Lecture 5

Topics covered in this lecture session

1. Remainder and Factor Theorems.
2. Polynomial Division.
3. Polynomial Factorisation.

Division Process (for numbers)

Example $26 \div 3$

$$\frac{26}{3} \Rightarrow \begin{array}{r} 8 \\ 3\,\big)\overline{\,26\,} \\ 24 \\ \hline 2 \end{array}$$

8 → Quotient

3 → Divisor

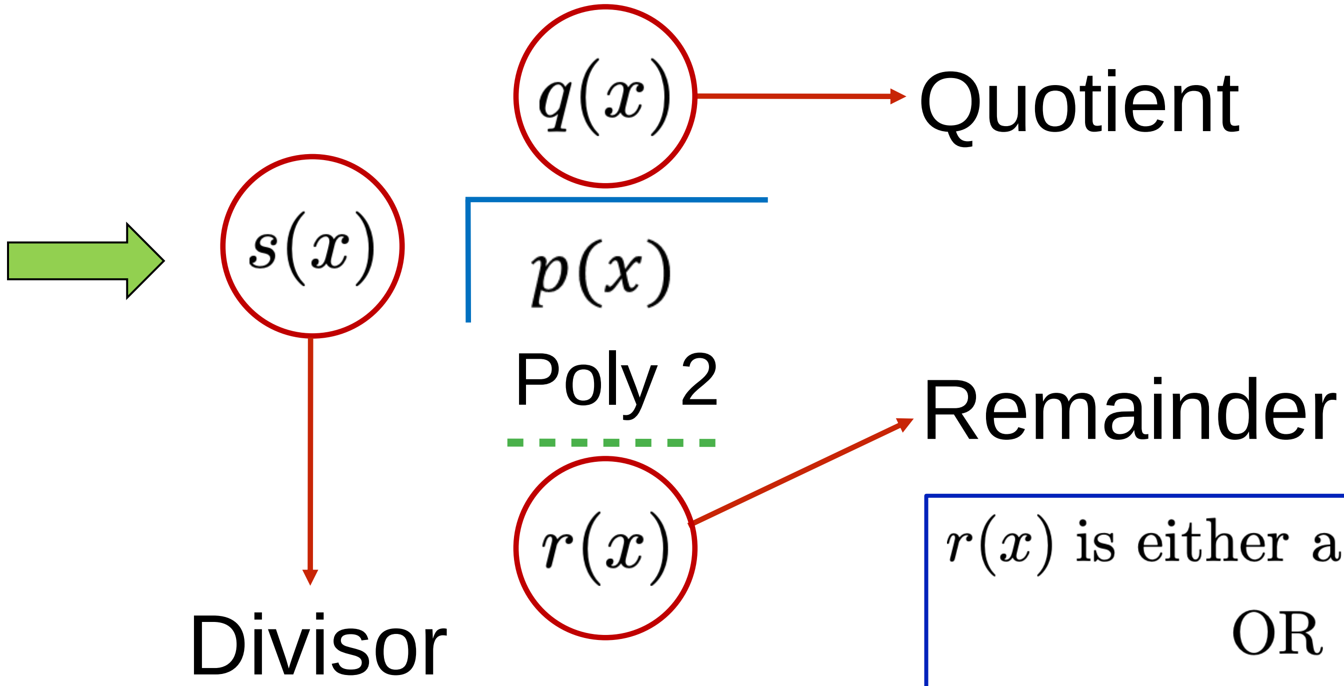
2 → Remainder

$$\boxed{\therefore \frac{26}{3} = 8 + \frac{2}{3} = Quotient + \frac{Remainder}{Divisor}}$$

Division of polynomials (Analogous result)

e.g. $p(x) \div s(x)$ where $s(x) \neq 0$

$$\frac{p(x)}{s(x)}$$



$r(x)$ is either a constant

OR

$\deg(r(x)) < \deg(s(x))$

Division of polynomials

Thus, $\dfrac{p(x)}{s(x)} = q(x) + \dfrac{r(x)}{s(x)} \quad \Rightarrow \quad p(x) = s(x)\,q(x) + r(x)$

where, $q(x)$ is the quotient, and

$r(x)$ is the remainder - which is either a constant (r)

or $\deg(r(x)) < \deg(s(x))$.

In particular, when $p(x)$ is divided by $(x-c)$, the remainder must be some constant r.

Remainder Theorem

$$\text{i.e. } \frac{p(x)}{(x-c)} = q(x) + \frac{r}{(x-c)}$$

$$\Rightarrow \quad p(x) = (x-c)\,q(x) + r$$

$$\Rightarrow \quad p(c) = r$$

Remainder Theorem

If a polynomial $p(x)$ is divided by $(x-c)$, then the remainder is $p(c)$.

Example

If $x^2 - 7x + k$ has a remainder 1 when divided by $(x+1)$, find k.

Solution: $(x+1) \equiv (x-c) \Rightarrow c = -1$

By Remainder Theorem, $p(c) = r$

$$\Rightarrow \; p(-1) = 1$$

$$\Rightarrow \; (-1)^2 - 7(-1) + k = 1$$

$$\Rightarrow \; k + 8 = 1 \quad \Rightarrow \quad k = -7.$$

Factor Theorem

Factorising a polynomial $p(x)$ means to write it as a product of lower-degree polynomials - called factors of $p(x)$.

For $s(x)$ to be a factor of $p(x)$, there must be no remainder when $p(x)$ is divided by $s(x)$.

i.e. $\dfrac{p(x)}{s(x)} = q(x) + 0$ **or** $p(x) = s(x)\,q(x)$

Factor Theorem

In particular, when $(x-c)$ is a factor of the polynomial $p(x)$, $p(x)$ can be expressed as

$$p(x) = (x-c)\,q(x) \quad \text{i.e.} \quad p(c) = 0.$$

Factor Theorem

A polynomial $p(x)$ has a factor $(x-c)$, if any only if $p(c) = 0$.

Note: $p(c) = r$ is the Remainder Theorem

$p(c) = 0$ is the Factor Theorem

Example

If $(x-2)$ is a factor of $ax^2 - 12x + 4$, find a.

Solution: Here, $(x-c) = (x-2) \Rightarrow c = 2$

By Factor theorem, $p(c) = 0.$

$\Rightarrow p(2) = 0 \Rightarrow a\,(2)^2 - 12\,(2) + 4 = 0$

$\Rightarrow 4a - 24 + 4 = 0$

$\Rightarrow 4a = 20. \quad \Rightarrow a = 5.$

Polynomial Division

1. Method of Long Division (or actual division)

The process of long division for dividing polynomials is similar to that of division of numbers.

Suppose, we want to determine

$$\frac{27x^3+9x^2-3x-10}{3x-2}$$

$$
\begin{array}{r|l}
 & 9x^2 + 9x + 5 \\
\hline
3x-2 & 27x^3 + 9x^2 - 3x - 10 \\
 & \underline{27x^3 - 18x^2} \\
 & \qquad\quad 27x^2 - 3x \\
 & \qquad\quad \underline{27x^2 - 18x} \\
 & \qquad\qquad\qquad 15x - 10 \\
 & \qquad\qquad\qquad \underline{15x - 10} \\
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

$$\text{Thus, } \frac{27x^3+9x^2-3x-10}{3x-2} = 9x^2+9x+5$$

Polynomial Division

2. Method of Synthetic Division

The method of Synthetic Division is a powerful alternative to the Method of Long Division.

We study this method only for linear divisors of the form $(x - c)$.

To understand the method, let us consider the example:

Example: If $\dfrac{x^3 - 9x^2 - 20}{(x-3)} = q(x) + \dfrac{r(x)}{(x-3)}$, find $q(x)$ and $r(x)$.

Method of Synthetic Division

Step 1

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ & & & & \\ \hline & & & & \end{array}$$

Write the coefficients of the polynomial to be divided at the top. Put zero as coefficient for unseen power(s) of x.

Step 2

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ 3 & & & & \\ \hline & & & & \end{array}$$

$$\frac{x^3-9x^2-20}{x-3}$$

Negate the constant term in the divisor, and write-in on the left side, that is, if $(x-a)$ is the divisor, write a on the left side.

Method of Synthetic Division

Step 3

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ 3 & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Drop the first coefficient after the bar to the last row.

Step 4

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ 3 & \downarrow & 3 & & \\ \hline & 1 & & & \end{array}$$

Multiply the dropped number with the number before the bar, and place it in the next column.

Method of Synthetic Division

Step 5

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ 3 & \downarrow & 3 & & \\ \hline & 1 & -6 & & \end{array}$$

Perform addition in the next column.

Repeat the previous two steps to obtain the following.

$$\begin{array}{c|cccc} & 1 & -9 & 0 & -20 \\ 3 & \downarrow & 3 & -18 & -54 \\ \hline & 1 & -6 & -18 & \mathbf{-74} \end{array}$$

Thus,

$$\frac{x^3 - 9x^2 - 20}{(x-3)} = (x^2 - 6x - 18) + \frac{-74}{(x-3)}$$

Factorising Polynomials
(with at least one integer zero)

Result:

Let $p(x) = c_n x^n + c_{n-1} x^{n-1} + ... + c_1 x + c_0$ be a polynomial with integer coefficients. Then, r is an integer zero of $p(x)$, if r is a divisor of the constant term c_0.

Examples:

Solved in Class