Lecture 8

Topics covered in this lecture session

1. Matrix - Introduction
2. Algebra of matrices.
3. Inverse matrix.
4. Solving systems of linear equations using matrices.
5. More definitions.

Matrix - Introduction

A matrix is a rectangular array (table) of numbers in rows (horizontal) and columns (vertical).

In general, we denote a matrix by

$$A = (a_{ij})_{m\times n} = \begin{array}{cc}
\begin{array}{ccccccccc}
\text{Col } 1 & \text{Col } 2 & \text{Col } 3 & \cdots & \cdots & \text{Col } j & \cdots & \cdots & \text{Col } n \\
\downarrow & \downarrow & \downarrow & \cdots & \cdots & \downarrow & \cdots & \cdots & \downarrow
\end{array} & \\
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1j} & \cdots & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2j} & \cdots & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{i1} & a_{i2} & a_{i3} & \cdots & \cdots & a_{ij} & \cdots & \cdots & a_{in} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mj} & \cdots & \cdots & a_{mn}
\end{pmatrix} &
\begin{array}{l}
\longleftarrow \text{Row } 1 \\
\longleftarrow \text{Row } 2 \\
\qquad \vdots \\
\longleftarrow \text{Row } i \\
\qquad \vdots \\
\longleftarrow \text{Row } m
\end{array}
\end{array}$$

Matrix - Introduction

Each a_{ij} is called an element (entry) of the matrix.

The order (or size or dimension) of a matrix is defined as

number of rows x number of columns.

e.g. The matrix $B = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \end{pmatrix}$ is a (rectangular)

matrix of order 2 x 3.
i.e. Matrix B has 2 rows and 3 columns.

Application areas of matrices

- **In Physics**:
  in the study of electrical circuits (e.g. in solving problems using Kirchoff's laws).

- **In robotics and automation**:
  as base elements for the robot movements.

- **In computers**:
  in the projection of 3D image into a 2D screen.

- **In Google search engine**:
  to rank the webpages.

- **In Online banking**:
  by encrypting message codes/passwords.

Algebra of Matrices

1. Equality of Matrices

Two matrices A and B of the same order are equal if their corresponding elements are equal.

e.g. Matrices $A = \begin{pmatrix} 1 & a \\ b & 2 \end{pmatrix}$ and $B = \begin{pmatrix} c & -2 \\ 0 & -a \end{pmatrix}$ are equal

$$\Rightarrow \; a = -2, \; b = 0, \; \text{and} \; c = 1.$$

Algebra of matrices

Note: To add/subtract two matrices, they must be of the same order.

1. Addition of Matrices

The sum of two matrices of the same order is defined as the matrix formed by adding its corresponding elements.

2. Difference of Matrices

The difference of two matrices of the same order is defined as the matrix formed by subtracting its corresponding elements.

Algebra of matrices

e.g. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -2 & 3 \\ 0 & 9 \end{pmatrix}$, then

$$A + B = \begin{pmatrix} 1+2 & 2+(-1) \\ 3+(-2) & 4+3 \\ 5+0 & 6+9 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 7 \\ 5 & 15 \end{pmatrix}$$

and

$$A - B = \begin{pmatrix} 1-2 & 2-(-1) \\ 3-(-2) & 4-3 \\ 5-0 & 6-9 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 5 & 1 \\ 5 & -3 \end{pmatrix}.$$

Algebra of matrices

3. Multiplication of a Matrix by a scalar

Multiplication of a matrix by a scalar k is defined as multiplying each element of the matrix by that number k.

e.g. If $A = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$, then

$$\mathbf{3}A = \begin{pmatrix} \mathbf{3} \times 1 & \mathbf{3} \times 2 \\ \mathbf{3} \times 0 & \mathbf{3} \times (-3) \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 0 & -9 \end{pmatrix}.$$

Algebra of matrices

4. Multiplication of Matrices

The product of two matrices $A = (a_{ik})_{m \times p}$ and $B = (b_{kj})_{p \times n}$ is a matrix $C = (c_{ij})_{m \times n}$ where the elements c_{ij} of the product matrix C are defined by:

$$c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj} \quad \text{where} \quad \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix} \quad ; \quad i, j, k \in \mathbb{N}$$

Algebra of matrices

Clearly, Matrix multiplication is a complex process in comparison to addition and subtraction of matrices.

So, we understand the process with a couple of worked examples.

Algebra of matrices

1. Given matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$, find AB.

Solution:

$$C = \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} a_{11} \times b_{11} + a_{12} \times b_{21} \\ a_{21} \times b_{11} + a_{22} \times b_{21} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \times 7 + 2 \times 8 \\ 3 \times 7 + 4 \times 8 \end{pmatrix} = \begin{pmatrix} 23 \\ 53 \end{pmatrix}.$$

Algebra of matrices

2. Given matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$,

find AB.

Solution: $C = AB = \begin{pmatrix} \boxed{1 \quad 2} \\ \boxed{3 \quad 4} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 5 \\ 7 \end{matrix}} & \boxed{\begin{matrix} 6 \\ 8 \end{matrix}} \end{pmatrix}$

$$= \begin{pmatrix} \boxed{1 \times 5 + 2 \times 7} & \boxed{1 \times 6 + 2 \times 8} \\ \boxed{3 \times 5 + 4 \times 7} & \boxed{3 \times 6 + 4 \times 8} \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}.$$

Algebra of matrices

Ex.1 Find AB and BA for the matrices,

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$. Is $AB = BA$?

Some definitions

1. Row Matrix

A matrix that consists of only one row is called a row matrix.

e.g. $\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$ is a row matrix (of order 1×4).

2. Column Matrix

A matrix that consists of only one column is called a column matrix. e.g. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a column matrix (of order 3×1).

Some definitions

3. Square Matrix

A square matrix is a matrix with the same number of row as columns.

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is a 3×3 square matrix.

It is also called an order 3 matrix.

Some definitions

4. Upper triangular matrix

A square matrix is called upper triangular if all the entries below the main diagonal are zero.

e.g. The matrix $U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & 9 \\ 0 & 0 & -2 \end{pmatrix}$ is upper triangular.

Some definitions

5. Lower triangular matrix

A square matrix is called lower triangular if all the entries above the main diagonal are zero.

e.g. The matrix $L = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ 5 & -3 & \mathbf{0} \\ 9 & -7 & -2 \end{pmatrix}$ is lower triangular.

Some definitions

6. Diagonal matrix

A matrix that is both upper and lower triangular is called a diagonal matrix.

e.g.
$$D = \mathrm{diag}\,(-1, 4, 8) = \begin{pmatrix} -1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 8 \end{pmatrix}$$

is a diagonal matrix.

Some definitions

7. Identity matrix

A diagonal matrix with all its main diagonal entries as 1

is called a Unit or Identity matrix. It is denoted by I or I_n.

e.g. $$I = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is a unit matrix of order 3.

Inverse Matrix

Given a square matrix A, if there exists a matrix B such that $AB = BA = I$, then the matrix B is said to be the inverse of matrix A, and is denoted by A^{-1}.

Here, the Identity matrix I is of the same order as matrices A and B.

Thus, $A\,A^{-1} = A^{-1}\,A = I$.

Inverse Matrix

Method to find the inverse of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

To find inverse of a matrix, we need a number called determinant.

Determinant ($\det(A)$) of a 2×2 matrix is a number given by:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Note: For inverse matrix to exists, $\det(A)$ must be non-zero.

Inverse Matrix

The Method:

Step 1: Find $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Step 2: Interchange elements of the principal diagonal.
i.e. a and d.

Step 3: Change the signs of elements on the secondary diagonal. i.e. change signs of elements b and c.

Step 4: Divide the matrix so obtained by $\det(A)$.

Inverse Matrix

Thus, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boxed{A^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

e.g. For $A = \begin{pmatrix} 3 & -2 \\ 6 & 5 \end{pmatrix}$,

$$\det(A) = \begin{vmatrix} 3 & -2 \\ 6 & 5 \end{vmatrix} = 15 + 12 = 27 \neq 0.$$

$$\therefore \; A^{-1} = \frac{1}{27} \begin{pmatrix} 5 & 2 \\ -6 & 3 \end{pmatrix}.$$

Systems of linear equations

A system of linear equations (or linear system) is a collection of linear equations involving the same set of variables.

e.g.

$$\left.\begin{aligned} x + 2y &= 13 \\ 2x - 5y &= 8 \end{aligned}\right\}$$

is a system of linear equations in 2 variables (x and y).

$$\left.\begin{aligned} x + 2y + 4z &= 9 \\ 2x - 5y - z &= 14 \\ 3x - y + 2z &= 7 \end{aligned}\right\}$$

is a system of linear equations in 3 variables (x , y , and z).

Systems of linear equations

There are various methods to solve the linear systems, such as:

a) Method of Substitution

b) Method of Elimination

c) Cramer's Rule

d) Iteration Methods.

Here, we study the Matrix method for solving a 2 x 2 linear system.

Systems of linear equations in Matrix form

To study the method, first we need to put the linear system of equations in Matrix form, $AX = B$.

where, $A \equiv$ (Square) matrix of the coefficients

$X \equiv$ (Column) matrix of the unknowns (variables)

$B \equiv$ (Column) matrix of the constants on the Right-hand side

Systems of linear equations in Matrix form

$\boxed{\text{Form } AX = B}$

$\Downarrow$

e.g.

$$\left.\begin{array}{rcr} x+2y &=& 13 \\ 2x-5y &=& 8 \end{array}\right\} \Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \end{pmatrix}$$

$$\left.\begin{array}{rcr} x+2y+4z &=& 9 \\ 2x-5y-z &=& 14 \\ 3x-y+2z &=& 7 \end{array}\right\} \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 2 & -5 & -1 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 14 \\ 7 \end{pmatrix}$$

Matrix method for solving a 2x2 linear system

We assume that the inverse of matrix A exist and use it to find the solution matrix X.

$$AX = B \quad \Rightarrow A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow \left(A^{-1}A\right) X = A^{-1}B$$

$$\Rightarrow (I)\, X = A^{-1}B$$

$$\Rightarrow X = A^{-1}B$$

Thus, $\boxed{AX = B \quad \Rightarrow \quad X = A^{-1}B}$

Matrix method for solving a 2x2 linear system

The Method

Step 1

$$\left.\begin{array}{rcr} x + 2y &=& 13 \\ 2x - 5y &=& 8 \end{array}\right\} \Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \end{pmatrix}$$

Step 2

Here, $A = \begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix} \Rightarrow \begin{cases} \det A = -5 - 4 = -9 \neq 0 \\ \therefore A^{-1} \text{ (and hence unique solution) exist.} \end{cases}$

Step 3

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} -5 & -2 \\ -2 & 1 \end{pmatrix} = \frac{-1}{9}\begin{pmatrix} -5 & -2 \\ -2 & 1 \end{pmatrix}$$

Matrix method for solving a 2x2 linear system

Step 4

$$\therefore\; X = \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}B = \frac{-1}{9}\begin{pmatrix} -5 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 13 \\ 8 \end{pmatrix}$$

$$= \frac{-1}{9}\begin{pmatrix} (-5)\times 13 + (-2)\times 8 \\ (-2)\times 13 + (1)\times 8 \end{pmatrix}$$

$$= \frac{-1}{9}\begin{pmatrix} -81 \\ -18 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

$\therefore\; x = 9$ and $y = 2$ is the required solution.

More definitions

8. Transpose matrix

The transpose of matrix A is the matrix formed by interchanging the rows and corresponding columns of A.

It is denoted by A^T.

e.g. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$,

then its transpose matrix is: $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

More definitions

9. Trace of a square matrix

The trace of a square matrix A is defined as the sum of the elements on the main (leading or principal) diagonal of A.

i.e. $\mathrm{trace}\,(A) = a_{11} + a_{22} + a_{33} + \ldots\ldots + a_{nn} = \sum_{k=1}^{n} a_{kk}$

e.g. Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $\mathrm{trace}(A) = 1 + (-5) + 9$

$= 5.$

More definitions

10. Zero matrix

A zero (or null) matrix is a matrix with all its entries as zero.

It is denoted by O.

e.g. $$O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad O_{1\times 2} = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad O_{2\times 1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

are all zero matrices.

Note that a zero matrix can also be rectangular.

More definitions

11. Symmetric matrix

A symmetric matrix is a square matrix that is equal to its transpose.

i.e. A is symmetric if $A^T = A$.

e.g. $A = \begin{pmatrix} 1 & 2 & -7 \\ 2 & 8 & 3 \\ -7 & 3 & 6 \end{pmatrix}$ is a symmetric matrix.

More definitions

12. Skew-symmetric (anti-symmetric) matrix

An skew-symmetric matrix is a square matrix whose

transpose is its negative. i.e. A is symmetric if $A^T = -A$.

e.g. $A = \begin{pmatrix} 0 & -2 & 7 \\ 2 & 0 & -3 \\ -7 & 3 & 0 \end{pmatrix}$ is a skew-symmetric matrix.

Note that for an antisymmetric matrix, the entries on its main diagonal are all zero.

More definitions

13. Non-singular matrix

A square matrix A is called non-singular if its inverse exists.

i.e. A is non-singular, if $\det(A) \neq 0$.

e.g. $A = \begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}$ is non-singular.

14. Singular matrix

A square matrix A is called singular if its inverse does not exists.

i.e. A is singular, if $\det(A) = 0$.

e.g. $A = \begin{pmatrix} 8 & 4 \\ 6 & 3 \end{pmatrix}$ is singular.