Foundation Calculus and Mathematical Techniques (CELEN037)

Problem Sheet 4 Topics: Applications of Differentiation

Topic 1: Classification of Stationary Points

1. Find and classify the stationary points for the following functions:

   (i) $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x$ (ii) $f(x) = 2x^3 - 3x^2 - 12x + 5$

   (iii) $f(x) = x^4 - \frac{8}{3}x^3 - 2x^2 + 8x + 1$ (iv) $f(x) = \ln x + \frac{1}{x}$

2. Given that $g(t) = 3t^4 - 4t^3 - 72t^2 + 7$:

   (i) Find and classify the stationary points of $g(t)$;

   (ii) Find the global maximum and minimum values of $g(t)$ on $[-2, 1]$.

Topic 2: Optimisation Problems

3. (i) Suppose that $r(x) = 9x$ is the revenue function and $c(x) = x^3 - 6x^2 + 15x$ is the cost function, where x represents millions of MP4 players produced. Is there a production level that maximizes profit? If so, what is it? (Hint: Profit = Revenue − Cost.)

   (ii) What is the smallest perimeter possible of a rectangle whose area is 16 m^2, and what are its dimensions?

   (iii) A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

   (iv) You are designing a rectangular poster to contain 50 cm^2 of printing with a 4-cm margin at the top and bottom and a 2-cm margin at each side. What overall dimensions will minimize the amount of paper used?

   (v) Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximu volume?

Topic 3: Related Rates

4. (i) A spherical ballon is inflated with helium at the rate of 100π m^3/min. How fast is the ballon's radius increasing at the instant the radius is 5 m? How fast is the surface area increasing?

(ii) Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

(iii) A rectangular swimming pool is being filled with water at a rate of 5 m^3/min. The length of the pool is 10 m and the width is 4 m. How fast is the height of the water increasing?

(iv) When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

Topic 4: Simple Integration

5. Evaluate the following integrals:

(i) $\displaystyle\int (5\sqrt{x} - 8x^2 + e^5)\ dx$ (ii) $\displaystyle\int (3x-2)^3\ dx$

(iii) $\displaystyle\int \left(5^x - e^x + \frac{1}{x}\right)\ dx$ (iv) $\displaystyle\int \left(\frac{8}{x} - \frac{5}{x^2} + \frac{6}{x^3}\right)\ dx$

(v) $\displaystyle\int \left(\sqrt{x} + \frac{1}{3\sqrt{x}}\right)\ dx$ (vi) $\displaystyle\int \left(\frac{1}{x^2+9} + \frac{1}{x^2-9}\right)\ dx$

(vii) $\displaystyle\int \left(\frac{4}{16-x^2} + \frac{2}{\sqrt{16-x^2}}\right)\ dx$ (viii) $\displaystyle\int \frac{1}{x\sqrt{x}}\ dx$

(ix) $\displaystyle\int (\cos x - \sec^2 x)\ dx$ (x) $\displaystyle\int \sin x \cdot \left(1 + \frac{1}{\cos^2 x}\right)\ dx$

Answers

1. (i) $\left(-1, -\frac{5}{6}\right)$ is the point of minimum value, $\left(-2, -\frac{2}{3}\right)$ is the point of maximum value.

(ii) $(2, -15)$ is the point of minimum value, $(-1, 12)$ is the point of maximum value.

(iii) $\left(-1, -\frac{16}{3}\right)$ and $\left(2, \frac{11}{3}\right)$ are the points of minimum value, $\left(1, \frac{16}{3}\right)$ is the point of maximum value.

(iv) $(1, 1)$ is the point of minimum value.

2. (i) $(-3, -290)$ and $(4, -633)$ are the points of minimum value, $(0, 7)$ is the point of maximum value.

(ii) Global maximum is 7 and global minimum is -201.

3. (i) $x = 2 + \sqrt{2}$.

   (ii) Smallest perimeter $= 16$ m; length $=$ width $= 4$ m.

   (iii) Largest area $= 32$; length $= 4$ and width $= 8$.

   (iv) Length $= 9$ cm and width $= 18$ cm.

   (v) Height $= \dfrac{20\sqrt{3}}{3}$ cm and radius $= \dfrac{10\sqrt{6}}{3}$ cm; maximum volume $= \dfrac{4000\sqrt{3}}{9}\pi$ cm^3.

4. (i) $\dfrac{dr}{dt} = 1$ m/min; $\dfrac{dS}{dt} = 40\pi$ m^2/min.

   (ii) $\dfrac{dA}{dt} = 48$ cm^2/s.

   (iii) $\dfrac{dh}{dt} = \dfrac{1}{8}$ m/min.

   (iv) $\dfrac{dA}{dt} = \pi$ cm^2/min.

5. (i) $\dfrac{10}{3}x^{\frac{3}{2}} - \dfrac{8}{3}x^3 + e^5x + C$

   (ii) $\dfrac{27}{4}x^4 - 18x^3 + 18x^2 - 8x + C$

   (iii) $\dfrac{5^x}{\ln 5} - e^x + \ln|x| + C$

   (iv) $8\ln|x| + \dfrac{5}{x} - \dfrac{3}{x^2} + C$

   (v) $\dfrac{2}{3}x^{\frac{2}{3}} + \dfrac{2}{3}x^{\frac{1}{2}} + C$

   (vi) $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x}{3}\right) + \dfrac{1}{6}\ln\left|\dfrac{x-3}{x+3}\right| + C$

   (vii) $\dfrac{1}{2}\ln\left|\dfrac{x+4}{x-4}\right| + 2\sin^{-1}\left(\dfrac{x}{4}\right) + C$

   (viii) $-\dfrac{2}{\sqrt{x}} + C$

   (ix) $\sin x - \tan x + C$

   (x) $-\cos x + \sec x + C$