Lecture 9

Topics covered in this lecture session

1. Complex numbers - Introduction
   - Algebra of complex numbers.
   - Square root of a complex number.
2. Polar form of a complex number.
3. Algebraic operations on Argand diagram.

Complex Numbers - Introduction

In solving quadratic equations $ax^2 + bx + c = 0$ using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, if the discriminant $\Delta < 0$, then no real root exist.

With Complex Numbers, we can explore further into the possibility of finding roots even when $\Delta = b^2 - 4ac < 0.$

For that, we first define *Imaginary Numbers*.

Complex Numbers - Introduction

An imaginary number is the one whose square is a negative real number.

e.g. $\sqrt{-1}, \quad \sqrt{-7}, \quad \sqrt{-8}, \quad \sqrt{-25}, \quad \sqrt{-1.21},$ etc.

are all imaginary numbers, because their squares $-1,\ -7,\ -8,\ -25,\ -1.1$ are all negative real numbers.

We use the notation $\sqrt{-1} = i$ to represent imaginary numbers. e.g. $\sqrt{-7} = \sqrt{7}\, i, \ \ \sqrt{-25} = 5\, i,$ and so on.

Complex Numbers - Introduction

Note: $i = \sqrt{-1} \Rightarrow i^2 = -1.$

$\begin{aligned} i^3 &= i^2 \cdot i \\ &= (-1) \cdot i \\ &= -i \end{aligned}$	$\begin{aligned} i^4 &= i^2 \cdot i^2 \\ &= (-1) \cdot (-1) \\ &= 1 \end{aligned}$
$\begin{aligned} i^5 &= i^4 \cdot i \\ &= (1) \cdot i \\ &= i \end{aligned}$	$\begin{aligned} \frac{1}{i} = \frac{i}{i^2} &= \frac{i}{-1} \\ &= -i \end{aligned}$

Imaginary numbers

Using the notation $i = \sqrt{-1}$, it is now possible to solve quadratic equations with negative discriminants.

e.g. $x^2 - 2x + 2 = 0 \;\Rightarrow\; x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2}$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\,i}{2}$$

$$= 1 \pm i$$

$$= \boxed{(1) \pm i \cdot (1)}$$

Form: $a + i\,b$ where $a, b \in \mathbb{R}$
and $i^2 = -1.$

Complex Numbers

A Complex Number is of the form: $\boxed{a + i\,b \text{ where } a, b \in \mathbb{R} \text{ and } i^2 = -1.}$

a is called the Real part of the complex number z and is denoted by $Re(z)$.

b is called the Imaginary part of the complex number z and is denoted by $Im(z)$.

Thus, $\boxed{z = Re(z) + i\,Im(z).}$

Algebra of Complex Numbers

1. Equality

Two complex numbers are equal if and only if their real and imaginary parts are equal.

i.e. $z_1 = x_1 + i\,y_1 \quad \text{and} \quad z_2 = x_2 + i\,y_2 \quad \text{are equal}$

$$\Leftrightarrow \quad x_1 = x_2 \quad \text{and} \quad y_1 = y_2$$

Algebra of Complex Numbers

2. Addition and Subtraction

For two complex numbers z_1 and z_2, the operations of addition and subtraction are defined by

<u>Addition</u>

$$
\begin{aligned}
z_1 + z_2 &= (x_1 + i\,y_1) + (x_2 + i\,y_2) \\
&= (x_1 + x_2) + i\,(y_1 + y_2)
\end{aligned}
$$

<u>Subtraction</u>

$$
\begin{aligned}
z_1 - z_2 &= (x_1 + i\,y_1) - (x_2 + i\,y_2) \\
&= (x_1 - x_2) + i\,(y_1 - y_2)
\end{aligned}
$$

Algebra of Complex Numbers

3. Multiplication

Multiplication of complex numbers is carried out in a similar way to expanding brackets, and then replacing i^2 by -1.

Multiplication

$$z_1 \cdot z_2 = (x_1 + i\, y_1) \cdot (x_2 + i\, y_2)$$

$$= x_1\, x_2 + i\, x_1\, y_2 + i\, x_2\, y_1 + i^2 y_1\, y_2$$

$$= (x_1\, x_2 - y_1\, y_2) + i\, (x_1\, y_2 + x_2\, y_1)$$

Algebra of Complex Numbers

4. Division

To define division of complex numbers, we first need to define the conjugate complex number.

Conjugate Complex Number

For $z = a + i\,b$, the conjugate complex number, denoted by $\bar{z}$, is defined by $\bar{z} = a - i\,b$.

Clearly, $\bar{\bar{z}} = \overline{a - i\,b} = a + i\,b = z \;\Rightarrow$

z and $\bar{z}$ are conjugates of each other.

Algebra of Complex Numbers

<u>Division</u>

$$\frac{z_1}{z_2} = \frac{x_1 + i\,y_1}{x_2 + i\,y_2} = \left(\frac{x_1 + i\,y_1}{x_2 + i\,y_2}\right) \cdot \left(\frac{x_2 - i\,y_2}{x_2 - i\,y_2}\right)$$

(Multiply and Divide by the Conjugate of the Denominator)

$$= \frac{x_1\,x_2 - i\,x_1\,y_2 + i\,x_2\,y_1 - i^2\,y_1\,y_2}{x_2^2 - i^2\,y_2^2}$$

$$= \frac{x_1\,x_2 - i\,x_1\,y_2 + i\,x_2\,y_1 - i^2\,y_1\,y_2}{x_2^2 + y_2^2} \qquad (\because\ i^2 = -1)$$

$$= \left(\frac{x_1\,x_2 + y_1\,y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2\,y_1 - x_1\,y_2}{x_2^2 + y_2^2}\right)$$

Square root of a complex number

Example: Find $\sqrt{5-12\,i}$

Suppose $\sqrt{5-12\,i} = a + i\,b$

$$\Rightarrow 5 - 12\,i = a^2 + 2\,i\,ab + i^2\,b^2$$

$$\Rightarrow 5 - 12\,i = (a^2 - b^2) + i\,(2ab)$$

Equating real and imaginary parts $\Rightarrow a^2 - b^2 = 5 \;\text{ and }\; 2ab = -12$

which upon solving gives: $a = 3,\; b = -2 \quad \textbf{or} \quad a = -3,\; b = 2$

Thus, $\sqrt{5-12\,i} \;=\; 3 - 2\,i \qquad \textbf{or} \qquad -3 + 2\,i$

Argand diagram

A complex number can be represented on the Argand diagram; where

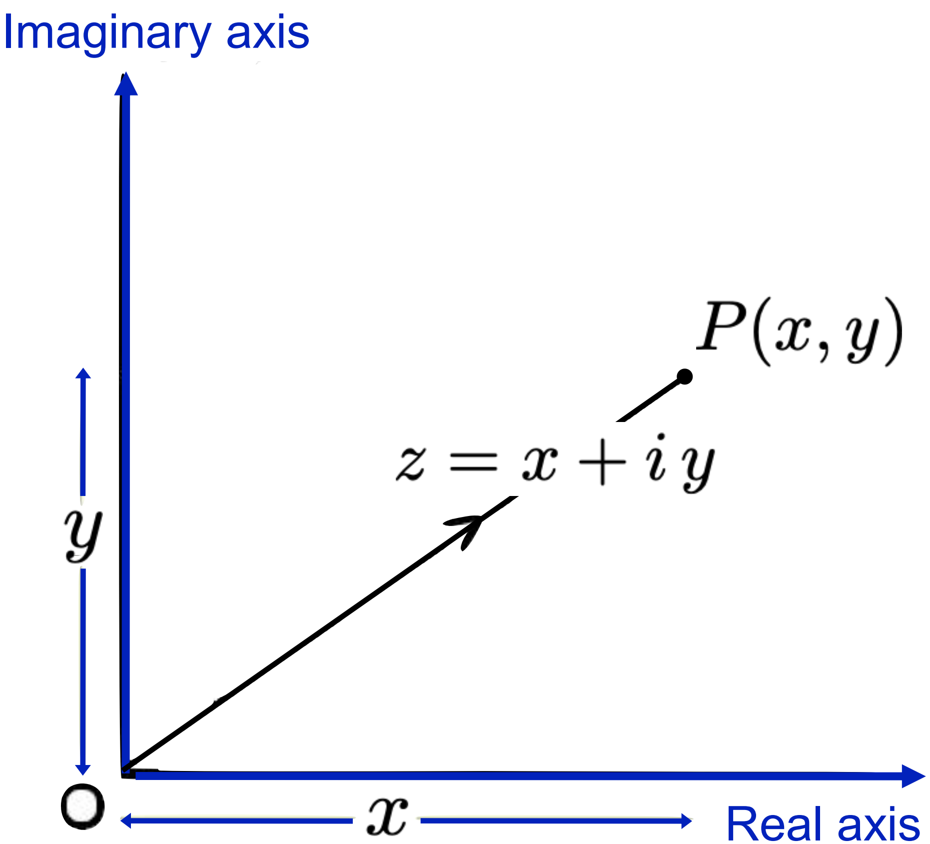
- Real numbers are represented on the X-axis (called real axis);

and

- Imaginary numbers are represented on the Y–axis (called imaginary axis).

Argand diagram

Thus, a general complex number $z = x + i\,y$ is represented by the vector $\overrightarrow{OP}$ where $P(x, y)$ is the point (x, y) in the XY-plane (called the Argand plane or complex plane).



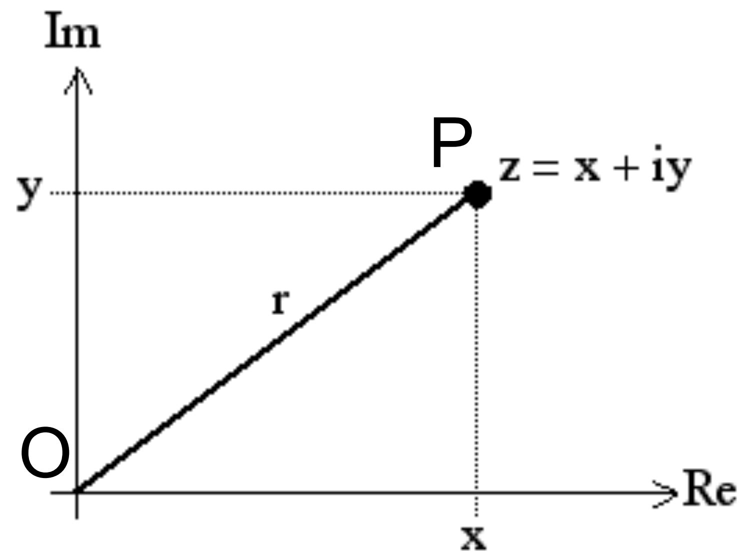
Modulus of a complex number

The length of $\overline{OP}$ is called the modulus of the complex number $z = x + i\,y$ and is denoted by:

$$r = |\,z\,| = |\,x + i\,y\,|$$

$$= \sqrt{x^2 + y^2}$$

i.e. $|\,z\,| = \sqrt{[Re(z)]^2 + [Im(z)]^2}$



Properties of Modulus

1) $$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

2) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

3) $$|z_1 + z_2| \leq |z_1| + |z_2|$$

4) $$|z_1 - z_2| \geq |z_1| \sim |z_2|$$ ($\sim$ denotes positive difference)

Worked Examples

1) Find $|-4+7i|$

$$|-4+7i| = \sqrt{(-4)^2+(7)^2} = \sqrt{65}$$

2) Find $\left|\dfrac{2-3i}{4+\sqrt{2}i}\right|$

$$\left|\frac{2-3i}{4+\sqrt{2}i}\right| = \frac{|2-3i|}{|4+\sqrt{2}i|} = \frac{\sqrt{2^2+(-3)^2}}{\sqrt{4^2+(\sqrt{2})^2}} = \sqrt{\frac{13}{18}}$$

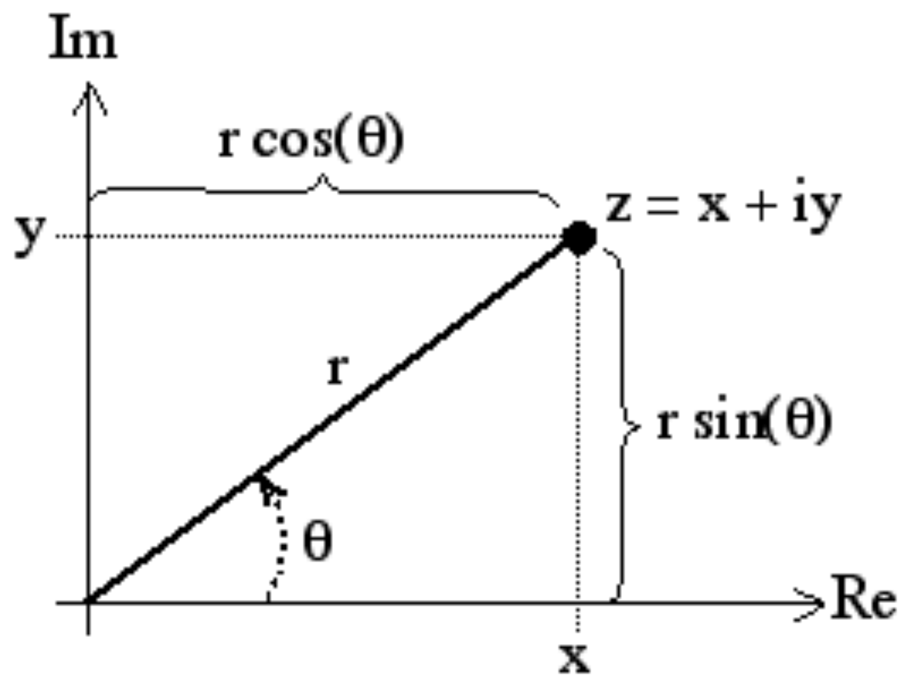
Polar form of a complex number

There is another way of representing a complex number using Polar coordinates (r, θ), and is called the Polar form of a complex number.

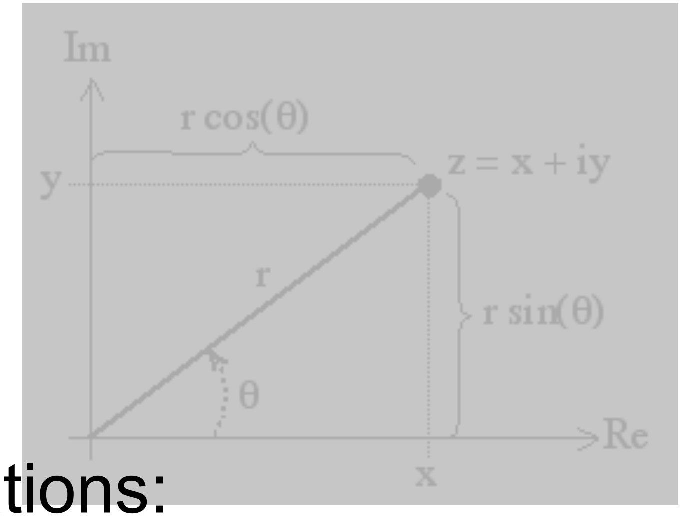
Suppose, the complex number $z = x + i\,y$ is represented in Cartesian form on the Argand diagram, by the point $P(x, y)$.

Polar form of a complex number

The same point P can be located by using its distance r from the origin O, and the angle θ made by the line $\overrightarrow{OP}$ with the real axis (X-axis).



Polar form of a complex number



Thus, $P(x, y)$ becomes the point

$$P(r, \theta) \equiv (r\cos\theta, r\sin\theta)$$

where, $r = \sqrt{x^2 + y^2}$

and θ is found from the set of equations:

$$\cos\theta = \frac{x}{r} \quad ; \quad \sin\theta = \frac{y}{r}.$$

Thus, $z = x + i\,y = r\cos\theta + i\,r\sin\theta$

Argument of a complex number

The angle θ is called the argument of the complex number

$$z = x + i\,y = r\cos\theta + i\,r\sin\theta$$

It is written as $Arg(x + i\,y)$, and obtained from the set of equations: $\cos\theta = \dfrac{x}{r} \quad ; \quad \sin\theta = \dfrac{y}{r}$.

As there are infinite number of angles that satisfy the above set of equations, the definition needs to be tightened so that everyone gets the same answer.

(Principal) Argument of a complex number

We denote the principal value of the argument by

$$\arg(z) = \theta \qquad \text{if} \qquad -\pi < \theta \leq \pi.$$

Example

Express the following complex numbers in polar form and show them on the Argand diagram:

$(i)\quad z_1 = 1 + i$

$(ii)\quad z_2 = -1 + i$

$(iii)\quad z_3 = -1 - i$

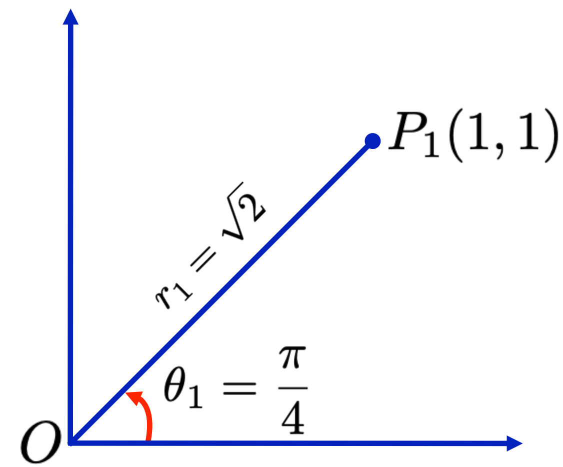
$(iv)\quad z_4 = 1 - i$

(Principal) Argument of a complex number

$$(i) \quad z_1 = 1 + i \;\equiv\; x + i\,y \;\Rightarrow\; x = 1,\; y = 1$$

$$\therefore\; r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{aligned} \cos\theta &= \frac{x}{r} = \frac{1}{\sqrt{2}} \\ \sin\theta &= \frac{y}{r} = \frac{1}{\sqrt{2}} \end{aligned}\right\} \Rightarrow\; \theta = \frac{\pi}{4}$$



$$\boxed{-\pi < \theta \leq \pi}$$

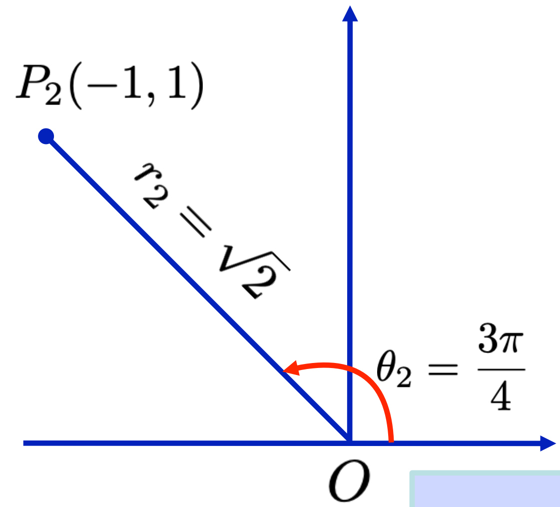
Thus, $z_1 = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right]$

(Principal) Argument of a complex number

$$(ii) \quad z_2 = -1 + i \equiv x + i\,y \Rightarrow x = -1,\; y = 1$$

$$\therefore\; r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{aligned} \cos\theta &= \frac{x}{r} = \frac{-1}{\sqrt{2}} \\ \sin\theta &= \frac{y}{r} = \frac{1}{\sqrt{2}} \end{aligned}\right\} \Rightarrow \theta = \frac{3\pi}{4}$$



$$\boxed{-\pi < \theta \leq \pi}$$

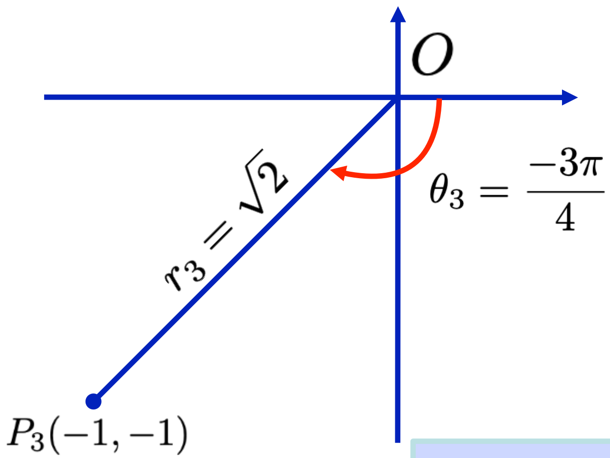
Thus, $z_2 = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\,\sin\left(\frac{3\pi}{4}\right)\right]$

(Principal) Argument of a complex number

$$(iii) \quad z_3 = -1 - i \equiv x + iy \Rightarrow x = -1, \; y = -1$$

$$\therefore \; r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{aligned} \cos\theta &= \frac{x}{r} = \frac{-1}{\sqrt{2}} \\ \sin\theta &= \frac{y}{r} = \frac{-1}{\sqrt{2}} \end{aligned}\right\} \Rightarrow \theta = \frac{-3\pi}{4}$$



$$-\pi < \theta \leq \pi$$

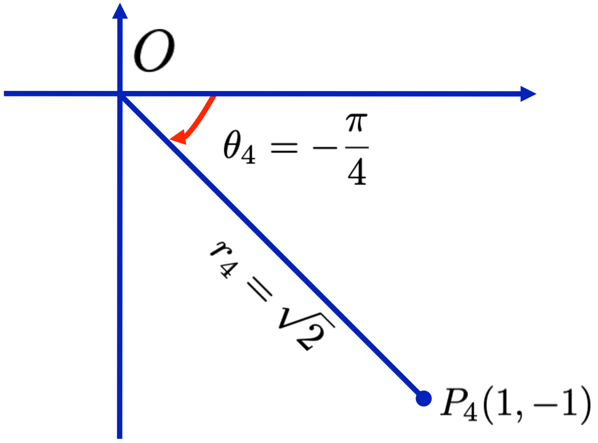
Thus, $z_3 = \sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i\,\sin\left(\frac{-3\pi}{4}\right)\right]$

(Principal) Argument of a complex number

$$(iv) \quad z_4 = 1 - i \ \equiv \ x + i\,y \ \Rightarrow \ x = 1, \ y = -1$$

$$\therefore \ r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{aligned} \cos\theta &= \frac{x}{r} = \frac{1}{\sqrt{2}} \\ \sin\theta &= \frac{y}{r} = \frac{-1}{\sqrt{2}} \end{aligned}\right\} \Rightarrow \ \theta = -\frac{\pi}{4}$$



$$-\pi < \theta \leq \pi$$

Thus, $z_4 = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\,\sin\left(-\frac{\pi}{4}\right)\right]$

Finding arg(z) using a calculator

Quadrant	First	Second	Third	Fourth
Interval	$\left(0, \frac{\pi}{2}\right)$	$(0, \pi)$	$\left(\pi, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
Signs of x and y	$x > 0,\ y > 0$	$x < 0,\ y > 0$	$x < 0,\ y < 0$	$x > 0,\ y < 0$
Principal argument $\theta = \arg(z)$	$\tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$\pi - \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$-\pi + \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$-\tan^{-1}\left\lvert \frac{y}{x} \right\rvert$

Example

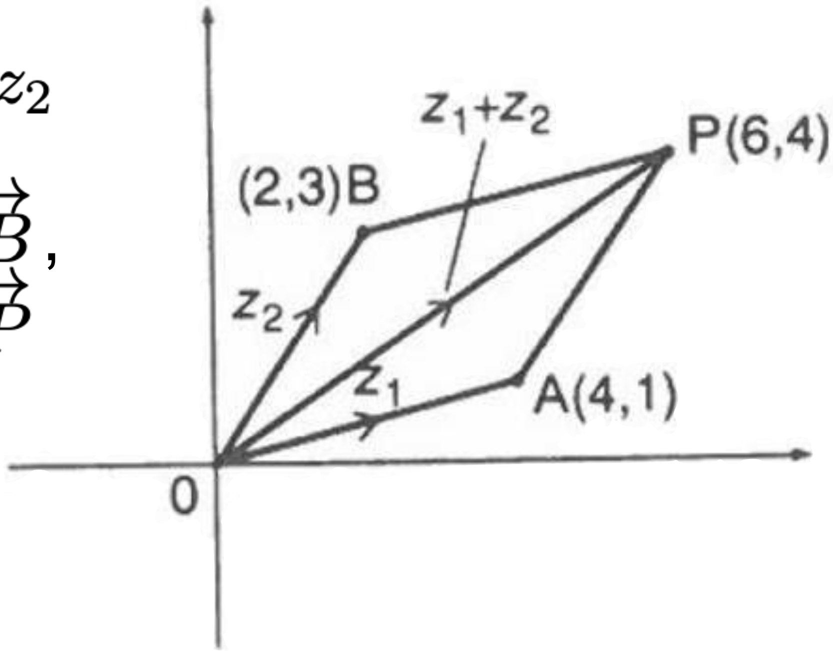
Express the complex number $z = -2 + 5\,i$ in polar form

$z = r\,(\cos\theta + i\,\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$.

Algebraic operations on Argand diagram

1. Addition

If the complex numbers z_1 and z_2 are shown by sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then $z_1 + z_2$ is the diagonal $\overrightarrow{OP}$ of the parallelogram $OAPB$.



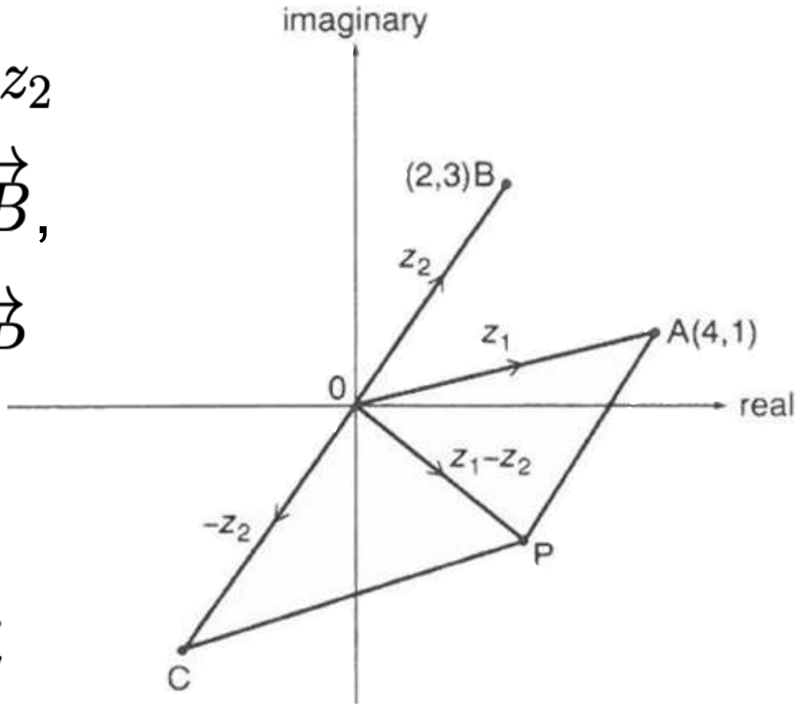
e.g. $z_1 = 4 + i$ and $z_2 = 2 + 3\,i$

then, $z = z_1 + z_2 = 6 + 4\,i$ is the point $P(6,4)$.

Algebraic operations on Argand diagram

2. Subtraction

If the complex numbers z_1 and z_2 are shown by sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then $z_1 - z_2$ is the diagonal $\overrightarrow{OP}$ of the parallelogram $OAPC$.



e.g. $z_1 = 4 + i$ and $z_2 = 2 + 3\,i$

then, $z = z_1 - z_2 = 2 - 2\,i$ is the point $P(2, -2)$.

Algebraic operations on Argand diagram

3. Multiplication

To show the product of complex numbers on the Argand plane, it is useful to first represent them in polar form.

Let $z_1 = r_1\,(\cos\theta_1 + i\,\sin\theta_1)$ and $z_2 = r_2\,(\cos\theta_2 + i\,\sin\theta_2)$

be two complex numbers in polar form.

Then, $z_1 \cdot z_2 = r_1\,(\cos\theta_1 + i\,\sin\theta_1) \cdot r_2\,(\cos\theta_2 + i\,\sin\theta_2)$

Algebraic operations on Argand diagram

$$\therefore\ z_1 \cdot z_2 = r_1 \cdot r_2 \left(\underline{\cos\theta_1\ \cos\theta_2} + \underline{i\ \cos\theta_1\ \sin\theta_2}\right.$$
$$\left. + \underline{i\ \sin\theta_1\ \cos\theta_2} + \underline{i^2\ \sin\theta_1\ \sin\theta_2}\right)$$

$$= r_1 \cdot r_2 \left(\underline{\cos\theta_1\ \cos\theta_2 - \sin\theta_1\ \sin\theta_2}\right.$$
$$\left. \underline{+ i\left(\sin\theta_1\ \cos\theta_2 + \cos\theta_1\ \sin\theta_2\right)}\right)$$

$$= r_1 \cdot r_2\ \left[\cos\left(\theta_1 + \theta_2\right) + i\ \sin\left(\theta_1 + \theta_2\right)\right]$$

Thus,

$$\boxed{z_1 \cdot z_2 \equiv R\ \left(\cos\theta + i\ \sin\theta\right) \text{ where } R = r_1 \cdot r_2\,, \quad \theta = \theta_1 + \theta_2}$$

Algebraic operations on Argand diagram

e.g. $z_1 = 2 + 2\,i$

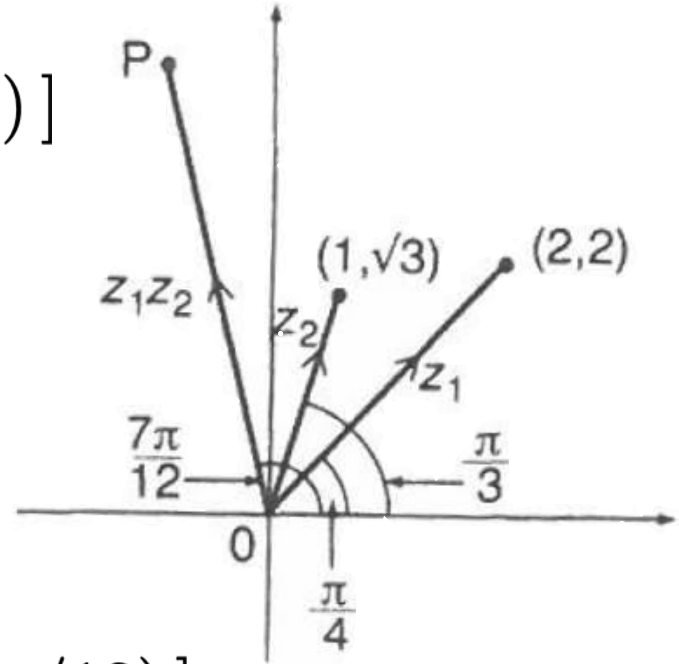
$$= 2\sqrt{2}\,\left[\cos(\pi/4) + i\,\sin(\pi/4)\right]$$

$$z_2 = 1 + \sqrt{3}\,i$$

$$= 2\,\left[\cos(\pi/3) + i\,\sin(\pi/3)\right]$$

then, $z = z_1 \cdot z_2$

$$= 4\sqrt{2}\,\left[\cos(7\pi/12) + i\,\sin(7\pi/12)\right]$$



Algebraic operations on Argand diagram

4. Division

In a similar way, it can be shown that:

$$\frac{z_1}{z_2} = \frac{r_1\,(\cos\theta_1 + i\,\sin\theta_1)}{r_2\,(\cos\theta_2 + i\,\sin\theta_2)} = \frac{r_1}{r_2}\,[\cos(\theta_1 - \theta_2) + i\,\sin(\theta_1 - \theta_2)]$$

Thus,

$\dfrac{z_1}{z_2} \equiv R\,(\cos\theta + i\,\sin\theta)$ where $R = \dfrac{r_1}{r_2}$ and $\theta = \theta_1 - \theta_2$.

Suggested Reading

College Algebra by J. W. Coburn

Chapter 1 (page 104 – 112)

Foundation Algebra by P. Gajjar

Chapter 11