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COMP2054 Tutorial Session 4: Recurrence Relations

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Session outcomes

- Solve recurrence relations to provide exact solutions.
- Use induction to prove recurrence relation definitions.

Exact Solutions

Resolving exact solutions from recurrence relations

Q1. $T(n) = T(n-1) + 1$ and $T(1) = 1$

- Given $T(n) = T(n-1) + 1$ and $T(1) = 1$, give the solution of $T(n)$.
  - $T(1) = 1$
  - $T(2) = T(1) + 1 = 1 + 1$
  - $T(3) = T(2) + 1 = 1 + 1 + 1$
  - $T(4) = T(3) + 1 = 1 + 1 + 1 + 1$

$$T(n) = n$$

Q2. $\boldsymbol{T(n) = 2 \cdot T(n-1)}$ and $\boldsymbol{T(1) = 1}$

- Given $T(n) = 2 \cdot T(n-1)$ and $T(1) = 1$, give the solution of $T(n)$.
    - $T(1) = 1$
    - $T(2) = 2 \cdot T(1) = 2 \times 1$
    - $T(3) = 2 \cdot T(2) = 2 \times 2 \times 1$
    - $T(4) = 2 \cdot T(3) = 2 \times 2 \times 2 \times 1$

$$T(n) = 2^{n-1}$$

Q3. $T(n) = 2 \cdot T(n/2)$ and $T(1) = 1$

- Given $T(n) = 2 \cdot T(n/2)$ and $T(1) = 1$, give the solution of $T(n)$.
  - $T(2) = T(2^1) = 2 \cdot T(1) = 2 \times 1$
  - $T(4) = T(2^2) = 2 \cdot T(2) = 2 \cdot 2 \cdot T(1) = 2 \times 2 \times 1$
  - $T(8) = T(2^3) = 2 \cdot T(4) = 2 \cdot 2 \cdot 2 \cdot T(1) = 2 \times 2 \times 2 \times 1$

$$T(2^k) = 2^k$$

Here we are dividing ‘n’ by a constant, in this case 2, so we want to evaluate the values of ‘n’ that are Powers of this constant. Here we are doing base case 1 = 2^0, and then $2^1, 2^2, 2^3$ hence we keep the term $T(2^k)$ as it is generally easier for when we do the proofs.

Resolve the exact solutions for the following:

- Q4. $T(n) = 3 \cdot T(n-1)$ and $T(1) = 1$
- Q5. $T(n) = 3 \cdot T(n/3)$ and $T(1) = 1$
- Q6. $T(n) = 2 \cdot T(n/4)$ and $T(1) = 1$

Resolve the exact solutions for the following:

- Q4. $T(n) = 3 \cdot T(n-1)$ and $T(1) = 1$ $T(n) = 3^{n-1}$
- Q5. $T(n) = 3 \cdot T(n/3)$ and $T(1) = 1$ $T\left(3^k\right) = 3^k$
- Q6. $T(n) = 2 \cdot T(n/4)$ and $T(1) = 1$ $T\left(4^k\right) = 2^k$

Recurrence Proofs

Proving exact solutions are the same as their recursive definitions

Q1. Proof

Given: $T(n) = T(n-1) + 1$ and $T(1) = 1$

Assume "thing we need to prove" is true for $n = k$.

Prove: $T(n) = n$

Base case:

- $T(1) = 1$

From "thing we need to prove" with $n = 1$.
Matches base case from definition. ✓

Induction step:

Assume induction hypothesis is true for $n = k$ such that $T(k) = k$ and prove for $n = k + 1$:

- $T(k+1) = T(k+1-1) + 1$ Using original definition with $n = k + 1$
- $= T(k) + 1$ Simplify $T(k+1-1)$ to $T(k)$
- $= k + 1$ Use the equivalence from the induction hypothesis to replace (rewrite) $T(k)$ with k.

QED We wanted to prove $T(n) = n$ and have shown that $T(k+1) = k + 1$ in the step case, hence, we have finished.

Q2. Proof

Given: $T(n) = 2 \cdot T(n-1)$ and $T(1) = 1$
Prove: $T(n) = 2^{n-1}$

Base case:

- $T(1) = 2^0 = 1$

Induction step:

Assume induction hypothesis is true for $n = k$ such that $T(k) = 2^{k-1}$ and prove for $n = k + 1$:

- $T(k+1) = 2 \cdot T(k + 1 - 1)$
- $= 2 \cdot T(k)$
- $= 2 \cdot 2^{k-1}$
- $= 2^k = 2^{(k+1)-1}$

QED

Proof follows the same structure as the previous Q1 proof. Refer to previous slide.

Q3. Proof

In this proof, we are doing induction on the 'k' as opposed to 'n' more generally as seen in the previous questions.

Given: $T(n) = 2 \cdot T(n/2)$ and $T(1) = 1$

Prove: $T(2^k) = 2^k$

Base case:

- $T(1) = 2^0 = 1$

Induction step:

Assume induction hypothesis is true for $n = k$ such that $T(2^k) = 2^k$ and prove for $k + 1$:

- $T(2^{k+1}) = 2 \cdot T\left(\frac{2^{k+1}}{2}\right)$
- $= 2 \cdot T(2^k)$
- $= 2 \cdot 2^k$
- $= 2^{k+1}$

QED

Notice here that because we are doing the induction on 'k' we are now proving that this holds for $T(2^{k+1})$; not for $T(2^k + 1)$.

We wanted to prove $T(2^k) = 2^k$ and have shown that $T(2^{k+1}) = 2^{k+1}$ in the step case, hence, we have finished.

Recurrence proofs

- Q4. Given $T(n) = 3 \cdot T(n-1)$ and $T(1) = 1$

  Prove that $T(n) = 3^{n-1}$
- Q5. $T(n) = 3 \cdot T(n/3)$ and $T(1) = 1$

  Prove that $T(3^k) = 3^k$
- Q6. $T(n) = 2 \cdot T(n/4)$ and $T(1) = 1$

  Prove that $T(4^k) = 2^k$

Q4. Proof

Given: $T(n) = 3 \cdot T(n-1)$ and $T(1) = 1$
Prove: $T(n) = 3^{n-1}$

Base case:

- $T(1) = 3^0 = 1$

Induction step:

Assume IH is true for $n = k$ such that $T(k) = 3^{k-1}$ and prove for $k + 1$:

- $T(k+1) = 3 \cdot T(k)$
- $= 3 \cdot 3^{k-1}$
- $= 3^k$
- $= 3^{(k+1)-1}$

QED

Q5. Proof

Given: $T(n) = 3 \cdot T(n/3)$ and $T(1) = 1$
Prove: $T(3^k) = 3^k$
Base case:
- $T(1) = 3^0 = 1$

Induction step:
Assume IH is true for $n = k$ such that $T(3^k) = 3^k$ and prove for $k + 1$:
- $T(3^{k+1}) = 3 \cdot T\left(\frac{3^{k+1}}{3}\right)$
- $= 3 \cdot T(3^k)$
- $= 3 \cdot 3^k$
- $= 3^{(k+1)}$

QED

Q6. Proof

Given: $T(n) = 2 \cdot T(n/4)$ and $T(1) = 1$

Prove: $T\left(4^k\right) = 2^k$

Base case:

- $T(1) = 4^0 = 1 = 2^0$

Induction step:

Assume IH is true for $n = k$ such that $T\left(4^k\right) = 2^k$ and prove for $k + 1$:

- $T\left(4^{k+1}\right) = 2 \cdot T\left(\frac{4^{k+1}}{4}\right)$
- $= 2 \cdot T\left(4^k\right)$
- $= 2 \cdot 2^k$
- $= 2^{(k+1)}$

QED

Additional Practice Questions

For each of the following:

1. Find the exact solution
2. Prove by induction

Assume you are given $T(1) = 1$

- Q7. $T(n) = 4 \cdot T(n/4)$
- Q8. $T(n) = 4 \cdot T(n/2)$
- Q9. $T(n) = T(n-1) + n$
- Q10. $T(n) = 2 \cdot T(n/2) + 1$
- Q11. $T(n) = n \cdot T(n-1)$

Q7. Exact solution

$T(n) = 4 \cdot T(n/4)$

$T(1) = 1$

$T(4) = 4 \cdot T(1) = 4 \times 1 = 4$

$T(16) = 4 \cdot T(4) = 4 \times 4 \times 1 = 16$

$T\left(4^k\right) = 4^k$

Q7. Proof

Given: $T(n) = 4 \cdot T(n/4)$ and $T(1) = 1$

Prove: $T(4^k) = 4^k$

Base case:

- $T(1) = 4^0 = 1$

Induction step:

Assume IH is true for $n = k$ such that $T(4^k) = 4^k$ and prove for $n = k + 1$:

- $T(4^{k+1}) = 4 \cdot T\left(\frac{4^{k+1}}{4}\right)$
- $= 4 \cdot T(4^k)$
- $= 4 \cdot 4^k$
- $= 4^{(k+1)}$

QED

Q8. Exact solution

$T(n) = 4 \cdot T(n/2)$

$T(1) = 1$

$T(2) = 4 \cdot T(1) = 4 \times 1 = 4$

$T(4) = 4 \cdot T(2) = 4 \times 4 \times 1 = 16$

$T(8) = 4 \cdot T(4) = 4 \times 4 \times 4 \times 1 = 64$

$T\left(2^k\right) = 4^k$

Q8. Proof

Given: $T(n) = 4 \cdot T(n/2)$ and $T(1) = 1$

Prove: $T(2^k) = 4^k$

Base case:

- $T(1) = T(2^0) = 4^0 = 1$

Induction step:

Assume IH is true for $n = k$ such that $T(2^k) = 4^k$ and prove for $n = k + 1$:

- $T(2^{k+1}) = 4 \cdot T\left(\frac{2^{k+1}}{2}\right)$
- $= 4 \cdot T(2^k)$
- $= 4 \cdot 4^k$
- $= 4^{(k+1)}$

QED

Q9. Exact solution

$T(n) = T(n-1) + n$

$T(1) = 1$

$T(2) = T(1) + 2 = 1 + 2 = 2$

$T(3) = T(2) + 3 = 1 + 2 + 3 = 6$

$T(4) = T(3) + 4 = 1 + 2 + 3 + 4 = 10$

[Arithmetic series]

$T(n) = \frac{n(n+1)}{2}$

Q9. Proof

Given: $T(n) = T(n-1) + n$ and $T(1) = 1$

Prove: $T(n) = \frac{n(n+1)}{2}$

Base case:

- $T(1) = \frac{1 \cdot 2}{2} = 1$

Induction step:

Assume IH is true for $n = k$ such that $T(k) = \frac{k(k+1)}{2}$ and prove for $n = k + 1$:

- $T(k+1) = T(k) + k + 1$
- $= \frac{k(k+1)}{2} + k + 1$
- $= \frac{k(k+1)+2k+2}{2}$
- $= \frac{k^2+3k+2}{2}$
- $= \frac{(k+1)(k+2)}{2}$

QED

Q10. Exact solution

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 1$$

$$T(1) = 1$$

$$T(2) = 2 \cdot T(1) + 1 = (2 \times 1) + 1 = 3$$

$$T(4) = 2 \cdot T(2) + 1 = 2 \times \big((2 \times 1) + 1\big) + 1 = 7$$

$$T(8) = 2 \cdot T(4) + 1 = 15$$

$$T\left(2^k\right) = 2^{k+1} - 1$$

Q10. Proof

Given: $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 1$ and $T(1) = 1$

Prove: $T\left(2^k\right) = 2^{k+1} - 1$

Base case:

- $T(1) = T(2^0) = 2^1 - 1 = 1$

Induction step:

Assume IH is true for $n = k$ such that $T\left(2^k\right) = 2^{k+1} - 1$ and prove for $n = k + 1$:

- $T\left(2^{k+1}\right) = 2 \cdot T\left(\frac{2^{k+1}}{2}\right) + 1$
- $= 2 \cdot T\left(2^k\right) + 1$
- $= 2 \cdot \left(2^{k+1} - 1\right) + 1$
- $= 2^{k+1+1} - 1$

QED

Q11. Exact solution

$T(n) = T(n) = n \cdot T(n-1)$

$T(1) = 1$

$T(2) = 2 \cdot T(1) = 2 \times 1 = 2$

$T(3) = 3 \cdot T(2) = 3 \times 2 \times 1 = 6$

$T(4) = 4 \cdot T(3) = 4 \times 3 \times 2 \times 1 = 24$

$T(n) = n!$

Q11. Proof

Given: $T(n) = n \cdot T(n-1)$

Prove: $T(n) = n!$

Base case:

- $T(1) = 1! = 1$

Induction step:

Assume IH is true for $n = k$ such that $T(n) = n!$ and prove for $n = k + 1$:

- $T\big((n+1)\big) = (n+1) \times T(n)$
- $= (n+1) \times n!$
- $= 1 \times \cdots \times n \times (n+1) = (n+1)!$
- **QED**

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Thank you