Lecture 3

Lecture Content

- Parametric differentiation
- Higher order Derivatives
- McLaurin's Series
- Revisiting topics from coordinate geometry
- Equation of a tangent line
- Newton-Raphson method
- Increasing, Decreasing, and Constant Functions

Parametric Differentiation

When we define $\dfrac{dy}{dx}$, the variable x is independent,

and y is dependent on x.

i.e. with change in x, the value of y changes;

and the rate of change of y with change in x is given by $\dfrac{dy}{dx}$.

We now consider the case where both x and y depend on

an independent third variable t, usually thought of as *time*.

i.e. $\quad x = f(t) \qquad$ and $\qquad y = g(t)$

Parametric Differentiation

To find $\dfrac{dy}{dx}$, we find $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ separately

and then use the following formula:

$$\frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)}$$

Note: Remember to put $\left(\dfrac{dy}{dt}\right)$ in the numerator.

This method of finding $\dfrac{dy}{dx}$ for parametric equations is called Parametric Differentiation.

Example

Find $\dfrac{dy}{dx}$, if the parametric equations of a curve are:

$$x = t - 3\sin t, \qquad y = 4 - 3\cos t \qquad ; \qquad t \in \mathbb{R}$$

Solution:

$$x = t - 3\sin t \qquad\qquad y = 4 - 3\cos t$$

$$\Rightarrow \frac{dx}{dt} = 1 - 3\cos t \qquad\qquad \Rightarrow \frac{dy}{dt} = 3\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{3\sin t}{1 - 3\cos t}$$

Second Order Derivative

The derivative $\frac{dy}{dx}$ is called the second order derivative of $y = f(x)$ and it is denoted by $\frac{d^2y}{dx^2}$.

$$\boxed{\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x) = \frac{d^2y}{dx^2}}$$

Example

Given $y = A \sin mx + B \cos mx$, where A and B are constants.

Show that $\frac{d^2y}{dx^2} + m^2 y = 0$.

Solution:

$$\frac{dy}{dx} = Am \cos mx + Bm\,(-\sin mx)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = Am^2\,(-\sin mx) - Bm^2\,(\cos mx) = -\,m^2 y$$

$$\boxed{\therefore \quad \frac{d^2y}{dx^2} + m^2 y = 0}$$

Example

Given $x = e^{mt}$ where m is constant. Show that $\frac{d^2x}{dt^2} - m^2x = 0$.

Solution:

$$x = e^{mt}$$

$$\Rightarrow \frac{dx}{dt} = m\, e^{mt}$$

$$\Rightarrow \frac{d^2x}{dt^2} = m^2e^{mt} = m^2x.$$

$$\boxed{\therefore \frac{d^2x}{dt^2} - m^2\, x = 0}$$

Higher Order Derivatives

For a continuously differentiable function $y = f(x)$, the first and the successive derivatives are:

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \ldots\ldots, \quad \frac{d^ny}{dx^n}$$

or $$f'(x), \ f''(x), \ f'''(x), \ldots\ldots, \ f^{(n)}(x)$$

The value of the n^{th} derivative when $x = a$ is denoted by:

$$\boxed{\left.\frac{d^ny}{dx^n}\right|_{x=a} \quad \text{or} \quad f^{(n)}(a).}$$

Example

Given $y = \sin^2 x$. Show that $\left.\frac{d^3y}{dx^3}\right|_{x=\frac{\pi}{4}} = -4$.

Solution:

$$y \quad = sin^2 x$$

$$\Rightarrow \frac{dy}{dx} = 2\sin x \cos x = \sin 2x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\cos 2x$$

$$\Rightarrow \frac{d^3y}{dx^3} = -4\, sin\, 2x$$

$$\Rightarrow \left.\frac{d^3y}{dx^3}\right|_{x=\frac{\pi}{4}} = -4\, sin\left(\frac{\pi}{2}\right) = -4$$

Maclaurin's series

The Maclaurin's series of a continuously differentiable function f is an infinite series given by:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

provided that $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, ..., $f^{(n)}(0)$ all have finite values.

Example

Expand $f(x) = \cos x$ using Maclaurin's series.

Solution:

Maclaurin's series $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \ldots\ldots$

$$\left.\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(iv)}(x) &= \cos x \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= -1 \\ f'''(0) &= 0 \\ f^{(iv)}(0) &= 1 \end{aligned}$$

Example

Expand $f(x) = \cos x$ using Maclaurin's series.

$$f(0) = 1 \qquad f'(0) = 0 \qquad f''(0) = -1$$

$$f'''(0) = 0 \qquad f^{(iv)}(0) = 1$$

Now, Maclaurin's series expansion of f is:

$$\boxed{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \ldots\ldots}$$

$$\cos x = f(x)$$

$$= 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \ldots\ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$$

Example

Show that for small values of x, $e^{2x} - e^{-x} \approx 3x + \frac{3}{2}x^2$.

Solution:

To find the expansion for e^x, let $f(x) = \mathrm{e}^x$.

$$\left.\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f''(x) &= e^x \\ &\dots\dots\dots\dots\dots \\ f^{(n)}(x) &= e^x \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= 1 \\ f''(0) &= 1 \\ &\dots\dots\dots\dots\dots \\ f^{(n)}(0) &= 1 \end{aligned}$$

Example

$$1 = f(0) = f'(0) = f''(0) \ldots \ldots .$$

$$\boxed{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \ldots\ldots}$$

$$e^x = f(x)$$

$$= 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1) + \ldots\ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$$

Example

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$$

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \ldots\ldots = 1 + 2x + 2x^2 + \ldots\ldots$$

and $e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \ldots\ldots = 1 - x + \frac{x^2}{2!} - \ldots\ldots$

$$\therefore \quad e^{2x} - e^{-x} = 3x + \frac{3}{2}x^2 + \ldots\ldots$$

$\therefore$ For small values of x,

$$e^{2x} - e^{-x} \approx 3x + \frac{3}{2}x^2.$$

Example

(i) Given $f(x) = \sqrt{1-x}$, find $f(0)$, $f'(0)$ and $f''(0)$.

(ii) Use the Maclaurin's series expansion formula to show that

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x - \frac{1}{8}x^2.$$

(iii) By substituting $x = \frac{1}{50}$, find $\sqrt{2}$ correct to 3 decimal places.

Example: McLaurin's Series

(i) Given $f(x) = \sqrt{1-x}$, find $f(0)$, $f'(0)$ and $f''(0)$.

$$\left.\begin{aligned} f(x) &= \sqrt{1-x} = (1-x)^{\frac{1}{2}} \\ f'(x) &= -\frac{1}{2} \cdot (1-x)^{-\frac{1}{2}} \\ f''(x) &= -\frac{1}{4}(1-x)^{-\frac{3}{2}} \end{aligned}\right\} \Rightarrow \begin{aligned} f(0) &= 1 \\ f'(0) &= -\frac{1}{2} \\ f''(0) &= -\frac{1}{4} \end{aligned}$$

Example: McLaurin's Series

(ii) Use the Maclaurin's series expansion formula to show that

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x - \frac{1}{8}x^2.$$

$$f(x) = \sqrt{1-x} = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$

$$= 1 + x\left(-\frac{1}{2}\right) + \frac{x^2}{2!}\left(-\frac{1}{4}\right) + \ldots\ldots$$

$$\therefore \quad \sqrt{1-x} \approx 1 - \frac{x}{2} - \frac{x^2}{8}$$

Example: McLaurin's Series

(iii) By substituting $x = \frac{1}{50}$, find $\sqrt{2}$ correct to 3 decimal places.

$$\sqrt{1 - \left(\frac{1}{50}\right)} \approx 1 - \left(\frac{1}{50}\right)\left(\frac{1}{2}\right) - \frac{1}{8}\left(\frac{1}{50}\right)^2$$

$$\frac{7}{\sqrt{50}} \approx 1 - \left(\frac{1}{100}\right) - \frac{1}{8}\left(\frac{1}{2500}\right) = 1 - 0.01 - 0.00005 = 0.98995$$

$$\therefore \frac{7}{\sqrt{50}} \approx \mathbf{0.98995}$$

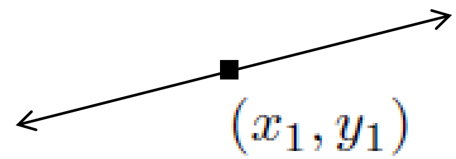
$$\sqrt{50} \approx \frac{7}{0.98995} = 7.071$$

$$\sqrt{2} \approx \frac{7.071}{5} = \mathbf{1.414} \qquad (\because \sqrt{50} = 5\sqrt{2})$$

Revisiting Topics from Coordinate Geometry

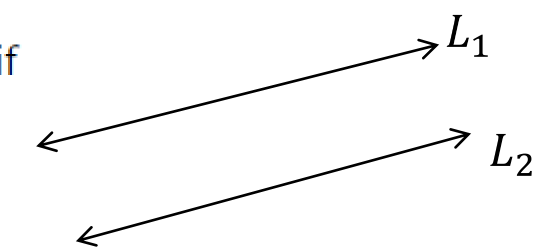
1. The equation of a straight line passing through the point (x_1, y_1) and having slope m is:

$$y - y_1 = m(x - x_1)$$



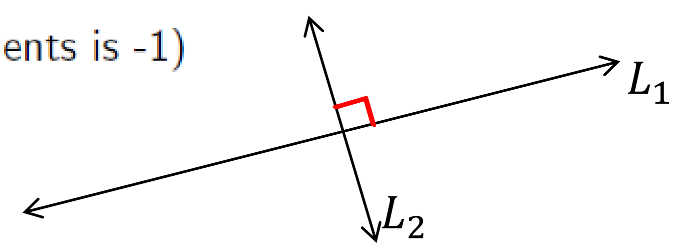
2. Two lines $L_1 : y = m_1 x + c_1$ and $L_2 : y = m_2 x + c_2$ are **parallel** if

$$m_1 = m_2 \qquad \text{(i.e. slopes/gradients are equal)}$$



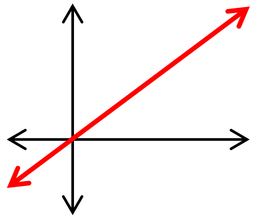
3. Two lines $L_1 : y = m_1 x + c_1$ and $L_2 : y = m_2 x + c_2$ are **perpendicular** (orthogonal) if

$$m_1 \cdot m_2 = -1 \qquad \text{(i.e. product of slopes/gradients is -1)}$$

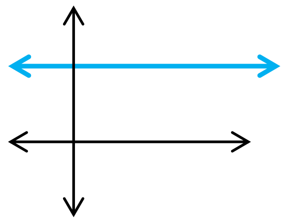


Revisiting Topics from Coordinate Geometry

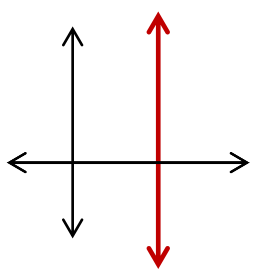
4. If the line $y = mx + c$ **passes through the origin** then, $c = 0$.



5. The slope of a **horizontal** line (i.e. line parallel to the x-axis) is 0.



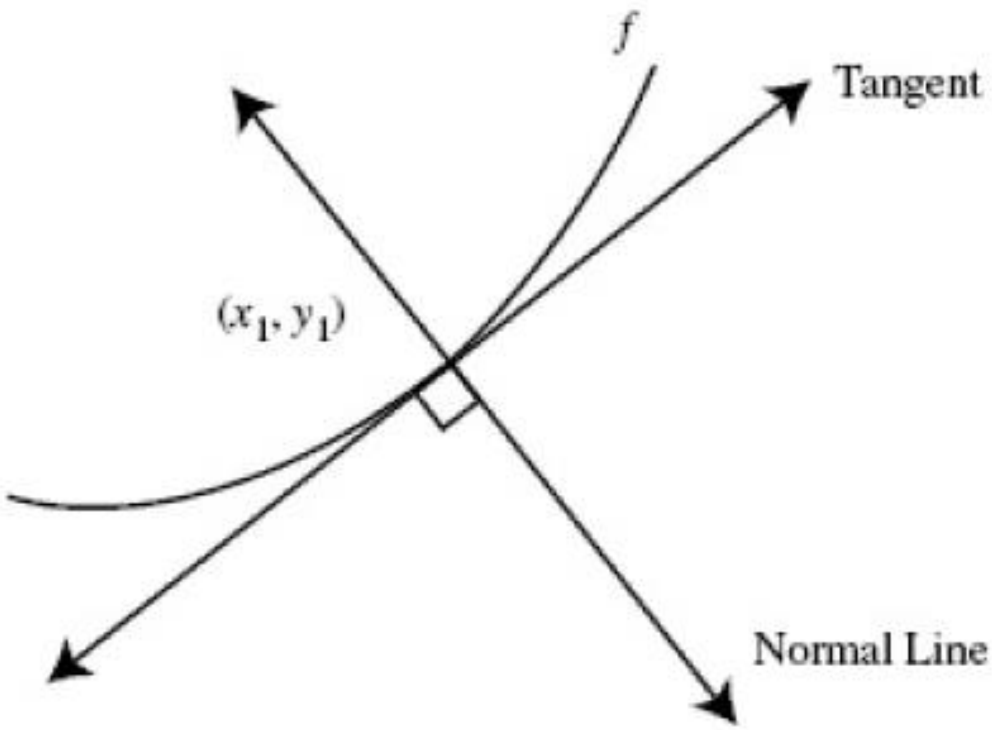
6. The slope of a **vertical** line (i.e. line parallel to the y-axis) is not defined.



7. If the line/curve $y = f(x)$ **cuts**/intersects the x-axis, put $y = 0$.

8. If the line/curve $y = f(x)$ **cuts**/intersects the y-axis, put $x = 0$.

Tangent and Normal Lines



The point $(x_1\,, y_1)$ must be ***on the curve****.*

For tangents from points outside the curve, the method is different.

A **normal** line is a line that is perpendicular (orthogonal) to the tangent line at the point of contact.

Equation of a Tangent Line

The equation of a tangent line at the point (x_1, y_1) to the curve $y = f(x)$ is given by

$$y - y_1 = m\,(x - x_1)$$

where $m = \left.\dfrac{dy}{dx}\right|_{(x_1, y_1)}$

$=$ slope of the tangent line at point (x_1, y_1).

Equation of a Normal Line

The equation of a normal line at the point (x_1, y_1) to the curve $y = f(x)$ is given by

$$y - y_1 = n\,(x - x_1)$$

where $n = \dfrac{-1}{\left.\dfrac{dy}{dx}\right|_{(x_1, y_1)}} = -\left.\dfrac{dx}{dy}\right|_{(x_1, y_1)}$

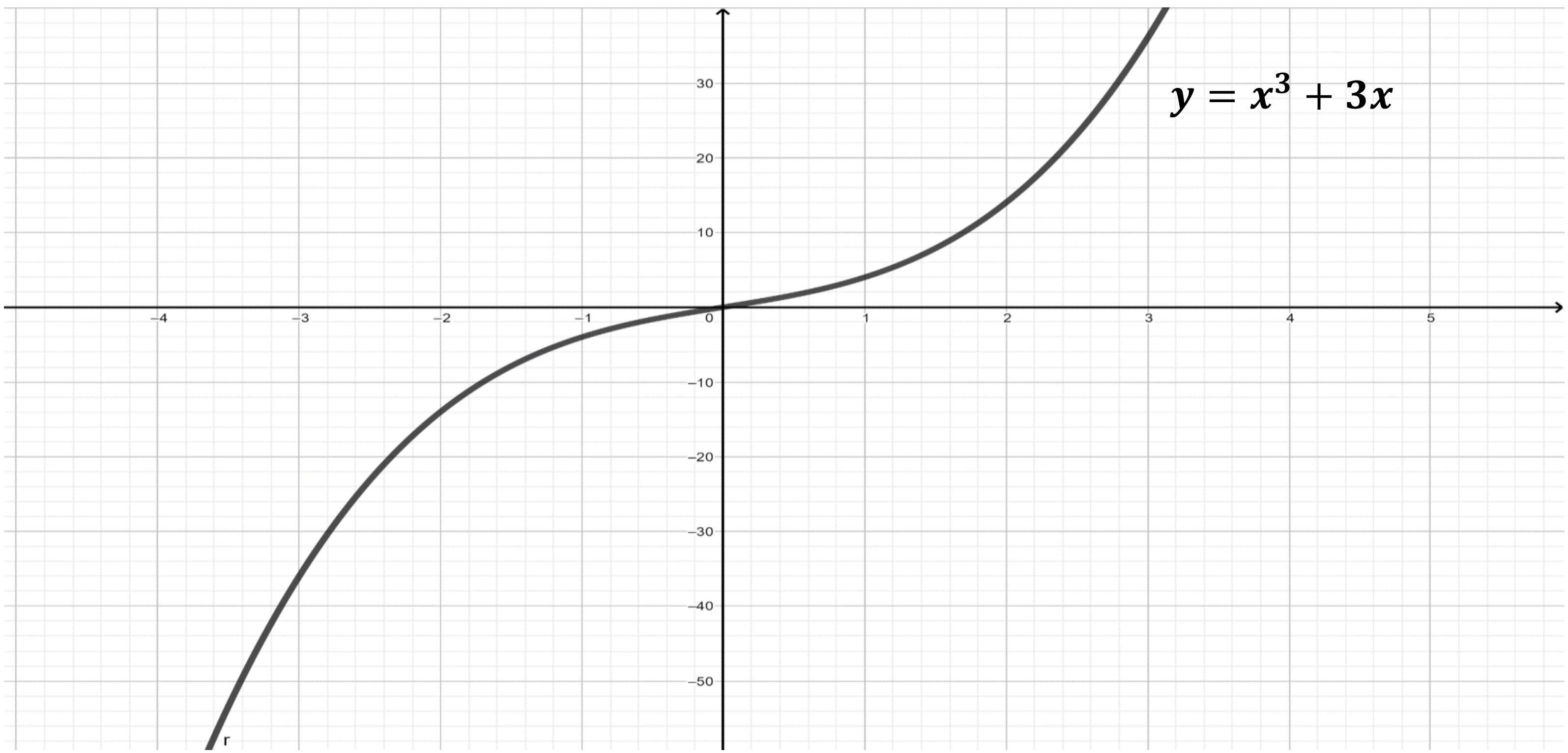
$=$ slope of the normal line at point (x_1, y_1).

Practice Problem

Find the equations of the tangents to $y = x^3 + 3x$ which are parallel to the line $y = 15x + 2$

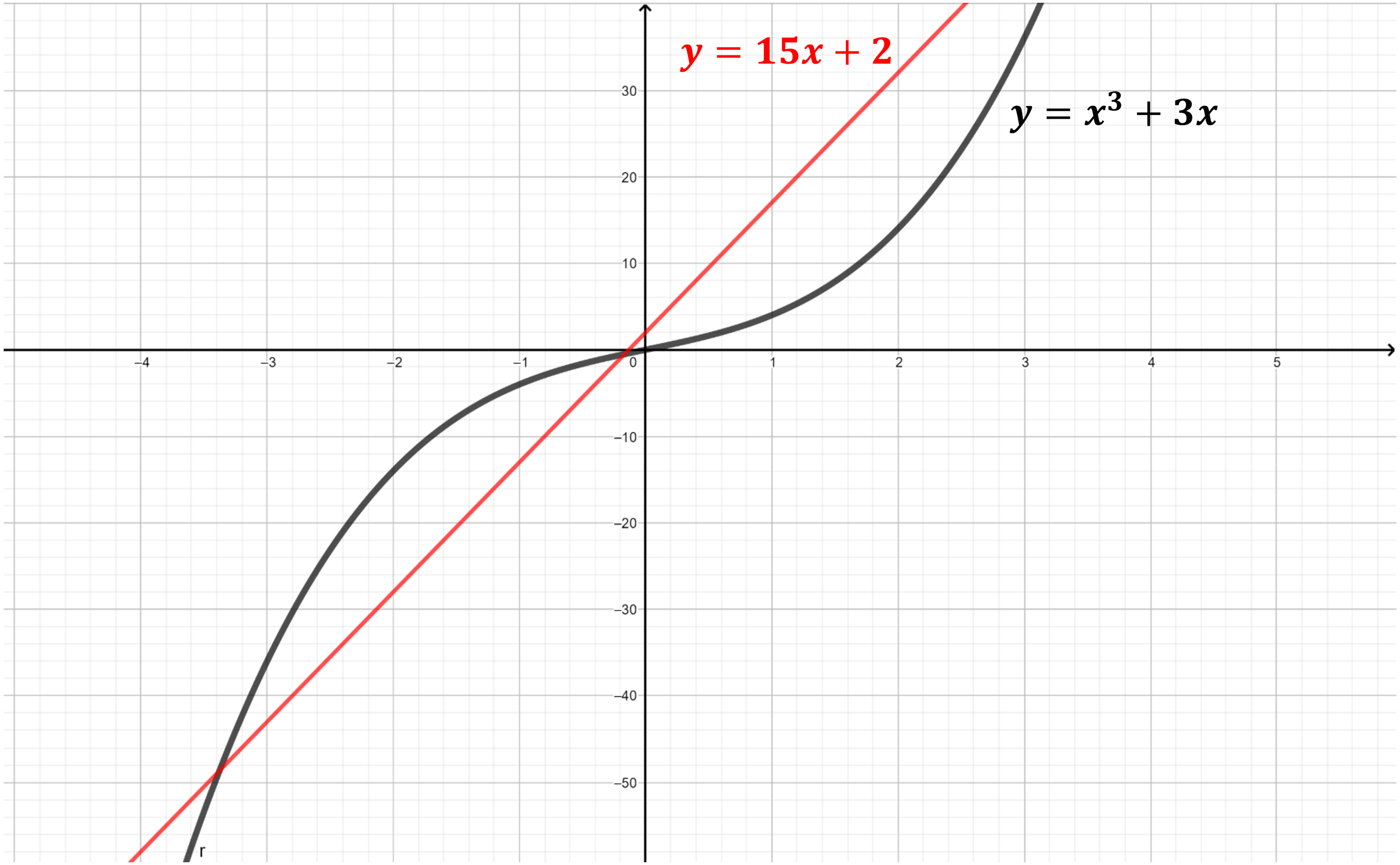
Graphical Illustration

Equations of the tangents to $y = x^3 + 3x$ which are parallel to the line $y = 15x + 2$



Graphical Illustration

Equations of the tangents to $y = x^3 + 3x$ which are parallel to the line $y = 15x + 2$



Graphical Illustration

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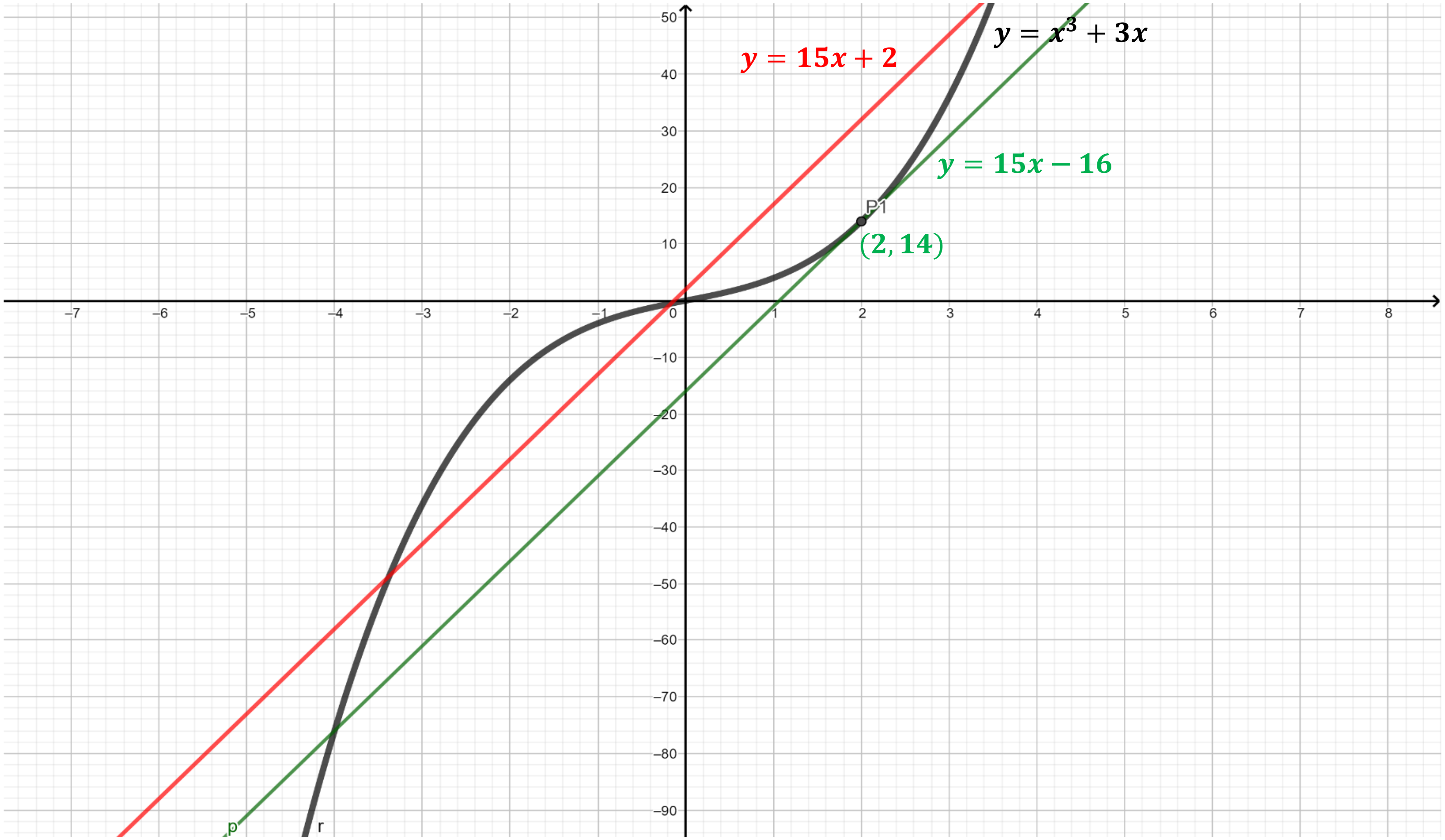
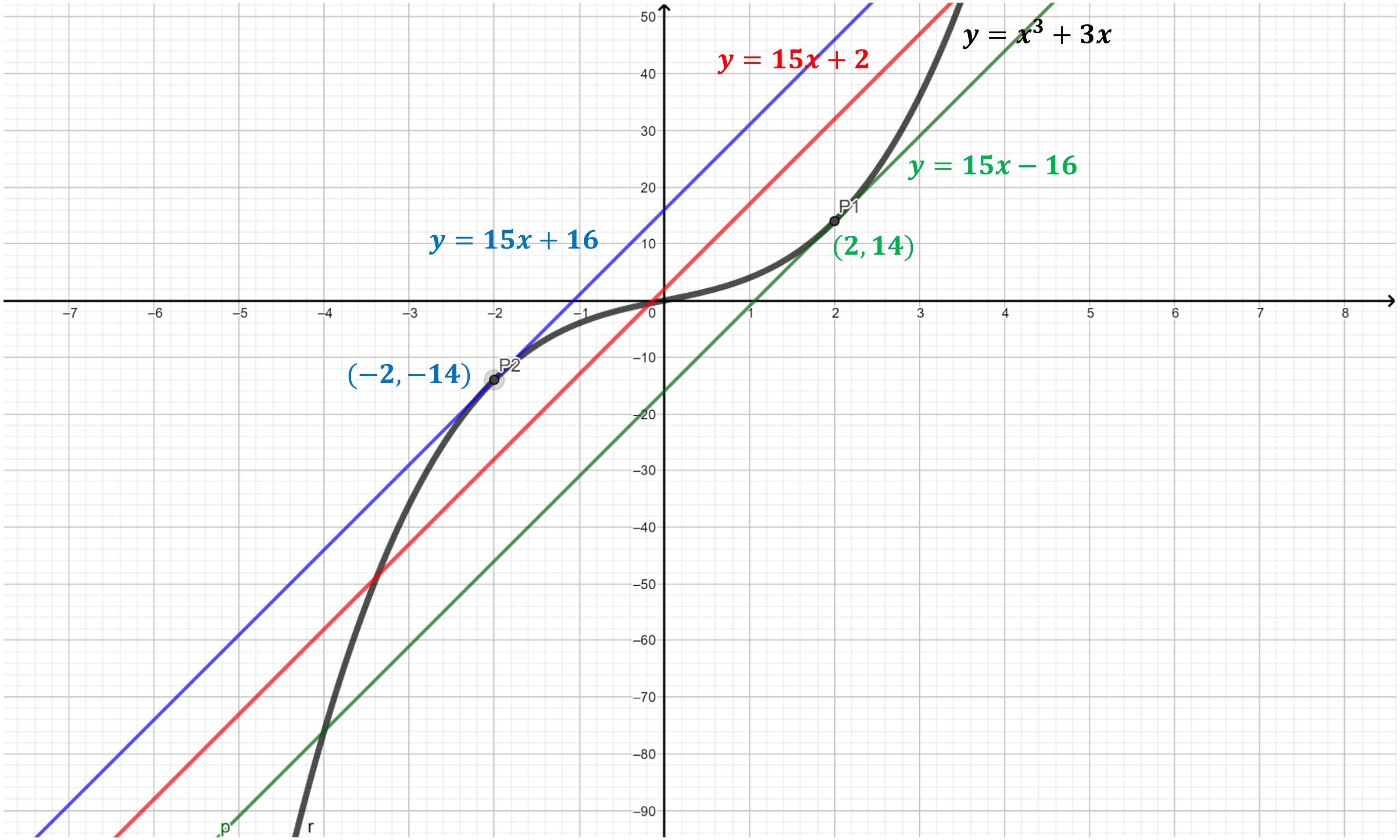


Graphical Illustration

Equations of the tangents to $y = x^3 + 3x$ which are parallel to the line $y = 15x + 2$



Newton-Raphson Method

Recap :: Methods for solving equations

Linear equation: $ax + b = 0 \quad \Rightarrow \quad x = -\frac{a}{b}$

Quadratic equation: $ax^2 + bx + c = 0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubic equation: Cardano's method

Quartic equation: Ferrari's method

Equation of degree : ≥ 5

analytical solution impossible.

Numerical Methods:
Bisection method
Fixed-point iteration method

(You learnt last semester)

Newton Raphson Method: uses the concept of derivatives to find roots

Newton-Raphson Method

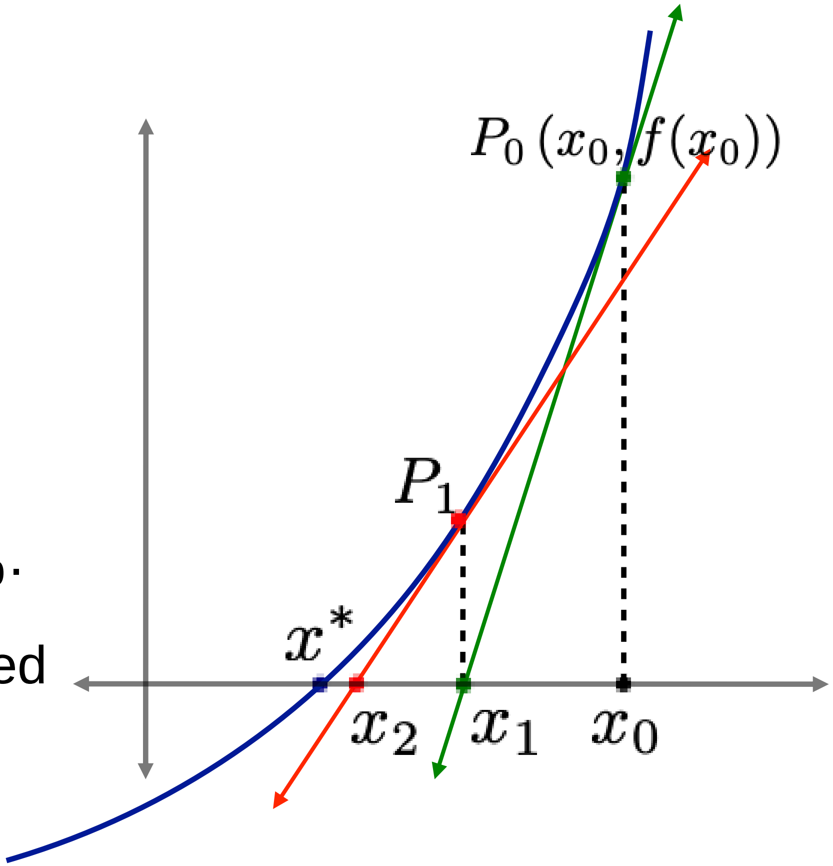
Let x_0 be the initial approximation.

As a first step, we approximate the curve by a tangent line at $P_0(x_0, f(x_0))$.

Let the tangent line at P_0 intersect the X-axis at x_1.

Then, x_1 is a better approximation than x_0.

Repeat the process until the root of desired accuracy is obtained.



Newton-Raphson Method

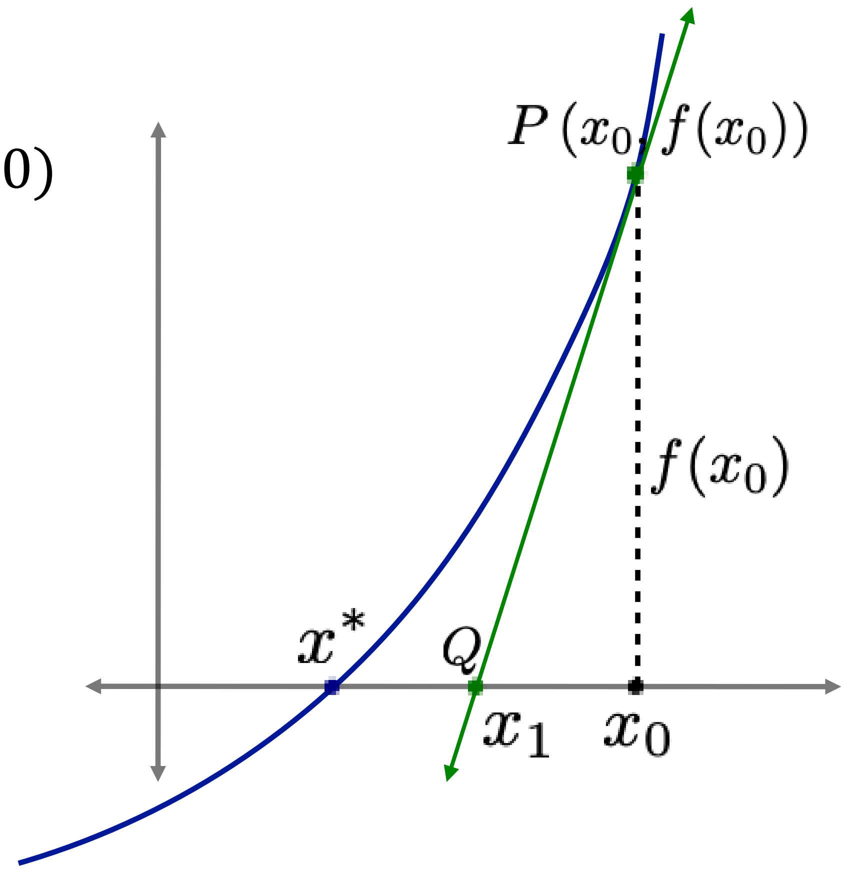
Formulation

Here, $P \leftrightarrow (x_0, f(x_0))$ and $Q \leftrightarrow (x_1, 0)$

By definition,

$$\text{Slope of } \overleftrightarrow{PQ} = \frac{d}{dx} f(x) \Big|_{x = x_0}$$

$$\Rightarrow \frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0)$$



Newton-Raphson Method

$$\frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0) \quad \Rightarrow \quad x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Generalising the result, we have

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; \quad \text{where } n = 0, 1, 2, 3, \ldots\ldots}$$

Example

Apply the Newton-Raphson method to approximate $\sqrt{2}$ correct to 6 d.p.

Solution:

$$x = \sqrt{2}$$
$$\Rightarrow x^2 - 2 = 0$$

$\therefore$ We have to solve $f(x) = x^2 - 2 = 0$

Differentiating $f(x)$, we have $f'(x) = 2x$

The Newton-Raphson formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_n - \frac{x_n^2 - 2}{2x_n} \Rightarrow \frac{x_n^2 + 2}{2x_n}$$

Example

Starting with $x_0 = 1$, successive approximations are as shown in the adjacent table.

n	x_n
0	1.000000
1	1.500000
2	1.416667
3	1.414216
4	1.414214
5	1.414214

$\therefore \quad \sqrt{\mathbf{2}} \;=\; \mathbf{1.414214}$ **(6 d.p.)**

Clearly, this method is simpler and faster than the fixed point iteration method.

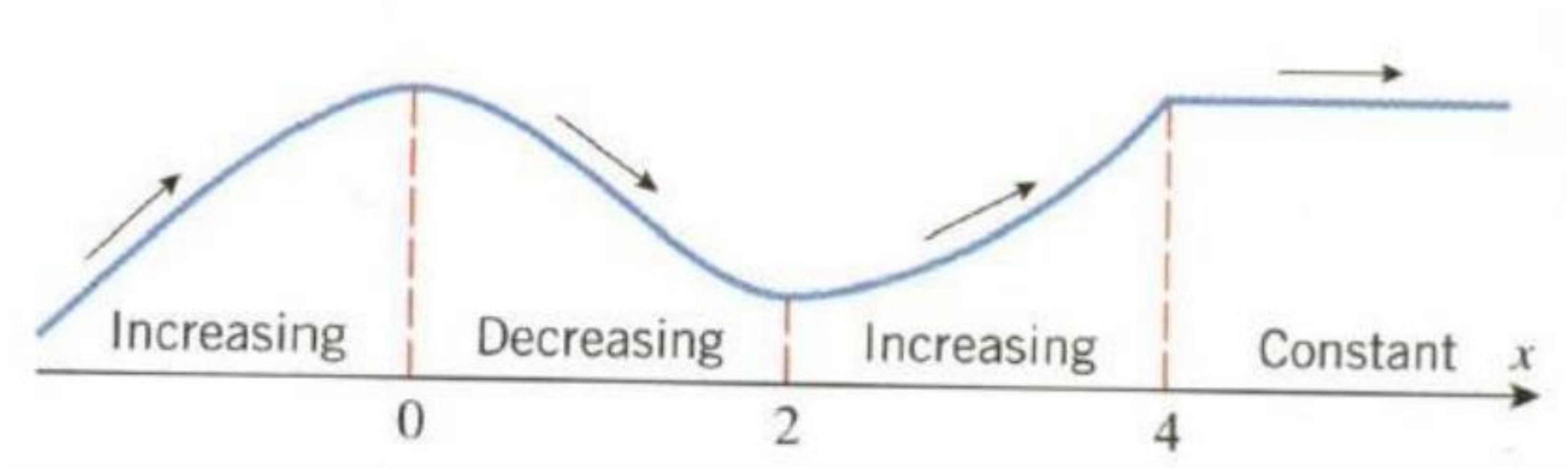
Applications of Derivatives

We have applied Derivatives to revisit coordinate geometry, to find tangent of lines, and to find roots of equations, here we apply derivatives to:

Analyse functions and their graphs, and to determine maximum/minimum values

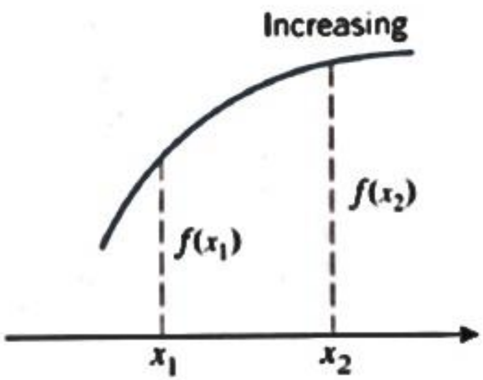
Increasing, Decreasing and Constant Functions

The terms increasing, decreasing, and constant are used to describe the behaviour of a function as we travel from left to right along its graph.



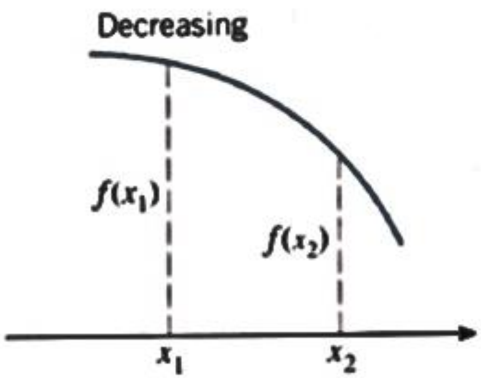
Increasing, Decreasing and Constant Functions

Let f be defined on an interval and, let x_1 and x_2 denote points in that interval.



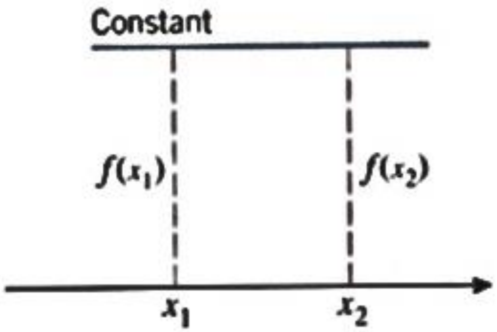
(a) f **is increasing on the interval if**

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$



(b) f **is decreasing on the interval if**

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$



(c) f **is constant on the interval if for all**

$x_1, x_2 \Rightarrow f(x_1) = f(x_2)$

Increasing, Decreasing and Constant Functions

Result

Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).

(a) If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.

(c) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

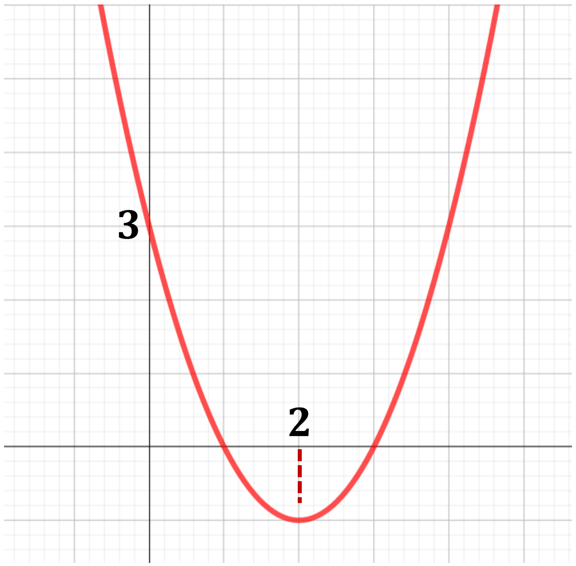
Increasing, Decreasing and Constant Functions

Example

Find the interval on which

$$f(x) = x^2 - 4x + 3$$

is increasing and the intervals on which it is decreasing.



$$f'(x) = 2x - 4 = 2(x - 2).$$

$\therefore\ x < 2 \ \Rightarrow\ f'(x) < 0$

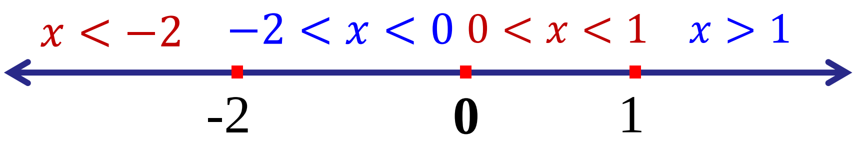
$\Rightarrow\ f$ **is decreasing**

$\therefore\ x > 2 \ \Rightarrow\ f'(x) > 0$

$\Rightarrow\ f$ **is increasing**

Increasing, Decreasing and Constant Functions

Find the interval on which $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is increasing and the intervals on which it is decreasing.



$$f'(x) = 12x^3 + 12x^2 - 24x$$
$$= 12x(x+2)(x-1).$$

$$f'(x) = 0 \;\Rightarrow\; x = -2, x = 0, x = 1$$

$$x < -2 \;\Rightarrow\; x,\;\; x+2, x-1 < 0$$
$$\Rightarrow\; f'(x) < 0$$
$\Rightarrow$ f **is decreasing.**

$$-2 < x < 0 \;\Rightarrow\; x,\;\; x-1 < 0, x+2 > 0$$
$$\Rightarrow\; f'(x) > 0$$
$\Rightarrow$ f **is increasing.**

$$0 < x < 1 \;\Rightarrow\; x,\;\; x+2 > 0, x-1 < 0$$
$$\Rightarrow\; f'(x) < 0$$
$\Rightarrow$ f **is decreasing.**

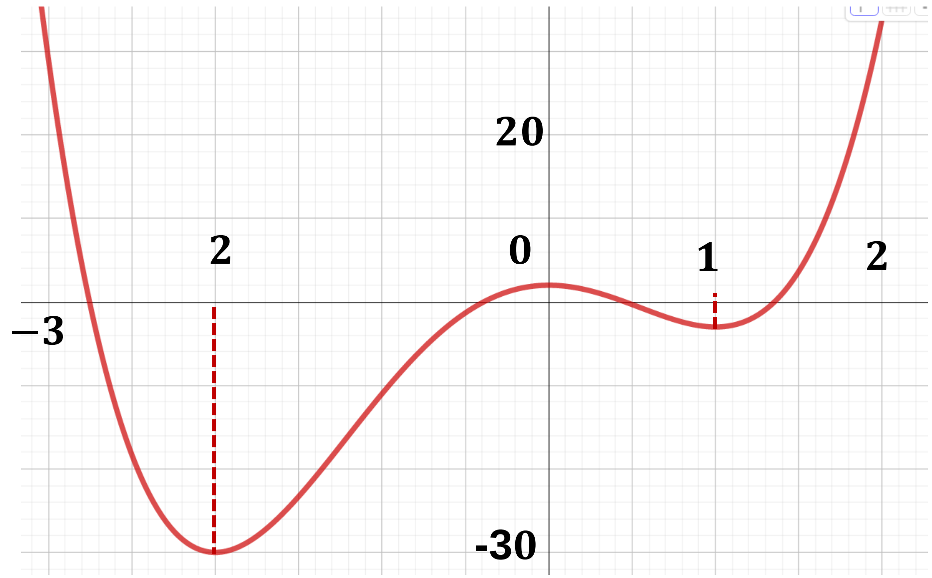
Increasing, Decreasing and Constant Functions

$$f'(x) = 12x(x+2)(x-1).$$

$$x > 1 \;\Rightarrow\; x,\; x+2, x-1 > 0$$

$$\Rightarrow\; f'(x) > 0$$

$\Rightarrow$ f **is increasing.**



decreasing. Increasing .decreasing Increasing

-2 **0** 1

Practice Problems

1. Use the Newton-Raphson method with starting value $x_0 = 1$ to approximate one root of the equation

$$x \tan x = 4,$$

correct to 9 decimal places.

2. Find the equation of the normal to the curve $y = x^2 + 3x - 2$ at the point where the curve cuts the y-axis.

3. Find the equation of the normal to $y = x^2 - 4$ which is parallel to $x + 3y - 1 = 0$.