Foundation Calculus and Mathematical Techniques (CELEN037)

Problem Sheet 3

Topic 1: Parametric Differentiation

1. Using parametric differentiation, find $\dfrac{dy}{dx}$, where
   (i) $x = 2\cos\theta + \cos 2\theta + 1, y = 2\sin\theta + \sin 2\theta$
   (ii) $x = \cos^{-1} t$, $y = \sin^{-1} t$
   (iii) $x = \dfrac{1}{2}\ln(1+t^2), y = \tan^{-1} t$
   (iv) $x = e^t \cos t, y = e^t \sin t$

2. Find $\left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{4}}$, where $x = \sec^2\theta$ and $y = \sec\theta \cdot \tan\theta$.

Topic 2: Maclaurin's Series

3. Find the Maclaurin series expansions of the following functions:
   (i) $f(x) = e^x \cdot \ln|1+x|$
   (ii) $f(x) = x \cdot \sin x$
   (iii) $f(x) = \sin 2x + \cos 2x$
   (iv) $f(x) = e^{\cos x}$ (up to terms x^4)

4. (a) Find the Maclaurin series for $(1+x)^m$, where m is not necessarily an integer.
   (b) Use your answer to find the expansion of $\dfrac{1}{\sqrt{1-x^2}}$ up to the term in x^6.

5. (a) Find the Maclaurin expansion of $\cos x$;
   (b) Use the result above to show that $\cos^2 x = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^{2n-1} \cdot x^{2n}}{(2n)!}$.

Topic 3: Equations of Tangents and Normal Lines

6. Find the equation of each tangent of the function $f(x) = x^3+x^2+x+1$ which is perpendicular to the line $2y + x + 5 = 0$.

7. Find the equation of the tangent and equation of the normal to the curve $y = x^e + e^x + \ln x$ at the point where $x = e$.

8. Find the points on the curve $y(x)$ given by $y = x^3 - 6x^2 + x + 3$ where the tangents are parallel to the line $y = x + 5$.

9. The equation of a curve is given by $y(x)$ given by $x^2 - 4y^2 = 12$. Obtain the equations of (a) the tangent line and (b) the normal line to the curve at point $P(4, 1)$.

Topic 4: Newton-Raphson Method

10. Use the Newton-Raphson method with starting value $x_0 = 4$ to approximate the root of $x^2 + \ln x - 16 = 0$, correct to 9 d.p..

11. Use the Newton-Raphson method with starting value $x_0 = 1$ to approximate the root of $\cos x = x$, correct to 9 d.p..

12. The function $f(x) = x - 2 + \ln|x|$ has a root near $x = 1.5$, use the Newton-Raphson method to obtain a better estimate (up to 4 d.p.).

13. Use the Newton-Raphson method to approximate the value of $\sqrt[3]{3}$, correct to 8 d.p., by starting with $x_0 = 1.5$.

Topic 5: Increasing and Decreasing Functions

14. Find the intervals where $f(x) = x^3 + 4x^2 + 4x + 9$ is increasing and decreasing.

15. Find the intervals where $f(x) = x^2 \cdot e^{-3x}$ is increasing and decreasing.

16. Find the intervals where $f(x) = -x^3 + 2x^2 + 12$ is increasing and decreasing.

17. Find the intervals where $f(x) = x^2 \cdot \ln x$ is increasing and decreasing.

Answers

1. (i) $-\dfrac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$ (ii) -1 (iii) $\dfrac{1}{t}$ $\dfrac{\sin t + \cos t}{\cos t - \sin t}$

2. $\dfrac{3\sqrt{2}}{4}$

3. (i) $f(x) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{3x^5}{40} + \cdots \; ; \; -1 < x \leq 1$

   (ii) $f(x) = x^2 - \dfrac{x^4}{3!} + \dfrac{x^6}{5!} - \dfrac{x^8}{7!} + \cdots$

   (iii) $f(x) = 1 + 2x - 2x^2 - \dfrac{4}{3}x^3 + \dfrac{2}{3}x^4 + \cdots$

   (iv) $f(x) = e\left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{6} + \cdots\right)$

4. (a) $1 + mx + \dfrac{m(m-1)}{2!}x^2 + \dfrac{m(m-1)(m-2)}{3!}x^3 + \cdots$

(b) $1+\frac{1}{2}x^2+\frac{3x^4}{8}+\frac{5x^6}{16}+\cdots$

5. (a) $\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n}}{(2n)!}$

   (b) Hint: $\cos^2 x=\frac{1+\cos 2x}{2}$

6. $y=2x+2$, $y=2x+\frac{22}{27}$

7. Equation of the tangent line: $\left(2e^e+\frac{1}{e}\right)x-y=2e^e(e+1)$;

   Equation of the normal line: $ex+(2e^{e+1}+1)y=4e^{2e+1}+2e^{e+1}+2e^e+e^2+1$

8. The two points are: $(0,3)$, and $(4,-25)$

9. (a) $x-y-3=0$

   (b) $x+y-5=0$

10. $x\approx 3.828514482$

11. $x\approx 0.739085133$

12. $x\approx 1.5571$

13. $\sqrt[3]{3}\approx 1.44224957$

14. f is increasing on $(-\infty,-2)$ and $\left(-\frac{2}{3},+\infty\right)$; decreasing on $\left(-2,-\frac{2}{3}\right)$.

15. f is decreasing on $(-\infty,0)$; increasing on $\left(0,\frac{2}{3}\right)$.

16. f is increasing on $\left(0,\frac{4}{3}\right)$; decreasing on $(-\infty,0)$ and $\left(\frac{4}{3},+\infty\right)$.

17. f is decreasing on $\left(0,\frac{1}{\sqrt{e}}\right)$; increasing on $\left(\frac{1}{\sqrt{e}},+\infty\right)$.