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Foundation Calculus & Mathematical Techniques — CELEN037

Weekly Worksheet-6 **Topics:** Standard substitutions

Type 1: Integrals of the form $\int \sin^m x \cos^n x \, dx$.

1. Evaluate the following integrals:

(i) $\int \sin^4 x \cos^3 x \, dx$ (ii) $\int \sin^3 x \cos^7 x \, dx$

(iii) $\int \sin^3 x \cos^2 x \, dx$ (iv) $\int \sin^2 x \cos^5 x \, dx$

Type 2: Integrals of the form $\int \frac{f'(x)}{f(x)} \, dx$.

2. Evaluate the following integrals by using given substitutions:

(i) $\int \frac{\sin 2x}{1+\sin^2 x} \, dx$ (ii) $\int \frac{\sin 2x}{1+\cos^2 x} \, dx$

(iii) $\int \frac{x}{1+x^2} \, dx$ (iv) $\int \frac{x^2}{1+x^3} \, dx$

(v) $\int \tan x \, dx$ (vi) $\int \cot x \, dx$

(vii) $\int \sec x \, dx = \int \frac{\sec x \cdot (\sec x - \tan x)}{(\sec x - \tan x)} \, dx$ (viii) $\int \text{cosec } x \, dx = \int \frac{\text{cosec } x \cdot (\text{cosec } x - \cot x)}{(\text{cosec } x - \cot x)} \, dx$

(ix) $\int \frac{e^x (1+x)}{x \, e^x} \, dx$ (x) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$

(xi) $\int \frac{e^{2x}-1}{e^{2x}+1} \, dx$ (xii) $\int \frac{e^{2x}+1}{e^{2x}-1} \, dx$

Type 3: Integration after completing the square in the denominator (D^r).

Note: **Useful formulae**

$\int \frac{1}{x^2+a^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$	$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$	$\int \frac{1}{\sqrt{x^2-a^2}} \, dx = \ln\left\lvert x + \sqrt{x^2-a^2}\right\rvert + C$
$\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \ln\left\lvert\frac{x+a}{x-a}\right\rvert + C$	$\int \frac{1}{\sqrt{x^2+a^2}} \, dx = \ln\left\lvert x + \sqrt{x^2+a^2}\right\rvert + C$

3. Evaluate the following integrals by completing the square for the term in the denominator:

(i) $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$ (ii) $\displaystyle\int \frac{1}{x^2+2x+10}\,dx$

(iii) $\displaystyle\int \frac{1}{x^2+4x+9}\,dx$ (iv) $\displaystyle\int \frac{1}{9-x^2-4x}\,dx$

(v) $\displaystyle\int \frac{1}{8-x^2+2x}\,dx$ (vi) $\displaystyle\int \frac{1}{\sqrt{x^2+4x-5}}\,dx$

(vii) $\displaystyle\int \frac{1}{\sqrt{x^2+4x+3}}\,dx$ (viii) $\displaystyle\int \frac{1}{\sqrt{x^2+9x-5}}\,dx$

(ix) $\displaystyle\int \frac{1}{\sqrt{5-x^2+4x}}\,dx$ (x) $\displaystyle\int \frac{1}{\sqrt{4x^2+4x+3}}\,dx$

Type 4: The method of t-substitution

$$\boxed{\tan\left(\frac{x}{2}\right)=t \quad \Rightarrow \quad dx=\frac{2\,dt}{1+t^2}}$$

$$\boxed{\sin x = \frac{2\,t}{1+t^2} \quad \textbf{or} \quad \cos x = \frac{1-t^2}{1+t^2} \quad \textbf{or} \quad \tan x = \frac{2\,t}{1-t^2}}$$

4. Evaluate the following integrals by using the method of t-substitution.

(i) $\displaystyle\int \frac{1}{2+\cos x}\,dx$ (ii) $\displaystyle\int \frac{1}{1+2\cos x}\,dx$

(iii) $\displaystyle\int \frac{1}{2+3\cos x}\,dx$ (iv) $\displaystyle\int \frac{1}{2-3\cos x}\,dx$

(v) $\displaystyle\int \frac{1}{4\cos x+1}\,dx$ (vi) $\displaystyle\int \frac{1}{2-\cos x}\,dx$

(vii) $\displaystyle\int \frac{1}{2+\sin x}\,dx$ (viii) $\displaystyle\int \frac{1}{3\cos x+4\sin x+5}\,dx$

Type 5: Integrals of the form $\displaystyle\int \frac{1}{a\cos^2 x+b\sin^2 x+c}\,dx.$

Process:

(i) Divide numerator and denominator by $\cos^2 x$, and simplify.

(ii) Substitute $\tan x = t$.

5. Evaluate the following integrals:

(i) $\displaystyle\int \frac{1}{1+2\,\cos^2 x}\,dx$ (ii) $\displaystyle\int \frac{1}{1+\cos^2 x}\,dx$

(iii) $\displaystyle\int \frac{1}{3\sin^2 x+2}\,dx$ (iv) $\displaystyle\int \frac{1}{2\cos^2 x-1}\,dx$

(v) $\displaystyle\int \frac{1}{1+\sin^2 x}\,dx$ (vi) $\displaystyle\int \frac{1}{4\cos^2 x+\sin^2 x}\,dx$