CELEN037 Seminar 7

University of Nottingham
UK | CHINA | MALAYSIA

Topics

- Integration using Partial Fractions
- Integration by Parts
- Evaluating Definite Integrals

Integration by Partial Fractions

Type 1: Non-repeated linear factors

$$\frac{p(x)}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

$$\Rightarrow \quad \int \frac{p(x)}{(x+a)(x+b)}\,dx = \int \frac{A}{x+a}\,dx + \int \frac{B}{x+b}\,dx$$

Type 2: Non-repeated quadratic factor

$$\frac{p(x)}{(x+a)(x^2+b)} = \frac{A}{x+a} + \frac{Bx+C}{x^2+b} = \frac{A}{x+a} + \frac{Bx}{x^2+b} + \frac{C}{x^2+b}$$

$$\Rightarrow \quad \int \frac{p(x)}{(x+a)(x^2+b)}\,dx = \int \frac{A}{x+a}\,dx + \int \frac{Bx}{x^2+b}\,dx + \int \frac{C}{x^2+b}\,dx$$

Type 3: Repeated linear factor

$$\frac{p(x)}{(x+a)^2(x+b)} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{x+b}$$

$$\Rightarrow \quad \int \frac{p(x)}{(x+a)^2(x+b)}\,dx = \int \frac{A}{x+a}\,dx + \int \frac{B}{(x+a)^2}\,dx + \int \frac{C}{x+b}\,dx$$

Type 1: Non-repeated linear factors

$$\frac{p(x)}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} \Rightarrow \int \frac{p(x)}{(x+a)(x+b)}\,dx = \int \frac{A}{x+a}\,dx + \int \frac{B}{x+b}\,dx$$

Example: Evaluate $\displaystyle\int \frac{13}{(3x-2)(2x+3)}dx$

Solution: Use partial fractions to decompose the integrand:

$$\text{Let} \quad \frac{13}{(3x-2)(2x+3)} = \frac{A}{3x-2} + \frac{B}{2x+3}$$

$$\text{Then} \quad 13 = A(2x+3) + B(3x-2)$$

$$x = \frac{2}{3} \quad \Rightarrow \quad A = 3$$

$$x = -\frac{3}{2} \quad \Rightarrow \quad B = -2$$

Hence $\displaystyle\int \frac{13}{(3x-2)(2x+3)}\,dx$

$$= \int \frac{3}{3x-2}\,dx - \int \frac{2}{2x+3}\,dx$$

$$= \ln|3x-2| - \ln|2x+3| + C$$

Practice Problems on Worksheet:

1. Q1(iii)
2. Q1(iv)
3. Q1(v)
4. Q1(vi)

Answers:

1: $\ln|x-1| + 2\ln|x+2| + C$

2: $2\ln|x+5| - \ln|x-2| + C$

3: $4\ln|x+4| + \ln|x-3| + C$

4: $\dfrac{8}{7}\ln|x-3| + \dfrac{13}{7}\ln|x+4| + C$

University of Nottingham UK | CHINA | MALAYSIA

Type 2: Non-repeated quadratic factor

$$\frac{p(x)}{(x+a)(x^2+b)} = \frac{A}{x+a} + \frac{Bx+C}{x^2+b} = \frac{A}{x+a} + \frac{Bx}{x^2+b} + \frac{C}{x^2+b}$$

$$\Rightarrow \quad \int \frac{p(x)}{(x+a)(x^2+b)}\,dx = \int \frac{A}{x+a}\,dx + \int \frac{Bx}{x^2+b}\,dx + \int \frac{C}{x^2+b}\,dx$$

Example: Evaluate $\displaystyle\int \frac{3}{(x+1)(x^2+2)}dx$

Solution: Use partial fractions to decompose the integrand:

Let $\displaystyle\frac{3}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2} \quad \Rightarrow \quad 3 = A(x^2+2) + (Bx+C)(x+1)$

$x=-1 \quad \Rightarrow \quad A=1$

$x=0 \quad \Rightarrow \quad C=1$

$x=1 \quad \Rightarrow \quad B=-1$

Hence $\displaystyle\int \frac{3}{(x+1)(x^2+2)}\,dx$

$$= \int \frac{1}{x+1}\,dx - \int \frac{x}{x^2+2}\,dx + \int \frac{1}{x^2+2}\,dx$$

$$= \ln|x+1| - \frac{1}{2}\ln\left(x^2+2\right) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

Practice Problems on Worksheet:

1. Q2(iii)
2. Q2(iv)

Answers:

1: $\ln|x-4| - \frac{1}{2}\ln\left(x^2+1\right) - 4\tan^{-1}x + C$

2: $\ln|x-1| - \frac{1}{2}\ln\left(x^2+9\right) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

Integration using Partial Fractions

Type 3: Repeated linear factor

$$\frac{p(x)}{(x+a)^2(x+b)} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{x+b}$$

$$\Rightarrow \quad \int \frac{p(x)}{(x+a)^2(x+b)}\,dx = \int \frac{A}{x+a}\,dx + \int \frac{B}{(x+a)^2}\,dx + \int \frac{C}{x+b}\,dx$$

Example: Evaluate $\displaystyle\int \frac{1}{(x+5)^2(x-1)}dx$

Solution: Let $\displaystyle\frac{1}{(x+5)^2(x+1)} = \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{x-1}$

Then $\quad 1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2$

$x = 1 \quad \Rightarrow \quad C = \dfrac{1}{36}$

$x = -5 \quad \Rightarrow \quad B = -\dfrac{1}{6}$

$x = 0 \quad \Rightarrow \quad A = -\dfrac{1}{36}$

$$\int \frac{1}{(x+5)^2(x-1)}\,dx$$

$$= -\frac{1}{36}\int \frac{1}{x+5}\,dx - \frac{1}{6}\int \frac{1}{(x+5)^2}\,dx + \frac{1}{36}\int \frac{1}{x-1}\,dx$$

$$= -\frac{1}{36}\ln|x+5| + \frac{1}{6(x+5)} + \frac{1}{36}\ln|x-1| + C$$

Practice Problems on Worksheet:

1. Q3(i)
2. Q3(ii)

Answers:

1: $-\ln|x-3| - \dfrac{5}{x-3} + \ln|x+2| + C$

2: $-\ln|x-2| - \dfrac{3}{x-2} + \ln|x+1| + C$

Integration by Parts

Result

$$\int u \cdot \frac{dv}{dx}\, dx = u \cdot v - \int v \cdot \frac{du}{dx}\, dx$$

LIATE Rule: Choose the function that appears first in the following list as u and the other as $\frac{dv}{dx}$

- **L:** **L**ogarithmic functions ($\ln x$, $\log_a x$, etc.)
- **I :** **I**nverse trigonometric functions ($\sin^{-1} x$, $\tan^{-1} x$, etc.)
- **A:** **A**lgebraic functions (x^2, x^n, etc.)
- **T:** **T**rigonometric functions ($\sin x$, $\cos x$, $\tan x$, etc.)
- **E:** **E**xponential functions (e^x, a^x, etc.)

Example: Evaluate $I = \int x^2 \ln x \; dx$

Solution: x^2: **A**lgebraic; $\ln x$: **L**ogarithmic. **L** before **A**:

$$u = \ln x, \qquad \frac{dv}{dx} = x^2$$

$$v = \frac{x^3}{3}, \qquad \frac{du}{dx} = \frac{1}{x}$$

$$\Rightarrow \quad I = \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} \; dx$$

$$= \frac{x^3 \ln x}{3} - \int \frac{x^2}{3} \; dx$$

$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$$

Practice Problems on Worksheet:

1. Q4(iii)
2. Q4(iv)
3. Q4(v)
4. Q4(vi)

Answers:

1: $\dfrac{x^3 \sin^{-1} x}{3} - \dfrac{\left(1-x^2\right)^{\frac{3}{2}}}{9} + \dfrac{\sqrt{1-x^2}}{3} + C$

2: $x \cos^{-1} x - \sqrt{1-x^2} + C$

3: $\dfrac{\left(x^2+1\right) \tan^{-1} x}{2} - \dfrac{x}{2} + C$

4: $x \tan x + \ln \left| \cos x \right| + C$

Sometimes, we need to first use appropriate substitutions on the integrand before applying the method of integration by parts.

Example: Evaluate $I = \int \sin(\ln x)\ dx$

Solution: Let $\ln x = t$. Then $x = e^t \Rightarrow dx = e^t\ dt$.

$$\Rightarrow \quad I = \int e^t \sin t\ dt$$

Let $u = \sin t, \quad \dfrac{dv}{dt} = e^t$

$$\Rightarrow \quad v = e^t, \quad \frac{du}{dt} = \cos t$$

$$I = e^t \sin t - \int e^t \cos t\ dt$$

For $\int e^t \cos t\ dt$, use integration by parts again.

Let $u = \cos t, \quad \dfrac{dv}{dt} = e^t$

$$\Rightarrow \quad v = e^t, \quad \frac{du}{dt} = -\sin t$$

$$\int e^t \cos t\ dt = e^t \cos t + \int e^t \sin t\ dt$$

$$\Rightarrow \quad \int e^t \sin t\ dt = e^t \sin t - e^t \cos t - \int e^t \sin t\ dt$$

$$\int e^t \sin t\ dt = \frac{1}{2} e^t (\sin t - \cos t) + C$$

$$\Rightarrow \int \sin(\ln x)\ dx = \frac{1}{2} x\big(\sin(\ln x) - \cos(\ln x)\big) + C$$

Practice Problems on Worksheet:

1. Q5(i)
2. Q5(ii)

Answers:

1: $\dfrac{e^{x^2}\left(x^2-1\right)}{2}+C$

2: $4\sqrt{x}\left(\ln\sqrt{x}-1\right)+C$

University of Nottingham
UK | CHINA | MALAYSIA

Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = \Big[F(x)\Big]_a^b$$

Example: Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + 2}\,dx$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + 2}\,dx \qquad \left(\frac{f'(x)}{f(x)}\right)$$

$$= \Big[\ln\big|\sin x + 2\big|\Big]_0^{\frac{\pi}{2}}$$

$$= \Big[\ln(\sin x + 2)\Big]_0^{\frac{\pi}{2}}$$

$$= \ln(\sin\frac{\pi}{2} + 2) - \ln(\sin 0 + 2)$$

$$= \ln 3 - \ln 2$$

Evaluating Definite Integrals

Practice Problems on Worksheet:

1. Q6(i)
2. Q6(ii)
3. Q6(iii)

Answers:

1: $\ln 2 + 1$

2: $1 - \dfrac{\pi}{4}$

3: $\ln 2$

Office hours:

Day	Time	Venue
Tuesday	12:00 to 14:00	Trent 314a
	13:00 to 14:00	Trent 322
Thursday	16:30 to 17:30	TB 417
	17:00 to 18:00	IAMET 315
Friday	14:00 to 15:00	PB 330
	17:00 to 18:00	TB 417

Weekly drop-in session: Wednesday 4 – 5 pm in PB-115.