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COMP2054-ADE
The "Master Theorem"

Master Theorem (MT)

Consider recurrence relations of the form

$$T(n) = a\,T(n/b) + f(n)$$

- Designed for "divide and conquer" in which problems are divided into 'a' instances of a problem of size n/b.
- Aim is to be able to quickly express the "big-Oh family" behavior of T(n) for various cases of the values of a and b, and the scaling behavior of f(n).

It does not cover all cases, but does cover many useful cases.

Master Theorem

- No proof needed in this module.
  - Though you are expected to have some understanding of its structure, and why it is reasonable
- Need to learn it, and how to apply it!
- Suggest to generate and try many examples

Motivations

- Consider the special case that $f(n) = 0$
- $T(n) = a\,T(n/b)$ with $T(1) = 1$
  - $T(b) = a$
  - $T(b^2) = a^2$
  - $T(b^3) = a^3$
- So $T(b^k) = a^k$
- Exercise (offline): prove by induction

Motivations

- Consider the special case that $f(n) = 0$
  - $T(n) = a\,T(n/b)$ with $T(1) = 1$
  - Gives $T(b^k) = a^k$
- Put $n = b^k$ then
    - (for clarity in the superscripts write "$\log_b$" as "log_b")
  - $a^k = (b^{\text{log\_b(a)}})^k$
    - Note: used the identity that $a = b^{\text{log\_b(a)}}$
    - Then we have (swapping the order of the exponents)
  - $a^k = (b^k)^{\text{log\_b(a)}}$
  - $a^k = (n)^{\text{log\_b(a)}}$
- So we finally get $T(n) = n^{\text{log\_b(a)}}$

Motivations

- Consider the special case that $f(n) = 0$
  - $T(n) = a\,T(n/b)$ with $T(1) = 1$
  - Gives $T(n) = n^{\log\_b(a)}$
- But now suppose $f(n) = n^c$ for some c
- We can ask which term dominates
  - the recurrence or the f(n)?
- So need to compare the values of c and $\log_b(a)$

Master Theorem (MT): Cases

T(n) = a T(n/b) + f(n)

Results are split into 3 cases, according to comparing the growth rate of f(n) to $n\hat{}(\log_b(a))$

- Case 1: f(n) "grows slower". Recurrence term dominates. "Solution ignores f"
- Case 2: f(n) grows same – up to log factors – "mix of recurrence with a,b, and also the f(n) term"
- Case 3: f(n) grows faster. "Solution ignores recurrence terms and a,b"

MT: Case 1

T(n) = a T(n/b) + f(n)

f(n) is O(n^c) with $c < \log_b a$

Note: it is "<" not "<=" and it is a "big-Oh"

Then T(n) is $\Theta(n\hat{}(\log_b a))$

That is, T(b^k) grows as $b^{k \log\_b a} = a^k$,

as expected from earlier

MT: Case 1: Example

$T(n) = 2\ T(n/2) + d$

$a = 2$, $b=2$ so $\log_b(a) = \log_2(2) = 1$

$f(n)$ is $O(1)$ which is $O(n^c)$ with $c = 0$
and so we have that $c < \log_b(a)$

Hence MT gives that $T(n)$ is $\Theta(n)$

MT: Case 2

T(n) = a T(n/b) + f(n)

if

f(n) is $\Theta(n^c (\log n)^\wedge k)$

with c = $\log_b$ a and $k \geq 0$

(Note: it is "c =" and Big-Theta)

Then T(n) is $\Theta(n^c\,(\log n)^\wedge(k+1)\,)$

Note the growth depends on both the recurrence, a,b, and also depends on f via k.

MT: Case 2: Example

$T(n) = 2\,T(n/2) + 3\,n + 5$

$f(n)$ is $\Theta(\,n\,(\log n)^k\,)$

with

$c = \log_2 2 = 1$,

and

$k=0$

Then $T(n)$ is $\Theta(n \log n)$

(Same as merge sort of previous lecture)

MT: Case 3

$T(n) = a\,T(n/b) + f(n)$

f(n) is $\Omega(n^c)$ with $c > \log_b a$

Notice: it is "c > .." and big-Omega!

And f(n) satisfies the "regularity condition"

$a\,f(n/b) <= k\,f(n)$ for some $k < 1$

Then T(n) is $\Theta(f(n))$.

Growth is dominated by f(n) and so a,b of the recurrence are not used.

MT: Case 3 : Example

$T(n) = 2\ T(n/2) + n^2$

$f(n)$ is $\Omega(n^c)$ with $c = 2 > \log_b a = \log_2 2 = 1$

Also, $f(n)$ satisfies the "regularity condition"

$$2\ f(n/2) = 2\ (n/2)^2 <= k\ f(n) \quad \text{with } k=1/2$$

Then $T(n)$ is $\Theta(n^2)$

Regularity Condition (case 3)

- $a\, f(n/b) <= k\, f(n)$ for some $k < 1$
- Suppose $f(n) = d\, n^c$ then we need
  - $a\, d\, (n/b)^c <= k\, d\, n^c$ for some $k < 1$
  - $a / b^c <= k$ for some $k < 1$
  - $a / b^c < 1$
  - $a < b^c$
  - Now take log_b of both sides
  - Need $\log_b(a) < c$
  - Which is already satisfied for case 3 to apply.
  - So is not a new condition in this case (needed for more complex cases)
- (Note: this is included for completeness; usually not relevant)

MT Example

- $T(n) = 4\ T(n/2) + d\ n$ with $T(1)=1$
- $a=4,\ b=2$
- so $\log_b a = \log_2 4 = \log_2 2^2 = 2$

- Note that $f(n)$ is $O(n^c)$ with $c=1$
  - Hence $c < 2$, and is big-Oh
- Hence is case 1.

- Hence is $\Theta(n^2)$
  - Matches the exact solution in previous lecture.

Expectations

- Know and understand the Master Theorem (MT)
  - Be able to apply it to examples
  - (Do not need to be able to prove the MT itself!)
- May well be asked to solve a recurrence relation exactly, and then also to solve it using the Master Theorem