2015-16

1 (a) (i) Roots are equal $\Rightarrow \Delta = 0$

$\Rightarrow (k+4)^2 - 4(1)(k+7) = 0$

$\Rightarrow k^2 + 8k + 16 - 4k - 28 = 0$

$\Rightarrow k^2 + 4k - 12 = 0$

$\Rightarrow (k+6)(k-2) = 0.$

$\Rightarrow k = 2$ or $-6.$

But $k > 0$ $\quad \therefore \underline{k = 2.}$

(ii) $f(x) = 0 \Rightarrow x^2 + 6x + 9 = 0$

$\Rightarrow (x+3)^2 = 0$

$\Rightarrow x = -3.$

(b) $y = f(x) = \ln x + 3$

$\Rightarrow y - 3 = \ln x$

$\Rightarrow x = e^{y-3}$

$\Downarrow$

$f^{-1}(x) = e^{x-3}.$

(c). $\log_2(x-3) + \log_2(x-5) = 3$

$\Rightarrow \log\left[(x-3)(x-5)\right] = 2^3$

$\Rightarrow x^2 - 8x + 15 = 8$

$\Rightarrow x^2 - 8x + 7 = 0$

$\Rightarrow (x-1)(x-7) = 0.$

$\Rightarrow x = 1 \quad \text{or} \quad x = 7$

But $x > 5$.

$\therefore \underline{x = 7}$

(d) $|3x - 4) \geqslant 5$

$\Rightarrow \pm(3x - 4) \geqslant 5$

$\Rightarrow 3x - 4 \geqslant 5$

$\Rightarrow 3x \geqslant 9$

$\Rightarrow \underline{x \geqslant 3}$

OR

$-3x + 4 \geqslant 5$

$-3x \geqslant 1$

$\underline{x \leqslant -\frac{1}{3}}$.

2 (a) $\quad LHS = \dfrac{(1+\sin x)^2 - (1-\sin x)^2}{1-\sin^2 x}$

$$= \frac{\cancel{1} + 2\sin x + \cancel{\sin^2 x} \cancel{-1} + 2\sin x - \cancel{\sin^2 x}}{\cos^2 x}$$

$$= \frac{4\sin x}{\cos^2 x}$$

$$= 4\sec x \cdot \tan x$$

$$= RHS.$$

(b). $\quad 10\sin^2\theta - 3\cos\theta = 6$

$\Rightarrow 10(1-\cos^2\theta) - 3\cos\theta = 6$

$\Rightarrow 10 - 10\cos^2\theta - 3\cos\theta - 6 = 0$

$\Rightarrow 10\cos^2\theta + 3\cos\theta - 4 = 0.$

$\Rightarrow 10\cos^2\theta + 8\cos\theta - 5\cos\theta - 4 = 0$

$\Rightarrow 2\cos\theta(5\cos\theta + 4) - 1(5\cos\theta + 4) = 0$

$\Rightarrow (2\cos\theta - 1)(5\cos\theta + 4) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ $\quad$ or $\quad$ $\cos\theta = -4/5$

But $\theta \in (0, \pi/2) \Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \underline{\theta = \pi/3}.$

3

(C) (i) $f(x) = \cos x - 2\sin x \equiv a\cos x + b\sin x$

$\therefore a = 1, \quad b = -2$

$\therefore R = \sqrt{a^2+b^2} = \sqrt{5}$

$\cos\theta = \frac{a}{R} = \frac{1}{\sqrt{5}} > 0$, $\sin\theta = \frac{b}{R} = \frac{-2}{\sqrt{5}} < 0$

$\therefore \theta$ lies in the 4^{th} Quadrant.

Now, $\tan\theta = \frac{-2}{1} \Rightarrow \theta = \tan^{-1}(-2)$

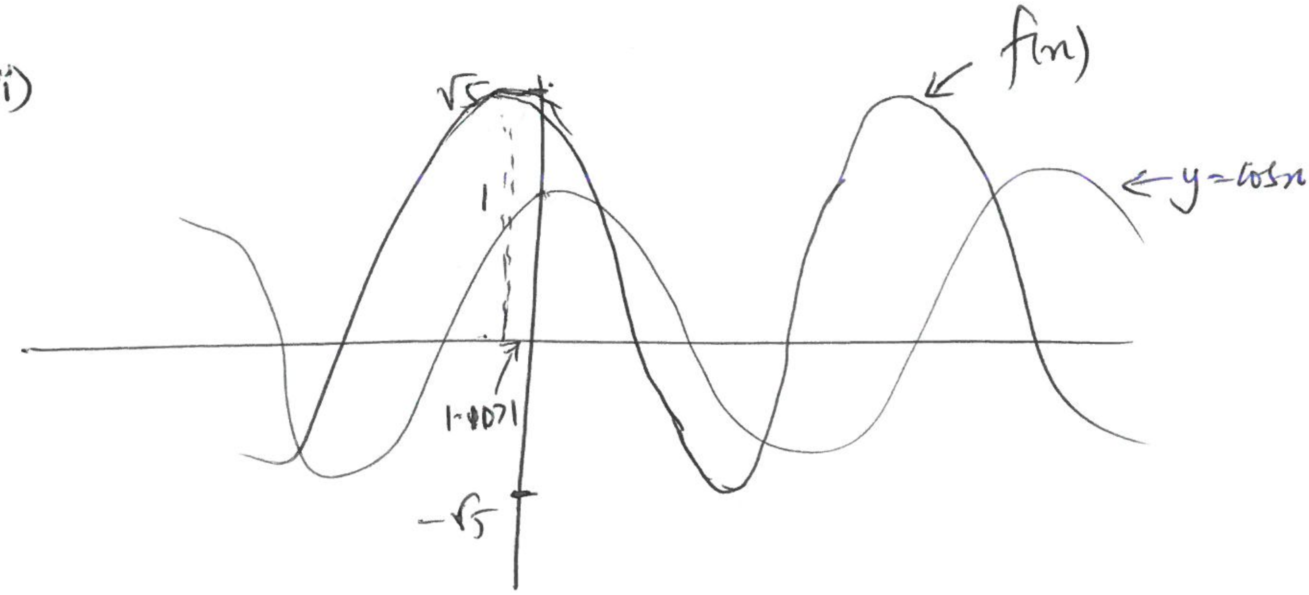
$= -1.1071$ radians

$\therefore f(x) = \cos x - 2\sin x$

$= \sqrt{5}\cos(x - (-1.1071))$

$= \sqrt{5}\cos(x + 1.1071)$.

(ii)



(d)

$$LHS = \frac{\tan 70^\circ + \tan 50^\circ}{1 - \tan 70^\circ \tan 50^\circ} + \sqrt{3}$$

$$= \tan(70^\circ + 50^\circ) + \sqrt{3}$$

$$= \tan 120^\circ + \sqrt{3}$$

$$= -\sqrt{3} + \sqrt{3}$$

$$= 0.$$

$$= RHS.$$

3 (a) (i)

$$p(x) = 2x^3 + 7x^2 - 7x - 12$$

$$\begin{array}{r|rrrr}
 & 2 & 7 & -7 & -12 \\
-4 & & -8 & 4 & 12 \\
\hline
 & 2 & -1 & -3 & \boxed{0}
\end{array}$$

$\Rightarrow (x+4)$ is a factor of $p(x)$.

(ii)

Other factor is

$$2x^2 - x - 3$$

$$= 2x^2 - 3x + 2x - 3$$

$$= (2x-3)(x+1)$$

$$\therefore \underline{p(x) = (x+4)(x+1)(2x-3).}$$

(b). Remainder is 2 when f(x) is divided by $(x-1)$

$\Rightarrow p(1) = 2$

$\Rightarrow 3 + a + b = 2$

$\Rightarrow a + b + 1 = 0$ ——— (1)

Remainder is 6 when f(x) is divided by $(x+1)$

$\Rightarrow p(-1) = 6$

$\Rightarrow 3(-1)^2 + a(-1) + b = 6.$

$\Rightarrow -a + b + 3 = 6$

$\Rightarrow a - b + 3 = 0.$ ——— (2)

(1) + (2) gives $2a + 4 = 0$

$\Rightarrow a = -2.$

$\therefore b = -1 - a = -1 + 2 = 1$

$\therefore a = -2$ & $b = 1$

(c)

$$\frac{13}{(3x-2)(2x+3)} = \frac{A}{3x-2} + \frac{B}{2x+3}$$

$$\Rightarrow \quad A(2x+3) + B(3x-2) = 13$$

$$x = \frac{2}{3} \Rightarrow \quad A\left(\frac{4}{3}+3\right) = 13 \quad \Rightarrow \quad A\left(\frac{13}{3}\right) = 13$$

$$\therefore \underline{A = 3}$$

$$\& \quad x = -\frac{3}{2} \Rightarrow \quad B\left(-\frac{9}{2}-2\right) = 13 \quad \Rightarrow \quad B\left(\frac{-13}{2}\right) = 13$$

$$\Rightarrow \underline{B = -2}$$

Thus, $$\frac{13}{(3x-2)(2x+3)} = \frac{3}{(3x-2)} - \frac{2}{(2x+3)}.$$

4. (a). $f(x) = x^3 - 4x - 5$

$\therefore f(2) = 8 - 8 - 5 < 0$

& $f(3) = 27 - 12 - 5 > 0$

$\therefore f(2) \cdot f(3) < 0$

$\Rightarrow$ Root lies betn. ~~2~~ 2 & 3.

(b). $x^3 - 4x - 5 = 0$

$\Rightarrow x(x^2 - 4) = 5$

$\Rightarrow x^2 - 4 = \frac{5}{x}$

$\Rightarrow x^2 = 4 + \frac{5}{x}$

$\Rightarrow x = \sqrt{4 + \frac{5}{x}}$

$\therefore x_{n+1} = \sqrt{4 + \frac{5}{x_n}}$.

(c)

$x_0 = 2.5$

$\therefore x_1 = \sqrt{4 + \frac{5}{2.5}} = 2.457894$

Subsequent values are:

2.456474,

2.456713

2.45668.

$\therefore$ Root correct to 5 d.p. is $\underline{2.45668}$

or $\underline{2.45667}$

(d).

n	a	b	f(a)	f(b)	c	f(c)	Decision
0	2	3	<0	>0	2.5	>0	b by c.
1	2	2.5	<0	>0	2.25	<0	a by c
2	2.25	2.5	<0	>0	2.375		

$\therefore x_1 = 2.25$

& $x_2 = 2.375$.

5 (a)

$$\left(2-\frac{3x^2}{5}\right)^8 = 2^8+\binom{8}{1}2^7\left(-\frac{3x^2}{5}\right)+\binom{8}{2}2^6\left(-\frac{3x^2}{5}\right)^2$$

$$+\underbrace{\binom{8}{3}(2)^5\cdot\left(-\frac{3x^2}{5}\right)^3}+\cdots+\left(-\frac{3x^2}{5}\right)^8$$

$\therefore$ Term with x^6 is

$$\binom{8}{3}\;2^5\cdot\left(-\frac{3x^2}{5}\right)^3 = \frac{8\cdot 7\cdot \cancel{6}}{1\cdot\cancel{2}\cdot\cancel{3}}\cdot 32\cdot\frac{(-27)\cdot x^6}{125}$$

$$= -\frac{48384}{125}\cdot x^6.$$

(b). $$(2-ax)^5 = 2^5+\binom{5}{1}2^4\cdot(-ax)+\binom{5}{2}2^3(-ax)^2$$

$$+\cdots+(-ax)^5.$$

$\therefore$ Coefficient of $x^2 = \binom{5}{2}\cdot 2^3\cdot(-a)^2$

$\Rightarrow \quad 20 = 10\cdot 8\cdot a^2$

$\Rightarrow \quad a^2 = \frac{1}{4}$

$\therefore a = \pm\frac{1}{2}$

But $a>0 \Rightarrow \underline{a=\frac{1}{2}}.$

(c) (i)

$$(1+2x)^{1/3} = 1 + \frac{1}{3}(2x) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(2x)^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(2x)^3 + \dots$$

$$\approx 1 + \frac{2x}{3} + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)}{\cancel{2}} \cdot 4x^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)8x^3}{{}_3\cancel{6}}.$$

$$= 1 + \frac{2x}{3} - \frac{4}{9}x^2 + \frac{40}{81}x^3.$$

(ii) Put $x = -\frac{1}{100}$

$$\Rightarrow \left(1 - \frac{2}{100}\right)^{1/3} \cong 1 + \frac{2}{3}\left(-\frac{1}{100}\right) - \frac{4}{9}\left(-\frac{1}{100}\right)^2 + \frac{40}{81}\left(-\frac{1}{100}\right)^3$$

$$\Rightarrow \underline{(0.98)^{1/3} \approx 0.9933.}$$

$\therefore (0.98)^{1/3} \approx 0.993$ (3 dp).

(d)

$$A = \frac{\sqrt{3}}{4} l^2 \quad \text{and} \quad \Delta l = 1.2\ \%\ \text{of}\ l$$
$$= \left(\frac{1.2}{100}\right) l$$
$$= 0.012\, l$$

Now,
$$A + \Delta A = \frac{\sqrt{3}}{4} (l + \Delta l)^2$$
$$= \frac{\sqrt{3}}{4} \left[l + 0.012\, l \right]^2$$
$$= \frac{\sqrt{3}}{4} l^2 (1 + 0.012)^2$$
$$\approx \frac{\sqrt{3}}{4} l^2 \left[1 + 2(0.012) + \frac{2(1)(0.012)^2}{2} \right]$$
$$= A\left[1 + 0.024144\right]$$
$$\therefore \Delta A = 2.4144\ \%\ \text{of}\ A$$

6 (a)

$$5C^T - 4AB = 5\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} - 4\begin{pmatrix} 1 & -4 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & -5 \\ 10 & 5 \end{pmatrix} - 4\begin{pmatrix} 10 & -19 \\ 8 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 15-40 & -5+76 \\ 10-32 & 5-16 \end{pmatrix}$$

$$= \begin{pmatrix} -25 & 71 \\ -22 & -11 \end{pmatrix}$$

(b). (i) $AX = B \Rightarrow \begin{pmatrix} 7 & -8 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

(ii) $$A^{-1} = \frac{1}{(-35+32)}\begin{pmatrix} -5 & 8 \\ -4 & 7 \end{pmatrix}$$

$$= \frac{1}{-3}\begin{pmatrix} -5 & 8 \\ -4 & 7 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 & -8 \\ 4 & -7 \end{pmatrix}$$

(iii) $$X = A^{-1}B \Rightarrow X = \frac{1}{3}\begin{pmatrix} 5 & -8 \\ 4 & -7 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 35-8 \\ 28-7 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 27 \\ 21 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

$\therefore x = 9$ & $y = 7$

(c)

A does not have an inverse

if $|A| = 0$

$$\Rightarrow \begin{vmatrix} -2 & k-2 \\ k+3 & 3 \end{vmatrix} = 0$$

$$\Rightarrow -6 - (k-2)(k+3) = 0$$

$$\Rightarrow k^2 + k - 6 = -6$$

$$\Rightarrow k^2 + k = 0$$

$$\Rightarrow \underline{k = 0 \text{ or } k = -1.}$$

7 (a). $z = 1 - i^2 - i^3 + i^4 - i^5$

$= 1 - (-1) - (-i) + 1 - i$

$= 1 + 1 + \cancel{i} + 1 - \cancel{i}$

$= 3$

$= 3 + i(0)$

$\therefore \underline{a = 3, \quad b = 0.}$

(b). $(2-i)x - (1+3i)y - 7 = 0$

$\Rightarrow \quad 2x - ix - y - 3iy - 7 = 0$

$\Rightarrow \quad (2x - y - 7) + i(-x - 3y) = 0.$

$\Rightarrow \quad 2x - y - 7 = 0$

& $\quad x + 3y = 0. \Rightarrow x = -3y$

$\therefore \quad 2x - y - 7 = 0 \Rightarrow -6y - y - 7 = 0$

$\Rightarrow 7y = -7$

$\Rightarrow \underline{y = -1.}$

& $\therefore \underline{x = 3.}$

16 (ii) $|x + iy| = \sqrt{9+1} = \sqrt{10}.$

(c)

(i) $z = z_1 \cdot z_2$

$= (4-3i)(3+4i)$

$= 12 + 16i - 9i - 12i^2$

$= 24 + 7i$

$\Rightarrow \underline{a = 24, \; b = 7.}$

(ii) $|z| = \sqrt{a^2+b^2} = \sqrt{24^2+7^2} = 25.$

~~(iii)~~ & $a > 0, \; b > 0 \; \therefore \; \theta = \arg(z)$

$= \tan^{-1}\left|\frac{7}{24}\right|$

$= 0.2838$ radian

(iii) $\therefore$ Polar form is

$z = 25\left(\cos(0.2838) + i\sin(0.2838)\right)$

(d). $\left|\frac{z_1 \cdot \overline{z_3}}{z_2^2}\right| = \frac{|z_1| \cdot |\overline{z_3}|}{|z_2|^2}$

$= \frac{\sqrt{2} \cdot \cancel{\sqrt{25}}}{(\cancel{\sqrt{5}})^{\cancel{2}}} = \underline{\underline{\sqrt{2}}}$ Ans.

8. (a).

10^{th} term $= 15 \Rightarrow a + 9d = 15.$

$$S_{19} = \frac{19}{2}\left[2a + (19-1)d\right]$$

$$= \frac{19}{2}\left[2a + 18d\right]$$

$$= 19(a + 9d)$$

$$= 19(15) = \cancel{195}.\ 285.$$

(b). $162, 54, 18, 6, \ldots$

(i) $\therefore a = 162, \quad r = \frac{1}{3}.$

$$\therefore a_n = a \cdot r^{n-1}$$

$$= 162 \cdot \left(\frac{1}{3}\right)^{n-1}$$

$$= 162 \cdot \left(3^{-n+1}\right)$$

$$= \underline{486 \cdot (3)^{-n}}$$

(ii) $S_5 = \frac{a(1-r^5)}{1-r} = \frac{162\left(1 - \frac{1}{243}\right)}{1 - \frac{1}{3}} = \frac{162 \cdot (242)}{\left(\frac{2}{3}\right)(243)}$

$$= \underline{\underline{242}}$$

(c). $\sum f(n) = \sum (6n^2+4n-1)$

$= 6\sum n^2 + 4\sum n - \sum 1$

$= \frac{\cancel{6}\, n(n+1)(2n+1)}{\cancel{6}} + \frac{\cancel{4}^{2}\, n(n+1)}{\cancel{2}} - n$

$= n\left[(2n+1)(n+1) + 2(n+1) - 1\right]$

$= n\left[2n^2+3n+1+2n+2-1\right]$

$= n(2n^2+5n+2)$

$= n(2n^2+4n+n+2)$

$= \underline{n(n+2)(2n+1).}$

(ii) $\sum_{1}^{10} f(n) = 10(10+2)(20+1) = 2520.$

$\sum_{1}^{20} f(n) = 20(22)(41) = 18040.$

$\therefore$ (iii) $\sum_{11}^{20} f(n) = \sum_{1}^{20} f(n) - \sum_{1}^{10} f(n)$

$= 18040 - 2520$

$\therefore \sum_{11}^{20} (6n^2+4n-1) = \underline{\underline{15520.}}$

Answer.