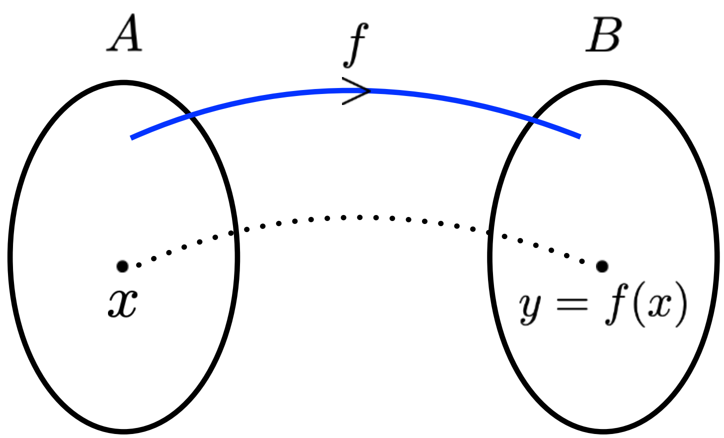
Lecture 1

Topics covered in this lecture session

1. Functions
2. Graphing functions
3. Inequalities.
4. General review of functions

Functions

- A function f is a rule that associates a **unique** output with **each** input.



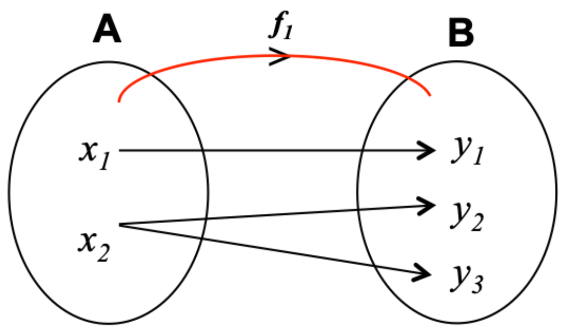
x is Input **(Independent variable)**

y is Output **(Dependent variable)**

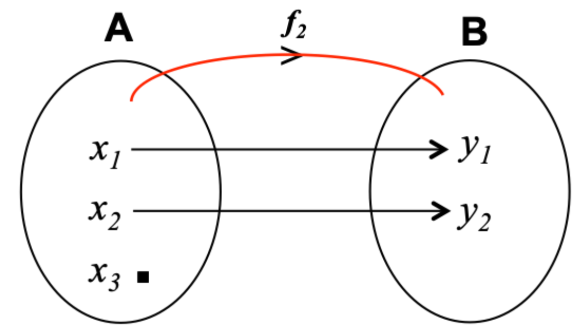
- All x in A must have exactly one mapped value in B.

- All x in A must be mapped.

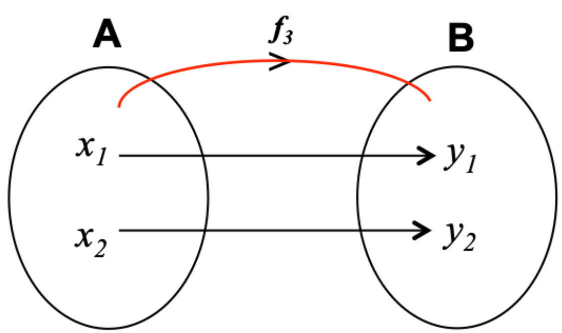
Which of the following mappings are functions?



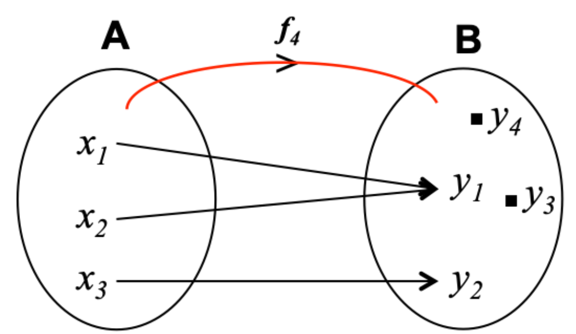
f_1 is not a function because the element x_2 of A is **NOT** mapped uniquely.



f_2 is not a function because the element x_3 of A is **NOT** mapped.



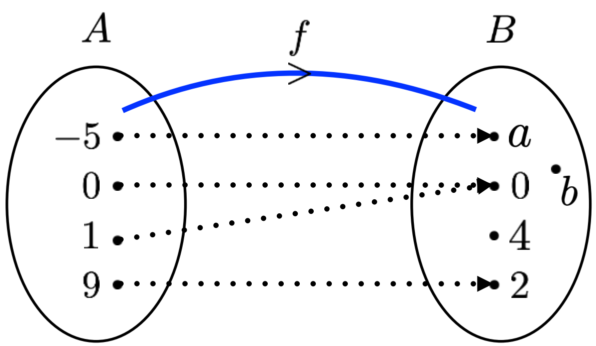
f_3 is a function (Type: One-one & onto)



f_4 is a function (Type: Many one & into).

Functions can be represented by five common methods:

Using Venn-diagrams



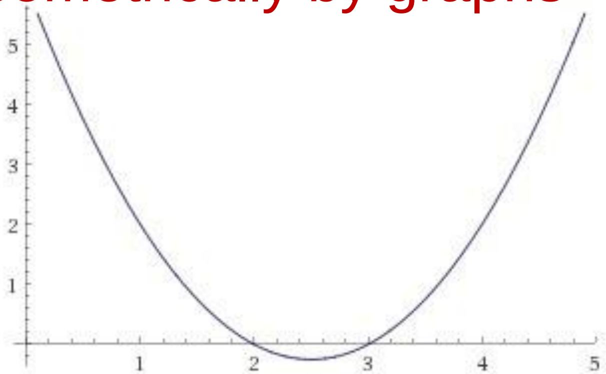
Numerically by tables

x	0	1	2	3
$y = f(x)$	3	4	-1	6

Algebraically by formulas

$$y = f(x) = x^2 - 5x + 6$$

Geometrically by graphs



Verbally (i.e. described in words)

e.g. Newton's law of Universal Gravitation.

The gravitational force of attraction between two bodies in the Universe is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

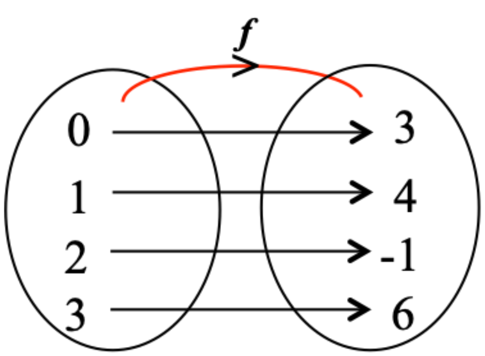
Domain and Range of a function

Domain is the set of allowable inputs (i.e. x values)

Range is the set of outputs (i.e. y values) when x varies over the domain.

e.g. For the function f defined by

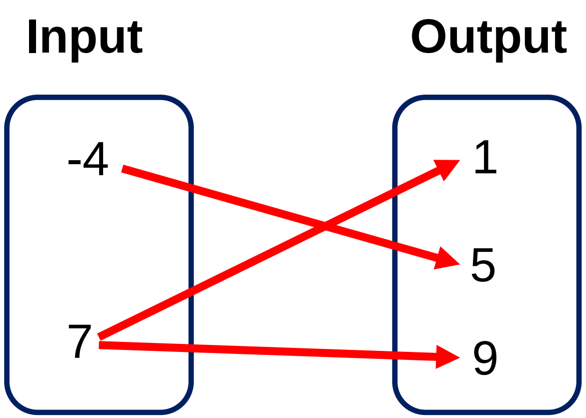
x	0	1	2	3
$y = f(x)$	3	4	-1	6

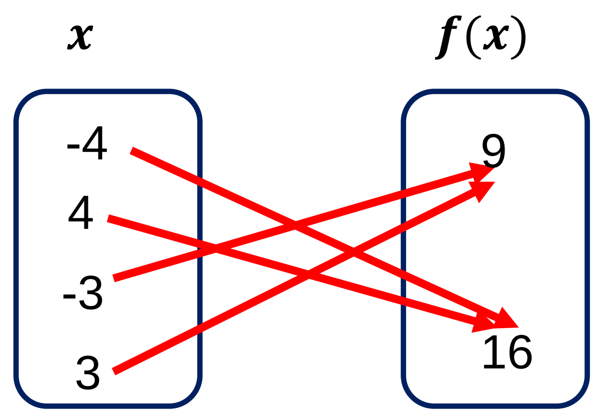


Domain of f is $D_f = \{0, 1, 2, 3\}$

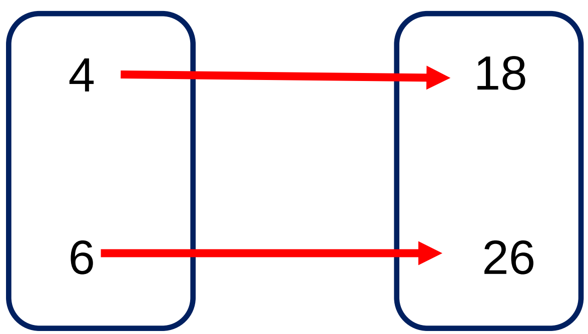
Range of f is $R_f = \{-1, 3, 4, 6\}$

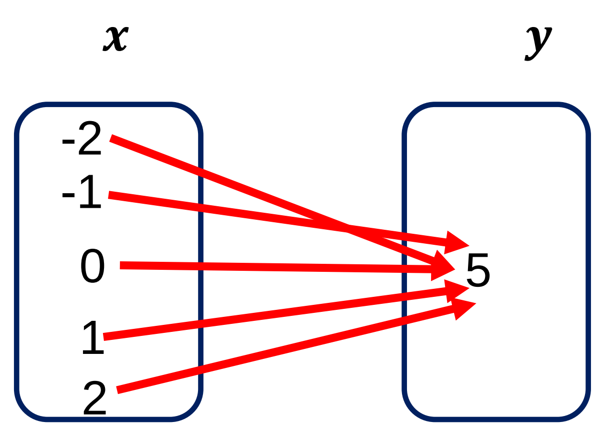
Which of the following is not a function, and for those that are functions, what type are they (Many-one, One-One, and Into, Onto)





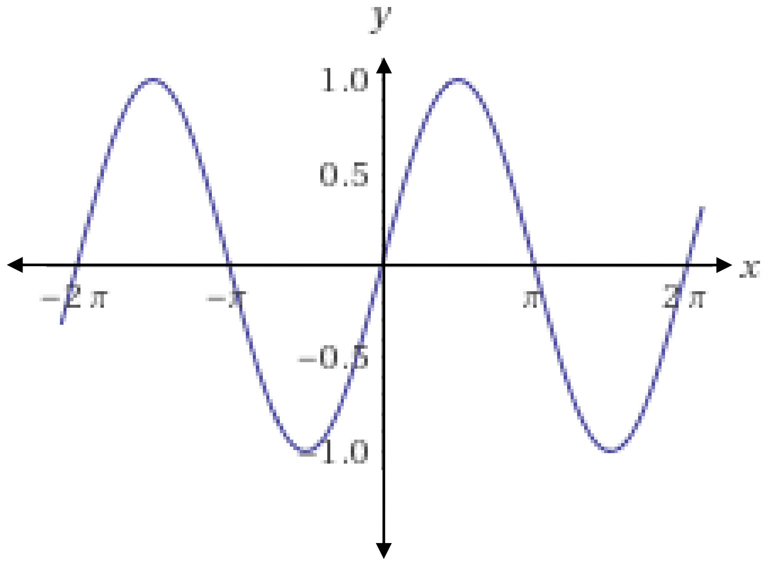
Two more than quadruple a number



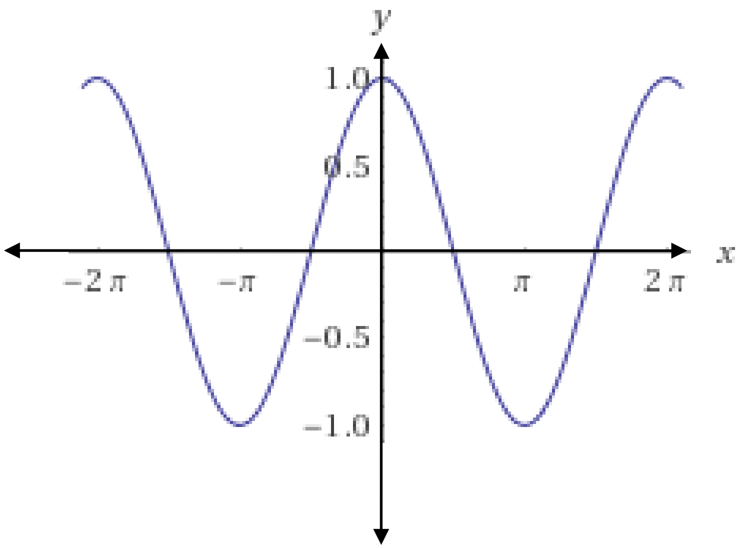


Graphs of standard functions and curves

1. Trigonometric functions



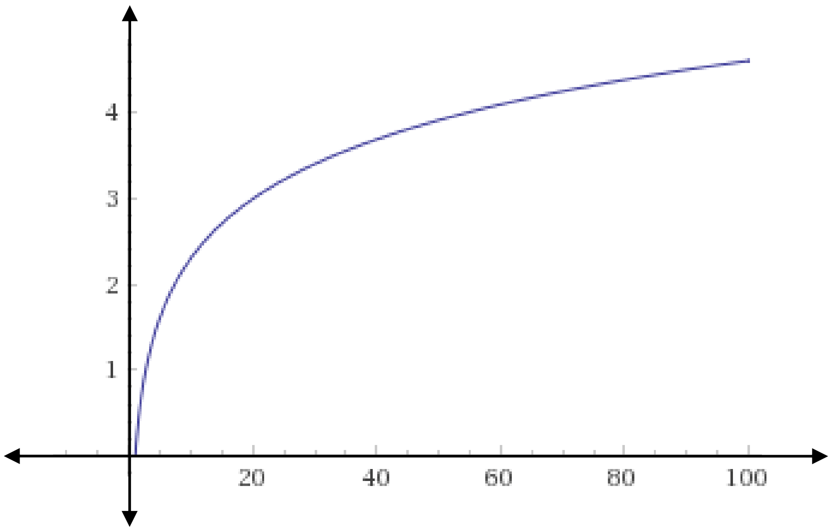
$$f(x) = y = \sin x$$

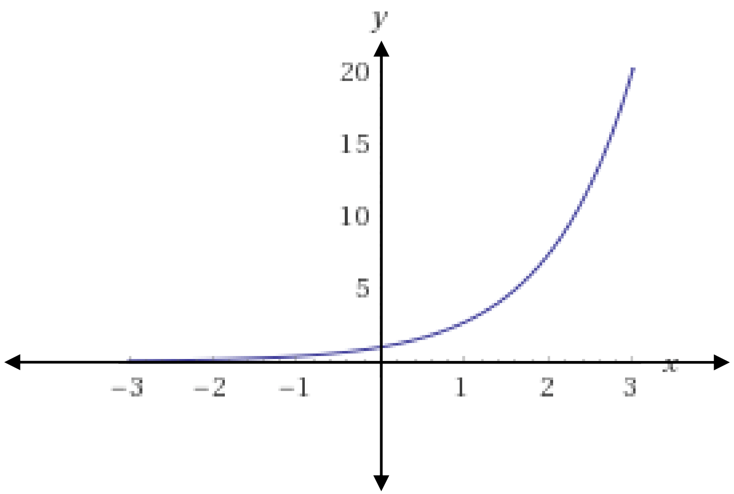


$$g(x) = y = \cos x$$

Graphs of standard functions and curves

2. Logarithmic and exponential functions



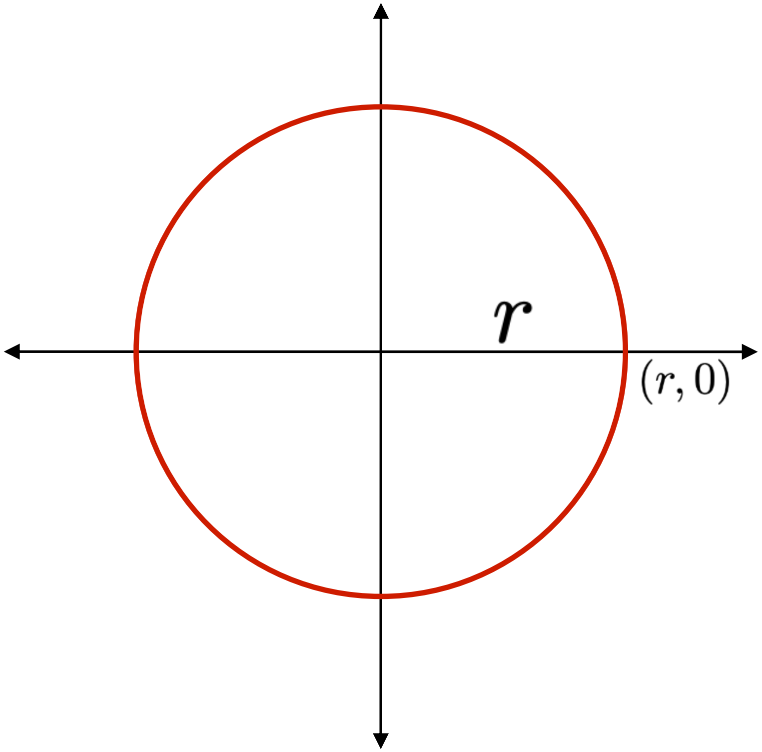


$$p(x) = y = \ln x$$

$$q(x) = y = e^x$$

Graphs of standard functions and curves

3. Circle with centre at origin



Its equation is:

$$x^2 + y^2 = r^2$$

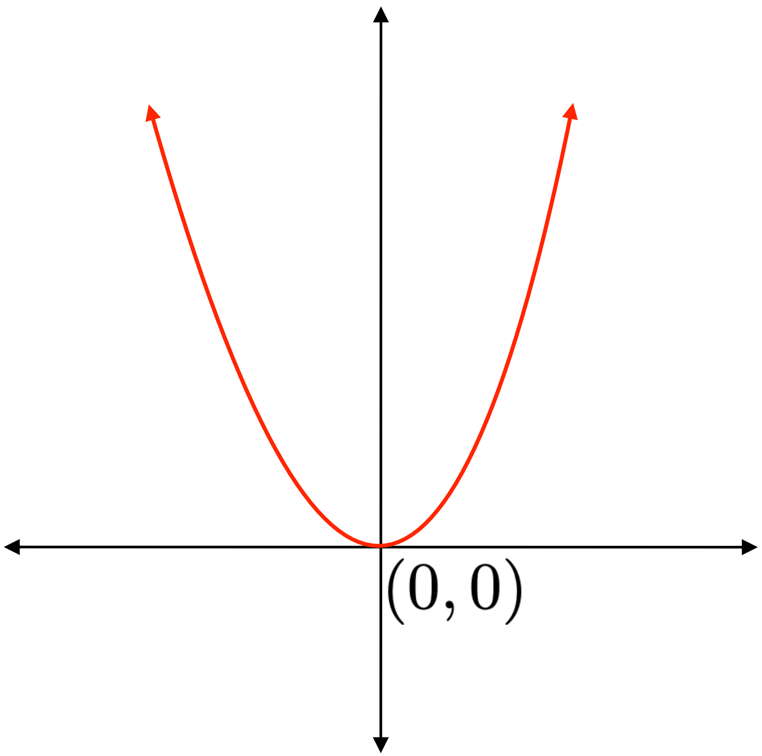
This equation is a combination of two functions

$f_1(x) = y = +\sqrt{r^2 - x^2}$ upper semicircle

and $f_2(x) = y = -\sqrt{r^2 - x^2}$ lower semicircle

Graphs of standard functions and curves

4. Parabola



Its equation is:

$$y = x^2$$

What type of function is:

$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$?

Answer: Many-one and into.

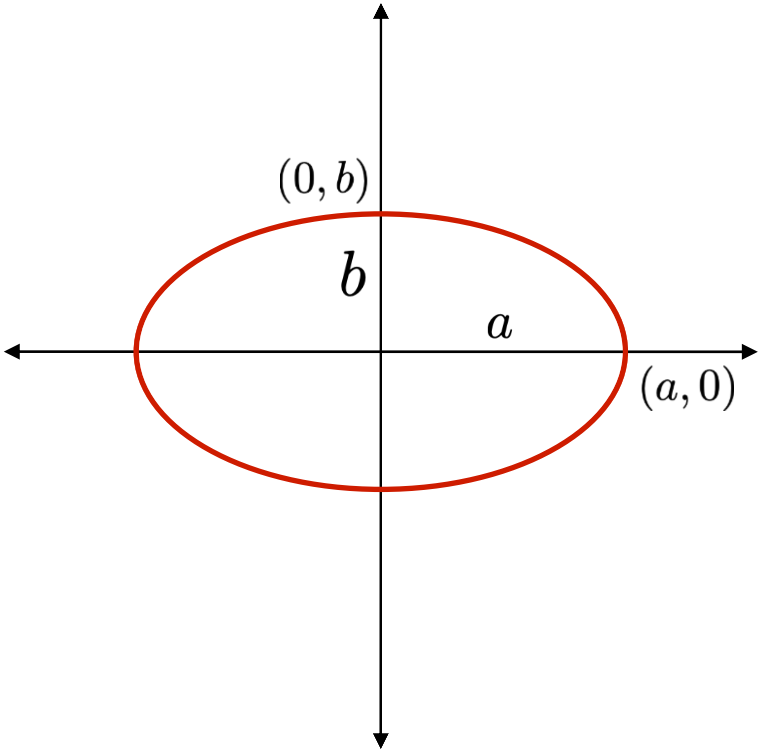
What type of function is:

$f : \mathbb{R} \to \mathbb{R}^+, \quad f(x) = x^2$?

Answer: Many-one and onto.

Graphs of standard functions and curves

5. Ellipse



Its equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

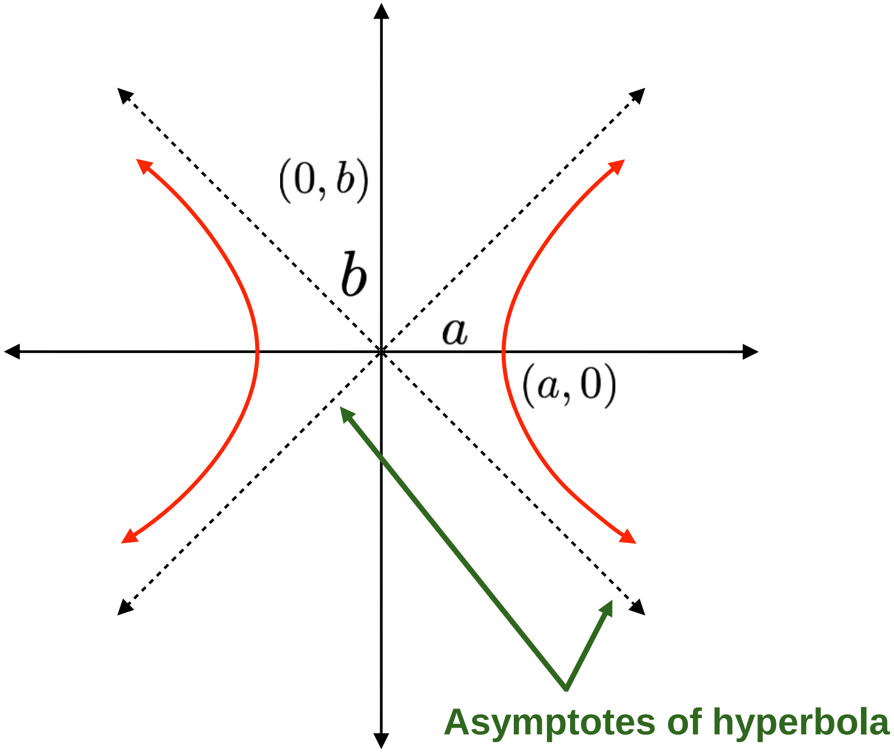
This equation is a combination of two functions

$$f_1(x) = y = +\frac{b}{a}\sqrt{a^2 - x^2}$$ upper semi-ellipse

and $$f_2(x) = y = -\frac{b}{a}\sqrt{a^2 - x^2}$$ lower semi-ellipse

Graphs of standard functions and curves

6. Hyperbola



Its equation is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

If asymptotes are at right angle, the curve is called rectangular hyperbola.

Its equation is: $x^2 - y^2 = a^2$

Sketching graphs of functions (Translations)

For $c > 0$, to obtain the graph of

1) $y = f(x) + c$, shift the graph of $y = f(x)$

   by a distance of c units upward.

2) $y = f(x) - c$, shift the graph of $y = f(x)$

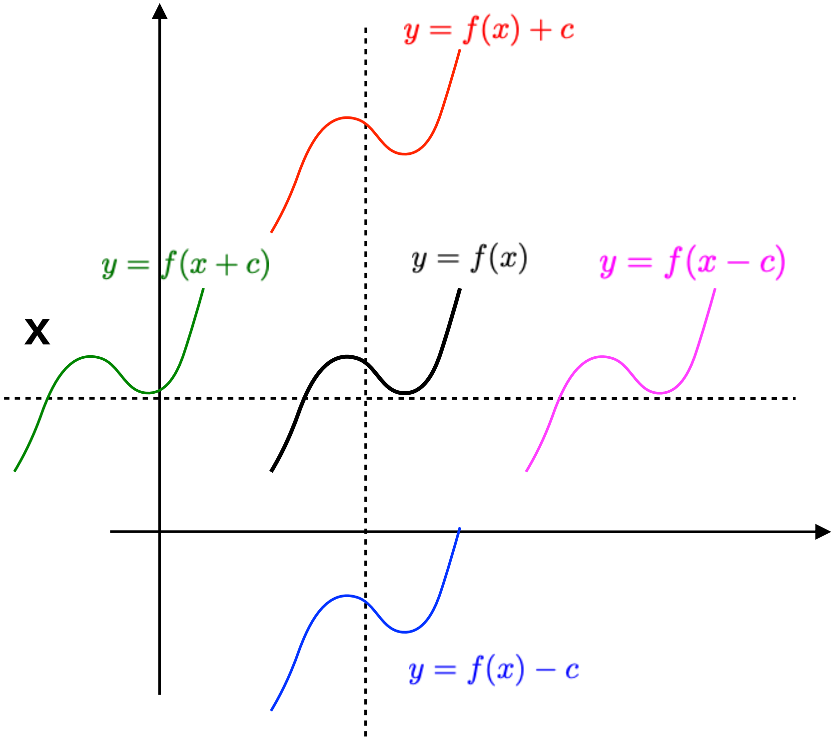
   by a distance of c units downward.

3) $y = f(x - c)$, shift the graph of $y = f(x)$

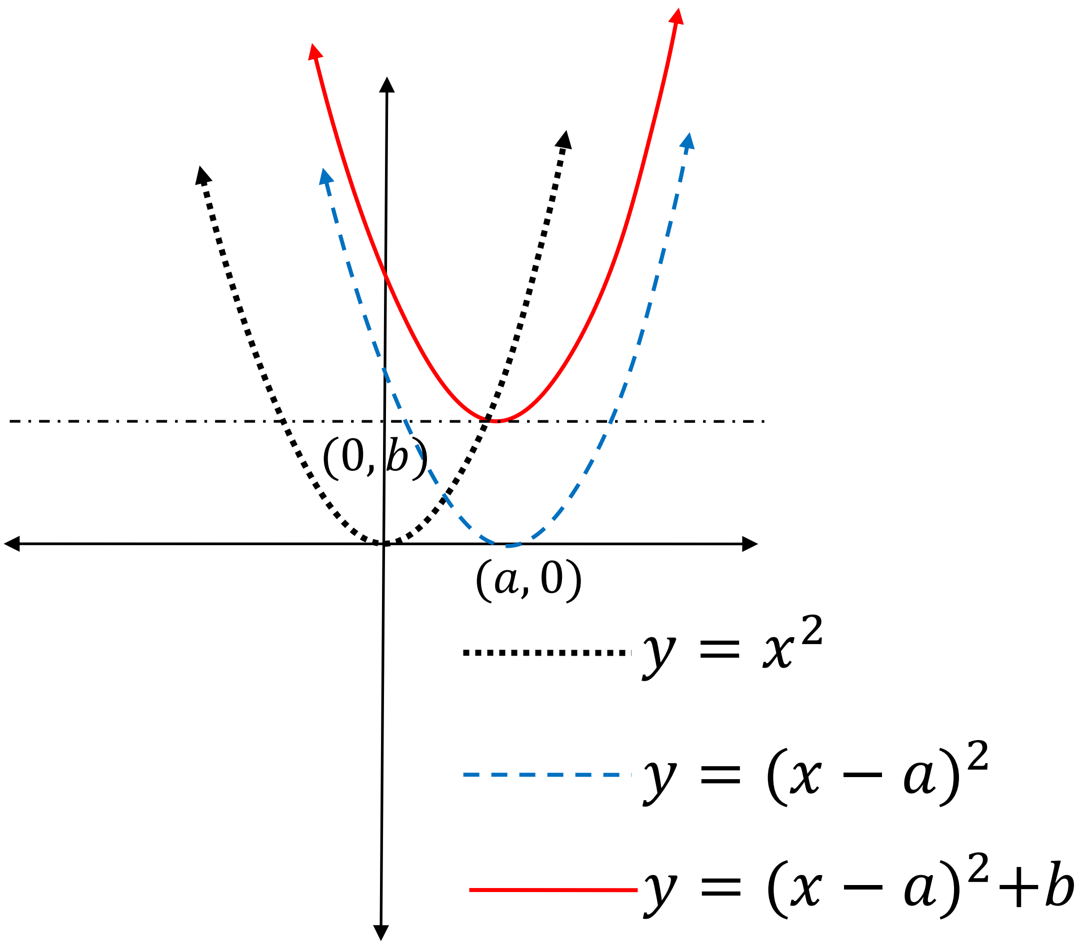
   by a distance of c units to the right.

4) $y = f(x + c)$, shift the graph of $y = f(x)$

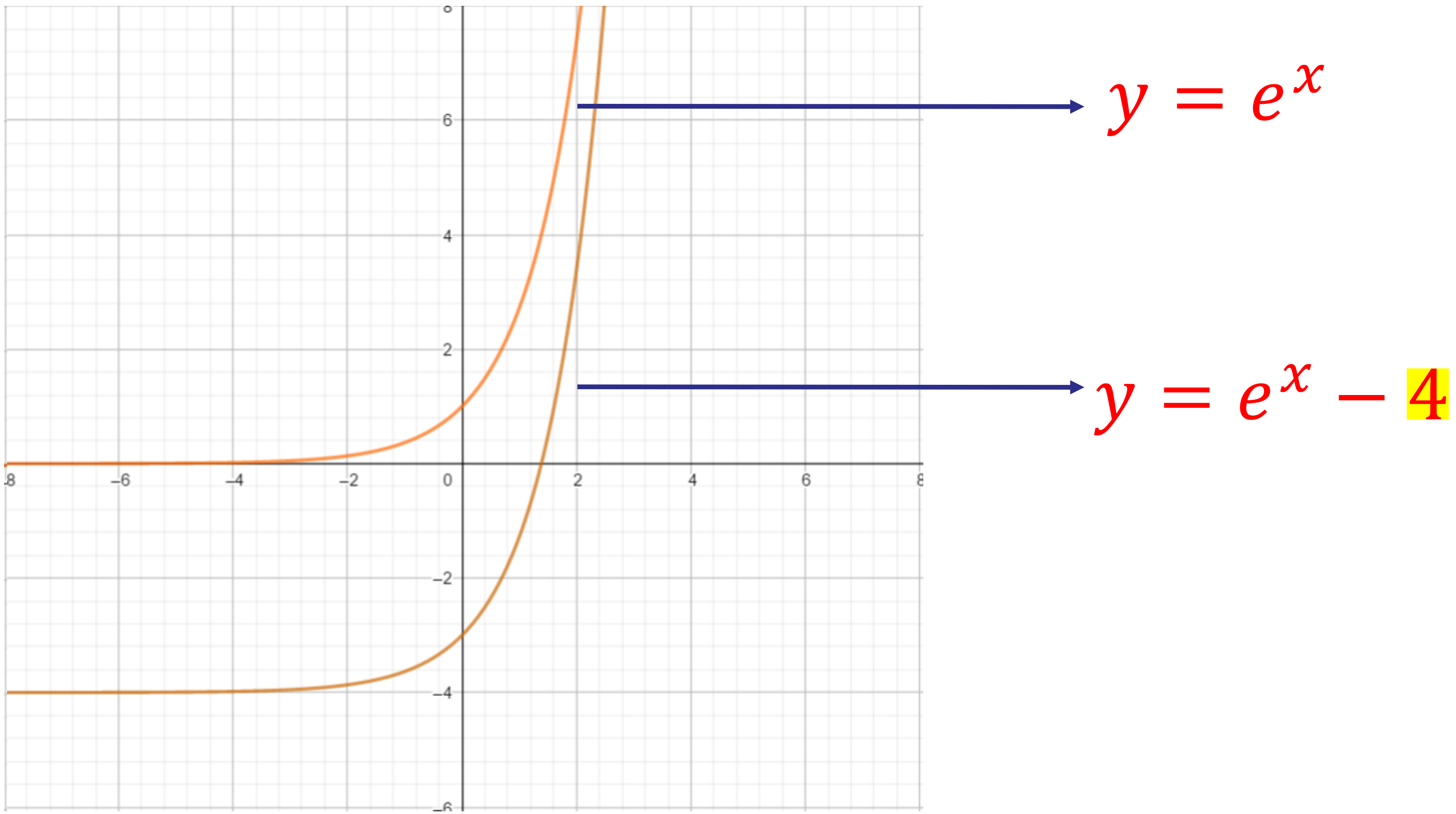
   by a distance of c units to the left.



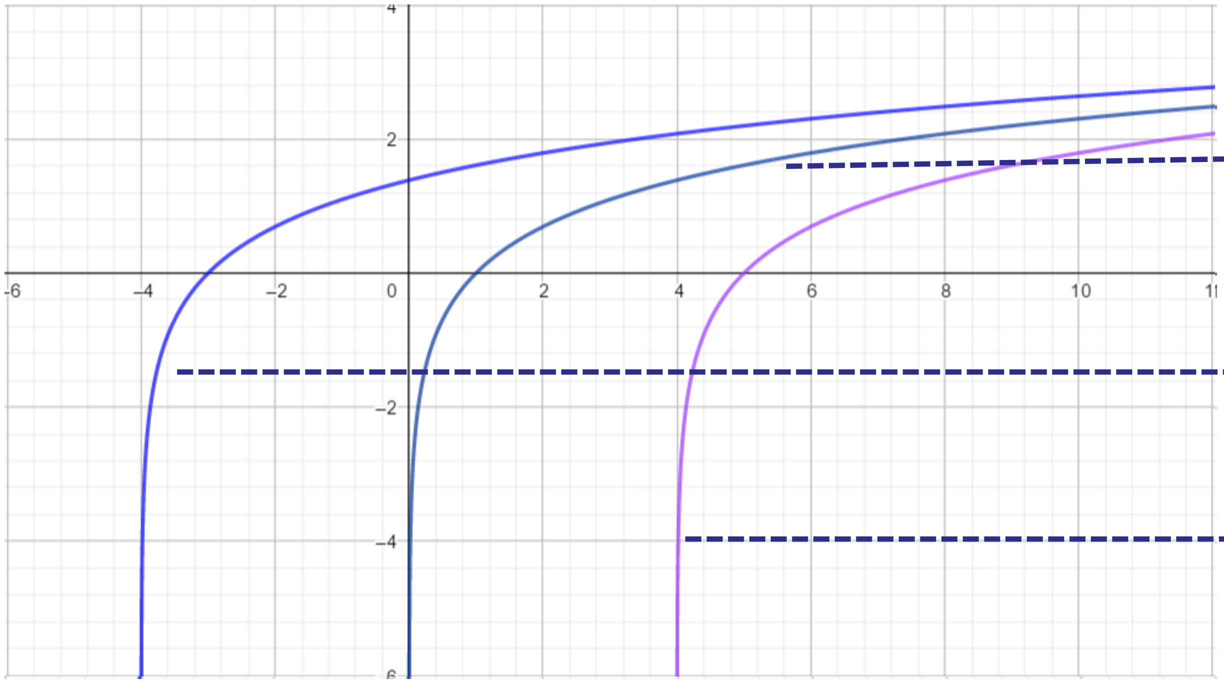
Sketching graphs of functions (Translations)



Sketching graphs of functions (Translations)



Sketching graphs of functions (Translations)



$y = \ln(x)$

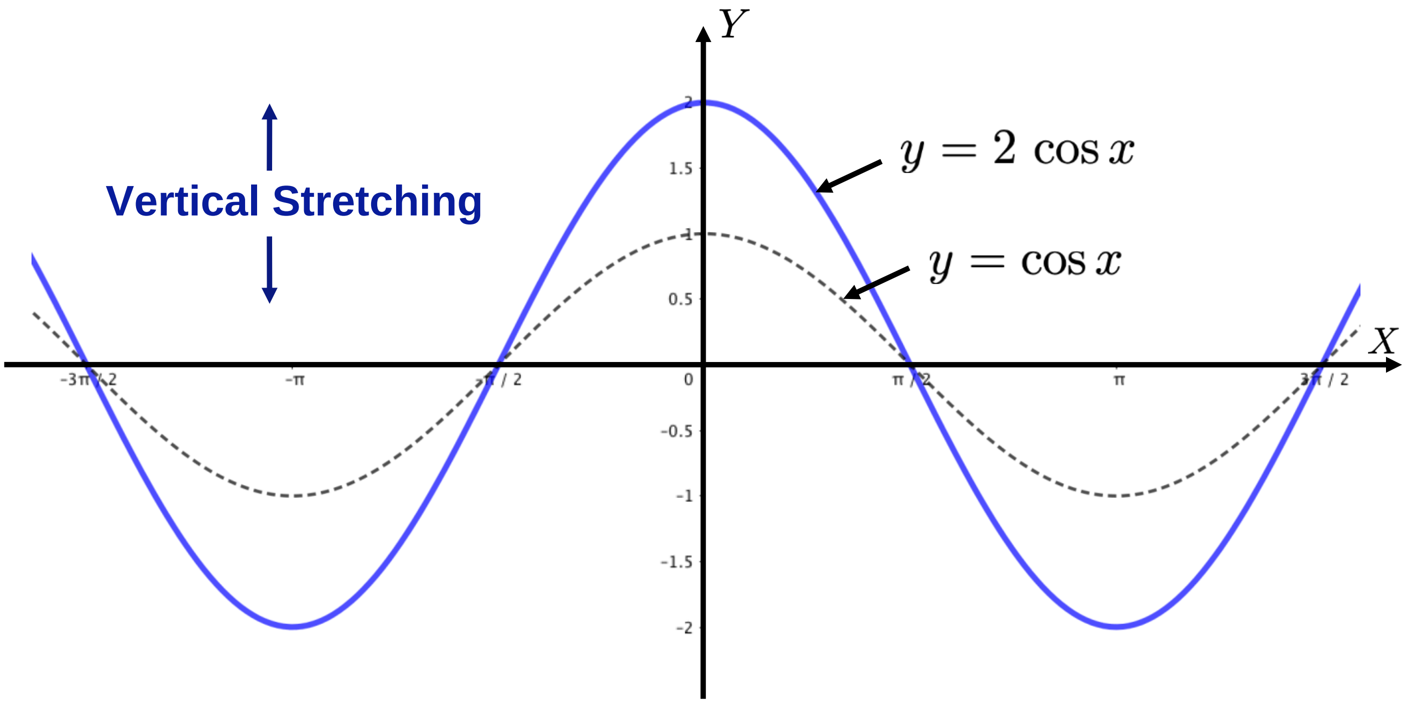
$y = \ln(x+4)$

$y = \ln(x-4)$

Sketching graphs of functions

Stretches and Compressions

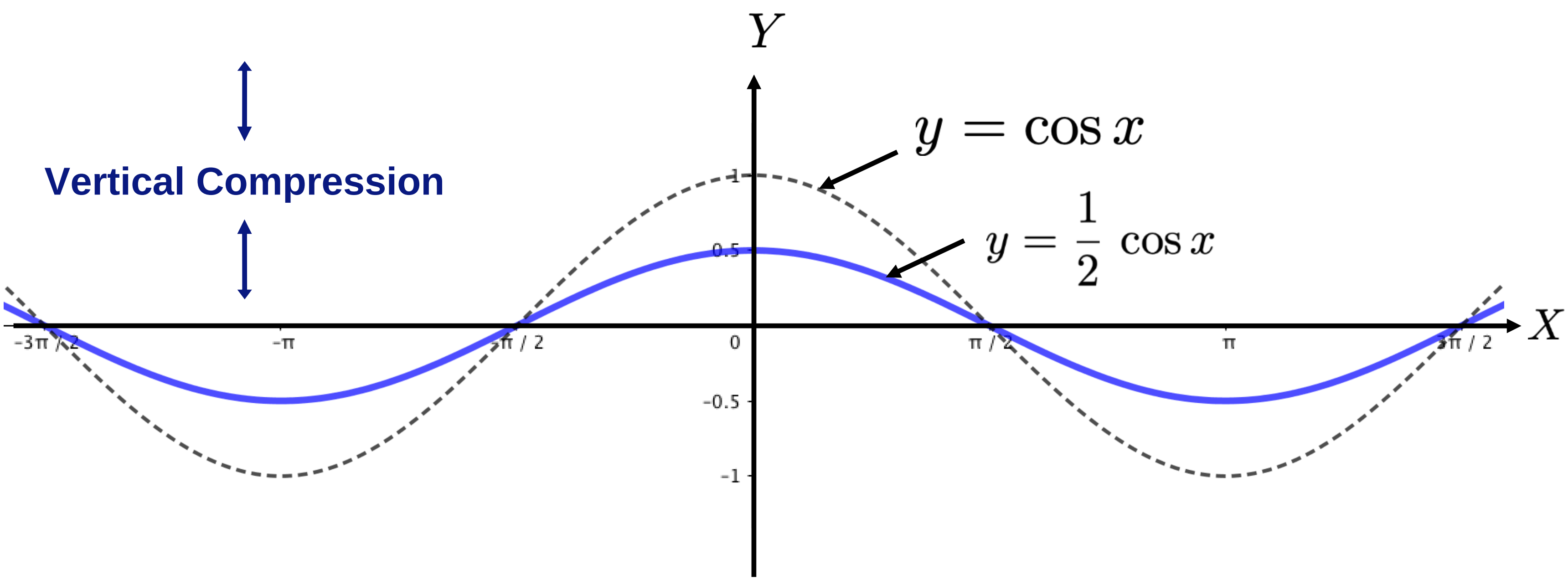
1. $f(x)$ and $k \cdot f(x)$ $(k > 1)$



Sketching graphs of functions

Stretches and Compressions

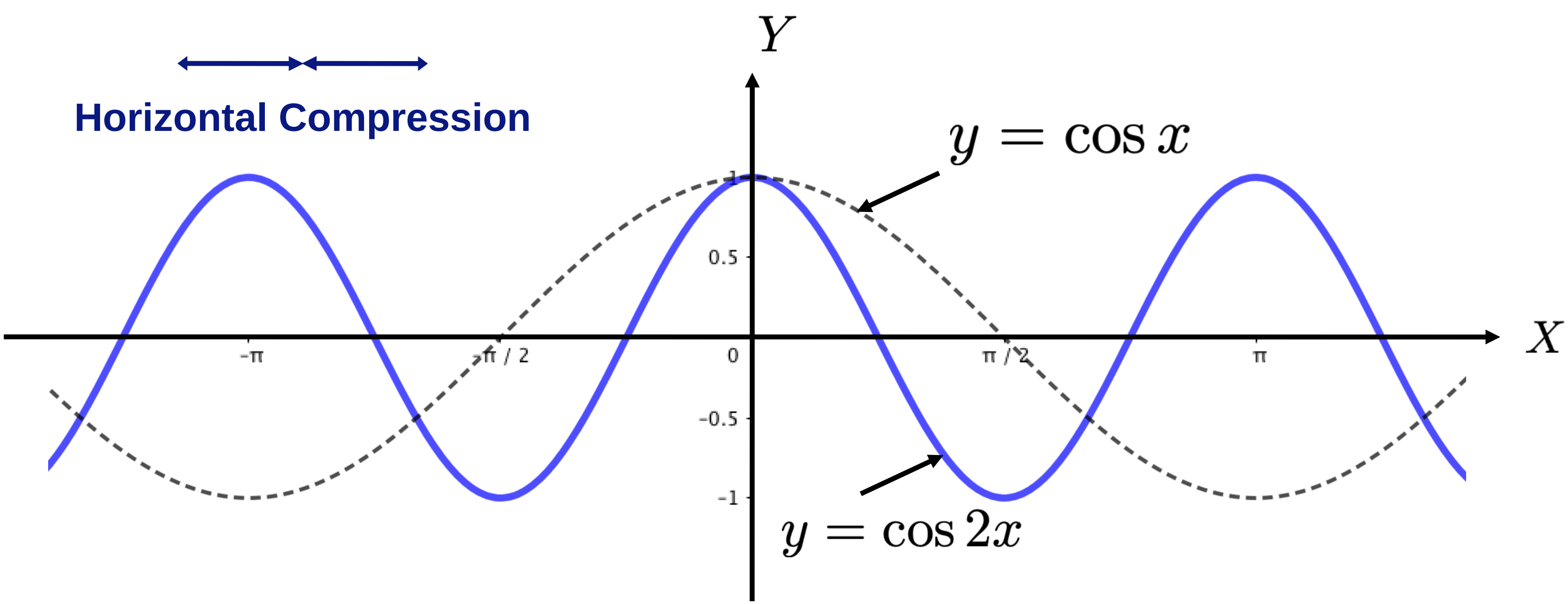
$$2. \quad f(x) \quad \text{and} \quad k \cdot f(x) \quad (k < 1)$$



Sketching graphs of functions

Stretches and Compressions

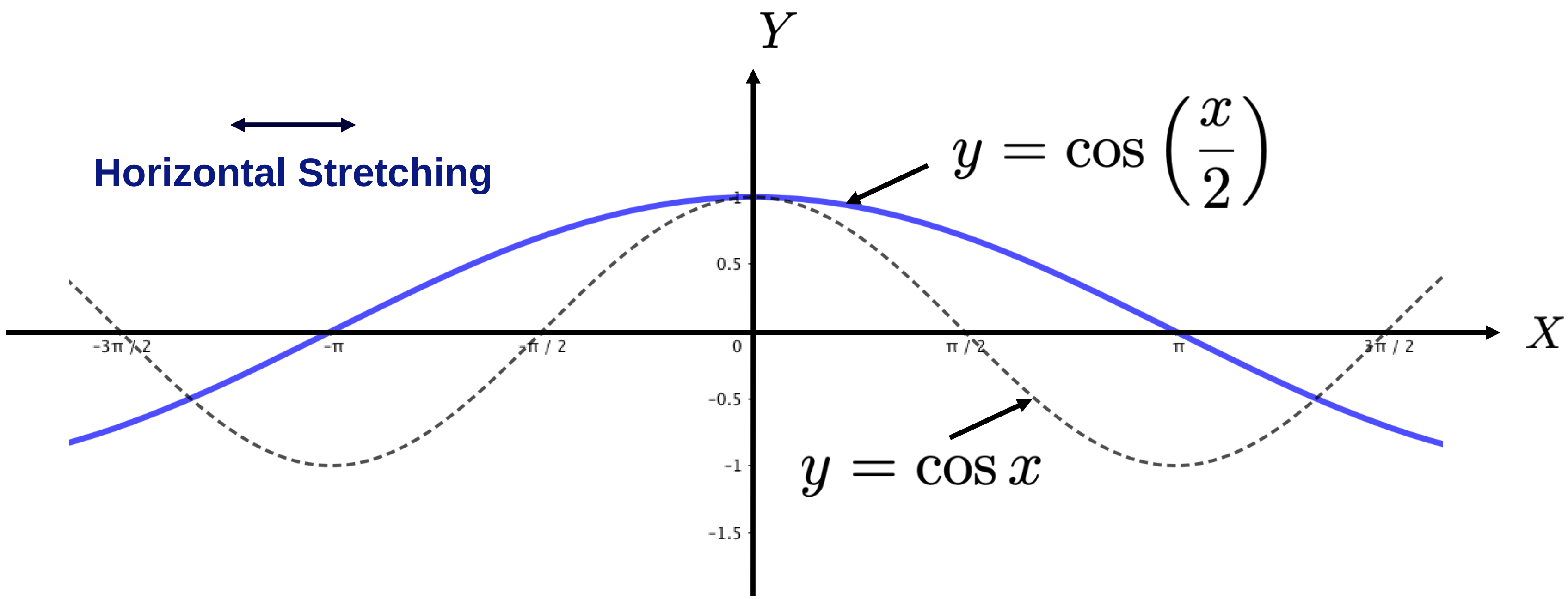
3. $f(x)$ and $f(kx)$ $(k > 1)$



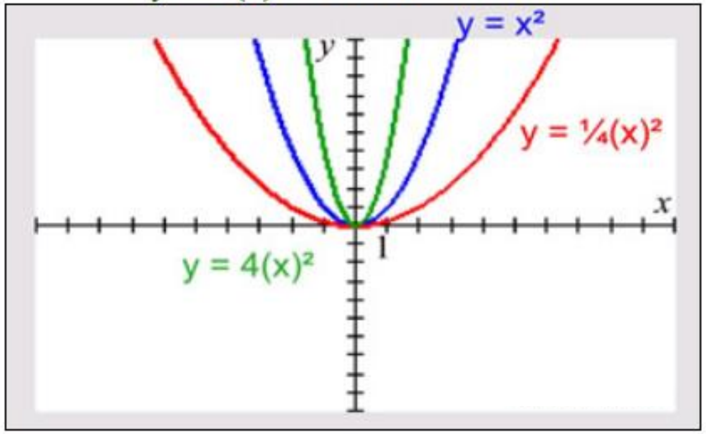
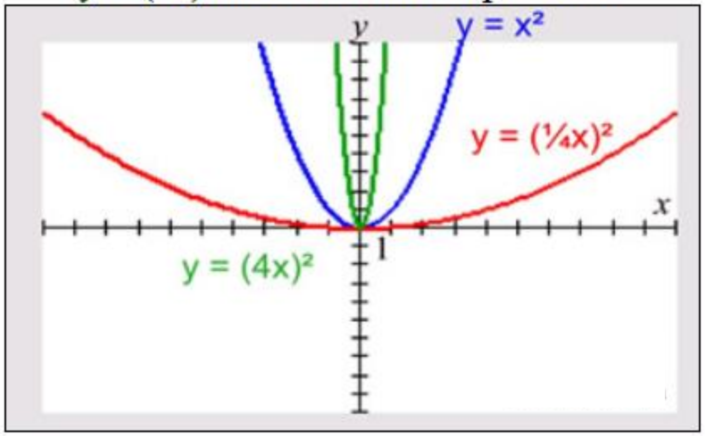
Sketching graphs of functions

Stretches and Compressions

4. $f(x)$ and $f(kx)$ $(k<1)$



Sketching graphs of functions

Stretch or Compress Vertically	Stretch or Compress Horizontally
stretches away from the x-axis or compresses toward the x-axis $y = a \bullet x^2$ $\lvert a \rvert > 1$ is a stretch; $0 < \lvert a \rvert < 1$ is a compression $y = x^2$ parent graph $y = \frac{1}{4}(x)^2$ vertical compression $y = 4(x)^2$ vertical stretch	*stretches away from the y-axis or compresses toward the y-axis* $y = (a \bullet x)^2$ $\lvert a \rvert > 1$ is a compression by factor of $1/a$; $0 < \lvert a \rvert < 1$ is a stretch by factor of $1/a$ $y = x^2$ parent graph $y = (\frac{1}{4}x)^2$ horizontal stretch $y = (4x)^2$ horizontal compression
	

Polynomial Function

- A polynomial in x is a function f that can be expressed as a sum of **finitely** many terms of the form ax^n, where a is constant and n is a non-negative integer.
- Its general form is $P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$
- The degree (or order) of the polynomial is defined as the highest power of x that occurs in a polynomial.

e.g. $4x^5 - 15x^4 + 7x^3 + x$ is a polynomial of degree 5.

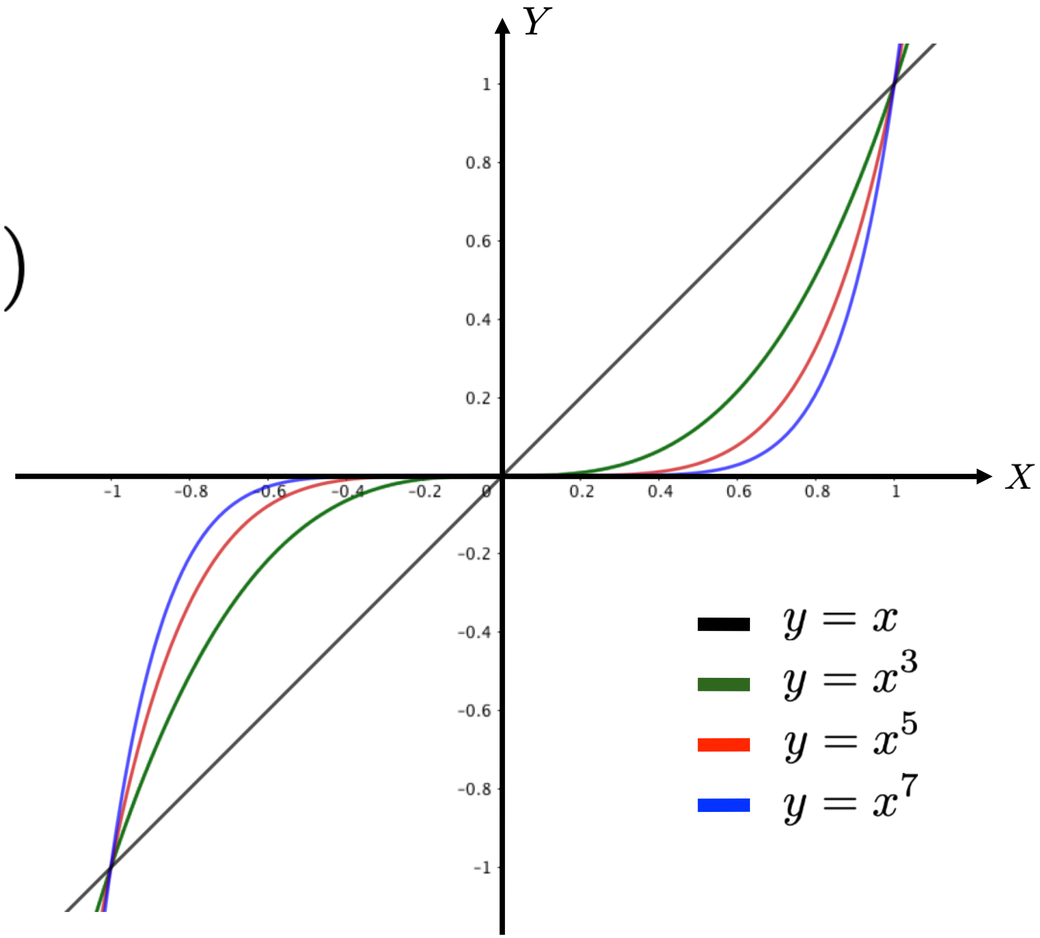
Standard Polynomial names

Degree	Name	General Form	Example
0	constant	c	9
1	Linear	$ax+b$	$2x+3$
2	Quadratic	ax^2+bx+c	$5x^2-2x+3$
3	Cubic	ax^3+bx^2+cx+d	$2x^3-2x+3$
4	Quartic	$ax^4+bx^3+cx^2+dx+e$	$3x^4+4x^3-x^2+2x+7$
5	Quintic	$ax^5+bx^4+cx^3+dx^2+ex+f$	x^5-2x^2+3x+9

Graphs of standard polynomial functions

Graphs of

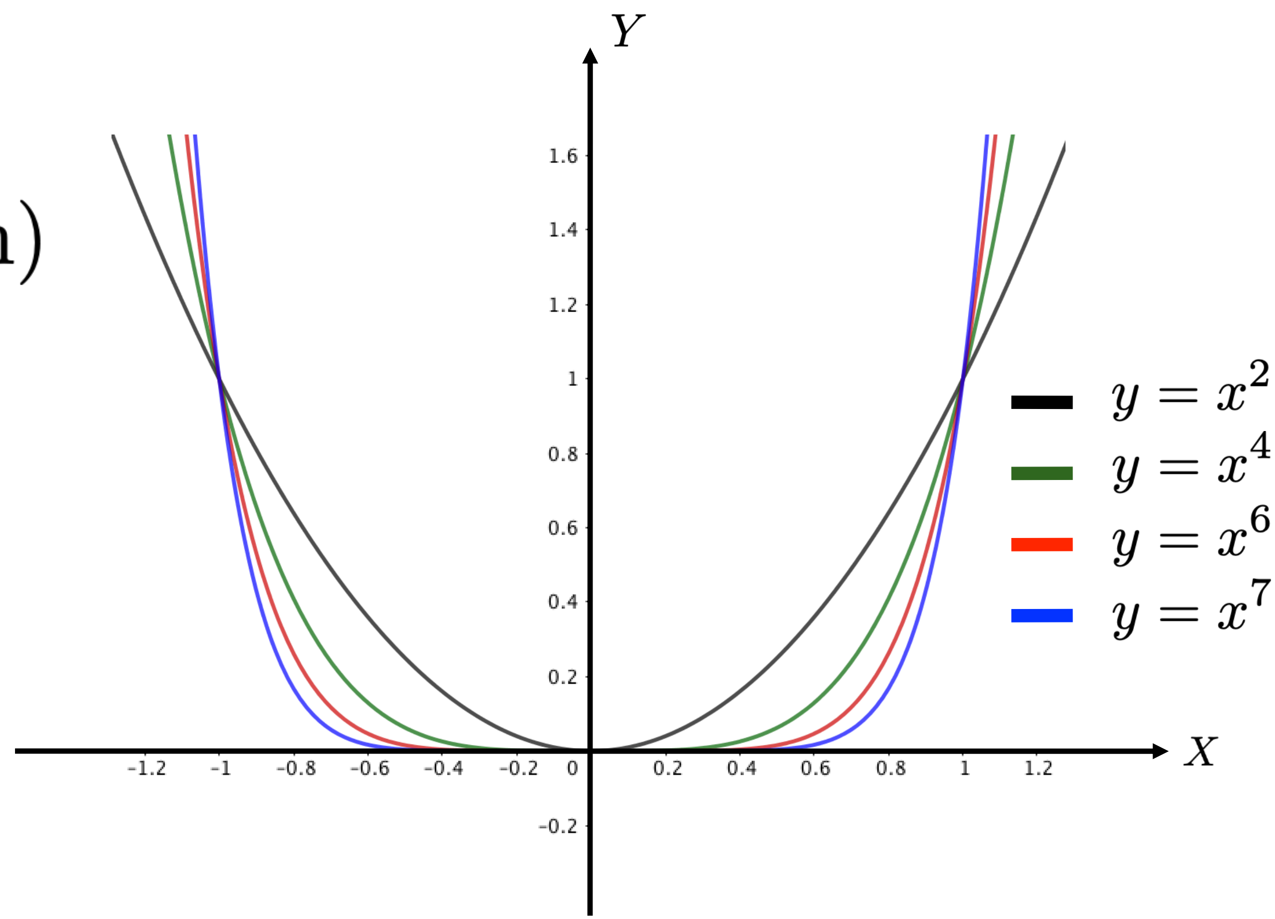
$$x^n \quad (n \text{ is odd})$$



Graphs of standard polynomial functions

Graphs of

$x^n \quad (n \text{ is even})$

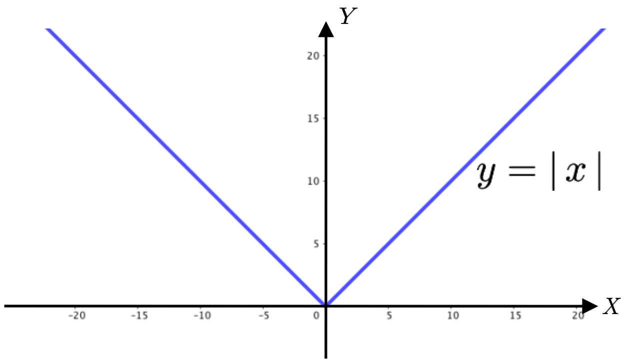


Modulus Function

The modulus function for $x \in \mathbb{R}$ is defined by

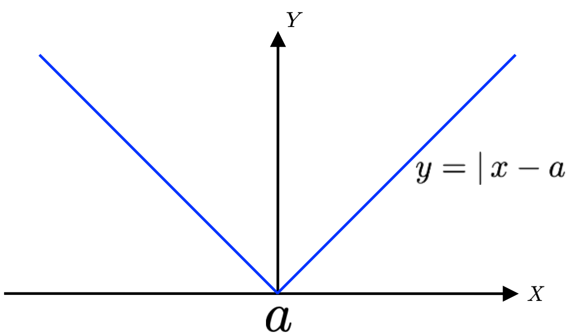
$$|\,x\,| = \begin{cases} x & ; \quad x \geq 0 \\ -x & ; \quad x < 0. \end{cases}$$

e.g. $|\,5\,| = 5$ and $|-5\,| = -(-5) = 5$



In general,

$$|\,x - a\,| = \begin{cases} x - a & ; \quad x \geq a \\ a - x & ; \quad x < a \end{cases}$$



Inequalities

Inequality	Meaning
$a > b$	a is greater than b
$a \geq b$	a is greater than or equal to b
$a < b$	a is less than b
$a \leq b$	a is less than or equal to b

Properties of inequalities

(1) $a > b \;\Leftrightarrow\; a + c > b + c \quad ; \quad c \in \mathbb{R}$

(2) $a > b \;\Leftrightarrow\; ac > bc \quad ; \quad c > 0$

i.e. Inequality will NOT change if both sides are multiplied by a positive number.

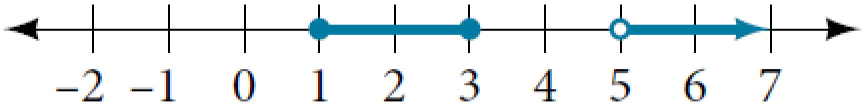
(3) $a > b \;\Leftrightarrow\; ac < bc \quad ; \quad c < 0$

i.e. Inequality will change if both sides are multiplied by a negative number.

(4) $|\,x - a\,| < b \;\Leftrightarrow\; a - b < x < a + b$

Example: Solve $|\,2x - 1\,| \geq 5.$

Some Notations using inequalities

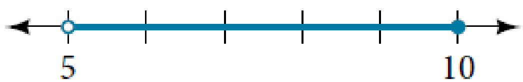
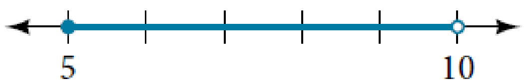
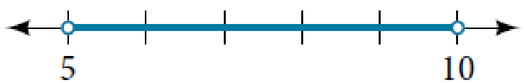
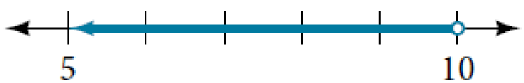
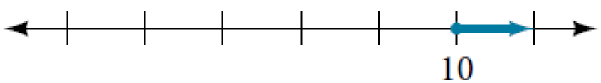
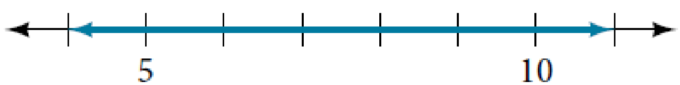


Inequality	$1 \leq x \leq 3$ or $x > 5$
Set-builder notation	$\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$
Interval notation	$[1, 3] \cup (5, \infty)$

Some Notations using inequalities

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, \text{a})$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

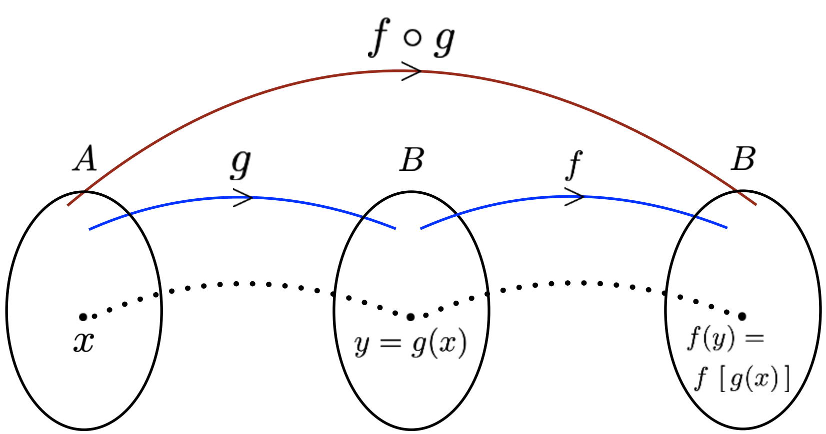
Some Notations using inequalities

	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Composition of functions

The composition of functions f and g is defined by

$$(f \circ g)(x) = f\,(\,g(x)\,)$$



e.g. $f(x) = x^2 + 3, \quad g(x) = \sqrt{x},$ then

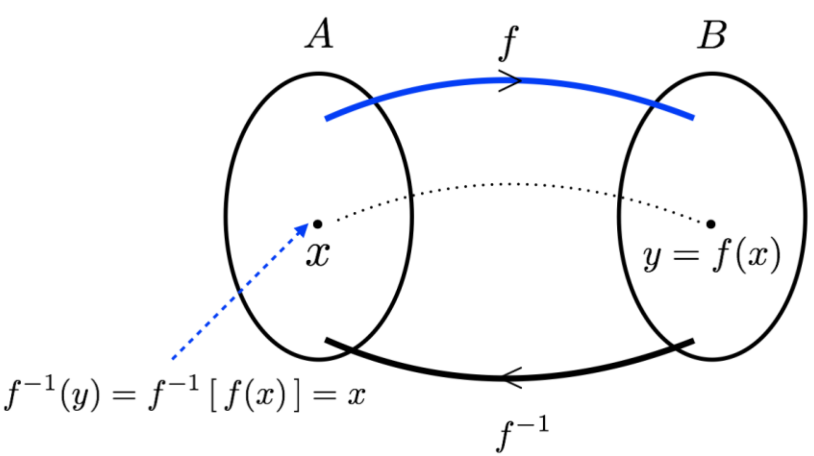
$$\begin{aligned}(f \circ g)(x) &= f\,(g(x)) \\ &= f\,(\sqrt{x}\,) \\ &= (\sqrt{x}\,)^2 + 3 \\ &= x + 3.\end{aligned}$$

and

$$\begin{aligned}(g \circ f)(x) &= g\,(f(x)) \\ &= g\,(x^2 + 3) \\ &= \sqrt{x^2 + 3}\end{aligned}$$

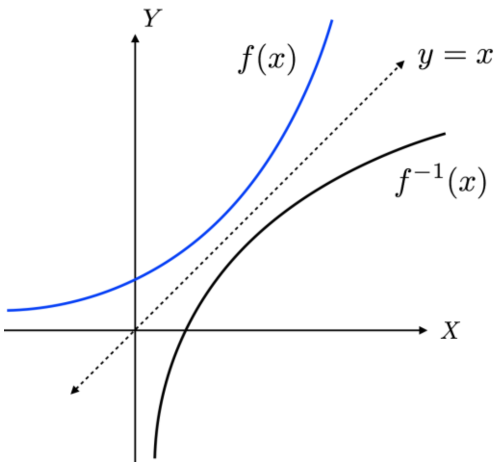
Inverse Function

- The inverse function performs the opposite operation to the function f.

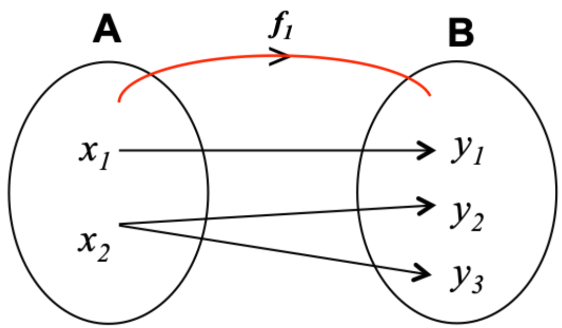


- $f\, f^{-1}(x) = f^{-1}\, f(x) = x$

- Inverse function only exists for functions that are one-one and onto.
- The graph of $f^{-1}(x)$ is a reflection of $f(x)$ in the line $y = x$.



Which of the following mappings are functions?



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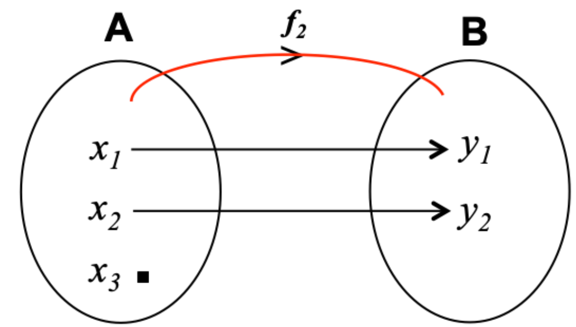
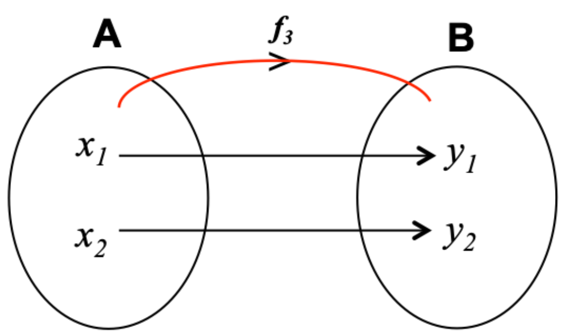
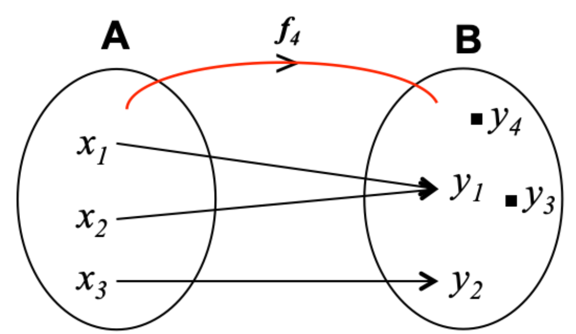


f_2 is not a function because the element x_3 of A is **NOT** mapped.



f_3 is a function (Type: One-one & onto)



f_4 is a function (Type: Many one & into).

Finding the inverse function

Example: Given function $f:\mathbb{R}^+ \cup \{0\} \to \{y \in \mathbb{R} \,/\, y \geq 2\}$,

$$f(x) = 3x^2 + 2, \text{ find } f^{-1}(x).$$

- Step 1 Let $y = f(x) = 3x^2 + 2$
- Step 2 Express x as a function of y.

$$\Rightarrow 3x^2 = y - 2 \quad \Rightarrow x = \sqrt{\frac{y-2}{3}}$$

- Step 3 $f^{-1}(x)$ is obtained by replacing y by x on the **RHS**.

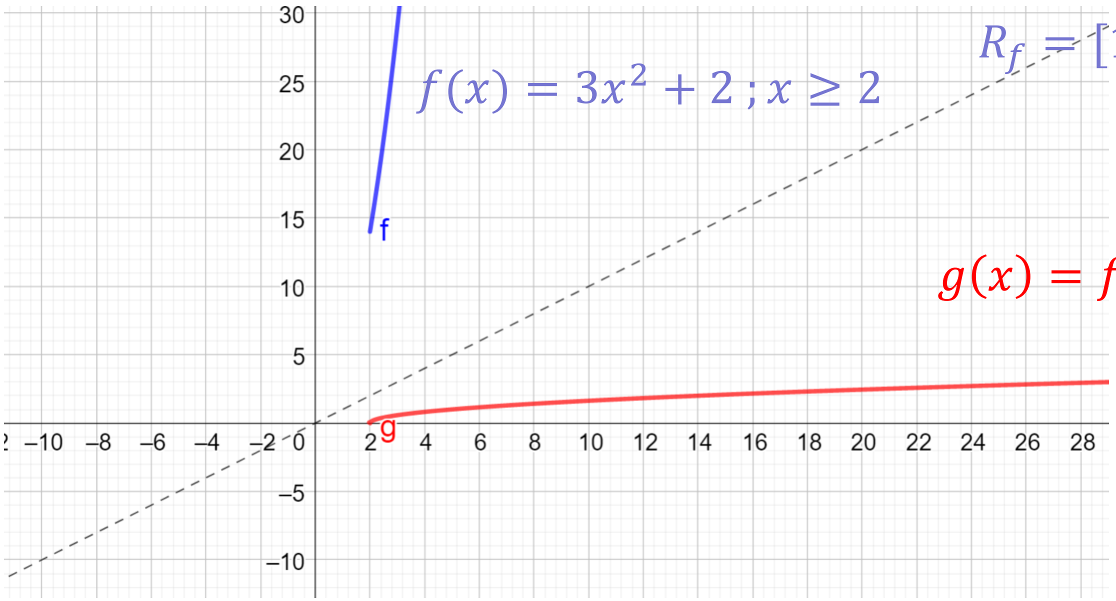
$$\therefore \quad f^{-1}(x) = \sqrt{\frac{x-2}{3}}$$

General Review of functions

Range and Domain of a function

$$D_f = [2, \infty)$$
$$R_f = [14, \infty)$$

$$f(x) = 3x^2 + 2\,; x \geq 2$$

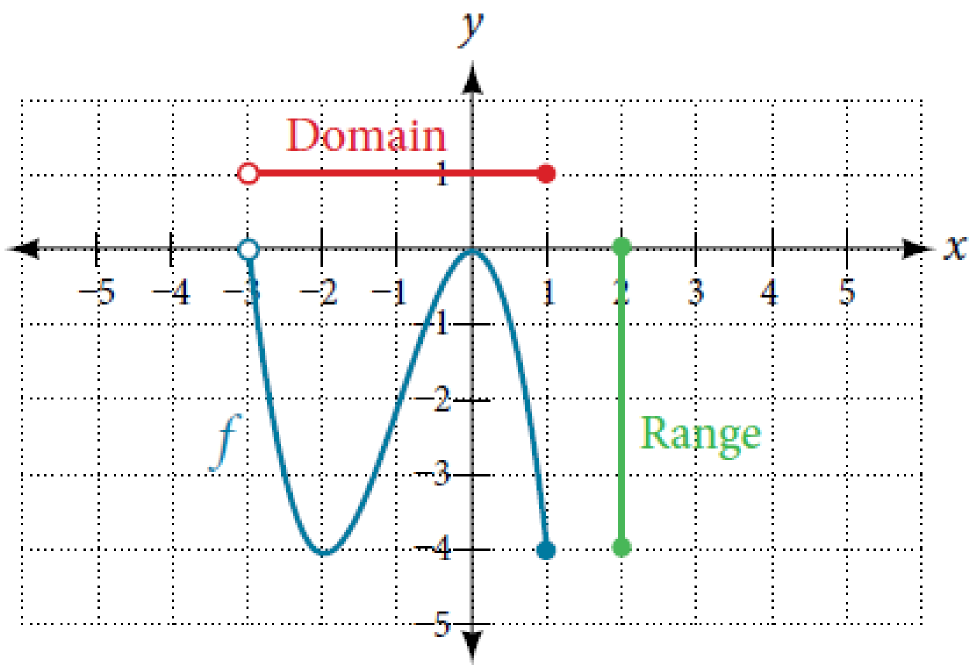


$$g(x) = f^{-1}(x) = \sqrt{\frac{x-2}{3}}\,; x \geq 2$$

$$D_f = [2, \infty)$$
$$R_f = [0, \infty)$$

General Review of functions

Range and Domain of a function



$D_f = [-3, 1)$

$R_f = [0, -4]$

General review of functions

Example

Find the range and domain of the following functions

1. $$f(x) = \frac{2}{x+1}$$

2. $$f(x) = 2\sqrt{x+4}$$

General review of functions

Piecewise function: is a function in which more than one formula is used to define the output. Each formula has its own domain.

Hence the domain of the **piecewise function** is the union of all these smaller domains.

$$f(x) = \begin{cases} formula\ 1 & if\ x\ is\ in\ domain\ D_1 \\ formula\ 2 & if\ x\ is\ in\ domain\ D_2 \\ formula\ 3 & if\ x\ is\ in\ domain\ D_3 \end{cases}$$

Range and Domain of Functions

Piecewise function

Example

Sketch the graph of the following functions

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \le 1 \\ 3 & \text{if} \quad 1 < x \le 2 \\ x & \text{if} \quad x > 2 \end{cases}$$

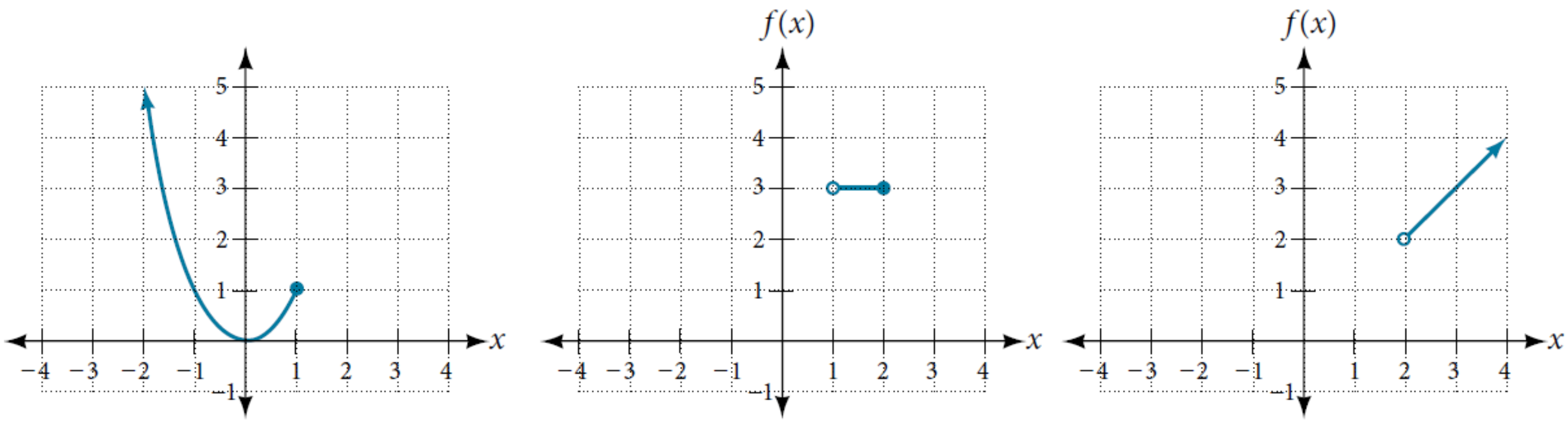
Range and Domain of Functions

Piecewise function

$$f(x)=\begin{cases} x^2 & \text{if} \quad x \le 1 \\ 3 & \text{if} \quad 1 < x \le 2 \\ x & \text{if} \quad x > 2 \end{cases}$$

(a)
(b)
(c)

Each function presented in separate graphs



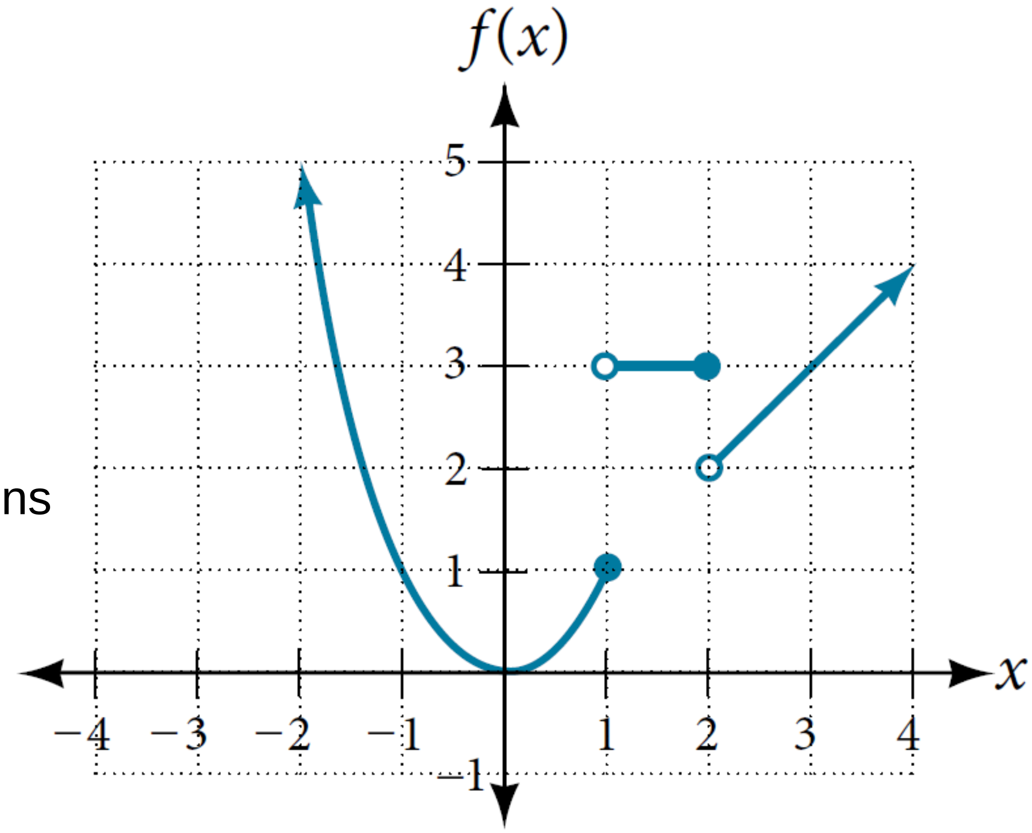
(a) (b) (c)

Range and Domain of Functions

Piecewise function

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \le 1 \\ 3 & \text{if} \quad 1 < x \le 2 \\ x & \text{if} \quad x > 2 \end{cases}$$

Join the plots of the separate functions to form the piecewise function



Further Reading (click on links)

College Algebra by J. W. Coburn & J. P. Coffelt (3rd edition)

(Section 2.4 to 2.5)

Foundation Algebra by P. Gajjar.

(Chapter 2)

Introduction to functions from OpenStax™

Range and domain of functions from OpenStax™

THANKS FOR YOUR ATTENTION