Lecture 7

Lecture Content

- Integration by Partial fractions
- Integration by Parts
- (Definite) Integration as a limit of sum
- Evaluating definite integrals

The method of Partial fractions

$$\deg\left(p(x)\right) < \deg\left(q(x)\right) \;\; ; \; q(x) \text{ can be factorised}$$

Type 1: Non-repeated linear factors

$$\boxed{\frac{1}{(x+a)\,(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}}$$

Example Evaluate $\displaystyle\int \frac{1}{x^2+x-2}\;dx$

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)\,(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

The method of Partial fractions

$$\Rightarrow \; A\,(x+2) + B\,(x-1) = 1 \;\Rightarrow\; A = \frac{1}{3} \text{ and } B = \frac{-1}{3}$$

$$\therefore \; I = \int \frac{1}{x^2+x-2}\, dx = \frac{1}{3} \cdot \int \frac{1}{(x-1)} - \frac{1}{3} \cdot \int \frac{1}{(x+2)}$$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{3} \ln|x+2| + C$$

$$= \boxed{\frac{1}{3} \ln\left|\frac{x-1}{x+2}\right| + C}$$

The method of Partial fractions

$$\deg\left(p(x)\right) < \deg\left(q(x)\right) \;\; ; \; q(x) \text{ can be factorised}$$

Type 2: Repeated linear factors

$$\boxed{\frac{1}{(x+a)^2\,(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+b)}}$$

Example Evaluate $\displaystyle\int \frac{x+2}{x^3-2x^2}\,dx$

$$\frac{x+2}{x^3-2x^2} = \frac{x+2}{x^2\,(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

The method of Partial fractions

$$\Rightarrow \; x+2 = A\,x\,(x-2) + B\,(x-2) + C\,x^2$$

$$\therefore \; A=-1, \; B=-1, \; C=1$$

$$= \int \frac{-1}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{1}{x-2}\,dx$$

$$= -\ln|x| + \frac{1}{x} + \ln|x-2| + C = \frac{1}{x} + \ln\left|\frac{x-2}{x}\right| + C$$

The method of Partial fractions

$$\deg\left(p(x)\right) < \deg\left(q(x)\right) \;\; ; \; q(x) \text{ can be factorised}$$

Type 3: Non-repeated Quadratic factors

$$\boxed{\frac{1}{(x^2+a)\,(x+b)} = \frac{Ax+B}{(x^2+a)} + \frac{C}{(x+b)}}$$

Example Evaluate $\displaystyle\int \frac{13}{(2x+3)\;(x^2+1)}\; dx$

$$\frac{13}{(2x+3)\;(x^2+1)} = \frac{A}{2x+3} + \frac{Bx+C}{x^2+1}$$

The method of Partial fractions

$$\Rightarrow \; A(x^2+1)+(Bx+C)(2x+3)=13$$

$$\therefore \; A=4, \quad B=-2, \quad C=3$$

$$\therefore \; \frac{13}{(2x+3)\;(x^2+1)}=\frac{4}{2x+3}+\frac{-2x+3}{x^2+1}$$

$$\therefore \; I=\int \frac{4}{2x+3}\,dx-\int \frac{2x}{x^2+1}\,dx+3\int \frac{1}{x^2+1}\,dx$$

$$=2\;\ln|\,2x+3\,|-\ln(x^2+1)+3\;\tan^{-1}x+C$$

Integration by Parts

Here, we study how to integrate the products between <u>two functions</u>.

We start by taking derivative of the product and then use the definition of anti-derivative.

$$\frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$\Rightarrow \; u \cdot \frac{dv}{dx} = \frac{d}{dx}[u \cdot v] - v \cdot \frac{du}{dx}$$

Integration by Parts

Integrating the LHS and RHS

$$\Rightarrow \int u \cdot \frac{dv}{dx} \, dx = \int \frac{d}{dx}[u \cdot v] \, dx - \int v \cdot \frac{du}{dx} \, dx$$

$$\Rightarrow \boxed{\int u \cdot \frac{dv}{dx} \, dx = u \cdot v - \int v \cdot \frac{du}{dx} \, dx}$$

The method is called integration by parts.

Integration by Parts

$$\int u \cdot \frac{dv}{dx} \; dx = u \cdot v - \int v \cdot \frac{du}{dx} \; dx$$

Example

Evaluate $\displaystyle\int x \cos x \; dx$

Let $u = x$

and $\dfrac{dv}{dx} = \cos x$

$$\Rightarrow \left[\begin{array}{l} \dfrac{du}{dx} = 1 \\ v = \displaystyle\int \cos x \, dx = \sin x \end{array} \right.$$

$$\Rightarrow I = x \cdot \sin x - \int \sin x \cdot (1) \; dx$$

$$= x \cdot \sin x + \cos x + C$$

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Integration by Parts

How to choose u and $\dfrac{dv}{dx}$?

Rule 1: Choose $\dfrac{dv}{dx}$ such that it is readily integrable.

$$\text{e.g.} \quad \text{In} \int x \cdot \ln x \; dx$$ choose $\dfrac{dv}{dx} = x$

Rule 2: Choose u to be the function whose category occurs earlier in the list (LIATE).

Integration by Parts

How to choose u and $\dfrac{dv}{dx}$?

L	Logarithmic (e.g:. $\ln x$, $\log_a x$)
I	Inverse Trigonometric functions (e.g.: $\sin^{-1} x$, $\cos^{-1} x$)
A	Algebraic (e.g.: x^2, x^n)
T	Trigonometric (e.g.: $\cos x$, $\sin x$, $\tan x$)
E	Exponential (e.g.: e^x, 2^x, a^x)

Integration by Parts

Example

$$\boxed{\int u \cdot \frac{dv}{dx}\ dx = u \cdot v - \int v \cdot \frac{du}{dx}\ dx}$$

Evaluate $\displaystyle\int x \ln x \ dx$

L
I
A
T
E

Let $u = \ln x$

and $\dfrac{dv}{dx} = x$

$$\Rightarrow \left[\begin{array}{l} \dfrac{du}{dx} = \dfrac{1}{x} \\ v = \displaystyle\int x \ dx = \dfrac{x^2}{2} \end{array}\right.$$

$$\Rightarrow I = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \left(\frac{1}{x}\right) \ dx = \frac{x^2}{2} \cdot \ln x - \frac{x^2}{4} + C$$

Integration by Parts

$$\int u \cdot \frac{dv}{dx}\ dx = u \cdot v - \int v \cdot \frac{du}{dx}\ dx$$

Example

Evaluate $\int x \cdot e^x\, dx$

L
I
A
T
E

Let $u = x$

and $\frac{dv}{dx} = e^x$

$$\Rightarrow \left[\begin{array}{l} \frac{du}{dx} = 1 \\ v = \int e^x\ dx = e^x \end{array} \right.$$

$$\Rightarrow I = x \cdot e^x - \int e^x \cdot (1)\ dx = e^x \cdot (x-1) + C$$

Integration by Parts

$$\int u \cdot \frac{dv}{dx} \, dx = u \cdot v - \int v \cdot \frac{du}{dx} \, dx$$

Example

Evaluate $\displaystyle \int \ln x \, dx = \int (1) \cdot \ln x \, dx$

L
I
A
T
E

Let $u = \ln x$

and $\dfrac{dv}{dx} = 1$

$$\Rightarrow \left[\begin{array}{l} \dfrac{du}{dx} = \dfrac{1}{x} \\ v = \displaystyle\int 1 \, dx = x \end{array} \right.$$

$$\Rightarrow I = \ln x \cdot (x) - \int x \cdot \left(\frac{1}{x}\right) \, dx = x \, (\ln x - 1) + C$$

Integration by Parts

$$\int u \cdot \frac{dv}{dx}\ dx = u \cdot v - \int v \cdot \frac{du}{dx}\ dx$$

Evaluate $\displaystyle\int e^{x} \cos x \ dx$

L
I
A
T
E

Let $u = \cos x$

and $\dfrac{dv}{dx} = e^{x}$

$$\Rightarrow \left[\begin{array}{l} \dfrac{du}{dx} = -\sin x \\ \\ v = \displaystyle\int e^{x}\ dx = e^{x} \end{array} \right.$$

$$\Rightarrow I = \cos x \cdot e^{x} - \int e^{x} \cdot (-\sin x)\ dx$$

Integration by Parts

$$\therefore\ I = e^x \cdot \cos x + \boxed{\int e^x \cdot \sin x \ dx}$$

Again integrating by parts (in integral on the right)

Let $u = \sin x$

and $\dfrac{dv}{dx} = e^x$

$$\Rightarrow \left[\begin{array}{l} \dfrac{du}{dx} = \cos x \\ \\ v = \displaystyle\int e^x \ dx = e^x \end{array} \right.$$

$$\boxed{\int u \cdot \frac{dv}{dx} \ dx = u \cdot v - \int v \cdot \frac{du}{dx} \ dx}$$

$$\therefore\ I = e^x \cdot \cos x + \sin x \cdot e^x - \int e^x \cdot \cos x \ dx$$

Integration by Parts

$$\therefore\ I = e^x \cdot \cos x + \sin x \cdot e^x - \int e^x \cdot \cos x \ \ dx$$

The integral on the right, is now identical to the original integral, and as such can be re-written as ***I***

$$\therefore\ I = e^x \cdot \cos x + e^x \cdot \sin x - I$$

$$\text{i.e.} \quad 2\,I = e^x \cdot (\cos x + \sin x)$$

$$\therefore\ I = \int e^x \cos x \ \ dx = \frac{e^x}{2} \cdot (\cos x + \sin x) + C$$

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Integration by Parts

Note

For integrals of the form $$\int p(x)\, f(x)\ dx,$$

where $p(x)$ is a polynomial

the following short method works well.

Example

Evaluate $$\int \underbrace{(x^2 - x)}_{p(x)}\ \underbrace{\cos x}_{f(x)}\ dx$$

Integration by Parts

Example

Alternate sign	Repeated Derivatives of $u = x^2 - x$	Repeated Integration of $\int \frac{dv}{dx} = \cos x$
	Start with u	**Start with** $v = \int \frac{dv}{dx} dx$
$+$	$x^2 - x$	$\sin x$
$-$	$2x - 1$	$-\cos x$
$+$	2	$-\sin x$
$-$	0	

$$\therefore\ I = +(x^2 - x)\sin x - (2x-1)(-\cos x) + 2(-\sin x) + C$$

Integration by Parts

$$\therefore\ I = \int (x^2 - x)\ \cos x\ \ dx$$

$$= \quad (x^2 - x)\ \sin x + (2x - 1)\ \cos x - 2\ \sin x + C$$

Example

Evaluate $\displaystyle\int x^2\ \sqrt{x-1}\ \ dx$

Answer

$$\int x^2\ \sqrt{x-1}\ \ dx\ = \frac{2x^2}{3}\ (x-1)^{3/2} - \frac{8x}{15}\ (x-1)^{5/2} + \frac{16}{105}\ (x-1)^{7/2} + C$$

Integration by Parts (Substitution)

Example

Evaluate $\displaystyle\int x^3 \cdot e^{x^2}\ dx$

In some problems, appropriate substitutions should be carried out before the method of integration by parts is applied.

$$I = \int x^2 \cdot e^{x^2} \cdot x\ dx$$

Let $x^2 = t$

$$\Rightarrow x\,dx = \frac{1}{2}\,dt$$

$$\therefore\ I = \int t \cdot e^t\,\frac{1}{2}\,dt$$

$$= \frac{1}{2} \cdot \left[t \cdot e^t - \int e^t \cdot (1)\ dt \right]$$

$$= \frac{1}{2} \cdot \left[e^t \cdot (t-1) \right] + C$$

$$= \frac{e^{x^2}}{2} \cdot (x^2 - 1) + C$$

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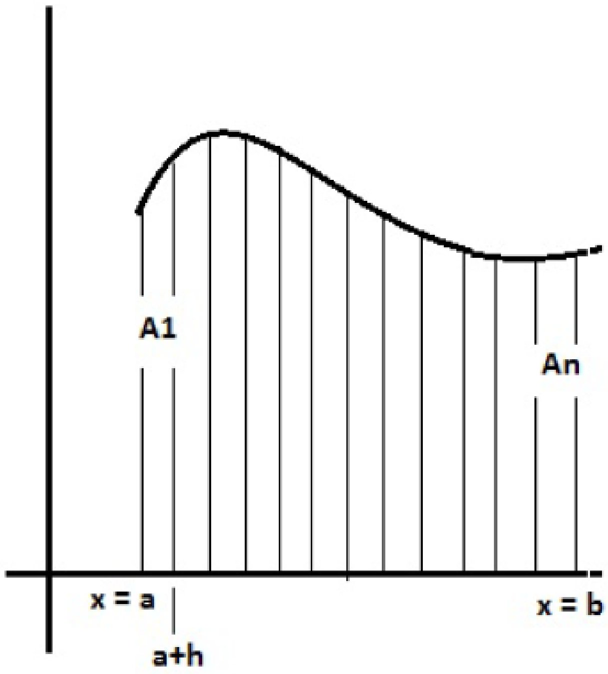
(Definite) Integration as a limit of sum

Suppose we want to find the area of region R bounded by the curve $y = f(x)$ and the lines $x = a$ and $x = b$.

Let the region R be subdivided into n thin strips of equal width h (say).



$$\therefore \; h = \frac{b-a}{n}$$

$$\equiv \quad \text{width of each strip}$$

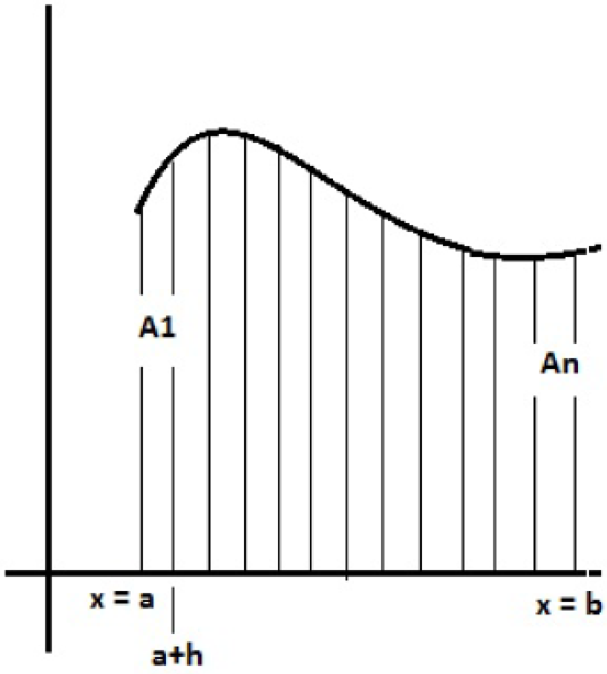
(Definite) Integration as a limit of sum

Now,

Total area $\boldsymbol{A}$ = sum of areas $\boldsymbol{A_1, A_2, A_3, \dots \dots, A_n}$

$$= \sum_{i=1}^{n} A_i$$

$$\approx \sum_{i=1}^{n} h \cdot f(a+ih)$$



(Definite) Integration as a limit of sum

If the number of strips are infinitely many, then

$$A = \lim_{\substack{n \to \infty \\ h \to 0}} \sum_{i=1}^{n} h \cdot f(a + ih)$$

which is same as

$$\int_a^b f(x)\, dx$$

upper limit: b

lower limit: a

and is called **definite integral** from a to b.

(Definite) Integration as a limit of sum

Thus, integration as a limit of sum is defined by

$$\boxed{\int_a^b f(x)\ dx = \lim_{\substack{n \to \infty \\ h \to 0}} \sum_{i=1}^{n} h \cdot f(a+ih)}$$

$$\text{where} \quad h = \frac{b-a}{n}$$

(Definite) Integration as a limit of sum

Example

Evaluate $\int_0^1 x\,dx$ as a limit of sum

$$\int_a^b f(x)\ dx$$

$$\boxed{\begin{aligned} a &= 0 \\ b &= 1 \end{aligned} \quad \Rightarrow \quad h = \frac{b-a}{n} = \frac{1}{n}}$$

$$f(x) = x \quad \Rightarrow \quad f(a+ih) = f(0+ih) = ih$$

$$\boxed{\int_a^b f(x)\ dx = \lim_{\substack{n \to \infty \\ h \to 0}} \sum_{i=1}^{n} h \cdot f(a+ih)}$$

$$\therefore \int_0^1 x\ dx = \lim_{\substack{n \to \infty \\ h \to 0}} \sum_{i=1}^{n} h \cdot (ih)$$

(Definite) Integration as a limit of sum

$$= \lim_{\substack{n \to \infty \\ h \to 0}} \sum_{i=1}^{n} h \cdot (ih)$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^2} \frac{n(n+1)}{2} \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^2} \frac{n^2 \left(1 + \frac{1}{n}\right)}{2} \right]$$

$$= \frac{(1+0)}{2} = \frac{1}{2}$$

Thus, $\displaystyle\int_0^1 x \, dx = \frac{1}{2}$

(Definite) Integration as a limit of sum

Example

Evaluate $\int_0^4 x^3\,dx$ as a limit of sum

In solving definite integration as limit of sum, take note of the following formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Evaluating Definite integrals

Fundamental Theorem of Calculus

If f is continuous on any interval $[a,b]$ and F is any antiderivative of f in $[a,b]$, then

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a)}$$

Example

$$\int_0^1 x\ dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

Evaluating Definite integrals

Examples

$$(i) \quad \int_{1}^{4} 2 \; dx = [\,2x\,]_1^4 = 8 - 2 = 6$$

$$(ii) \quad \int_{-1}^{2} x^2 \; dx = \left[\frac{x^3}{3}\right]_{-1}^{2} = \frac{2^3}{3} - \frac{(-1)^3}{3} = 3$$

$$(iii) \quad \int_{0}^{\pi/2} \cos x \; dx = [\,\sin x\,]_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin 0 = 1$$

Sample Practice Problems

1. Evaluate $\displaystyle\int \frac{x^2+2}{x(x+2)(x-1)}\,dx$

2. Evaluate $\displaystyle\int x^2 \tan^{-1} x\,dx$

3. Find k given that $\displaystyle\int_2^4 (x-k)\,dx = 2$