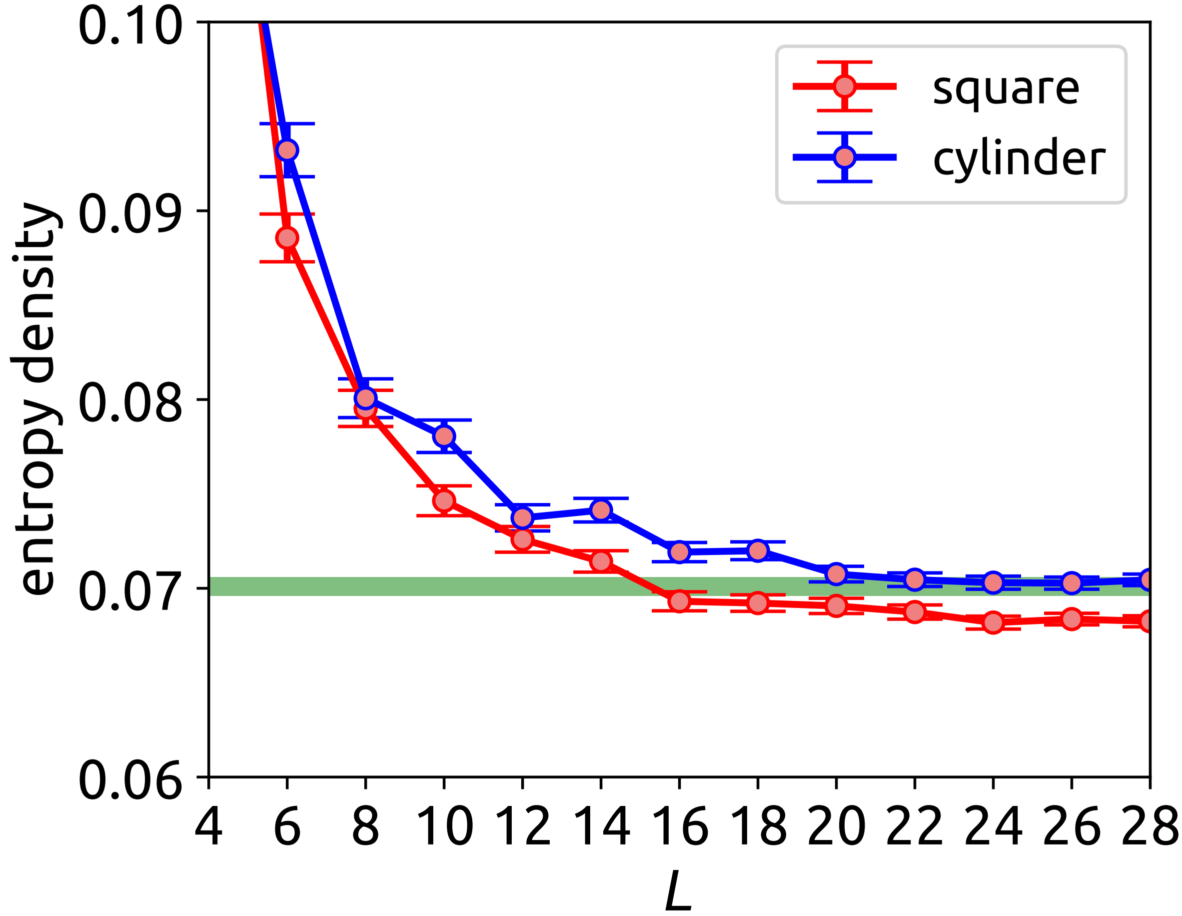

Figure 1: Residual entropy of 2D square lattice spinglass.

Letter to the Editor

Dear Dr. Samindranath,

Thank you for your handling and careful consideration of our manuscript "Tropical Tensor Network for Ground States of Spin Glasses", for sending it for in–depth review, and for giving us the opportunity to address the reviewers' comments and to refine our manuscript. We thank both referees for their careful readings and constructive questions which helped us deepen understandings on the topic and improve the presentation of the manuscript.

We are very happy about reviewers' very positive assessments. We have revised the manuscript according to the reviewers' constructive suggestions. Below, we provide detailed point-to-point response to all of the reviewers' comments, as well as a summary of the changes we have introduced in the main text, figures, and supplementary materials. Thank you again for your consideration!

Best Wishes,
Jin-Guo Liu, Lei Wang and Pan Zhang

List of Main Modifications in the Manuscript

1. The Section ...
2. A new figure, ...
3. We have added a paragraph at the ...
4. In the revised Supplementary Material, a Section ...
5. In the revised Supplementary Material, the Section ...
6. The section ...

Responses to the first Referee

0. The authors studied spin-glass problems by unified exact tensor network approach to compute the ground state energy, identify the optimal configuration, and count the number of solutions. As the authors admitted, the approach is not conceptually new, but they combined several concepts from graphical models, tensor networks, differentiable programming, and quantum circuit simulation, including newly developed techniques, such as tropical tensor network, automatic differentiation. They computed the exact ground state energy of Ising spin glasses with random Gaussian couplings on the square lattice. They also calculated the exact ground state energy of $\pm$ J Ising spin glasses on the chimera graph, and investigated the residual entropy. They used both GPU and D-wave quantum annealer.

   The research is comprehensive, and combining several new techniques will open the way for future research in several directions, not only in spin-glass problems, but also in general optimization problems. This work may appeal to general interest. I support the publication in Phys, Rev. Lett. The authors may pay attention to the following points before the paper is ready for publication.

   Response: We thank the reviewer for the very positive assessment.

1. 1. The residual entropy on the chimera graph is smaller than that of the square lattice. Is there a simple reason for that? The references of the residual entropy of the square lattice are rather old. Can they calculate the residual entropy of the square lattice with modern technique?

   Response:

2. 2. minor points: wit (p.3, right, 5th lines from the bottom) intimidate (p.3, right, 4th lines from the bottom) 3. arxiv of Ref.[66] is missing.

   Response:

3. Additional remarks in response to a query from the editor:

   Thank you for your message regarding LV1718. I read the manuscript again.

   The subject of the research is the ground-state property of 2D spin-glasses. They used several numerical concepts, such as graphical models, tensor networks, differentiable programming, and quantum circuit simulation. The tensor networks are recent hot topics. They admitted that each method is not newly developed by them. They combined several techniques using fashionable hardware such as GPU and D-Wave.

   The obtained physical results of the exact ground states and the degeneracy of states (residual entropy) are interesting as a ground-state search problem, or optimization problem, but limited.

   I evaluated the comprehensive numerical study using many sophisticated techniques, which may contribute to the advance in the field. Depending on how important it is for PRL that a tool should be "new", other PR journals may be more appropriate.

   Response:

Responses to the second Referee

0. In this manuscript, the authors introduce a tensor network formulation of spin glass systems based on the "tropical algebra". This formulation allows for the computation of the ground state energy and its degeneracy through a single contraction of the network. By using the technique of automatic differentiation, the authors can go further and also obtain ground states as well. The authors identify an interesting connection between the ground state computations and state evolution via a quantum circuit, which allows them to make use of optimized software developed for the latter problem.

   I find the ideas presented in the manuscript very ingenious. The manuscript is very well written and even someone who is not an expert in the specific techniques used by the authors can follow the work without much trouble. This could make the manuscript palatable and attractive to a broader audience of readers seeking novel numerical methods in the area of many-body spin problems.

   Response:

1. The application to the 2D Ising spin glass is impressive, so is the application to the Chimera lattice. But this is where I have doubts whether the manuscript brings something sufficiently new to warrant publication in PRL or not.

   Response: Thanks for the notice. In the revised manuscript we have made more effort to make the situation clear, forcing on:

   - Our method is more general. The state-of-the-art methods, the branch-and-cut, heavily rely on the bound technique. For spin glasses on 2D lattice, a particular circle inequality applies so the branch-and-cut is efficient, particularly for Gaussian couplings. However on other topology such as 3D lattices, the bounding method is less power, hence the branch-and-cut method becomes less efficient. Furthermore, for more general problems, such as the 2D Potts glass, linear programming does not even provide a good bound, one has to adopt other immature bounding techniques such as the semi-definite programming, this leads to poor performance of the branch-and-cut. Actually in the revised manuscript we show that our method siginificantly outperforms typical branch-and-cut method using the SDP bounding.
   - Our method can compute both ground-state energy and entropy. Although the existing methods on exact solution of 2D Ising spin glass (based on a particular branch-and-cut technique) works to a larger lattice size, they can only provide ground-state energy. In contrast, our method gives both ground-state energy and ground state entropy together.
   - The branch-and-bound method can be used to compute entropy, but is very in-efficient. In the revised manuscript we have shown that our method is significantly superior. For example, as listed in Table 1 (in the page 147) of reference [3], although the branch-and-cut can give ground-state-energy of 2D $\pm J$ spin glasses on 50×50 lattices, the branch-and-bound works only for 8×8 lattice. In contrast, our method works out the ground-state entroyp of $\pm J$ spin glass on 32×32 lattices in minutes.

2. If the tools developed by the authors do advance our capabilities to solve Ising spin glass, can they address some physics that was previously inaccessible?

   Response: In the revised manuscript we have added more applications to show superiory of our method. These include the 2D Potts glass, 3D Ising spin glass, and the spin glass on the random graphs, where our method significantly improves over the state-of-the-art results.

3. This point is far from clear after reading the manuscript. While solving for the ground state of a 32x32 Ising spin glass lattice in about 590 seconds is very good, I know that previous work using an entirely different method managed to reach lattices as large as 100x100 [see De Simone et al., J. Stat. Phys. vol. 80, p. 487 (1995)]. This may not even be the current record. Could the authors compare their results to the current state-of-the-art and demonstrate their superiority?

Response: Thanks for mentioning the De Simone et al 1995 paper [1]. We have also noticed the other paper of De Simone et al on 1996, but on the $\pm J$ spin glasses [2]. In De Simon 1995 paper, the authors proposed the branch-and-bound algorithm equipped with the sophisticated "circle inequality" to compute ground state energy ....

In Table 1 (in the page 147) of reference [3], results on the ground state energy and entropy are listed for $\pm J$ spin glasses on lattices, using various methods. We can see that although the branch-and-cut method can give ground state energy for 50×50 2D lattice, it does not give any entropy value. The branch-and-bound gives ground-state entropy, but in the table only results 8×8 lattices reported, reflecting the fact that computing entropy is much harder than only computing ground-state energy. We also notice in the table that for 3-D lattice, only $4 \times 4 \times 4$ results are reported, using branch and bound method.

4. In view of these considerations, I do not find the manuscript suitable to PRL in its current form.

   Response: With these new results on spin glasses and optimization problems on regular random graphs and 3-d lattice, as well as Potts model on lattices, we are now more confident that the revised manuscript is suitable for publication in the Physical Review Letter. Thank you for your kind consideration!

References

[1] Caterina De Simone, Martin Diehl, Michael Jünger, Petra Mutzel, Gerhard Reinelt, and Giovanni Rinaldi. Exact ground states of ising spin glasses: New experimental results with a branch-and-cut algorithm. *Journal of Statistical Physics*, 80(1-2):487–496, 1995.

[2] Caterina De Simone, Martin Diehl, Michael Jünger, Petra Mutzel, Gerhard Reinelt, and Giovanni Rinaldi. Exact ground states of two-dimensional±j ising spin glasses. *Journal of Statistical Physics*, 84(5-6):1363–1371, 1996.

[3] Allon Percus, Gabriel Istrate, and Cristopher Moore. *Computational complexity and statistical physics*. OUP USA, 2006.