COMP2054-ADE

The Vector Data Type, and Amortised Analysis

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Vector ADT

- The “Vector” is an Abstract Data Type, ADT, corresponding to generalising the notion of the “Array” (Concrete Data Type, CDT)

- Key idea:
  - The “index” of an entry in an array can be thought of as the “number of elements preceding it”
  - E.g. in an array A, then A[2] has two elements, A[0], A[1] that precede it
  - In these lectures it is then called “rank”
  - The notion of “rank” can be more general than the idea of index:
    - Not necessarily implemented as “C-style with pointers” A[i]=*(A+i)

The Vector ADT

- The Vector ADT is based on the array CDT, and stores a sequence of arbitrary objects
- An element can be accessed, inserted or removed by specifying its **rank, i.e. the number of elements preceding it**
- An exception is thrown if an incorrect rank is specified (e.g., a negative rank)
- Main vector operations:
  - object elemAtRank(integer r): returns the element at rank r without removing it
  - object replaceAtRank(integer r, object o): replace the element at rank with o and return the old element
  - insertAtRank(integer r, object o): insert a new element o to have rank r
  - object removeAtRank(integer r): removes and returns the element at rank r
- Additional operations size() and isEmpty()

Vector as a Stack

- A common usage of a Vector is as a “Stack”
  - add and remove elements at the end
  - i.e. maximum rank values:
  - push(o) - insertAtRank(size(), object o) – hence the new element is after all the existing ones
    pop() - removeAtRank(size()) – hence the element removed is after all the existing ones

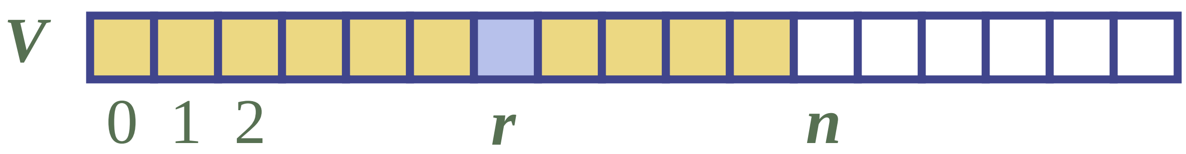
- This is very useful if reading elements from a file, and not knowing how many there will be

Applications of Vectors

- There is not an automatic limit on the storage size
  - unlike arrays of a fixed size
- Direct applications
  - Sorted collection of objects (elementary database)
- Indirect applications
  - Auxiliary data structure for many algorithms
  - Components of other data structures

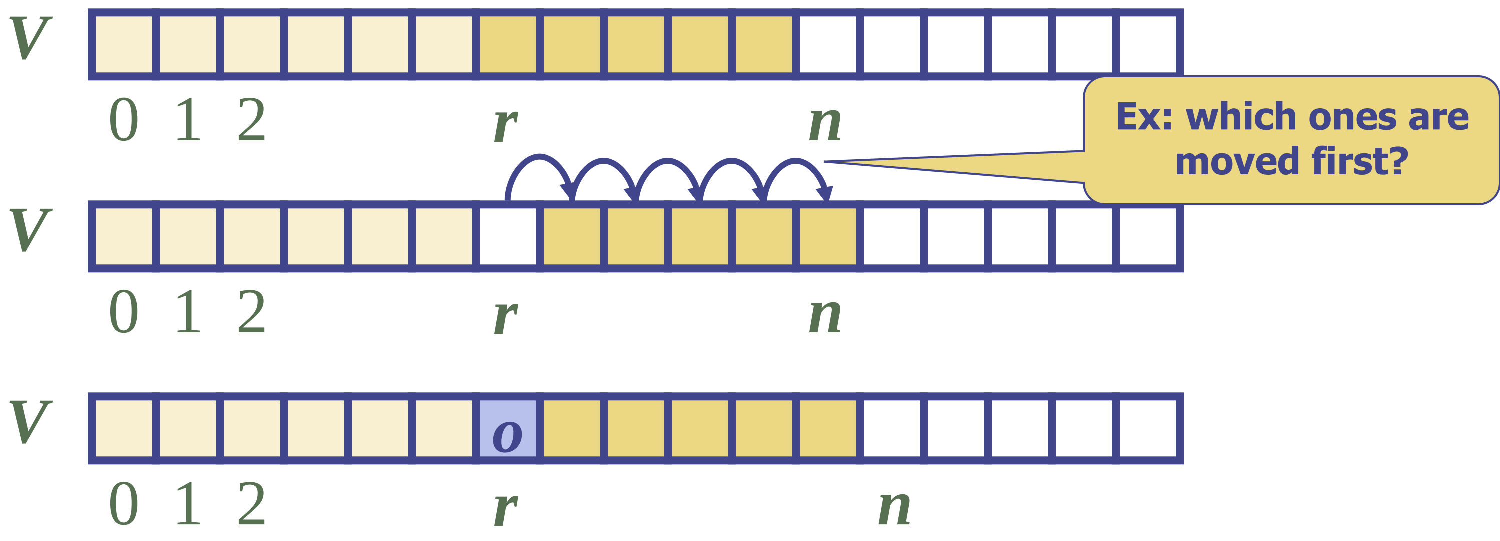
Array-based Vector

- Use an array V of size N as the CDT
- A variable n keeps track of the size of the vector (number of elements currently stored)
- Operation ***elemAtRank***(r) is implemented in $O(1)$ time by simply returning $V[r]$



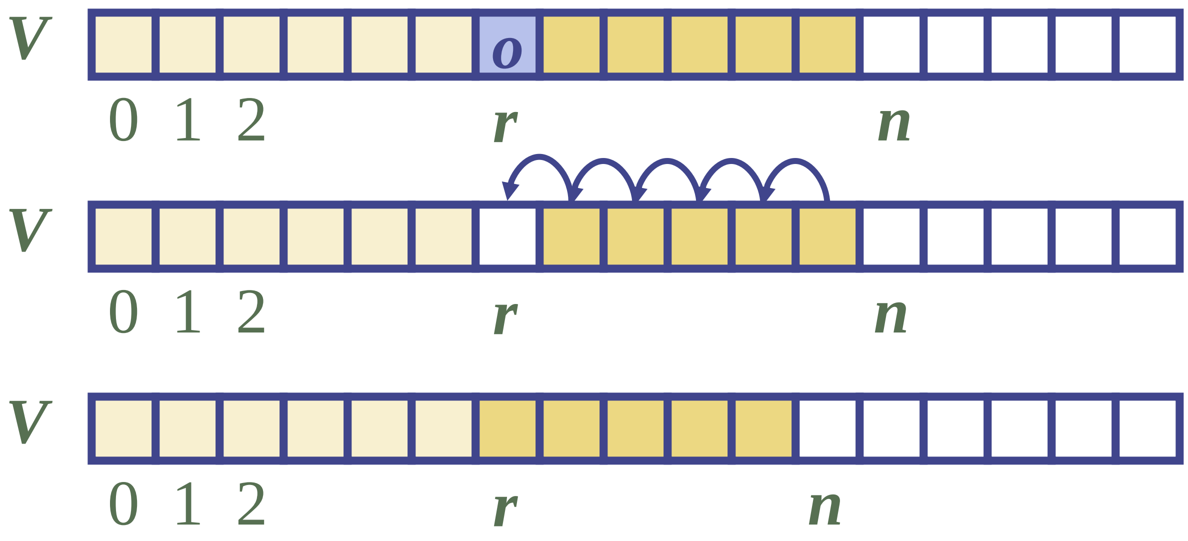
Insertion

- In operation ***insertAtRank***(r, o), we need to make room for the new element by shifting forward the $n - r$ elements $V[r], \ldots, V[n-1]$
- In the worst case ($r = 0$), this takes $O(n)$ time



Deletion

- In operation ***removeAtRank***(r), we need to fill the hole left by the removed element by shifting backward the $n - r - 1$ elements $V[r + 1]$, …, $V[n - 1]$
- In the worst case ($r = 0$), this takes $O(n)$ time



Performance

- In the array-based implementation of a Vector
  - The space used by the data structure is $O(n)$
  - ***size***, ***isEmpty***, ***elemAtRank*** and ***replaceAtRank*** run in $O(1)$ time
  - ***insertAtRank*** and ***removeAtRank*** run in $O(n)$ time
  - ***push*** runs in O(1) time, as do not need to move elements
    - **unless need to resize the array**
  - ***pop*** runs in O(1) time
- In an ***insertAtRank*** operation, when the array is full, instead of throwing an exception, we can replace the array with a larger one

Growable Array-based Vector

- In a push (insertAtRank(n)) operation, when the array is full, instead of throwing an exception, we can replace the array with a larger one
  - But the resizing has a high cost – O(n) as need to copy at all n elements
- How large should the new array be?
  - **incremental strategy**: increase size by a constant c
  - **doubling strategy**: double the size

Algorithm ***push***(***o***)
  if $n = V.length - 1$ **then**
    $A \leftarrow$ new array of size …
    for $i \leftarrow 0$ **to** t **do**
      $A[i] \leftarrow V[i]$
    $V \leftarrow A$
    $n \leftarrow n + 1$
    $V[t] \leftarrow o$

Comparison of the Strategies

- We compare the incremental strategy and the doubling strategy by analyzing the total time $T(n)$ needed to perform a series of n push operations
  - "push" means "add an element at the end" – treat it as a stack
- We assume that we start with an empty stack represented by an array of size 1
- We call **amortized** time of a push operation the average time taken by a push over the series of operations, i.e., $T(n)/n$

Meaning of “Amortize”

- See http://www.thefreedictionary.com/amortize or similar if you are not familiar with the word.
- It refers to writing off, or paying off, debts over a period of time.
- Similar to the way a mortgage for a house is paid back over many years, as opposed to needing to pay all in one go.

Remarks on Amortised Analysis

- Suppose some individual operation (such as ‘push’) takes time T in the worst-case
- Suppose do a **sequence** of operations:
  - Suppose s such operations take total time T_s
  - Then sT is an upper-bound for the total time T_s
  - But, such an upper-bound might not ever occur.
- The time T_s might well be o(sT) even in the worst-case
  - the average time per operation, T_s /s can be the most relevant quantity in practice
  - E.g. if ‘push’ing a long sequence of elements into a Vector – e.g. when reading from a file

Question (pause and try):

- Why is amortised analysis different from the average case analysis?

Question:

- Why is amortised analysis different from the average case analysis?
- Answer:
  - “Amortised”: (long) **real sequence** of dependent operations
  - vs.
  - “Average”: **Set** of (possibly independent) operations

Describing amortised costs

Note the usual big-Oh family is still used to describe amortised analysis

- Recall Big-oh is just describing functions
  - It is not limited to “worst case of an algorithm”!
- We have different measures of the runtime cost:
  - Worst-case “cost per operation of a sequence”

  not just
  - “Worst case of a single operation”

Incremental: Example

- Take c=3, with start capacity of 3, then a sequence of pushes might have costs for each push as follows:

1,1,1,3+1,1,1,1,6+1,1,1,1,9+1,1,1,1,12+1,1,1,1,15+ 1,1,1,18+1,...

- A constant fraction, 1/3, of the pushes have cost O(n).
- Average, per push operation, is O(n).
- Much worse than O(1) cost without resizing!

Example: In detail

- [- , - , -] n=0 Starting point
  - empty, but capacity=3
  - "-" means "not yet used"
- Suppose do a sequence of "push(0)", we get the sequence of costs (primitive operations):

1. [0 , - , -] n=1, **cost=1**
2. [0 , 0 , -] n=2, **cost=1**
3. [0 , 0 , 0] n=3, **cost=1**
4. [0 , 0 , 0, 0 , - , -] n=4, **cost=3+1**
   - cost= "3 for the copy" + "1 for the push(0)"
5. [0 , 0 , 0, 0 , 0 , -] n=5, **cost=1**
6. [0 , 0 , 0, 0 , 0 , 0] n=6, **cost=1**
7. [0 , 0 , 0, 0 , 0 , 0 , 0 , - , -] n=7, **cost=6+1**
   - cost= "6 for the copy" + "1 for the push(0)"
8. etc, etc...

Incremental Strategy Analysis

- We replace the array $k = n/c$ times
- Each "replace" costs the current size
- The total time $T(n)$ of a series of n push operations is proportional to

$$n + c + 2c + 3c + 4c + \ldots + kc =$$
$$n + c(1 + 2 + 3 + \ldots + k) =$$
$$n + ck(k + 1)/2$$

- Since c is a constant, $T(n)$ is $O(n + k^2)$, i.e., $O(n^2)$
- The amortized time of a push operation is $O(n)$
- This is bad as the normal cost of a push is O(1).

Doubling: Example

- With start capacity of 3, then a sequence of pushes starting from an empty vector, might have costs for each push in turn of

1,1,1,3+1,1,1,6+1,1,1,1,1,1,12+1,1,1,1,1,1,1,...

- The 3,6,12,... are the costs for the resizing
- The fraction of pushes having cost O(n) drops with n.
- What is the average?

Doubling: Example

- With start capacity of 2, then a sequence of pushes might have costs

1,1,2+1,1,4+1,1,1,1,8+1,1,1,1,1,1,1,1,1,1,…,16+1,…

- For every push of cost O(n) we will be able to do another O(n) pushes of cost O(1) before having to resize again.
- The O(n) cost on resizing can be ‘amortised’ (spread) over n other O(1) operations
- Gives an average of O(1) per operation.

Doubling: Example

- With start capacity of 2, then a sequence of pushes might have costs

  1,1,2+1,1,4+1,1,1,1, ...

- The cost of the doubling can be “spread” over the later operations and so might be counted as

  1,1,1+1,1+1,1,1+1,1+1,1+1,1+1,...

where the red is a cost that has been moved.

Analogy: save £1/day “for a rainy day”

- This ‘view’ makes it clearer that the net effect will just be a (rough) doubling of the original costs

Doubling Strategy Analysis

- Question: for ***n*** ‘pushes’ how many times is the array grown?
- For simplicity assume n is a power of 2
  - We replace the array ***k*** = $\log_2$ ***n*** times

- The total time ***T***(***n***) of a series of ***n*** push operations is proportional to

$$n + 1 + 2 + 4 + 8 + \ldots + 2^{k-1}$$

Examples

- n=1 #doublings=0
  - log2 n = 0
- n=2 #doublings=1
  - log2 n = 1
- n=4 #doublings=2
  - log2 n = 2

Example: n=4

‘|’ is ‘End of stack’ marker

- start [| _]
  - capacity=1, size=0
- push(A) [A |]
- push(B) [A B |]
  - needed: 1 double – 1 copies
- push(C) [A B C | _]
  - needed: 1 double – 2 copies
- push(D) [A B C D |]

Exercise (offline): Do this is full detail; as done earlier for the incremental strategy.

Recall: Geometric sums

Want to find S

$$S = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{\boldsymbol{k}}$$

Standard trick:

$$2S = \quad 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1}$$

So

$$2S - S = 2^{k+1} - 1$$

Hence $S = 2^{k+1} - 1$

I.e. "the next term minus one"

Doubling Strategy Analysis

- We replace the array $k = \log_2 n$ times
- The total time $T(n)$ of a series of n push operations is proportional to

$$n + 1 + 2 + 4 + 8 + \ldots + 2^{k-1} =$$

$$n + 2^k - 1 = 2n - 1$$

$T(n)$ is $O(n)$

- Amortized time of a single push operation is $O(1)$
- That is, no worse than if all the needed memory was pre-assigned!
  - (Big-Oh hides the constant factor '2' extra cost.)

Offline Exercises/Discussion

- When might you still want to use incremental increase rather than doubling?
- Why is it doubling rather than tripling? Or increasing by some other constant factor?
  - Try redoing the analysis with a arbitrary growth factor *b*
  - *As ever: implement it all!*

C/C++ comments

- Doubling vector size might be made even more efficient if use realloc rather than malloc or new
  - Internally it can (often not always) just extend the space allocated to the array, and so avoid the need for a copy
  - It can use "memcpy" which is direct copy rather than via individuals

Expectations

- The “Vector” data structure
- The big-Oh costs of various single operations
- The amortised cost of a sequence of operations
- The amortised complexities of different strategies for resizing of the underlying array