Lecture 7

Topics covered in this lecture session

1. The Binomial Theorem.
   - Pascal's Triangle and Binomial coefficients.
   - Factorial notation and formula for $\binom{n}{r}$.
2. General expansion formula $(a+b)^n$; $n \in \mathbb{R}$.
3. Applications of Binomial Theorem in approximation and error analysis.

Binomial Theorem - Introduction

Consider the expansion formulae:

$$
\begin{aligned}
(1+x) &= 1 + x \\
(1+x)^2 &= 1 + 2x + x^2 \\
(1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\
(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4
\end{aligned}
$$

Binomial coefficients

In the above expansions, the numbers

$$\underline{1\ ,\ 1} \qquad \underline{1\ ,\ 2\ ,\ 1} \qquad \underline{1\ ,\ 3\ ,\ 3\ ,\ 1} \qquad \underline{1\ ,\ 4\ ,\ 6\ ,\ 4\ ,\ 1}$$

are coefficients of powers of x, are called Binomial coefficients.

These numbers are in a fixed pattern.

If we go on writing them, the pattern so formed is the Pascal's Triangle.

Binomial coefficients

$$
\begin{array}{ccccccccccccccccccc}
&&&&&&&&&1&&&&&&&&& \\
&&&&&&&&1&&1&&&&&&&& \\
&&&&&&&1&&2&&1&&&&&&& \\
&&&&&&1&&3&&3&&1&&&&&& \\
&&&&&1&&4&&6&&4&&1&&&&& \\
&&&&1&&5&&10&&10&&5&&1&&&& \\
&&&\boxed{1}&&\boxed{6}&&\boxed{15}&&\boxed{20}&&\boxed{15}&&\boxed{6}&&\boxed{1}&&& \\
&&1&&7&&21&&35&&35&&21&&7&&1&& \\
&1&&8&&28&&56&&70&&56&&28&&8&&1& \\
1&&9&&36&&84&&126&&126&&84&&36&&9&&1
\end{array}
$$

e.g. $(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

Binomial coefficients

What if we want to expand further? (i.e. higher order terms)

e.g. $(1+x)^{10}$

It is definitely not meaningful to continue writing rows of the Pascal's Triangle.

In such cases, we rely on a useful formula based on factorial function/notation.

Factorial Function

By definition,

$$0\,! = 1,$$

and

$$\begin{aligned} n\,! &= n\,(n-1)\,(n-2)\,\ldots\ldots \times 3 \times 2 \times 1 \\ &= 1 \times 2 \times 3 \times \ \ldots\ldots (n-2)\,(n-1)\,n. \end{aligned}$$

For example, $5\,! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$

Combinations - An important formula

$$nC_k \equiv \binom{n}{k} = \frac{n\,!}{k\,!\,(n-k)\,!}$$

For example,

$$\binom{10}{2} = \frac{10\,!}{2\,!\,(10-2)\,!}$$

$$= \frac{10 \times 9 \times 8\,!}{2 \times 8\,!} = \frac{90}{2} = 45$$

Because of their appearance as coefficients in a Binomial expansion, the numbers

$$\binom{n}{k}$$

are called Binomial coefficients.

Binomial Theorem

Using this notation, we expand $(1+x)^n$ as:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \ldots\ldots\ldots\ldots + x^n$$

By writing $x = \dfrac{b}{a}$ and simplifying, we get

a general formulation for Binomial Theorem as:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots\ldots + b^n$$

Binomial Theorem

Examples:

1. Expand $(x+7)^5$ using Binomial Theorem.

Here, $a = x$, $\quad b = 7$, $\quad$ and $\quad n = 5$.

Using $(a+b)^n$

$$= \quad a^n + \binom{n}{1} a^{n-1}\, b + \binom{n}{2} a^{n-2}\, b^2 + \binom{n}{3} a^{n-3}\, b^3 + \ldots\ldots + b^n$$

Binomial Theorem

$$\Rightarrow (x+7)^5 = x^5 + \binom{5}{1} x^4 (7) + \binom{5}{2} x^3 (7)^2 + \binom{5}{3} x^2 (7)^3$$

$$+ \binom{5}{4} x (7)^4 + \binom{5}{5} x^0 (7)^5$$

$$= x^5 + 5\,x^4 (7) + 10\,x^3 (49) + 10\,x^2 (343)$$

$$+ 5\,x\,(2401) + (1)\,x^0\,(16807)$$

$$= x^5 + 35\,x^4 + 490\,x^3 + 3430\,x^2 + 12005\,x + 16807$$

Binomial Theorem

2. Expand $(1-3x)^4$ using Binomial Theorem.

Here, $a = 1, \quad b = -3x, \quad$ and $\quad n = 4.$

Using $(a+b)^n$

$$= \quad a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots.. + b^n$$

$$\Rightarrow \; (1-3x)^4$$

$$= \quad 1^4 + \binom{4}{1} 1^4 (-3x) + \binom{4}{2} 1^3 (-3x)^2 + \binom{4}{3} 1^2 (-3x)^3 + \binom{4}{4} (-3x)^4$$

$$= \quad 1 + 4(-3x) + 6(9x^2) + 4(-27x^3) + (1)(81x^4)$$

$$= \quad 1 - 12x + 54x^2 - 108x^3 + 81x^4.$$

Finding coefficient of x^n

Find the coefficient of x^3 in the expansion of $\left(3-\frac{2x}{5}\right)^5$.

$$\left(3-\frac{2x}{5}\right)^5 = 3^5 + \binom{5}{1}\cdot 3^4\cdot\left(\frac{-2x}{5}\right) + \binom{5}{2}\cdot 3^3\cdot\left(\frac{-2x}{5}\right)^2 + \boxed{\binom{5}{3}\cdot 3^2\cdot\left(\frac{-2x}{5}\right)^3} + \binom{5}{4}\cdot 3^1\cdot\left(\frac{-2x}{5}\right)^4 + \binom{5}{5}\cdot 3^0\cdot\left(\frac{-2x}{5}\right)^5$$

$\therefore$ **Term with x^3 is:** $\binom{5}{3}\cdot 3^2\cdot\left(\frac{-2x}{5}\right)^3$

$\therefore$ The coefficient of x^3 is:

$$\binom{5}{3}\cdot 3^2\cdot\left(\frac{-2}{5}\right)^3 = 10\cdot 9\cdot\left(\frac{-8}{125}\right)$$
$$= -\frac{144}{25}$$

General expansion formula

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \frac{n(n-1)(n-2)}{3\,!}x^3 + \ldots\ldots\ldots$$

where $n \in \mathbb{R}$ and $|\, x \,| < 1$.

Note: $$\binom{n}{2} = \frac{n\,!}{2\,!\cdot(n-2)\,!}$$

$$= \frac{n\cdot(n-1)\cdot(n-2)\,!}{2\,!\cdot(n-2)\,!} = \frac{n\cdot(n-1)}{2\,!}$$

Similarly other terms can be obtained.

Generalised Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \frac{n(n-1)(n-2)}{3\,!}x^3 + \dots\dots\dots\dots$$

where $n \in \mathbb{R}$ and $|\,x\,| < 1.$

Ex.1 Expand $(1+x)^{-3}$

$$(1+x)^{-3} = 1 + (-3)x + \frac{(-3)(-3-1)}{2\,!}x^2 + \frac{(-3)(-3-1)(-3-2)}{3\,!}x^3 + \cdots$$

$$= 1 - 3x + \frac{(-3)(-4)}{2\cdot 1}x^2 + \frac{(-3)(-4)(-5)}{3\cdot 2}x^3 + \frac{(-3)(-4)(-5)(-6)}{4\cdot 3\cdot 2}x^4 + \cdots$$

$$= 1 - 3x + 6x^2 - 10x^3 + 15x^4 + \cdots$$

Approximation using Binomial Theorem

Approximate $(1.02)^{-1}$ using Binomial Theorem.

$$(1.02)^{-1} = (1 + 0.02)^{-1}$$

So, $a = 1$, $x = 0.02$, and $n = -1$ **NOT a positive Integer**

General expansion formula for $n \in \mathbb{R}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}\,x^2 + \frac{n(n-1)(n-2)}{3\,!}\,x^3 + \ldots\ldots\ldots\ldots$$

apply where $n \in \mathbb{R}$ and $|\,x\,| < 1.$

Approximation using Binomial Theorem

As, $|\, x \,| = |\, 0.02 \,| < 1.$

Using $(1+x)^n$

$$= \quad 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \frac{n(n-1)(n-2)}{3\,!}x^3 + \ldots\ldots\ldots\ldots$$

$$\Rightarrow \quad (1+0.02)^{-1}$$

$$= \quad 1 + (-1)(0.02) + \frac{(-1)(-1-1)}{2\,!}(0.02)^2$$

$$+ \frac{(-1)(-1-1)(-1-2)}{3\,!}(0.02)^3 + \ldots.$$

Approximation using Binomial Theorem

$$\approx 1 - 0.02 + 0.0004 - 0.000008$$

Approximate sign is introduced because we are terminating the infinite series.

$$= \quad 0.980392.$$

Thus, $(1.02)^{-1} \approx 0.980392.$

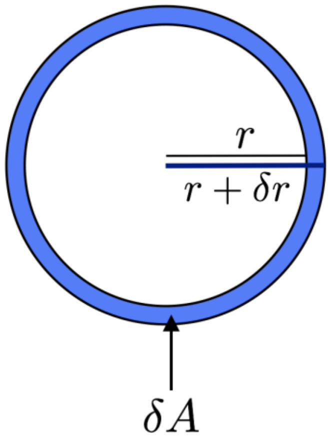
Error Analysis using Binomial approximation

Example:

The radius of a circle is measured as r, with an error of δr = 1.5% of r.

The area of the circle $A = \pi r^2$ is then calculated using the measured r.

Find the resulting error, δA, in the area calculated.



Note:

Using approximation, $(1+x)^n \approx 1+nx$.

Error Analysis using Binomial approximation

given that $\delta r = 1.5\%$ of $r \quad \Rightarrow \quad \delta r = 0.015r$.

Now, $A \;=\; \pi r^2$

$$
\begin{aligned}
\Rightarrow \cancel{A} + \delta A &= \pi\,(r+\delta r)^2 = \pi\,(r+0.015r)^2 \\
&= \pi\, r^2\,(1+0.015)^2 \\
&\approx A\,(1+2\times 0.015) \quad \text{using approximation } (1+x)^n \approx 1+nx \\
&= A(1+0.03) \\
&= \cancel{A} + 0.03\,A
\end{aligned}
$$

$$
\boxed{\begin{aligned} \therefore\ \delta A &\approx 0.03\,A \\ \text{i.e. } \delta A &\approx 3\% \text{ of } A. \end{aligned}}
$$

Further Reading (click on links)

College Algebra by J. W. Coburn & J. P. Coffelt (3rd edition)

(Page 724 to 725, & Page 728 to 729)

Foundation Algebra by P. Gajjar.

(Chapter 9)