CELEN037 Seminar 10

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Topics

- Solutions of Ordinary Differential Equations (ODE)
- Solving ODEs of Variable-Separable Form
- Solving Initial Value Problems (IVP) of Variable-Separable Form
- Applications of Differential Equations

Definition

A function $f(x)$ is called a solution of a differential equation if the differential equation is satisfied when $y = f(x)$ and its derivatives are substituted into the given differential equation.

Example 1: Show that $y = C_1 \sin 4x + C_2 \cos 4x$ is a solution of the ODE $\dfrac{d^2y}{dx^2} + 16y = 0.$

Proof:

$$
\begin{aligned}
y = C_1 \sin 4x + C_2 \cos 4x \quad &\Rightarrow \quad \frac{dy}{dx} = 4C_1 \cos 4x - 4C_2 \sin 4x \\
&\Rightarrow \quad \frac{d^2y}{dx^2} = -16C_1 \sin 4x - 16C_2 \cos 4x = -16y \\
&\Rightarrow \quad \frac{d^2y}{dx^2} + 16y = 0
\end{aligned}
$$

Thus $y = C_1 \sin 4x + C_2 \cos 4x$ is a solution of the given ODE.

Definition

A function $f(x)$ is called a solution of a differential equation if the differential equation is satisfied when $y = f(x)$ and its derivatives are substituted into the given differential equation.

Example 2: Show that $y = e^{-x} + ax + b$ is a solution of the ODE $e^x \cdot \dfrac{d^2y}{dx^2} - 1 = 0$.

Proof:

$$y = e^{-x} + ax + b \quad \Rightarrow \quad \frac{dy}{dx} = -e^{-x} + a$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = e^{-x}$$

$$\Rightarrow \quad e^x \cdot \frac{d^2y}{dx^2} - 1 = e^x \cdot e^{-x} - 1 = 1 - 1 = 0$$

Thus $y = e^{-x} + ax + b$ is a solution of the given ODE.

Practice Problems on Worksheet:

1. Q3(ii)
2. Q3(iii)
3. Q3(iv)
4. Q3(vi)

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Result

The differential equation in variable-separable form can be written in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \qquad \text{i.e.} \qquad g(y)\,dy = f(x)\,dx$$

Integrating both sides:

$$\int g(y)\,dy = \int f(x)\,dx$$

$$\Rightarrow \qquad G(y) = F(x) + C$$

where $G(y)$ and $F(x)$ denote the antiderivatives of $g(y)$ and $f(x)$, respectively.

Example 1: Solve the following ODE:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Solution:

$$\begin{aligned}
\frac{dy}{dx} = -\frac{x}{y} \quad &\Rightarrow \quad y\,dy = -x\,dx \\
&\Rightarrow \quad \int y\,dy = -\int x\,dx \\
&\Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C' \\
&\Rightarrow \quad x^2 + y^2 = C
\end{aligned}$$

Thus $x^2 + y^2 = C$ is the general solution of the given ODE.

Example 2: Solve the following ODE:

$$\ln(\sin x)\frac{dy}{dx} = \cot x$$

Solution:

$$\ln(\sin x)\frac{dy}{dx} = \cot x \quad \Rightarrow \quad dy = \frac{\cot x}{\ln(\sin x)}\,dx = \frac{1}{\ln(\sin x)}\frac{\cos x}{\sin x}\,dx$$

$$\Rightarrow \int dy = \int \frac{1}{\ln(\sin x)}\frac{\cos x}{\sin x}\,dx$$

$$\text{Let } \ln(\sin x) = t \quad \Rightarrow \quad \frac{\cos x}{\sin x}\,dx = dt$$

$$\Rightarrow \quad y = \int \frac{1}{t}\,dt = \ln|\ln(\sin x)| + C$$

Thus $y = \ln|\ln(\sin x)| + C$ is the general solution of the given ODE.

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Example 3: Solve the following ODE:

$$\frac{dy}{dx} = \frac{xe^x}{y\sqrt{1+y^2}}$$

Solution:

$$\frac{dy}{dx} = \frac{xe^x}{y\sqrt{1+y^2}} \quad \Rightarrow \quad y\sqrt{1+y^2}\,dy = xe^x\,dx$$

$$\Rightarrow \int y\sqrt{1+y^2}\,dy = \int xe^x\,dx$$

$$\Rightarrow \quad \frac{1}{3}(1+y^2)^{\frac{3}{2}} = xe^x - \int e^x\,dx$$

$$\Rightarrow \quad \frac{1}{3}(1+y^2)^{\frac{3}{2}} = xe^x - e^x + C$$

$$\Rightarrow \quad (1+y^2)^{\frac{3}{2}} = 3xe^x - 3e^x + C$$

Thus $(1+y^2)^{\frac{3}{2}} = 3xe^x - 3e^x + C$ is the general solution of the given ODE.

Practice Problems on Worksheet:

1. Q1(ii)
2. Q1(iv)
3. Q1(vii)
4. Q1(x)

Answers:

1: $\ln|y| = \ln|x| + C$

2: $y + \dfrac{y^3}{3} = x + \dfrac{x^3}{3} + C$

3: $\tan^{-1} y = \dfrac{x^3}{3} + C$

4: $2\ln|\cos x| + \ln(1 + y^2) = C$

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Practice Problems on Worksheet (Cont'ed):

1. Q1(xiv)
2. Q1(xv)
3. Q1(xviii)
4. Q1(xxii)

Answers:

1: $\ln|\tan y| = \ln|x| + C$

2: $y = \ln\left(e^{x} + e^{-x}\right) + C$

3: $\ln|y| = \ln(1 + \sin x) + C$

4: $\ln|x| = \ln|\ln y| + C$

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Example 1: Use the method of separation of variables to solve the following IVP:

$$\frac{dy}{dx} = \frac{x^2}{y^2}; \quad y(0) = 2$$

Solution:

$$\begin{aligned}
\frac{dy}{dx} = \frac{x^2}{y^2} \quad &\Rightarrow \quad y^2\,dy = x^2\,dx \\
&\Rightarrow \int y^2\,dy = \int x^2\,dx \\
&\Rightarrow \quad \frac{y^3}{3} = \frac{x^3}{3} + C \\
y(0) = 2 \quad &\Rightarrow \quad \frac{2^3}{3} = \frac{0^3}{3} + C \\
&\Rightarrow \quad C = \frac{8}{3} \\
&\Rightarrow \quad y^3 = x^3 + 8 \quad \left(\text{or } y = \sqrt[3]{x^3+8}\right)
\end{aligned}$$

Example 2: Use the method of separation of variables to solve the following IVP:

$$e^{\frac{dy}{dx}} = x + 1 \quad (x > -1); \quad y(0) = 3$$

Solution:

$$\begin{aligned}
e^{\frac{dy}{dx}} = x + 1 \quad &\Rightarrow \quad \frac{dy}{dx} = \ln(x+1) \\
&\Rightarrow \quad dy = \ln(x+1)\,dx \\
&\Rightarrow \int dy = \int \ln(x+1)\,dx \\
&\Rightarrow \quad y = x\ln(x+1) - \int \frac{x}{x+1}\,dx \quad \text{(integration by parts)} \\
&\Rightarrow \quad y = x\ln(x+1) - x + \ln(x+1) + C \\
y(0) = 3 \quad &\Rightarrow \quad C = 3 \\
&\Rightarrow \quad y = (x+1)\ln(x+1) - x + 3
\end{aligned}$$

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Practice Problems on Worksheet:

1. Q2(ii)
2. Q2(iii)
3. Q2(iv)
4. Q2(vi)

Answers:

1: $y = \dfrac{1}{2x^2 + 1}$

2: $2 \ln y + y^2 + 2 \cos x = 3$

3: $y = \sec x$

4: $y = \sqrt{2 - \sqrt{x^2 + 1}}$

Example: The rate of decay of certain chemical is proportional to the amount (m) of the material at that time.

(i) Formulate a differential equation to show that the amount of the material at time t is $m = m_0 \cdot e^{kt}$, where $k < 0$ is a constant and m_0 is the initial amount.

(ii) Assume that 20 grams becomes 5 grams in 1 hour. How much will remain after 3 hours?

Solution: (i):

$$\frac{dm}{dt} = km \quad \Rightarrow \quad \frac{dm}{m} = k\,dt$$

$$\Rightarrow \int \frac{dm}{m} = \int k\,dt$$

$$\Rightarrow \quad \ln m = kt + C$$

$$t = 0,\ m = m_0 \quad \Rightarrow \quad \ln m_0 = C$$

$$\ln m = kt + \ln m_0 \Rightarrow \quad m = e^{kt + \ln m_0}$$

$$m = e^{\ln m_0} e^{kt} \Rightarrow \quad m = m_0 \cdot e^{kt}$$

(ii):

$$m_0 = 20 \quad \Rightarrow \quad m = 20e^{kt}$$

$$m(1) = 5 \quad \Rightarrow \quad m(1) = 20e^{k} = 5$$

$$\Rightarrow \quad e^{k} = \frac{1}{4}$$

$$\Rightarrow \quad m(3) = 20e^{k \cdot 3} = 20\left(\frac{1}{4}\right)^3$$

$$= 0.3125$$

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Practice Problems on Worksheet:

1. Q4(i)
2. Q4(ii)
3. Q4(iii)

Answers:

2b: 87.06

3: 34.657