parity nodes

The BP-algorithm requires the quantification of how certain we are about the bit being flipped or not. This is done using **log-likelihood ratios** (LLR).

**Definition.** Instead of probabilities, BP uses **log-likelihood ratios** (LLR) for numerical stability:

$$\mathrm{LLR}(e_j) = \log \frac{P(e_j = 0)}{P(e_j = 1)} \tag{8}$$

- $\mathrm{LLR} > 0$ means $e_j = 0$ is more likely (bit probably correct)
- $\mathrm{LLR} < 0$ means $e_j = 1$ is more likely (bit probably flipped)
- $|\mathrm{LLR}|$ indicates confidence level

For the channel with error probability $p$, the **channel LLR** is:

$$p_l = \log \frac{1-p}{p} \tag{9}$$

Since $p < 0.5$ in practice, we have $p_l > 0$.

### 3.3 BP Algorithm: Step-by-Step

**Step 1: Initialization**

Set the channel LLR:

$$p_l = \log \frac{1-p}{p} \tag{10}$$

Initialize all messages from data nodes to parity nodes with the channel prior:

$$m_{v_j \to u_i} := p_l \quad \text{for all edges } (v_j, u_i) \tag{11}$$

**Why?** Before any message passing, the only information we have about each bit is from the **channel itself**. Since each bit flips independently with probability $p$, the initial belief is simply the channel's prior: "this bit is probably correct" (because $p < 0.5$, so $p_l > 0$).

**Code:** The implementation adds small epsilon values for numerical stability:

```
# Initialize channel LLRs
channel_llr = torch.log((1 - channel_probs + 1e-10) / (channel_probs + 1e-10))

# Initialize qubit-to-check messages with channel prior
msg_q2c = channel_llr[qubit_edges].unsqueeze(0).expand(batch_size, -1).clone()
msg_c2q = torch.zeros(batch_size, num_edges, device=device)
```

### Step 2: Parity-to-Data Messages

Each parity node $u_i$ sends a message to each connected data node $v_j$:

- **Min-sum form:**

$$m_{u_i \to v_j} = (-1)^{s_i} \cdot \alpha \cdot \prod_{v_j' \in V(u_i) \setminus v_j} \operatorname{sign}\left(m_{v_j' \to u_i}\right) \cdot \min_{v_j' \in V(u_i) \setminus v_j} |m_{v_j' \to u_i}| \tag{12}$$

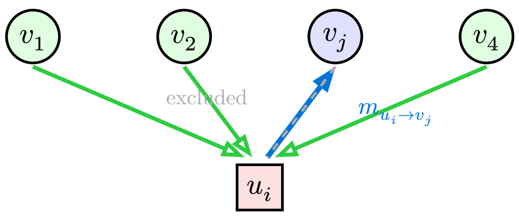
- **Sum-Product form:**

$$m_{u_i \to v_j} = (-1)^{s_i} \cdot 2 \tanh^{-1}\left( \prod_{v_j' \in V(u_i) \setminus v_j} \tanh\left(\frac{m_{v_j' \to u_i}}{2}\right)\right) \tag{13}$$

Where:

- $s_i$ = the $i$-th syndrome bit (given as input, either 0 or 1)
- $V(u_i) \setminus v_j$ = all neighbors of $u_i$ except $v_j$ (defined in Section 3.5)
- $\operatorname{sign}(x) = +1$ if $x \geq 0$, else $-1$
- $\alpha = 1 - 2^{-t}$ is a **damping factor** at iteration $t$ (helps convergence)

**Why?** A parity check enforces that XOR of all connected bits equals the syndrome bit $s_i$. The check node tells $v_j$: "Based on what I know about the **other** bits, here's how likely **you** are to be flipped." If $s_i = 0$, the parity check says "even number of flipped bits." If the other bits all look correct (positive LLR), then $v_j$ should also be correct. If one other bit looks flipped (negative LLR), then $v_j$ should be correct to maintain even parity. The formula computes this XOR-like logic in LLR form.




Check node collects info from $v_1, v_2, v_4$
to compute message to $v_j$ (excluding $v_j$'s own message)

Figure 4: Parity-to-data message: $u_i$ uses info from all neighbors **except** $v_j$ to tell $v_j$ what it should be

**Code:** The implementation computes signs and magnitudes separately, using a sorting trick to efficiently find the minimum excluding each edge:

```
def _check_to_qubit_minsum(self, msg_q2c, syndromes):
    for c in range(num_checks):
        edges = self.check_to_edges[c]
        incoming = msg_q2c[:, edges]  # (batch, degree)

        # Separate signs and magnitudes
        signs = torch.sign(incoming)  # (batch, degree)
        mags = torch.abs(incoming)    # (batch, degree)

        # Product of all signs
        total_sign = torch.prod(signs, dim=1, keepdim=True)
```

```python
        # Apply syndrome: flip sign if syndrome is 1
        syndrome_sign = 1 - 2 * syndromes[:, c:c+1]
        total_sign = total_sign * syndrome_sign

        # For each edge, divide out its sign to get product of others
        outgoing_signs = total_sign / (signs + 1e-10)

        # Min magnitude excluding each edge (second minimum trick)
        sorted_mags, _ = torch.sort(mags, dim=1)
        min_mag = sorted_mags[:, 0:1]
        second_min = sorted_mags[:, 1:2] if sorted_mags.shape[1] > 1 else min_mag

        # If edge has the min, use second_min; else use min
        is_min = (mags == min_mag)
        outgoing_mags = torch.where(is_min, second_min, min_mag)

        # Apply scaling factor
        scaling = 0.625
        msg_c2q[:, edges] = scaling * outgoing_signs * outgoing_mags
```

**Key Point. Second Minimum Trick:** To compute $\min_{v'_j \neq v_j} |m_{v'_j}|$ efficiently, we sort all magnitudes once. For each edge: if it holds the minimum, use the second minimum; otherwise use the first minimum. This avoids recomputing $O(d)$ minimums for each of $d$ edges.

**Sum-Product Code:** The sum-product variant uses the tanh identity for exact computation:

```python
def _check_to_qubit_sumproduct(self, msg_q2c, syndromes):
    for c in range(num_checks):
        edges = self.check_to_edges[c]
        incoming = msg_q2c[:, edges]  # (batch, degree)

        # Compute tanh(LLR/2) - clamp for numerical stability
        half_llr = torch.clamp(incoming / 2, min=-20, max=20)
        tanh_vals = torch.tanh(half_llr)  # (batch, degree)

        # Product of all tanh values
        total_prod = torch.prod(tanh_vals, dim=1, keepdim=True)

        # Apply syndrome: flip sign if syndrome is 1
        syndrome_sign = 1 - 2 * syndromes[:, c:c+1]
        total_prod = total_prod * syndrome_sign

        # For each edge, divide out its tanh contribution
        outgoing_prod = total_prod / (tanh_vals + 1e-10)

        # Clamp to valid range for atanh (-1, 1)
        outgoing_prod = torch.clamp(outgoing_prod, min=-1+1e-7, max=1-1e-7)

        # Convert back: 2 * atanh(prod)
        msg_c2q[:, edges] = 2 * torch.atanh(outgoing_prod)
```

**Key Point. Min-Sum vs Sum-Product:** Min-sum is an **approximation** of sum-product, not an equivalent algorithm. The relationship comes from the identity:

$$2\tanh^{-1}\left(\prod_i \tanh(x_i/2)\right) \approx \text{sign}\left(\prod_i x_i\right) \cdot \min_i |x_i| \tag{14}$$

This approximation holds because $\tanh(x/2) \approx \text{sign}(x)$ for large $|x|$, and the product of tanh values is dominated by the smallest magnitude input. The scaling factor $\alpha = 0.625$ in min-sum compensates for the systematic overestimation of confidence that results from this approximation [6].

| **Algorithm** | **Formula** | **Trade-off** |
|---|---|---|
| Sum-Product | $2\tanh^{-1}(\prod \tanh(x/2))$ | Exact but slower (tanh/atanh) |
| Min-Sum | $\alpha \cdot \text{sign}(\prod x) \cdot \min\lvert x\rvert$ | Approximate but faster |

Table 2: Comparison of check-to-qubit message algorithms

**Step 3: Data-to-Parity Messages**

Each data node $v_j$ sends a message to each connected parity node $u_i$:

$$m_{v_j \to u_i} = p_l + \sum_{u_i' \in U(v_j) \setminus u_i} m_{u_i' \to v_j} \tag{15}$$

Where $U(v_j) \setminus u_i$ = all parity neighbors of $v_j$ except $u_i$.

**Why?** A data node collects evidence from multiple parity checks. Each check provides independent information about whether this bit is flipped. The data node sums up all this evidence (in LLR, multiplication of probabilities becomes addition).

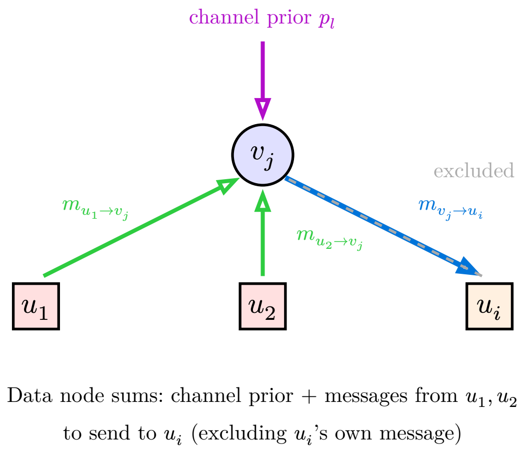



Figure 5: Data-to-parity message: $v_j$ combines channel prior with info from other checks

**Intuition:** Why exclude $u_i$? To avoid **echo effects**. If $v_j$ included the message it previously received from $u_i$, that information would bounce back, creating a feedback loop. On a **tree-structured graph**, this exclusion ensures each piece of evidence is counted exactly once, making BP exact. On graphs with cycles, this is an approximation.

**Code:** The implementation efficiently computes this by computing the total sum and subtracting each edge's contribution:

```
def _qubit_to_check(self, msg_c2q, channel_llr):
    for q in range(num_qubits):
        edges = self.qubit_to_edges[q]
        incoming = msg_c2q[:, edges]  # (batch, degree)

        # Sum of all incoming messages plus channel prior
        total_sum = incoming.sum(dim=1, keepdim=True) + channel_llr[q]

        # For each edge, subtract its own contribution
        msg_q2c[:, edges] = total_sum - incoming
```

**Key Point. Why subtract?** Instead of computing $n-1$ sums for each of $n$ edges (expensive), we compute one total sum and subtract each term. This reduces complexity from $O(d^2)$ to $O(d)$ per qubit.

**Step 4: Compute Soft Decisions**

For each bit $j$, compute the total belief (sum of all evidence):

$$P_1(e_j) = p_l + \sum_{u_i \in U(v_j)} m_{u_i \to v_j} \tag{16}$$

**Why?** Unlike Step 3, here we include **all** incoming messages (no exclusion). This is the final belief about bit $j$, combining the channel prior with evidence from **every** connected parity check. The result is the log-posterior probability ratio.

**Code:** The implementation sums all incoming messages (no exclusion) and converts to probability:

```
def _compute_marginals(self, msg_c2q, channel_llr):
    for q in range(num_qubits):
        edges = self.qubit_to_edges[q]

        # Total LLR = channel + sum of ALL incoming
        total_llr = channel_llr[q] + msg_c2q[:, edges].sum(dim=1)

        # Convert LLR to probability: P(1) = sigmoid(-LLR)
        marginals[:, q] = torch.sigmoid(-total_llr)
```

**Key Point. Sigmoid conversion:** The relationship between LLR and probability is:

$$\text{LLR} = \log \frac{P(0)}{P(1)} \Rightarrow P(1) = \frac{1}{1+e^{\text{LLR}}} = \sigma(-\text{LLR}) \tag{17}$$

PyTorch's `torch.sigmoid(-total_llr)` computes this efficiently.

**Step 5: Make Hard Decisions**

Convert soft decisions to a binary estimate:

$$e_j^{\mathrm{BP}} = \begin{cases} 1 \text{ if } P_1(e_j) < 0 & \text{(more likely flipped)} \\ 0 \text{ otherwise} & \text{(more likely correct)} \end{cases} \tag{18}$$

This gives us the BP estimate $\boldsymbol{e}^{\mathrm{BP}} = (e_1^{\mathrm{BP}}, e_2^{\mathrm{BP}}, ..., e_n^{\mathrm{BP}})$.

**Why?** The sign of LLR directly tells us the most likely value: $P_1 > 0$ means $P(e_j = 0) > P(e_j = 1)$, so the bit is probably correct. $P_1 < 0$ means the bit is probably flipped.

**Step 6: Check Convergence**

Verify if the estimate satisfies the syndrome equation:

$$H \cdot \boldsymbol{e}^{\mathrm{BP}} = \boldsymbol{s} \quad ? \tag{19}$$

- **If yes:** BP has **converged**. Return $\boldsymbol{e}^{\mathrm{BP}}$ and soft decisions $P_1$.
- **If no:** Go back to Step 2 and repeat.
- **If max iterations reached:** BP has **failed to converge**.

**Why iterate?** On graphs with cycles, a single pass doesn't propagate information globally. Each iteration allows beliefs to travel further through the graph. Eventually, if the error is correctable, the hard decisions will satisfy all parity checks.

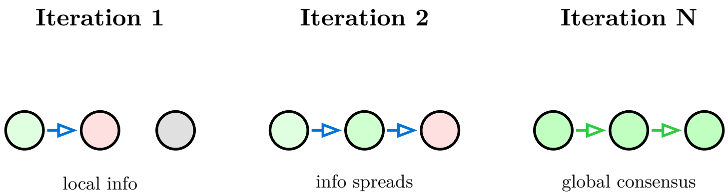



Figure 6: Information propagates further with each iteration until convergence

### 3.3.1 Damping for Convergence

**Theory:** Damping prevents oscillation by mixing old and new messages:

$$m^{(t+1)} = \gamma \cdot m^{(t)} + (1 - \gamma) \cdot m_{\mathrm{new}} \tag{20}$$

where $\gamma \in [0, 1)$ is the damping factor.

**Code:** Applied after computing new check-to-qubit messages:

```
# Main iteration loop
for _ in range(max_iter):
    # Compute new check-to-qubit messages
    msg_c2q_new = self._check_to_qubit_minsum(msg_q2c, syndromes)

    # Damping: blend old and new messages
    msg_c2q = damping * msg_c2q + (1 - damping) * msg_c2q_new

    # Update qubit-to-check messages
    msg_q2c = self._qubit_to_check(msg_c2q, channel_llr)
```

| Damping Value | Behavior | Use Case |
|---|---|---|
| $\gamma = 0$ | No damping (full update) | Simple graphs, fast convergence |
| $\gamma = 0.2$ | Light damping | Typical default |
| $\gamma = 0.5$ | Strong damping | Graphs with short cycles |

Table 3: Effect of damping factor on BP convergence

### 3.3.2 Summary: Complete BP Iteration

Putting it all together, one BP iteration consists of:

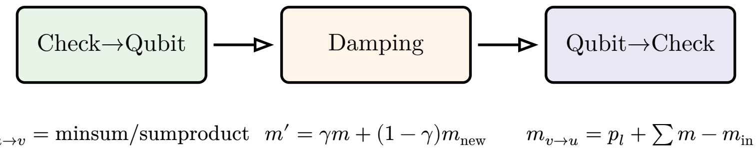



Figure 7: One BP iteration: message updates with damping

## 3.4 Why BP Fails on Quantum Codes

### 3.4.1 Signal Dilution in High-Weight Checks

Consider a check node with syndrome $s = 1$ and channel error probability $p = 0.1$ (LLR $\approx$ 2.2). The check-to-variable message strength depends on the number of neighbors:

| Check Weight | Message Strength | Net LLR | Result |
|---|---|---|---|
| 2-bit (classical) | $-2.2$ | 0 | Correct uncertainty |
| 4-bit (quantum) | $-1.1$ | $+1.1$ | **False confidence** |

With 4-bit plaquettes, the check message is too weak to overturn the channel prior. The hard decision outputs $\boldsymbol{e} = \mathbf{0}$, which violates the syndrome constraint $0 + 0 + 0 + 0 = 0 \neq 1$.

### 3.4.2 The Degeneracy Problem

**The Degeneracy Problem**

In quantum codes, **multiple distinct error patterns** produce the same syndrome and differ only by a stabilizer. BP cannot distinguish between these equivalent patterns.

**Definition.** Two errors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are **degenerate** if:

$$H \cdot \boldsymbol{e}_1 = H \cdot \boldsymbol{e}_2 = \boldsymbol{s} \quad \text{and} \quad \boldsymbol{e}_1 + \boldsymbol{e}_2 \in \text{Stabilizers} \tag{21}$$

When BP encounters symmetric degenerate errors:
1. Beliefs split equally between solutions ($P(e_i) \approx 0.5$)
2. Hard decision outputs either $\mathbf{0}$ or the stabilizer $\boldsymbol{e}_1 + \boldsymbol{e}_2$
3. Neither satisfies the syndrome: $H \cdot \boldsymbol{e}^{\text{BP}} \neq \boldsymbol{s}$

**Key Point.** For quantum codes with high degeneracy (like the Toric code), **BP alone shows no threshold** — increasing code distance makes performance worse, not better. This motivates post-processing techniques like OSD.

# 4 Ordered Statistics Decoding (OSD)

## 4.1 The Key Insight

The parity check matrix $H$ (size $m \times n$ with $n > m$) has more columns than rows and cannot be directly inverted. However, we can select a subset of $r = \text{rank}(H)$ linearly independent columns to form an invertible $m \times r$ submatrix.

**Definition.** For an $m \times n$ matrix $H$ with $\text{rank}(H) = r$:
- **Basis set** $[S]$: indices of $r$ linearly independent columns
- **Remainder set** $[T]$: indices of the remaining $k' = n - r$ columns
- $H_{[S]}$: the $m \times r$ submatrix of columns in $[S]$ (this is invertible!)
- $H_{[T]}$: the $m \times k'$ submatrix of columns in $[T]$

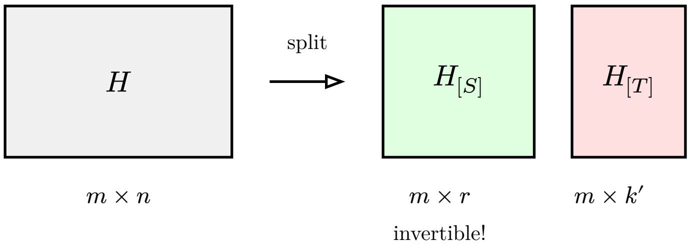



Figure 8: Splitting $H$ into basis and remainder parts

OSD then resolves split beliefs and **forcing a unique solution**. It first choose the basis selection $[S]$ through BP soft decisions guide. and then calculate the matrix inversion on $H_{[S]}$ eliminates ambiguity.

## 4.2 OSD-0: The Basic Algorithm

Ordered Statistics Decoding (OSD) was introduced by Fossorier and Lin as a soft-decision decoding algorithm for linear block codes that approaches maximum-likelihood performance [7]. The algorithm was later extended with computationally efficient variants [8]. For quantum LDPC codes, the BP+OSD combination was shown to be remarkably effective by Panteleev and Kalachev, and further developed by Roffe et al. [9].

**Definition. OSD-0** finds a solution by:
1. Choosing a “good” basis $[S]$ using BP soft decisions $P_1$
2. Solving for the basis bits via matrix inversion
3. Setting all remainder bits to zero

**Algorithm 2: OSD-0**

**Input:**
- Parity matrix $H$ (size $m \times n$, rank $r$)
- Syndrome $\boldsymbol{s}$
- BP soft decisions $P_1(e_1), ..., P_1(e_n)$ (from Algorithm 1)

**Steps:**

1. **Rank bits by probability:** Sort bit indices by $P_1$ values: most-likely-flipped first. Result: ordered list $[O_{\mathrm{BP}}] = (j_1, j_2, ..., j_n)$
2. **Reorder columns:** $H_{[O_{\mathrm{BP}}]}$ = matrix $H$ with columns reordered by $[O_{\mathrm{BP}}]$
3. **Select basis:** Scan left-to-right, select first $r$ linearly independent columns. Basis indices: $[S]$. Remainder indices: $[T]$ (size $k' = n - r$)
4. **Solve on basis:** $\boldsymbol{e}_{[S]} = H_{[S]}^{-1} \cdot \boldsymbol{s}$
5. **Set remainder to zero:** $\boldsymbol{e}_{[T]} = \boldsymbol{0}$ (zero vector of length $k'$)
6. **Remap to original ordering:** Combine $\left(\boldsymbol{e}_{[S]}, \boldsymbol{e}_{[T]}\right)$ and undo the permutation

**Output:** $\boldsymbol{e}^{\text{OSD-0}}$ satisfying $H \cdot \boldsymbol{e}^{\text{OSD-0}} = \boldsymbol{s}$

Figure 9: OSD-0 algorithm

The GPU-accelerated implementation in `batch_osd.py` realizes OSD-0 through Gaussian elimination rather than explicit matrix inversion. Below is the mapping between the theoretical definition and the code logic.

### 4.2.1 Step 1: Choosing a "Good" Basis (Sorting)

The definition requires selecting basis columns $[S]$ based on BP soft decisions. For quantum error correction, the implementation sorts by **probability descending** rather than reliability $|p - 0.5|$.

```
# Sort by probability descending (highest probability first)
# High-probability errors become pivots; low-probability errors become free variables
sorted_indices = np.argsort(probs)[::-1]
```

**Key Point. Why probability-based sorting for quantum codes?**

Traditional OSD uses reliability $|p - 0.5|$, but this fails for quantum codes where:

- Most qubits have $p \approx 0$ (no error) $\to$ reliability $\approx 0.5$
- Identified errors have $p \approx 1 \to$ reliability $\approx 0.5$

Both have similar reliability despite being very different! Sorting by probability directly places likely errors in $[S]$ (pivots) and unlikely errors in $[T]$ (free variables set to 0).

### 4.2.2 Step 2: Solving for Basis Bits (RREF)

Instead of computing $H_{[S]}^{-1}$ explicitly, the code computes the **Reduced Row Echelon Form** (RREF) of the augmented matrix $[H_{\text{sorted}} \mid \boldsymbol{s}]$. This is numerically stable and solves the linear system in one pass. For example, consider a parity check matrix $H$ with 3 checks and 6 variables, and syndrome $\boldsymbol{s} = (1, 0, 1)^T$. Assume BP has already sorted columns by probability (column 0 = highest probability error). Then the initial augmented matrix $[H_{\text{sorted}} \mid \boldsymbol{s}]$ writes:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array}\right) \tag{22}$$

**Iteration 1: Process column 0**

- Find pivot: Row 0 has a 1 in column 0 ✓
- Eliminate: Row 1 also has 1 in column 0, so XOR row 1 with row 0

$$\left(\begin{array}{cccccc|c} \mathbf{1} & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array}\right) \quad \text{pivot\_cols} = [0] \tag{23}$$

**Iteration 2: Process column 1**

- Find pivot: Row 1 has a 1 in column 1 ✓
- Eliminate: Row 2 also has 1 in column 1, so XOR row 2 with row 1

$$\left(\begin{array}{cccccc|c} \mathbf{1} & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & \mathbf{1} & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right) \quad \text{pivot\_cols} = [0, 1] \tag{24}$$

**Iteration 3: Process column 2**

- Find pivot: No rows have a 1 in column 2 below the current pivot row $\times$
- Skip this column (it becomes a free variable)

**Iteration 4: Process column 3**

- Find pivot: Row 2 has a 1 in column 3 ✓
- Eliminate: Rows 0 and 1 have 1s in column 3, XOR both with row 2

$$\left(\begin{array}{cccccc|c} \mathbf{1} & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & \mathbf{1} & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \end{array}\right) \quad \text{pivot\_cols} = [0, 1, 3] \tag{25}$$

**Final Result:** Pivot columns $[S] = \{0, 1, 3\}$ (basis variables) and free columns $[T] = \{2, 4, 5\}$ (remainder variables).

In the code, the H matrix is stored as a 2D 'int8' array to fully utilized the GPU's integer arithmetic capabilities. The fuctiion 'compute_rref' implements Gaussian elimination over GF(2):

```python
def _get_rref_cached(self, sorted_indices: np.ndarray, syndrome: np.ndarray):
    # Reorder columns by sorted indices
    H_sorted = self.H[:, sorted_indices]
    # Build augmented matrix [H_sorted | s]
    augmented = np.hstack([H_sorted, syndrome.reshape(-1, 1)]).astype(np.int8)
    # Compute RREF in-place
    pivot_cols = self._compute_rref(augmented)
    return augmented, pivot_cols

def _compute_rref(self, M: np.ndarray) -> List[int]:
    m, n = M.shape
    pivot_row = 0
    pivot_cols = []

    for col in range(n - 1):  # Don't pivot on syndrome column
        if pivot_row >= m:
            break
        # Find a row with 1 in this column
        candidates = np.where(M[pivot_row:, col] == 1)[0]
        if len(candidates) == 0:
            continue  # No pivot in this column

        # Swap to bring pivot to current row
        swap_r = candidates[0] + pivot_row
        if swap_r != pivot_row:
            M[[pivot_row, swap_r]] = M[[swap_r, pivot_row]]

        pivot_cols.append(col)

        # Eliminate all other 1s in this column (XOR in GF(2))
        rows_to_xor = np.where(M[:, col] == 1)[0]
        rows_to_xor = rows_to_xor[rows_to_xor != pivot_row]
        if len(rows_to_xor) > 0:
            M[rows_to_xor, :] ^= M[pivot_row, :]

        pivot_row += 1

    return pivot_cols
```

- `pivot_cols`: The column indices where pivots were found. These form the basis set $[S]$.
- After RREF, the basis submatrix has an identity-like structure, and the syndrome column contains the solution values.

#### 4.2.3 Step 3: Reading the OSD-0 solution:

Continue from the previous example, the result can be read from the pivot collumns and syndrome column $\boldsymbol{s}$, as illustated by the following steps:

- Set free variables to zero: $e_2 = e_4 = e_5 = 0$
- Read pivot values from the transformed syndrome column:
  - Row 0: pivot at column 0, syndrome value $= 1 \to e_0 = 1$
  - Row 1: pivot at column 1, syndrome value $= 1 \to e_1 = 1$
  - Row 2: pivot at column 3, syndrome value $= 0 \to e_3 = 0$
- Solution in sorted order then must be $\boldsymbol{e}_{\text{sorted}} = (1, 1, 0, 0, 0, 0)$, and can be verified through:

$$H_{\text{sorted}} \cdot \boldsymbol{e}_{\text{sorted}} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ \mathbf{0} \\ 0 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 1 \oplus 0 \\ 1 \oplus 1 \\ 0 \oplus 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{s} \quad \checkmark \tag{26}$$

In the code, the solution is extracted by initializing the solution vector to zeros and updating **only** the pivot positions using the transformed syndrome.

```python
# OSD-0 Solution: Initialize all bits to 0 (remainder bits stay 0)
solution_base = np.zeros(self.num_errors, dtype=np.int8)

# Build pivot-to-row mapping and extract solution
pivot_row_map = {}
for r in range(augmented.shape[0]):
    row_pivots = np.where(augmented[r, :self.num_errors] == 1)[0]
    if len(row_pivots) > 0:
        col = row_pivots[0]
        if col in pivot_cols:
            pivot_row_map[col] = r
            # Pivot bit = transformed syndrome value
            solution_base[col] = augmented[r, -1]
```

- `solution_base = np.zeros(...)`: Ensures $\boldsymbol{e}_{[T]} = \mathbf{0}$ (OSD-0 constraint).
- `augmented[r, -1]`: The transformed syndrome value. Since the basis submatrix is now identity-like, this directly gives $\boldsymbol{e}_{[S]}$.

#### 4.2.4 Step 4: Inverse Mapping (Unsort)

Finally, the solution is mapped back to the original column ordering:

```python
# Remap from sorted order back to original order
estimated_errors = np.zeros(self.num_errors, dtype=int)
estimated_errors[sorted_indices] = final_solution_sorted
return estimated_errors
```

| Theoretical Step | Code Realization (`batch_osd.py`) |
|---|---|
| 1. Sort by soft decisions | `np.argsort(probs)[::-1]` (probability descending) |
| 2. Select basis $[S]$ | `pivot_cols` from `_compute_rref` |
| 3. Matrix inversion | RREF transforms basis to identity structure |
| 4. Solve $\boldsymbol{e}_{[S]}$ | `solution_base[col] = augmented[r, -1]` |
| 5. Set $\boldsymbol{e}_{[T]} = \mathbf{0}$ | `np.zeros(...)` initialization |
| 6. Unsort | `estimated_errors[sorted_indices] = solution` |

Table 4: Mapping OSD-0 theory to `batch_osd.py` implementation

## 4.3 Higher-Order OSD (OSD-$\lambda$)

OSD-0 assumes the remainder error bits are zero ($\boldsymbol{e}_{[T]} = \mathbf{0}$). While this provides a valid solution, it forces all "correction" work onto the basis bits $[S]$, which may result in a high-weight (improbable) error pattern.

**Definition. Higher-order OSD** improves this by testing non-zero configurations for the remainder bits $\boldsymbol{e}_{[T]}$. For any chosen hypothesis $\boldsymbol{e}_{[T]}$, the corresponding basis bits $\boldsymbol{e}_{[S]}$ are uniquely determined to satisfy the syndrome:

$$\boldsymbol{e}_{[S]} = H_{[S]}^{-1} \cdot \left(\boldsymbol{s} + H_{[T]} \cdot \boldsymbol{e}_{[T]}\right) \tag{27}$$

It is straightforward to show that the constructed error $\boldsymbol{e} = \left(\boldsymbol{e}_{[S]}, \boldsymbol{e}_{[T]}\right)$ always satisfies the parity check equation $H \cdot \boldsymbol{e} = \boldsymbol{s}$ and the OSD-0 is a special case of OSD-$\lambda$ when $\lambda = 0$.

$$\begin{aligned} H \cdot \boldsymbol{e} &= \left(H_{[S]} \;\; H_{[T]}\right) \cdot \begin{pmatrix} \boldsymbol{e}_{[S]} \\ \boldsymbol{e}_{[T]} \end{pmatrix} = H_{[S]} \cdot \boldsymbol{e}_{[S]} + H_{[T]} \cdot \boldsymbol{e}_{[T]} \\ &= H_{[S]} \cdot \left[H_{[S]}^{-1} \cdot \left(\boldsymbol{s} + H_{[T]} \cdot \boldsymbol{e}_{[T]}\right)\right] + H_{[T]} \cdot \boldsymbol{e}_{[T]} \\ &= I \cdot \left(\boldsymbol{s} + H_{[T]} \cdot \boldsymbol{e}_{[T]}\right) + H_{[T]} \cdot \boldsymbol{e}_{[T]} \\ &= \boldsymbol{s} + H_{[T]} \cdot \boldsymbol{e}_{[T]} + H_{[T]} \cdot \boldsymbol{e}_{[T]} = \boldsymbol{s} + \mathbf{0} = \boldsymbol{s} \end{aligned} \tag{28}$$

Then the problem change to find the minimum soft-weight solution for the remainder bits $\boldsymbol{e}_{[T]}$. A naive way to do this is to implement an exhaustive search (OSD-E) testing on all $2^{k'}$ patterns. This guarantees finding the minimum weight solution. Unfortunately, the remainder set $[T]$ has size $k' = n - r$, which is exponentially large in the code parameters $n - r$. To make this feasible, we restrict the search to the search depth $\lambda$, i.e., the **most suspicious** $\lambda$ bits in $[T]$ (those with highest error probability among free variables) and accelerate this serch process by using the GPU. The `batch_osd.py` implementation accelerates OSD-E by evaluating all $2^{\lambda}$ candidates in parallel on GPU. Here is how it works:

**Step 1: Identify Search Columns**

Select the $\lambda$ free variables with highest error probability (most suspicious):

```
# Get free columns (not pivots)
all_cols = set(range(self.num_errors))
free_cols = sorted(list(all_cols - set(pivot_cols)))

# Sort free columns by probability (highest first = most suspicious)
free_cols_with_prob = [(col, probs[sorted_indices[col]]) for col in free_cols]
```

```
free_cols_with_prob.sort(key=lambda x: -x[1])

# Select top osd_order free variables for search
search_cols = [col for col, _ in free_cols_with_prob[:osd_order]]
```

### Step 2: Generate All $2^\lambda$ Candidates

```
# Exhaustive: Generate all 2^k combinations using bit manipulation
num_candidates = 1 << len(search_cols)  # 2^k
candidates_np = np.array([
    [(i >> j) & 1 for j in range(len(search_cols))]
    for i in range(num_candidates)
], dtype=np.int8)
```

### Step 3: Parallel Evaluation on GPU

All candidates are evaluated simultaneously using batched matrix operations:

```
def _evaluate_candidates_gpu(self, candidates, augmented, search_cols, probs_sorted,
pivot_cols):
    num_candidates = candidates.shape[0]

    # Transfer to GPU
    M_subset = torch.from_numpy(augmented[:, search_cols]).float().to(self.device)
    syndrome_col = torch.from_numpy(augmented[:, -1]).float().to(self.device)

    # Compute modified syndromes for ALL candidates in parallel
    # target_syndrome = (s + M @ e_T) % 2
    target_syndromes = (syndrome_col.unsqueeze(0) + candidates.float() @ M_subset.T)
% 2

    # Initialize solution matrix (num_candidates × n)
    cand_solutions = torch.zeros(num_candidates, self.num_errors, device=self.device)

    # Set free variable values from candidates
    cand_solutions[:, search_cols] = candidates.float()

    # Solve for pivot variables using the RREF structure
    for r in range(augmented.shape[0]):
        row_pivots = torch.where(augmented_torch[r, :] == 1)[0]
        if len(row_pivots) > 0:
            pivot_c = row_pivots[0].item()
            if pivot_c in pivot_cols:
                # Pivot value = modified syndrome for this row
                cand_solutions[:, pivot_c] = target_syndromes[:, r]

    # Compute soft-weighted costs and return best solution
    costs = self._compute_soft_weight_gpu(cand_solutions, probs_sorted)
    best_idx = torch.argmin(costs)
    return cand_solutions[best_idx]
```

### Step 4: Soft-Weight Cost Function

In our code, the OSD uses the **soft-weighted cost** based on log-probabilities [9]:

> **Definition. Soft-Weighted Cost (Log-Probability Weight).** For an error pattern $\boldsymbol{e} = (e_1, ..., e_n)$ with bit-wise error probabilities $p_i = P(e_i = 1)$, the soft-weighted cost is:

$$W_{\text{soft}(\boldsymbol{e})} = \sum_{i:e_i=1} (-\log p_i) = -\sum_{i=1}^{n} e_i \cdot \log p_i \tag{29}$$

Lower cost indicates a more probable error pattern.

There actually are several other cost functions that can be used for OSD, such as the Hamming weight, Euclidean distance, and LLR-based weight, as listed in the following table:

| **Cost Function** | **Formula** | **Properties** |
|---|---|---|
| Hamming Weight [10] | $W_{H(\boldsymbol{e})} = \sum_i e_i$ | Counts flipped bits; ignores probabilities |
| Soft Weight (Log-Prob) [9] | $W_{\text{soft}(\boldsymbol{e})} = -\sum_i e_i \log p_i$ | Weights by $-\log p_i$; approximates ML |
| Euclidean Distance [11] | $d_E^2 = \sum_i (r_i - c_i)^2$ | For AWGN channels with continuous signals |
| LLR-Based Weight [12] | $W_{\text{LLR}(\boldsymbol{e})} = \sum_i e_i \, \lvert L_i \rvert$ | Uses log-likelihood ratios $L_i = \log\left(\frac{p_i}{1-p_i}\right)$ |

Table 5: Comparison of cost functions for selecting the best error pattern

Then why soft weight is preferred for BP+OSD? This is because the soft weight approximates the Maximum-Likelihood Decoding (ML) objective [13]. The ML decoder selects the error pattern $\boldsymbol{e}^*$ that maximizes the posterior probability

$$\boldsymbol{e}^* = \arg\max_{\boldsymbol{e}} P(\boldsymbol{e} \mid \text{syndrome}). \tag{30}$$

Taking the logarithm, which is a monotonic transformation, and suppose the error $e_i$ are independent, each with probability $P(e_i = 1) = p_i$, this becomes:

$$\boldsymbol{e}^* = \arg\max_{\boldsymbol{e}} \sum_i [e_i \log p_i + (1 - e_i) \log(1 - p_i)] \tag{31}$$

For sparse errors where most $e_i = 0$, minimizing $W_{\text{soft}(\boldsymbol{e})} = -\sum_i e_i \log p_i$ closely approximates the ML objective.

**Key Point. Question:** Why not use the Hamming weight or Euclidean distance as the cost function?

In the code, the soft-weight cost function is implemented as follows:

```
def _compute_soft_weight_gpu(self, solutions, probs):
    # Clip to avoid log(0)
    probs_clipped = torch.clamp(probs, 1e-10, 1 - 1e-10)
    # Log-probability weights: -log(p) penalizes flipping low-probability bits
    log_weights = -torch.log(probs_clipped)
    # Total cost = sum of weights for flipped bits
    costs = (solutions * log_weights).sum(dim=1)
    return costs
```

## 4.4 Combination Sweep Strategy (OSD-CS)

To allow for a larger search depth (e.g., $\lambda \approx 50 - 100$) without exponential cost, we use the **combination sweep** strategy, first proposed for reducing error floors in classical LDPC codes and adapted for quantum codes by Roffe et al. [9].

> **Definition. OSD-CS** assumes the true error pattern on the remainder bits is **sparse**. Instead of checking **all** $2^\lambda$ patterns on $\lambda$ most suspicious bits, it only checks those with low Hamming weight (w = 0,1,2). Exausted on those strings only take $C_\lambda^0 + C_\lambda^1 + C_\lambda^2 = 1 + \lambda + \frac{\lambda(\lambda-1)}{2}$ candidates. With $\lambda = 60$: approximately $1 + k + 1770$ configurations (vs $2^k$ for exhaustive search!)
>
> **Algorithm Steps:**
>
> 1. **Sort:** Select the $\lambda$ most suspicious positions in $[T]$ (highest probability among free variables).
> 2. **Sweep:** Generate candidate vectors $\boldsymbol{e}_{[T]}$ with:
>    - **Weight 0:** The zero vector (equivalent to OSD-0).
>    - **Weight 1:** All single-bit flips among the $\lambda$ bits.
>    - **Weight 2:** All pairs of bit flips among the $\lambda$ bits.
> 3. **Select:** Calculate $\boldsymbol{e}_{[S]}$ for each candidate, compute the soft-weighted cost, and pick the best one.

| Method | Complexity | Use Case |
|---|---|---|
| OSD-0 | $O(1)$ | Fastest, baseline performance |
| OSD-E (Exhaustive) | $O(2^\lambda)$ | Optimal for small $\lambda$ ($\leq 15$) |
| OSD-CS (Comb. Sweep) | $O(\lambda^2)$ | Near-optimal for large $\lambda$ ($\approx 60$) |

Table 6: Comparison of OSD Search Strategies

**Why OSD-CS works for Quantum Codes?** In the low-error regime relevant for QEC, it is statistically very unlikely that the optimal solution requires flipping 3+ bits in the specific subset of "uncertain" remainder bits. Checking only weights 0, 1, and 2 captures the vast majority of likely error configurations while reducing complexity from exponential to polynomial (quadratic) [9].

> **Definition. Combination sweep** is a greedy search testing configurations by likelihood:
>
> 1. **Sort remainder bits:** Order bits in $[T]$ by error probability (most likely first)
> 2. **Test weight-0:** The zero vector (OSD-0 baseline)
> 3. **Test weight-1:** Set each single bit in $\boldsymbol{e}_{[T]}$ to 1 (all $k$ possibilities)
> 4. **Test weight-2:** Set each pair among the first $\lambda$ bits to 1
>
> Keep the minimum soft-weight solution found.

### 4.4.1 OSD-CS Implementation in `batch_osd.py`

The GPU-accelerated implementation realizes OSD-CS by explicitly generating sparse error patterns (weights 0, 1, and 2) instead of iterating through all binary combinations.

**Step 1: Generating Sparse Candidates**

The `_generate_osd_cs_candidates` method generates candidate vectors $\boldsymbol{e}_{[T]}$ with structured loops:

```python
def _generate_osd_cs_candidates(self, k: int, osd_order: int) -> np.ndarray:
    """Generate OSD-CS (Combination Sweep) candidate strings."""
    candidates = []

    # Weight 0: Zero vector (OSD-0 baseline)
    candidates.append(np.zeros(k, dtype=np.int8))

    # Weight 1: Single-bit flips (k candidates)
    for i in range(k):
        candidate = np.zeros(k, dtype=np.int8)
        candidate[i] = 1
        candidates.append(candidate)

    # Weight 2: Two-bit flips (limited to osd_order)
    for i in range(min(osd_order, k)):
        for j in range(i + 1, min(osd_order, k)):
            candidate = np.zeros(k, dtype=np.int8)
            candidate[i] = 1
            candidate[j] = 1
            candidates.append(candidate)

    return np.array(candidates, dtype=np.int8)
```

- `np.zeros(k)`: The "baseline" hypothesis (OSD-0 solution).
- `range(k)` loop: Adds $k$ candidates, each with a single bit flip at index `i`.
- Nested `range(limit)` loop: Adds $\binom{\lambda}{2}$ candidates, representing pairs of flips at indices $(i, j)$.

### Step 2: Integration into the Solve Loop

The `solve` method switches between OSD-E and OSD-CS based on the `osd_method` parameter:

```python
# Generate candidates based on method
if osd_method == 'combination_sweep':
    # OSD-CS: O(λ²) sparse candidates
    candidates_np = self._generate_osd_cs_candidates(len(search_cols), osd_order)
else:
    # Exhaustive: O(2^k) all combinations
    num_candidates = 1 << len(search_cols)
    candidates_np = np.array([[(i >> j) & 1 for j in range(len(search_cols))]
                              for i in range(num_candidates)], dtype=np.int8)

# Transfer to GPU and evaluate all candidates in parallel
candidates = torch.from_numpy(candidates_np).to(self.device)
best_solution_sorted = self._evaluate_candidates_gpu(
    candidates, augmented, search_cols, probs_sorted, pivot_cols
)
```

Both methods use the same GPU evaluation function—only the candidate generation differs.

### Complexity Comparison

| **Method** | **Candidates** | **Example ($\lambda = 15$)** |
|---|---|---|
| OSD-E | $2^\lambda$ | 32768 |
| OSD-CS | $1 + k + \binom{\lambda}{2}$ | $\approx 1 + k + 105$ |

Table 7: Number of candidates for OSD-E vs OSD-CS

| **Weight Class** | **Code Realization** |
|---|---|
| Weight 0 | `candidates.append(np.zeros(k))` |
| Weight 1 | `for i in range(k): candidate[i] = 1` |
| Weight 2 | `for i in range(limit): for j in range(i+1, limit): ...` |

Table 8: Mapping OSD-CS theory to `batch_osd.py` implementation

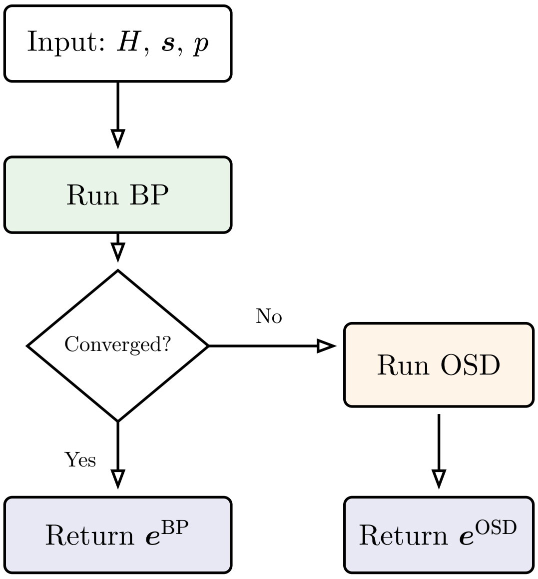



Figure 10: BP+OSD decoder flowchart

**Key Point.**

- If BP succeeds (converges): use BP result — fast!
- If BP fails: use OSD to resolve degeneracy — always gives valid answer

# 5 Tropical Tensor Network

In this section, we introduce a complementary approach to decoding: **tropical tensor networks**. While BP+OSD performs approximate inference followed by algebraic post-processing, tropical tensor networks provide a framework for **exact** maximum a posteriori (MAP) inference by reformulating the problem in terms of tropical algebra.

The key insight is that finding the most probable error configuration corresponds to an optimization problem that can be solved exactly using tensor network contractions in the tropical semiring. This approach is particularly powerful for structured codes where the underlying factor graph has bounded treewidth.

## 5.1 Tropical Semiring

**Definition.** The **tropical semiring** (also called the **max-plus algebra**) is the algebraic structure $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ where:

- **Tropical addition**: $a \oplus b = \max(a, b)$
- **Tropical multiplication**: $a \otimes b = a + b$ (ordinary addition)
- **Additive identity**: $-\infty$ (since $\max(a, -\infty) = a$)
- **Multiplicative identity**: $0$ (since $a + 0 = a$)

**Key Point.** The tropical semiring satisfies all semiring axioms:

- Associativity: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
- Commutativity: $a \oplus b = b \oplus a$
- Distributivity: $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

This algebraic structure allows us to replace standard summation with maximization while preserving the correctness of tensor contractions.

The tropical semiring was first systematically studied in the context of automata theory and formal languages [14]. Its connection to optimization problems makes it particularly useful for decoding applications.

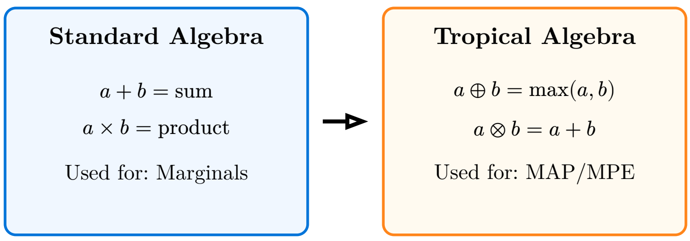



Figure 11: Standard algebra vs tropical algebra: switching the algebraic structure transforms marginalization into optimization

## 5.2 From Probabilistic Inference to Tropical Algebra

Recall that the MAP (Maximum A Posteriori) decoding problem seeks:

$$\boldsymbol{e}^* = \arg\max_{\boldsymbol{e}: H\boldsymbol{e} = \boldsymbol{s}} P(\boldsymbol{e}) \tag{32}$$

For independent bit-flip errors with probability $p$, the probability factors as:

$$P(\boldsymbol{e}) = \prod_{i=1}^{n} P(e_i) = \prod_{i=1}^{n} p^{e_i}(1-p)^{1-e_i} \tag{33}$$

Taking the logarithm transforms products into sums:

$$\log P(\boldsymbol{e}) = \sum_{i=1}^{n} \log P(e_i) = \sum_{i=1}^{n} [e_i \log p + (1 - e_i) \log(1 - p)] \tag{34}$$

**Key Point.** In the log-probability domain:

- **Products become sums**: $\log(P \cdot Q) = \log P + \log Q$
- **Maximization is preserved**: $\arg\max_x f(x) = \arg\max_x \log f(x)$

This means finding the MAP estimate for a function $\prod_f \varphi_f(\boldsymbol{e}_f)$ is equivalent to:

$$\boldsymbol{e}^* = \arg \max_{\boldsymbol{e}:H\boldsymbol{e}=\boldsymbol{s}} \sum_f \log \varphi_f(\boldsymbol{e}_f) \tag{35}$$

where each factor $\varphi_f$ contributes additively in log-space.

The connection to tropical algebra becomes clear: if we replace standard tensor contractions (sum over products) with tropical contractions (max over sums), we transform marginal probability computation into MAP computation [2].

| **Operation** | **Standard (Marginals)** | **Tropical (MAP)** |
|---|---|---|
| Combine factors | $\varphi_a \cdot \varphi_b$ | $\log \varphi_a + \log \varphi_b$ |
| Eliminate variable | $\sum_x$ | $\max_x$ |
| Result | Partition function $Z$ | Max log-probability |

Table 9: Correspondence between standard and tropical tensor operations

**Example:** Consider a simple Markov chain with three binary variables $x_1, x_2, x_3 \in \{0, 1\}$ and two factors:

$$P(x_1, x_2, x_3) = \varphi_1(x_1, x_2) \cdot \varphi_2(x_2, x_3) \tag{36}$$

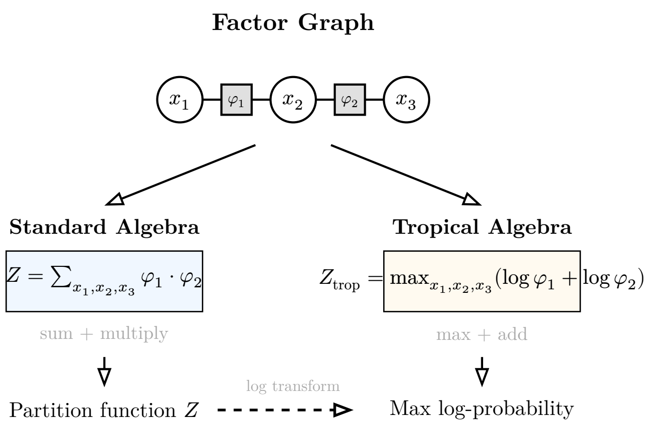



Figure 12: Standard vs tropical contraction of a Markov chain. The same factor graph structure supports both marginal computation (standard algebra) and MAP inference (tropical algebra).

The partition function in standard algebra sums over all configurations:

$$Z = \sum_{x_1,x_2,x_3} \varphi_1(x_1, x_2) \cdot \varphi_2(x_2, x_3) \tag{37}$$

The same structure in tropical algebra computes the maximum log-probability:

$$Z_{\text{trop}} = \max_{x_1,x_2,x_3} [\log \varphi_1(x_1, x_2) + \log \varphi_2(x_2, x_3)] \tag{38}$$

**Key Point. Beyond a Change of Language** [15]: Tropical tensor networks provide computational capabilities unavailable in traditional approaches:

1. **Automatic Differentiation for Configuration Recovery**: Backpropagating through tropical contraction yields gradient "masks" that directly identify optimal variable assignments $\boldsymbol{x}^*$—no separate search phase is needed.
2. **Degeneracy Counting via Mixed Algebras**: By tracking $(Z_{\text{trop}}, n)$ where $n$ counts multiplicities, one simultaneously finds the optimal value AND counts all solutions achieving it in a single contraction pass.
3. **GPU-Accelerated Tropical BLAS**: Tropical matrix multiplication maps to highly optimized GPU kernels, enabling exact ground states for 1024-spin Ising models and 512-qubit D-Wave graphs in under 100 seconds.

## 5.3 Tensor Network Representation

A tensor network represents the factorized probability dis tribution as a graph where nodes of tensors correspond to factors $\varphi_f$ and the edges of correspond to functions that contract the variables.

**Definition.** Given a factor graph with factors $\{\varphi_f\}$ and variables $\{x_i\}$, the corresponding **tensor network** consists of:

- A tensor $T_f$ for each factor, with indices corresponding to the variables in $\varphi_f$
- The **contraction** of the network computes: $\sum_{x_1,\dots,x_n} \prod_f T_f(\boldsymbol{x}_f)$

In the tropical semiring, this becomes: $\max_{x_1,\dots,x_n} \sum_f T_f(\boldsymbol{x}_f)$

The efficiency of tensor network contraction depends critically on the **contraction order**—the sequence in which variables are eliminated.

**Key Point.** The **treewidth** of the factor graph determines the computational complexity:

- A contraction order exists with complexity $O(n \cdot d^{w+1})$ where $w$ is the treewidth
- For sparse graphs (like LDPC codes), treewidth can be small, enabling efficient exact inference
- Tools like `omeco` find near-optimal contraction orders using greedy heuristics

**Factor Graph → Tensor Network**

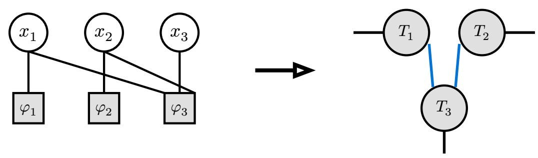



Figure 13: Factor graph representation as a tensor network. Edges between tensors represent indices to be contracted (summed/maximized over).

The contraction process proceeds by repeatedly selecting a variable to eliminate:

```python
# Conceptual contraction loop (simplified)
for var in elimination_order:
    bucket = [tensor for tensor in tensors if var in tensor.indices]
    combined = tropical_contract(bucket, eliminate=var)
    tensors.update(combined)
```

### 5.4 Backpointer Tracking for MPE Recovery

A critical challenge with tensor network contraction is that it only computes the **value** of the optimal solution (the maximum log-probability), not the **assignment** that achieves it.

> **Definition.** A **backpointer** is a data structure that records, for each max operation during contraction:
>
> - The indices of eliminated variables
> - The $\arg\max$ value for each output configuration
>
> Formally, when computing $\max_x T(y, x)$, we store: $\mathrm{bp}(y) = \arg\max_x T(y, x)$

The recovery algorithm traverses the contraction tree in reverse:

**Contraction Tree with Backpointers**

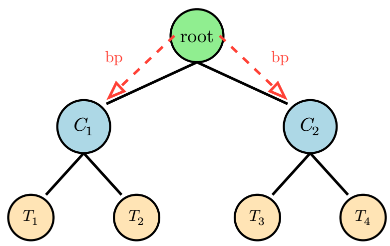



Figure 14: Contraction tree with backpointers. During contraction (bottom-up), backpointers record argmax indices. During recovery (top-down, dashed arrows), backpointers are traced to reconstruct the optimal assignment.

The implementation in the `tropical_in_new/` module demonstrates this pattern:

```python
# From tropical_in_new/src/primitives.py
@dataclass
class Backpointer:
    """Stores argmax metadata for eliminated variables."""
    elim_vars: Tuple[int, ...]       # Which variables were eliminated
    elim_shape: Tuple[int, ...]      # Domain sizes
    out_vars: Tuple[int, ...]        # Remaining output variables
    argmax_flat: torch.Tensor        # Flattened argmax indices

def tropical_reduce_max(tensor, vars, elim_vars, track_argmax=True):
    """Tropical max-reduction with optional backpointer tracking."""
    # ... reshape tensor to separate kept and eliminated dimensions ...
    values, argmax_flat = torch.max(flat, dim=-1)
    if track_argmax:
        backpointer = Backpointer(elim_vars, elim_shape, out_vars, argmax_flat)
    return values, backpointer
```

The recovery algorithm traverses the tree from root to leaves:

```python
# From tropical_in_new/src/mpe.py
def recover_mpe_assignment(root) -> Dict[int, int]:
    """Recover MPE assignment from a contraction tree with backpointers."""
```

```python
    assignment: Dict[int, int] = {}

    def traverse(node, out_assignment):
        assignment.update(out_assignment)
        if isinstance(node, ReduceNode):
            # Use backpointer to recover eliminated variable values
            elim_assignment = argmax_trace(node.backpointer, out_assignment)
            child_assignment = {**out_assignment, **elim_assignment}
            traverse(node.child, child_assignment)
        elif isinstance(node, ContractNode):
            # Propagate to both children
            elim_assignment = argmax_trace(node.backpointer, out_assignment)
            combined = {**out_assignment, **elim_assignment}
            traverse(node.left, {v: combined[v] for v in node.left.vars})
            traverse(node.right, {v: combined[v] for v in node.right.vars})

    # Start from root with initial assignment from final tensor
    initial = unravel_argmax(root.values, root.vars)
    traverse(root, initial)
    return assignment
```

### 5.5 Application to Error Correction Decoding

For quantum error correction, the MAP decoding problem is:

$$\boldsymbol{e}^* = \arg \max_{\boldsymbol{e}:\boldsymbol{H}\boldsymbol{e}=\boldsymbol{s}} P(\boldsymbol{e}) \tag{39}$$

The syndrome constraint $\boldsymbol{H}\boldsymbol{e} = \boldsymbol{s}$ can be incorporated as hard constraints (factors that are $-\infty$ for invalid configurations and 0 otherwise) [16].

| **Aspect** | **BP+OSD** | **Tropical TN** |
|---|---|---|
| Inference type | Approximate marginals | Exact MAP |
| Degeneracy handling | OSD post-processing | Naturally finds one optimal |
| Output | Soft decisions $\to$ hard | Direct hard assignment |
| Complexity | $O(n^3)$ for OSD | Exp. in treewidth |
| Parallelism | Iterative | Highly parallelizable |

Table 10: Comparison of BP+OSD and tropical tensor network decoding approaches

**Key Point. Advantages of tropical tensor networks for decoding:**

- **Exactness**: Guaranteed to find the MAP solution (no local minima)
- **No iterations**: Single forward pass plus backtracking
- **Natural for structured codes**: Exploits graph structure via contraction ordering

**Limitations:**

- Complexity grows exponentially with treewidth
- For dense or high-treewidth codes, may be less efficient than BP+OSD
- Requires careful implementation of backpointer tracking

The tensor network approach is particularly well-suited to codes with local structure, such as topological codes where the treewidth grows slowly with system size [17].

## 5.6 Complexity Considerations

The computational complexity of tropical tensor network contraction is governed by the **treewidth** of the underlying factor graph.

> **Definition.** The **treewidth** $w$ of a graph is the minimum width of any tree decomposition, where width is one less than the size of the largest bag. Intuitively, it measures how "tree-like" the graph is.

| **Code Type** | **Treewidth** | **Contraction Complexity** |
|---|---|---|
| 1D repetition | $O(1)$ | $O(n)$ |
| 2D toric | $O(\sqrt{n})$ | $O\left(n \cdot 2^{\sqrt{n}}\right)$ |
| LDPC (sparse) | $O(\log n)$ to $O(\sqrt{n})$ | Varies |
| Dense codes | $O(n)$ | $O(2^n)$ – intractable |

Table 11: Treewidth and complexity for different code families

> **Key Point.** For LDPC codes used in quantum error correction:
> - The sparse parity check matrix leads to bounded-degree factor graphs
> - Greedy contraction order heuristics (like those in `omeco`) often find good orderings
> - The practical complexity is often much better than worst-case bounds suggest
>
> The tropical tensor network approach provides a systematic way to exploit code structure for efficient exact decoding when the treewidth permits.

# 6 Results and Performance

## 6.1 Error Threshold

**Definition.** The **threshold** $p_{\text{th}}$ is the maximum physical error rate below which the logical error rate decreases with increasing code distance.

- If $p < p_{\text{th}}$: Larger codes → exponentially better protection
- If $p > p_{\text{th}}$: Larger codes → worse protection (error accumulates faster than correction)

For rotated surface codes under circuit-level depolarizing noise, the theoretical threshold is approximately **0.7%** [18].

## 6.2 Threshold Plots

The threshold behavior is visualized by plotting logical error rate versus physical error rate for different code distances. At the threshold, curves for different distances **cross**.

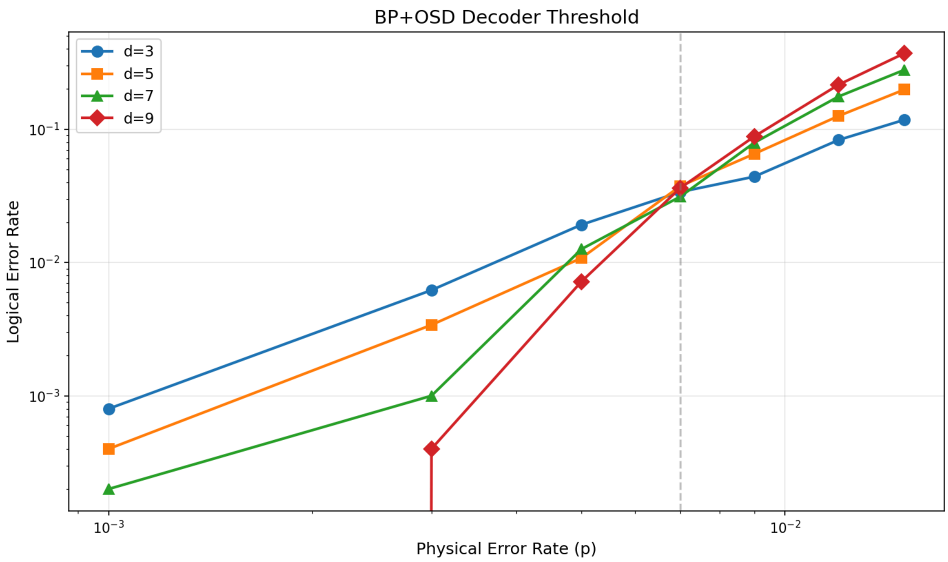



Figure 15: Logical error rate vs. physical error rate for rotated surface codes ($d = 3, 5, 7$). The crossing point indicates the threshold $p_{\text{th}} \approx 0.7\%$.

Below threshold, increasing distance $d$ exponentially suppresses the logical error rate:

$$p_L \approx A \left( \frac{p}{p_{\text{th}}} \right)^{\frac{d+1}{2}} \tag{40}$$

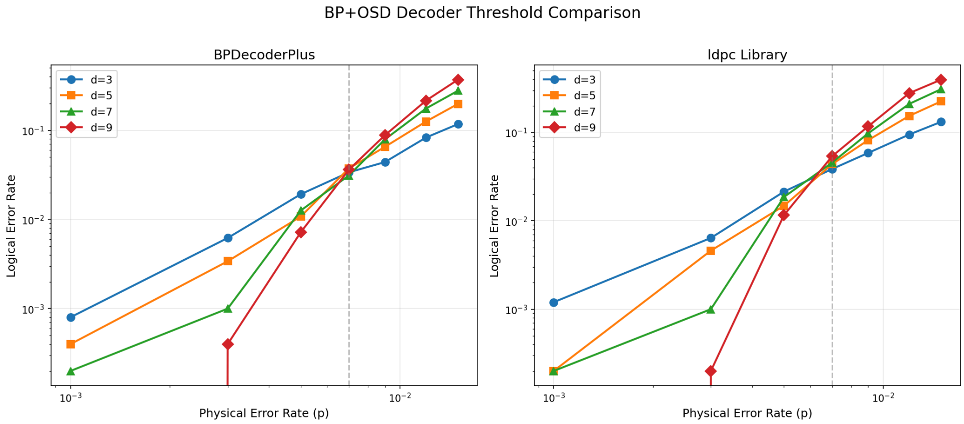



Figure 16: Comparison of BP+OSD decoder performance across different code distances. Error bars show statistical uncertainty from finite sampling.

## 6.3 BP Failure and OSD Recovery

The following plots demonstrate why OSD post-processing is essential for quantum codes:

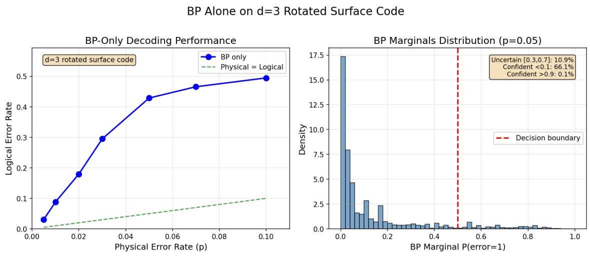



Figure 17: BP alone fails due to degeneracy: the decoder outputs an invalid solution that does not satisfy the syndrome.

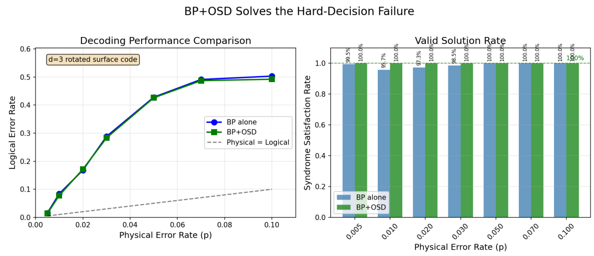



Figure 18: BP+OSD successfully decodes: OSD resolves the split belief by forcing a unique valid solution via matrix inversion.

## 6.4 Reference Results from Literature

For comparison, the original BP+OSD paper [9] reports thresholds for phenomenological noise:

| Code Family | BP Only | BP+OSD-0 | BP+OSD-CS |
|---|---|---|---|
| Toric | N/A (fails) | 9.2% | **9.9%** |
| Semi-topological | N/A (fails) | 9.1% | **9.7%** |
| Random QLDPC | 6.5% | 6.7% | **7.1%** |

Table 12: Thresholds under phenomenological noise [9]. Circuit-level thresholds are typically lower (0.5–1%).

## 6.5 Complexity Analysis

| Component | Complexity | Notes |
|---|---|---|
| BP (per iteration) | $O(n)$ | Linear in block length |
| OSD-0 | $O(n^3)$ | Dominated by Gaussian elimination |
| OSD-$w$ combination sweep | $O(\binom{\lambda}{w})$ | $\lambda \approx 60$, $w = 2$ gives 1770 trials |
| **Total (OSD-0)** | $O(n^3)$ | Practical for $n < 10^4$ |

Table 13: Complexity analysis of BP+OSD decoder

**Key Point. Practical performance:** For a distance-$d$ rotated surface code with $n = d^2$ data qubits:

- $d = 3$: $n = 9$, decoding time $< 1$ ms
- $d = 7$: $n = 49$, decoding time $\sim 10$ ms
- $d = 15$: $n = 225$, decoding time $\sim 1$ s

# 7 Summary

## 7.1 Key Takeaways

1. **Classical BP** computes marginal probabilities via message passing on factor graphs
2. **Quantum codes suffer from degeneracy**: multiple errors can produce the same syndrome, causing BP to output invalid solutions (split beliefs)
3. **OSD resolves degeneracy** by selecting a basis guided by BP soft decisions, then solving via matrix inversion to get a unique valid solution
4. **Combination sweep** efficiently improves OSD-0 by testing low-weight configurations of the remainder bits
5. **BP+OSD is general**: works for Toric codes, semi-topological codes, and random QLDPC codes, achieving near-optimal thresholds

## 7.2 The BP+OSD Recipe

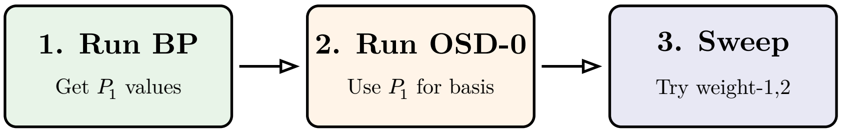



Figure 19: BP+OSD in three steps

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*End of Lecture Note*