COMP2054 Tutorial Session 5: Master Theorem

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Session outcomes

- Identify which recurrences can the Master Theorem be applied to.
- Prove runtime complexities of recurrences using the M.T.

Master Theorem

Master Theorem cases and complexity proofs

Master Theorem "Cheat Sheet"

For a given recurrence of the form $T(n) = a \cdot T(n/b) + f(n)$ the M.T. can tell us the growth rate of $T(n)$ according to three cases:

Case 1: Recurrence dominates (plus special case that $f(n) = 0$)

IF $f(n)$ is $O(n^c)$ with $c < \log_b a$ THEN $T(n)$ is $\Theta\left(n^{\log_b a}\right)$

Case 2: Neither term dominates

IF $f(n)$ is $\Theta\left(n^c (\log n)^k\right)$ with $c = \log_b a$ and $k \geq 0$ THEN $T(n)$ is $\Theta\left(n^c (\log n)^{k+1}\right)$

Case 3: f(n) dominates

IF $f(n)$ is $\Omega(n^c)$ with $c > \log_b a$ THEN $T(n)$ is $\Theta\left(f(n)\right)$

In this module we assume the regularity condition for case 3 holds but you should be aware of it!

Q1. $T(n) = 2 \cdot T(n/2)$ and $T(1) = 1$

- Case 1 (special case f(n) = 0) growth depends on the recurrence.
  - For $\mathrm{T(n) = a \cdot T(n/}b)$ and $T(1) = 1$
  - $T(n)$ is $\Theta\left(n^{\log_b a}\right)$
  - $a = 2; b = 2$
  - $T(n)$ is $\Theta\left(n^{\log_2 2}\right)$
  - Hence is $\Theta(n)$

Q2. $T(n) = 2 \cdot T(n/2) + \mathbf{n}$ and $T(1) = 1$

- For $T(n) = a \cdot T(n/b) + f(n)$
- $a = 2; b = 2; f(n) = n$
- $c = \log_2 2 = 1$
- $f(n)$ is at least $O(n^1)$ and $1 \nless \log_2 2$; hence not case 1.
- Similarly, $f(n)$ is at most $\Omega(n^1)$ and $1 \ngtr \log_2 2$; hence not case 3.
- Case 2 – growth depends on both recurrence and f(n).
- $f(n)$ is $\Theta(n^1(\log n)^0)$, hence $f(n)$ is $\Theta\left(n^c(\log n)^k\right)$ with $c = \log_b a$ and $k \geq 0$
- $\therefore T(n)$ is $\Theta\left(n^c(\log n)^{k+1}\right) = \Theta(n^1(\log n)^1) = \Theta(n \log n)$

Q3. $T(n) = 2 \cdot T(n/4)$+n and $T(1) = 1$

- $a = 2; b = 4; f(n) = n$
- $c = \log_4 2 = \frac{1}{2}$
- $1 \nleq \frac{1}{2}$ hence is not case 1 or 2.
- Case 3 - growth depends on f(n)
- $f(n)$ is $\Omega(n^1)$ with $1 > 0.5$.
- $\therefore T(n)$ is $\Theta(n)$.

Q4. $T(n) = T(n-1) + 1$ and $T(1) = 1$

- $T(n)$ not in the form $a \cdot T(n/b) + f(n)$ hence
- M.T. does not apply, sorry.
- Need to prove only by induction.

Identify which case of the Master Theorem each applies to (if any) and find the scaling behaviour

With the base case $T(1) = 1$:

- Q5. $T(n) = 2 \cdot T(n/4) + 1$
- Q6. $T(n) = 4 \cdot T(n/2) + n^2$
- Q7. $T(n) = 2 \cdot T(n-1)$
- Q8. $T(n) = 3 \cdot T(n/3) + n \log n$
- Q9. $T(n) = 2 \cdot \mathrm{T(n/2)} + 2n^2$
- Q10. $T(n) = 2 \cdot T(n/2) + n(\log n)^2$

Identify which case of the Master Theorem each applies to (if any) and find the scaling behaviour

- Q5. $T(n) = 2 \cdot T(n/4) + 1$
- $f(n) = 1$ hence is $O(1) = O(n^0)$
- $c = 0$ and $0 < \log_4 2$ hence case 1 applies.
- $T(n)$ is $\Theta\left(n^{\log_b a}\right) = \Theta(n^{0.5})$

Identify which case of the Master Theorem each applies to (if any) and find the scaling behaviour

- Q6. $T(n) = 4 \cdot T(n/2) + n^2$
- $a = 4; b = 2; f(n) = n^2$
- $f(n)$ is $O(n^2)$ hence $c \geq 2$ but $2 \nless \log_2 4$ so is not case 1.
- $f(n)$ is $\Theta(n^2(\log n)^0)$ with $2 = \log_2 4$ and $0 \geq 0$ hence is case 2 with $T(n)$ is $\Theta(n^2 \log n)$

Identify which case of the Master Theorem each applies to (if any) and find the scaling behaviour

- Q7.$T(n) = 2 \cdot T(n-1)$
- $a = 2; b = \text{not defined}$
- M.T. not applicable – solve by induction.
- $T(n) = 2^{n-1}$
- Base case: $T(1) = 2^0 = 1$
- IH: $T(k) = 2^{k-1}$
- Prove: $T(k+1) = 2 \cdot T(k)$
- $= 2 \cdot 2^{k-1}$
- $= 2^{k+1-1}$
- QED.

Identify which case of the Master Theorem each applies to (if any) and find the scaling behaviour

- Q8. $T(n) = 3 \cdot T(n/3) + n \log n$
- $f(n) = n \log n$
- Does not match case 1
- Case 2: $f(n)$ is $\Theta(n^1 (\log n)^1)$ with $c = \log_b a$ $(1 = \log_3 3)$
- $\therefore$ $T(n)$ is $\Theta\left(n^c (\log n)^{k+1}\right) = \Theta(n(\log n)^2)$

Identify which case of the Master Theorem each applies to (if any) and find the scaling behaviour

- Q9.$T(n) = 2 \cdot \mathrm{T(n/2)} + 2n^2$
- $f(n) = 2n^2$
- Not case 1 or 2 since $2 \nleq 1$
- Case 3:
  - $T(n)$ is $\Theta\big(f(n)\big)$ if $f(n)$ is $\Omega(n^c)$ with $c > \log_b a$
  - $f(n)$ is $\Omega(n^2)$ and $2 > \log_2 2$
  - $\therefore T(n)$ is $\Theta(n^2)$

Identify which case of the Master Theorem each applies to (if any) and find the scaling behaviour

- Q10.$T(n) = 2 \cdot T(n/2) + n(\log n)^2$
- $f(n) = n(\log n)^2$
- $c\ (1) = \log_b a\ \ (1)$ hence M.T. case 2 with $k = 2$:
- $T(n)$ is $\Theta(n(\log n)^3)$

Additional Practice Questions

If you would like some additional practice with the Master Theorem, check the MT Additional Practice Questions document on Moodle.

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Thank you