COMP2054 Tutorial Session 7: DP, Change Giving and MSTs

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Session outcomes

- Understand how dynamic programming can be used to solve (min-coin) change giving problem.
- Know what an MST is.
- Calculate the MST from a graph.

Dynamic Programming

(Min-Coins) Change Giving

Dynamic Programming

A technique used to solve a larger problem by solving smaller "decomposed" subproblems and building-up to the original problem.

Example: Subset Sum (decision problem) – given a multiset of integers, x[0] … x[n-1], is there a subset that sums to a given value, K?

- Smallest subproblem has a multiset of cardinality zero.
- Build up to the original problem by adding the next element from the multiset until all elements are added (or the target value is confirmed).

Subset Sum Example

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Given $X = \{1,2,2,5\}$ and $K = 4$:

Value/index	0	1	2	3	4
Possible?	0	0	0	0	0

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Given $X = \{1,2,2,5\}$ and $K = 4$: **Considering {}**

Value/index	0	1	2	3	4
Possible?	0	0	0	0	0

Value/index	0	1	2	3	4
Possible?	1	0	0	0	0

Subset Sum Example

Given $X = \{1, 2, 2, 5\}$ and $K = 4$:

Value/index	0	1	2	3	4
Possible?	1	0	0	0	0

Considering $\{1\}$

Value/index	0	1	2	3	4
Possible?	1	1	0	0	0

Subset Sum Example

Given $X = \{1,2,2,5\}$ and $K = 4$:

Value/index	0	1	2	3	4
Possible?	1	0	0	0	0

Value/index	0	1	2	3	4
Possible?	1	1	0	0	0

Considering $\{1,2\}$

Value/index	0	1	2	3	4
Possible?	1	1	1	1	0

Subset Sum Example

Given $X = \{1,2,2,5\}$ and $K = 4$:

Value/index	0	1	2	3	4
Possible?	1	0	0	0	0

⇓

Value/index	0	1	2	3	4
Possible?	1	1	0	0	0

⇓

Value/index	0	1	2	3	4
Possible?	1	1	1	1	0

⇓

Value/index	0	1	2	3	4
Possible?	1	1	1	1	0

Considering `{1,2,2}`

2 + x[2] == 4; return success;

DP for Min-Coins-Change-Giving

Given a multiset of coins, S, and a target value, K, find the subset of S which sums to K **with the smallest cardinality**.

```
input: x[0], …, x[n-1] and K
init: Y[0] = 0, Y[m] = -1 (∀m > 0)
for(i=0; i<n; i++) {
  for(m=K-x[i]; m>=0; m--) {
    if(Y[m] >= 0) {
      if(Y[m+x[i]] == -1) {
        Y[m+x[i]] = Y[m]+1
      } else {
        Y[m+x[i]] = min(Y[m+x[i]], Y[m]+1)
      }
    }
  }
}
```

DP for Min-Coins-Change-Giving

Given a multiset of coins, S, and a target value, K, find the subset of S which sums to K **with the smallest cardinality**.

Basic idea:

- For each element in S.
- Scan backwards over an array Y which tells us whether each value was possible using some number of coins for the subset already considered.
- Introducing a coin of value $x[i]$ means we can give $m + x[i]$ change using $\min(Y[m + x[i]], Y[m] + 1)$ coins.

```
input: x[0], …, x[n-1] and K
init: Y[0] = 0, Y[m] = -1 (∀m > 0)
for(i=0; i<n; i++) {
  for(m=K-x[i]; m>=0; m--) {
    if(Y[m] >= 0) {
      if(Y[m+x[i]] == -1) {
        Y[m+x[i]] = Y[m]+1
      } else {
        Y[m+x[i]] = min(Y[m+x[i]], Y[m]+1)
      }
    }
  }
}
```

DP for Min-Coins-Change-Giving

- Given $S = \{\textcolor{red}{\mathbf{2}}, 3,5\}$ and $K = 5$:
- Initialise $Y = [0, -1, -1, -1, -1]$
- First iteration ($\textcolor{red}{\boldsymbol{i = 0}}$):

Do nothing for m=3 to m=1 as Y[m]<0

0	1	2	3	4	5
0	-1	-1	-1	-1	-1

m=3

```
input: x[0], …, x[n-1] and K
init: Y[0] = 0, Y[m] = -1 (∀m > 0)
for(i=0; i<n; i++) {
  for(m=K-x[i]; m>=0; m--) {
    if(Y[m] >= 0) {
      if(Y[m+x[i]] == -1) {
        Y[m+x[i]] = Y[m]+1
      } else {
        Y[m+x[i]] = min(Y[m+x[i]], Y[m]+1)
      }
    }
  }
}
```

DP for Min-Coins-Change-Giving

- Given $S = \{\mathbf{2}, 3,5\}$ and $K = 5$:
- Initialise $Y = [0, -1, -1, -1, -1]$
- First iteration ($\boldsymbol{i = 0}$):

Coins for sum “0+2” = 0+1 = 1

0	1	2	3	4	5
0	-1	~~-1~~ **1**	-1	-1	-1

m=0

```
input: x[0], …, x[n-1] and K
init: Y[0] = 0, Y[m] = -1 (∀m > 0)
for(i=0; i<n; i++) {
  for(m=K-x[i]; m>=0; m--) {
    if(Y[m] >= 0) {
      if(Y[m+x[i]] == -1) {
        Y[m+x[i]] = Y[m]+1
      } else {
        Y[m+x[i]] = min(Y[m+x[i]], Y[m]+1)
      }
    }
  }
}
```

DP for Min-Coins-Change-Giving

- Given $S = \{2, \mathbf{3}, 5\}$ and $K = 5$:
- Second iteration ($\boldsymbol{i} = \mathbf{1}$):

Coins for sum "2+3" = 1+1 = 2

0	1	2	3	4	5
0	-1	1	-1	-1	~~-1~~ **2**

m=2

```
input: x[0], …, x[n-1] and K
init: Y[0] = 0, Y[m] = -1 (∀m > 0)
for(i=0; i<n; i++) {
  for(m=K-x[i]; m>=0; m--) {
    if(Y[m] >= 0) {
      if(Y[m+x[i]] == -1) {
        Y[m+x[i]] = Y[m]+1
      } else {
        Y[m+x[i]] = min(Y[m+x[i]], Y[m]+1)
      }
    }
  }
}
```

DP for Min-Coins-Change-Giving

- Given $S = \{2, \mathbf{3}, 5\}$ and $K = 5$:
- Second iteration ($\boldsymbol{i = 1}$):

Coins for sum “0+3” = 0+1 = 1

0	1	2	3	4	5
0	-1	1	~~-1~~ **1**	-1	~~-1~~ **2**

m=0

We found that we **can** make change of value 5 using 2 coins, but can we stop here? (Min coins)

```
input: x[0], …, x[n-1] and K
init: Y[0] = 0, Y[m] = -1 (∀m > 0)
for(i=0; i<n; i++) {
  for(m=K-x[i]; m>=0; m--) {
    if(Y[m] >= 0) {
      if(Y[m+x[i]] == -1) {
        Y[m+x[i]] = Y[m]+1
      } else {
        Y[m+x[i]] = min(Y[m+x[i]], Y[m]+1)
      }
    }
  }
}
```

DP for Min-Coins-Change-Giving

- Given $S = \{2,3,\mathbf{5}\}$ and $K = 5$:
- Second iteration ($\boldsymbol{i = 2}$):

Coins for sum "0+5"
= min(Y[5], Y[0] + 1)
= min(2, 1)
= 1

0	1	2	3	4	5
0	-1	1	1	-1	~~2~~ **1**

m=0

```
input: x[0], …, x[n-1] and K
init: Y[0] = 0, Y[m] = -1 (∀m > 0)
for(i=0; i<n; i++) {
  for(m=K-x[i]; m>=0; m--) {
    if(Y[m] >= 0) {
      if(Y[m+x[i]] == -1) {
        Y[m+x[i]] = Y[m]+1
      } else {
        Y[m+x[i]] = min(Y[m+x[i]], Y[m]+1)
      }
    }
  }
}
```

Questions

Q1. Given the set of coins $\{1,1,3,5,9\}$, what is the minimum number of coins required to give change of 10?

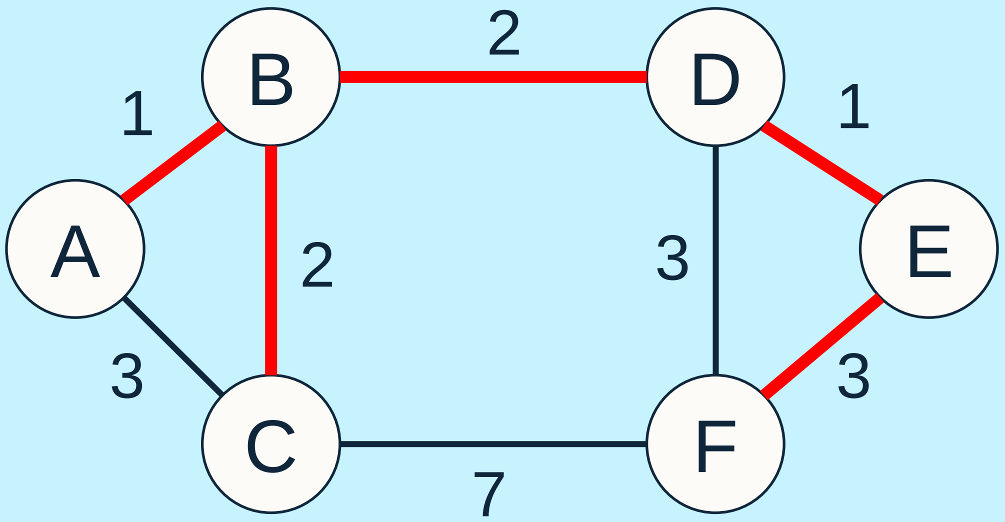
Q2. Given the set of coins $\{2,2,2,5\}$, what is the minimum number of coins required to give change of 6?

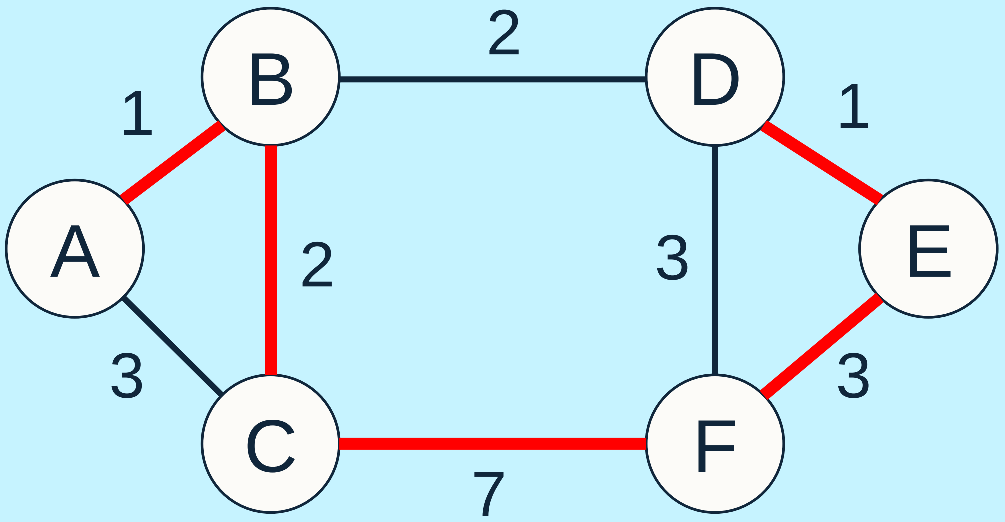
Q3. Does the DP algorithm still work if we scan the array, Y, forwards instead of backwards? If yes, explain; if no, give a counter example.

Minimum Spanning Trees

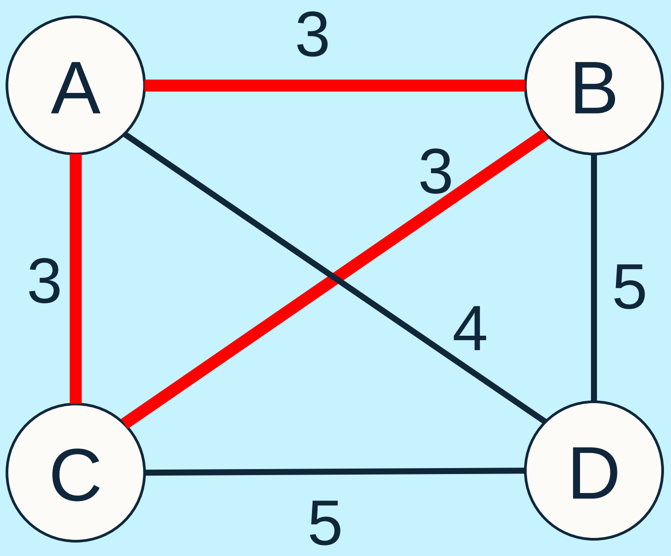
Prim's Algorithm: An optimal greedy approach

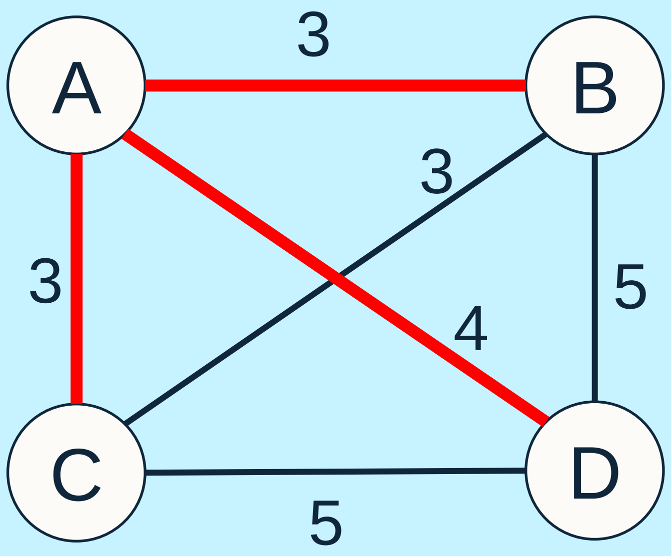
Which of the following is an MST?





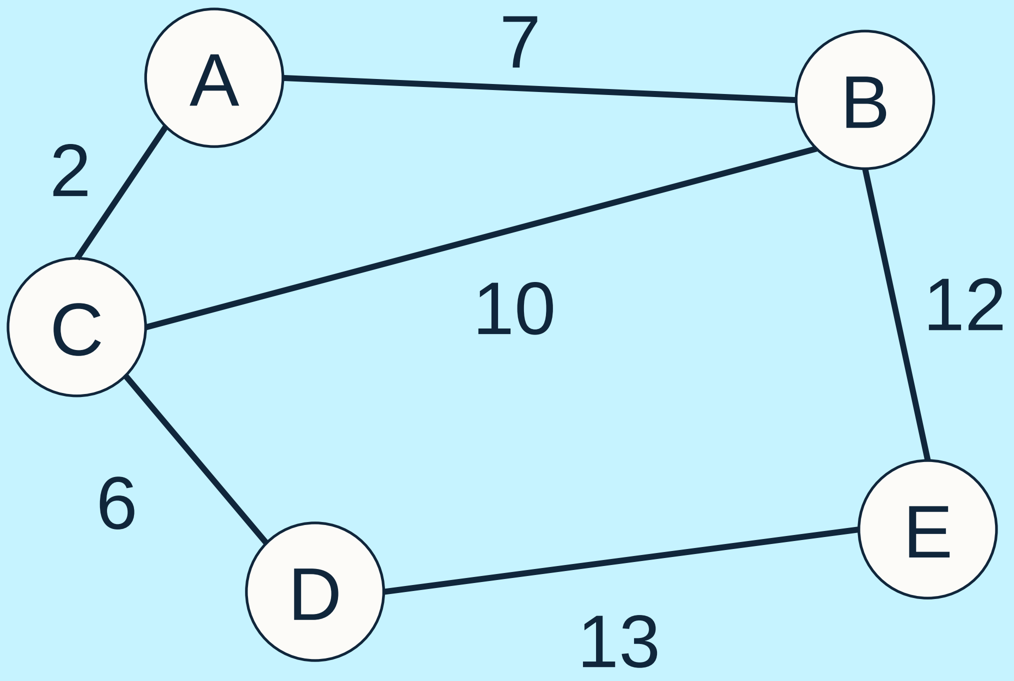
Which of the following is an MST?





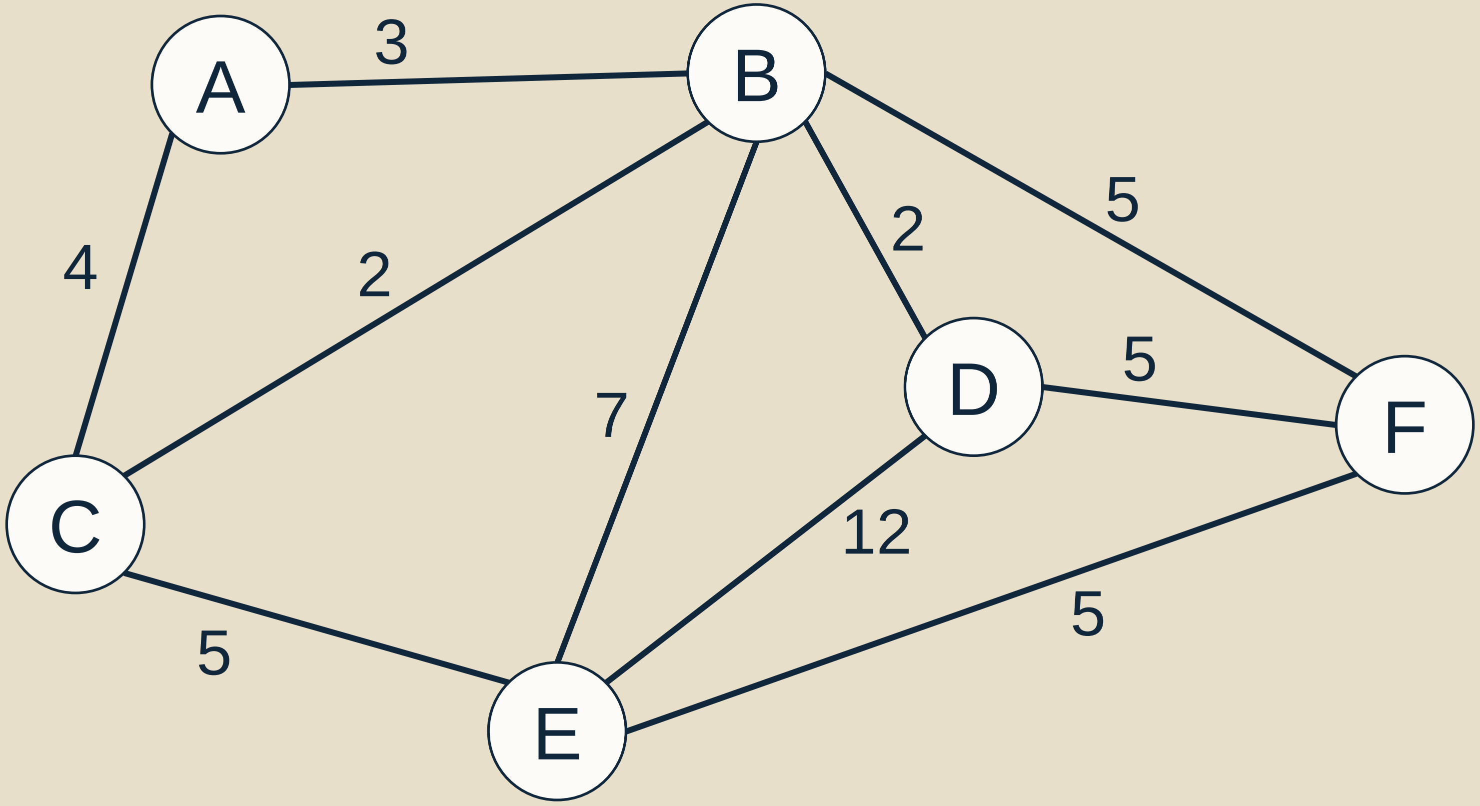
MST: Prim’s Algorithm

Given the below graph and starting node A, create an associated minimum spanning tree. You should show all steps and give the final total cost of the MST.



MST Question

Given the below graph and **starting node D**, create an associated minimum spanning tree. You should show all steps and give the final total cost of the MST.



Thank you

Next week – Floyd-Warshall Algorithm