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COMP2054 2023-24 ADE Coursework TWO (12.5%)
Mon. 18-MAR-2024

Time: 30 minutes.
Do not turn over page until instructed.
Answer ALL questions for a potential total of 25 marks.
Calculators are not permitted.
Write your answers on these sheets within the spaces provided.
Please write clearly.
Write your name & ID in the box below CLEARLY AND IN UPPER CASE LETTERS.
Circle the first initial of your family name in the column on the left.

FAMILY NAME:	**ANSWERS**
FIRST NAME(S):	
Student ID number:	
Signature:	

Information that might, or might not, be helpful:

Geometric series: $1 + 2 + 2^2 + 2^3 + \ldots + 2^p = 2^{p+1} - 1$

Powers of 2:

n	0	1	2	3	4	5	6
$\mathbf{2^n}$	1	2	4	8	16	32	64

Reminders of properties of logs:

$a^0 = 1$ $\qquad$ log_2 (2^a) = a

log_b (a) = log_2 (a) / log_2 (b) $\qquad$ log_b (a) = 1 / log_a (b)

Master theorem: Given T(n) = a T(n/b) + f(n) and T(1)=1.

Case 1: If f(n) is O(n^c) for some c, with c < log_b(a)
then T(n) is Θ($n^{\log\_b(a)}$)

Case 2: If f(n) is Θ(n^c (log n) ^{k}) for some k ≥ 0, and with c = log_b (a)
then T(n) is Θ(n^c (log n)$^{k+1}$)

Case 3: If f(n) is Ω(n^c) for some c, with c > log_b(a)
then T(n) is Θ(f(n))
(strictly, we need f to satisfy a "regularity condition", which you can ignore here)

For completion by markers:
Total mark (out of 25):

Question 1. "Vectors and Amortised complexity" [4 marks]

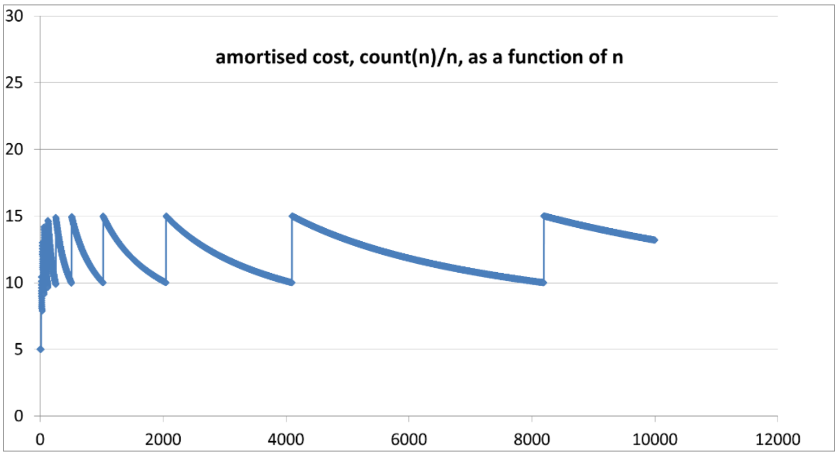
An empirical study of the amortised complexity of insertions into the Vector data structure is considered. The study consists of starting from a Vector data structure with just a small array and then inserting n extra elements one at a time – using a 'push' operation. An estimate of the total number of primitive operations is maintained:

> count(n) = estimate of the number of primitive operations (i.e. an estimate of the runtime) needed to push n elements starting from a small fixed size (a measure of the T(n) used in lectures)

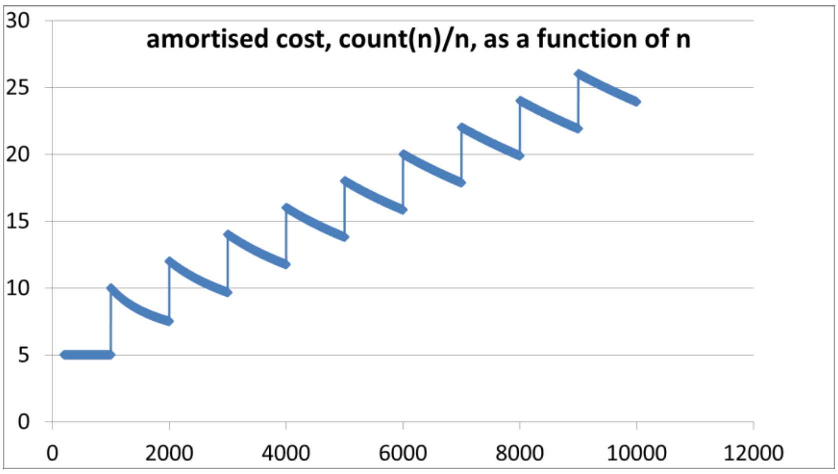
The amortised cost is then the function: count(n) / n
Two "resizing strategies" are studied for resizing the array when it is full:
- "**Incremental**" – increase the size by some constant number
- "**Doubling**" - double the size of the array each time

Two graphs below are obtained from plotting count(n)/n (y-axis) as a function of n (x-axis). You only have to identify which is which:

Graph A: Circle one: "Incremental" or "Doubling" ?



Graph B: Circle one: "Incremental" or "Doubling" ?



These are directly from the labs. And the scaling matches the lecture.

Question 2. Recurrence – Master Theorem (MT) [9 marks]

Using the Master Theorem, state the value of log_b(a), identify the MT case, and solve for the Big-Theta behaviour of T(n) for the following three recurrence relations. In all three problems, you can assume T(1)=1. If using "Case 3" then you can assume that the regularity condition is satisfied. There is no need (or point) to justify your answers.

Q2.a **$T(n) = 2\,T(n/16) + n^2$**

Give the value of log_b (a): **log_16 (2) = 1 / log_2 (16) = 1 / 4**

Circle the case: Case 1 Case 2 Case 3

Circle the correct Theta behaviour of T(n):

A. $\Theta(n^2)$

B. $\Theta(n^2 \log(n))$

C. $\Theta(n^3)$

D. $\Theta(n^3 \log(n))$

E. $\Theta(n^4)$

Q2.b **$T(n) = 8\,T(n/2) + n^3 \log n$**

Give the value of log_b (a): **log_2 (8) = 3**

Circle the case: Case 1 Case 2 Case 3

Circle the correct Theta behaviour of T(n):

A. $\Theta(n^2 \log(n))$

B. $\Theta(n^3)$

C. $\Theta(n^3 \log(n))$

D. $\Theta(n^3 \log(n^2))$

E. $\Theta(n^3 (\log(n))^2)$

Q2.c **$T(n) = 16\ T(n/4) + n$**

Give the value of log_b (a): **log_4 (16) = 2**

Circle the case: Case 1 Case 2 Case 3

Circle the correct Theta behaviour of T(n):

A. $\Theta(n)$

B. $\Theta(n \log(n))$

C. $\Theta(n^2)$

D. $\Theta(n^2 \log(n))$

E. $\Theta(n^3)$

The above are just direct applications of the Master Theorem.

Question 3. Recurrence relations – induction proof [6 marks]

Consider the following recurrence relation:

$$T(n) = 3\,T(\,n / 3\,) + 2 \qquad\qquad T(1) = 1$$

The exact solution is claimed to be $T(\,3^k\,) = 2 * 3^k - 1$

Use induction to prove that this solution is correct, and show your working.

Base case: Complete the following with the value of k that is used: k= **0**

Proof: **The exact solution gives**
T(3^0) = 2 * 3^0 – 1 which is
T(1) = 2 * 1 – 1 = 1 as required.

Step case: Complete the following using the exact solution above as the induction hypothesis. For each step of your proof state whether the step follows from:
A. the recurrence relation, or
B. the induction hypothesis; or
C. simplifying or rearranging
Proof:

$T(\,3^{k+1}\,) =$

3 * T (3^(k+1) / 3) + 2 **A. using the recurrence relation**

= 3 * T (3^k) + 2 **C. simplifying**

= 3 * (2 * 3^k - 1) + 2 **.B. using the induction hypothesis**

= 2 * 3^(k+1) – 3 + 2 **C. simplifying**
= 2 * 3^(k+1) – 1 **C. simplying.**
This matches the exact solution at k+1.

Question 4. Recurrence relations – find solution [6 marks]

Consider the following recurrence relation:

$T(n) = 2\ T(\ n\ /\ 3\ ) + 1 \qquad\qquad T(1) = 1$

Compute the exact solution for $T(\ 3^k\ )$ as some function of k.
Note that you are not required to prove it correct.
Hint: compute values of $T(\ 3^k\ ) + 1$ for some small values of k.
Show your answer by completing the following:

$T(\ 3^k\ )\ =$ **2^(k+1) - 1**

Show your working below:

T(3^0) = T(1) = 1 **T(3^0) + 1 = 2 = 2^1**

T(3^1) = T(3) = 2 T(1) + 1 = 3 **T(3^1) + 1 = 4 = 2^2**

T(3^2) = T(9) = 2 T(3) + 1 = 7 **T(3^2) + 1 = 8 = 2^3**

T(3^3) = T(27) = 2 T(9) + 1 = 15 **T(3^3) + 1 = 16 = 2^4**

So the pattern is T(3^k) + 1 = 2^(k+1)