Foundation Calculus & Mathematical Techniques — CELEN037

Weekly Worksheet — **Topics:** Integration, Substitution, Standard Integrals, Integrals of the form $\displaystyle\int f(ax+b)\,dx$ and $\displaystyle\int \sin mx\,\cos nx\,dx$

Type 1: Simple Integration

1. Evaluate the following integrals:

(i) $\displaystyle\int \left(x+\frac{1}{x}\right)^2 dx$ (ii) $\displaystyle\int \frac{(2x+1)^2}{x}\,dx$

(iii) $\displaystyle\int \left(\frac{x^6+x^4+1}{x^2}\right) dx$ (iv) $\displaystyle\int \left(\frac{x^4-1}{x^2-1}\right) dx$

(v) $\displaystyle\int \left(\frac{x^4-8x}{x-2}\right) dx$ (vi) $\displaystyle\int \left(\frac{x-\dfrac{1}{x}}{\sqrt{x}+\dfrac{1}{\sqrt{x}}}\right) dx$

(vii) $\displaystyle\int (e^x+x^e+e+x)\,dx$ (viii) $\displaystyle\int \left(\frac{5\sin x+2}{\cos^2 x}\right) dx$

(ix) $\displaystyle\int \left(\frac{\cos 2x}{\cos^2 x\,\sin^2 x}\right) dx$ (x) $\displaystyle\int \left(\frac{1}{\cos^2 x\,\sin^2 x}\right) dx$

(xi) $\displaystyle\int \left[\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right]^2 dx$ (xii) $\displaystyle\int \left[2^x+2\,\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)\right] dx$

(xiii) $\displaystyle\int \left(\frac{\sec^2 x}{\operatorname{cosec}^2 x}\right) dx$ (xiv) $\displaystyle\int \sec x\,(\sec x+\tan x)\,dx$

Type 2: The method of substitution for Integration

2. Evaluate the following integrals by using given substitutions:

(i) $\displaystyle\int \frac{1}{x\,(\ln x)^2}\,dx$ Let $\ln x = t$ (ii) $\displaystyle\int \frac{\sec^2 x}{1+2\,\tan x}\,dx$ Let $1+2\,\tan x = t$

(iii) $\displaystyle\int \frac{\sin 3x}{1+\cos 3x}\,dx$ Let $1+\cos 3x = t$ (iv) $\displaystyle\int \frac{\cos 4x}{(1+2\sin 4x)^4}\,dx$ Let $1+2\sin 4x = t$

(v) $\displaystyle\int x^2\,\cos(1-x^3)\,dx$ Let $1-x^3 = t$ (vi) $\displaystyle\int x^3\,(1-x^4)^7\,dx$ Let $1-x^4 = t$

(vii) $\displaystyle\int x^3\cdot e^{(x^4)}\,dx$ Let $x^4 = t$ (viii) $\displaystyle\int \frac{1}{\sqrt{x}\cdot e^{(2\sqrt{x})}}\,dx$ Let $\sqrt{x} = t$

3. Evaluate the following integrals by using appropriate substitutions:

(i) $\displaystyle\int x\,\sec^2\left(x^2\right)\,dx$

(ii) $\displaystyle\int \frac{\sec^2\left(\sqrt{x}\,\right)}{\sqrt{x}}\,dx$

(iii) $\displaystyle\int \frac{\sin\left(1/x\right)}{3\,x^2}\,dx$

(iv) $\displaystyle\int e^{\tan x}\cdot\sec^2 x\,dx$

(v) $\displaystyle\int \sec^3 x\,\tan x\,dx$

(vi) $\displaystyle\int \frac{1}{x}\cdot(\ln x)^n\,dx \quad ; \quad n\in\mathbb{N}$

(vii) $\displaystyle\int (2x+7)\,\left(x^2+7x+3\right)^{\frac{4}{5}}\,dx$

(viii) $\displaystyle\int \frac{5\,x^4}{\left(x^5+1\right)^2}\,dx$

(ix) $\displaystyle\int e^x\cdot\left(e^x+2\right)^4\,dx$

(x) $\displaystyle\int (x+1)\cdot(x+3)^5\,dx$

(xi) $\displaystyle\int \frac{1}{x\cdot(\ln x+1)^2}\,dx$

(xii) $\displaystyle\int \frac{e^x}{1+e^x}\,dx$

(xiii) $\displaystyle\int \frac{\sec^2 x}{1-\tan x}\,dx$

(xiv) $\displaystyle\int x^3\cdot\sqrt{5+x^4}\,dx$

Note: In this type of integrals, let the term inside the square root sign $=\ t^2$.

(xv) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$

(xvi) $\displaystyle\int \frac{x}{\sqrt{1+x}}\,dx$

(xvii) $\displaystyle\int \frac{1+4x}{\sqrt{1+x+2x^2}}\,dx$

(xviii) $\displaystyle\int \frac{\sin x}{\sqrt{5+\cos x}}\,dx$

Type 3: More standard integrals

Formulae:

$\displaystyle\int \frac{1}{x^2+a^2}\,dx=\frac{1}{a}\,\tan^{-1}\left(\frac{x}{a}\right)+C$	$\displaystyle\int \frac{1}{x^2+1}\,dx=\tan^{-1}(x)+C$
$\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)+C$	$\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx=\sin^{-1}(x)+C$
$\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx=\frac{1}{a}\,\sec^{-1}\left(\frac{x}{a}\right)+C$	$\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx=\sec^{-1}(x)+C$

4. Evaluate the following integrals:

(i)	$\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$	(ii)	$\displaystyle\int \frac{1}{x\,\sqrt{x^2-25}}\,dx$
(iii)	$\displaystyle\int \left(2\,x+5\,(1-x^2)^{-\frac{1}{2}}\right)\,dx$	(iv)	$\displaystyle\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$
(v)	$\displaystyle\int \frac{1}{1+\sin^2 x}\cos x\,dx$	(vi)	$\displaystyle\int \frac{\sin x}{1+\cos^2 x}\,dx$
(vii)	$\displaystyle\int \frac{x}{1+x^4}\,dx$	$(viii)$	$\displaystyle\int \frac{x^2}{1+x^6}\,dx$
(ix)	$\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$	(x)	$\displaystyle\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$
(xi)	$\displaystyle\int \frac{1}{\sqrt{x}\cdot(1+x)}\,dx$	(xii)	$\displaystyle\int \frac{1}{x\cdot\sqrt{1-(\ln x)^2}}\,dx$

Type 4: Integrals of the form $\displaystyle\int f(ax+b)\,dx$

$$\boxed{\int f(ax+b)\,dx = \frac{1}{a}\,F(ax+b)+C, \text{ where } F \text{ is the antiderivative of } f.}$$

5. Evaluate the following integrals:

(i)	$\displaystyle\int \cos 2x\,dx$	(ii)	$\displaystyle\int \sin(2x+5)\,dx$
(iii)	$\displaystyle\int e^{3x+7}\,dx$	(iv)	$\displaystyle\int e^{-x}\,dx$
(v)	$\displaystyle\int e^{-2x}\,dx$	(vi)	$\displaystyle\int \sec^2(4x+1)\,dx$
(vii)	$\displaystyle\int \sin x\,\cos x\,dx$	$(viii)$	$\displaystyle\int \left(e^x+e^{-x}\right)^2\,dx$

6. Evaluate the following integrals:

(i)	$\displaystyle\int \sin 5x\,\cos 3x\,dx$	(ii)	$\displaystyle\int \sin 3x\,\cos 2x\,dx$
(iii)	$\displaystyle\int \cos 4x\,\sin 2x\,dx$	(iv)	$\displaystyle\int \cos 4x\,\cos 3x\,dx$
(v)	$\displaystyle\int \sin 4x\,\sin x\,dx$	(vi)	$\displaystyle\int \cos 5x\,\sin 4x\,dx$