Lecture 11

Topics covered in this lecture session

1. Partial sums and the sigma notation.
2. Series (Arithmetic series, Geometric series).
3. The sum of an infinite Geometric series.
4. Power Series.
5. Method of differences.

Partial Sums

Let $a_1,\ a_2,\ a_3, \ldots.$ be a given sequence.

Define the sums:

$$
\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \quad \vdots \quad \vdots \\
S_n &= a_1 + a_2 + a_3 + \ldots\ldots + a_n
\end{aligned}
$$

The sums defined as above are called partial sums.

Sigma notation

$$\sum_{k=1}^{n} a_k = \sum_{1}^{n} a_n = a_1 + a_2 + a_3 + \dots. + a_n$$

$$\sum_{1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots. + a_n + \dots\dots\dots..$$

Using sigma notation, the partial sums for sequence $\{a_n\}$ is:

$$S_n = \sum_{k=1}^{n} a_k$$

Sigma notation

Some examples on the use of sigma notation:

$$\sum_{1}^{5} r^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$$

$$1 - x + x^2 - x^3 + ... = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$2 - 4 + 8 - 16 + ... + 128 = \sum_{n=1}^{7} (-1)^{n+1} 2^n$$

Series

A series $\{S_n\}$ is a sequence whose terms are partial sums of terms of a given sequence $\{a_n\}$.

e.g. if a given sequence is 2, 4, 6, 8, 10, ….

then the corresponding (associated) series is:

$$2, \quad 2+4, \quad 2+4+6, \quad 2+4+6+8, \quad .....$$

i.e. $2, \quad 6, \quad 12, \quad 20, \quad .....$

Series

n^{th} term of series $\{S_n\}$ can be obtained from sequence $\{a_n\}$, using

$$S_n = \sum_{k=1}^{n} a_k$$

On the other hand,

$$\begin{aligned} S_n - S_{n-1} &= (a_1 + a_2 + ... + a_{n-1} + a_n) \\ &\qquad - (a_1 + a_2 + ... + a_{n-1}) \\ &= a_n \end{aligned}$$

$$\therefore \boxed{a_n = S_n - S_{n-1}}$$

Arithmetic Series

Consider the sum S_n of the first n terms of an A.P.

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \ldots\ldots + [a+(n-2)d] + [a+(n-1)d]$$

Writing $l = a + (n-1)\,d$

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \ldots\ldots + (l-d) + l$$

Reversing the sum

$$S_n = l + (l-d) + (l-2d) + (l-3d) + \ldots\ldots + (a+d) + a$$

Adding

$$2S_n = (a+l) + (a+l) + (a+l) + (a+l) + \ldots\ldots + (a+l) + (a+l)$$

n times

Arithmetic Series

$$\therefore\ 2S_n = n(a+l) \Rightarrow S_n = \frac{n}{2}(a+l) = \frac{n}{2}[a+a+(n-1)d]$$

Thus, the sum of the first n terms of an A.P. is:

$$\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$$

Example:

The eighth term of an A.P. is 23 and its 24th term is 103.

Find the sum of its first 30 terms.

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$$8^{th}\text{ term} = 23 \rightarrow a + 7d = 23$$

$$24^{th}\text{ term} = 103 \rightarrow a + 23d = 103$$

$$\therefore 16d = 80$$

$$\boxed{d = 5}$$

$$a = 23 - 7d = 23 - 35$$

$$\boxed{a = -12}$$

Example:

The eighth term of an A.P. is 23 and its 24th term is 103.

Find the sum of its first 30 terms.

$$\boxed{S_n = \frac{n}{2}\left[2a + (n-1)\,d\right]}$$

Sum of first 30 terms i.e. $S_{30} = \frac{30}{2}[2(-12) + (30-1)5]$

$$\boxed{S_{30} = 1815}$$

Geometric Series

The sum of the first n terms of a G.P. is:

$$\boxed{S_n = \begin{cases} na & ; \quad r = 1 \\ a\left(\dfrac{1-r^n}{1-r}\right) & ; \quad r \neq 1 \end{cases}}$$

Example:

If $r = \dfrac{1}{3}$, $\quad S_4 = 150$, find the first term a.

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For a G.P. : $S_n = a\left(\frac{1-r^n}{1-r}\right) \quad r \neq 1$

$$r = \frac{1}{3}; S_4 = 150; \ a = ?$$

$$150 = a\left(\frac{1-\left(\frac{1}{3}\right)^4}{1-\frac{1}{3}}\right)$$

$$101.25 + 33.75 + 11.25 + 3.75 + \cdots \qquad \boxed{a = 101.25}$$

Sum of Infinite Geometric Series

$$S_n = a\left(\frac{1-r^n}{1-r}\right) \quad ; \quad r \neq 1$$

If $|\,r\,| < 1$ then, $\lim\limits_{n\to\infty} r^n = 0$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[a\left(\frac{1-r^n}{1-r}\right)\right] = \frac{a}{1-r} \Rightarrow \boxed{S = \frac{a}{1-r}}$$

Example: Find the sum of the infinite geometric series:

$$5 + 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots\ldots$$

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$$5 + 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots\ldots$$

From the given series: $a = 5$ and $r = \frac{1}{5}$

$$S = \frac{a}{1-r} = \frac{5}{1-\frac{1}{5}} \quad \therefore \boxed{S = \frac{25}{4}}$$

Harmonic Series

The harmonic series is the divergent infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots.$$

Power Series

If $k \in \mathbb{N}$, the series:

$$1^k + 2^k + 3^k + \ldots\ldots + n^k = \sum_{n=1}^{n} n^k$$

is called the Power Series.

Power Series

- When $k = 1$,

$$1 + 2 + 3 + \ldots\ldots + n = \sum_{n=1}^{n} n = \frac{n(n+1)}{2}$$

- When $k = 2$,

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \sum_{n=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$$

- When $k = 3$,

$$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \sum_{n=1}^{n} n^3 = \frac{n^2(n+1)^2}{4}$$

Power Series (Worked Example)

Find the sum: $1 + (1+2) + (1+2+3) + ...\quad$ (up to n terms)

Solution:

$$\boxed{\text{Sum} \;=\; \sum n^{th}\,\text{term}}$$

$$\therefore \text{Sum} = \sum (1+2+3+....+n)$$

$$= \sum \left(\sum n \right) \;= \sum \frac{n(n+1)}{2}$$

Power Series (Worked Example)

$$\therefore \text{Sum} = \frac{1}{2}\sum (n^2+n)$$

$$= \frac{1}{2}\left(\sum n^2 + \sum n\right)$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)(n+2)}{6}$$ (upon simplification)

Method of differences

Find the sum: $\displaystyle\sum_{1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$

Solution:

$$=\left(\frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}+\cancel{\ldots}+\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

Method of differences

Find the sum: $\displaystyle\sum_{1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)$

Solution:

$$=\lim_{n\to\infty}\left(\frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}+\cancel{\ldots}+\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

$$=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1-0=1$$

Suggested Reading

Foundation Algebra by P. Gajjar.

Chapter 13: To review this week's lecture

Chapters 12 and 14: For this week's seminar

No Seminar next week: Review Chapter 13 as Self-study (Seminar Slides would be uploaded next week)

Next Week: Revision Lecture