Foundation Calculus & Mathematical Techniques   CELEN037
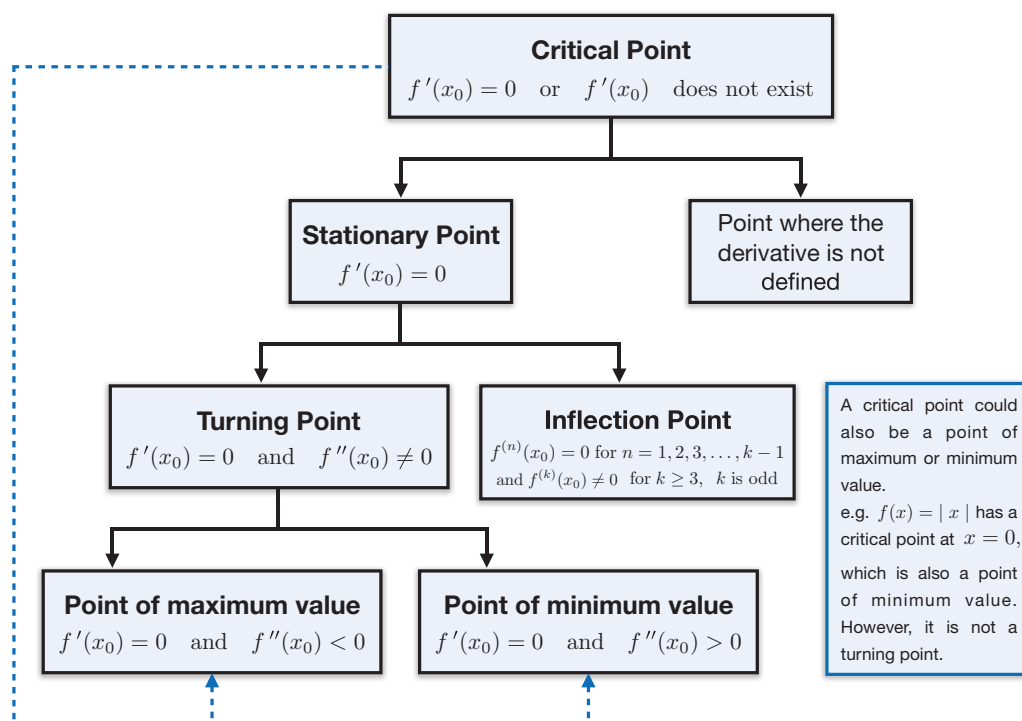
Weekly Worksheet-4   **Topics:** Stationary points, Optimisation problems, Related rates, Simple Integration.

Type 1: Stationary points and their Classification

Critical Point
$f'(x_0) = 0$ or $f'(x_0)$ does not exist

Stationary Point
$f'(x_0) = 0$

Point where the derivative is not defined

Turning Point
$f'(x_0) = 0$ and $f''(x_0) \neq 0$

Inflection Point
$f^{(n)}(x_0) = 0$ for $n = 1, 2, 3, \ldots, k-1$
and $f^{(k)}(x_0) \neq 0$ for $k \geq 3$, k is odd

A critical point could also be a point of maximum or minimum value.
e.g. $f(x) = |x|$ has a critical point at $x = 0$, which is also a point of minimum value. However, it is not a turning point.

Point of maximum value
$f'(x_0) = 0$ and $f''(x_0) < 0$

Point of minimum value
$f'(x_0) = 0$ and $f''(x_0) > 0$

1. Find and classify the stationary points for the following functions.

   Also sketch the curve $y = f(x)$.

   (i) $f(x) = 8x^3 - 3x^2 - 10$ (ii) $f(x) = 2x^3 - 9x^2 - 24x + 6$

   (iii) $f(x) = 3x^4 - 8x^3 - 18x^2 + 25$ (iv) $f(x) = x^4 + 4x^3 - 8x^2 + 1$

   (v) $f(x) = x^4 - 2x^3 - 2x^2$ (vi) $f(x) = x^3 + x^2 - 8x - 15$

Type 2: Optimisation problems

(i) Find the dimensions of the rectangle with largest area whose diagonal is 10 m.

(ii) Find the point on the curve $y = x^2$ that is nearest to the point $(18, 0)$.

(iii) Find the largest area of the rectangle inscribed in a semicircle of radius 8 cm whose one side is on the diameter of the semicircle.

Type 3: Related rates

(i) The volume of a right circular cone is given by $V = \dfrac{1}{3}\pi r^2 h$, where r is the radius and h is the height of the cone. If the height of the cone is increasing at the rate of 3 cm/sec, find the rate at which its volume is increasing when the radius is 5 cm.

(ii) A spherical balloon is inflated by a machine which pumps-in air at a rate of 10 cm^3/sec. Find the rate at which its radius is increasing when its radius is 10 cm.

(iii) An aircraft is climbing at a $30°$ angle to the horizontal. Determine how fast the aircraft is gaining altitude if its speed is 700 km/h.

(iv) The area and circumference of a circle with radius r are given by πr^2 and $2\pi r$ respectively. Given that the area of the circle is decreasing at a rate of 0.5 cm^2/sec, find the rate at which the circumference is decreasing when the radius is 2 cm.

Type 4: Simple Integration

Evaluate the following integrals:

(i) $\displaystyle\int (4x^3 - 5x + 6)\, dx$

(ii) $\displaystyle\int \left(x^7 - \dfrac{1}{x^5} + \sqrt{x} \right) dx$

(iii) $\displaystyle\int (2x + 1)^3 \, dx$

(iv) $\displaystyle\int \left(6\sqrt{x^3} - \dfrac{1}{x^2} - \dfrac{5}{x} \right) dx$

(v) $\displaystyle\int \left(e^x - \dfrac{2}{\sqrt{x}} \right) dx$

(vi) $\displaystyle\int \left(\sqrt[3]{x^2} + 3\, e^x + \dfrac{1}{x^2} \right) dx$

(vii) $\displaystyle\int (\cos x - 2\sin x)\, dx$

$(viii)$ $\displaystyle\int (x^4 - \sec^2 x)\, dx$

(ix) $\displaystyle\int \left(\dfrac{1 + \cos^2 \theta}{\cos^2 \theta} \right) dx$

(x) $\displaystyle\int (\cos x - \cot^2 x)\, dx$

(xi) $\displaystyle\int \left(e^x - \dfrac{\sin x}{\cos^2 x} \right) dx$

(xii) $\displaystyle\int \left(\dfrac{\sin x}{\sin^2 x - 1} \right) dx$

$(xiii)$ $\displaystyle\int \left(\dfrac{1 - \cos 2x}{2\cos^2 x} \right) dx$

(xiv) $\displaystyle\int \dfrac{1 + x + x^3}{1 + x^2}\, dx$

(xv) $\displaystyle\int (a^x - x^a)\, dx \quad ; \quad a > 0$