Foundation Calculus & Mathematical Techniques CELEN037

Weekly Worksheet-8 **Topics:** Definite integrals and their properties, Area calculation

Type 1: Evaluating definite integrals:

1. Evaluate the following definite integrals:

$$(i)\ \int_0^1 \frac{1}{1+x^2}\ dx \qquad (ii)\ \int_0^1 \frac{1}{4-x^2}\ dx \qquad (iii)\ \int_0^4 \frac{1}{\sqrt{x^2+9}}\ dx$$

$$(iv)\ \int_0^3 \frac{1}{\sqrt{9-x^2}}\ dx \qquad (v)\ \int_1^e \frac{1}{x}\ dx \qquad (vi)\ \int_{\pi/12}^{\pi/9} \operatorname{cosec}^2(3x)\ dx$$

Type 2: Definite Integrals using substitution.

Remember to change the limits of integration for the transformed integral

2. Evaluate the following definite integrals by using appropriate substitutions:

$$(i)\ \int_0^{\sqrt{\pi}} 5\,x\ \cos\left(x^2\right)\ dx \qquad (ii)\ \int_0^{\pi/4} \tan^2 x\ \sec^2 x\ \ dx$$

$$(iii)\ \int_{1/3}^1 \frac{1}{\sqrt{x}\,(x+1)}\ dx \qquad (iv)\ \int_{-1}^1 \frac{x^2}{\sqrt{x^3+9}}\ dx$$

$$(v)\ \int_{\pi^2}^{4\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\ dx \qquad (vi)\ \int_0^{\pi/2} \frac{\sin x}{\sqrt{5+\cos x}}\ dx$$

$$(vii)\ \int_1^2 \frac{1}{x\sqrt{1-(\ln x)^2}}\ dx \qquad (viii)\ \int_0^1 \frac{e^x\,(x+1)}{\cos^2(x\,e^x)}\ dx$$

Type 3: Integration by parts for definite integrals

$$\boxed{\int_a^b u\cdot\frac{dv}{dx}\ dx = [\,u\,v\,]_a^b - \int_a^b v\cdot\frac{du}{dx}\ dx}$$

3. Evaluate the following definite integrals by using integration by parts:

$$(i)\ \int_1^e x^2\ \ln x\ \ dx \qquad (ii)\ \int_{-1}^1 \ln(x+2)\ \ dx$$

$$(iii)\ \int_0^2 \ln\left(x^2+1\right)\ dx \qquad (iv)\ \int_1^4 \sec^{-1}\left(\sqrt{x}\right)\ dx$$

Type 4: Use of properties for evaluating definite integrals

- If f is integrable on a closed interval containing three points a, b, and c, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

- If f is EVEN integrable function on $[-a, a]$, then $\displaystyle\int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx.$
- If f is ODD integrable function on $[-a, a]$, then $\displaystyle\int_{-a}^{a} f(x)\ dx = 0.$
- If f is integrable on $[0, a]$, then $\displaystyle\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx.$
- If f is integrable on $[a, b]$, then $\displaystyle\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx.$

4. Use properties of definite integrals to evaluate the following:

(i) $\displaystyle\int_0^3 f(x)\ dx$ where $f(x) = \begin{cases} x^2 & ; \quad x < 2 \\ 3x-2 & ; \quad x \geq 2 \end{cases}$

(ii) $\displaystyle\int_0^2 \left|(1-x^2)\right|\ dx$ **Note**: Use definition of modulus function.

(iii) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\ dx$ (iv) $\displaystyle\int_0^4 \frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}\ dx$

(v) $\displaystyle\int_1^8 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\ dx$ (vi) $\displaystyle\int_{-1}^1 \frac{x^4\sqrt{\cos(x^2)}}{(\sin x - \tan x)}\ dx$

Type 5: Area calculation using definite integrals.

Type 5A: Area of region bounded by the curve and axis.

- The area of the region bounded by the curve $y = f(x)$, lines $x = a$, $x = b$, and the X-axis is:

$$\boxed{A = \int_a^b y\ dx = \int_a^b f(x)\ dx}$$

- The area of the region bounded by the curve $x = g(y)$, lines $y = c$, $y = d$, and the Y-axis is:

$$\boxed{A = \int_c^d x\ dy = \int_c^d g(y)\ dy}$$

5. (i) Find the area of the region bounded by the curve $y = x^3$, lines $x = -2$, $x = 2$, and the X-axis.

   (ii) Find the area of the region bounded by the curve $y = x^2 - 4x + 5$, lines $x = 1$, $x = 2$ and the X-axis.

   (iii) Find the area of the region bounded by $x = y^3 - 3y + 7$, lines $y = -1$, $y = 3$, and the Y-axis.

   (iv) Find the area of the region bounded by the curve $y = 2\,x\,e^x$ and the X-axis, where $x \in [\,0, 1\,]$.

   (v) Find the area of the region bounded by the curve $y = e^{\sqrt{x}}$ and the X-axis, where $0 \le x \le 1$.

   (vi) Find the area of the region bounded by the curve $y = e^{\sin x} \cdot \sin 2x$ and the X-axis, where $x \in \left[0, \dfrac{\pi}{2}\right]$.

Type 5B: Area of region bounded by two curves.

- The area of the region bounded by the curves $y = f_1(x)$, $y = f_2(x)$, and lines $x = a$ and $x = b$ is:

$$\boxed{A \;=\; \left| \int_a^b \left[\, f_1(x) - f_2(x) \,\right] \; dx \right|}$$

- The area of the region bounded by the curves $x = g_1(y)$, $x = g_2(y)$, and lines $y = c$ and $y = d$ is:

$$\boxed{A \;=\; \left| \int_c^d \left[\, g_1(y) - g_2(y) \,\right] \; dy \right|}$$

6. (i) Find the area of the region bounded by the curves $y = x^2$, $y = \sqrt{x}$ and lines $x = \dfrac{1}{4}$ and $x = 1$.

   (ii) Find the area of the region bounded by the curves $y = \sec^2 x$, $y = 2$ and lines $x = -\dfrac{\pi}{4}$ and $x = \dfrac{\pi}{4}$.