The University of Nottingham Ningbo China

Centre for English Language Education

Semester Two, 2016-2017

FOUNDATION CALCULUS & MATHEMATICAL TECHNIQUES

Time allowed 1 Hour 30 Minutes

Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced.

This paper contains EIGHT questions which carry equal marks.

An indication is given of the weighting of each subsection of a question by means of a figure enclosed by square brackets, eg. [12], *immediately following that subsection.*

Only silent, self-contained calculators with a Single-line Display or Dual-line Display are permitted in this examination.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

Do NOT turn examination paper over until instructed to do so.

ADDITIONAL MATERIAL: *Formula Sheet* (attached to the back of the question paper)

INFORMATION FOR INVIGILATORS:

1. *Please give a 15 minute warning.*
2. *Please collect Answer Booklets, Question Papers, and Formula Sheet at the end of the exam.*

This page is intentionally blank

1 (a) Given $y = e^{-x}$. Use first principles to find $\dfrac{dy}{dx}$. [2]

(b) (i) Given $y = \sin\left(\cos\left(e^{2x+3}\right)\right)$.

Use the substitutions $v = e^{2x+3}$ and $u = \cos v$, to find $\dfrac{dy}{dx}$.

(ii) Given $y = C_1\, e^{mx} + C_2\, e^{-mx}$, where C_1, C_2 and m are constants.

Show that $\dfrac{d^2y}{dx^2} - m^2\, y = 0$. [5]

(c) Given $x = y\sqrt{1-y^2}$. Use the definition of the derivative of an inverse function

to show that

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1-2\,y^2}\,.$$ [3]

2 (a) The equation of a curve is given by $x = a\,\cos\theta\,,\ y = b\,\sin\theta \quad ; \quad \theta \in \mathbb{R}$.

(i) Use parametric differentiation to find $\dfrac{dy}{dx}$.

(ii) Find the gradient of tangent to the curve at $\theta = \dfrac{\pi}{4}$.

(iii) Show that the equation of the tangent line to the curve at $\theta = \dfrac{\pi}{4}$ is:

$$\frac{x}{a} + \frac{y}{b} = \sqrt{2}\,.$$

(iv) Obtain the equation of the normal line to the curve at $\theta = \dfrac{\pi}{4}$.

[5]

(b) Given $y = (\ln x)^x$. Use logarithmic differentiation to find $\dfrac{dy}{dx}$.

[2]

(c) Given $x^3 + x^2y + xy^2 + y^3 = 4xy$. Use the method of implicit differentiation to find $\dfrac{dy}{dx}$.

[3]

3 (a) Given that the equation

$$f(x) = x^3 + x^2 - 8\,x - 15 = 0. \tag{3.1}$$

has only one real root which is positive.

(i) Find the stationary points of f.

(ii) Use the second derivative test to classify the stationary points obtained in 3(a)(i) as a point of maximum or minimum value.

(iii) Draw a rough sketch of $y = f(x)$.

(iv) Show that the iterative formula obtained by applying the Newton-Raphson method to (3.1) is

$$x_{n+1} \;=\; \frac{2\,x_n^3 + x_n^2 + 15}{3\,x_n^2 + 2\,x_n - 8}\,. \tag{3.2}$$

(v) By choosing appropriate x_0 value and using the iterative formula (3.2), obtain the approximate positive root of (3.1), correct to 6 decimal places. [9]

(b) The volume of a right circular cone is given by

$V = \dfrac{1}{3}\,\pi\,r^2\,h,$ where r is the radius and h is the height of the cone.

If the height h of the cone is increasing at the rate of 3 cm/sec, find the rate at which its volume is increasing when the radius is 5 cm.

[1]

4 (a) Evaluate the following integrals:

(i) $\displaystyle\int \frac{1 + x + x^2}{x^2}\; dx$ (ii) $\displaystyle\int \left(e^{2\,x} + \sin x\,\sec^2 x\right)\; dx$

(iii) $\displaystyle\int \frac{1}{x\,[\,1 + (\,\ln x\,)^2\,]}\; dx$ by using appropriate substitution. [4]

(b) Evaluate the following integrals:

(i) $\displaystyle\int \sin 7x\,\cos 3x\;\; dx$ (ii) $\displaystyle\int \sin^7 x\,\cos^3 x\;\; dx$ [4]

(c) Evaluate the integral $\displaystyle\int \frac{2x+3}{\sqrt{1 + 3x + x^2}}\; dx.$ [2]

5 (a) Express $f(x) = \dfrac{8}{(x-3)(3x-1)}$ as a sum of partial fractions.

Hence evaluate $\displaystyle\int_1^2 f(x)\,dx$. [4]

(b) (i) Use the method of integration by parts to show that

$$\int e^x \sin x \, dx = \frac{e^x}{2}(\sin x - \cos x) + C. \qquad (5.1)$$

(ii) Use (5.1) to calculate the area of the region enclosed by the curve $y = e^x \sin x$, vertical lines $x = 0,\ x = \dfrac{\pi}{2}$ and the X-axis. [4]

(c) The region bounded by $y = \tan x\ ;\ x \in \left[0, \dfrac{\pi}{4}\right]$ is rotated around the X-axis.

Find the volume of the solid of revolution formed. [2]

6 (a) Use appropriate substitution to evaluate the definite integral

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx.$$ [2]

(b) (i) Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$.

(ii) Use the method of t-substitution and result in 6(b)(i) to show that

$$\int \frac{1}{2\cos x + \sin x + 3}\,dx = \tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)+1}{2}\right) + C.$$ [5]

(c) Evaluate the definite integral $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx$ by using Simpson's rule, by dividing $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$ into 4 sub-intervals of equal width. Obtain the answer correct to 6 decimal places. [3]

7 (a) Use the method of separation of variables to solve the following initial value problems:

(*i*) $\operatorname{cosec} y\, dx + \cos^2 x\, dy = 0 \quad ; \quad y(\pi/4) = 0$

(*ii*) $\dfrac{dy}{dx} = \dfrac{y\, \cos x}{1 + \sin x} \quad ; \quad y(0) = 1.$ [5]

(b) Show that $y = a\, \cos^{-1} x + b$ (a, b are arbitrary constants) is a solution of the differential equation $(1 - x^2)\, \dfrac{d^2y}{dx^2} - x\, \dfrac{dy}{dx} = 0.$ [2]

(c) In an experiment of culture of bacteria in a laboratory, the differential equation model governing the growth of bacteria is given by

$$\frac{dm}{dt} = k\, m, \qquad \text{where } k > 0 \text{ is constant, and } t \text{ is time.} \tag{7.1}$$

(*i*) Show that the general solution of (7.1) is given by

$$m = m_0\, e^{k\,t}, \qquad \text{where } m_0 = m(0) \equiv \text{number of bacteria at } t = 0.$$

(*ii*) With $k = 1.5$/hour, use the result in 7(c)(*i*) to estimate the number of bacteria at the end of 8 hours. [3]

8 (a) Given $f(x) = \ln(1 + x)\ ;\ x \in \mathbb{R}^+$.

(*i*) Obtain the derivatives of f up to the fourth order and hence find the values of

$$f(0),\ f'(0),\ f''(0),\ f'''(0), \quad \text{and} \quad f^{(iv)}(0).$$

(*ii*) Hence, obtain the Maclaurin's series expansion of $f(x)$ up to the term with x^4.

(*iii*) Use the Maclaurin's series obtained in 8(a)(*ii*) to obtain the series expansion of

$$\ln\left(\frac{1+x}{1-x}\right) \quad ; \quad 0 < x < 1.$$

[7]

(b) Show that the Maclaurin's series expansion of $f(x) = \dfrac{1}{1-x} \quad ; \quad x \neq 1$ is:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots. \tag{8.1}$$ [3]