University of Nottingham
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Foundation Algebra (CELEN036)

Problem Sheet 2 **Topics: Quadratics, exponentials and logarithms**

Topic 1: Quadratic equations

1. By completing the square, find the range of the following functions for $x \in \mathbb{R}$:

   (i) $f(x) = x^2 - 2x - 8$ (ii) $g(x) = 6 - 3x - x^2$ (iii) $h(x) = 2x^2 - 6x + 1$

2. Given $f(x) = 6 + 2x - x^2 \equiv p - (q - x)^2$; $x \in \mathbb{R}$. Find p, q and the range of f.

3. Completing the square for $f(x) = x^2 + x - 2$ and hence sketch the graph of $f(x)$.

4. Solve the following quadratic equations.

   (i) $2x^2 - 6x + 4 = 0$ (ii) $x^2 + 4x - 8 = 0$ (iii) $2x^2 + 7x + 3 = 0$

   (iv) $x^2 - 2ax + b = 0$ (v) $6x^2 + 11x - 35 = 0$ (vi) $x^2 + x - 4 = 0$

   (vii) $2x^2 - 5x + 1 = 0$ (viii) $x^2 + x - 6 = 0$ (ix) $4x^2 - 10x - 7 = 0$

5. Determine the nature of the roots of the following quadratic equations (without finding the roots).

   (i) $4x^2 - 7x + 3 = 0$ (ii) $x^2 + ax + a^2 = 0$ (iii) $x^2 - px - q^2 = 0$

   (iv) $x^2 - 6x + 9 = 0$ (v) $x^2 - 6x + 10 = 0$ (vi) $2x^2 - 5x + 3 = 0$

   (vii) $3x^2 + 4x + 2 = 0$ (viii) $4x^2 - 12x + 9 = 0$ (ix) $4x^2 - 12x - 9 = 0$

   (x) $5x^2 - 2x - 7 = 0$ (xi) $9x^2 + 24x + 16 = 0$ (xii) $x^2 + 6x - 5 = 0$

(xiii) $x^2 + x - 5 = 0$ (xiv) $x^2 - x + 5 = 0$ (xv) $x^2 - 5x + 6 = 0$

(xvi) $x^2 - 16x + 64 = 0$ (xvii) $x^2 + 2x - 1 = 0$ (xviii) $x^2 + 4x + 5 = 0$

6. Find the value(s) of k if the roots of the equation $3x^2 + kx + 12 = 0$ are equal (ie. repeated roots).

7. Find the value(s) of k if $9x^2 + kx + 16$ is a perfect square.

8. Find k if the quadratic equation $2x^2 - kx + 8 = 0$ has equal roots.

9. Show that $kx^2 + 2x - (k - 2) = 0$ has real roots for any $k \in \mathbb{R}$.

10. Show that the roots of $ax^2 + (a + b)x + b = 0$ are real for all values of a and b.

11. Find the relation between p and q if the roots of $px^2 + qx + 1 = 0$ are equal.

12. Find k if the roots of $kx^2 + (k - 1)x + (k - 1) = 0$ are equal. Also find the equal roots for the obtained value(s) of k.

13. For what values of m, will the polynomial $m^2x^2 + 2(m + 1)x + 4$ have exactly one zero?

Topic 2: Exponentials

14. Prove that $\dfrac{1}{1 + x^{a-b} + x^{a-c}} + \dfrac{1}{1 + x^{b-c} + x^{b-a}} + \dfrac{1}{1 + x^{c-a} + x^{c-b}} = 1.$

15. Simplify $(1 + x^{a-b})^{-1} + (1 + x^{b-a})^{-1}$.

16. A function f is defined by $f(x) = \dfrac{1}{2}(10^x + 10^{-x})$; $x \in \mathbb{R}$. Show that:

(i) $2[f(x)]^2 = f(2x) + 1$ (ii) $2f(x)f(y) = f(x + y) + f(x - y)$

Topic 3: Logarithms

17. Simplify:

    (i) $\ln 3x^2 + \ln 2x - \ln 6x^3$

    (ii) $\log 5x^2 - \log 10x^2 + \log 4x$

    (iii) $3\log x - \log x^2$

    (iv) $\log x - 3\log 2x + 2\log 4x$

    (v) $2\ln 3x - \frac{1}{2}\ln 16x^2$

    (vi) $\frac{1}{3}\left(\log 9x + \log 3x^2\right)$

18. Using the change of base rule and the power rule, simplify: $\dfrac{\log_2 128 \cdot \log_9 243}{\log_{125} 625}$.

19. Prove that $\log\left(\frac{a^2}{bc}\right) + \log\left(\frac{b^2}{ac}\right) + \log\left(\frac{c^2}{ab}\right) = 0$.

20. Prove that $\log\left(\sqrt{x^2+1} + x\right) + \log\left(\sqrt{x^2+1} - x\right) = 0$.

21. Prove that $\ln\left(y - \sqrt{y^2-1}\right) = -\ln\left(y + \sqrt{y^2-1}\right)$.

22. Prove that $\dfrac{1}{\log_2 n} + \dfrac{1}{\log_2 n} + \dfrac{1}{\log_4 n} + \cdots + \dfrac{1}{\log_{100} n} = \dfrac{1}{\log_{100!} n}$.

    Note: For $n \in \mathbb{R}$, $n! = n \times (n-1) \times (n-2) \cdots 3 \times 2 \times 1$.

23. Prove that $\log_a x + \log_{a^2} x^2 + \log_{a^3} x^3 + \cdots + \log_{a^n} x^n = \log_a x^n$.

24. Solve for $x \in \mathbb{R}$:

    (i) $\log(x-1) + \log(x+1) = 2$

    (ii) $\ln(2x+5) = \ln(14-x)$

    (iii) $\dfrac{\log 2x}{\log x} = 2$

    (iv) $\ln\left(\dfrac{x^2}{2}\right) - \ln x = 0.7$

25. Solve the following equations:

    (i) $e^{2x+3} = 500$

    (ii) $2e^x - 1 = 83$

    (iii) $e^{5-x} = 10$

    (iv) $4e^{3x+2} = 78$

    (v) $e^{4x+1} = e^{1-x}$

    (vi) $10^x = 59$

26. By substituting $e^x = t$, solve the following equations:

(i) $e^{2x} - 5e^x + 6 = 0$

(ii) $e^x + e^{-x} = 2$

(iii) $12e^{2x} + 6 = 17e^x$

(iv) $3e^{2x} - 5e^x = 2$

27. Given $f(x) = 2(2^x) + 2$. Find c such that $f(c) = 34$.

28. Sketch the graphs of the following functions:

(i) $y = e^x + 1$

(ii) $3 - e^x$

(iii) $\ln(3 - x)$

(iv) $3 + \ln(x + 2)$

Answers

1. (i) $y \geq -9,\ y \in \mathbb{R}$ (ii) $y \leq \frac{33}{4},\ y \in \mathbb{R}$ (iii) $y \geq -\frac{7}{2},\ y \in \mathbb{R}$

2. $p = 7, q = 1,$ range is $\{y \in \mathbb{R} / y \leq 7\}$

3. $\left(x + \frac{1}{2}\right)^2 - \frac{7}{4}$

4. (i) $1,\ 2$ (ii) $-2 \pm 2\sqrt{3}$ (iii) $-\frac{1}{2},\ -3$
   (iv) $a \pm \sqrt{a^2 - b}$ (v) $\frac{5}{3},\ -\frac{7}{2}$ (vi) $\frac{-1 \pm \sqrt{17}}{2}$
   (vii) $\frac{5 \pm \sqrt{17}}{4}$ (viii) $2,\ -3$ (ix) $\frac{5 \pm \sqrt{53}}{4}$

5. Case 1: Roots are real and distinct (i), (iii), (vi), (ix), (x), (xii), (xiii), (xv), (xvii)

   Case 2: Roots are real and equal (iv), (viii), (xi), (xvi)

   Case 3: No real roots (ii), (v), (vii), (xiv), (xviii)

6. ± 12

7. ± 24

8. ± 8

11. $q^2 = 4p$

12. $k = 1$ or $-\frac{1}{3}$, $x = 0$ or -2

13. $m = 1$ or $-\frac{1}{3}$

15. 1

17. (i) 0 (ii) $\log 2x$ (iii) $\log x$
    (iv) $\log 2$ (v) $\log\left(\frac{9x}{4}\right)$ (vi) $\log 3x$

18. $\frac{105}{8}$

24. (i) 10.05 (ii) 3 (iii) 2 (iv) 4.03

25. (i) 1.61 (ii) 3.74 (iii) 2.70
    (iv) 0.32 (v) 0 (vi) 1.77

26. (i) $\ln 2$ or $\ln 3$ (ii) 0 (iii) $\ln\left(\frac{2}{3}\right)$ or $\ln\left(\frac{3}{4}\right)$ (iv) $\ln 2$

27. 4