Lecture 4

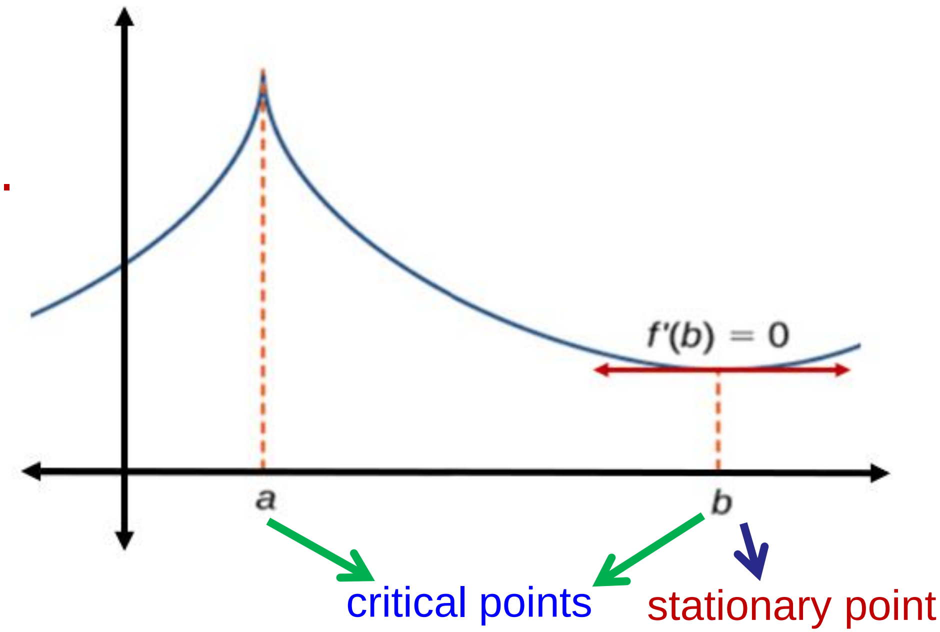
Lecture Content

- Stationary and Critical points, Inflection and Turning points
- Maximum and Minimum values of a function
- Application of Derivatives in:
  - Solving Optimization problems
  - Problems on related rates
- Integration – An Introduction
  - Integration as an antiderivative
  - Integration as the area under a curve
- Integration of standard functions

Stationary and Critical Points

A stationary point is a point on the graph of a function where the derivative is zero.

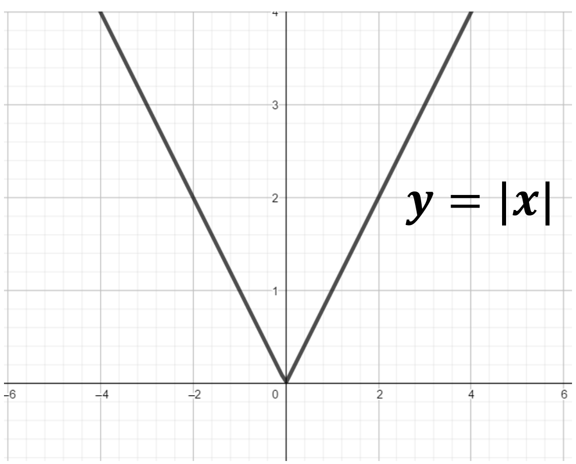
Thus, for a stationary point x, $f'(x) = 0$.

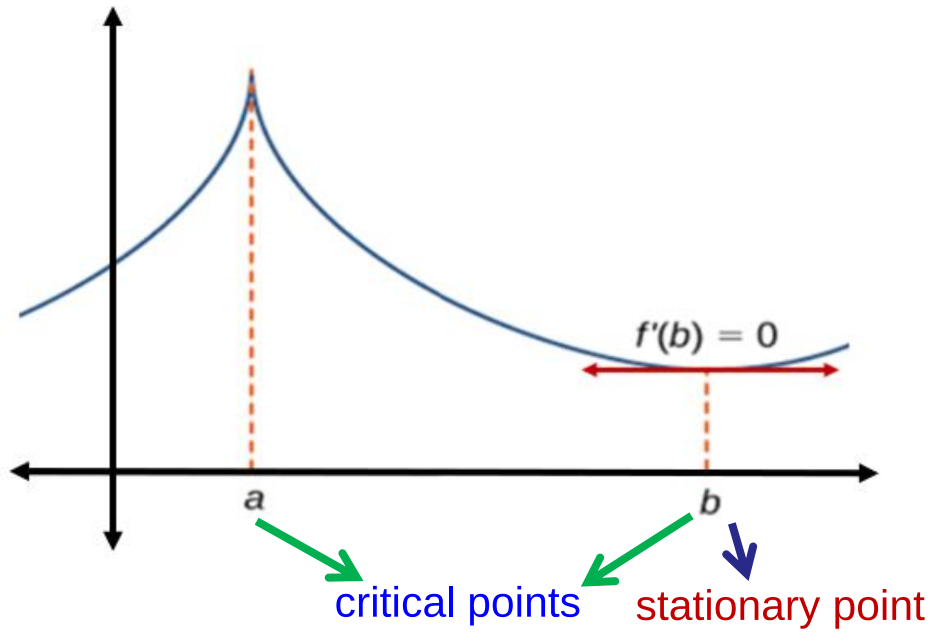


A critical point of a function of a real variable is any value in the domain where either the function is not differentiable or its derivative is 0.

Stationary and Critical Points

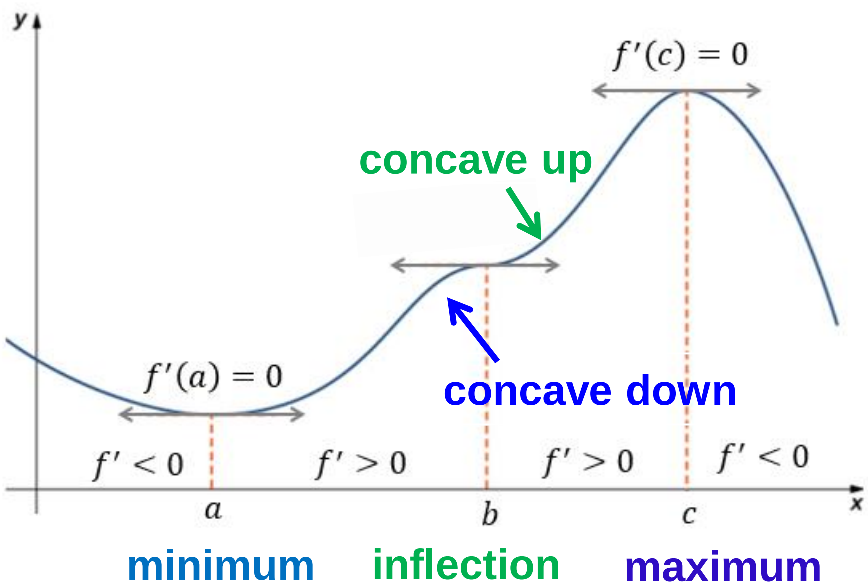
A critical point of a function of a real variable is any value in the domain where either the function is not differentiable or its derivative is 0.

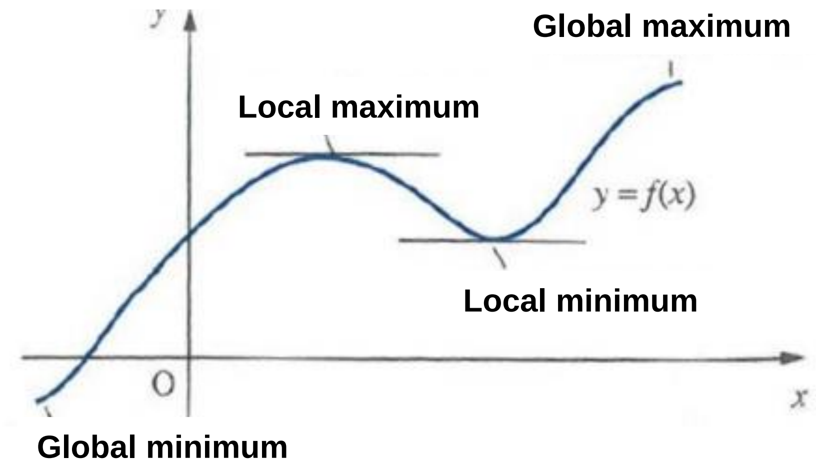




at $x = 0, y = |x|$ is not differentiable, hence there is a **critical** point at $x = 0$.

Turning Point and Point of Inflection



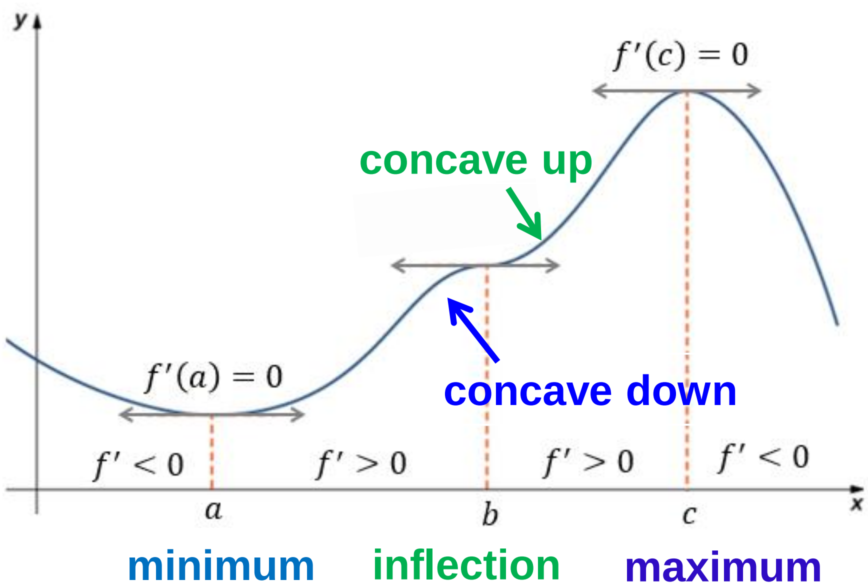


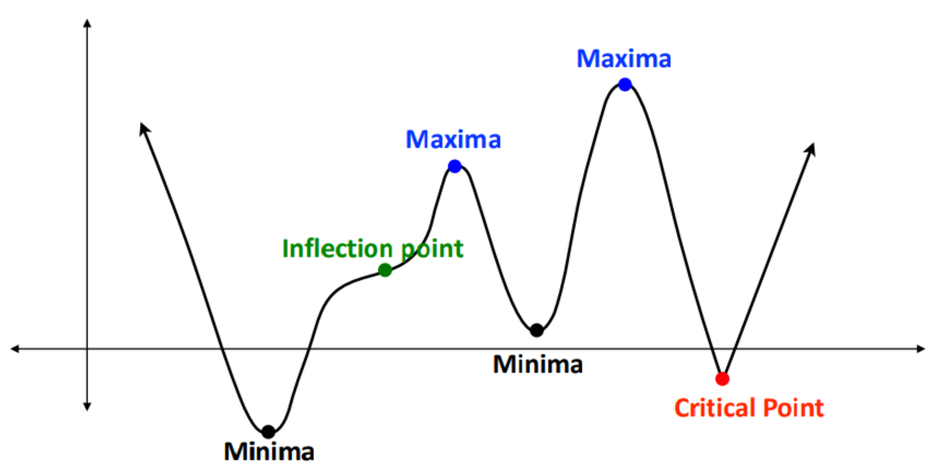
A turning point is a point at which the derivative changes sign.

A turning point may be either a local maximum or a minimum point.

A point at which concavity changes from concave upwards to concave downwards or vice versa is the point of inflection.

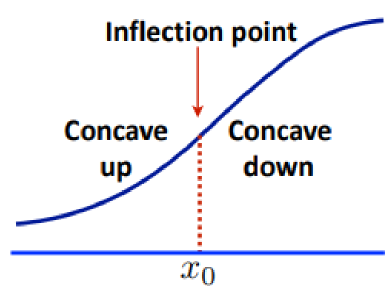
Turning Point and Point of Inflection

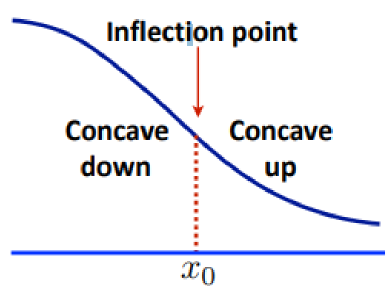




Inflection points are stationary points

that are not turning points.





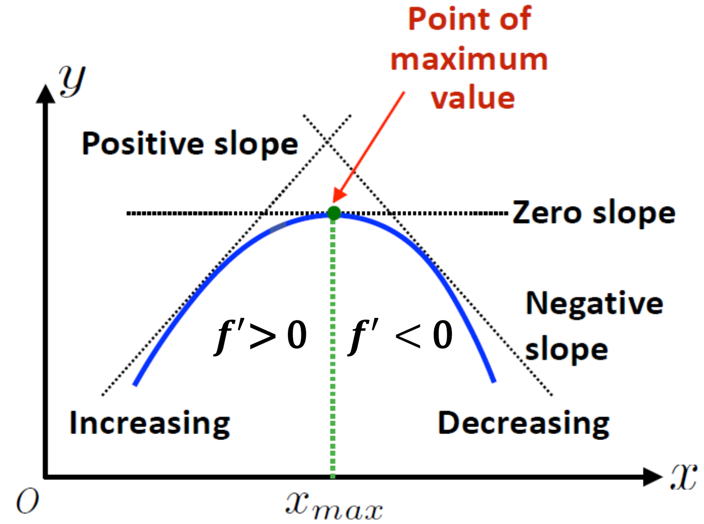
Maxima and Minima

There are two methods to determine whether the point of curve is a point of maxima / minima / inflection.

Method 1 (First Derivative Test)

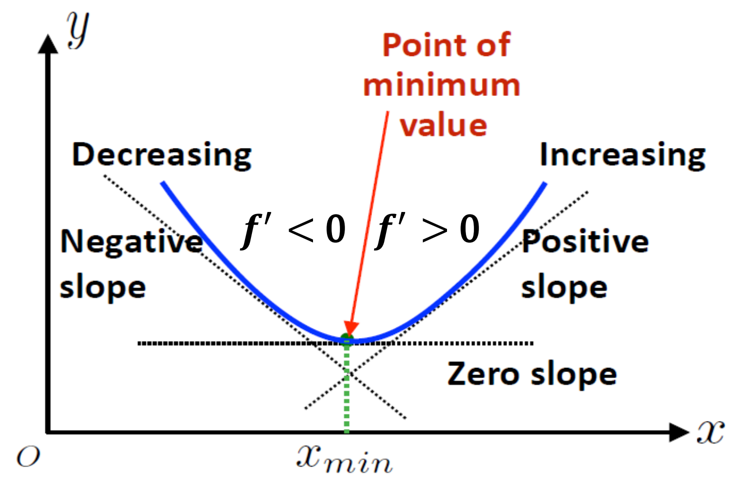
Based on the signs of the derivative $f'(x)$ or $\dfrac{dy}{dx}$ on either side of the critical point.

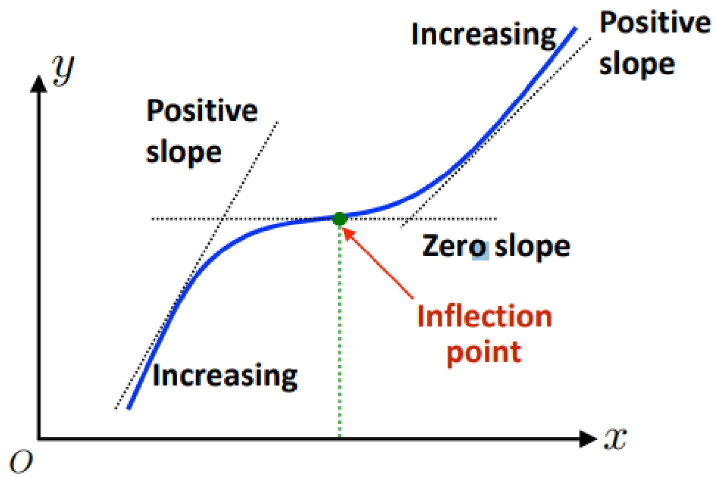
(a) If the value of $f'(x)$ changes from positive to negative as we pass from left to right through a stationary point then it is a point of local maximum.



Maxima and Minima

(b) If the value of $f'(x)$ changes from negative to positive as we pass from left to right through a stationary point then it is a point of local minimum.

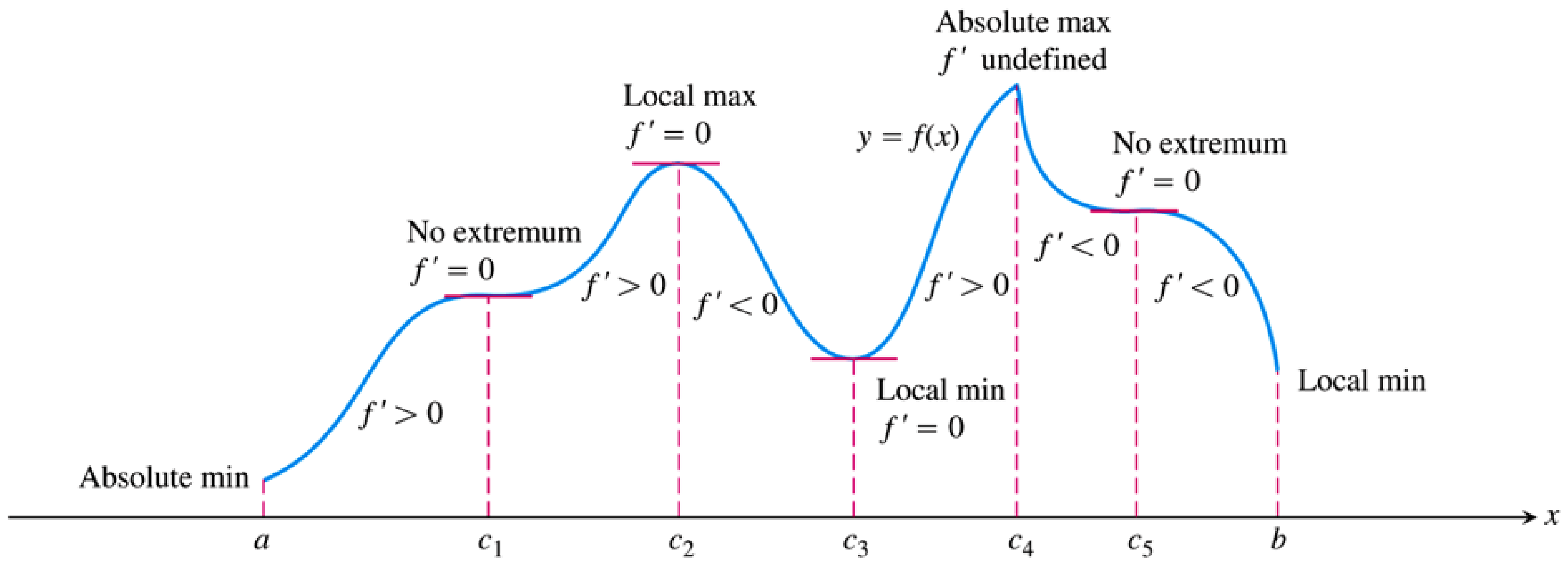




(c) If $f'(x)$ does not change sign as we pass through a stationary point then it is a point of inflection.

Maxima and Minima

Summary of First derivative test



Maxima and Minima

Method 2 (Second Derivative Test)

Based on the signs of the second derivative $f''(x)$ or $\frac{d^2y}{dx^2}$.

Suppose that f is twice differentiable at the point x_0.

(i) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a local minimum at x_0.

(ii) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a local maximum at x_0.

(iii) If $f''(x_0) = 0$ or is undefined, then f has a point of inflection at x_0.

Maxima and Minima

Global Maximum and Minimum

In finding optimal values of a function it is important to check the end points (if any) of the domain of the function.

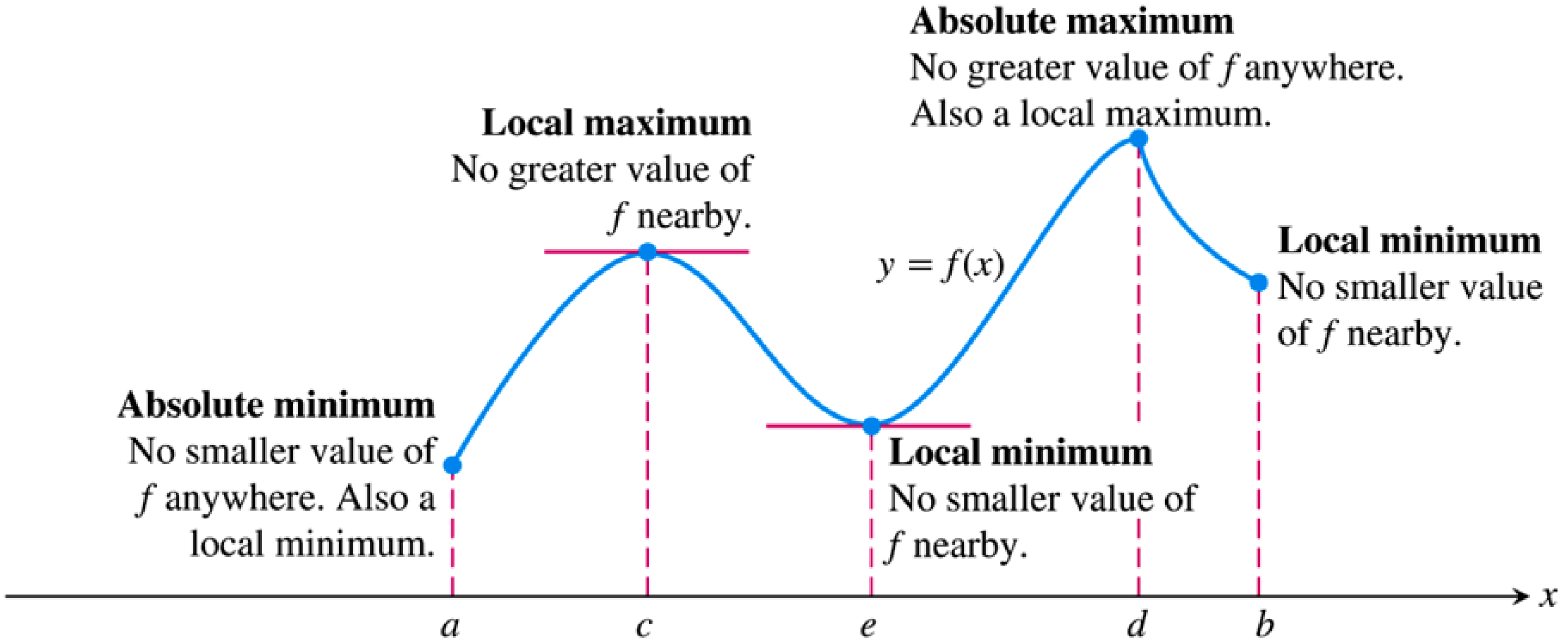
Consider an interval in the domain of a function f and a point x_0 in that interval. We say that:

(a) f has a global maximum at x_0 if $f(x) \leq f(x_0)$ for all x in the interval.

(b) f has a global minimum at x_0 if $f(x) \geq f(x_0)$ for all x in the interval.

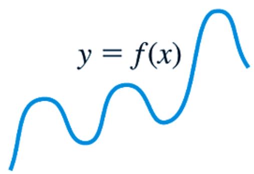
Maxima and Minima

Identifying types of maxima and minima for a function with domain $a \leq x \leq b$

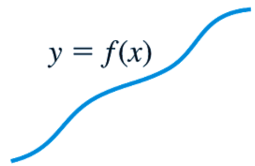


Maxima and Minima

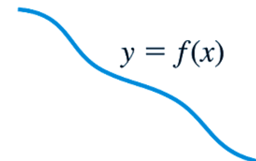
Different profiles for describing properties of curves



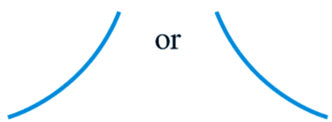
Differentiable $\Longrightarrow$ smooth, connected; graph may rise and fall



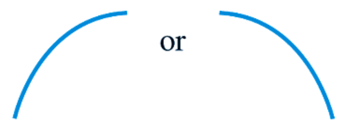
$y' > 0 \Longrightarrow$ rises from left to right; may be wavy



$y' < 0 \Longrightarrow$ falls from left to right; may be wavy

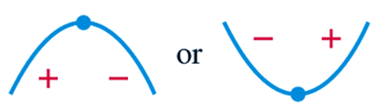


$y'' > 0 \Longrightarrow$ concave up throughout; no waves; graph may rise or fall or both

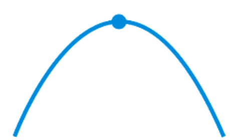


$y'' < 0 \Longrightarrow$ concave down throughout; no waves; graph may rise or fall or both

y'' changes sign at an inflection point



y' changes sign $\Longrightarrow$ graph has local maximum or local minimum



$y' = 0$ and $y'' < 0$ at a point; graph has local maximum

$y' = 0$ and $y'' > 0$ at a point; graph has local minimum

Maxima and Minima

Example Given $f(x) = 2x^3 - 7x^2 + 4x - 5$

Find and classify the stationary points of f. Using:

(i) the first derivative test

(ii) the second derivative test. Also, sketch the graph of $y = f(x)$.

Solution:

$$f'(x) = 6x^2 - 14x + 4$$

$$\therefore f'(x) = 0 \Rightarrow 3x^2 - 7x + 2 = 0$$

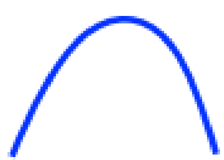
$\Rightarrow (3x - 1) \cdot (x - 2) = 0 \Rightarrow x = \frac{1}{3}, 2$ are stationary points.

Maxima and Minima

(i) The first derivative test

$f'(x) = 6x^2 - 14x + 4$	
$x < \frac{1}{3}$	$f'(x) > 0$
$x > \frac{1}{3}$	$f'(x) < 0$

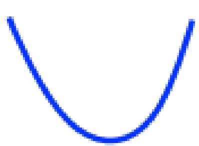
$f'(x)$ changes from positive to negative



$\Rightarrow \quad x = \dfrac{1}{3}$ is a maximum.

$f'(x) = 6x^2 - 14x + 4$	
$x < 2$	$f'(x) < 0$
$x > 2$	$f'(x) > 0$

$f'(x)$ changes from negative to positive



$\Rightarrow \quad x = 2$ is a minimum.

Maxima and Minima

(ii) The second derivative test

$$f'(x) = 6x^2 - 14x + 4 \qquad f''(x) = 12x - 14$$

$$f''(x)\Big|_{x=\frac{1}{3}} = 12\left(\frac{1}{3}\right) - 14$$

$$\because \ f''(x)\Big|_{x=\frac{1}{3}} < 0$$

$$\boldsymbol{x = \frac{1}{3}}$$ is a maximum.

$$f''(x)\Big|_{x=2} = 12(2) - 14$$

$$\because \ f''(x)\Big|_{x=2} > 0$$

$$\boldsymbol{x = 2}$$ is a minimum.

Maxima and Minima

We know: $f(x) = 2x^3 - 7x^2 + 4x - 5$

$$\Rightarrow \; f\left(\frac{1}{3}\right) = 2 \cdot \frac{1}{27} - 7 \cdot \frac{1}{9} + 4 \cdot \frac{1}{3} - 5$$

$$= -\frac{118}{27} \approx -4.37$$

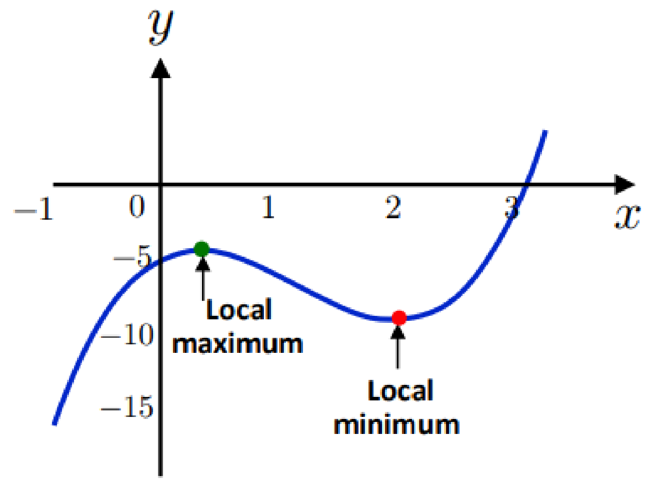
$\therefore \left(\frac{1}{3}, -\frac{118}{27}\right)$ is a local maximum.

and

$$f(2) = 2 \cdot (8) - 7 \cdot (4) + 4 \cdot (2) - 5$$

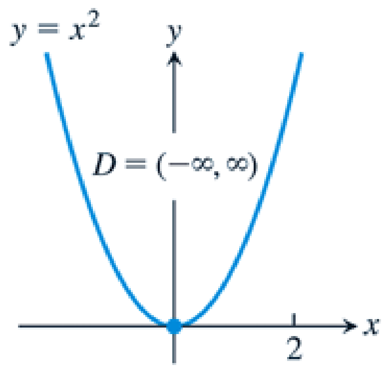
$$= -9$$

$\therefore$ $(2, -9)$ is a local minimum.

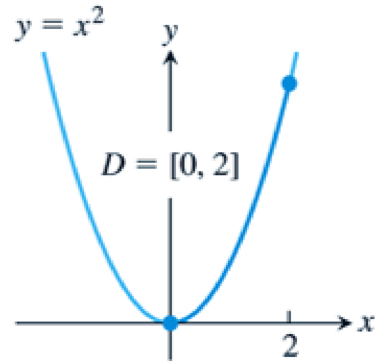


Maxima and Minima

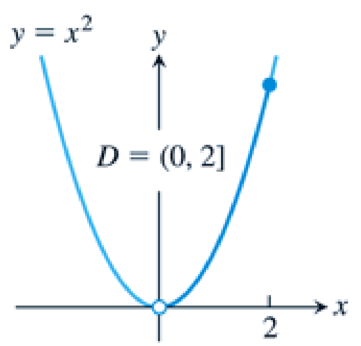
Describing maximum and minimum points within a given domain $a \leq x \leq b$



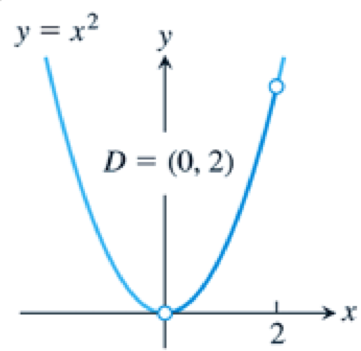
(a) abs min only



(b) abs max and min



(c) abs max only



(d) no max or min

Maxima and Minima

Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Find all critical points of f on the interval.
2. Evaluate f at all critical points and endpoints.
3. Take the largest and smallest of these values.

Maxima and Minima (Summary)

Critical Point: $f'(x) = 0$ or $f'(x)$ does not exist

- **Stationary Point**: $f'(x) = 0$
  - **Turning Point**: $f'(x) = 0$ and $f''(x) \neq 0$
    - **Point of Maxima**: $f'(x) = 0$ and $f''(x) < 0$
    - **Point of Minima**: $f'(x) = 0$ and $f''(x) > 0$
  - **Inflection Point**: $f'(x) = 0$, $f''(x) = 0$ but, $f'''(x) \neq 0$
- **Point where derivative is not defined**

A Critical point could also be a point of maxima or minima.

e.g. $f(x) = |x|$ has a critical point at $x = 0$, which is also a point of minima. However, it is not a turning point.

Solving Optimization (maximum/minimum) Problems

1. Read the problem carefully and identify the quantity to be optimized.

2. Use a diagram to understand the given problem if it is required.

3. Express the quantity to be optimized as a function of one variable.

   If the function has more than one variable use the given information to convert it to a function of one variable.

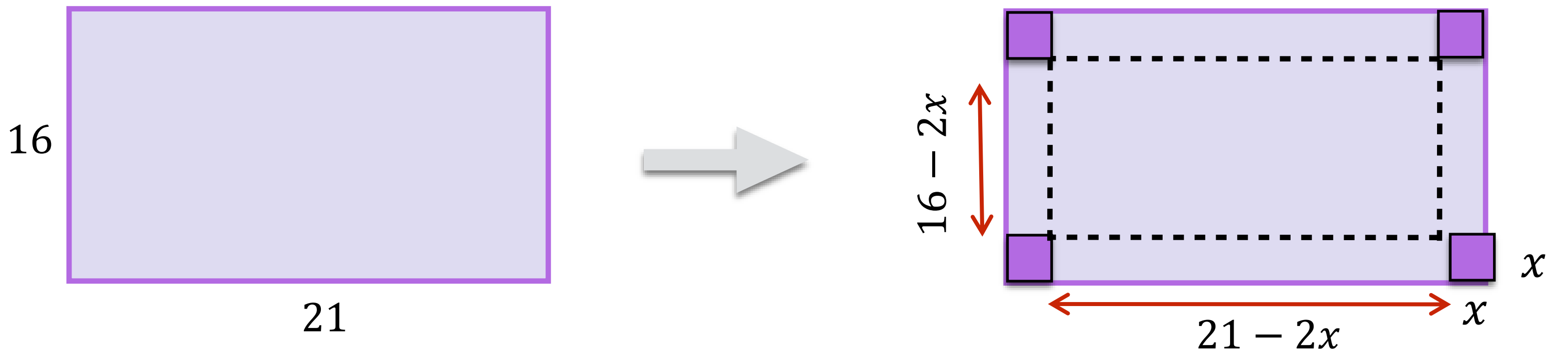
4. Find maximum/minimum value by using first or second derivative tests.

Solving Optimisation Problems

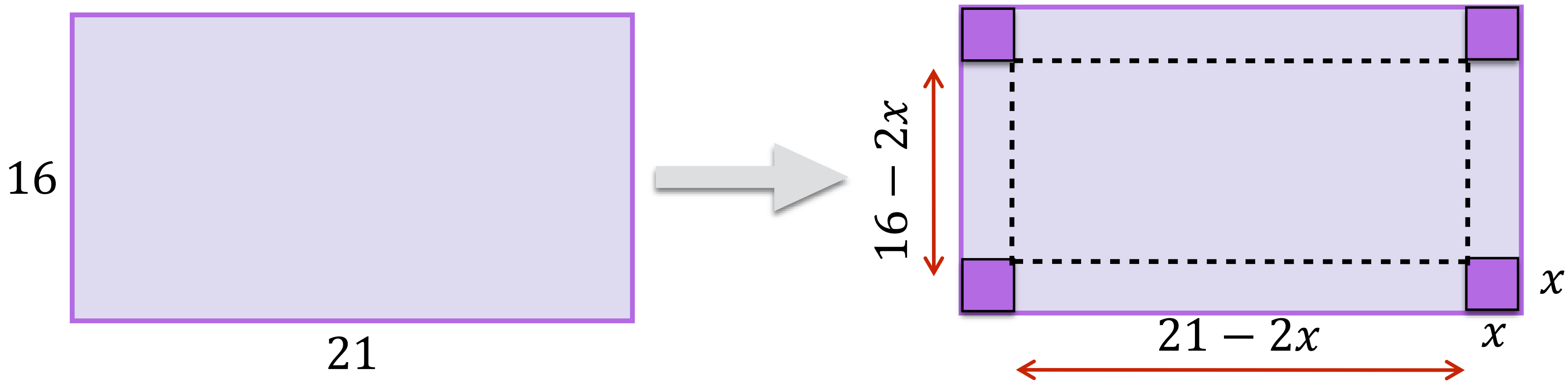
Example

An open-top box with a rectangular base is to be constructed from a rectangular sheet of metal with dimensions 16 x 21 inches, by cutting the same size square from each corners and then bending up the resulting sides. Find the size of the corner square to be removed so as to maximize the volume of the box so formed.

Solution



Solving Optimisation Problems



$f(x) = \text{Volume}$

$= x(21 - 2x)(16 - 2x)$ is to be maximized.

$$f(x) = x(21 - 2x)(16 - 2x)$$
$$= x(336 - 74x + 4x^2)$$
$$= 4x^3 - 74x^2 + 336x$$

Solving Optimisation Problems

$$\Rightarrow \quad f'(x) = 12x^2 - 148x + 336$$

$\therefore \; f'(x) = 0 \Rightarrow x = 3$ or $\frac{28}{3}$

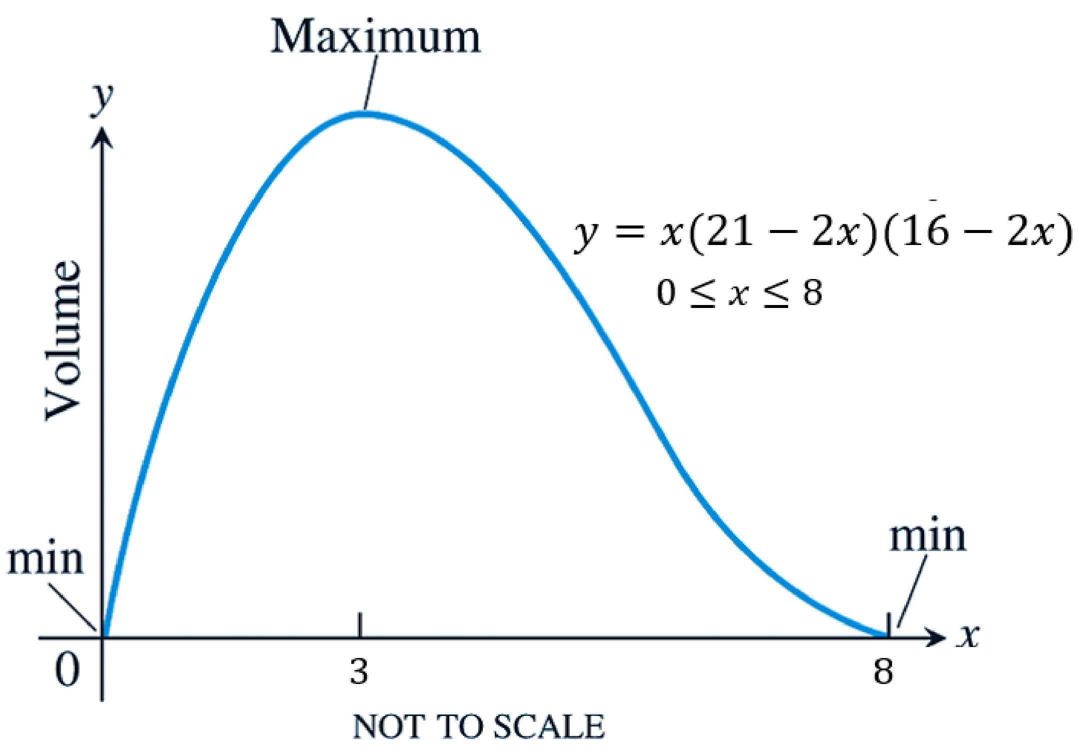
$\therefore \; x = 3$ (as $\frac{28}{3}$ is not possible.)

and $f''(x)\,|_{x=3} = 24x - 148\,|_{x=3} = -76 < 0$

$\therefore$ V is maximum, when $x = 3$.

Solving Optimisation Problems

Graphical illustration of $V(x)$.

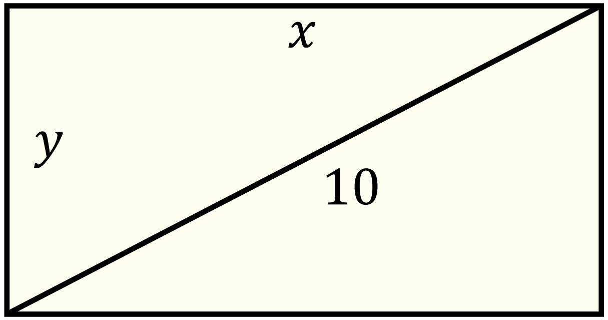


Solving Optimisation Problems

Example

Find the dimensions of the rectangle with the largest area whose diagonal is $10m$.

Ans: $x = y = 5\sqrt{2}\,m$



Area $= A = xy$ is to be maximized.

From figure, $x^2 + y^2 = 100$

$$\Rightarrow \quad y = \sqrt{100 - x^2}$$

i.e. $f(x) = x\sqrt{100 - x^2}$ is to be maximized.

Solving Optimization Problem

i.e. $f(x) = x\sqrt{100 - x^2}$ is to be maximized.

$$f'(x) = \sqrt{100 - x^2} + x \cdot \frac{1}{2}\frac{1}{\sqrt{100 - x^2}} \cdot (-2x) = \sqrt{100 - x^2} - \frac{x^2}{\sqrt{100 - x^2}}$$

$$\Longrightarrow \quad \sqrt{100 - x^2} - \frac{x^2}{\sqrt{100 - x^2}} = 0 \qquad \therefore x = \sqrt{50} = 5\sqrt{2}$$

Hence the length of the other side is: $\sqrt{100 - x^2} = y = 5\sqrt{2}$

$$f''(x) = \frac{d}{dx}\left(\frac{100 - 2x^2}{\sqrt{100 - x^2}}\right) = \frac{\sqrt{100 - x^2}(-4x) - (100 - 2x^2) \cdot \frac{1}{2}\frac{1}{\sqrt{100 - x^2}} \cdot (-2x)}{100 - x^2}$$

Second derivative test, shows that: $f''(\sqrt{50}) = \frac{\sqrt{50}(-4\sqrt{50}) - 0}{50} < 0$, i.e., the point: $(\sqrt{50},\ 50)$ is a maximum

$\therefore$ the rectangle with diagonal $10\ m$ with both sides $\mathbf{y} = \boldsymbol{x} = \sqrt{\mathbf{50}}$ has the largest area.

Generally: the square is the rectangle with largest area.

Problems on related rates

Let $y = f(x)$, if x changes with time at a rate of $\frac{dx}{dt}$, then y correspondingly changes with time at a rate of $\frac{dy}{dt}$:

$$\text{and} \quad \frac{dy}{dt} = \frac{d}{dx}f(x) \cdot \frac{dx}{dt}$$

Example:

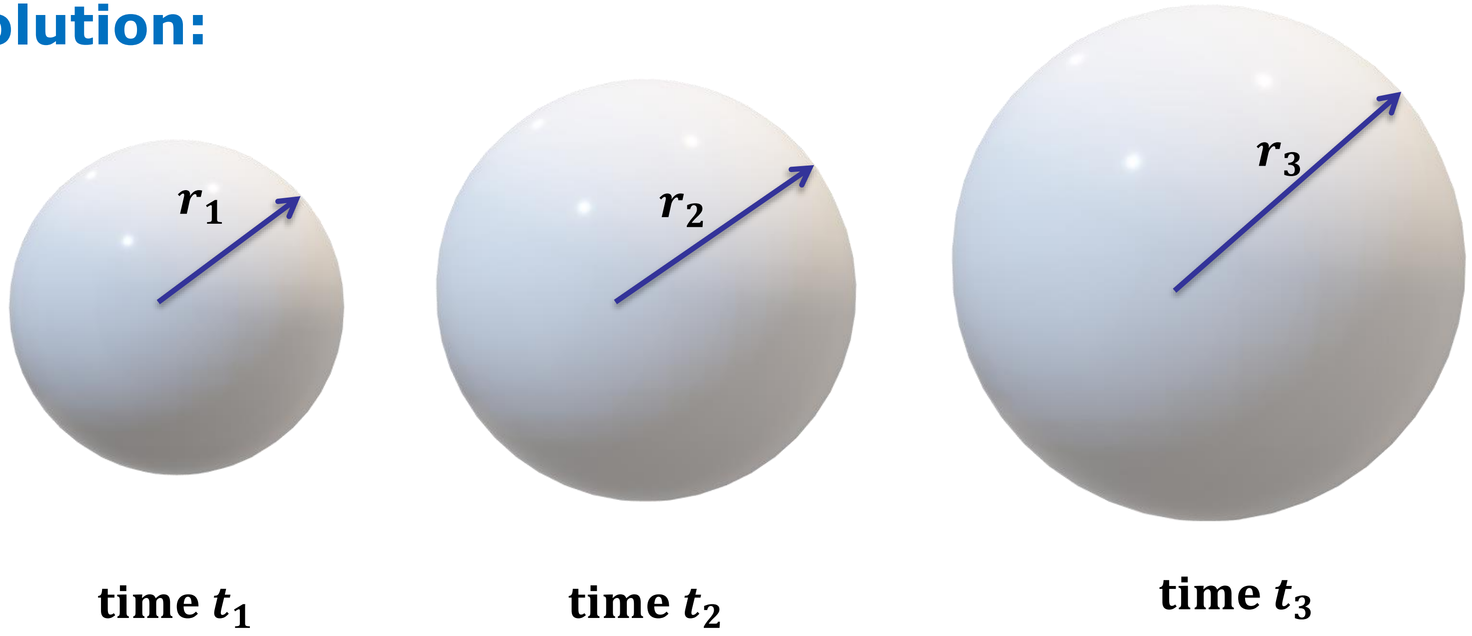
The radius of a sphere of radius 5 cm changes at the rate of 0.25 cm/s, find the rate at which the volume is changing.

Problems on related rates

Example:

The radius of a sphere of radius 5 cm changes at the rate of 0.25 cm/s, find the rate at which the volume is changing.

Solution:



Problems on related rates

Example:

The radius of a sphere of radius 5 cm changes at the rate of 0.25 cm/s, find the rate at which the volume is changing.

Solution:

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{d}{dx}(V) \cdot \frac{dx}{dt} \Rightarrow \frac{dV}{dt} = \frac{d}{dx}\left(\frac{4}{3}\pi r^3\right) \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dV}{dt} = (4\pi r^2) \cdot \frac{dx}{dt} \Rightarrow \frac{dV}{dt} = (4\pi(5)^2) \cdot (0.25)$$

$$= 25\pi\ cm/s$$

Integration as an Antiderivative

differentiate s

with respect to time t

given

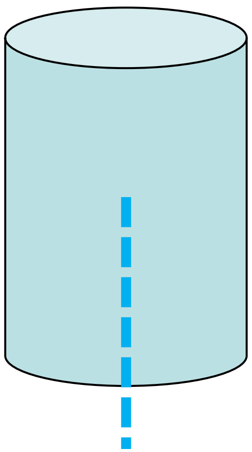
displacement function s

velocity $v = \dfrac{ds}{dt}$

?

given v, can we find s?

Integration as an Antiderivative



variable rate at which
water is leaking from a tank

differentiate V
with respect to time t

given V

$$\frac{dV}{dt}$$

?

given $\dfrac{dV}{dt}$, how can we find V?

Integration as an Antiderivative

Differentiation

Take Derivative

$$F(x) = \frac{1}{3}x^3$$

$$\frac{d}{dx}(F(x)) = \frac{1}{3}(3x^2)$$

$$= x^2 = f(x).$$

Take Anti-derivative

Integration (anti-differentiation)

Integration as an Antiderivative

In all the cases stated above, the question is:

Find a function F whose derivative is a known function f.

If such a function F exists, it is called an antiderivative of f.

A function F is called an **antiderivative** of a function f if

$$\frac{d}{dx}(F(x)) = f(x) \qquad \forall\, x \in D_f.$$

Integration as an Antiderivative

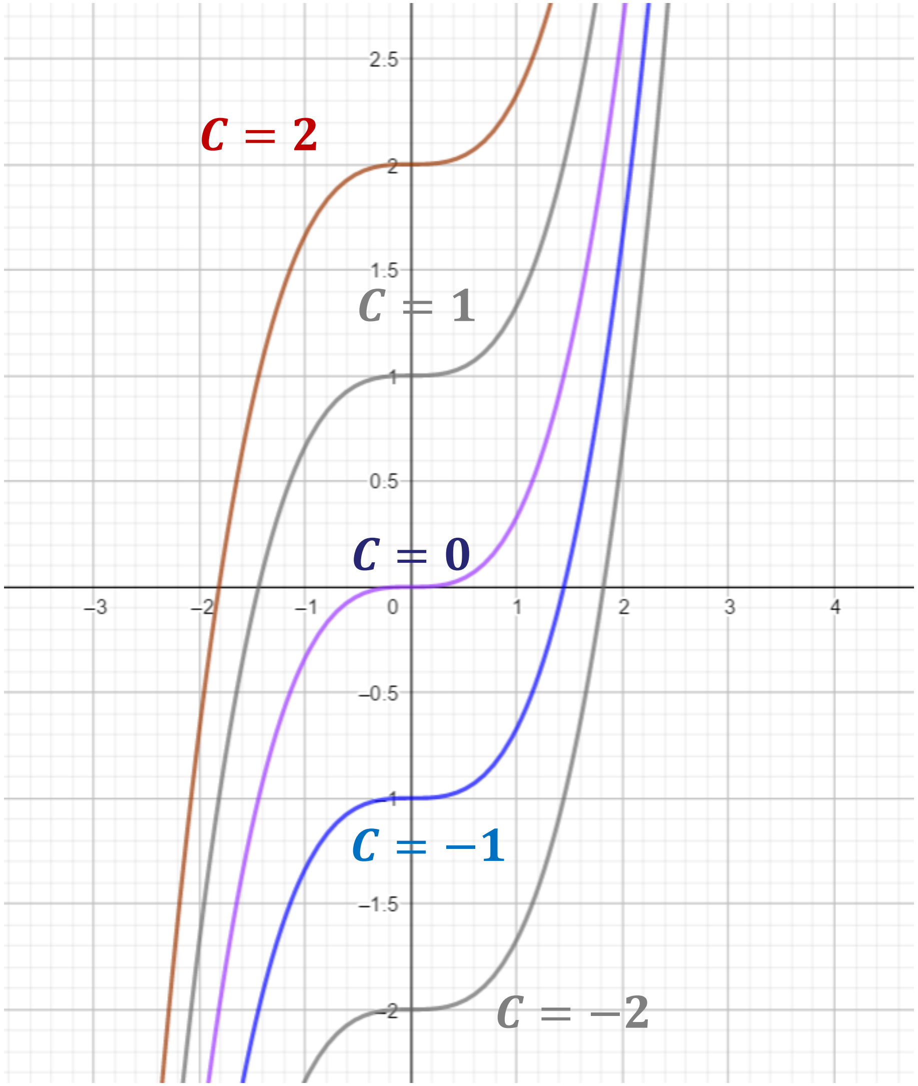
We saw that $$\frac{d}{dx}(F(x)) = \frac{1}{3}(3x^2) = x^2 = f(x).$$

However, $F(x) = \frac{1}{3}x^3$ is **NOT** the only antiderivative of f

$$G(x) = \frac{1}{3}x^3 + C \text{ is also an antiderivative of } f$$

since

$$\frac{d}{dx}(G(x)) = \frac{d}{dx}\left[\frac{1}{3}x^3 + C\right] = \frac{1}{3}(3x^2) + 0 = x^2 = f(x).$$



Family of curves defined by

$$G(x) = \frac{1}{3}x^3 + C$$

Integration as an Antiderivative

<u>Result</u>

If $F(x)$ is any antiderivative of $f(x)$, then for any constant C the function $F(x)+C$ is also an antiderivative.

Thus, $$\frac{d}{dx}\left(F(x)+C\right) = f(x)$$

$$\int f(x)\,dx = F(x) + C$$

C is an arbitrary constant.

(Constant of Integration)

Integral sign

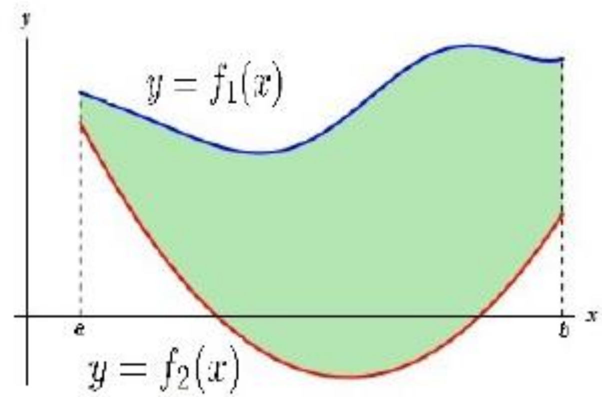
Integrand

Variable

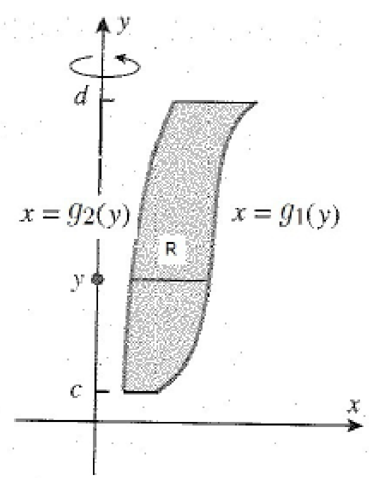
Integration or Antiderivative of f

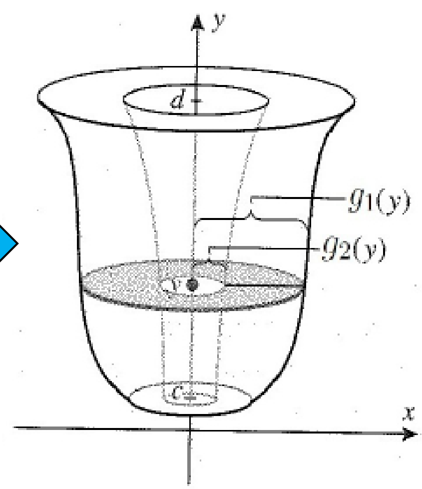
Some uses of Integration

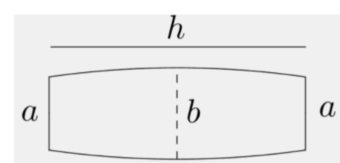
- To calculate area of regions with curved boundaries.



- To calculate volumes of solid of non-standard shapes (i.e. other than sphere, cylinder, etc.)



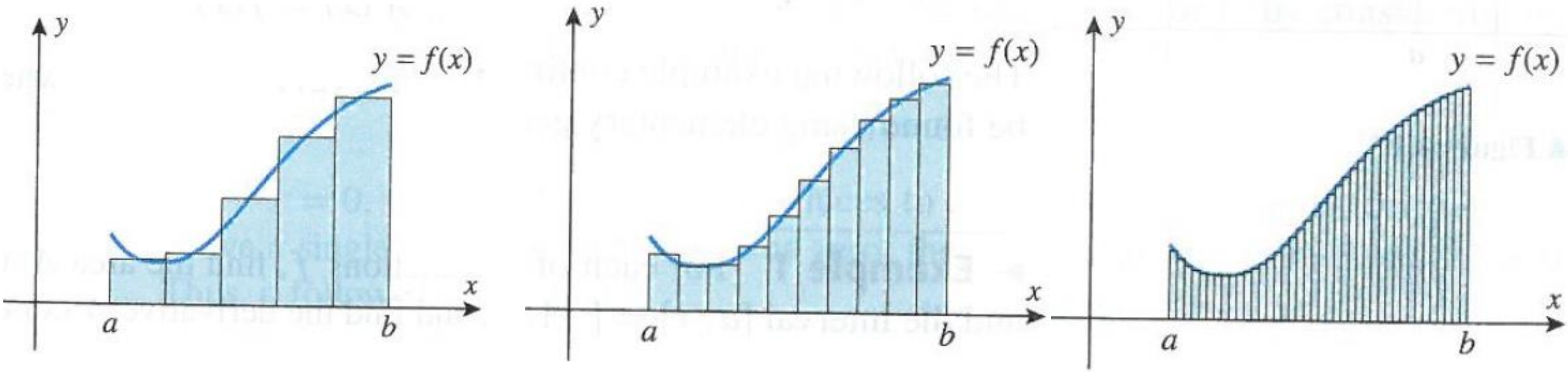




Integration as the Area under the curve

How to find areas of regions with curved boundaries?

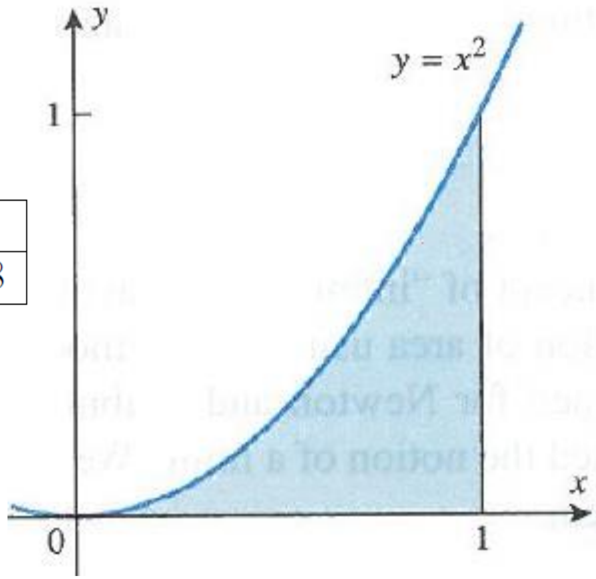
The Rectangle Method for finding areas:



$$A = \lim_{n \to \infty} A_n$$

Integration as the Area under the curve

For example, given $y = f(x) = x^2$, the total area under the curve, above the x-axis and bounded by $[0, 1]$ is:



n	4	10	100	1000	10000	100000
A_n	0.468750	0.385000	0.338350	0.333834	0.333383	0.333338

In fact, this area is same as $\frac{1}{3} = \int_0^1 x^2 dx$

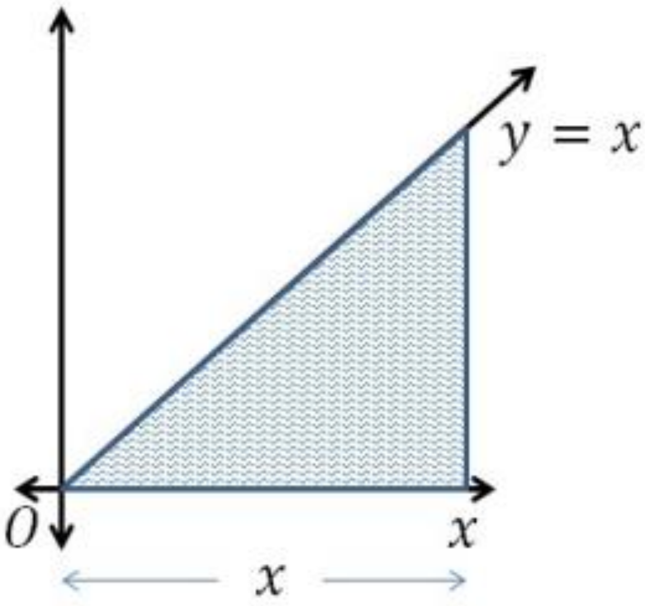
This will be established later, when we study applications of integration.

Thus, integration gives area under the curve.

$\therefore$ **Area $\equiv$ antiderivative of $f(x)$.**

Example Consider $y = f(x) = x$. What is the area bounded by: $y = f(x)$ and vertical lines at $x = 0$ and $x = x$?

Solution:



$$\boxed{\text{Area} =} A(x) = \frac{1}{2} \cdot x \cdot x = \frac{x^2}{2}$$

and $$\boxed{\int f(x)\, dx =} \int x\, dx = \frac{x^2}{2}$$

because $$\frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{1}{2} \cdot (2x) = x$$

$\therefore$ **Area $\equiv$ antiderivative of $f(x)$.**

Integration of Standard Functions

$\frac{d}{dx} F(x) = f(x)$	$\int f(x)\,dx = F(x) + C$
$\frac{d}{dx}(x) = 1$	$\int 1\,dx = x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + C$

Properties of Indefinite Integrals

If $\int f(x)\,dx = F(x)$ and $\int g(x)\,dx = G(x)$ k is a constant. Then:

$$(a) \quad \int k \cdot f(x)\,dx = k \int f(x)\,dx = k \cdot F(x) + C$$

$$(b) \quad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$= F(x) + G(x) + C$$

$$(c) \quad \int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

$$= F(x) - G(x) + C$$

Worked Examples

$$(i)\ \int 4\cos x\,dx = 4\int \cos x\,dx = 4\sin x + C$$

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \Rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad ;\ n \neq -1$$

$$(ii)\ \int \left(x^2 + x\right)\,dx = \int x^2\,dx + \int x\,dx = \frac{x^3}{3} + \frac{x^2}{2} + C$$

$$(iii)\ \int \left(3x^6 - 2x^2 + 7x + 1\right)\,dx = \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C$$

Worked Examples

$$\boxed{\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \Rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \;;\; n \neq -1}$$

$$(iv)\quad \int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\,dx = \int \operatorname{cosec} x \,\cot x\,dx$$

$$= -\operatorname{cosec} x + C$$

$$(v)\quad \int \frac{2x^4 - x^2}{x^4}\,dx = 2\int 1\,dx - \int \frac{1}{x^2}\,dx = 2x + \frac{1}{x} + C$$

$$(vi)\quad \int \frac{x^2}{x^2+1}\,dx = \int \frac{(x^2+1)-1}{x^2+1}\,dx = \int 1\,dx - \int \frac{1}{x^2+1}\,dx$$

$$= x - \tan^{-1} x + C$$

Worked Examples

$$\boxed{\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \Rightarrow \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \; ; \; n \neq -1}$$

$$(vii) \quad \int \frac{(x+1)^2}{x^2} \, dx = \int \frac{x^2 + 2x + 1}{x^2} \, dx$$

$$= \int 1 \, dx + 2 \int \frac{1}{x} \, dx + \int \frac{1}{x^2} \, dx$$

$$= x + 2 \ln x - \frac{1}{x} + C$$

Worked Examples

$$\boxed{\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \Rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \;\; ; \; n \neq -1}$$

$(viii)$

$$\int \left(3 \cdot 2^x + x^2 + e^x + x^e\right)\, dx$$

$$= 3 \cdot \int 2^x\, dx + \int x^2\, dx + \int e^x\, dx + \int x^e\, dx$$

$$= 3 \cdot \frac{2^x}{\ln 2} + \frac{x^3}{3} + e^x + \frac{x^{e+1}}{e+1} + C$$

Worked Examples

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \Rightarrow \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \ ; \ n \neq -1$$

(ix)
$$\int \frac{1 - x^2 - x^4}{1 + x^2} \, dx = \int \frac{1 - x^2 \cdot (1 + x^2)}{(1 + x^2)} \, dx$$

$$= \int \frac{1}{1 + x^2} \, dx - \int x^2 \, dx$$

$$= \tan^{-1} x - \frac{x^3}{3} + C$$

Worked Examples

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \Rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \;;\; n \neq -1$$

(x) $$\int \frac{t^2 - 2t^4}{t^4}\,dt = \int \frac{1}{t^2}\,dt - 2\int 1\,dt$$

$$= -\frac{1}{t} - 2t + C$$

(xi) $$\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx \Rightarrow \int \left[\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right]dx$$

$$= \int 1\,dx - \int \frac{1}{x^2+1}\,dx \Rightarrow x - \tan^{-1} x + C$$