Lecture 6

Lecture Content

- Integrals of the form $\int \sin^m x \cdot \cos^n x \, dx$
- Some useful results of indefinite integration
- Integrating Algebraic fractions
- The method of Partial fractions
- t-Substitution
- A special Integral: $\displaystyle\int \frac{1}{a\cos^2 x + b\sin^2 x + c}\, dx$

Integrals of the form $\int \sin^m x \cdot \cos^n x \, dx \,; m, n \in \mathbb{N}$

Case I: If m, n are odd.

In this case, it is convenient to take

$\sin x = t$ if $m > n$

$\cos x = t$ if $m < n$

Example

Evaluate $\int \cos^{3} x \sin^{7} x dx$

Here: $m > n$

Solution

$$\sin x = t \to \cos x \, dx = dt$$

$$I = \int \cos^3 x \sin^7 x dx \to \int \sin^7 x \cos^2 x \cos x dx$$

$$\to I = \int t^7 (1 - t^2) dt \to \int t^7 dt - \int t^9 dt$$

$$\to I = \frac{t^8}{8} - \frac{t^{10}}{10} + C$$

$$\therefore I = \frac{\sin^8 x}{8} - \frac{\sin^{10} x}{10} + C$$

Integrals of the form $\int \sin^m x \cdot \cos^n x \, dx \, ; m, n \in \mathbb{N}$

Case II:

$\boxed{\text{If } m \text{ is odd} \quad and \quad n \text{ is even, let} \quad \cos x = t}$

If m is even $\quad and \quad n$ is odd, let $\quad \sin x = t$

Example

Evaluate $\int \sin^5 x \cos^4 x dx$

Here: m is odd

n is even

Solution

$\cos x = t \to -\sin x \, dx = dt$

$$I = -\int \sin^4 x \cos^4 x(-\sin x) \, dx \to -\int (1 - t^2)^2 t^4 dt$$

$$\to -\int (1 - 2t^2 + t^4) t^4 dt = -\int t^4 dt + 2\int t^6 dt - \int t^8 dt$$

$$\to I = -\frac{t^5}{5} + \frac{2t^7}{7} - \frac{t^9}{9} + C$$

$$\therefore I = -\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + C$$

Integrals of the form $\int \sin^m x \cdot \cos^n x \, dx \,; m, n \in \mathbb{N}$

Case II:

If m is odd $\quad and \quad$ n is even, let $\quad \cos x = t$

$\boxed{\text{If } m \text{ is even} \quad and \quad n \text{ is odd, let} \quad \sin x = t}$

Example

Evaluate $\int \cos^3 x dx$

Solution

$\sin x = t \rightarrow cos\ x\ dx = dt$

$$I = \int \cos^2 x \cos x \, dx \rightarrow \int (1 - t^2) dt$$

$$\rightarrow \int dt - \int t^2 dt$$

$$\rightarrow I = t - \frac{t^3}{3} + C$$

$$\therefore I = \sin x - \frac{\sin^3 x}{3} + C$$

Integrals of the form $\int \sin^m x \cdot \cos^n x \, dx \,;\, m, n \in \mathbb{N}$

Case III: If m, n are both even, then transform the integrand using

$$\cos^2 x = \frac{1+\cos 2x}{2} \quad \text{and} \quad \sin^2 x = \frac{1-\cos 2x}{2}$$

Practice Exercises

1. Evaluate $\int \sin^2 x \cos^2 x \, dx$ **Hint:** $\int (\sin x \cos x)^2 dx = \int \left(\frac{1}{2}\sin 2x\right)^2 dx$

2. Evaluate $\int \sin^4 x \cos^2 x \, dx$ Apply Case III above.

Some useful Results of Integration

Some useful results can be obtained from the method of integration by substitution:

Some useful Results of Integration

1. $$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

Example

Evaluate $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx$

Let $f(x) = 1 + \sin x$

so that $f'(x) = \cos x$

$$\therefore\ I = \int \frac{f'(x)}{f(x)}\,dx$$

$$= \ln|f(x)| + C \quad = \ln|1+\sin x| + C$$

Some useful Results of Integration

1. $$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

Example

Evaluate $\displaystyle\int \frac{x^2+2x}{x^3+3x^2-5}\,dx$

Let $f(x) = x^3 + 3x^2 - 5$

so that $f'(x) = 3x^2 + 6x$

$$I = \frac{1}{3}\int \frac{3x^2+6x}{x^3+3x^2-5}\,dx$$

Some useful Results of Integration

1. $$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

Example

$$\therefore\ I = \frac{1}{3}\int \frac{f'(x)}{f(x)}\,dx \quad = \frac{1}{3}\ln|f(x)| + C$$

$$= \frac{1}{3}\ln\left|x^3 + 3x^2 - 5\right| + C$$

Some useful Results of Integration

1. $$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C}$$

Example

Evaluate $$\int \frac{e^{2x}+1}{e^{2x}-1}\, dx$$

Let $f(x) = e^{x} - e^{-x}$

so that $f'(x) = e^{x} + e^{-x}$

$$I = \int \frac{\cancel{e^{x}}\,(e^{x}+e^{-x})}{\cancel{e^{x}}\,(e^{x}-e^{-x})}\, dx = \int \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\, dx$$

$$\therefore\ I = \int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C = \ln\left|e^{x} - e^{-x}\right| + C$$

Formulas derived using $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

$$\int \cot x \; dx = \ln|\sin x| + C$$

$$\int \tan x \; dx = \ln|\sec x| + C$$

$$\int \sec x \; dx = \ln|\sec x + \tan x| + C$$

$$\int \operatorname{cosec} x \; dx = \ln|\operatorname{cosec} x - \cot x| + C$$

Some useful Results of Integration

1. $$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C}$$

Example

Show that $\displaystyle\int \sec x \, dx = \ln|\sec x + \tan x| + C$

$$I = \int \sec x \, dx = \int \frac{\sec x \cdot (\sec x + \tan x)}{(\sec x + \tan x)}\, dx$$

Some useful Results of Integration

1. $$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

Let $f(x) = (\sec x + \tan x)$

so that $f'(x) = \sec x \cdot (\sec x + \tan x)$

$$\therefore\ I = \int \frac{f'(x)}{f(x)}\,dx \ = \ln|f(x)| + C \ = \ln|\sec x + \tan x| + C$$

Some useful Results of Integration

2. $$\boxed{\int [f(x)]^n \ f'(x) \ dx = \frac{1}{n+1} \ [f(x)]^{n+1} + C}$$

Example

Evaluate $\displaystyle\int (x^2+1)^{50} \ 2x \ dx$

Let $f(x) = (x^2+1)$ so that $f'(x) = 2x$

$$\therefore \ I = \int [f(x)]^n \ f'(x) \ dx \quad = \frac{1}{51} \cdot (x^2+1)^{51} + C$$

Some useful Results of Integration

3. $$\boxed{\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C}$$

Example

Evaluate $\displaystyle\int \frac{4-2x}{\sqrt{5-x^2+4x}}\,dx$

Let $f(x) = 5 - x^2 + 4x$ so that $f'(x) = 4 - 2x$

$$\therefore\ I = \int \frac{f'(x)}{\sqrt{f(x)}}\,dx \quad = 2\sqrt{f(x)} + C \quad = 2\sqrt{5-x^2+4x} + C$$

Integrating Algebraic fractions

Algebraic fractions of the form $\frac{p(x)}{q(x)}$ are of 3 types:

(1) $deg(p(x)) \geq deg(q(x))$

We apply method of actual division.

(2) $deg(p(x)) < deg(q(x))$; $q(x)$ can be factorised

We apply method of partial fraction.

(3) $deg(p(x)) < deg(q(x))$; $q(x)$ can not be factorised

We apply method of completing the square in Denominator.

Integrating Algebraic fractions

$\mathbf{(1)} \quad \boldsymbol{deg(p(x)) \geq deg(q(x))}$

Evaluate $\displaystyle \int \frac{x^2+4}{x-5}\, dx$.

$$\therefore\ I = \int \boxed{\left[(x+5) + \frac{29}{x-5} \right]}\, dx$$

$$= \frac{x^2}{2} + 5x + 29 \ln|x-5| + C$$

$$x+5$$

$$x-5 \,\big)\, \cancel{x^2} + 4$$

$$\overset{-}{\cancel{x^2}} \overset{+}{-} 5x$$

$$4 + \cancel{5x}$$

$$\overset{-}{\cancel{5x}} \overset{+}{-} 25$$

$$29$$

$$\therefore \frac{x^2+4}{x-5}$$

$$= \boxed{(x+5) + \frac{29}{x-5}}$$

Integrating Algebraic fractions

(2) The method of Partial fractions

$$\deg\left(p(x)\right) < \deg\left(q(x)\right) \;\; ; \; q(x) \text{ can be factorised}$$

This would be treated in Lecture 7

Integrating Algebraic fractions

(3) $\boldsymbol{deg(p(x)) < deg(q(x))}$; $\boldsymbol{q(x)}$ **cannot be factorised**

The method also works if $\boldsymbol{q(x)}$ can be factorised.

In this case, we use the method of completing the square to express the term in the denominator as a sum/difference of two squares.

The method can also be used when with $\boldsymbol{\sqrt{q(x)}}$ in the denominator.

First, we list some useful integration formulae (shown in the next slide).

Integrating Algebraic fractions

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \ln|x + \sqrt{x^2 - a^2}| + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \ln|x + \sqrt{x^2 + a^2}| + C$$

$$\int \frac{1}{\sqrt{x^2 + k}}\, dx = \ln|x + \sqrt{x^2 + k}| + C$$

$$\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \ln\left|\frac{x + a}{x - a}\right| + C$$

$$\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \ln\left|\frac{x - a}{x + a}\right| + C$$

Integrating Algebraic fractions

Example Show that $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x+\sqrt{x^2+a^2}\right| + C$

Let $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$

$$\therefore\ I = \int \frac{1}{\sqrt{a^2\tan^2\theta + a^2}} \cdot a\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{\sec^2\theta}} \cdot \sec^2\theta\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C'$$

$$= \ln\left|\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right| + C'$$

$$= \ln\left|x+\sqrt{x^2+a^2}\right| - \ln|a| + C'$$

$$= \ln\left|x+\sqrt{x^2+a^2}\right| + C$$

Integrating Algebraic fractions

Example

Evaluate $\displaystyle\int \frac{1}{x^2+2x+3}\,dx$

$$\therefore I = \int \frac{1}{(x^2+2x+1)+2}\,dx = \int \frac{1}{(x+1)^2+\left(\sqrt{2}\right)^2}\,dx$$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\cdot\tan^{-1}\left(\frac{x}{a}\right)+C} = \frac{1}{\sqrt{2}}\cdot\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+C$$

Integrating Algebraic fractions

Example

Evaluate $$\int \frac{1}{x^2+2x-3}\,dx$$

Here, we use the method of completing the square.

$$\therefore\ I = \int \frac{1}{(x^2+2x+1)-4}\,dx \ = \int \frac{1}{(x+1)^2-(2)^2}\,dx$$

$$\boxed{\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \implies}$$

$$= \frac{1}{2(2)}\ln\left|\frac{(x+1)-2}{(x+1)+2}\right| + C$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C$$

This example can also be solved using the method of partial fractions.

Integrating Algebraic fractions

Example Evaluate $\displaystyle\int \frac{3}{4x^2 - 25}\,dx$

This example can also be solved using method of partial fractions

$$I = \int \frac{3}{4\left(x^2 - \frac{25}{4}\right)}\,dx = \frac{3}{4}\int \frac{1}{x^2 - \left(\frac{5}{2}\right)^2}\,dx$$

$$\boxed{\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C} \quad \Rightarrow \quad = \frac{3}{20}\ln\left|\frac{2x-5}{2x+5}\right| + C$$

Integrating Algebraic fractions

Example

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+2x-3}}\,dx$

$$\therefore\ I = \int \frac{1}{\sqrt{(x^2+2x+1)-4}}\,dx = \int \frac{1}{\sqrt{(x+1)^2-(2)^2}}\,dx$$

$$\boxed{\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right|+C}$$

$$= \ln\left|(x+1)+\sqrt{(x+1)^2-(2)^2}\right|+C$$

$$= \ln\left|(x+1)+\sqrt{x^2+2x-3}\right|+C$$

Integrating Algebraic fractions

Example

Evaluate $\displaystyle\int \frac{1}{\sqrt{5 - x^2 + 4x}}\,dx$

$$\therefore I = \int \frac{1}{\sqrt{9 - (x^2 - 4x + 4)}}\,dx \quad = \int \frac{1}{\sqrt{(3)^2 - (x-2)^2}}\,dx$$

$$\boxed{\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C} \qquad = \sin^{-1}\left(\frac{x-2}{3}\right) + C$$

Integrating Algebraic fractions

Example

Evaluate $\displaystyle\int \frac{x^2+3}{x^2-3}\,dx$

Simplify integrand into standard (integrable) form

$$I = \int \frac{(x^2-3)+6}{(x^2-3)}\,dx = \int 1\,dx + 6\int \frac{1}{x^2-(\sqrt{3})^2}\,dx$$

$$\boxed{\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C}$$

$$= x + 6\cdot\frac{1}{2\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C$$

$$= x + \sqrt{3}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C$$

t-substitution (A special substitution)

For integrands of the form

$$\frac{1}{a + b\cos x + c\sin x} \quad \text{OR} \quad \frac{1}{a + b\cos x} \quad \text{OR} \quad \frac{1}{a + b\sin x}$$

we use a special substitution, called *t*-substitution:

$$\boxed{\tan\left(\frac{x}{2}\right) = t} \Rightarrow \sec^2\left(\frac{x}{2}\right)\cdot\frac{1}{2} = \frac{dt}{dx} \Rightarrow dx = \frac{2\,dt}{1 + \tan^2\left(\frac{x}{2}\right)}$$

$$\therefore\ dx = \frac{2\,dt}{1+t^2} \quad \text{where} \quad t = \tan\left(\frac{x}{2}\right)$$

t-substitution (A special substitution)

and use the following trigonometric formulae in integrand

$$\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)} = \frac{2t}{1-t^2}$$

so as to transform the integrand as a function of t.

t-substitution (A special substitution)

Example Evaluate $\displaystyle\int \frac{1}{3\cos x - 4\sin x + 5}\,dx$

Let $\tan\left(\frac{x}{2}\right) = t \Rightarrow dx = \dfrac{2\,dt}{1+t^2}$

$\sin x = \dfrac{2\,t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$ where $t = \tan\left(\frac{x}{2}\right)$

$$\therefore\ I = \int \frac{\left(\frac{2\,dt}{1+t^2}\right)}{3\left(\frac{1-t^2}{1+t^2}\right) - 4\left(\frac{2t}{1+t^2}\right) + 5}$$

t-substitution (A special substitution)

$$= \int \frac{2}{3 - 3t^2 - 8t + 5 + 5t^2} \, dt$$

$$= \int \frac{2}{2t^2 - 8t + 8} \, dt$$

$$= \int \frac{1}{t^2 - 4t + 4} \, dt$$

$$= \int \frac{1}{(t-2)^2} \, dt$$

$$= \frac{-1}{(t-2)} + C$$

$$\boxed{= \frac{1}{2 - \tan\left(\frac{x}{2}\right)} + C}$$

32

A special integral $$\int \frac{1}{a\cos^2 x + b\sin^2 x + c}\,dx$$

In this type of integrals, divide numerator and denominator by $\cos^2 x$ and then substitute $\tan x = t$.

Example

Evaluate $$\int \frac{1}{1+3\sin^2 x}\,dx$$

$$\therefore\ I = \int \frac{\left(\dfrac{1}{\cos^2 x}\right)}{\left(\dfrac{1+3\sin^2 x}{\cos^2 x}\right)}\,dx = \int \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x}\,dx$$

33

A special integral $\displaystyle\int \frac{1}{a\cos^2 x + b\sin^2 x + c}\,dx$

$$= \int \frac{\sec^2 x}{1 + 4\tan^2 x}\,dx$$

Let $\tan x = t \;\Rightarrow\; \sec^2 x\,dx = dt$

$$\therefore\; I = \int \frac{dt}{1 + 4t^2}$$

$$= \int \frac{dt}{1 + (2t)^2}$$

$$= \frac{\tan^{-1}(2t)}{2} + C$$

$$= \boxed{\frac{1}{2}\cdot\tan^{-1}(2\,\tan x) + C}$$

Sample Practice Problem

1. Evaluate: $\displaystyle\int \frac{1}{1+\cos x}\,dx$

Using t substitution; let: $\tan\left(\frac{x}{2}\right) = t \rightarrow dx = \frac{2dt}{1+t^2}; \cos x = \frac{1-t^2}{1+t^2}$

$$I = \int \frac{1}{1+\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2} \rightarrow \int \frac{2dt}{1+t^2+1-t^2} = \int dt$$

$$= t + C$$

$$= \boxed{\tan\left(\frac{x}{2}\right) + C}$$

Sample Practice Problem

2. Evaluate: $\displaystyle\int \frac{\sin x}{\sin 3x}\,dx$

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$I = \int \frac{\sin x}{3\sin x - 4\sin^3 x}\,dx \rightarrow \int \frac{\sin x}{\sin x(3 - 4\sin^2 x)}\,dx$$

$$= \int \frac{1}{3 - 4\sin^2 x}\,dx$$

Divide through by $\cos^2 x$

$$I = \int \frac{\frac{1}{\cos^2 x}}{\frac{3}{\cos^2 x} - \frac{4\sin^2 x}{\cos^2 x}}\,dx \rightarrow \int \frac{\sec^2 x}{3\sec^2 x - 4\tan^2 x}\,dx$$

Sample Practice Problem

$$\Rightarrow \quad \sec^2 x = 1 + \tan^2 x$$

$$\therefore\ 3\sec^2 x - 4\tan^2 x = 3 - \tan^2 x$$

$$\text{and } I = \int \frac{\sec^2 x}{3 - \tan^2 x}\, dx$$

Let: $\tan x = t \rightarrow \sec^2 x\, dx = dt$

$$\therefore\ I = \int \frac{dt}{3 - t^2} = \int \frac{dt}{\sqrt{3}^2 - t^2}$$

$$\boxed{\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \ln\left|\frac{x+a}{x-a}\right| + C}$$

$$\therefore\ I = \frac{1}{2\sqrt{3}} \ln\left|\frac{t+\sqrt{3}}{t-\sqrt{3}}\right| + C$$

$$= \boxed{\frac{1}{2\sqrt{3}} \ln\left|\frac{\tan x+\sqrt{3}}{\tan x-\sqrt{3}}\right| + C}$$