COMP2054-ADE
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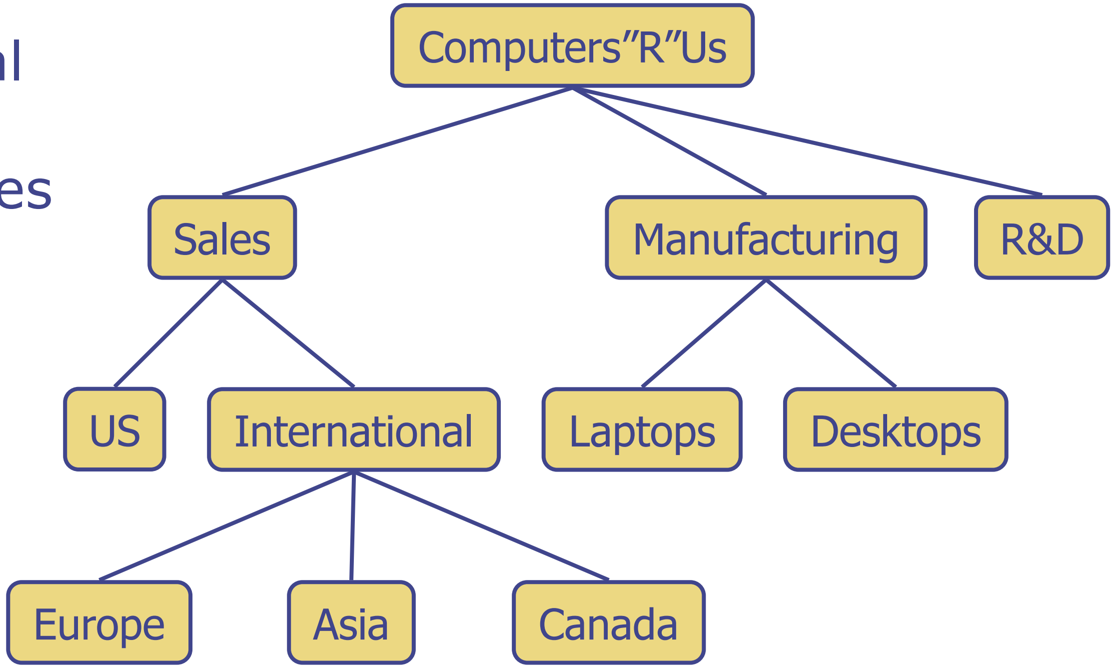
Trees : Terminology, Traversals, Representations, and Properties

What is a (Rooted) Tree

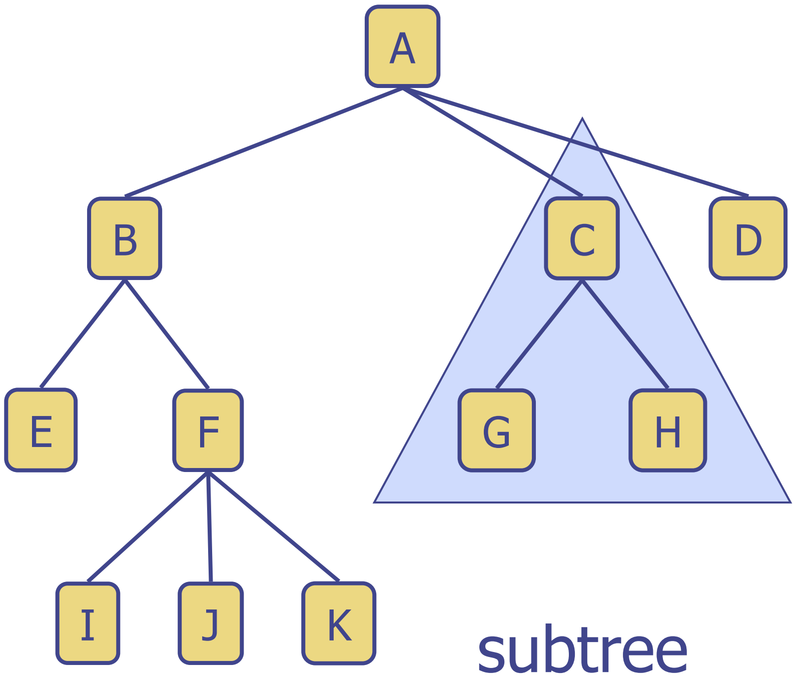
- In computer science, a tree is an abstract model of a hierarchical structure
- A tree consists of nodes with a parent-child relation (at most one parent!)
- Applications:
  - Organization charts
  - File systems
  - Programming environments
- **As data structures**
  - **Heaps**
  - **Search trees**



Tree Terminology

- **Root**: node without parent (A)
- **Internal** node: node with at least one child (A, B, C, F)
- **External** node (a.k.a. leaf): node without children (E, I, J, K, G, H, D)
- **Ancestors** of a node: parent, grandparent, grand-grandparent, etc.
- **Depth** of a node: number of ancestors (not counting itself)
- **Height** of a tree: maximum depth of any node = length of longest path from root to a leaf
  - Height of tree on right = 3
- **Descendant** of a node: child, grandchild, grand-grandchild, etc.

- **Subtree**: tree consisting of a node and its descendants



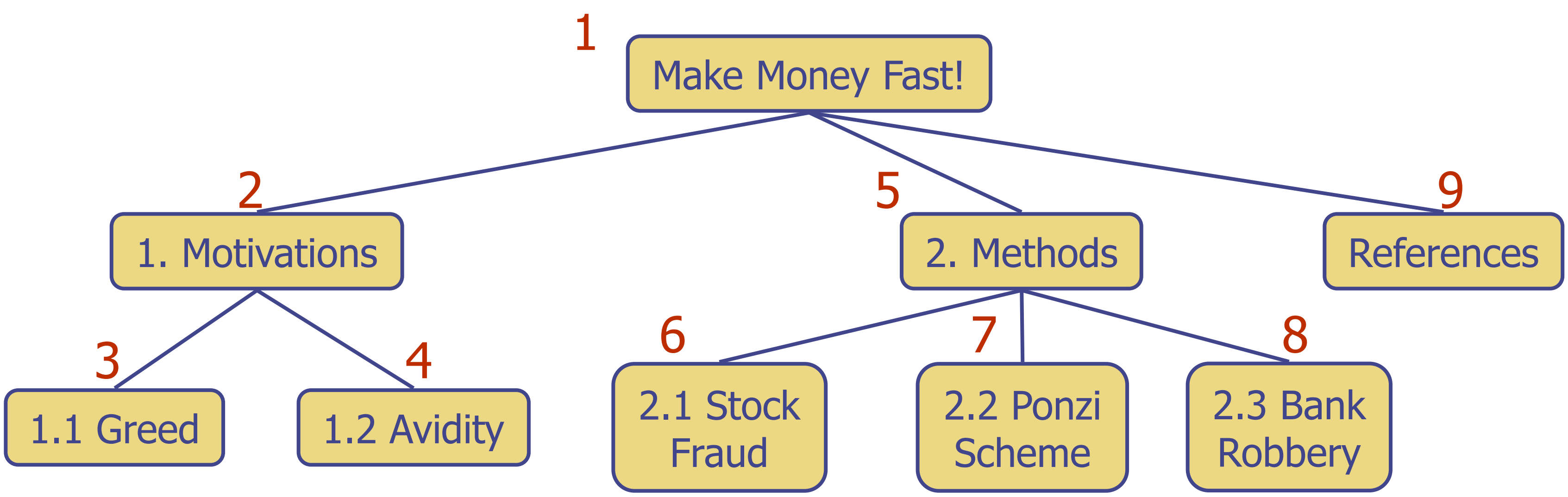
Traversals

- Given a data structure, a common task is to traverse all elements
  - visit each element precisely once
  - visit in some systematic and meaningful order
  - Note “visit” means “process the contents” but does not include just “passing through using the links”
- For an array, or linked list, the natural way is a forwards scan
  - For trees there are more options:

Preorder Traversal

- In a preorder traversal, a node is visited **before** its descendants
- Application: print a structured document

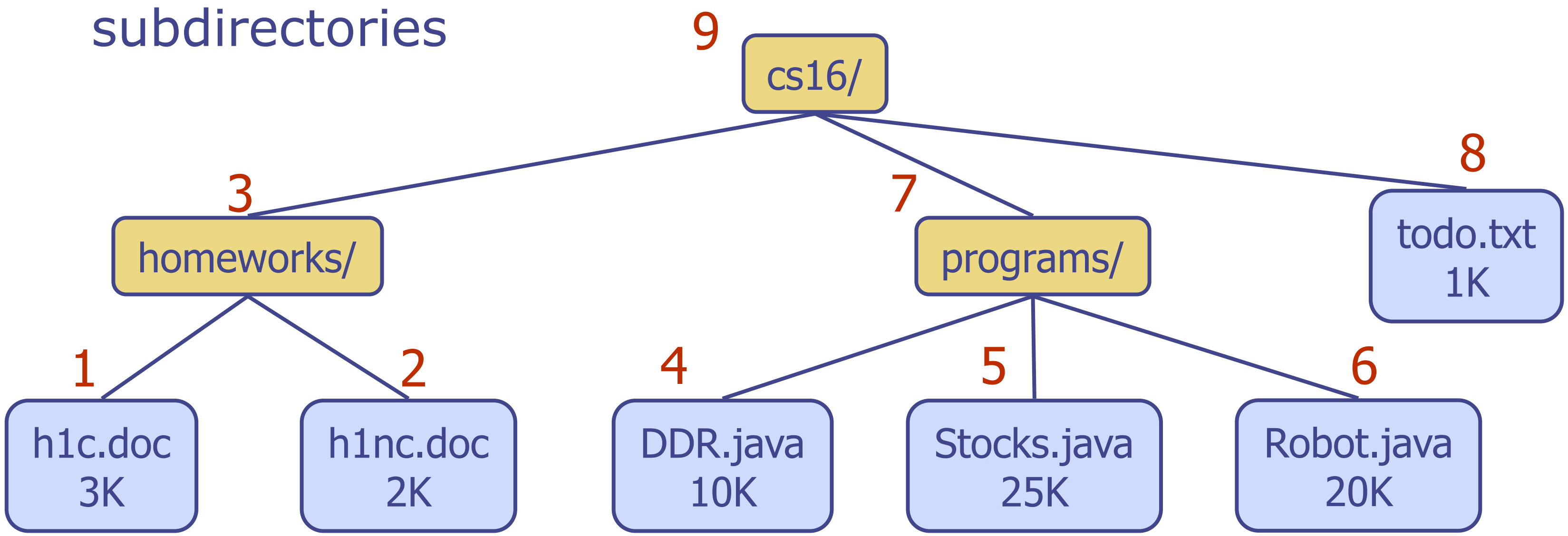
Algorithm ***preOrder***(*v*)
    visit(*v*)
    for each child *w* of *v*
        preorder (*w*)



Postorder Traversal

- In a postorder traversal, a node is visited **after** its descendants
- Application: compute space used by files in a directory and its subdirectories

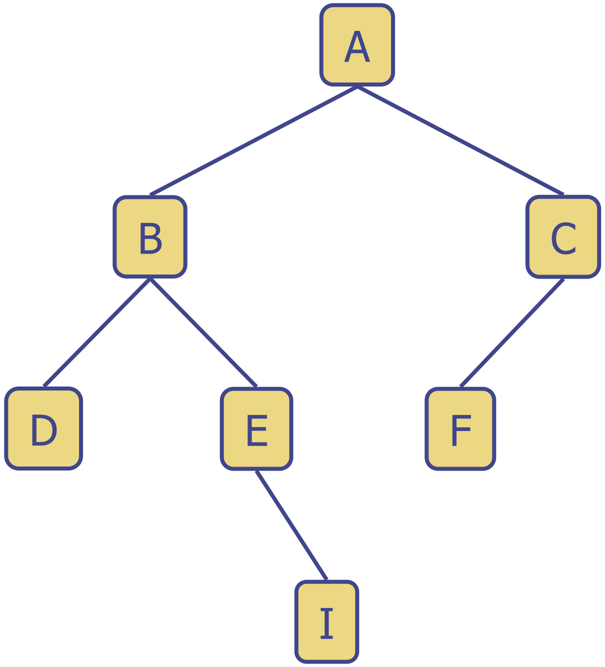
Algorithm *postOrder*(v)
  for each child w of v
    postOrder (w)
  visit(v)



Binary Trees

- Applications:
  - searching

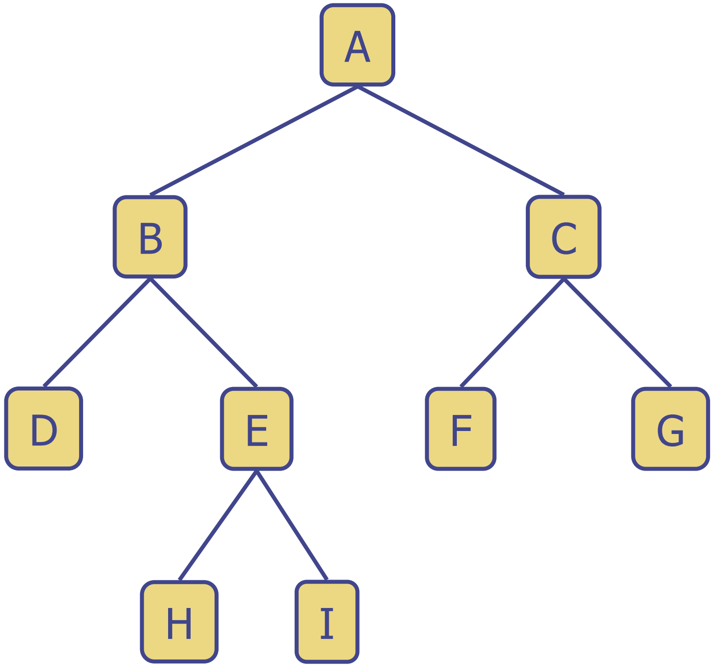
- A binary tree is a tree with the following properties:
  - Each internal node has at most two children
  - The children of a node are an ordered pair - though one might be “missing”
- We call the children of an internal node left child and right child
- Alternative recursive definition: a binary tree is either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of “children”, each of which is missing (a null) or is the root of a binary tree



Proper Binary Trees

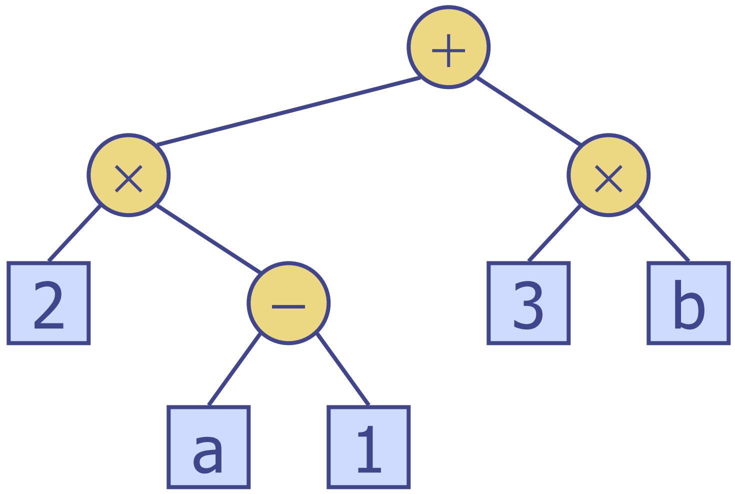
- A proper binary tree is a tree with the following properties:
  - Each internal node has either two children or no children
  - The children of a node are an ordered pair
- We call the children of an internal node left child and right child
- Alternative recursive definition: a binary tree is either
  - a tree consisting of a single node, or
  - a tree whose root has an ordered pair of children, each of which is a root of a binary tree

- Applications:
  - arithmetic expressions
  - decision processes



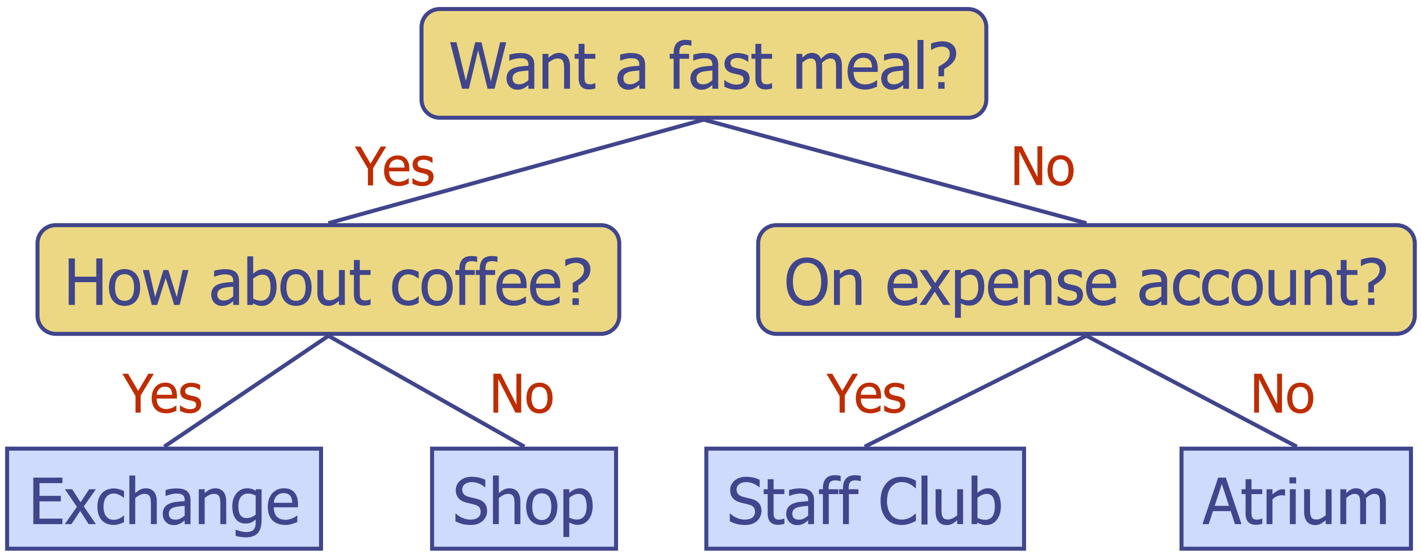
Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression
  - internal nodes: (binary) operators
  - external nodes: operands
- Example: arithmetic expression tree for the expression $(2 \times (a - 1) + (3 \times b))$



Decision Tree

- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
    - hence a proper tree
  - external nodes: decisions
- Example: dining decision



Inorder Traversal

- **In an inorder traversal a node is visited after its left subtree and before its right subtree**
- Application: draw a binary tree by (x,y) coords:
  - x(v) = inorder rank of v
  - y(v) = depth of v

Algorithm *inOrder*(*v*)

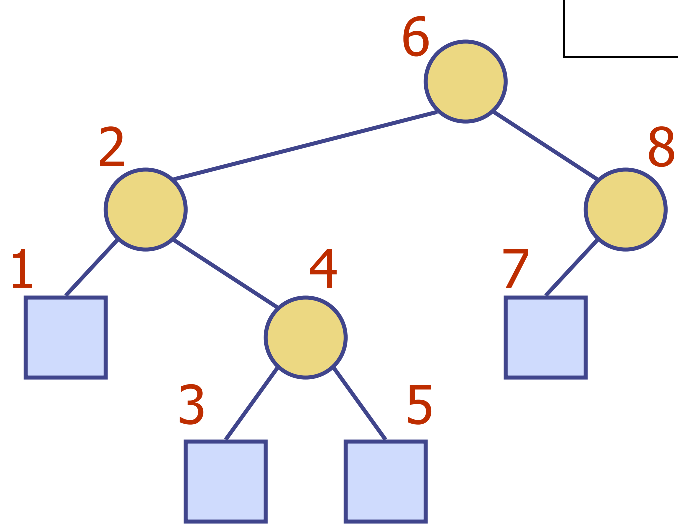
  if *hasLeft* (*v*)

    inOrder (*left* (*v*))

  visit(*v*)

  if *hasRight* (*v*)

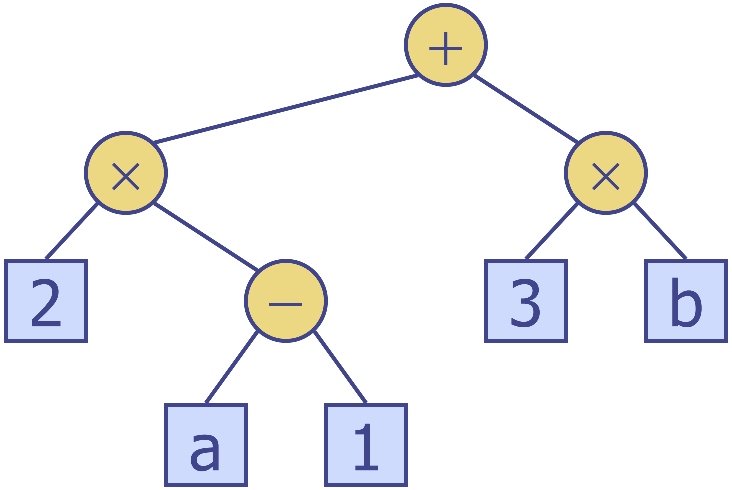
    inOrder (*right* (*v*))



Print Arithmetic Expressions

- Specialization of an inorder traversal
  - print operand or operator when visiting node
  - print "(" before traversing left subtree
  - print ")" after traversing right subtree

Algorithm *printExpression*(*v*)
    if *hasLeft* (*v*)
      print("(")
      printExpression (*left*(*v*))
    print(*v.element* ())
    if *hasRight* (*v*)
      printExpression(*right*(*v*))
      print (")")

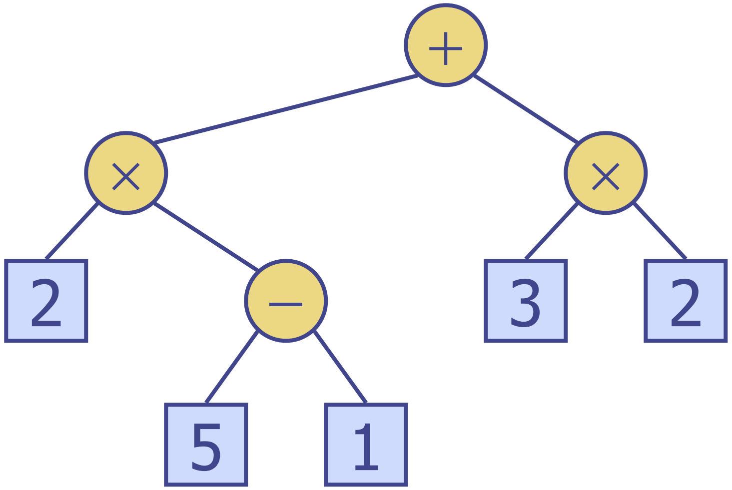


$((2 \times (a - 1)) + (3 \times b))$

Evaluate Arithmetic Expressions

- Specialization of a postorder traversal:
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees

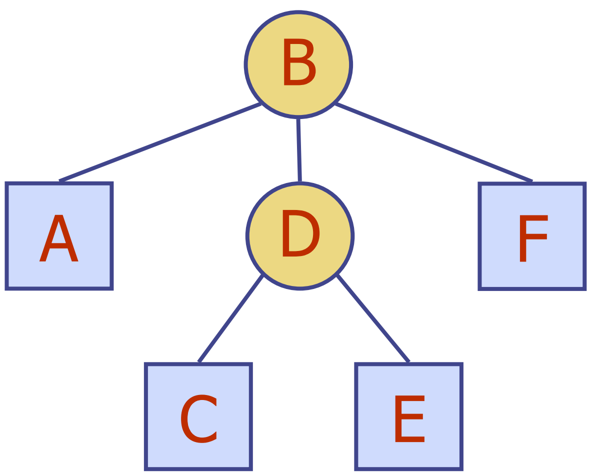
Algorithm *evalExpr*(v)
    if *isExternal* (v)
        return *v.element* ()
    else
        $x \leftarrow$ ***evalExpr*(*leftChild* (v))**
        $y \leftarrow$ ***evalExpr*(*rightChild* (v))**
        ◊ $\leftarrow$ operator stored at $\boldsymbol{v}$
        return x ◊ y

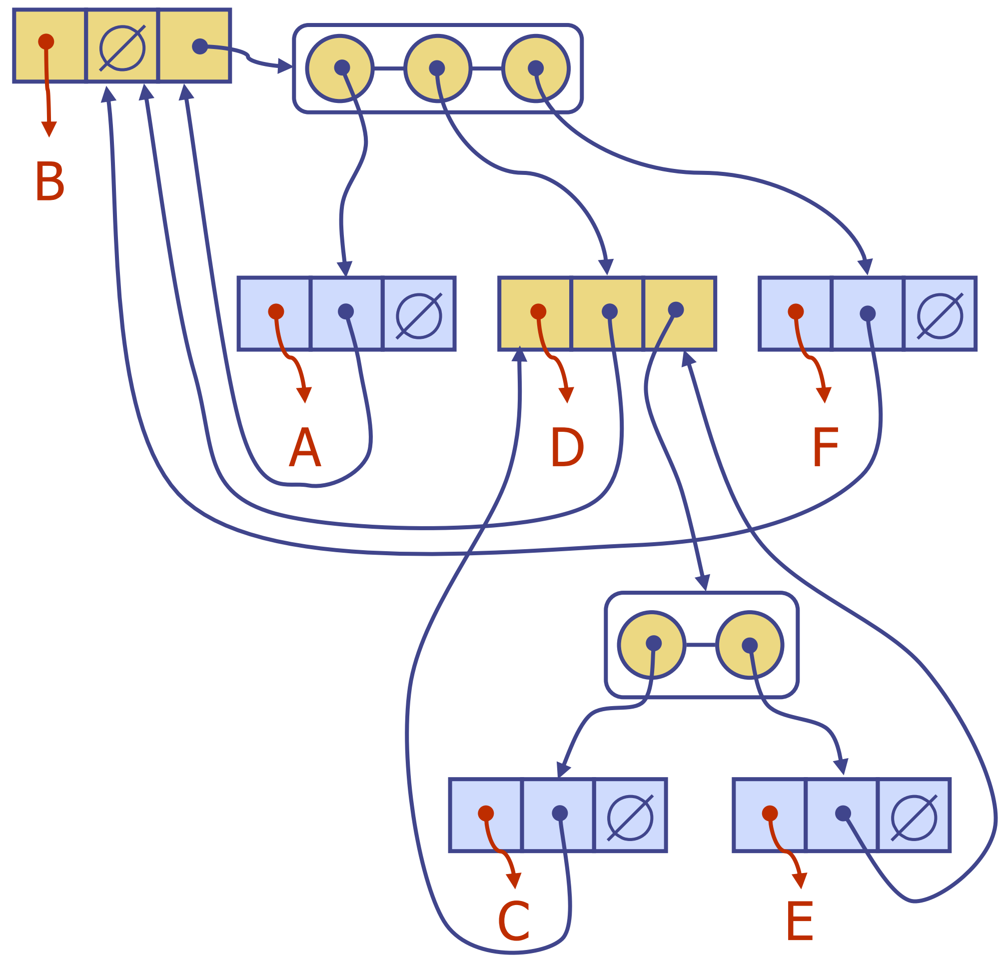


- Exercise: what is the value?
- Exercise: Which traversal ?
- Post- In- or Pre- ?

Linked Structure for Trees

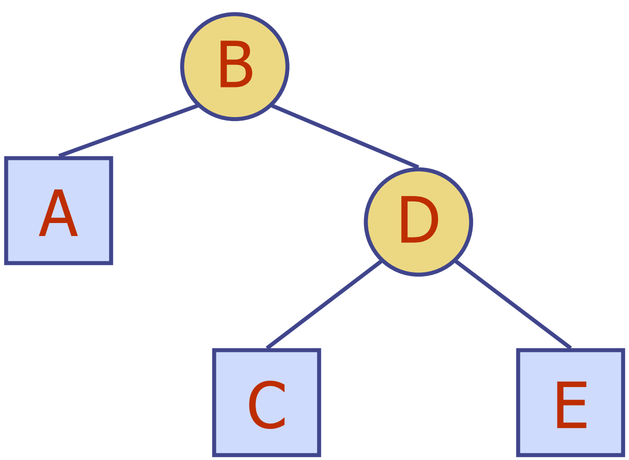
- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes

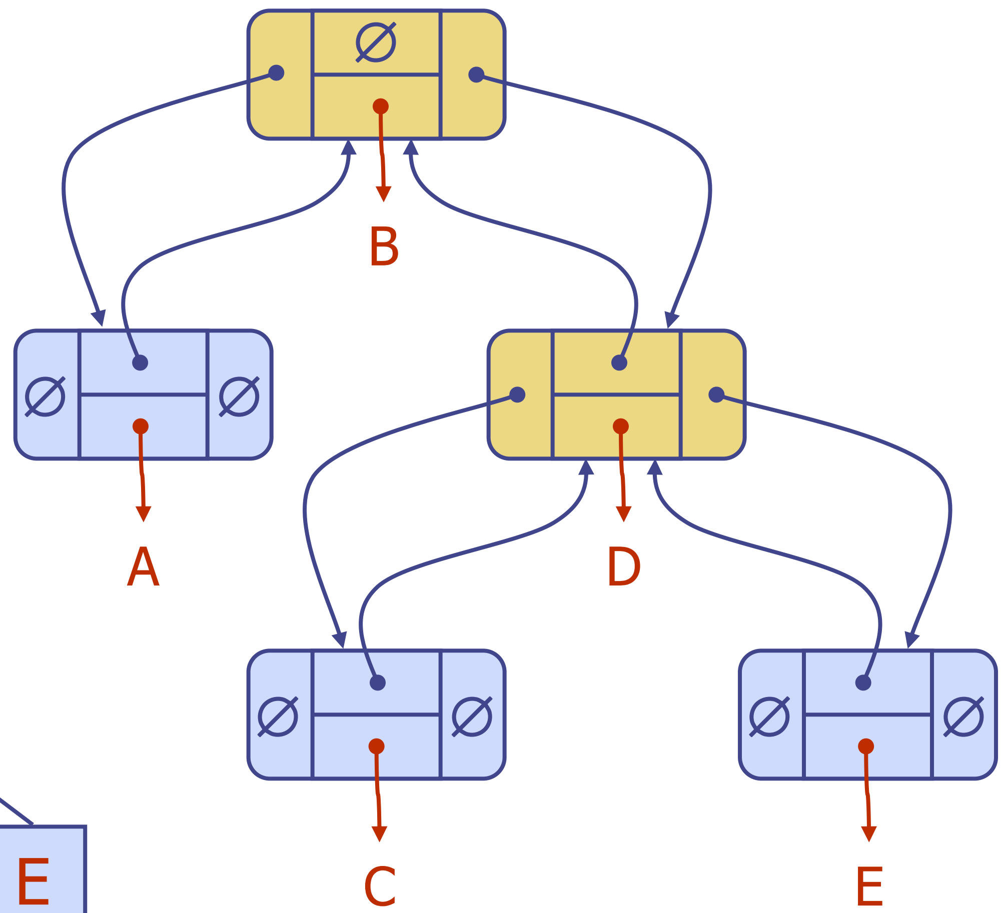




Linked Structure for Binary Trees

- A node is represented by an object storing
  - Element
  - Parent node
  - Left child node
  - Right child node





Abstract Data Types (ADTs)

- An abstract data type (ADT) is an abstraction of a data structure
- **An ADT specifies:**
  - **Data stored**
  - **Operations on the data**
  - **Error conditions associated with operations**
- **An ADT does <u>not</u> specify the implementation itself - hence "abstract"**

Abstract Data Types (ADTs)

- Example: ADT modeling a simple stock trading system
  - The data stored are buy/sell orders
  - The operations supported are
    - order buy(stock, shares, price)
    - order sell(stock, shares, price)
    - void cancel(order)
  - Error conditions:
    - Buy/sell a nonexistent stock
    - Cancel a nonexistent order

Concrete Data Types (CDTs)

- The actual date structure that we use
  - Possibly consists of Arrays or similar
- An ADT might be implemented using different choices for the CDT
  - The choice of CDT will not be apparent from the interface: "data hiding" "encapsulation" – e.g. see 'Object Oriented Methods'
  - The choice of CDT will affect the runtime and space usage – and so is a major topic of this module

ADT & Efficiency

- Often the ADT comes with efficiency requirements expressed in big-Oh notation, e.g.
  - "cancel(order) must be O(1)"
  - "sell(order) must be O(log(|orders|))"
- However, such requirements do not automatically force a particular CDT.
  - The underlying implementation is still not specified
- This is typical of many "library functions"
- Note that such efficiency specifications rely on using the big-Oh family.

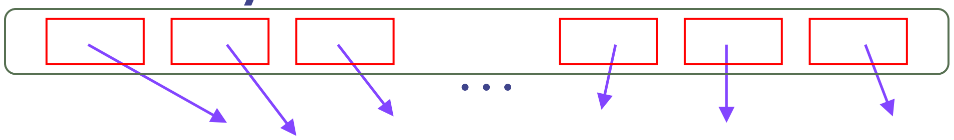
Tree ADT

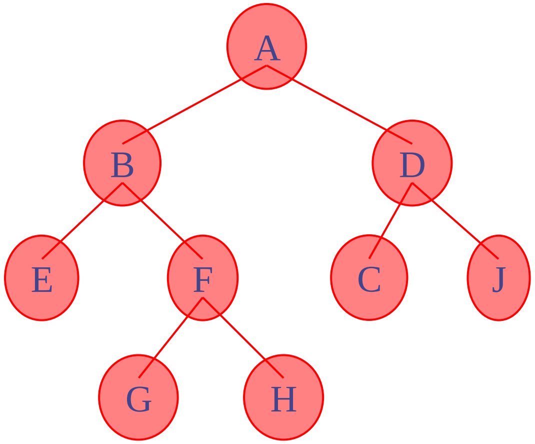
- We can use "positions", p, to abstract nodes
- Generic methods:
  - integer size()
  - boolean isEmpty()
  - Iterator iterator()
  - Iterator positions()
- Accessor methods:
  - position root()
  - position parent(p)
  - Iterator children(p)
- Query methods:
  - boolean isInternal(p)
  - boolean isExternal(p)
  - boolean isRoot(p)
- Update method:
  - object replace (p, o)
- Additional update methods may be defined by data structures implementing the Tree ADT

BUT the CDT can be quite different !!

Array-Based Representation of Binary Trees

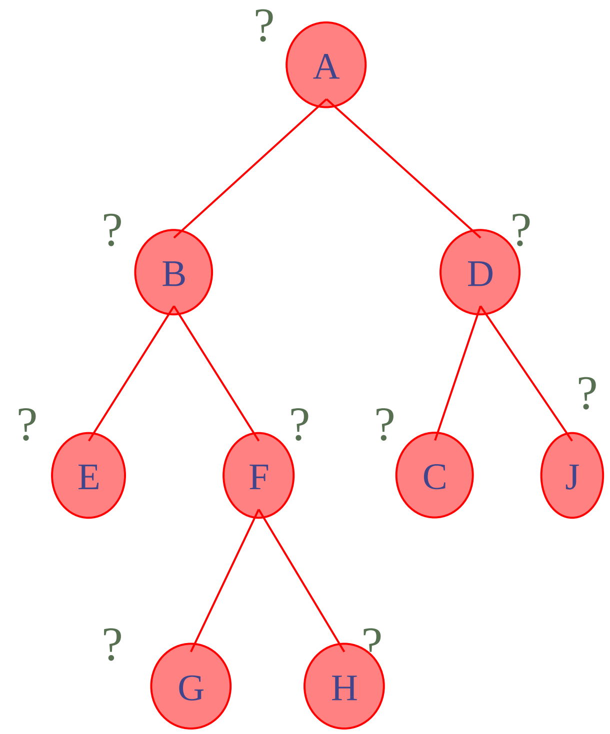
- nodes are stored in an array





- let rank(node), or index in the array, be defined as follows:
  - rank(root) = 1
  - if node is the left child of parent(node),
    rank(node) = 2*rank(parent(node))
  - if node is the right child of parent(node),
    rank(node) = 2*rank(parent(node))+1

Array-Based Representation of Binary Trees



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Exercise (online): fill in the array with nodes, i.e. find the index for each node of the tree

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

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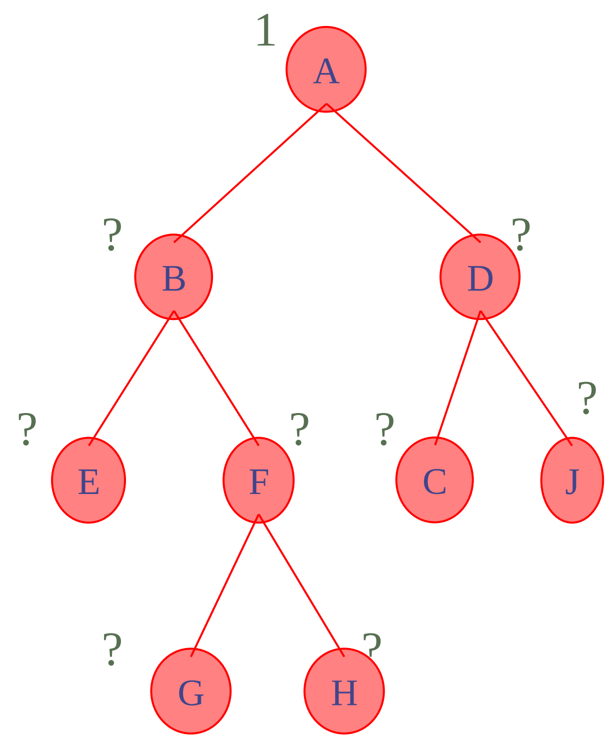

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	A															

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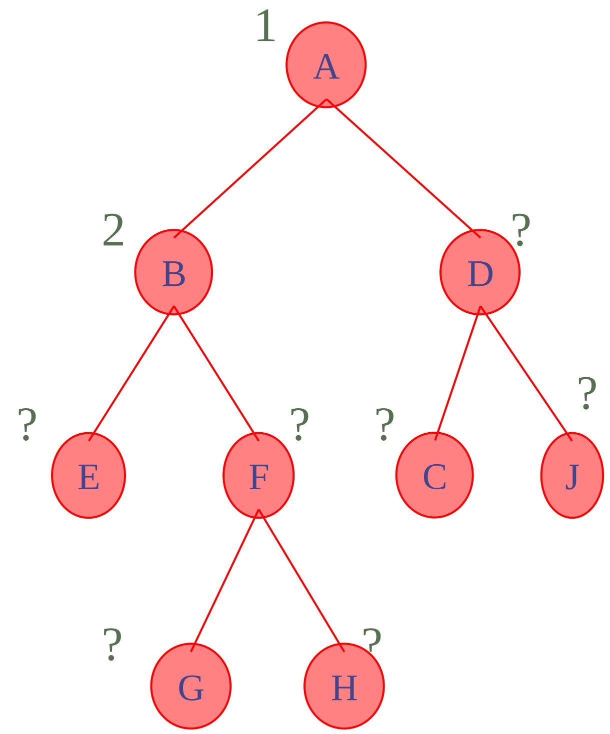

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	A	B														

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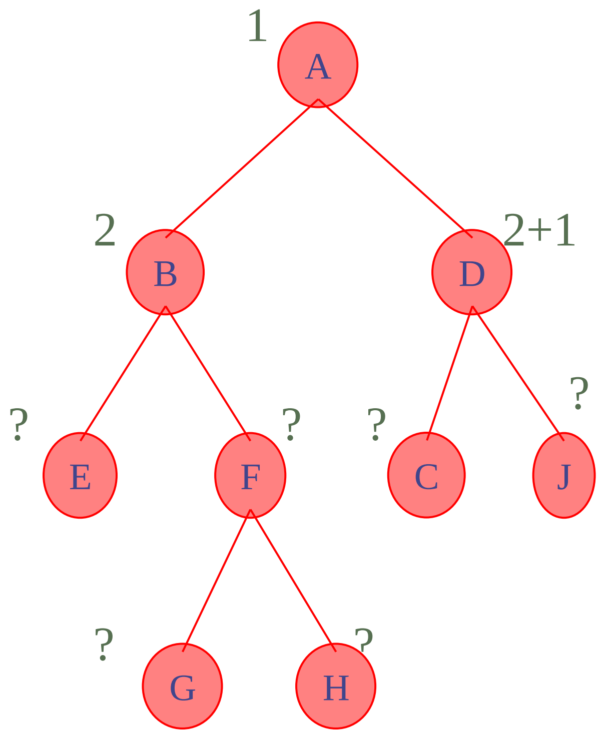
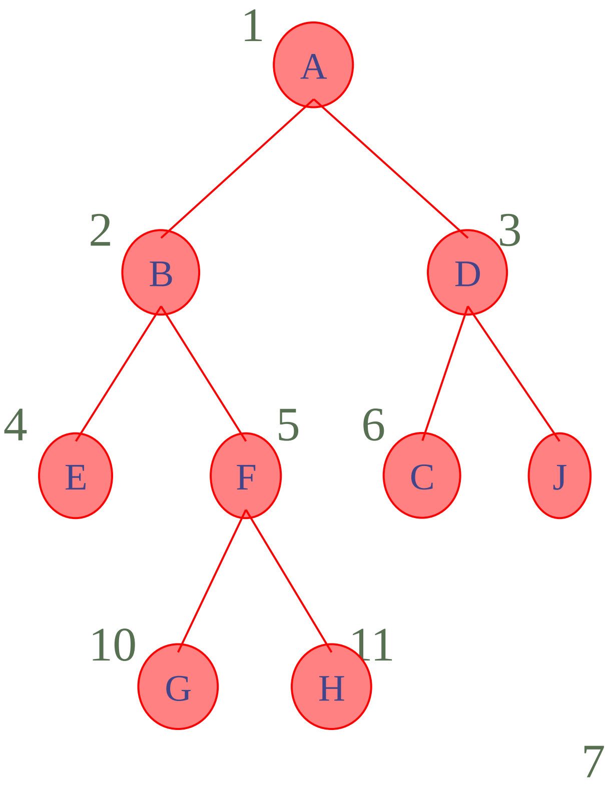


0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	A	B	D													

Array-Based Representation of Binary Trees



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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	A	B	D	E	F	C	J			G	H					

Implemention

- Remember that if think of the rank, r(n) of node n, as a binary number then
  - "times 2" is a left shift "<<1"
- r(n) = r(par(n))<<1 + 0 for left
- r(n) = r(par(n))<<1 + 1 for right
- E.g. r(par(n)) = 101 gives children at
  - "101"+"0" = 1010
  - "101"+"1" = 1011
- Going to the parent is a right shift
- Hence, implementations of this can be very fast – used in binary heaps later.

Exercise (offline)

- Why is this representation correct?
  - I.e. does the mapping between array satisfy needed uniqueness properties?
  - Is it true that each element of the array corresponds to a unique node of the tree?
  - E.g. If I claim that it is incorrect, then how would you convince me otherwise?
- Hint: from the previous slide think of the rank written as binary number
  - Realise that it describes the “L” vs. “R” decisions on going from the root.
  - Relate to each number having a unique binary representation

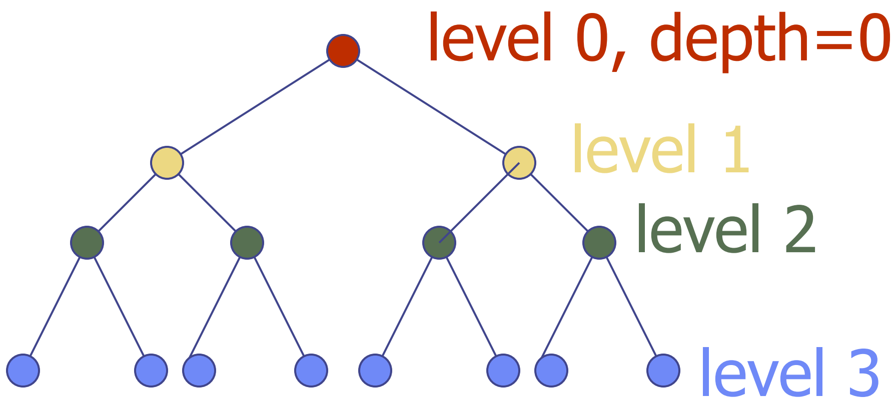
Misc. Comments

- Advantages of the tree-as-array structure:
  - Saves space as do not have to store the pointers – they are replaced by fast computations
  - The storage can be more compact – “better memory locality” and this can be good because of cache and memory hierarchies – when an array element is accessed then other entries can be pulled into the cache, and so access becomes faster.

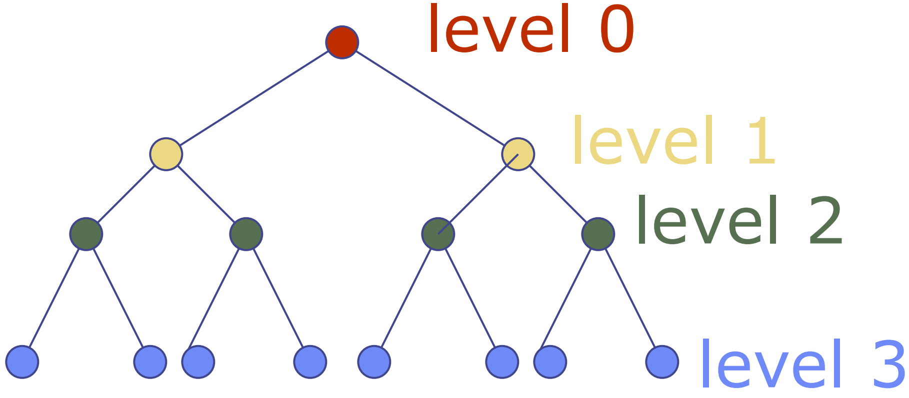
Properties of perfect binary trees

- A binary tree is said to be "**proper**" (a.k.a. "**full**") if every internal node has exactly 2 children
- It is "**perfect**" if it is proper and all leaves are at the same depth; hence all levels (depths) are full.

Perfect binary tree of height 3:



Properties of perfect binary trees



d	at d	at d or less
0	1	1
1	2	3
2	4	7
3	8	15

Counting suggests numbers of nodes are:

- 2^d at level d
- $2^{d+1}-1$ at level d or less

Can formally prove using induction

Numbers of nodes are exponential in the depth

Height (h) is logarithmic in size (n)

- This is a very important property of perfect binary trees
  - Exercise: is this true for all trees?
- Tree algorithms often work by going down the tree level by level – following a path from root to a leaf
  - Their running time depends on the number of levels
  - hence are O(height)
- But we usually only know the number of nodes, n, and so need to convert height, h, to a function of n
- Let us prove that for perfect binary trees

  $$\boldsymbol{h} = \log_2(\boldsymbol{n}+1) - 1$$

  where n in the number of nodes. Or (same thing):

  number of levels = $\log_2(\boldsymbol{n}+1)$.

How many nodes at level k

- First, it is useful to find out how many nodes are at a certain level in perfect binary tree
- Let us count levels from 0. This way level k contains nodes which have depth k.

How many nodes at level k

- Claim: level (depth) k contains 2^k nodes.
- Proof: by induction on k.
  - (basis of induction) if $k = 0$, the claim is true:

  $2^0 = 1$, and we only have one node (root) at level 0.
  - (inductive step): suppose the claim is true for k-1: level k-1 contains 2^{k-1} nodes.
  - We need to prove that then the claim holds for k: level k holds 2^k nodes.
  - Since each node at level k-1 has 2 children, there are twice as many nodes at level k.
  - So, level k contains $2 * 2^{k-1} = 2^k$ nodes. QED

How many nodes in a tree of height h?

Theorem: A perfect binary tree of height h contains

$2^{h+1} - 1$ nodes.

Proof: by induction on h

- (basis of induction): h=0. The tree contains $2^1 - 1 = 1$ node.
- (inductive step): assume a tree of height h-1 contains $2^h - 1$ nodes. A tree of depth h has one more level (h) which contains 2^h nodes. The total number of nodes in the tree of height h is: $2^h - 1 + 2^h = 2 * 2^h - 1 = 2^{h+1} - 1$. QED

What is the height of a (perfect) binary tree of size n (with n nodes)?

We know that $n = 2^{h+1} - 1$.

So, $2^{h+1} = n + 1$.

$h + 1 = \log_2(n+1)$

$h = \log_2(n+1) - 1$.

So, the height of the tree is logarithmic in the size of the tree.

The size of the tree is exponential in the height (number of levels) of the tree.

What is the height of an arbitrary binary tree of size n (with n nodes)?

- If the tree is perfect, then it has height that is logarithmic in the size of the tree: $\Theta(log(n))$
- If it is imperfect, then for the same n it must have at least this height: $\Omega(\log(n))$
- However, consider a simple “chain” – basically a linked list
  - each non-leaf node has just one child. It is still a binary tree – just a special case.
  - It has height n-1 and this is “obviously” maximal height
  - Hence, trees have height O(n).
- **Hence, for a general binary tree on n nodes, the height is $\boldsymbol{\Omega(log(n))}$ and $\boldsymbol{O(n)}$**

Minimum Expectations

- Definitions associated with trees
- **Post- Pre- and In-order traversal and their usages**
- Implementation methods
  - nodes
  - array based
- Binary Trees – meaning of proper, perfect
- **Sizes and heights of binary trees**