1 (a) $$y = 2x^3 + \sqrt{x} + x + \frac{2}{x}$$

$$\therefore \frac{dy}{dx} = 6x^2 + \frac{1}{2\sqrt{x}} + 1 - \frac{2}{x^2}.$$

(b) i) $$y = \frac{5x^2 - 10x + 9}{(x-1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(x-1)^{\cancel{2}} \cdot (10x - 10) - (5x^2 - 10x + 9) \cdot 2\cancel{(x-1)}}{(x-1)^{\cancel{4}}}.$$

$$= \frac{10(x-1)^2 - 2(5x^2 - 10x + 9)}{(x-1)^3}$$

$$= \frac{\cancel{10x^2} - \cancel{20x} + 10 - \cancel{10x^2} + \cancel{20x} - 18}{(x-1)^3}$$

$$= \frac{-8}{(x-1)^3}.$$

(ii) $$y = \ln\left(\frac{x-1}{x+1}\right)$$

$$= \ln(x-1) - \ln(x+1)$$

$$\therefore \frac{dy}{dx} = \frac{1}{x-1} - \frac{1}{x+1}$$

$$= \frac{x+1-x+1}{x^2-1}$$

$$= \frac{2}{(x-1)^2}.$$

iii) $y = e^{kx}$

$$\therefore \frac{dy}{dx} = k e^{kx}$$

$$\frac{d^2y}{dx^2} = k^2 e^{kx}$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = k^n \cdot e^{kx}.$$

(c)

(i) $x = 2\cos t + \sin 2t$

$$\therefore \frac{dx}{dt} = -2\sin t + 2\cos 2t$$

$y = \cos t - 2\sin t$

$$\therefore \frac{dy}{dt} = -\sin t - 2\cos t$$

(ii)

$$\therefore \left.\frac{dy}{dx}\right|_{t=\pi/4} = \left.\frac{-(\sin t + 2\cos t)}{-2\sin t + 2\cos 2t}\right|_{t=\pi/4}$$

$$= \frac{-\left(\frac{1}{\sqrt{2}} + 2\cdot\frac{1}{\sqrt{2}}\right)}{-\frac{2}{\sqrt{2}} + 2.(0)}$$

$$= \frac{-3}{-2} = \frac{3}{2}.$$

2(a) $x^2 + 3xy^2 - y^3 = 9$

$$\therefore\ 2x + 6xy\frac{dy}{dx} + 3y^2 - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{2x + 3y^2}{3y^2 - 6xy}$$

$$\therefore\ \left.\frac{dy}{dx}\right|_{(2,1)} = \frac{4+3}{3-12} = -\frac{7}{9}.$$

(b) $y = (\sin x)^{\sin x}$

$$\therefore\ \ln y = \sin x \cdot (\ln(\sin x))$$

$$\therefore\ \frac{1}{y}\frac{dy}{dx} = \cancel{\sin x}\cdot\frac{1}{\cancel{\sin x}}\cdot\cos x + \ln(\sin x)\cdot\cos x$$

$$\therefore\ \frac{dy}{dx} = y\left[\cos x + \cos x\cdot\ln(\sin x)\right]$$

$$\therefore\ \frac{dy}{dx} = (\sin x)^{x}\cdot\cos x\,(1 + \ln(\sin x)).$$

(c) $y = x^3 - 3x^2 + 2x - 1$

$$\therefore\ \frac{dy}{dx} = 3x^2 - 6x + 2$$

$$\therefore\ \left.\frac{dy}{dx}\right|_{(3,5)} = 27 - 18 + 2 = 11 = m.$$

i) $\therefore$ Tgt line is

$$y-5 = 11(x-3)$$

$$\Rightarrow \underline{11x - y - 28 = 0}.$$

ii) Normal line is

$$y-5 = -\frac{1}{11}(x-3)$$

$$\Rightarrow 11y - 55 = -x+3$$

$$\Rightarrow \underline{x + 11y - 58 = 0}$$ (proved).

3 (a)

$$A = \pi r^2$$

$$\therefore \frac{dA}{dt} = 2\pi . r \frac{dr}{dt}$$

$$\therefore \left.\frac{dA}{dt}\right|_{r=3} = 2\pi(3) . \left.\frac{dr}{dt}\right|_{r=3}$$

$$= 6\pi(5)$$

$$= 30\pi \quad cm^2/sec.$$

(b). (i) $f'(x) = 0 \Rightarrow 1 - \frac{9}{x^2} = 0$

$$\Rightarrow x^2 = 9$$

$$\Rightarrow x = \pm 3.$$

ii) $f''(x) = \frac{18}{x^3} \Rightarrow f''(x)\Big|_{x=3} > 0$

$\therefore$ $\therefore x = 3$ is a point of min. value.

OR $(3, 6)$

& $f''(x)\Big|_{x=-3} < 0 \Rightarrow x = -3$ or $(-3, -6)$ is a point of max. value.

(c). $f(x) = x^4 + x^2 - 80$

i) $\Rightarrow f'(x) = 4x^3 + 2x$

$$\therefore x_{n+1} = x_n - \frac{(x_n^4 + x_n^2 - 80)}{4x_n^3 + 2x_n}$$

$$= \frac{4x_n^4 + 2x_n^2 - x_n^4 - x_n^2 + 80}{4x_n^3 + 2x_n}$$

$$\therefore x_{n+1} = \frac{3x_n^4 + x_n^2 + 80}{4x_n^3 + 2x_n}.$$

ii)

n	x_n
0	3
1	2.9122
2	2.9083
3	2.9083

$\therefore x^* = 2.9083$ (4. d.p.)

(d) $f'(x) = 5x^4 + 3x^2 + 1$

$> 0 \quad \forall\, x \in R$

$\therefore f$ is always increasing.

4 (a) $f(x) = \ln(1+x) \Rightarrow f(0) = 0$

i) $f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = \frac{1}{1} = 1$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$

$f^{(iv)}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{(iv)}(0) = -6.$

$\therefore \ln(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

ii) $\therefore \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

$\therefore \ln(1+x) - \ln(1-x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$

$\therefore \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

$\Rightarrow \ln\sqrt{\frac{1+x}{1-x}} = x + x^3/3 + x^5/5 + \cdots$

4 (b) $f(x) = x \cdot \cos x$

Let $g(x) = \cos x \Rightarrow g(0) = 1$

$\therefore g'(x) = -\sin x \Rightarrow g'(0) = 0$

$g''(x) = -\cos x \Rightarrow g''(0) = -1$

$g'''(x) = \sin x \Rightarrow g'''(0) = 0$

$g^{(iv)}(x) = \cos x \Rightarrow g^{(iv)}(0) = 1$

$$\therefore g(x) = \cos x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots$$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\therefore f(x) = x \cdot \cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots$$

(c) Given $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\therefore e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$

$$\& \quad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\therefore e^{2x} - e^{-x} = 3x + \frac{3x^2}{2!} + \frac{9x^3}{3!} + \cdots$$

$$\therefore e^{2x} - e^{-x} \approx 3x + \frac{3}{2}x^2.$$

5 (a)

i)

$$I = \int \frac{1}{x}\, dx + \int \tan^2 x\, dx$$

$$= \ln|x| + \int (\sec^2 x - 1)\, dx$$

$$= \ln|x| + \tan x - x + C$$

ii)

$$\int \frac{x^2}{1+x^2}\, dx$$

$$= \int \frac{x^2+1-1}{x^2+1}\, dx$$

$$= \int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= x - \tan^{-1} x + C.$$

iii)

$$I = \int \frac{e^x\, dx}{(e^x)^2 + 9}$$

Let $e^x = t \Rightarrow e^x\, dx = dt$

$$\therefore I = \int \frac{dt}{t^2+9}$$

$$= \frac{1}{3} \tan^{-1}\left(t/3\right) + C$$

$$= \frac{1}{3} \tan^{-1}\left(\frac{e^x}{3}\right) + C.$$

(b) i) Let $f(x) = t^2$

$\therefore f'(x)\,dx = 2t\,dt$

$\therefore \int \frac{f'(x)}{\sqrt{f(x)}}\,dx = \int \frac{2t\,dt}{t}$

$= \int 2\,dt$

$= 2t + C$

$= 2\sqrt{f(x)} + C$

ii) $I = \int \frac{x}{\sqrt{x^2+1}}\,dx$

$= \frac{1}{2} \int \frac{2x}{\sqrt{x^2+1}}\,dx$

$= \frac{1}{2} \left(2\sqrt{x^2+1}\right) + C$

$= \sqrt{x^2+1} + C.$

c) i) $I = \int \cos^4 x \cdot \sin^3 x\,dx$

$= \int \cos^4 x \cdot \sin^2 x \cdot \sin x \cdot dx$

Let $\cos x = t \Rightarrow \sin x\,dx = -dt$

$= \int t^4 (1-t^2)(-dt)$

$= \int (t^6 - t^4)\,dt = \frac{t^7}{7} - \frac{t^5}{5} + C = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$

ii) $$I = \int \cos 4x \cdot \sin 3x \, dx$$

$$= \frac{1}{2} \int 2 \cos 4x \sin 3x \, dx$$

$$= \frac{1}{2} \int (\sin 7x - \sin x) \, dx$$

$$= \frac{1}{2} \left[\frac{-\cos 7x}{7} - (-\cos x) \right] + C$$

$$= \frac{1}{2} \left[\cos x - \frac{\cos 7x}{7} \right] + C.$$

6 (a) $$\frac{9}{(2x-1)(x+4)} = \frac{A}{2x-1} + \frac{B}{x+4}$$

$$\therefore \quad A(x+4) + B(2x-1) = 9$$

$x = -4 \Rightarrow B(-9) = 9 \Rightarrow \underline{B = -1}$

$x = \frac{1}{2} \Rightarrow A\left(\frac{9}{2}\right) = 9 \Rightarrow \underline{A = 2}$

i) $$\therefore \int f(x)\, dx = \int \frac{2}{2x-1} dx - \int \frac{1}{x+4} dx$$

$$= \frac{\cancel{2} \ln|2x-1|}{\cancel{2}} - \ln|x+4| + C$$

$$= \ln\left|\frac{2x-1}{x+4}\right| + C$$

ii) $$\int_1^2 f(x)\, dx = \left[\ln\left|\frac{2x-1}{x+4}\right| \right]_1^2 = \ln\left(\frac{3}{6}\right) - \ln\left(\frac{1}{5}\right) = \ln\left(\frac{5}{2}\right).$$

(b). $I = \int_0^{\pi/2} x \sin x \, dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x \Rightarrow v = \int \sin x \, dx = -\cos x$

$\therefore I = \left[-x \cos x\right]_0^{\pi/2} - \int_0^{\pi/2} (-\cos x) \cdot (1) \, dx$

$= 0 + \left[\sin x\right]_0^{\pi/2}$

$= 1.$ (proved).

c) $\text{Area} = \int_0^{\pi/6} \cos x \cdot \sqrt{1 - 2\sin x} \, dx$

Let $1 - 2\sin x = t^2$

$\Rightarrow -2\cos x \, dx = 2t \, dt$

$\Rightarrow \cos x \, dx = -t \, dt$

x	0	$\pi/6$
t	1	0

$\therefore \text{Area} = \int_1^0 \sqrt{t^2} \, (-t \, dt)$

$= -\left[\frac{t^3}{3}\right]_1^0 = \frac{1}{3}.$ Answer.

7 (a) $$V = \pi \int_0^{\ln 3} (e^x)^2 \, dx$$

$$= \pi \left[\frac{e^{2x}}{2} \right]_0^{\ln 3}$$

$$= \frac{\pi}{2} \left(e^{2\ln 3} - e^0 \right)$$

$$= \frac{\pi}{2} \left(e^{\ln 9} - 1 \right)$$

$$= \frac{\pi}{2} (9-1) = 4\pi. \quad \text{cubic units.}$$

(b) $$\int_0^2 f(x) \, dx = \int_0^1 x \, dx + \int_1^2 x^2 \, dx$$

$$= \left[\frac{x^2}{2} \right]_0^1 + \left[\frac{x^3}{3} \right]_1^2$$

$$= \frac{1}{2} + \frac{8}{3} - \frac{1}{3}$$

$$= \frac{3+16-2}{6} = \frac{17}{6}.$$

(c) $$I = \int_1^2 \frac{1}{\sqrt{4x - x^2}} \, dx$$

$$= \int_1^2 \frac{1}{\sqrt{-(x^2-4x+4)+4}} \, dx$$

$$= \int_1^2 \frac{1}{\sqrt{2^2-(x-2)^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x-2}{2}\right)\right]_1^2$$

$$= \sin^{-1}(0) - \sin^{-1}\left(-\tfrac{1}{2}\right)$$

$$= \sin^{-1}\left(\tfrac{1}{2}\right)$$

$$= \pi/6 .$$

(d) $$I = \int \frac{1}{5+4\cos x}\,dx$$

let $\tan(x/2) = t$

$\therefore\ dx = \dfrac{2\,dt}{1+t^2}$ & $\cos x = \dfrac{1-t^2}{1+t^2}$

$$\therefore I = \int \frac{\frac{2\,dt}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int \frac{2\,dt}{5+5t^2+4-4t^2}$$

$$= \int \frac{2\,dt}{t^2+9}$$

$$= \frac{2}{3}\tan^{-1}(t/3) + C = \frac{2}{3}\tan^{-1}\left(\frac{\tan(x/2)}{3}\right) + C$$

8 (a)

i) $$\frac{dy}{dx} = x \cdot e^{y}$$

$$\Rightarrow \int e^{-y}\,dy = \int x\,dx$$

$$\Rightarrow -e^{-y} = \frac{x^2}{2} + C$$

ii) $$\sin^2 y \,\frac{dy}{dx} = \frac{\cos^2 y}{\cos^2 x}$$

$$\therefore \int \tan^2 y\,dy = \int \sec^2 x\,dx$$

$$\therefore \int (\sec^2 y - 1)\,dy = \int \sec^2 x\,dx$$

$$\Rightarrow \tan y - y = \tan x + C.$$

(b) i) $$\frac{dy}{dx} = 2 + \sin 3x$$

$$\Rightarrow \int dy = \int (2 + \sin 3x)\,dx$$

$$\Rightarrow y = 2x - \frac{\cos 3x}{3} + C$$

Now $y(\pi/2) = 0 \Rightarrow 0 = 2\left(\frac{\pi}{2}\right) - \frac{\cos 3\pi/2}{3} + C$

$$\Rightarrow C = -\pi$$

$$\therefore y = 2x - \frac{\cos 3x}{3} - \pi.$$

ii) $y = A \sin kx + B \cos kx$

$\Rightarrow \dfrac{dy}{dx} = Ak . \cos kx - Bk \sin kx$

$\Rightarrow \dfrac{d^2y}{dx^2} = Ak^2 (-\sin kx) - Bk^2 \cos kx$

$= -k^2 (A \sin kx + B \cos kx)$

$= -k^2 . y$

$\therefore \dfrac{d^2y}{dx^2} + k^2 . y = 0.$ (proved).

c) $\dfrac{dm}{dt} \propto m$

$\therefore \displaystyle\int \frac{dm}{m} = \int k \, dt$ $(k < 0)$

$\therefore \ln m = kt + c$

When $t = 0$, $m = m_0$.

$\therefore \ln m_0 = k(0) + c \Rightarrow c = \ln(m_0)$.

$\therefore \ln m = kt + \ln m_0$

$\therefore \ln\left(\dfrac{m}{m_0}\right) = kt$

$\Rightarrow \dfrac{m}{m_0} = e^{kt}$

$\therefore \underline{m = m_0 . e^{kt}}$ (proved).