# The University of Nottingham Ningbo China

Centre for English Language Education

Semester Two, 2017-2018

**FOUNDATION CALCULUS & MATHEMATICAL TECHNIQUES**

Time allowed 1 Hour 30 Minutes

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*Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced.*

**This paper contains EIGHT questions which carry equal marks.**

*An indication is given of the weighting of each subsection of a question by means of a figure enclosed by square brackets, eg.* [12]*, immediately following that subsection.*

*Only silent, self-contained calculators with a Single-line Display or Dual-line Display are permitted in this examination.*

*Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.*

*No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.*

***Do NOT turn examination paper over until instructed to do so.***

**ADDITIONAL MATERIAL:** *Formula Sheet* (attached to the back of the question paper)

**INFORMATION FOR INVIGILATORS:**

1. *Please give a 15 minute warning.*
2. *Please collect Answer Booklets, Question Papers, and Formula Sheet at the end of the exam.*

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1 (a) Given $y = 2x^3 + \sqrt{x} + \left(\dfrac{x^3+2x}{x^2}\right)$. Find $\dfrac{dy}{dx}\cdot$ [2]

(b) $(i)$ Given $y = \dfrac{5x^2 - 10x + 9}{(x-1)^2} \quad ; \quad x \neq 1.$

Use the quotient rule for derivatives to show that $\dfrac{dy}{dx} = \dfrac{-8}{(x-1)^3}\cdot$

$(ii)$ Given $y = \ln\left(\dfrac{x-1}{x+1}\right)$. Find $\dfrac{dy}{dx}\cdot$

$(iii)$ Given $y = e^{kx}$, where $k$ is constant. Find $\dfrac{d^n y}{dx^n} \ (n \in \mathbb{N})\cdot$ [5]

(c) The equation of a curve is given by $x = 2\cos t + \sin 2t\,, \quad y = \cos t - 2\sin t\ ;\ t \in (0, \pi)$.

$(i)$ Find $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ in terms of $t$.

$(ii)$ Find $\left.\dfrac{dy}{dx}\right|_{t=\pi/4}$.

[3]

2 (a) Given $x^2 + 3xy^2 - y^3 = 9$.

Use the method of implicit differentiation to find the gradient of the curve at point $(2, 1)$.

[3]

(b) Given $y = (\sin x)^{\sin x}$.

Use logarithmic differentiation to find $\dfrac{dy}{dx}$.

[3]

(c) Given $y = x^3 - 3x^2 + 2x - 1$.

$(i)$ Find the equation of the tangent line to the curve at the point $(3, 5)$.

$(iv)$ Show that the equation of the normal line to the curve at the point $(3, 5)$ is

$$x + 11y - 58 = 0.$$

[4]

3 (a) Given that the area of the circle is $A = \pi r^2$, where $r$ is the radius of the circle.

If the radius $r$ increases at the rate of $5$ cm/sec, find the rate at which its area is increasing when the radius is $3$ cm.

[2]

(b) Given that the equation $f(x) = x + \dfrac{9}{x} \quad ; \quad x \in \mathbb{R} - \{0\}$.

($i$) Solve the equation $f'(x) = 0$.

($ii$) Use the second derivative test to determine the points of maximum and minimum values of $f$.

[3]

(c) Given $f(x) = x^4 + x^2 - 80$.

($i$) Use the Newton-Raphson iteration formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ for the given function $f$ to obtain $x_{n+1}$.

($i$) Starting with $x_0 = 3$, use the Newton-Raphson method to approximate the root of $f(x) = 0$, correct to $4$ decimal places.

[4]

(d) Show that the function defined by $f(x) = x^5 + x^3 + x$ is always increasing. [1]

4 (a) Given $f(x) = \ln(1 + x)\ ;\ x \in \mathbb{R}^+$.

($i$) Obtain the Maclaurin's series expansion of $f(x)$ up to the term with $x^4$.

($ii$) Hence show that

$$\ln\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots \quad ; \quad -1 < x < 1.$$

[5]

(b) Obtain the Maclaurin's series expansion of $f(x) = x \cdot \cos x$.

[3]

(c) Use the Maclaurin's series expansion of $e^x = 1 + x + \dfrac{x^2}{2!} + + \dfrac{x^3}{3!} + \ldots\ldots$

to show that for small values of $x$, $e^{2x} - e^{-x} \approx 3x + \dfrac{3}{2}x^2$.

[2]

5 (a) Evaluate the following integrals:

$(i)$ $\displaystyle\int \left(\frac{1}{x} + \tan^2 x\right) dx$ $(ii)$ $\displaystyle\int \left(\frac{x^2}{1+x^2}\right) dx$

$(iii)$ $\displaystyle\int \left(\frac{e^x}{9+e^{2x}}\right) dx$ by using appropriate substitution. [5]

(b) $(i)$ Use the substitution $f(x) = t^2$ to show that

$$\int \frac{f'(x)}{\sqrt{f(x)}}\ dx = 2\sqrt{f(x)} + C. \tag{5.1}$$

$(ii)$ Use the result in (5.1) to evaluate $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\ dx.$ [2]

(c) Evaluate the following integrals:

$(i)$ $\displaystyle\int \cos^4 x \cdot \sin^3 x\ \ dx$

$(ii)$ $\displaystyle\int \cos 4x\ \sin 3x\ \ dx$

[3]

6 (a) Given $f(x) = \dfrac{9}{(2x-1)(x+4)}$.

$(i)$ Show that $\displaystyle\int f(x)\ dx\ =\ \ln\left|\frac{2x-1}{x+4}\right| + C.$

$(ii)$ Hence evaluate $\displaystyle\int_1^2 f(x)\ dx.$

[4]

(b) Use the method of integration by parts to show that

$$\int_0^{\pi/2} x \sin x\ dx\ =\ 1.$$

[3]

(c) Given $f(x) = \cos x \cdot \sqrt{1 - 2\sin x}$.

Find the area of the region bounded by $y = f(x)$, lines $x = 0$, $x = \dfrac{\pi}{6}$ and the $X$-axis.

[3]

7 (a) The region bounded by $y = e^x$, $X-$axis and lines $x = 0$ and $x = \ln 3$ is revolved about $X-$axis. Find the volume of the solid of revolution. [2]

(b) Use the property $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ of definite integrals to evaluate

$$\int_0^2 f(x)\ dx \quad \text{where} \quad f(x) = \begin{cases} x & ; \quad 0 \le x \le 1 \\ x^2 & ; \quad 1 \le x \le 2 \end{cases}$$

[2]

(c) Complete the square and evaluate the integral $\int_1^2 \frac{1}{\sqrt{4x - x^2}}\ dx.$ [3]

(d) Use the substitution $\tan\left(\frac{x}{2}\right) = t$ to evaluate the integral $\int \frac{1}{5 + 4\cos x}\ dx.$ [3]

8 (a) Solve the following differential equations using the method of separation of variables:

($i$) $\frac{dy}{dx} = x\, e^y$

($ii$) $\sin^2 y\ \frac{dy}{dx} = \frac{\cos^2 y}{\cos^2 x}.$ [4]

(b) ($i$) Solve the initial value problem $\frac{dy}{dx} = 2 + \sin 3x \quad ; \quad y\left(\frac{\pi}{2}\right) = 0.$

($ii$) Show that $y = A\sin kx + B\cos kx$, where $A$, $B$ and $k$ are constants is a solution of the differential equation $\frac{d^2y}{dt^2} + k^2 y = 0.$ [4]

(c) The rate of decay of a radio active material is proportional to the amount $(m)$ of material present at that time.

Formulate a differential equation to show that the amount of material at time $t$ is

$$m = m_0 \cdot e^{kt}$$

where $k < 0$ is constant and $m_0$ is the initial amount.

[2]