Foundation Calculus and Mathematical Techniques (CELEN037)

Problem Sheet 10 Topics: Differential Equations

Topic 1: Variable Separable Equations

1. Solve the following ODEs.

(i) $\dfrac{dy}{dx} = \dfrac{5x}{7y}$

(ii) $\dfrac{dy}{dx} = \dfrac{3y-1}{4x}$

(iii) $\dfrac{dy}{dx} = ky$

(iv) $\dfrac{dy}{dx} = \dfrac{\sin x}{\cos y}$

(v) $\dfrac{dy}{dx} = ay(1-by)$

(vi) $\dfrac{dy}{dx} = \dfrac{\sin y}{\cos x}$

(vii) $x - y^2\dfrac{dy}{dx} = 0$

(viii) $\dfrac{dy}{dx} = \dfrac{\cos^2 x}{\sin^2 y}$

(ix) $y\dfrac{dy}{dx} - (1+y)x^2 = 0$

(x) $y^3\dfrac{dy}{dx} - (1+y^2)x^2 = 0$

(xi) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$

(xii) $(1+x^2)e^{\sqrt{3}y}\dfrac{dy}{dx} = 2x$

(xiii) $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$

Topic 2: Initial Value Problems

2. Use the method of separation of variables to solve the following IVPs:

(i) $\dfrac{dy}{dx} = 5x^3y^{\frac{1}{3}};\;\; y(1) = 1$

(ii) $xy\dfrac{dy}{dx} = x^2 + 1;\;\; y(1) = 0$

(iii) $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^3};\;\; y(0) = 1$

(iv) $\dfrac{d\theta}{dt} = \dfrac{t\sec\theta}{\theta e^{t^2}};\;\; \theta(0) = 0$

(v) $\dfrac{dp}{dt} = t^2p - p + t^2 - 1;\;\; p(1) = 0$

(vi) $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u};\;\; u(0) = -5$

(vii) $\dfrac{dx}{dy} = \dfrac{\sqrt{5-x^2}}{5-y^2};\;\; y(0) = 2\sqrt{5}$

(viii) $\dfrac{dy}{dx} = e^{2y}\cdot\ln x;\;\; y(1) = 0$

Topic 3: Exponential Growth and Decay

3. Applications of differential equations.

(i) A bacteria culture grows exponentially so that the initial number has doubled in 2 hours. How many times the initial number will be present after 8 hours?

(ii) A certain chemical decomposes exponentially. Assume that 400 grams becomes 100 grams in 1 hour. How much will remain after 3 hours?

(iii) Show that, when a quantity grows or decays exponentially, the rate of increase over a fixed time interval is a constant (that is, it depends only on the time interval, not on the time at which the interval begins.)

(iv) If the population of the world in 1980 was 4.5 billion and if it is growing exponentially with a growth rate $K = 0.02 \ln 3$, find the population in the year 2040.

(v) If a quantity y grows exponentially with a growth constant K and if during each unit of time there is an increase in y of r percent, find the relationship between K and r.

(vi) If a population is increasing exponentially at the rate of 2 percent per year, what will be the percentage increase over a period of 8 years?

(vii) If an amount of money y_0 is invested at a rate of r percent per year, compounded n times per year, what is the amount of money that will be available after k years?

(viii) An amount of money y_0 earning r percent per year is compounded continuously (that is, assume that it is compounded n times per year when n approaches infinity). How much is available after k years?

(ix) If an amount of money earning 10 percent per year is compounded quarterly, what is the equivalent annual rate of return?

(x) If an amount of money earning 10 percent per year is compounded 10 times per year, what is the equivalent annual rate of return?

(xi) If an amount of money receiving interest of 10 percent per year is compounded continuously, what is the equivalent annual rate of return?

(xii) A sum of money, compounded continuously, is multiplied by 5 in 10 years. If it amounts to $\$20,000$ after 20 years, what was the initial sum of money?

(xiii) If a quantity of money, earning interest compounded continuously, is worth 40 times the initial amount after 100 years, what was the annual rate of interest?

(xiv) Assume that a quantity y decays exponentially, with a decay constant K. The *half-life* T is defined to be the time interval after which half of the original quantity remains. Find the relationship between K and T.

(xv) The *half-life* of radium is 1690 years. If 20 percent of an original quantity of radium remains, how long ago was the radium created?

Answers

1.

(i) $y^2 = \frac{5}{7}x^2 + C$

(ii) $3y - 1 = Cx^{\frac{3}{4}}$

(iii) $y = Ce^{kx}$

(iv) $\sin y = -\cos x + C$

(v) $y = \frac{1}{b + Ce^{-ax}}$

(vi) $\ln|(\tan x + \sec x)(\csc y + \cot y)| = C$

(vii) $y^3 = \frac{3}{2}x^2 + C$

(viii) $y - \frac{1}{2}\sin 2y = x + \frac{1}{2}\sin 2x + C$

(ix) $y - \ln|1 + y| = \frac{1}{3}x^3 + C$

(x) $\frac{1}{2}[1 + y^2 - \ln(1 + y^2)] = \frac{1}{3}x^3 + C$

(xi) $y = \frac{x + C}{1 - Cx}$

(xii) $e^{\sqrt{3}y} = \sqrt{3}\ln(1 + x^2) + C$

(xiii) $y = x\cos C + \sqrt{1 - x^2} \cdot \sin C$

2.

(i) $6y^{\frac{2}{3}} = 5x^4 + 1$

(ii) $y^2 = x^2 + 2\ln|x| - 1$

(iii) $\ln|y| + \frac{1}{3}y^3 = \sin x + \frac{1}{3}$

(iv) $\theta\sin\theta + \cos\theta = -\frac{1}{2}e^{-t^2} + \frac{3}{2}$

(v) $\ln|p + 1| = \frac{1}{3}t^3 - t + \frac{2}{3}$

(vi) $u^2 = t^2 + \tan t + 25$

(vii) $\ln\left|\frac{y + \sqrt{5}}{y - \sqrt{5}}\right| = 2\sqrt{5} \cdot \sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + \ln 3$

(viii) $x(1 - \ln x) = \frac{1}{2}(1 + e^{-2y})$

3.

(i) 16

(ii) 6.25 grams

(iii)

(iv) 16.817 billion

(v) $r = 100 \cdot (e^K - 1)$

(vi) 17.351 percent

(vii) $y_0 \cdot \left(1 + \frac{r}{100n}\right)^{kn}$

(viii) $y_0 \cdot e^{\frac{rk}{100}}$

(ix) 10.381 percent

(x) 10.462 percent

(xi) 10.517 percent

(xii) $\$800$

(xiii) 3.76 percent

(xiv) $e^{KT} = \frac{1}{2}$

(xv) 3924 years