Foundation Algebra (CELEN036)

Problem Sheet 6 **Topics: Numerical Methods**

Topic 1: Intermediate value theorem

1. Show that each of the following equation has a root in the given intervals.

   (i) $x^3 - x + 5 = 0\ ;\ -2 < x < -1$ (ii) $x^5 - 5x^3 - 10 = 0\ ;\ 2 < x < 3$

   (iii) $\sqrt[3]{x} - \cos x = 0\ ;\ 0.5 < x < 0.6$ (iv) $\sin x - \ln x = 0\ ;\ 2.2 < x < 2.3$

   (v) $e^{-x} = x^2\ ;\ 0.7 < x < 0.71$ (vi) $e^x + 2x - 3 = 0\ ;\ 0.5 < x < 0.6$

2. Show that $x - \sqrt{\sin x + \cos x} = 0\ ;\ 0 \leq x \leq \dfrac{3\pi}{4}$ has a root between $x = \dfrac{\pi}{3}$ and $x = \dfrac{\pi}{2}$.

Topic 2: Bisection method

3. Given that a root of the equation $x^3 - 3x^2 - 2x + 5 = 0$ lies between $x = 3$ and $x = 4$.

   Use the Bisection method to approximate this root, correct to 2 decimal places.

4. Use the Bisection method to solve the following equations (correct to 2 decimal places):

   (i) $x = \cos x$ (ii) $e^{-x} = \ln x$ (iii) $e^x + x^4 + x = 2$ (Root lies in $(0, 1)$)

Topic 3: Iteration method

5. The equation $x^3 - 5x - 2 = 0$ has a root between 2 and 3. Use the iterative formula

   $$x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 - 5}$$

   starting with $x_0 = 2$, to find this root correct to 5 decimal places.

6. Consider solving numerically the equation

$$2x^3 - x - 4 = 0. \tag{1}$$

Show that equation (1) can be rearranged to give the iterative formula

$$x_{n+1} = \sqrt{\frac{2}{x_n} + \frac{1}{2}}. \tag{2}$$

Use (2) with $x_0 = 1.35$ to find the root of (1), correct to 3 decimal places.

7. Use the iterative formula

$$x_{n+1} = 2\sin x_n$$

with $x_0 = 1$, to find the root of the equation $x = 2\sin x$, correct to 3 decimal places.

8. Given $f(x) = 5x - 4\sin x - 2$, where x is in radians.

   (i) Find $f(1.1)$ and $f(1.15)$ and state whether the equation $f(x) = 0$ has a root in the interval $(1.1, 1.15)$ or not.

   (ii) If the iteration formula is of the form $x_{n+1} = p\sin x_n + q$, find the constants p and q.

   (iii) Use this iteration formula with $x_0 = 1.1$ to find x_4 correct to 4 decimal places.

9. Show that the two possible arrangements of

$$x^3 - 4x + 1 = 0 \tag{3}$$

lead to the iterative formulae

$$x_{n+1} = \frac{1}{4}\left(x_n^3 + 1\right) \text{ and} \tag{4}$$

$$x_{n+1} = \sqrt[3]{4x_n - 1} \tag{5}$$

   (i) By taking $x_0 = 1$, use (4) to calculate the positive root of (3), correct to 3 d.p..

(ii) By taking $x_0 = 2$, use (5) to calculate the root of (3), correct to 3 d.p..

(iii) Show that with $x_0 = 2$, the iterative scheme (4) leads to a divergent sequence.

10. Show that the three possible arrangements of

$$x^3 - 6x - 2 = 0 \tag{6}$$

lead to the iterative formulae

$$x_{n+1} = \frac{x_n^3 - 2}{6} \tag{7}$$

$$x_{n+1} = \sqrt[3]{6x_n + 2} \tag{8}$$

$$x_{n+1} = \frac{6x_n + 2}{x_n^2} \tag{9}$$

Find the roots of (6), correct to 4 decimal places by starting with

(i) $x_0 = -2$ in (7)

(ii) $x_0 = 1$ in (8)

(iii) $x_0 = -2$ in (9)

11. By putting $x_{n+1} = x_n = x$ in the iterative formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{12}{x_n}\right) \tag{10}$$

show that the sequence converges to $\sqrt{12}$.

Use the iterative scheme (10) with $x_0 = 2$ to approximate $\sqrt{12}$, correct to 4 decimal places.

12. (i) Show that

$$x^3 - 3x^2 - 2x + 5 = 0 \tag{11}$$

has a root in the interval $3 < x < 4$.

(ii) Use the iteration formula $x_{n+1} = \sqrt{\dfrac{x_n^3 - 2x_n + 5}{3}}$ to find an approximation for the root, correct to 4 decimal places of the equation (11) by taking $x_0 = 3$.

(iii) What happens if you take starting value as $x_0 = 3.5$? (Compare with Q.3)

13 (i) Show that

$$e^x - x = 4 \tag{12}$$

has a root between 1 and 2.

(ii) Show that the iterative formula

$$x_{n+1} = \frac{e^{x_n}(x_n - 1) + 4}{e^{x_n} - 1} \tag{13}$$

leads to a solution of the equation (12).

(iii) Use $x_0 = 1$ in the iterative formula (13) to find a root, correct to 4 decimal places, of the equation (12).

(iv) Hence find a root of $e^{2\cos x} - 2\cos x = 4$, correct to 4 decimal places.

Answers

3. 3.13

4. (i) 0.74 (ii) 1.31 (iii) 0.43

5. 2.41421

6. 1.392

7. 1.895

8. (i) $f(1.1) = -0.0648,\ f(1.15) = 0.0989$ (ii) $p = \dfrac{4}{5}$ and $q = \dfrac{2}{5}$ (iii) 1.1200

9. (i) 0.254 (ii) 1.861 (iii) $x_5 = 365869.5225$, thus divergent

10. (i) -0.3399 (ii) 2.6016 (iii) -2.2618 (extremely slow convergence)

11. 3.4641

12. (ii) 1.2017 (iii) The sequence of approximations is divergent

13. (iii) 1.7490 (iv) 0.5064 radians