

Instruction level automatic differentiation on a reversible Turing machine

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This paper considers the instruction level adjoint mode automatic differentiation. Here, instruction level means, given the backward rules of basic instructions like +, -, * and /, one can differentiate an arbitrary program efficiently. In this paper, we review briefly why instruction level automatic differentiation is hard for traditional machine learning frameworks and propose a solution to these problems by back propagating a reversible Turing machine. This implementation can be used to generate the backward rules for functions from exp to unitary matrix multiplication and QR decomposition. Also we discuss the challenges that we face towards a rigorous reversible programming from the instruction and hardware perspective.

I. INTRODUCTION

There are two modes of automatic differentiation (AD) [1], the tangent mode and the adjoint mode. Consider a multi-in multi-out function $\vec{y} = f(\vec{x})$, the tangent mode computes one column of its Jacobian $\frac{\partial \vec{y}}{\partial x_i}$ efficiently, where x_i is one of the input variables. Whereas the adjoint mode computes one row of Jacobian $\frac{\partial y_i}{\partial \vec{x}}$ efficiently. Most popular automatic differentiation packages implement the adjoint mode AD. Because the adjoint mode is computational more efficient in variational applications, where the loss as output is always a scalar. However, implementing adjoint mode AD is harder than implementing its tangent mode counterpart. It requires a program's intermediate state for back propagation, which includes

1. the computational graph,
2. and input variables of nodes in computational graph.

A computational graph is a directed acyclic graph (DAG) that records the relation between data (edges) and functions (nodes). In Pytorch [2] and Flux [3], every variable has a tracker field that stores its parent information, i.e. the input data and function that generate this variable. TensorFlow [4] implements a static computational graph as a description of the program before actual computation happens. Source to source automatic differentiation package Zygote [3, 5] use an intermediate representation (IR) of a program, the static single assignment (SSA) form, as the computational graph in order to back propagate a native julia code. To cache intermediate states, it has to access a global storage.

Several limitations are observed in these AD implementations due to the recording and caching. First of all, these package requires a lot primitive functions with programmer defined backward rules. This is not necessary given the fact that, at the lowest level, these primitive functions are compiled to a finite set of instructions including '+', '-', '*', '/' and conditional jump. By defining backward rules for these basic

instructions, AD should just works. These machine learning packages can not use instructions as the computational graph for practical reasons. The cost of memorizing the computational graph and caching intermediate states is huge. It can decrease the performance for more than two orders when a program contains loops (as we will show latter). Even more, the memory consumption for caching intermediate results increases linearly as time. In many deep learning models like recurrent neural network [6] and residual neural networks [7], the depth can reach several thousand. The memory is often the bottleneck of these programs, which is known as the memory wall problem. [8] Secondly, inplace functions are not handled properly in the diagram of computational graph, because the notation of parent and child are not compatible with inplace operations. Even in Zygote that uses SSA as the computational graph, it is not trivial to handle inplace functions. On the other side, most functions in BLAS and LAPACK are implemented as inplace functions. All AD packages that use BLAS and LAPACK functions have to define their own backward rules for their non-inplace wrappers due to the lack of AD support to inplace functions. Thirdly, obtaining higher order gradients are not efficient in these packages. For example, in most machine learning packages, people back propagate the whole program of obtaining first order gradients to obtain second order gradients. The repeated use of back propagation cause an exponential overhead with respect to the order of gradients. A better approach is using Taylor propagation like in JAX [9]. However Taylor propagation requires writing backward rules for all primitives[JG: true?].

Our solution to these issues is making a program time reversible. It is not same as the work arounds in the machine learning field that use information buffer [10] and reversible activation functions to reduce the memory allocations in recurrent neural network [11] and residual neural networks [12]. Our approach is more general, we develop an embed domain specific language (eDSL) in Julia language that implements reversible Turing machine (RTM). [13, 14]. The gradient of any program written in this eDSL can be obtained in comparable time with the forward computation. The AD implementation is similar to ForwardDiff [15] but runs backward. In history, there has been some prototypes of reversible languages like Janus [16], R (not the popular one) [17], Erlang [18] and object oriented ROOPL [19]. These

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languages have reversible control flow that allows user to input an additional postcondition in control flows to help programs run backward. In the past, the main motivation of making a program time reversible is to support reversible devices. Reversible devices do not have a lower bound of energy consumption by Landauer principle [20]. However, people show less and less interest to reversible programming since 15 years ago since the energy efficiency of CMOS devices are still two orders [14] above this lower bound, this lower bound is not an urgent problem yet. The main contribution of our work is breaking the information barrier between machine learning community and reversible programming community, providing yet another strong motivation to develop reversible programming. Our eDSL borrows the design of reversible control flow in the Janus, meanwhile provides multiple dispatch based abstraction. With these features, the AD engine could be implemented in less than 100 lines. It generates native julia code, and is completely compatible with Julia language. Potential applications includes

1. generate AD rules for primitive functions like `exp`,
2. control problem in robotics [21] where tensor is not the dominating data type,
3. differentiating over reversible integrators [22] without intermediate state caching,
4. Stabilize the backward rules for linear algebra functions. Current backward rules for singular value decomposition (SVD) and eigenvalue decomposition (ED) [23–25] are vulnerable to spectrum degeneracy. The development of backward rules for these linear algebra functions can greatly change researches in physics [26, 27].

In this paper, we first introduce the design of our eDSL in Sec. II. In Sec. III, we show how to back propagation Jacobians and Hessians in a reversible programming language. We propose a self consistent training strategy in Sec. IV that utilizes reversibility directly, which does require gradients. In Sec. V, we show several examples. In Sec. VI, we discuss on several important issues, how time space tradeoff works, reversible instructions and hardwares and finally an outlook to some open problems to be solved.

II. LANGUAGE DESIGN

We introduce NiLang, a eDSL in Julia that simulates RTM. Its grammar is shown in Appendix A. Its main feature is contained in a single macro `@i`. It interprets a NiLang function to a native Julia function. At the same time, it generates the inverse of this function. The target instructions are closed under the inverse operation “ \sim ”, hence all functions defined in NiLang are also closed under “ \sim ” operations.

In a modern programming language, functions are pushed to a global stack for scheduling. The memory layout of a function is consisted of input arguments, a function frame with informations like return address and saved memory segments, local variables and working stack. After each call,

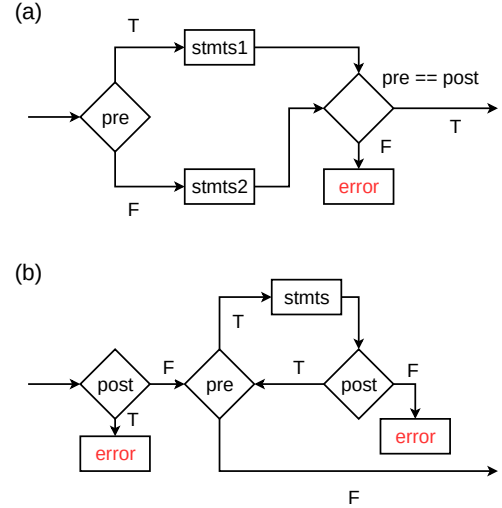


Figure 1. Flow chart for reversible (a) if statement and (b) while statement. “`stmts`”, “`stmts1`” and “`stmts2`” are statements, statements in true branch and statements in false branch respectively. “`pre`” and “`post`” are precondition and postconditions respectively. “`error`” refers to `InvertibilityError`.

the function clears the input arguments, function frame, local variables and working stack, only stores the return value. In the reversible programming style, this kind of design pattern is no longer the best practise, input variables can not be easily discarded after a function call, since discarding information may ruin reversibility. Hence, a function or a instruction changes inputs “inplace” in NiLang. To design such a eDSL, we first introduce a reversible IR that plays a central role in NiLang.

A. Reversible IR

In NiLang’s IR, a statement can be an instruction, a function call, a controlflow, a macrocall or the inverse statement \sim . With the reversible IR, the inverse of a statement can be defined easily as shown in in Table I.

The reversible control flow is different from the irreversible one that a condition expression in a `if` or a `while` statements is a two-element tuple that consist of a precondition and a postcondition. This design allows user putting additional postcondition in control flows to help reverse the program. For the `if` statement as shown in Fig. 1 (a), the program enters the branch specified by precondition. After executing this branch, the program checks the consistency of precondition and postcondition to make sure they are same. In the reverse pass, the program enters the branch specified by the postcondition. For the `while` statement as shown in Fig. 1 (b), before entering, the program check the postcondition to make sure it is false. After each iteration, the program asserts the postcondition to be true. The inverse function exchanges the precondition and postcondition. The definition of reversible `for` statement is similar to irreversible ones except it is more restrictive. The

statement	inverse
<f>(<args>...)	(~<f>)(<args>...)
<y!> += <f>(<args>...)	<y!> -= <f>(<args>...)
<y!> .+= <f>(<args>...)	<y!> .-= <f>(<args>...)
<y!> ∇= <f>(<args>...)	<y!> ∇= <f>(<args>...)
<y!> .∇= <f>(<args>...)	<y!> .∇= <f>(<args>...)
@anc <a> = <expr>	@deanc <a> = <expr>
begin <stmts> end	begin ~(<stmts>) end
if (<pre>, <post>) <stmts1> else <stmts2> end	if (<post>, <pre>) ~(<stmts>) else ~(<stmts>) end
while (<pre>, <post>) <stmts> end	while (<post>, <pre>) ~(<stmts>) end
for <i>=<m>:<s>:<n> <stmts> end	for <i>=<m>:-<s>:<n> ~(<stmts>) end
@safe <expr>	@safe <expr>

Table I. A collection of reversible statements.

program first stores the loop informations, start, step and stop. After the executing the loop, the program checks the values of these variables to make sure they are not changed. The reverse program exchanges start and stop and inverse the sign of step.

There is no assign statements in a reversible language, a reversible replacement is the macro @anc. @anc a = <expr> binds variable a to an initial value specified by <expr>. Its inverse @deanc a = <expr> deallocates the variable a. Before deallocating the variable, the program checks that the value of variable is same as the value of <expr>, otherwise throws an InvertibilityError. @anc and @deanc must appear in pairs inside a function call, a while statement or a for statement. @deanc will be added automatically. Similar designs in Janus and R are local/delocal statement and let statement. The additional check underlines the difference between the irreversible assign statement and reversible ancilla statement. The @safe macro can be followed by an arbitrary statement, it allows user to use external statements that does not break reversibility. For example, one can use @safe @show var for debugging.

B. Compiling

The interpretation of a reversible function consists three stages. The first stage preprocess human inputs to a reversible IR, The second stage generates the reversed IR according to table Table I. The third stage is translating this reversible IR to native Julia code. The following example shows how to compile an if statement

```
julia> using NiLangCore, MacroTools

julia> ex0 = :(if (x > 3, ~)
               -arr[3].value += x * y
             end);

julia> ex0 |> prettify
:(if (x > 3, ~)
  -(arr[3]).value += x * y
end)

julia> ex1 = NiLangCore.precom_ex(ex0,
  NiLangCore.PreInfo());

julia> ex1 |> prettify # after stage 1
:(if (x > 3, x > 3)
  -(arr[3]).value += x * y
else
end)

julia> ex2 = NiLangCore.dual_ex(ex1);

julia> ex2 |> prettify # after stage 2
:(if (x > 3, x > 3)
  -(arr[3]).value -= x * y
else
end)

julia> ex3 = NiLangCore.interpret_ex(ex2);

julia> ex3 |> prettify # after stage 3
quote
  wren = x > 3
  if wren
    @instr -(arr[3]).value -= x * y
  else
  end
  @invcheck x > 3 wren
end
```

In the first stage, the preprocessor expands the symbol ~ in postcondition field of if statement to the precondition as shown above. Besides, it adds missing @deanc to ensure @anc and @deanc statements appear in pairs and expands @routine macro. @routine r <stmt> records a statement to symbol r. When ~@routine r is called, the inverse statement is inserted to that position for uncomputing. We will use macro extensively in the example in Sec. V. In the last stage, the compiler adds @instr before each instruction and function call statement. The macro @instr assign the output of a function to the argument list of a function. We will explain this macro in detail in next subsection. It also adds statements to check the consistency between preconditions and postconditions to ensure reversibility. In the above example, @invcheck x > 3 wren will throw an error if the variable wren and x > 3 are not equivalent. Finally, at the end of a function body, it attaches a return statement that uses input variables as the output. Now the function is ready to execute on the host language.

C. Types and Dataviews

So far, the language design is not too different from a traditional reversible language. Next, we introduce type and dataviews that are important to the implementation of adjoint mode AD.

The constructor of a type is a reversible function. Its inverse function is a “destructor”, which does not deallocate memory directly but unpacks data. One can use `@iconstruct` to define a reversible constructor. For example, we can define a type that copies the input data to a new field.

```
using NiLangCore, Test

struct DVar{T}
    x::T
    g::T
end

@iconstruct function DVar(xx, gg=zero(xx))
    gg += identity(xx)
end

@test (~DVar)(DVar(0.5)) == 0.5
```

Here, the `@iconstruct` generates a reversible constructor with single parameters `xx` as input. The statement `gg = zero(xx)` initializes a new memory to be used. The body of function is a reversible program that modifies `xx` and `gg` reversibly. Finally it calls the default constructor `DVar(xx, gg)`. It is easy to define the inverse procedure that transform a `DVar{T}` instance to a `T` instance by reversing the statements. With the flexibility to use types, it is not necessary to use global stacks in our eDSL.

Before introducing dataviews, let's first consider the following line that appear in the last subsection

```
-arr[3].value += x * y
```

In Julia, this statement will raise a syntax error, since a function call `(-)` can not be assigned. Meanwhile `arr[3]` might be a immutable type. In our eDSL, we wish it works because every memory cell should be modifiable “in-place”. As we have mentioned, `-arr[3].value += x * y` is translated to `@instr -arr[3].value += x * y` at the third stage. `@instr` translate the statement to

```
1 res = PlusEq*(-arr[3].value, x, y)
2 arr[3] = chfield(arr[1], Val{:value},
3   chfield(arr[3].value, -, res[1]))
4 x = res[2]
5 y = res[3]
```

`PlusEq*(-arr[3].value, x, y)` computes the out-

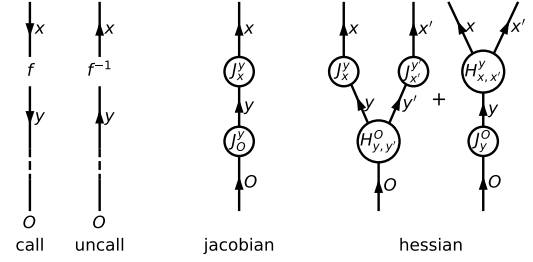


Figure 2. Adjoint rules for Jacobians and Hessians in tensor network language.

put, which is a tuple of length 3. The assignments in line 4 and 5 can be handled directly while the assignment in line 2-3 are not so, where `chfield` is used to modify a dataview. A dataview of a data can be data itself, a field of its view, an array element of its view, or a bijective mapping of its view. `chfield(x, Val{:value}, val)` can modify the value field of an instance given the default constructor of a type is not overwritten. A bijective mapping of a field can also be modified. To change `-x`, one simply overwrite the `chfield(x, -, res[1])` function.

III. AUTOMATIC DIFFERENTIATION

A. First order gradient

Given a node $\vec{y} = f(\vec{x})$ in a computational graph, the adjoint mode AD propagates the Jacobians in the reversed direction like

$$\begin{aligned} J_{O'}^O &= \delta_{O,O'} \\ J_x^O &= J_y^O J_x^y, \end{aligned} \quad (1)$$

where O represents the outputs of the program, $J_{x/y}^O$ is the adjoint to be propagated, and J_x^y is the local Jacobian matrix. Einstein's notation is used so that duplicated indices are summed over. This back propagation rule can be rewritten in the language of tensor networks [28] as shown in Fig. 2.

In reversible programming with multiple dispatch, the adjoint mode AD algorithms can be implemented as

Algorithm 1: Reversible programming AD

Result: $\text{grad}(\vec{x}_g)$

let `iloss` be the index of loss variable in \vec{x}

$\vec{y} = f(\vec{x})$

$\vec{y}_g = \text{GVar}(\vec{y})$

$\text{grad}(\vec{y}_g[\text{iloss}]) += 1.0$

$\vec{x}_g = f^{-1}(\vec{y}_g)$

Here, `GVar` is a reversible type. The constructor attaches a zero gradient field to a variable, which is similar to the dual number in tangent mode automatic differentiation [15]. If the input is an array, `GVar` will be broadcasted to each array element. The gradient field of a `GVar` instance can be accessed by the `grad` dataview. Its inverse `~GVar` deallocates the gradient

field safely and returns its value field. Here, "safely" means before deallocation, the program will check the gradient field to make sure its value is restored to 0. When an instruction `instruct` meets a `GVar`, besides computing its value field $\text{value}(\vec{y}) = \text{instruct}(\text{value}(\vec{x}))$, it also updates the gradient field $\text{grad}(\vec{y}) = [J_{\vec{x}}^{\vec{y}}]^{-1} \text{grad}(\vec{x})$, where $[J_{\vec{x}}^{\vec{y}}]^{-1}$ is the Jacobian of `instruct`⁻¹. One can define this gradient function on either `instruct` or `instruct`⁻¹, NiLang will generate the backward rule for the inverse function automatically. As an example, We bind the adjoint function of ROT to its reverse IROT by defining a new function that dispatch to `GVar`

```
@i function IROT(a!::GVar, b!::GVar, θ::GVar)
    IROT(value(a!), value(b!), value(θ))
    NEG(value(θ))
    value(θ) -= identity(π/2)
    ROT(grad(a!), grad(b!), value(θ))
    grad(θ) += value(a!) * grad(a!)
    grad(θ) += value(b!) * grad(b!)
    value(θ) += identity(π/2)
    NEG(value(θ))
    ROT(grad(a!), grad(b!), π/2)
end
```

The definition of ROT instruction could be found in Sec. B. This backward rule has been included in NiLang, one can check the gradients by typing in a Julia REPL

```
julia> using NiLang, NiLang.AD

julia> x, y, θ = GVar(0.5), GVar(0.6), GVar(0.9)
(GVar(0.5, 0.0), GVar(0.6, 0.0), GVar(0.9, 0.0))

julia> @instr grad(x) += identity(1.0)

julia> @instr ROT(x, y, θ)

julia> x, y, θ
(GVar(-0.1591911616411577, 0.6216099682706646),
 GVar(0.7646294357761403, 0.7833269096274833),
 GVar(0.8999999999999999, 0.6))
```

The implementation of Algorithm 1 is so short that we present the function definition as follows

```
@i function (g::Grad)(args...; kwargs...)
    @safe @assert count(x -> x isa Loss, args) == 1
    @anc iloss = 0
    @routine getiloss begin
        for i=1:length(args)
            if (tget(args,i) isa Loss, iloss==i)
                iloss += identity(i)
                (~Loss)(tget(args,i))
            end
        end
    end

    g.f(args...; kwargs...)
    GVar.(args)
    grad(tget(args,iloss)) += identity(1.0)
    (~g.f)(args...; kwargs...)

    ~@routine getiloss
end
```

Input variables must contain exactly one `Loss` instance. This program first checks the input parameters and locate the loss variable as `iloss`. Then `Loss` unwraps the loss variable. After computing the forward pass and backward pass, `@routine getiloss` uncomputes the ancilla `iloss` and returns the location information to the target variable. `tget(args, i)` is used to get the *i*-th element of a tuple. In NiLang, array indexing are supported. To avoid confusion, tuple indexing is forbidden deliberately.

The overhead of using `GVar` type can be removed thanks to Julia's multiple dispatch and type inference. Let's consider a simple example that accumulate 1.0 to a target variable *x* for *n* times

```
julia> using NiLang, NiLang.AD, BenchmarkTools

julia> @i function prog(x, one, n::Int)
    for i=1:n
        x += identity(one)
    end
end

julia> @benchmark prog'(Loss(0.0), 1.0, 10000)
BenchmarkTools.Trial:
  memory estimate: 1.05 KiB
  allocs estimate: 39
  -----
  minimum time:      35.838 s (0.00% GC)
  median time:       36.055 s (0.00% GC)
  mean time:         36.483 s (0.00% GC)
  maximum time:      185.973 s (0.00% GC)
  -----
  samples:            10000
  evals/sample:       1
```

We implement the same function with TensorFlow, PyTorch and Zygote for comparison. The codes could be found in our paper's github repository [29]. Benchmark results on CPU Intel(R) Xeon(R) CPU E5-2680 v4 @ 2.40GHz are

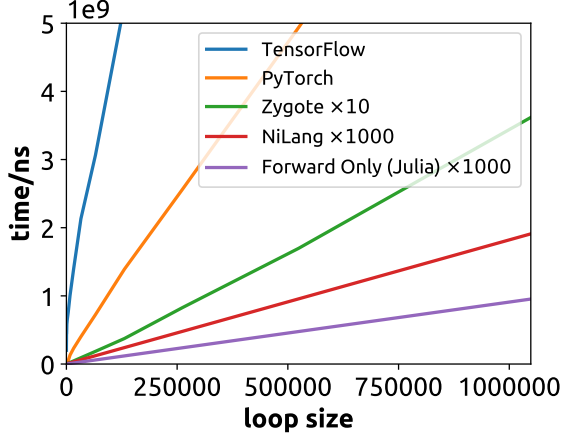


Figure 3. The time for obtaining gradient as function of loop size. $\times n$ in legend represents a rescaling of time.

shown in Fig. 3. One can see that the NiLang implementation is unreasonably fast, it is approximately two times the forward pass written in native Julia code. Reversible programming is not always as fast as its irreversible counterparts. In practical applications, a reversible program may have memory or computation overhead. We will discuss the details of time and space tradeoff in Sec. VIA.

B. Second order gradient

Second order gradients can be obtained in two different approaches.

1. Back propagating first order gradients

Back propagating the first order gradients is the most widely used approach to obtain the second order gradients. Suppose the function space is closed under gradient operation, one can obtain higher order gradients by recursively differentiating lower order gradients without defining new backward rules.

Fig. 4 show the four passes in computing Hessian. The first two passes (black lines) are obtaining gradients. Before entering the third pass, the program wraps each field in GVar with GVar. Then we pick a variable x_i and add 1 to $\text{grad}(\text{grad}(x_i))$ to compute the i -th row of Hessian. At the final stage, the $\sim\text{GVar}$ operation does not unwrap GVar directly because the second order gradient fields may not be zero in this case. Instead, we use $\text{Partial}\{x\}(\cdot)$ to safely compute $\sim\text{GVar}$ on data type $\text{GVar}\{<\text{GVar}, <\text{GVar}\}$. $\text{Partial}\{x\}$ takes the x field of an instance without deallocating memory. By repeating the above process for different x_i , one can obtain the Hessian matrix.

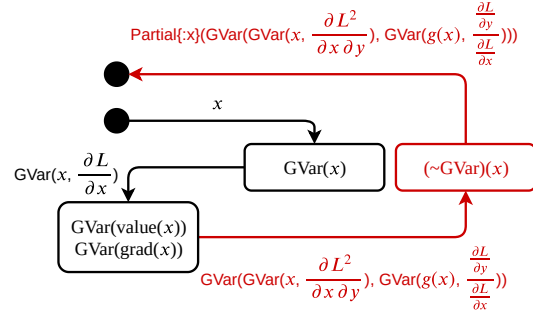


Figure 4. Obtaining the second order gradient with the reversible differentiation approach. Black lines are computing gradients, red lines are back propagating the process of obtaining the first order gradients. Annotations on lines are data types and their fields used in the computation.

2. Taylor propagation

A probably more efficient approach is back propagating Hessians directly [30] by

$$\begin{aligned} H_{O',O''}^O &= \mathbf{0}, \\ H_{x,x'}^O &= J_x^y H_{y,y'}^O J_{x'}^{y'} + J_y^O H_{x,x'}^y. \end{aligned} \quad (2)$$

Here, the Hessian tensor $H_{x,x'}^O$ is rank three, where the top index is often takes as a scalar and omitted. In tensor network language, the above equation can be represented as in Fig. 2. Hessian propagation is a special case of Taylor propagation. With respect to the order of gradients, Taylor propagation is exponentially more efficient in obtaining higher order gradients than differentiating lower order gradients recursively. However, the exhausted support to Taylor propagation [9] requires much more effort than Jacobian propagation, this is why most AD packages choose the recursive approach. Instruction level automatic differentiation has the advantage of having very limited primitives. It is more flexible in obtaining higher order gradients like Hessian. An example is provided in Sec. VB.

C. Gradient on ancilla problem

An ancilla can also carry gradient during computation. As a result, even if an ancilla can be uncomputed regoriously in the original program, its GVar version can not be safely uncomputed. In these case, we simply “drop” the gradient field instead of raising an error. In this subsection, we prove the following theorem

Theorem 1. *Dropping the gradient field of an ancilla at deallocation stage does not harm the reversibility of a gradient function.*

Proof. Consider a reversible function $\vec{y}, b = f(\vec{x}, a)$, where a and b are the input and output values of an ancilla. The

reversibility requires $b = a$ for any \vec{x} . So that

$$\frac{\partial b}{\partial \vec{x}} = \vec{0}. \quad (3)$$

In the backward pass, we discarded the gradient field of b . The gradient fields are derived from the values of variables, they should not have any effect to the value fields. The rest is to show changing the value of $\text{grad}(b)$ does not result in a different $\text{grad}(\cdot)$ in the backward pass. It can be seen from the expression the back propagation rule

$$\frac{\partial O}{\partial \vec{x}} = \frac{\partial O}{\partial \vec{y}} \frac{\partial \vec{y}}{\partial \vec{x}} + \frac{\partial O}{\partial b} \frac{\partial b}{\partial \vec{x}}, \quad (4)$$

where the second term with $\frac{\partial O}{\partial b}$ vanishes naturally. Hence one can assume any value in the gradient field of an ancilla when entering a function, it does not have to be the discarded value. \square

This theorem is very important to the reversibility of gradient functions, without it, the recursive differentiation scheme to obtain second order gradients can not work. On the other side, we emphasize that although the initial value of the gradient field of an ancilla can be randomly chosen, not having a gradient field at the beginning is a different story.

IV. LEARN BY CONSISTENCY

Consider a training task that with input \vec{x}^* and output \vec{y}^* , find a set of parameters \vec{p}_x that satisfy $\vec{y}^* = f(\vec{x}^*, \vec{p}_x)$. In traditional machine learning, we define a loss $\mathcal{L} = \text{dist}(\vec{y}^*, f(\vec{x}^*, \vec{p}_x))$ and minimize it with gradient $\frac{\partial \mathcal{L}}{\partial \vec{p}_x}$. This works only when the target function is locally differentiable.

Here we provide an alternative by making use of reversibility. We construct a reversible program $\vec{y}, \vec{p}_y = f_r(\vec{x}, \vec{p}_x)$, where \vec{p}_x and \vec{p}_y are “garbage” spaces which include trainable parameters and auxillary parameters. The algorithm can be summarized as

Algorithm 2: Learn by consistency

Result: \vec{p}_x
 Initialize \vec{x} to \vec{x}^* , garbage space \vec{p}_x to random.
if \vec{p}_y *is null* **then**
 $\vec{x}, \vec{p}_x = f_r^{-1}(\vec{y}^*)$
else
 $\vec{y}, \vec{p}_y = f_r(\vec{x}, \vec{p}_x)$
 while $\vec{y} \neq \vec{y}^*$ **do**
 $\vec{y} = \vec{y}^*$
 $\vec{x}, \vec{p}_x = f_r^{-1}(\vec{y}, \vec{p}_y)$
 $\vec{x} = \vec{x}^*$
 $\vec{y}, \vec{p}_y = f_r(\vec{x}, \vec{p}_x)$

Here, $\text{garbage}(\cdot)$ is a function for taking the garbage space. This algorithm utilizes the self-consistency relation

$$\vec{p}_x^* = \text{garbage}(f_r^{-1}(\vec{y}^*, \text{garbage}(f_r(\vec{x}^*, \vec{p}_x^*)))), \quad (5)$$

Similar idea of training by consistency is used in self-consistent meanfield theory [31] in physics. The difficult part

of self-consistent training is to find a self-consistency relation, here the reversibility provides a natural self-consistency relation. Learn by consistency can be used to handle discrete optimization. However, it is not a silver bullet, and should be used with caution. Let’s consider the following example

```
@i function f1(y!, x, p!)
    p! += identity(x)
    y! -= exp(x)
    y! += exp(p!)
end

@i function f2(y!, x!, p!)
    p! += identity(x!)
    y! -= exp(x!)
    x! -= log(-y!)
    y! += exp(p!)
end

function train(f)
    loss = Float64[]
    p = 1.6
    for i=1:100
        y!, x = 0.0, 0.3
        @instr f(y!, x, p)
        push!(loss, y!)
        y! = 1.0
        @instr (~f)(y!, x, p)
    end
    loss
end
```

Functions $f1$ and $f2$ computes $f(x, p) = e^{(p+x)} - e^x$ and stores the output in a new memory $y!$. The only difference is $f2$ “uncompute” x arithmetically. The task of training is to find a p that make the output value equal to target value 1. After 100 steps, $f2$ runs into the fixed point with x equal to 1 upto machine precision. However, $f1$ does not do any training. The training of $f2$ fails because this function actually computes $f1(y, x, p) = y + e^{(p+x)} - e^x, x, x + p$, where the training parameter p is completely determined by the garbage space on the output side $x \cup x + p$. As a result, shifting y directly is the only approach to satisfy the consistency relation. On the other side, $f2(y, x, p) = y + e^{(p+x)} - e^x, \tilde{0}, x + p$, the garbage space can not uniquely determine the input garbage space p and y . Here we use $\tilde{0}$ to denote the zero with rounding error.

By viewing \vec{x} and parameters in \vec{p}_x as variables, we can study the trainability from the information perspective.

Theorem 2. Only if the conditional entropy $S(\vec{y}|\vec{p}_y)$ is nonzero, algorithm 2 is trainable.

Proof. The above example reveals a fact that the training can not work when the output \vec{p}_y completely determines \vec{p}_x , that

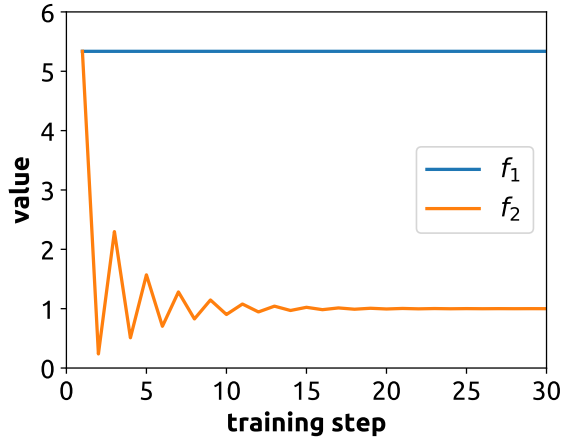


Figure 5. The value of x as a function of self-consistent training step.

is

$$\begin{aligned}
 S(\vec{p}_x | \vec{p}_y) &= S(\vec{p}_x \cup \vec{p}_y) - S(\vec{p}_y) \\
 &\leq S((\vec{p}_x \cup \vec{x}) \cup \vec{p}_y) - S(\vec{p}_y), \\
 &\leq S((\vec{p}_y \cup \vec{y}) \cup \vec{p}_y) - S(\vec{p}_y), \\
 &\leq S(\vec{y} | \vec{p}_y).
 \end{aligned} \tag{6}$$

The third line uses the bijectivity $S(\vec{x} \cup \vec{p}_x) = S(\vec{y} \cup \vec{p}_y)$. \square

This inequality shows that when the garbage space on the output side satisfies $S(\vec{y} | \vec{p}_y) = 0$, i.e. contains all information to determine the output field, the input parameters are also completely determined by this garbage space. In the above examples, it corresponds to the case $S(e^{(x+y)-e^x} | x \cup x + y) = 0$ in f_1 . One should remove the redundancy of information by uncomputing to make training by consistency work properly.

V. EXAMPLES

A. Computing Fibonacci Numbers

An example that everyone likes

```

@i function rfib(out, n::T) where T
    @anc n1 = zero(T)
    @anc n2 = zero(T)
    @routine init begin
        n1 += identity(n)
        n1 -= identity(1.0)
        n2 += identity(n)
        n2 -= identity(2.0)
    end
    if (value(n) <= 2, ~)
        out += identity(1.0)
    else
        rfib(out, n1)
        rfib(out, n2)
    end
    ~@routine init
end

```

The following example shows how to construct a reversible **while** statement. It computes the first Fibonacci number that greater or equal to 100

```

@i function rfib100(n)
    @safe @assert n == 0
    while (fib(n) < 100, n != 0)
        n += identity(1.0)
    end
end

```

In this example, the postcondition $n \neq 0$ is false before entering the loop, and becomes true in later iterations. In the reverse program, the **while** statement stops at $n == 0$.

B. exp function

An exp function can be computed using Taylor expansion

$$y!+ = \sum_n \frac{x^n}{\text{factorial}(n)} \tag{7}$$

This is a recursive algorithm that mimics the famous pebble game [13]. In each iteration, the accumulated term is $s_n \equiv \frac{x^n}{\text{factorial}(n)}$, the recursion relation is written as $s_n = \frac{x s_{n-1}}{n}$. There is no known constant memory and polynomial time algorithm to pebble game. Here the case is different. Notice \ast and $/$ are arithmetically reversible to each other, we can “uncompute” with the previous state with the relation $s_{n-1} = \frac{n s_n}{x}$ to deallocate the memory.


```

using NiLang, NiLang.AD

@i function iexp(y!, x::T; atol::Float64=1e-14)
    where T
    @anc anc1 = zero(T)
    @anc anc2 = zero(T)
    @anc anc3 = zero(T)
    @anc iplus = 0
    @anc expout = zero(T)

    y! += identity(1.0)
    @routine r1 begin
        anc1 += identity(1.0)
        while (value(anc1) > atol, iplus != 0)
            iplus += identity(1)
            anc2 += anc1 * x
            anc3 += anc2 / iplus
            expout += identity(anc3)
            # arithmetic uncompute
            anc1 -= anc2 / x
            anc2 -= anc3 * iplus
            SWAP(anc1, anc3)
        end
    end

    y! += identity(expout)

    ~@routine r1
end

```

The definition of SWAP instruction can be found in Appendix B. The two lines below the comment “# arithmetic uncompute” “uncomputes” variables anc1 and anc2 approximately. This uncomputation is only arithmetically true. As a result, the final output is not exact due to the rounding error. On the other side, the reversibility is not harmed since the inverse call at the last line of function uncomputes all ancilla bits rigorously. The while statement takes two conditions, the precondition and postcondition. Precondition $\text{val}(\text{anc1}) > \text{atol}$ indicates when to break the forward pass and post condition $\text{!isapprox}(\text{iplus}, 0.0)$ indicates when to break the backward pass.

To obtain the gradient, one can wrap the loss with Loss type and feed it into iexp' function

```

julia> y!, x = 0.0, 1.6
(0.0, 1.6)

julia> @instr iexp'(Loss(y!), x)

julia> grad(x)
4.9530324244260555

julia> y!, x = 0.0, 1.6
(0.0, 1.6)

julia> simple_hessian(iexp, (Loss(y!), x))
2×2 Array{Float64,2}:
 0.0  0.0
 0.0  4.95303

```

iexp' is a callable instance of type `Grad{typeof(iexp)}`. iexp'(\cdot) returns input variables with updated gradient field. The gradient function is implemented reversibly so that the gradient field of output can be differentiated again to obtain Hessians as shown in Fig. 4.

The back propagation approach can be more efficient in obtaining higher order gradients.

```

julia> y!, x = 0.0, 1.6
(0.0, 1.6)

julia> @instr iexp''(Loss(y!), x)

julia> collect_hessian()
2×2 Array{Float64,2}:
 0.0  0.0
 0.0  4.95303

```

iexp'' computes the second order gradients by wrapping variables with type BeijingRing [32]. Whenever an n -th variable or ancilla is created, we push a ring of size $2n - 1$ to a global tape. Whenever an ancilla is deallocated, we pop a ring from the top. The n -th ring stores $H_{i \leq n, n}$ and $H_{n, i < n}$. We didn't use the symmetry relation $H_{i,j} = H_{j,i}$ to save memory here. To simplify the implementation of backward rules described in the right most panel of Fig. 2. The final result can be collected by calling a global function collect_hessian.

C. QR decomposition

Let's consider a slightly non-trivial function, the QR decomposition

```

@i function iqr(Q, R, A::AbstractMatrix{T}) where T
    @anc anc_norm = zero(T)
    @anc anc_dot = zeros(T, size(A,2))
    @anc ri = zeros(T, size(A,1))
    for col = 1:size(A, 1)
        ri .+= identity.(A[:,col])
        for precol = 1:col-1
            idot(anc_dot[precol], Q[:,precol], ri)
            R[precol,col] +=
                identity(anc_dot[precol])
            for row = 1:size(Q,1)
                ri[row] -= anc_dot[precol] *
                    Q[row, precol]
            end
        end
        inorm2(anc_norm, ri)

        R[col, col] += anc_norm^0.5
        for row = 1:size(Q,1)
            Q[row,col] += ri[row] / R[col, col]
        end

        ~(ri .+= identity.(A[:,col]));
        for precol = 1:col-1
            idot(anc_dot[precol], Q[:,precol], ri)
            for row = 1:size(Q,1)
                ri[row] -= anc_dot[precol] *
                    Q[row, precol]
            end
        end
        end;
        inorm2(anc_norm, ri)
    end
end

```

```

@i function idot(out, v1::AbstractVector{T}, v2)
    where T
        @anc anc1 = zero(T)
        for i = 1:length(v1)
            anc1 += identity(v1[i])
            CONJ(anc1)
            out += v1[i]*v2[i]
            CONJ(anc1)
            anc1 -= identity(v1[i])
        end
    end

@i function inorm2(out, vec::AbstractVector{T})
    where T
        @anc anc1 = zero(T)
        for i = 1:length(vec)
            anc1 += identity(vec[i])
            CONJ(anc1)
            out += anc1*vec[i]
            CONJ(anc1)
            anc1 -= identity(vec[i])
        end
    end
end

```

One can easily check the gradient of this naive implementation of QR decomposition is correct

```

using Test
A = randn(4,4)
q = zero(A)
r = zero(A)

@i function test1(out, q, r, A)
    iqr(q, r, A)
    out += identity(q[1,2])
end

@i function test2(out, q, r, A)
    iqr(q, r, A)
    out += identity(r[1,2])
end

@test check_grad(test1, (Loss(0.0), q, r, A);
    atol=0.05, verbose=true)
@test check_grad(test2, (Loss(0.0), q, r, A);
    atol=0.05, verbose=true)

```

Here, the `check_grad` function is a gradient checker function defined in module `NiLangCore.ADCore`.

D. Unitary Matrices

This implementatin of QR decomposition is very naive that does not consider reorthogonalization. `idot` and `inorm2` are functions to compute dot product and vector norm. They are implemented as

Unitary matrices can be used to ease the gradient exploding and vanishing problem in recurrent networks [33–35]. One of the simplest way to parametrize a unitary matrix is representing a unitary matrix as a product of two-level unitary operations [35]. A real unitary matrix of size N can be parametrized

compactly by $N(N - 1)/2$ rotation operations [36]

$$\text{ROT}(a!, b!, \theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a! \\ b! \end{bmatrix}, \quad (8)$$

where θ is the rotation angle, $a!$ and $b!$ are target registers.

```
@i function umm!(x, )
  @anc k = 0
  @anc Nin = size(x, 2)
  @anc Nout = size(x, 1)
  for j=1:Nout
    for i=Nin-1:-1:j
      k += identity(1)
      ROT(x[i], x[i+1], [k])
    end
  end

  # uncompute k
  for j=1:Nout
    for i=Nin-1:-1:j
      k -= identity(1)
    end
  end
end
```

VI. DISCUSSION AND OUTLOOK

In this paper, we introduce a Julia eDSL NiLang that simulates a reversible Turing machine (RTM). We developed an instruction level automatic differentiation tool on this eDSL for differential programing. It can differentiate over any program to any order reliably and efficiently without sophisticated designs to memorize computational graph and intermediate states. Also, we introduce a new training strategy that does not rely on gradients, learn by consistency.

In the following, we discussed the paractical side of writing reversible programs, and several future directions to go.

A. Time Space Tradeoff

In history, there has been many other designs of reversible languages and instruction sets. One of the main reason why RTM is not so popular is it may have either a space overhead propotional to computing time T or a computational overhead that sometimes can be even exponential. In the simplest g-segment trade off scheme [37, 38],

$$\text{Time}(T) = \frac{T^{1+\epsilon}}{S^\epsilon} \quad (9)$$

$$\text{Space}(T) = \epsilon 2^{1/\epsilon} (S + S \log \frac{T}{S}) \quad (10)$$

with T and S the time and space usage on a irreversible Turing machine, ϵ is the control parameter. It is related to the g-segment trade off parameters by $g = k^n$, $\epsilon = \log_k(2k - 1)$ with $n \geq 1$ and $k \geq 1$. This section, we try to convince the readers

that the overhead of reversible computing is not as terrible as people thought.

First, at $\epsilon \rightarrow 0$, the resource used by a RTM is same as the caching strategy used in a traditional machine learning package that memorizing every inputs of primitives. Memorizing inputs always make a program reversible since it does not discard any information. For deep neural networks, people used checkpointing trick to trade time with space [39], which is also a widely used trick in reversible programming [13]. RTM just provides more alternatives to trade time and space.

Second, some computational overhead of running recursive algorithms with limited space resources can be mitigated by "pseudo uncomputing" without sacrificing reversibility like in the `iexp` example. With reversible floating point `*` and `/` operations [40], many primitives can be implemented with pure reversible functions, which may significant decrease the computation time and memory usage. We will review this point in Sec. VIB.

Third, making reversible programming an eDSL rather than a independant language allows flexible choices between reversibility and computational overhead. For example, in order to deallocate the gradient memory in a reversible language one has to uncompute the whole process of obtaining this gradient. As a reversible eDSL, we have the flexibility to deallocate the memory irreversibly, i.e. trade energy with time. To quantify the overhead of uncomputing, we introducing the concept

Definition 1 (program granularity). The logarithm of the ratio between the execution time of a reversible program and its irreversible counter part.

$$\log_2 \frac{\text{Time}(T)}{T} \quad (11)$$

Whenever the program uncomputes the memory, the program granularity increases by approximately one. In instruction design, defining primitive functions like `iexp`, and deallocating gradients used for training, we need uncomputing ancilla bits and increase the granularity. The overhead increase exponentially as the granuality increase. The granularity can be decreased by cleverer compilation of a program, since the uncomputing of ancillas can be executed at any level of abstraction.

B. Instructions and Hardwares

So far, our eDSL is not really compiled to instructions, instead, it runs on a irreversible host Julia. In the future, it can be compiled to low level instructions and is execute on a reversible devices. For example, the control flow defined in this NiLang can be compiled to reversible instructions like conditioned `goto` instruction. The reversibility requires the target instruction a `comefrom` instruction that specifying the postcondition. [41]

Arithmetic instructions should be redesigned to support better reversible programs. The major obstacle to exact reversibility programming is current floating point adders used in our computing devices are not exactly reversible.

There are proposals of reversible floating point adders and multipliers [40, 42–44] that introduces garbage bits to ensure reversibility. With these infrastructure, a reversible program can be executed without suffering from the irreversibility from rounding error and reduce the computational and memory overhead significantly. Notably, in machine learning field, information buffer is used to make multiplication operations [10] reversible and reduce memory cost.

Reversible programming is not necessarily related to reversible hardware. Reversible programs are a subset of irreversible programs, hence can be simulated efficiently on CMOS devices [41]. By using reversible hardware [1], the computation may cost zero energy by Landauer’s principle [20]. Reversible hardware is not necessarily related to reversible gates such as Toffoli gate and Fredkin gate. Devices with the ability of recovering signal energy is able to save energy, which is known as generalized reversible computing. [45, 46] In the following, we comment briefly on a special type of reversible device Quantum computer.

1. Quantum Computers

One of the fundamental difficulty of building a quantum computer is, unlike a classical state, an unknown quantum state can not be copied. Quantum random access memory [47] is very hard to design and implement, which is known to have many caveats [48]. A quantum state in an environment will decoherence and can not be recovered, this underlines the simulation nature of quantum devices. Reversible computing does not enjoy the quantum advantage, nor the quantum disadvantages of non-cloning and decoherence. The reversibility of quantum computing comes from the fact that microscopic processes are unitary. On the other side, the irreversibility is rare, it can come from interacting with classical devices. Irreversible processes include decaying, qubit state resetting, measurements and classical feedbacks to quantum devices. These are typically harder to implement on a quantum device.

Given the fundamental limitations of quantum decoherence and non-cloning and the reversible nature of microscopic world. It is reasonable to have a reversible computing device to bridge the gap between classical and universal quantum computing. By introducing entanglement little by little, we can accelerate some basic components. For example, quantum

Fourier transformation provides an interesting alternative to the adders and multipliers by introducing one additional CPHASE gate even though it is a classical-in classical-out algorithm. [49] By introducing quantum rotation gates $R_y(\theta)$ and $R_z(\theta)$ in the reversible programming, we make NiLang a path integral based universal quantum simulator as shown in Appendix ?? . The compiling theory developed for reversible programming will have profounding effect to quantum computing.

C. Outlook

So far NiLang is not full ready for productivity. It can be improved from multiple perspectives. We call for better compiling that decreases granularity and hence reduces overhead, rigorously reversible floating point arithmetics to let the reversibility free from rounding error. It is also interesting to see how it can be combined with a high performance quantum simulator like Yao [50]. It can provide control flow to Yao’s quantum block intermediate representation, while Yao can provide the quantum features for NiLang. Reversible programming is known to have advantage in parallel computing in handle asynchronous computing [51] and debugging with bidirectional move [52]. It is interesting to see how NiLang combines with other parts of Julia ecosystem like CUDANative [53] and Debugger.

VII. ACKNOWLEDGMENTS

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-
- [1] L. Hascoet and V. Pascual, *ACM Transactions on Mathematical Software (TOMS)* **39**, 20 (2013).
 - [2] A. Paszke, S. Gross, S. Chintala, G. Chanan, E. Yang, Z. DeVito, Z. Lin, A. Desmaison, L. Antiga, and A. Lerer, in *NIPS Autodiff Workshop* (2017).
 - [3] M. Innes, E. Saba, K. Fischer, D. Gandhi, M. C. Rudilosso, N. M. Joy, T. Karmali, A. Pal, and V. Shah, “Fashionable modelling with flux,” (2018), [arXiv:1811.01457 \[cs.PL\]](https://arxiv.org/abs/1811.01457).
 - [4] M. Abadi, A. Agarwal, P. Barham, E. Brevdo, Z. Chen, C. Citro, G. S. Corrado, A. Davis, J. Dean, M. Devin, S. Ghemawat, I. Goodfellow, A. Harp, G. Irving, M. Isard, Y. Jia, R. Jozefow-

- icz, L. Kaiser, M. Kudlur, J. Levenberg, D. Mané, R. Monga, S. Moore, D. Murray, C. Olah, M. Schuster, J. Shlens, B. Steiner, I. Sutskever, K. Talwar, P. Tucker, V. Vanhoucke, V. Vasudevan, F. Viégas, O. Vinyals, P. Warden, M. Wattenberg, M. Wicke, Y. Yu, and X. Zheng, “TensorFlow: Large-scale machine learning on heterogeneous systems,” (2015), software available from tensorflow.org.
- [5] M. Innes, A. Edelman, K. Fischer, C. Rackauckas, E. Saba, V. B. Shah, and W. Tebbutt, *CoRR abs/1907.07587* (2019), [arXiv:1907.07587](https://arxiv.org/abs/1907.07587).

- [6] Z. C. Lipton, J. Berkowitz, and C. Elkan, “A critical review of recurrent neural networks for sequence learning,” (2015), [arXiv:1506.00019 \[cs.LG\]](#).
- [7] K. He, X. Zhang, S. Ren, and J. Sun, **2016 IEEE Conference on Computer Vision and Pattern Recognition (CVPR)** (2016), 10.1109/cvpr.2016.90.
- [8] “Breaking the Memory Wall: The AI Bottleneck,” <https://blog.semi.org/semi-news/breaking-the-memory-wall-the-ai-bottleneck>.
- [9] J. Bettencourt, M. J. Johnson, and D. Duvenaud, (2019).
- [10] D. Maclaurin, D. Duvenaud, and R. Adams, in *Proceedings of the 32nd International Conference on Machine Learning*, Proceedings of Machine Learning Research, Vol. 37, edited by F. Bach and D. Blei (PMLR, Lille, France, 2015) pp. 2113–2122.
- [11] M. MacKay, P. Vicol, J. Ba, and R. B. Grosse, in *Advances in Neural Information Processing Systems 31*, edited by S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett (Curran Associates, Inc., 2018) pp. 9029–9040.
- [12] J. Behrmann, D. Duvenaud, and J. Jacobsen, **CoRR abs/1811.00995** (2018), [arXiv:1811.00995](#).
- [13] K. S. Perumalla, *Introduction to reversible computing* (Chapman and Hall/CRC, 2013).
- [14] M. P. Frank, **IEEE Spectrum** **54**, 3237 (2017).
- [15] J. Revels, M. Lubin, and T. Papamarkou, “Forward-mode automatic differentiation in julia,” (2016), [arXiv:1607.07892 \[cs.MS\]](#).
- [16] C. Lutz, “Janus: a time-reversible language,” (1986), *Letter to R. Landauer*.
- [17] M. P. Frank, *The R programming language and compiler*, Tech. Rep. (MIT Reversible Computing Project Memo, 1997).
- [18] I. Lanese, N. Nishida, A. Palacios, and G. Vidal, **Journal of Logical and Algebraic Methods in Programming** **100**, 7197 (2018).
- [19] T. Haulund, “Design and implementation of a reversible object-oriented programming language,” (2017), [arXiv:1707.07845 \[cs.PL\]](#).
- [20] R. Landauer, *IBM journal of research and development* **5**, 183 (1961).
- [21] M. Giffthaler, M. Neunert, M. Stäuble, M. Frigerio, C. Semini, and J. Buchli, **Advanced Robotics** **31**, 12251237 (2017).
- [22] D. N. Laikov, **Theoretical Chemistry Accounts** **137** (2018), 10.1007/s00214-018-2344-7.
- [23] M. Seeger, A. Hetzel, Z. Dai, E. Meissner, and N. D. Lawrence, “Auto-differentiating linear algebra,” (2017), [arXiv:1710.08717 \[cs.MS\]](#).
- [24] Z.-Q. Wan and S.-X. Zhang, “Automatic differentiation for complex valued svd,” (2019), [arXiv:1909.02659 \[math.NA\]](#).
- [25] C. Hubig, “Use and implementation of autodifferentiation in tensor network methods with complex scalars,” (2019), [arXiv:1907.13422 \[cond-mat.str-el\]](#).
- [26] J.-G. L. Hao Xie and L. Wang, [arXiv:2001.04121](#).
- [27] H.-J. Liao, J.-G. Liu, L. Wang, and T. Xiang, **Physical Review X** **9** (2019), 10.1103/physrevx.9.031041.
- [28] R. Orús, **Annals of Physics** **349**, 117158 (2014).
- [29] “Paper’s Github Repository,” <https://github.com/GiggleLiu/nilangpaper/tree/master/codes>.
- [30] J. Martens, I. Sutskever, and K. Swersky, “Estimating the hessian by back-propagating curvature,” (2012), [arXiv:1206.6464 \[cs.LG\]](#).
- [31] M. Bender, P.-H. Heenen, and P.-G. Reinhard, **Rev. Mod. Phys.** **75**, 121 (2003).
- [32] When people ask the location in Beijing, they will start by asking which ring.
- [33] M. Arjovsky, A. Shah, and Y. Bengio, **CoRR abs/1511.06464** (2015), [arXiv:1511.06464](#).
- [34] S. Wisdom, T. Powers, J. R. Hershey, J. L. Roux, and L. Atlas, “Full-capacity unitary recurrent neural networks,” (2016), [arXiv:1611.00035 \[stat.ML\]](#).
- [35] L. Jing, Y. Shen, T. Dubcek, J. Peurifoy, S. A. Skirlo, M. Tegmark, and M. Soljacic, **CoRR abs/1612.05231** (2016), [arXiv:1612.05231](#).
- [36] C.-K. LI, R. ROBERTS, and X. YIN, **International Journal of Quantum Information** **11**, 1350015 (2013).
- [37] C. H. Bennett, **SIAM Journal on Computing** **18**, 766 (1989), <https://doi.org/10.1137/0218053>.
- [38] R. Y. Levine and A. T. Sherman, **SIAM Journal on Computing** **19**, 673 (1990).
- [39] T. Chen, B. Xu, C. Zhang, and C. Guestrin, **CoRR abs/1604.06174** (2016), [arXiv:1604.06174](#).
- [40] T. Häner, M. Soeken, M. Roetteler, and K. M. Svore, “Quantum circuits for floating-point arithmetic,” (2018), [arXiv:1807.02023 \[quant-ph\]](#).
- [41] C. J. Vieri, *Reversible Computer Engineering and Architecture*, Ph.D. thesis, Cambridge, MA, USA (1999), aAI0800892.
- [42] M. Nachtigal, H. Thapliyal, and N. Ranganathan, in **10th IEEE International Conference on Nanotechnology** (2010) pp. 233–237.
- [43] M. Nachtigal, H. Thapliyal, and N. Ranganathan, in **2011 11th IEEE International Conference on Nanotechnology** (2011) pp. 451–456.
- [44] T. D. Nguyen and R. V. Meter, “A space-efficient design for reversible floating point adder in quantum computing,” (2013), [arXiv:1306.3760 \[quant-ph\]](#).
- [45] M. P. Frank, in **35th International Symposium on Multiple-Valued Logic (ISMVL’05)** (2005) pp. 168–185.
- [46] M. P. Frank, in *Reversible Computation*, edited by I. Phillips and H. Rahaman (Springer International Publishing, Cham, 2017) pp. 19–34.
- [47] V. Giovannetti, S. Lloyd, and L. Maccone, **Physical Review Letters** **100** (2008), 10.1103/physrevlett.100.160501.
- [48] S. Aaronson, **Nature Physics** **11**, 291 (2015).
- [49] L. Ruiz-Perez and J. C. Garcia-Escartin, **Quantum Information Processing** **16** (2017), 10.1007/s11128-017-1603-1.
- [50] X.-Z. Luo, J.-G. Liu, P. Zhang, and L. Wang, “Yao.jl: Extensible, efficient framework for quantum algorithm design,” (2019), [arXiv:1912.10877 \[quant-ph\]](#).
- [51] D. R. Jefferson, **ACM Transactions on Programming Languages and Systems (TOPLAS)** **7**, 404 (1985).
- [52] B. Boothe, **ACM SIGPLAN Notices** **35**, 299 (2000).
- [53] T. Besard, C. Foket, and B. D. Sutter, **CoRR abs/1712.03112** (2017), [arXiv:1712.03112](#).

Appendix A: NiLang Grammar

Terminologies

- *ident*, symbols

- *num*, numbers

- ϵ , empty statement

- *JuliaExpr*, native julia expression

```

<Stmts> ::=  $\epsilon$ 
          | <Stmt>
          | <Stmts> <Stmt>
<Stmt> ::= <BlockStmt>
          | <IfStmt>
          | <WhileStmt>
          | <ForStmt>
          | <InstrStmt>
          | <RevStmt>
          | <@anc> <Stmt>
          | <@routine> <Stmt>
          | <@safe> JuliaExpr
          | <CallStmt>
<BlockStmt> ::= begin <Stmts> end
<RevCond> ::= ( JuliaExpr , JuliaExpr )
<IfStmt> ::= if <RevCond> <Stmts> [else <Stmts>] end
<WhileStmt> ::= while <RevCond> <Stmts> end
<Range> ::= JuliaExpr : JuliaExpr [: JuliaExpr]
<ForStmt> ::= for ident = <Range> <Stmts> end
<CallStmt> ::= JuliaExpr ( [ <DataViews> ] )
<Constant> ::= num |  $\pi$ 
<InstrBinOp> ::= += | -= |  $\nabla$ =
<InstrTrailer> ::= [.] ( [ <DataViews> ] )
<InstrStmt> ::= <DataView> <InstrBinOp> ident [ <InstrTrailer> ]
<RevStmt> ::= ~ <Stmt>
<@routine> ::= @routine ident <Stmt>
<AncArg> ::= ident = JuliaExpr
<@anc> ::= @anc <AncArg>
           | @deanc <AncArg>
<@safe> ::= @safe JuliaExpr
<DataViews> ::=  $\epsilon$ 
               | <DataView>
               | <DataViews> , <DataView>
<DataView> ::= <DataView> [ JuliaExpr ]
               | <DataView> . ident
               | JuliaExpr ( <DataView> )
               | <DataView> '
               | - <DataView>
               | <Constant>
               | ident
               | ident ...

```

One can use `@i function <Stmts> end` to define a function and its inverse. All *JuliaExpr* is should be pure, otherwise the reversibility is not guranttred.

Dataview is a bijective mapping of an object or a field (or item) of an object. When modifying the dataview of an object, it changes the object directly with the `chfield` method.

- [], zero or one repetitions.

Appendix B: Instruction Table

The translation of instructions to Julia functions

instruction	translated	type
$y += f(args...)$	$\oplus(f)(args...)$	PlusEq
$y -= f(args...)$	$\ominus(f)(args...)$	MinusEq
$y \vee= f(args...)$	$\odot(f)(args...)$	XorEq

Table II. Instructions and their interpretation in NiLang.

The list of instructions implemented in NiLang

instruction	output
SWAP(a, b)	b, a
ROT(a, b, θ)	$a \cos \theta - b \sin \theta, b \cos \theta + a \sin \theta, \theta$
IROT(a, b, θ)	$a \cos \theta + b \sin \theta, b \cos \theta - a \sin \theta, \theta$
$y += a^b$	$y + a^b, a, b$
$y += \exp(x)$	$y + e^x, x$
$y += \log(x)$	$y + \log x, x$
$y += \sin(x)$	$y + \sin x, x$
$y += \cos(x)$	$y + \cos x, x$
$y += \text{abs}(x)$	$y + x , x$
NEG(y)	$-y$
CONJ(y)	y'

Table III. A collection of reversible instructions, “.” is the broadcast operations in Julia.