Foundation Algebra (CELEN036)

Problem Sheet 7   **Topics: Binomial theorem**

Topic 1: Expansion using the Binomial theorem

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \cdots\cdots + x^n, \quad n \in \mathbb{N}$$

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots\cdots + b^n, \quad n \in \mathbb{N}$$

1. Expand the following expressions using the Binomial theorem:

   (i) $(1+x)^7$ (ii) $(2x+5)^4$ (iii) $\left(1 + \dfrac{x}{4}\right)^5$ (iv) $\left(1 - \dfrac{x}{2}\right)^6$

   (v) $(4-3x)^4$ (vi) $(5x-6)^3$ (vii) $(2x+y)^4$ (viii) $(x^2 - y)^5$

Topic 2: To find the coefficient of certain term in the expansion

2. Find the coefficient of $x^7 y^2$ in the expansion of $(2y - x)^9$.

3. Find the coefficient of x^4 in the expansion of $(2x+3)^9$.

4. Find the coefficient of x^2 in the expansion of $(7 - 3x)^7$.

5. Find the coefficient of x^3 in the expansion of $\left(1 - \dfrac{x}{5}\right)^{15}$.

6. Find the coefficient of $x^4 y^4$ in the expansion of $(2x - y^2)^6$.

7. Find the coefficient of x^3 in the expansion of $\left(2x - \dfrac{1}{3x}\right)^9$.

8. Find the coefficient of x^6 in the expansion of $\left(2x - \dfrac{1}{3x}\right)^9$.

9. In the Binomial expansion of $(x + ky)^8$, the coefficient of $(x^5 y^3)$ is -1512. Find the value of $k \in \mathbb{R}$.

10. In the Binomial expansion of $\left(ax^3 + \dfrac{2}{x^2}\right)^4$, the coefficient of x^2 is 6. Find the value of $a \in \mathbb{R}$.

Topic 3: Application of the generalized Binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} \cdot x^2 + \frac{n(n-1)(n-2)}{3!} \cdot x^3 + \cdots\cdots \quad \text{where } n \in \mathbb{R} \text{ and } |x| < 1$$

11. Assuming that $|x| < 1$, obtain the Binomial expansions of the following expressions, up to the term with x^{-3} :

    (i) $(1+x)^{-2}$ 　　　　　 (ii) $(1+x)^{-6}$ 　　　　　 (iii) $(1-x)^{-3}$

12. Apply the Binomial theorem to approximate the following values. Use the first **four** terms in the expansion and round the answers to 4 decimal places.

    (i) $(1.01)^{-3}$ 　　 (ii) $(1.03)^{-2}$ 　　 (iii) $(1.03)^{\frac{1}{2}}$ 　　 (iii) $(1.02)^{-\frac{1}{3}}$

13. Use the first four terms of the Binomial expansion of $\left(1 - \dfrac{1}{50}\right)^{\frac{1}{2}}$ to derive an approximation of $\sqrt{2}$ to 5 decimal places.

14. Apply the Binomial theorem for $n \in \mathbb{N}$ to evaluate the following expressions:

    (i) $(2.1)^2$ 　　　　　 (ii) $(1.01)^3$ 　　　　　 (iii) $(3.02)^3$

15. The radius of a sphere is measured as r, with an error of $\delta r = 1.2\%$ of r. The volume of the sphere $V = \dfrac{4}{3}\pi r^3$ is then calculated using the measured r. Use the approximation

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2} \cdot x^2$$

to find the resulting error δV in the calculated volume.

Answers

1. (i) $1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$

   (ii) $16x^4 + 160x^3 + 600x^2 + 1000x + 625$

   (iii) $1 + \dfrac{5}{4}x + \dfrac{5}{8}x^2 + \dfrac{5}{32}x^3 + \dfrac{5}{256}x^4 + \dfrac{1}{1024}x^5$

   (iv) $1 - 3x + \dfrac{15}{4}x^2 - \dfrac{5}{2}x^3 + \dfrac{15}{16}x^4 - \dfrac{3}{16}x^5 + \dfrac{1}{64}x^6$

   (v) $256 - 768x + 864x^2 - 432x^3 + 81x^4$

   (vi) $125x^3 - 450x^2 + 540x - 216$

   (vii) $16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

   (viii) $x^{10} - 5x^8y + 10x^6y^2 - 10x^4y^3 + 5x^2y^4 - y^5$

2. -144

3. 489888

4. 3176523

5. $-\dfrac{91}{25}$

6. 240

7. $-\dfrac{1792}{9}$

8. 0

9. $k = -3$

10. $a = \pm\dfrac{1}{2}$

11. (i) $1 - 2x + 3x^2 - 4x^3 + \cdots$ (ii) $1 - 6x + 21x^2 - 56x^3 + \cdots$ (iii) $1 + 3x + 6x^2 + 10x^3 + \cdots$

12. (i) 0.9706 (ii) 0.9426 (iii) 1.0149 (iv) 0.9934

13. 1.414214

14. (i) 4.41 (ii) 1.030301 (iii) 27.543608

15. $\delta V \approx 3.6432\%$ of V