Lecture 2

Lecture Content

- Chain Rule for Derivatives of Composite functions
- Implicit Functions
- Implicit Differentiation
- Logarithmic Differentiation
- Derivative of $y = x^x$
- Derivative of an Inverse function
- Derivatives of Inverse Trigonometric functions

Chain Rule for Derivatives of Composite functions

If g is differentiable at x and f is differentiable at $g(x)$,

then the composition $f \circ g$ is differentiable at x.

In other words,

if $y = f\,(g(x))$ and $u = g(x)$ then $y = f\,(u)$ and

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Chain Rule

Generalizing, $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dw} \cdot \dfrac{dw}{dx}$

Examples: Chain Rule

Find $\dfrac{dy}{dx}$ where,

$(i) \quad y = \sin x^2 \qquad\qquad (ii) \quad y = \sin^2 x$

Solutions:

(i) Let, $y = \sin x^2$ **and** $x^2 = u$ so that,

$y = \sin u \quad \Rightarrow \quad \dfrac{dy}{du} = \cos u$ and, $\dfrac{du}{dx} = 2x.$

Examples: Chain Rule

Then, by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \cos u \cdot 2x$$

$$= 2x \cos x^2$$

Let, $y = \sin x^2$ **and** $x^2 = u$ so that,

$y = \sin u \quad \Rightarrow \quad \frac{dy}{du} = \cos u$ and, $\frac{du}{dx} = 2x$

Thus, $\boxed{\frac{d}{dx}(\sin x^2) = 2x \cos x^2}$

Examples: Chain Rule

(ii) Let, $y = \sin^2 x = (\sin x)^2$ **and** $\sin x = u$

so that, $y = u^2 \quad \Rightarrow \quad \dfrac{dy}{du} = 2u$ and, $\dfrac{du}{dx} = \cos x$.

Then, by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot \cos x$$

$$= 2 \sin x \cdot \cos x$$

$$= \sin 2x.$$

Thus, $\boxed{\dfrac{d}{dx}\left(\sin^2 x\right) = \sin 2x}$

Example

If $y = \cos(\sin(e^x))$, find $\dfrac{dy}{dx}$

Solution:

Let $v = e^x \quad \Rightarrow \quad \dfrac{dv}{dx} = e^x$

$u = \sin v \quad \Rightarrow \quad \dfrac{du}{dv} = \cos v$

$y = \cos u \quad \Rightarrow \quad \dfrac{dy}{du} = -\sin u$

Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= -\sin u \cdot \cos v \cdot e^x$$

$$= -\sin(\sin(e^x)) \cdot \cos(e^x) \cdot e^x$$

Fast Track Chain Rule

It is possible to use a 'faster' method to differentiate expressions using the chain rule.

This process involves the successive differentiation from the outer to the inner expression (or function).

Example

If $y = \sin(\ln(x^2 + 1))$, find $\dfrac{dy}{dx}$

Example

If $y = \sin(\ln(x^2+1))$, find $\dfrac{dy}{dx}$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\boxed{(\sin(\ln(x^2+1))} = \boxed{\cos(\ln(x^2+1))} \cdot \frac{d}{dx}\boxed{(\ln(x^2+1))}$$

$$= \cos(\ln(x^2+1)) \cdot \boxed{\frac{1}{x^2+1}} \cdot \frac{d}{dx}\boxed{(x^2+1)}$$

$$= \cos(\ln(x^2+1)) \cdot \frac{1}{x^2+1} \cdot \boxed{(2x+0)}$$

$$\therefore \frac{dy}{dx} = \frac{2x}{x^2+1}\cos(\ln(x^2+1))$$

Example

If $y = \sin\left(\sqrt{1+\cos x}\right)$ find $\dfrac{dy}{dx}$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\boxed{\left(\sin\sqrt{1+\cos x}\right)} = \boxed{\left(\cos\sqrt{1+\cos x}\right)} \cdot \frac{d}{dx}\boxed{\left(\sqrt{1+\cos x}\right)}$$

Note: $\dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{1}{2\sqrt{x}}$

$$= \cos\left(\sqrt{1+\cos x}\right) \cdot \boxed{\frac{1}{2\sqrt{1+\cos x}}} \cdot \frac{d}{dx}\boxed{(1+\cos x)}$$

$$= \cos\left(\sqrt{1+\cos x}\right) \cdot \frac{1}{2\sqrt{1+\cos x}} \cdot \boxed{(0-\sin x)}$$

$$\therefore \frac{dy}{dx} = -\frac{\sin x}{2\sqrt{1+\cos x}}\cos\left(\sqrt{1+\cos x}\right)$$

Implicit function

An equation of the form $y = f(x)$ is said to define y **explicitly** as a function of x because

- the variable y appears alone on one side of the equation
- y does not appear at all on the other side.

Cases of Implicit functions

1) In $yx + y + 1 = x$, the variable y is not alone on one side, i.e. equation is not of the form $y = f(x)$.

We say that such equation defines y implicitly as a function of x.

Implicit function

Cases of Implicit functions

2) An equation in x and y can define more than one functions of x.

e.g. if we solve the equation of the unit circle $x^2 + y^2 = 1$

we get two functions, namely

$$f_1(x) = \sqrt{1 - x^2} \qquad \text{and} \qquad f_2(x) = -\sqrt{1 - x^2}$$

We say that such equation defines y implicitly as a function of x.

Implicit function

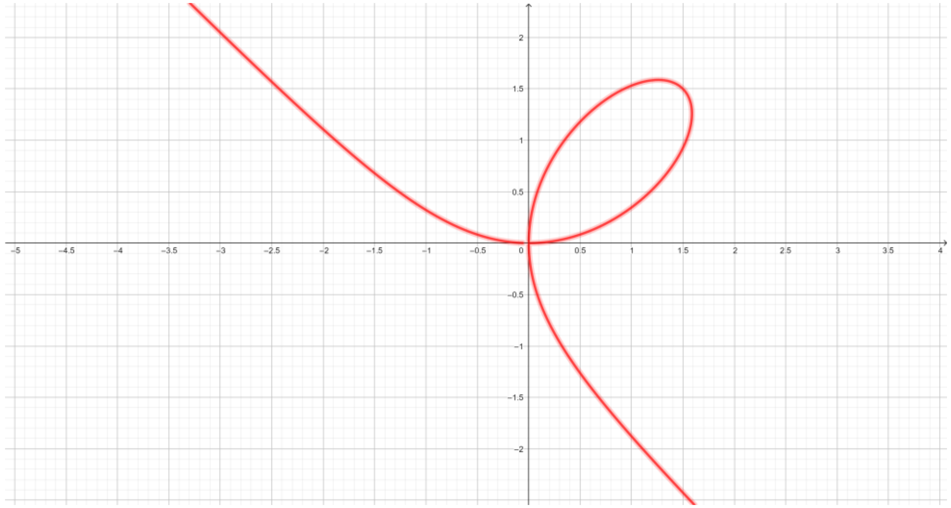
Cases of Implicit functions

3) It is sometimes too complicated or impossible to solve y in terms of x.

e.g.: $x^3 + y^3 = 3xy \Rightarrow$

or

$$sin(xy) = y$$



Folium of Descartes

We say that such equation defines y implicitly as a function of x.

Implicit Differentiation

To find derivatives of implicit functions, we differentiate both sides with respect to x (independent variable).

e.g. $xy = 1$, find $\frac{dy}{dx}$.

$$xy = 1$$

$$y = \frac{1}{x}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

$$\frac{d}{dx}[xy] = \frac{d}{dx}[1]$$

$$\therefore\ x\frac{d}{dx}[y] + y\frac{d}{dx}[x] = 0$$

$$\Rightarrow\ x\frac{dy}{dx} + y = 0$$

$$\Rightarrow\ \frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2}$$

Example

Given $x^3 + y^3 = 3xy$ find $\dfrac{dy}{dx}$

Solution:

Differentiate w.r.t x:

$$\frac{d}{dx}(x^3 + y^3) \Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(3xy)$$

$$\Rightarrow 3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3\left[x \cdot \frac{dy}{dx} + y \cdot \frac{dx}{dx}\right] \Rightarrow x^2 + y^2 \cdot \frac{dy}{dx} = x \cdot \frac{dy}{dx} + y$$

$$\Rightarrow (y^2 - x) \cdot \frac{dy}{dx} = y - x^2 \Rightarrow \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Example

Given $\cos(xy) = \sqrt{x + y}$ find $\dfrac{dy}{dx}$

Example

Find the gradient of $x^2 + 2xy - 2y^2 + x = 2$ at point $(-4, 1)$.

Solution:

Differentiate w.r.t x:

$$\frac{d}{dx}(x^2+2xy - 2y^2 + x) = \frac{d}{dx}(2)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) - 2\frac{d}{dx}(y^2) + \frac{d}{dx}(x) = \frac{d}{dx}(2)$$

$$\Rightarrow 2x + 2\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) - 2(2y) \cdot \frac{dy}{dx} + 1 = 0$$

Example

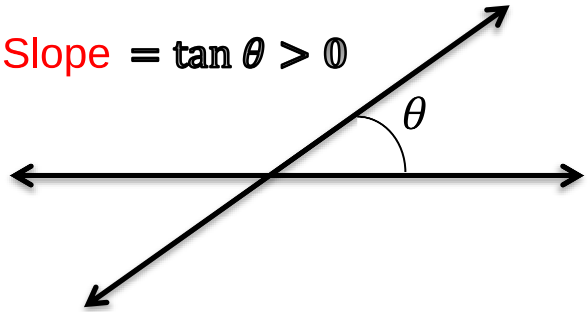
Note: Slope $= \tan\theta$

$$\Rightarrow\ 2x + 2x\frac{dy}{dx} + 2y - 4y\frac{dy}{dx} + 1 = 0$$

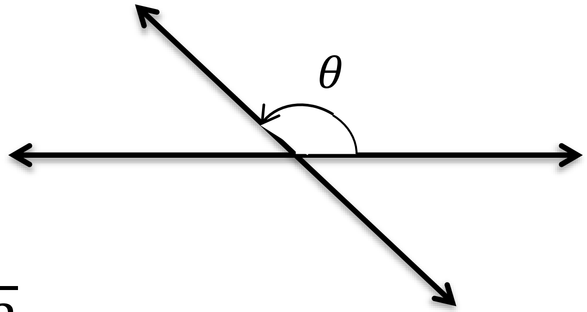
$$\Rightarrow\ (2x - 4y)\cdot\frac{dy}{dx} = -1 - 2x - 2y$$

$$\therefore \frac{dy}{dx} = \frac{-(1 + 2x + 2y)}{2x - 4y}$$

(Acute Angle) Slope $= \tan\theta > 0$



(Obtuse Angle) Slope $= \tan\theta < 0$



$\therefore$ Gradient (slope) is: $\left.\frac{dy}{dx}\right|_{(-4,1)} = \frac{-(1-8+2)}{-8-4} = -\frac{5}{12}$

(x, y)

Logarithmic Differentiation

Logarithmic differentiation means finding the derivative of a function after taking logarithms.

The method is useful when either

- the function is raised to the power of variables or functions.
- the function is composed of a product of a number of parts.

e.g. $(\sin x)^{\tan x}$ or $\left(\dfrac{\sqrt[3]{x^2-1}\ (1+e^x)^{2/3}}{(\sin x)^x}\right)$

The method relies on the chain rule and the properties of logarithms.

Derivative of $y = x^x$

(Particular case of finding derivative by taking logarithm)

We have, $y = x^x \Rightarrow \ln y = \ln(x^x)$

$$\Rightarrow \ln y = x \ln x$$

Differentiating with respect to x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + (1) \ln x = 1 + \ln x$$

$$\Rightarrow \frac{dy}{dx} = y \, (1 + \ln x) = x^x \, (1 + \ln x)$$

Thus, $\boxed{\frac{d}{dx}(x^x) = x^x \, (1 + \ln x)}$

Example

Given $y = \sin(x^x)$ find $\dfrac{dy}{dx}$

Solution:

Note: $\sin(x^x) \neq (\sin x)^x$ (Logarithmic differentiation not applicable)

Differentiate w.r.t x:

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(x^x)] \Rightarrow \cos(x^x) \cdot \frac{d}{dx}(x^x)$$ (Chain Rule)

$$\therefore \frac{dy}{dx} = \cos(x^x) \cdot x^x(1 + \ln x)$$

Note: $\boxed{\dfrac{d}{dx}(x^x) = x^x(1 + \ln x)}$

Example

Find $\dfrac{d}{dx}(\sin x)^{\tan x}$

Solution:

Let $y = (\sin x)^{\tan x}$.

Taking logarithms on both the sides.

$$\Rightarrow \quad \ln y = \tan x \cdot \ln(\sin x)$$

Differentiating with respect to x, gives:

$$\boxed{\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \tan x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \ln(\sin x) \cdot \frac{d}{dx}(\tan x)$$

Example

$$\frac{1}{y}\frac{dy}{dx} = \tan x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \ln(\sin x) \cdot \frac{d}{dx}(\tan x)$$

$$\therefore \frac{dy}{dx} = y\left[\tan x \cdot \frac{1}{\sin x} \cos x + \ln(\sin x) \sec^2 x\right]$$

$$= (\sin x)^{\tan x}\left[1 + \sec^2 x \ln(\sin x)\right].$$

Example

$$\text{Find } \frac{d}{dx}\left(\frac{\sqrt[3]{x^2-1}\ \left(1+e^x\right)^{2/3}}{(\sin x)^x}\right).$$

Solution:

$$\text{Let } y = \left(\frac{\sqrt[3]{x^2-1}\ \left(1+e^x\right)^{2/3}}{(\sin x)^x}\right)$$

Taking logarithms on both the sides.

$$\Rightarrow \quad \ln y \quad = \quad \ln(x^2-1)^{1/3} + \ln\left(1+e^x\right)^{2/3} - \ln\left(\sin x\right)^x$$

$$= \quad \frac{1}{3}\ \ln(x^2-1) + \frac{2}{3}\ \ln(1+e^x) - x\ \ln(\sin x)$$

Example

$$\ln y = \frac{1}{3}\ln(x^2-1) + \frac{2}{3}\ln(1+e^x) - x\ \ln(\sin x)$$

Differentiating with respect to x, gives:

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\cdot\frac{1}{(x^2-1)}\cdot\frac{d}{dx}(x^2-1)$$

$$+\frac{2}{3}\cdot\frac{1}{(1+e^x)}\cdot\frac{d}{dx}(1+e^x)$$

$$-x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x) - \ln(\sin x)\cdot\frac{dx}{dx}$$

Example

$$\therefore \frac{dy}{dx} = y\left[\frac{1}{3}\cdot\frac{1}{(x^2-1)}\cdot 2x + \frac{2}{3}\cdot\frac{1}{(1+e^x)}\cdot e^x - x\cdot\frac{1}{\sin x}\cdot\cos x - \ln(\sin x)\cdot(1)\right]$$

$$\therefore \frac{dy}{dx} = \left(\frac{\sqrt[3]{x^2-1}\;(1+e^x)^{2/3}}{(\sin x)^x}\right)\cdot\left[\frac{2x}{3}\,\frac{1}{(x^2-1)} + \frac{2}{3}\,\frac{e^x}{(1+e^x)} - x\cot x - \ln(\sin x)\right]$$

Example

Given $y = (\sin x)^{\tan x}$ find $\frac{dy}{dx}$

Derivatives of Inverse Functions

Let $y = f^{-1}(x)$, which is equivalent to writing $x = f(y)$.

Differentiating with respect to x, we obtain

$$\frac{dx}{dx} = \frac{d}{dy}\left[f(y)\right] \cdot \frac{dy}{dx} \qquad \text{(by Chain Rule)}$$

$$\Rightarrow \quad 1 = \frac{d}{dy}\left[f(y)\right] \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\frac{d}{dy}\left[f(y)\right]} \quad \Rightarrow \quad \frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{\frac{d}{dy}\left[f(y)\right]}$$

Derivatives of Inverse Functions

If f is a differentiable and one-to-one function, then

$$\boxed{\frac{d}{dx}\left[f^{-1}(x)\right]} = \frac{1}{\boxed{\frac{d}{dy}\left[f(y)\right]}} \qquad \text{where} \quad y = f^{-1}(x)$$

provided $\frac{d}{dy}\left[f(y)\right] \neq 0$

Also, $x = f(y)$ gives the alternative and more useful form

$$\boxed{\frac{dy}{dx}} = \frac{1}{\boxed{\left(\frac{dx}{dy}\right)}}$$

Example

Consider the function $f(x) = x^5 + x^3 + x$.

(i) Find a formula for the derivative of $f^{-1}(x)$.

(ii) Compute $\left.\dfrac{d}{dx}\left(f^{-1}(x)\right)\right|_{x=3}$

Example

(i) Find a formula for the derivative of $f^{-1}(x)$.

Solution:

Let $y = f^{-1}(x)$, so that $x = f(y)$.

Differentiating $x = f(y) = y^5 + y^3 + y$ with respect to x gives:

$$\frac{d}{dx}[x] = \frac{d}{dx}\left[y^5 + y^3 + y\right]$$

$$\frac{d}{dx}[x] = \frac{d}{dy}\left[y^5 + y^3 + y\right] \cdot \frac{dy}{dx}$$

$$\Rightarrow 1 = \left(5y^4 + 3y^2 + 1\right)\frac{dy}{dx}$$

$$\boxed{\Rightarrow \frac{dy}{dx} = \frac{1}{(5y^4 + 3y^2 + 1)}}$$

Example

$$(ii) \quad \text{Compute } \frac{d}{dx}\left(f^{-1}(x)\right)\Big|_{x=3}$$

Solution:

$$\begin{aligned}
\frac{d}{dx}\left(f^{-1}(x)\right)\Big|_{x=3} &= \frac{dy}{dx}\Big|_{x=3} \\
&= \frac{1}{(5y^4+3y^2+1)}\Big|_{x=3} \\
&= \frac{1}{(5y^4+3y^2+1)}\Big|_{y=1} \\
&= \frac{1}{9}
\end{aligned}$$

$$\boxed{x = f(y) = y^5 + y^3 + y}$$

Derivatives of Inverse Trigonometric Functions

1. $$\boxed{\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \quad ; \quad |\,x\,| < 1}$$

Proof:

As $|\,x\,| < 1$, $\exists\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $y = \sin^{-1} x$.

$$\therefore\ x = \sin y.$$

Differentiating with respect to y gives

$$\frac{dx}{dy} = \cos y$$

Derivatives of Inverse Trigonometric Functions

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\dfrac{dx}{dy}\right)} = \frac{1}{\cos y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

Thus, $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}} \quad ; \quad |\,x\,| < 1$

$$\boxed{\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}} \quad ; \quad |\,x\,| < 1}$$

Derivatives of Inverse Trigonometric Functions

$$\boxed{\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \quad ; \quad x \in \mathbb{R}}$$

$$\boxed{\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2} \quad ; \quad x \in \mathbb{R}}$$

$$\boxed{\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|\,x\,|\,\sqrt{x^2-1}} \quad ; \quad |\,x\,| > 1}$$

$$\boxed{\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = \frac{-1}{|\,x\,|\,\sqrt{x^2-1}} \quad ; \quad |\,x\,| > 1}$$

Example

Given $y = \tan^{-1}(\sin x)$ find $\dfrac{dy}{dx}$

Solution:

Differentiate w.r.t x:

$$\frac{dy}{dx} = \frac{d}{dx}\boxed{[\tan^{-1}(\sin x)]} \Rightarrow \boxed{\frac{1}{1+(\sin x)^2}} \cdot \frac{d}{dx}\boxed{(\sin x)} \qquad \text{(Chain Rule)}$$

$$\boxed{\frac{dy}{dx} = \frac{\cos x}{1+\sin^2 x}}$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

$$\therefore \frac{dy}{dx} = \frac{\cos x}{1+\sin^2 x}$$

Example

Given $y = \sin^{-1}(x^3)$ find $\dfrac{dy}{dx}$

Practice Problems

Given in class