Lecture 5

Early Module Feedback

Welcome to Foundation Calculus and Mathematical Techniques (CELEN037)

This module consolidates previous studies in mathematical techniques and introduce a range of mathematical topics used in the analysis of problems in engineering and physical sciences. The module will cover techniques and applications of differentiation, integration, and differential equations. Application to solving real life problems is also developed.

Announcements

Early Module Feedback

Your comments are useful to us, because it helps us to improve this module and to respond to any queries you may have.

Early Module Feedback

Respond to 4 questions and make relevant comments about the module

1. The module content was of sufficient quality to assist my learning on this module

2. Module materials were clear about what was expected of me

3. I was given sufficient opportunity to contact my teachers/faculty on this module

4. The overall experience of studying this module has contributed to my learning

5. In your opinion, what is working well on the module so far? If there are any suggestions for the remaining weeks on the module, please also leave your comments here.

Lecture Content

- The method of substitution

- Some useful substitutions

The method of substitution

The method of substitution follows from examining the chain rule from the viewpoint of anti-differentiation

Let F be an antiderivative of f, and g be a differentiable function.

$$\therefore \; F(x) = \int f(x)\; dx \quad \Rightarrow \quad \frac{dF}{dx} = f(x) \qquad \text{i.e. } F' = f$$

From Chain Rule for derivative ,

$$\frac{d}{dx}[F(g(x))] = F'\big(g(x)\big) \cdot g'(x) \; = \; f\,\big(g(x)\big) \cdot g'(x)$$

The method of substitution

$$\therefore \frac{d}{dx}\left[\, F(g(x)) \,\right] = f\big(g(x)\big)\cdot g'(x) \qquad (1)$$

$$\Rightarrow \int f\big(g(x)\big)\, g'(x)\, dx = F\big(g(x)\big) + C$$

Let $g(x) = t \quad \Rightarrow \quad g'(x) = \dfrac{dt}{dx}$

$$\Rightarrow \quad g'(x)\, dx = dt$$

$$\therefore (1) \Rightarrow \quad \int f(t)\; dt = F(t) + C$$

This process is called the method of substitution for integration.

The method of substitution: Proof by example

Evaluate $\int \cos(e^x) \cdot e^x dx$

Chain Rule: Let $F\big(g(x)\big) = \sin(e^x)\,;\, g(x) = e^x$

$$\rightarrow \frac{dF(g(x))}{dx} = \cos(e^x) \cdot \frac{d}{dx}(e^x) \rightarrow F'[g(x)] \cdot g'(x)$$

$$\therefore \int \cos(e^x) \cdot e^x dx = \int F'[g(x)] \cdot g'(x)\, dx$$

$$\int F'[g(x)] \cdot g'(x)\, dx = \int f[g(x)] \cdot g'(x)\, dx$$

Then: $\int f[g(x)] \cdot g'(x)\, dx = F\big(g(x)\big) + C$

$$\therefore \int \cos(e^x) \cdot e^x dx = \sin(e^x) + C$$

Note: This is an example of solving such problems using the chain rule.

The method of substitution is much simpler, and should always be used to solve problems of this sort.

The method of substitution: Actual Process

Evaluate $\int \cos(e^x) \cdot e^x dx$

Observe the integrand and take note of $g(x)$ and $g'(x)$

Substitution method:

Let $t = e^x \rightarrow \dfrac{dt}{dx} = e^x \quad \therefore dt = e^x dx$

Make relevant substitution: e.g. $t = g(x)$, and find the derivative $dt = g'(x)dx$

Thus: $I = \int \cos(e^x) \cdot e^x dx = \int \cos(t) \cdot dt$

Replace the integrand and the variable of integration with the parameter t and dt. **Note**: the new integrand and variable of integration must consist of only t and dt.

$\rightarrow \int \cos(t) \cdot dt = \sin(t) + C$

Solve the resulting expression using standard tables.

$\therefore I = \sin(e^x) + C$

Give your final answer in terms of the original variable.

The method of substitution

Integral	Substitution	Integral	Substitution
$\int f\left(g(x)\right)\ g'(x)\ dx$	$g(x) = t$	$\int f\left(\tan x\right)\ \sec^2 x\ dx$	$\tan x = t$
$\int f\left(x^n\right)\ x^{n-1}\ dx$	$x^n = t$	$\int f\left(\ln x\right)\ \frac{1}{x}\ dx$	$\ln x = t$
$\int f\left(x^3\right)\ x^2\ dx$	$x^3 = t$	$\int f\left(\sqrt{x}\right)\ \frac{1}{\sqrt{x}}\ dx$	$\sqrt{x} = t$
$\int f\left(\sin x\right)\ \cos x\ dx$	$\sin x = t$	$\int f\left(\tan^{-1} x\right)\ \frac{1}{1+x^2}\ dx$	$\tan^{-1} x = t$

Example

Evaluate $\int x \cdot \sec^2(x^2)\,dx$

Integral	Substitution
$\int f(x^n)\, x^{n-1}\, dx$	$x^n = t$

Solution:

Let $x^2 = t \Rightarrow xdx = \frac{1}{2}dt$

$$x^2 = t$$

$$\Rightarrow \frac{d}{dx}(x^2) = \frac{dt}{dx}$$

$$\Rightarrow 2x = \frac{dt}{dx}$$

$$\Rightarrow xdx = \frac{1}{2}dt$$

$$\Rightarrow I = \int \sec^2(t) \cdot \frac{1}{2}dt$$

$$\Rightarrow \frac{1}{2}\int \sec^2(t)\,dt = \frac{1}{2}\tan(t) + C$$

$$= \frac{1}{2}\tan(x^2) + C$$

Example

Evaluate $\int x^2 \cdot (x^3+2)^{1/2} dx$

Solution:

Let $x^3 + 2 = t \quad \Rightarrow \quad x^2 dx = \frac{1}{3} dt$

$$\Rightarrow \; I = \int t^{1/2} \frac{1}{3} dt = \frac{1}{3} \int t^{1/2} dt$$

$$= \left(\frac{1}{3}\right) \frac{t^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{2}{9} (x^3+2)^{3/2} + C$$

Integral	Substitution
$\int f(x^n)\, x^{n-1}\, dx$	$x^n = t$
$\int f(x^3)\, x^2\, dx$	$x^3 = t$

$$x^3 + 2 = t$$

$$\Rightarrow \; \frac{d}{dx}(x^3+2) = \frac{dt}{dx}$$

$$\Rightarrow \; 3x^2 = \frac{dt}{dx}$$

$$\Rightarrow \; x^2 dx = \frac{1}{3} dt$$

Example Evaluate $\int e^{\sin x} \cos x \, dx$

Integral	Substitution
$\int f(\sin x) \cos x \, dx$	$\sin x = t$

Solution:

Let $\sin x = t \quad \Rightarrow \quad \cos x \, dx = dt$

$$\Rightarrow \quad I = \int e^t \, dt$$

$$= e^t + C$$

$$= e^{\sin x} + C$$

$$\sin x = t$$

$$\Rightarrow \frac{d(\sin x)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \cos x = \frac{dt}{dx}$$

$$\Rightarrow \cos x \, dx = dt$$

Example

Evaluate $\displaystyle\int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx$

Integral	Substitution
$\int f\ (\tan x)\ \sec^2 x\ dx$	$\tan x = t$

Solution:

$$\int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx = \int \frac{1}{\sqrt{\tan x}} \cdot \sec^2 x\, dx$$

Let $\tan x = t \implies \sec^2 x dx = dt$

$$\Rightarrow I = \int \frac{1}{\sqrt{t}}\,dt = \int t^{-1/2} dt \quad \Rightarrow \frac{t^{1/2}}{\frac{1}{2}} + C \quad = 2\sqrt{\tan x} + C$$

Example

Evaluate $\displaystyle\int \frac{1}{x \cdot \ln x}\,dx$

Integral	Substitution
$\displaystyle\int f\,(\ln x)\,\frac{1}{x}\,dx$	$\ln x = t$

Solution:

$$\int \frac{1}{x \cdot \ln x}\,dx = \int \frac{1}{\ln x} \cdot \frac{1}{x}\,dx$$

Let $\ln x = t \;\Rightarrow\; \dfrac{1}{x}\,dx = dt$

$$\Rightarrow\; I = \int \frac{1}{t}\,dt = \ln|t| + C$$

$$= \ln|\ln x| + C$$

Example

Evaluate $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Integral	Substitution
$\displaystyle\int f\left(\sqrt{x}\right)\frac{1}{\sqrt{x}}\,dx$	$\sqrt{x}=t$

Solution:

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int e^{\sqrt{x}}\cdot\frac{1}{\sqrt{x}}\,dx$$

Let $\sqrt{x}=t \Rightarrow \dfrac{1}{2\sqrt{x}}\,dx = dt$

$$\Rightarrow \frac{1}{\sqrt{x}}\,dx = 2dt$$

$$\Rightarrow I = \int e^{\sqrt{x}}\cdot 2dt = 2\int e^{t}\,dt$$

$$\Rightarrow 2e^{t}+C = 2e^{\sqrt{x}}+C$$

Example

Evaluate $\int x^2 \cdot \sqrt{x-1}\,dx$

Integral	Substitution
$\int f\left(\sqrt{x}\right)\frac{1}{\sqrt{x}}\,dx$	$\sqrt{x}=t$

Solution:

Let $x-1=t^2 \quad \Rightarrow \quad x=t^2+1$

$$\Rightarrow \quad dx = 2t\,dt$$

$$\Rightarrow I = \int (t^2+1)^2 \cdot \sqrt{t^2} \cdot 2t\,dt$$

$$\Rightarrow I = \int 2t^2(t^4+2t^2+1)\,dt = 2\int (t^6+2t^4+t^2)\,dt$$

Example

$$\Rightarrow \quad I = 2\left(\frac{t^7}{7} + \frac{2t^5}{5} + \frac{t^3}{3}\right) + C$$

$$= 2\left(\frac{1}{7}(x-1)^{7/2} + \frac{2}{5}(x-1)^{5/2} + \frac{1}{3}(x-1)^{3/2}\right) + C$$

Example

Evaluate $\displaystyle\int \frac{(\tan^{-1} x)^2}{1+x^2}\,dx$

Integral	Substitution
$\displaystyle\int f\left(\tan^{-1} x\right) \frac{1}{1+x^2}\,dx$	$\tan^{-1} x = t$

Solution:

$$\int \frac{(\tan^{-1} x)^2}{1+x^2}\,dx = \int (\tan^{-1} x)^2 \cdot \frac{1}{(1+x^2)}\,dx$$

Let $\tan^{-1} x = t \implies \dfrac{1}{(1+x^2)}\,dx = dt$

$$\implies I = \int t^2 dt = \frac{t^3}{3} + C$$

$$= \int \frac{(\tan^{-1} x)^3}{3} + C$$

The method of substitution: Trigonometric Substitution

Integrand	Trigonometric substitution
$\dfrac{1}{\sqrt{a^2-x^2}}$	$x = a\sin\theta$
$\dfrac{1}{x^2+a^2}$	$x = a\tan\theta$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$x = a\sec\theta$

The method of substitution: Trigonometric Substitution

Example

Show that $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

Proof: Let $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}}\, a\cos\theta\, d\theta$$

$$= \int \frac{1}{a\sqrt{1 - \sin^2\theta}}\, a\cos\theta\, d\theta$$

The method of substitution: Trigonometric Substitution

$$\boxed{\begin{aligned} &\because\ x = a\sin\theta \\ \Rightarrow\quad &\theta = \sin^{-1}\left(\frac{x}{a}\right) \end{aligned}}$$

$$= \int \frac{1}{\cos\theta}\cos\theta\ d\theta = \int 1\ d\theta$$

$$= \theta + C = \boxed{\sin^{-1}\left(\frac{x}{a}\right)} + C$$

Following this method, a table for trigonometric substitutions can then be obtained as shown in next slide:

The method of substitution: Trigonometric Substitution

Formula	When $a = 1$
$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$	$\int \frac{1}{\sqrt{1 - x^2}}\, dx = \sin^{-1} x + C$
$\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$	$\int \frac{1}{x^2 + 1}\, dx = \tan^{-1} x + C$
$\int \frac{1}{\lvert x \rvert \sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left\lvert \frac{x}{a} \right\rvert + C$	$\int \frac{1}{\lvert x \rvert \sqrt{x^2 - 1}} = \sec^{-1} \lvert x \rvert + C$

Integrating functions with linear term

Result If $\int f(x)\,dx = F(x) + C$, then

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$$

Proof: Let $ax + b = t$

$$\Rightarrow \quad a\,dx = dt$$

$$\Rightarrow \quad dx = \frac{1}{a}\,dt$$

$$\therefore \quad I = \frac{1}{a}\int f(t)\,dt = \frac{1}{a}F(t) + C$$

$$= \frac{1}{a}F(ax+b) + C$$

Integrating functions with linear term

If $\displaystyle\int f(x)\,dx = F(x) + C$, then $\displaystyle\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$

Examples:

$$(i) \quad \int \cos(2x+3)\ dx = \frac{1}{2}\cdot\sin(2x+3) + C$$

$$(ii) \quad \int \frac{1}{(5x-7)}\ dx = \frac{1}{5}\cdot\ln(5x-7) + C$$

$$(iii) \quad \int e^{4x-9}\ dx = \frac{1}{4}\cdot e^{4x-9} + C$$

$$(iv) \quad \int \sec^2(3x+5)\ dx = \frac{1}{3}\cdot\tan(3x+5) + C$$

Integrating functions with linear term

Example

Evaluate $\displaystyle\int \cos 4x \, \cos 2x \; dx$

Solution:

$$\int \cos 4x \, \cos 2x \; dx$$

$$\boxed{\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]}$$

$$= \frac{1}{2} \int \cos 6x \; dx + \frac{1}{2} \int \cos 2x \; dx$$

$$= \frac{1}{2} \cdot \frac{\sin 6x}{6} + \frac{1}{2} \cdot \frac{\sin 2x}{2} + C = \frac{1}{12} \left(\sin 6x + 3 \sin 2x \right) + C$$

Integrating functions with linear term

Example

Evaluate $\int \sin mx \cos nx \, dx \quad ; \; m \neq \pm n, \quad m, n, \in N$

Solution:

$$\int \sin mx \cos nx \, dx$$

$$\boxed{\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]}$$

$$= \frac{1}{2}\int \sin(m+n)x \, dx + \frac{1}{2}\int \sin(m-n)x \, dx$$

$$= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right] + C$$