Lecture 10

Topics covered in this lecture session

1. Method of Partial Fractions.
2. Sequences
    1. Arithmetic sequence (A.P.)
    2. Geometric sequence (G.P.)
    3. Harmonic and Fibonacci sequence.

Partial Fractions

Process: Simplifying algebraic fractions

$$= \frac{(x+2)-(x+1)}{(x+1)\,(x+2)}$$

$$\frac{1}{(x+1)} - \frac{1}{(x+2)} \qquad\qquad = \frac{1}{(x^2+3x+2)}$$

Process: Finding partial fractions for a given expression

Partial Fractions

Thus, in the method of partial fraction, we decompose

a rational fraction

$$f(x) = \frac{p(x)}{q(x)} \quad ; \quad q(x) \neq 0,$$

where $p(x)$ and $q(x)$ are polynomials,

as a sum of several fractions with a

simpler denominator.

$$\frac{1}{(x^2+3x+2)}$$

$$\Downarrow$$

$$\frac{1}{(x+1)} - \frac{1}{(x+2)}$$

Partial Fractions

The method is applicable when the following conditions are satisfied.

a) $\deg[\,p(x)\,] < \deg[\,q(x)\,]$,

b) The expression in the denominator is factorable.

e.g. $$\frac{2x+3}{x^2+3x+2} \qquad \therefore \deg[p(x)] = 1 < 2 = \deg[q(x)]$$

$$\frac{3x+1}{(x-1)^2\,(x+2)} \qquad \therefore \deg[p(x)] = 1 < 3 = \deg[q(x)]$$

Forms of Partial Fractions

1. Non-repeated Linear Factors

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

e.g. $$\frac{3x}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

Forms of Partial Fractions

2. Non-repeated Quadratic Factors

$$\frac{1}{(x+a)(x^2+b)} = \frac{A}{(x+a)} + \frac{Bx+C}{(x^2+b)}$$

$$\frac{1}{(ax^2+bx+c)(x+d)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x+d)}$$

e.g. $$\frac{13}{(x^2+1)\,(2x+3)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(2x+3)}$$

Forms of Partial Fractions

3. Repeated Linear Factors

$$\frac{1}{(x+a)^2(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+b)}$$

$$\frac{1}{(x+a)^3(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{D}{(x+b)}$$

e.g. $$\frac{x}{(x-3)^2\,(2x+1)} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{(2x+1)}$$

Partial Fractions - The method

Non repeated linear factors

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

$$\Rightarrow \; A\,(x+b) + B\,(x+a) = 1$$

Put $x = -a$ to find the value of A

and then

put $x = -b$ to find the value of B.

Non repeated quadratic factor

$$\frac{1}{(x^2+a)(x+b)} = \frac{Ax+B}{(x^2+a)} + \frac{C}{(x+b)}$$

$$\Rightarrow \; (Ax+B)\,(x+b) + C\,(x^2+a) = 1$$

Put $x = -b$ to find the value of C

and then

equate the terms in x^2 or x

or constants, to find A and B.

Partial Fractions - The method

Step 1:

Express the given rational function of the form $\frac{p(x)}{q(x)}$ as a sum of partial fractions with constants A and B (and C).

Step 2:

Find the constants A and B (and C) as explained earlier.

Step 3:

Finally, write the given expression as a sum of partial fractions with obtained values of constants A and B (and C).

Examples: Partial Fractions

Express the following as a sum of partial fractions.

1. $\dfrac{2x}{(x-1)(x-3)}$

2. $\dfrac{1}{(x^2+1)(x-1)}$

3. $\dfrac{2x}{(x-1)(x+2)^2}$

Sequences - Introduction

A sequence is an ordered list of numbers (objects).

Mathematically,

A sequence is a function defined on a set of natural numbers.

that is,

A sequence is a function $f : \mathbb{N} \to A$, where A is any non-empty set of numbers (or objects).

Sequences - Introduction

Some examples of sequences are:

2, 4, 6, 8, 10, …. is a sequence of even numbers.

1, 2, 4, 8, 16, …. is a sequence of numbers of the form

$$2^{n-1} \quad ; \quad n = 1, 2, 3, 4, \ldots$$

4, 9, 16, 25, 36, …. is a sequence of numbers of the form

$$n^2 \quad ; \quad n = 2, 3, 4, \ldots$$

Sequences - Introduction

Each member of the set is called the term of the sequence, and is denoted by

$$a_1,\ a_2,\ a_3,\ ....\quad \text{or}\quad T_1,\ T_2,\ T_3,\ ....\quad \text{or}\quad f(1),\ f(2),\ f(3),\ .....$$

Thus, a sequence may be denoted by: $\{a_n\}_{n=1}^{n=k}$

If k is a finite number, the sequence is called finite sequence; otherwise infinite.

Sequences - Introduction

Some sequences have a general formula, some not.

e.g.

- The sequence of numbers

  2, 5, 8, 11, ….. has a general formula

  $$f(n) = 3n - 1 \quad ; \quad n \in \mathbb{N}.$$

- The sequence of primes

  2, 3, 5, 7, 11, ….. has no general formula.

Arithmetic Sequence/Progression (A.P.)

An Arithmetic Progression (A.P.) is a sequence in which difference between any two consecutive terms is constant.

e.g. 1, 5, 9, 13, 17, 21, ... is an A.P.

- The constant difference, called the common difference is denoted by d.
- The first term of the sequence is denoted by a.

Arithmetic Sequence/Progression (A.P.)

Some examples of A.P. are:

2, 4, 6, 8, 10, …. where $a = 2, \quad d = 2.$

5, 8, 11, 14, 17, …. where $a = 5, \quad d = 3.$

8, 5, 2, -1, - 4, …. where $a = 8, \quad d = -3.$

Thus, an A.P. takes the form:

$$a, \quad a + d, \quad a + 2d, \quad , \ldots.., \quad a + (n-1)\,d$$

$\therefore n^{th}$ term of an A.P. is: $\boxed{a_n = a + (n-1)\,d}$

Geometric Sequence/Progression (G.P.)

A Geometric Progression (G.P.) is a sequence in which ratio of any two consecutive terms is constant.

e.g. 4, 12, 36, 108, 324, ….. is a G.P.

- The constant ratio, called common ratio is denoted by r.
- The first term of the sequence is denoted by a.

Geometric Sequence/Progression (G.P.)

Some examples of G.P. are:

$$2, \quad 4, \quad 8, \quad 16, \quad 32, \; \ldots.. \qquad \text{where} \quad a = 2, \quad r = 2$$

$$1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \frac{1}{16}, \; \ldots.. \qquad \text{where} \quad a = 1, \quad r = \frac{1}{2}$$

Thus, a G.P. takes the form:

$$a, \quad a\,r, \quad a\,r^2, \quad a\,r^3, \quad , \ldots.., \quad a\,r^{n-1}$$

$\therefore n^{th}$ term of a G.P. is: $\boxed{a_n = a\,r^{n-1}}$

Worked Examples

1. The eighth term of an A.P. is 11 and its fifteenth term is 21. Find the common difference, the first term of the sequence, and the n^{th} term.
2. Find the G.P. of positive terms such that its first term is 4 and the fifth term is 324.
3. The fourth term of a G.P. is 24 and its ninth term is 768. Find its eleventh term.

Harmonic Sequence

A general harmonic progression (or harmonic sequence) is a progression formed by taking the reciprocals of an arithmetic progression.

i.e. it is a sequence of the form: $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, ...$

A particular case of general harmonic sequence is given by:

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{n}, ...$$

$\therefore n^{th}$ term of a Harmonic sequence is: $\boxed{f(n) = \frac{1}{n}}$

Fibonacci Sequence

- Fibonacci sequence is named after Leonardo Fibonacci.
- It consists of (Fibonacci) numbers in the following integer sequence: $0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots\ldots$

Mathematically, Fibonacci numbers is defined by the recurrence relation:

$$\boxed{f(n) = f(n-1) + f(n-2)} \quad ; \quad n \in \mathbb{N},\ n > 1$$

with $f(0) = 0$ and $f(1) = 1.$

Fibonacci Sequence and the Golden ratio

- The ratio of neighbouring Fibonacci numbers tends to the Golden ratio.

$$\phi = \frac{1+\sqrt{5}}{2}$$

$$= \boxed{1.6180339887}$$

Golden Ratio

2 / 1 = 2.0

3 / 2 = 1.5

5 / 3 = 1.67

8 / 5 = 1.6

13 / 8 = 1.625

21 / 13 = 1.615

34 / 21 = 1.619

55 / 34 = 1.618

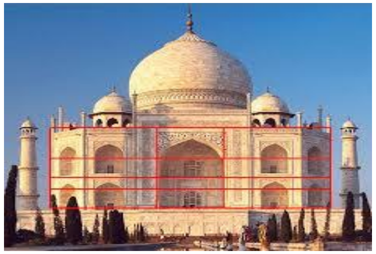
89 / 55 = 1.618

Fibonacci Sequence and the Golden ratio

Many artists and architects have been fascinated by the presumption that the golden rectangle (with length of sides as neighboring Fibonacci numbers) is considered aesthetically pleasing.

1.618

1



23

Fibonacci Sequence and the Golden ratio

- Universe has a 'golden ratio' that keeps everything in order, researchers claim.

More info:

Suggested Reading

Foundation Algebra by P. Gajjar.

(Chapters 12 and 14)