Foundation Algebra (CELEN036)

Problem Sheet 9 — Topics: Complex numbers

Topic 1: Real and imaginary parts of complex numbers

1. Find the real part and the imaginary part of the following complex numbers:

   (i) $z = 5 + 7i$ (ii) $z = -3 - 6i$ (iii) $z = 4$ (iv) $z = -\sqrt{5}i$

2. Solve the following equations for x and y, where $x, y \in \mathbb{R}$:

   (i) $x + iy = (3 + i)(2 - 3i)$ (ii) $x + iy = 3$

   (iii) $x + iy = 2i$ (iv) $2 + 3i = (x + iy)(1 - i)$

3. Find the real and imaginary parts of w defined by $w = \dfrac{1 + z}{1 - z}$, where $z = x + iy$ for some $x, y \in \mathbb{R}$.

Topic 2: Expressing complex numbers in the form $a + b\,i$

4. If $z_1 = 4 - i$ and $z_2 = 3 + 2i$, express the following in the form $a + b\,i$, where $a, b \in \mathbb{R}$:

   (i) $z_1 + z_2$ (ii) $z_1 - z_2$ (iii) $z_1 \cdot z_2$ (iv) $\dfrac{z_1}{z_2}$

5. Express the following complex numbers in the form $a + b\,i$, where $a, b \in \mathbb{R}$:

   (i) $(5 + 6i) + (7 + 3i)$ (ii) $(2 - 3i) - (5 + 2i)$ (iii) $(3 + 2i)(2 - i)$

   (iv) $(1 + i)^2$ (v) $\left(\sqrt{2} - \sqrt{2}i\right)^5$ (vi) $\dfrac{2 + i}{7 - 2i}$

   (vii) $\dfrac{9 - i}{1 + 3i}$ (viii) $\dfrac{(1 + 2i)(1 + 3i)}{1 + i}$ (ix) $(\overline{4 - 3i})(8 + i)$

   (x) $\dfrac{\overline{(3 + 4i)}}{5 - 2i}$ (xi) $\overline{\left(\dfrac{3 + 4i}{5 - 2i}\right)}$ (xii) $\dfrac{(1 + 3i)(1 + 2i)}{(1 + i)(2 - i)}$

6. Given $z = 1 + 3i$, express $z + \dfrac{2}{z}$ in the form $a + b\,i$, where $a, b \in \mathbb{R}$.

7. Evaluate $(1 + i)^6 - (1 - i)^3$.

8. Find the two square roots of $(-7 + 24i)$.

Topic 3: Solving equations

9. Solve the following equations for $z \in \mathbb{C}$:

   (i) $\dfrac{iz - 2}{z + 3i} = 1 + 2i$ (ii) $\dfrac{1}{z} + \dfrac{1}{2 + i} = \dfrac{1}{1 + 3i}$

10. Solve the following polynomial equations:

    (i) $x^2 + 6x + 10 = 0$

    (ii) $4x^2 + 25 = 0$

    (iii) $x^3 - 5x^2 + 17x - 13 = 0$, given that $x = 2 + 3i$ is a root.

    (iv) $x^3 + 6x^2 + 13x + 10 = 0$, given that $x = -2$ is a root.

11. Find all possible values of $z \in \mathbb{C}$ that satisfy the following equations:

    (i) $z\,\overline{z} + 2i \cdot z = 12 + 6i$ (ii) $z^2 + 2\,\overline{z} + 1 = 0$ (iii) $z^2 = \overline{z}$

Topic 4: Argand diagram

12. Plot the following complex numbers on the Argand diagram:

    (i) $3 - 4i$ (ii) $2 + 5i$ (iii) $-4 + 2i$ (iv) $1 - 6i$

    (v) $3i$ (vi) -7 (vii) $-i$ (viii) $-3 + 4i$

13. For the given complex numbers z_1 and z_2, plot $z_1 + z_2$ and $z_1 - z_2$ on the Argand diagram:

    (i) $z_1 = 3 + 4i, \quad z_2 = 2 - 3i$ (ii) $z_1 = 4 - 3i, \quad z_2 = 1 + 2i$

    (iii) $z_1 = 2 - i, \quad z_2 = 3 + i$ (iv) $z_1 = -4 - 3i, \quad z_2 = -4 + 3i$

Topic 5: Modulus and argument

14. Given $z_1 = 3 - 2i$, $z_2 = 1 + 4i$, and $z_3 = 4 + 5i$, find the following values:

    (i) $|z_1\, z_2|$ (ii) $\left|\dfrac{z_1\, z_3}{z_2}\right|$ (iii) $\left|\dfrac{z_1\, z_2 + z_3}{z_3}\right|$

15. Find the modulus and the principal value of the argument $(-\pi < \theta \le \pi)$ of the following complex numbers:

    (i) $z_1 = 1 + i$ (ii) $z_2 = 1 + \sqrt{3}\,i$ (iii) $z_3 = -1 + \sqrt{3}\,i$ (iv) $z_4 = 3i$

Topic 6: Polar form of complex numbers

16. Express the following complex numbers in polar form $r(\cos\theta + i\sin\theta) \quad (-\pi < \theta \le \pi)$:

    (i) $\sqrt{3} + i$ (ii) $1 - \sqrt{3}\,i$ (iii) $4i$ (iv) -1

    (v) $\sqrt{3} + \sqrt{3}\,i$ (vi) $\sqrt{3} - \sqrt{3}\,i$ (vii) $-1 + \sqrt{3}\,i$ (viii) $-\sqrt{3} - i$

17. Find the polar form of the following complex numbers:

    (i) $z_1 = 2 + 2i$ (ii) $z_2 = 2 - 2i$ (iii) $z_3 = i$.

    Hence find the modulus r and principal argument $(\theta \in (-\pi, \pi])$ of the complex numbers:

    (iv) $(2 + 2i)(2 - 2i)$ (v) $(2 + 2i)^2$ (vi) $i\,(2 + 2i)$.

18. Find the Cartesian form ($a + b\,i,\ a, b \in \mathbb{R}$) of the following complex numbers in the polar form (you may use the calculator):

    (i) $z = 2\sqrt{2}\left[\cos\left(\frac{7\pi}{12}\right) + i\,\sin\left(\frac{7\pi}{12}\right)\right]$ (ii) $z = 5\left[\cos\left(\frac{-3\pi}{7}\right) + i\,\sin\left(\frac{-3\pi}{7}\right)\right]$

19. Given complex numbers $z_1 = 8\,[\cos(2.24) + i\sin(2.24)]$ and $z_2 = 2\,[\cos(-1.43) + i\sin(-1.43)]$, find the polar form of the following complex numbers:

    (i) $z_1\,z_2$ (ii) $\frac{z_1}{z_2}$

Answers

1. (i) $Re(z) = 5,\ Im(z) = 7$ (ii) $Re(z) = -3,\ Im(z) = -6$

   (iii) $Re(z) = 4,\ Im(z) = 0$ (iv) $Re(z) = 0,\ Im(z) = -\sqrt{5}$

2. (i) $x = 9,\ y = -7$ (ii) $x = 3,\ y = 0$ (iii) $x = 0,\ y = 2$ (iv) $x = -0.5,\ y = 2.5$

3. (i) $Re(w) = \frac{1 - x^2 - y^2}{x^2 + y^2 - 2x + 1}$ $Im(w) = \frac{2y}{x^2 + y^2 - 2x + 1}$

4. (i) $7 + i$ (ii) $1 - 3i$ (iii) $14 + 5i$ (iv) $\frac{10}{13} - \frac{11}{13}i$

5. (i) $12 + 9i$ (ii) $-3 - 5i$ (iii) $8 + i$ (iv) $2i$

   (v) $-16\sqrt{2} + 16\sqrt{2}i$ (vi) $\frac{12}{53} + \frac{11}{53}i$ (vii) $\frac{3}{5} - \frac{14}{5}i$ (viii) $5i$

   (ix) $29 + 28i$ (x) $\frac{23}{29} - \frac{14}{29}i$ (xi) $\frac{7}{29} - \frac{26}{29}i$ (xii) $-1 + 2i$

6. $\frac{6}{5} + \frac{12}{5}i$

7. $2 - 6i$

8. $3 + 4i,\quad -3 - 4i$

9. (i) $\frac{1}{2} - \frac{7}{2}i$ (ii) $-3 + i$

10. (i) $-3 \pm i$ (ii) $\pm\frac{5}{2}i$ (iii) $2 \pm 3i,\quad 1$ (iii) $-2 \pm i,\quad -2$

11. (i) $3 + 3i$ or $3 - i$ (ii) -1 or $1 \pm 2i$ (iii) 0 or 1 or $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

14. (i) $\sqrt{221}$ (ii) $\sqrt{\frac{533}{17}}$ (iii) $15\sqrt{\frac{2}{41}}$

15. (i) $|z_1| = \sqrt{2}, \quad \arg(z_1) = \dfrac{\pi}{4}$ (ii) $|z_2| = 2, \quad \arg(z_2) = \dfrac{\pi}{3}$

(iii) $|z_3| = 2, \quad \arg(z_3) = \dfrac{2\pi}{3}$ (iv) $|z_4| = 3, \quad \arg(z_4) = \dfrac{\pi}{2}$

16. (i) $2\left[\cos\left(\dfrac{\pi}{6}\right) + i\sin\left(\dfrac{\pi}{6}\right)\right]$ (ii) $2\left[\cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right)\right]$

(iii) $4\left[\cos\left(\dfrac{\pi}{2}\right) + i\sin\left(\dfrac{\pi}{2}\right)\right]$ (iv) $\cos\pi + i\sin\pi$

(v) $\sqrt{6}\left[\cos\left(\dfrac{\pi}{4}\right) + i\sin\left(\dfrac{\pi}{4}\right)\right]$ (vi) $\sqrt{6}\left[\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right]$

(vii) $2\left[\cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)\right]$ (viii) $2\left[\cos\left(-\dfrac{5\pi}{6}\right) + i\sin\left(-\dfrac{5\pi}{6}\right)\right]$

17. (i) $2\sqrt{2}\left[\cos\left(\dfrac{\pi}{4}\right) + i\sin\left(\dfrac{\pi}{4}\right)\right]$ (ii) $2\sqrt{2}\left[\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right]$

(iii) $\cos\left(\dfrac{\pi}{2}\right) + i\sin\left(\dfrac{\pi}{2}\right)$ (iv) $r = 8, \quad \theta = 0$

(v) $r = 8, \quad \theta = \dfrac{\pi}{2}$ (vi) $r = 2\sqrt{2}, \quad \theta = \dfrac{3\pi}{4}$

18. (i) $-0.7321 + 2.7321i$ (ii) $1.1126 - 4.8746i$

19. (i) $16\left[\cos(0.81) + i\sin(0.81)\right]$ (ii) $4\left[\cos(3.67) + i\sin(3.67)\right]$