CELEN037 Seminar 8

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Topics

- Evaluating Definite Integrals
- Definite Integrals using Substitution
- Integration by Parts for Definite Integrals
- Use of Properties for Evaluating Definite Integrals
- Area Calculation using Definite Integrals

Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = \Big[F(x)\Big]_a^b$$

Example 1: Evaluate $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

Solution:

$$\begin{aligned}
I &= \int_0^1 \frac{1}{1+x^2}\,dx \\
&= \Big[\tan^{-1} x\Big]_0^1 \qquad \left(\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}\right) \\
&= \tan^{-1}(1) - \tan^{-1}(0) \\
&= \frac{\pi}{4}
\end{aligned}$$

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Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ on $[a,b]$, then

$$\int_a^b f(x)\ dx = F(b) - F(a) = \Big[F(x)\Big]_a^b$$

Example 2: Evaluate $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \operatorname{cosec}^2(3x)\ dx$

Solution:

$$I = \int_{\frac{\pi}{12}}^{\frac{\pi}{9}} \operatorname{cosec}^2(3x)\ dx$$

$$= -\frac{1}{3}\Big[\cot(3x)\Big]_{\frac{\pi}{12}}^{\frac{\pi}{9}}$$

$$= -\frac{1}{3}\left(\cot\left(3\cdot\frac{\pi}{9}\right) - \cot\left(3\cdot\frac{\pi}{12}\right)\right)$$

$$= -\frac{1}{3}\left(\cot\frac{\pi}{3} - \cot\frac{\pi}{4}\right)$$

$$= \frac{1}{3}\left(1 - \frac{1}{\sqrt{3}}\right)$$

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Practice Problems on Worksheet:

1. Q1(ii)
2. Q1(iii)
3. Q1(iv)
3. Q1(v)

Answers:

1: $\dfrac{\ln 3}{4}$

2: $\ln 3$

3: $\dfrac{\pi}{2}$

4: 1

Note

Remember to change the limits of integration for the transformed integral

Example 1: Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \; dx$

Solution: Let $\tan x = t \quad \Rightarrow \quad \sec^2 x \, dx = dt$

Integration limits:

x	0	$\frac{\pi}{4}$
t	0	1

Thus $$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \; dx = \int_0^1 t^2 \; dt$$

$$= \left[\frac{t^3}{3} \right]_0^1$$

$$= \frac{1}{3}$$

Note

Remember to change the limits of integration for the transformed integral

Example 2: Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{e^x(x+1)}{\cos^2(xe^x)}dx$

Solution: Let $xe^x = t$. Then $(e^x + xe^x)\,dx = e^x(x+1)dx = dt$

Integration limits:

x	0	$\frac{1}{2}$
t	0	$\frac{\sqrt{e}}{2}$

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{e^x(x+1)}{\cos^2(xe^x)}\,dx &= \int_0^{\frac{\sqrt{e}}{2}} \frac{1}{\cos^2 t}\,dt \\
&= \int_0^{\frac{\sqrt{e}}{2}} \sec^2 t\,dt \\
&= \Big[\tan t\Big]_0^{\frac{\sqrt{e}}{2}} \\
&= \tan\left(\frac{\sqrt{e}}{2}\right)
\end{aligned}$$

Practice Problems on Worksheet:

1. Q2(iii)
2. Q2(iv)
3. Q2(v)
4. Q2(vi)

Answers:

1: $\dfrac{\pi}{6}$

2: $\dfrac{2\sqrt{2}}{3}\left(\sqrt{5}-2\right)$

3: -4

4: $2\left(\sqrt{6}-\sqrt{5}\right)$

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Integration by parts

$$\int_a^b u \cdot \frac{dv}{dx}\, dx = [u \cdot v]_a^b - \int_a^b v \cdot \frac{du}{dx}\, dx$$

Example 1: Evaluate $\displaystyle\int_1^e x^2 \ln x\, dx$

Solution:

$$\text{Let} \quad u = \ln x \quad \text{and} \quad \frac{dv}{dx} = x^2$$

$$\Rightarrow \quad v = \frac{x^3}{3} \quad \text{and} \quad \frac{du}{dx} = \frac{1}{x}$$

$$\Rightarrow \quad \int_1^e x^2 \ln x\, dx = \left[\ln x \cdot \frac{x^3}{3}\right]_1^e - \int_1^e \frac{x^3}{3} \cdot \frac{1}{x}\, dx$$

$$= \left[\ln x \cdot \frac{x^3}{3}\right]_1^e - \left[\frac{x^3}{9}\right]_1^e$$

$$= \frac{2e^3 + 1}{9}$$

Integration by parts

$$\int_a^b u \cdot \frac{dv}{dx}\, dx = \left[uv\right]_a^b - \int_a^b v \cdot \frac{du}{dx}\, dx$$

Example 2: Evaluate $\displaystyle\int_1^4 \sec^{-1}\left(\sqrt{x}\right)\, dx$

Solution: $\sec^{-1}\left(\sqrt{x}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{x}}\right) \Rightarrow \displaystyle\int_1^4 \sec^{-1}\left(\sqrt{x}\right)\, dx = \int_1^4 \cos^{-1}\left(\frac{1}{\sqrt{x}}\right)\, dx$

Let $\quad u = \cos^{-1}\left(\dfrac{1}{\sqrt{x}}\right) \quad$ and $\quad \dfrac{dv}{dx} = 1$

$\Rightarrow \quad v = x \quad$ and $\quad \dfrac{du}{dx} = \dfrac{1}{2x\sqrt{x-1}}$

$$\Rightarrow \int_1^4 \sec^{-1}\left(\sqrt{x}\right)\, dx = \left[\cos^{-1}\left(\frac{1}{\sqrt{x}}\right)\cdot x\right]_1^4 - \int_1^4 \frac{1}{2x\sqrt{x-1}} \cdot x\, dx$$

$$= \left[\cos^{-1}\left(\frac{1}{\sqrt{x}}\right)\cdot x\right]_1^4 - \left[\sqrt{x-1}\right]_1^4$$

$$= \frac{4\pi}{3} - \sqrt{3}$$

Practice Problems on Worksheet:

1. Q3(ii)
2. Q3(iii)

Answers:

1: $3\ln 3 - 2$

2: $2\left(\ln 5 + \tan^{-1}(2) - 2\right)$

Properties for Evaluating Definite Integrals

1. If $f(x)$ is integrable on an interval I, and $a, b, c \in I$, then
$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$
2. If $f(x)$ is integrable and EVEN on $[-a, a]$, then
$$\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)\ dx$$
3. If $f(x)$ is integrable and ODD on $[-a, a]$, then
$$\int_{-a}^a f(x)\ dx = 0$$
4. If $f(x)$ is integrable on $[0, a]$, then
$$\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$$
5. If $f(x)$ is integrable on $[a, b]$, then
$$\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx$$

Example: Evaluate $\int_0^3 f(x)\ dx$ where $f(x) = \begin{cases} x^2, & x < 2 \\ 3x-2, & x \geq 2 \end{cases}$

Solution:

$$\begin{aligned}
\int_0^3 f(x)\ dx &= \int_0^2 f(x)\ dx + \int_2^3 f(x)\ dx \\
&= \int_0^2 x^2\ dx + \int_2^3 (3x-2)\ dx \\
&= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 \\
&= \left[\frac{8}{3} - 0\right] + \left[\left(\frac{27}{2} - 6\right) - (6-4)\right] \\
&= \frac{49}{6}
\end{aligned}$$

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Practice Problems on Worksheet:

1. Q4(iii)
2. Q4(iv)
3. Q4(v)
4. Q4(vi)

Answers:

1: $\dfrac{\pi}{4}$

2: 2

3: $\dfrac{7}{2}$

4: 0

Results

- The area of the region bounded by the curve $y = f(x)$, lines $x = a$, $x = b$, and the X-axis is:

$$A = \int_a^b y \; dx = \int_a^b f(x) \; dx$$

- The area of the region bounded by the curve $x = g(y)$, lines $y = c$, $y = d$, and the Y-axis is:

$$A = \int_c^d x \; dy = \int_c^d g(y) \; dy$$

Example 1: Find the area of the region bounded by the curve $y = x^3$, lines $x = -2$, $x = 2$, and the X-axis.

Solution: The function $y = x^3$ is an odd function.

$$\begin{aligned} A &= \left| \int_{-2}^{0} x^3 \ dx \right| + \int_{0}^{2} x^3 \ dx \\ &= -\int_{-2}^{0} x^3 \ dx + \int_{0}^{2} x^3 \ dx \\ &= 2 \int_{0}^{2} x^3 \ dx \\ &= 2 \left[\frac{x^4}{4} \right]_0^2 \\ &= 8 \end{aligned}$$

Example 2: Find the area of the region bounded by the curve $y = e^{\sin x} \sin 2x$ and the X-axis, where $x \in \left[0, \frac{\pi}{2}\right]$

Solution:

$$\begin{aligned}
A &= \int_0^{\frac{\pi}{2}} y \; dx \\
&= \int_0^{\frac{\pi}{2}} e^{\sin x} \sin 2x \; dx \\
&= 2\int_0^{\frac{\pi}{2}} \sin x \cdot e^{\sin x} \cos x \; dx \\
&= 2\int_0^{1} te^{t} \; dt \qquad (t = \sin x) \\
&= 2\left(\left[te^{t}\right]_0^1 - \int_0^1 e^{t} \; dt\right) \qquad (u = t, \quad dv/dt = e^{t}) \\
&= 2e - 2\left[e^{t}\right]_0^1 \\
&= 2
\end{aligned}$$

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Practice Problems on Worksheet:

1. Q5(ii)
2. Q5(iii)
3. Q5(iv)
4. Q5(v)

Answers:

1: $\dfrac{4}{3}$

2: 36

3: 2

4: 2

Office hours:

Day	Time	Venue
Tuesday	12:00 to 14:00	Trent 314a
	13:00 to 14:00	Trent 322
Thursday	16:30 to 17:30	TB 417
	17:00 to 18:00	IAMET 315
Friday	14:00 to 15:00	PB 330
	17:00 to 18:00	TB 417

Weekly drop-in session: Wednesday 4 – 5 pm in PB-115.