CELEN037

Seminar 9

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Topics

- Applications of Definite Integrals
  - Area Calculation using Definite Integrals
  - Volume Calculation using Definite Integrals
- Numerical Integration
  - Trapezoidal Rule
  - Simpson's Rule

Area of region bounded by two curves and X-axis

The area of the region bounded by the curves $y = f_1(x)$, $y = f_2(x)$, and lines $x = a$, $x = b$ is:

$$A = \left| \int_a^b [f_1(x) - f_2(x)]\, dx \right|$$

Area of region bounded by two curves and Y-axis

The area of the region bounded by the curves $x = g_1(y)$, $x = g_2(y)$, and lines $y = c$, $y = d$ is:

$$A = \left| \int_c^d [g_1(y) - g_2(y)]\, dy \right|$$

Area Calculation using Definite Integrals

Example 1: Find the area of the region bounded by the curves $y = \sec^2 x$, $y = 2$ and lines $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$.

Solution:

$$\begin{aligned}
Area &= \left| \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - 2)\, dx \right| \\
&= \left| \Big[\tan x - 2x \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \right| \\
&= \left| \left(\tan \frac{\pi}{4} - 2 \cdot \frac{\pi}{4} \right) - \left[\tan(-\frac{\pi}{4}) - 2 \cdot (-\frac{\pi}{4}) \right] \right| \\
&= \left| \left(1 - \frac{\pi}{2} \right) - \left(-1 + \frac{\pi}{2} \right) \right| \\
&= |2 - \pi| = \pi - 2
\end{aligned}$$

Example 2: Find the area of the region bounded by the curves $y = x^2$ and $y = 2x - x^2$.

Solution: First find the points of intersection

$$y = x^2 \text{ and } y = 2x - x^2 \quad \Rightarrow \quad x_1 = 0,\ y_1 = 0 \text{ or } x_2 = 1,\ y_2 = 1$$

$$\begin{aligned}
Area &= \left| \int_0^1 (x^2 - 2x + x^2)\, dx \right| \\
&= \left| 2 \int_0^1 (x^2 - x)\, dx \right| \\
&= \left| 2 \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^1 \right| \\
&= \left| 2\left(-\frac{1}{6} \right) \right| = \frac{1}{3}
\end{aligned}$$

Practice Problems on Worksheet:

1. Q1(i)
2. Q1(ii)

Answers:

1. $\dfrac{2(9\sqrt{3}-1)}{5}$
2. $\dfrac{9}{2}$

Volume of solids of revolution: Region bounded by one curve

- If the region bounded by the curve $y = f(x)$, lines $x = a$, $x = b$, and the X-axis is revolved about the X-axis, then the volume of the solid of revolution is:

$$V = \pi \int_a^b y^2 \, dx = \pi \int_a^b [f(x)]^2 \, dx$$

Example 1: Find the volume of solid of revolution when the region bounded by the curve $y = x$, lines $x = 0$, $x = 4$ and X-axis is revolved about X-axis.
Solution:

$$
\begin{aligned}
V &= \pi \int_0^4 y^2 \, dx \\
&= \pi \int_0^4 x^2 \, dx \\
&= \pi \left[\frac{x^3}{3} \right]_0^4 \\
&= \pi \left(\frac{4^3}{3} - \frac{0^3}{3} \right) = \frac{64\pi}{3}
\end{aligned}
$$

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Volume of solids of revolution: Region bounded by one curve

- If the region bounded by the curve $y = f(x)$, lines $x = a$, $x = b$, and the X-axis is revolved about the X-axis, then the volume of the solid of revolution is:

$$V = \pi \int_a^b y^2 \, dx = \pi \int_a^b [f(x)]^2 \, dx$$

- If the region bounded by the curve $x = g(y)$, lines $y = c$, $y = d$, and the Y-axis is revolved about the Y-axis, then the volume of the solid of revolution is:

$$V = \pi \int_c^d x^2 \, dy = \pi \int_c^d [g(y)]^2 \, dy$$

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Example 2: Find the volume of solid of revolution when the region bounded by the curve $x = 2y$, lines $y = 0$, $y = 2$ and Y-axis is revolved about Y-axis.
Solution:

$$V = \pi \int_0^2 x^2 \, dy$$

$$= \pi \int_0^2 4y^2 \, dy$$

$$= \pi \left[\frac{4y^3}{3} \right]_0^2$$

$$= \pi \left(\frac{4 \times 2^3}{3} - \frac{4 \times 0^3}{3} \right) = \frac{32\pi}{3}$$

Practice Problems on Worksheet:

1. Q2(ii)
2. Q2(iii)
3. Q2(iv)
4. Q2(v)

Answers:

1: $\dfrac{\pi^2}{2}$

2: $\dfrac{211\pi}{5}$

3: $\dfrac{32\pi}{5}$

4: $\dfrac{(e^2-1)\pi}{2}$

Volume of solids of revolution: Region bounded by two curves

- If the region bounded by two curves $y = f_1(x)$ and $y = f_2(x)$ between the points of intersection $x = a$, $x = b$ is revolved about the X-axis, then the volume of the solid of revolution is:

$$V = \pi \left| \int_a^b [f_1(x)]^2 - [f_2(x)]^2 \, dx \right|$$

Example 1: Find the volume of solid of revolution when the region enclosed by curves $y = x$ and $y = x^2$ is revolved about the X-axis.
Solution: First, find the points of intersection:

$$y = x \quad \text{and} \quad y = x^2 \quad \Rightarrow \quad x^2 - x = x(x-1) = 0$$

$$\Rightarrow \quad x_1 = 0,\ y_1 = 0 \quad \text{or} \quad x_2 = 1,\ y_2 = 1$$

$$\Rightarrow \quad V = \pi \left| \int_0^1 (x^2 - x^4)\, dx \right|$$

$$= \pi \left| \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \right|$$

$$= \pi \left| \left(\frac{1^3}{3} - \frac{1^5}{5} \right) - \left(\frac{0^3}{3} - \frac{0^5}{5} \right) \right| = \frac{2\pi}{15}$$

Volume of solids of revolution: Region bounded by two curves

- If the region bounded by two curves $y = f_1(x)$ and $y = f_2(x)$ between the points of intersection $x = a$, $x = b$ is revolved about the X-axis, then the volume of the solid of revolution is:

$$V = \pi \left| \int_a^b [f_1(x)]^2 - [f_2(x)]^2 \, dx \right|$$

- If the region bounded by two curves $x = g_1(y)$ and $x = g_2(y)$ between the points of intersection $y = c$, $y = d$ is revolved about the Y-axis, then the volume of the solid of revolution is:

$$V = \pi \left| \int_c^d [g_1(y)]^2 - [g_2(y)]^2 \, dy \right|$$

Example 2: Find the volume of solid of revolution when the region enclosed by curves $x = y^2$ and $x = y + 2$ is revolved about the Y-axis.

Solution: First, find the points of intersection:

$$x = y^2 \quad \text{and} \quad x = y + 2 \quad \Rightarrow \quad y^2 - y - 2 = (y-2)(y+1) = 0$$

$$\Rightarrow \quad x_1 = 1,\ y_1 = -1 \quad \text{or} \quad x_2 = 4,\ y_2 = 2$$

$$\Rightarrow \quad V = \pi \left| \int_{-1}^{2} \left[(y+2)^2 - y^4 \right] \, dy \right|$$

$$= \pi \left| \int_{-1}^{2} \left(-y^4 + y^2 + 4y + 4 \right) \, dy \right|$$

$$= \pi \left| \left[-\frac{y^5}{5} + \frac{y^3}{3} + 2y^2 + 4y \right]_{-1}^{2} \right|$$

$$= \frac{72\pi}{5}$$

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Practice Problems on Worksheet:

1. Q3(ii)
2. Q3(iii)

Answers:

1: $\dfrac{\pi}{2}$

2: $\dfrac{3\pi}{10}$

Trapezoidal Rule (also as Trapezium Rule)

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right], \quad h = \frac{b-a}{n}.$$

Example 1: Evaluate $I = \displaystyle\int_0^1 \frac{1}{1+x}\,dx$ by using the Trapezoidal rule. Divide $[0, 1]$ into 4 subintervals of equal width. Express the answer to 4 decimal places.
$(I = \ln 2 = 0.6932)$

Solution: Here $f(x) = \dfrac{1}{1+x}$ and $h = \dfrac{b-a}{n} = \dfrac{1-0}{4} = \dfrac{1}{4}$

x	f_n	$f(x) = \frac{1}{1+x}$
0	f_0	1
$1/4$	f_1	0.80000
$2/4$	f_2	0.66667
$3/4$	f_3	0.57143
1	f_4	0.50000

(i) Using the Trapezoidal rule:

$$\begin{aligned} I &\approx \frac{h}{2}\left[f_0 + 2\left(f_1 + f_2 + f_3\right) + f_4\right] \\ &= \frac{1}{8}\left[1 + 2(0.8 + 0.66667 + 0.57143) + 0.5\right] \\ &= 0.6970 \end{aligned}$$

Trapezoidal Rule (also as Trapezium Rule)

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right], \quad h = \frac{b-a}{n}.$$

Example 2: Evaluate $I = \int_0^2 e^{x^2}\,dx$ by using the Trapezoidal rule. Divide $[0,2]$ into 5 subintervals of equal width. Express the answer to 3 decimal places.

Solution: Here $f(x) = e^{x^2}$ and $h = \dfrac{b-a}{n} = \dfrac{2-0}{5} = \dfrac{2}{5}$

x	f_n	$f(x) = e^{x^2}$
0	f_0	1
$2/5$	f_1	1.1735
$4/5$	f_2	1.8965
$6/5$	f_3	4.2207
$8/5$	f_4	12.9358
2	f_5	54.5982

$$I \approx \frac{h}{2}\left[f_0 + 2\left(f_1 + f_2 + f_3 + f_4\right) + f_5\right]$$

$$= \frac{1}{5}\left[1 + 2(1.1735 + \cdots + 12.9358) + 54.5982\right]$$

$$= 19.010$$

Practice Problems on Worksheet:

1. Q4(i)
2. Q4(ii)

Answers:

1: 2.004

2: 17.306

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f_0 + 2(f_2 + f_4 + \cdots + f_{n-2}) + 4(f_1 + f_3 + \cdots + f_{n-1}) + f_n\right]$$

$h = \dfrac{b-a}{n}$, and n is an even number.

Example 1: Evaluate $I = \displaystyle\int_0^{\pi} \sin(x)\,dx$ by using the Simpson's rule. Divide $[0, \pi]$ into 6 subintervals of equal width. Express the answer to 3 decimal places. $(I = 2)$

Solution: Here $f(x) = \sin(x)$ and $h = \dfrac{b-a}{n} = \dfrac{\pi - 0}{6} = \dfrac{\pi}{6}$

x	f_n	$f(x) = \sin(x)$
0	f_0	0
$\pi/6$	f_1	0.5000
$2\pi/6$	f_2	0.8660
$3\pi/6$	f_3	1.0000
$4\pi/6$	f_4	0.8660
$5\pi/6$	f_5	0.5000
π	f_6	0

$$I \approx \frac{h}{3}\left[f_0 + 2\left(f_2 + f_4\right) + 4\left(f_1 + f_3 + f_5\right) + f_6\right]$$

$$= \frac{\pi}{18}\left[0 + 2(0.8660 + \cdots) + 4(0.5000 + \cdots) + 0\right]$$

$$= 2.001$$

Example 2: Evaluate $I = \int_0^1 \frac{1}{1+x}\,dx$ by using: (i) the Trapezoidal rule, and (ii) Simpson's rule. Divide $[0,1]$ into 4 subintervals of equal width. Express the answers to 4 decimal places. $(I = \ln 2 = 0.6932)$

Solution: Here $f(x) = \frac{1}{1+x}$ and $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

x	f_n	$f(x) = \frac{1}{1+x}$
0	f_0	1
$1/4$	f_1	0.80000
$2/4$	f_2	0.66667
$3/4$	f_3	0.57143
1	f_4	0.50000

(i) Using the Trapezoidal rule:

$$I \approx \frac{h}{2}\left[f_0 + 2\left(f_1 + f_2 + f_3\right) + f_4\right]$$

$$= \frac{1}{8}\left[1 + 2(0.8 + 0.66667 + 0.57143) + 0.5\right]$$

$$= 0.6970$$

(ii) Using Simpson's rule:

$$I \approx \frac{h}{3}\left[f_0 + 2\left(f_2\right) + 4\left(f_1 + f_3\right) + f_4\right]$$

$$= \frac{1}{12}\left[1 + 2(0.66667) + 4(0.8 + 0.57143) + 0.5\right]$$

$$= 0.6933$$

Practice Problems on Worksheet:

1. Q5(i)
2. Q5(iii)

Answers:

1: 0.230

2: 14.556

Trapezoidal Rule

We divide the interval $[a, b]$ into n subintervals of equal width $h = \dfrac{b-a}{n}$. Then

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\Big[f_0 + 2\,(f_1 + f_2 + \cdots + f_{n-1}) + f_n\Big]$$

$$\big(f_i = f(a + ih),\ i = 0, 1, 2, \ldots, n\big)$$

Simpson's Rule

We divide the interval $[a, b]$ into n (n is EVEN) subintervals of equal width $h = \dfrac{b-a}{n}$. Then

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\Big[f_0 + 2\,(f_2 + f_4 + \cdots + f_{n-2}) + 4\,(f_1 + f_3 + f_5 + \cdots + f_{n-1}) + f_n\Big]$$

$$\big(f_i = f(a + ih),\ i = 0, 1, 2, \ldots, n\big)$$