The University of Nottingham
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Science A Physics

Lecture 6: Circular Motion and Simple Harmonic Motion

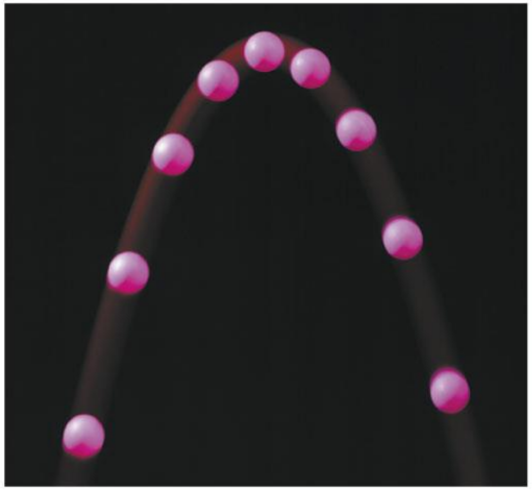
Aims of today's lecture

1. Circular Motion
2. Centripetal Acceleration and Force
3. Periodic/Oscillatory Motion
4. Simple Harmonic Motion
5. The Simple Pendulum

Types of Motion

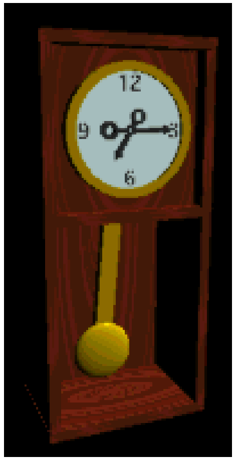
Linear motion

Circular motion



Projectile motion

Rotational motion

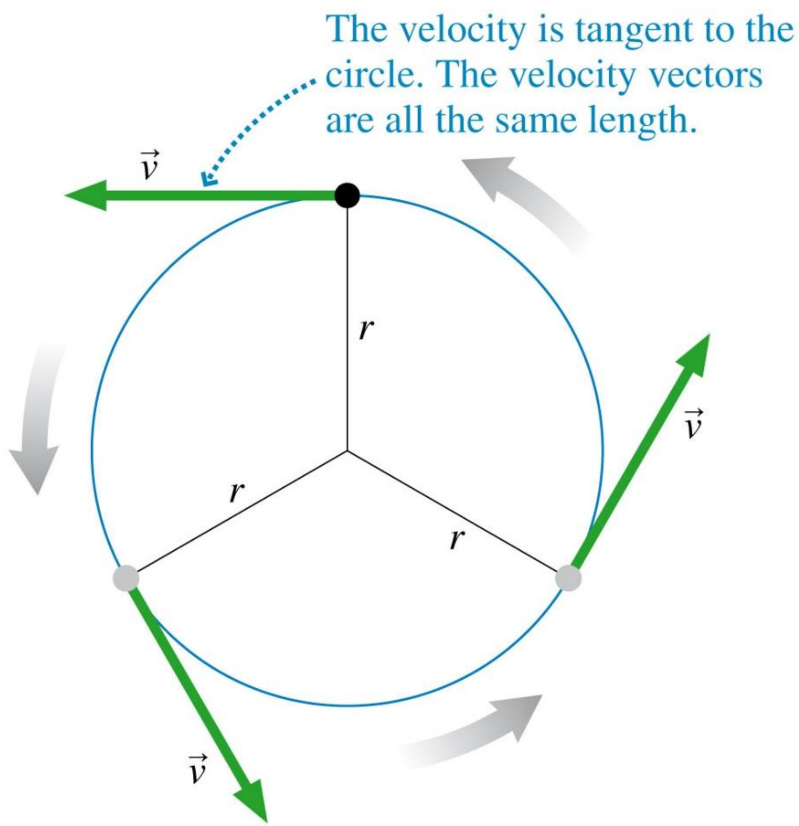


Periodic
(or oscillatory motion)

- Our focus for this course.

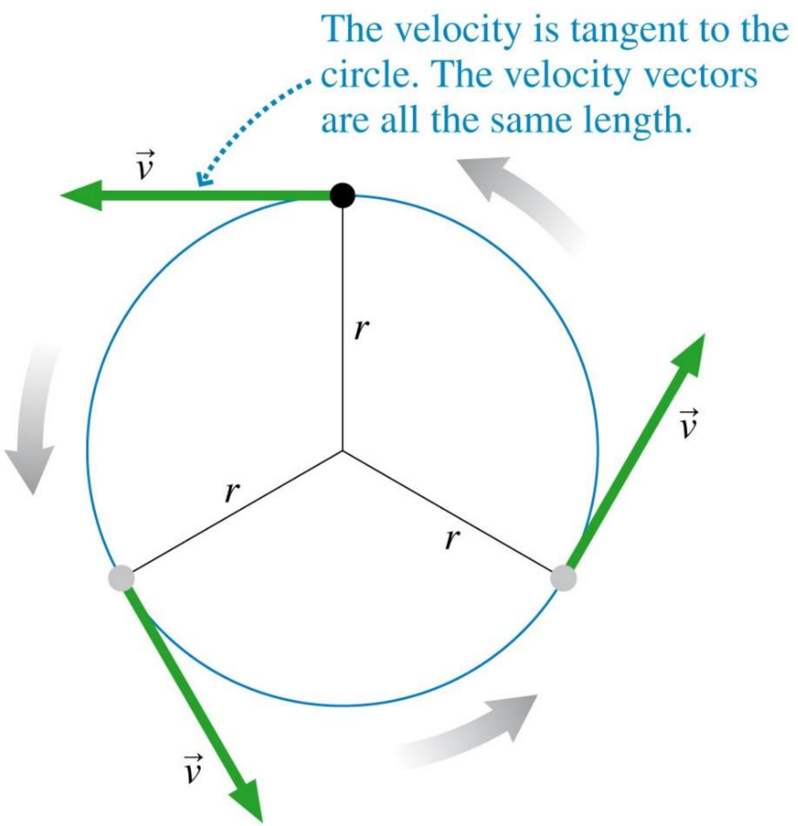
1. Circular Motion

Uniform Circular Motion



- If the object has a constant tangential velocity, we can say that circular motion is **uniform**.
- We can also say that the time interval to complete one revolution, called the period, T, is fixed.

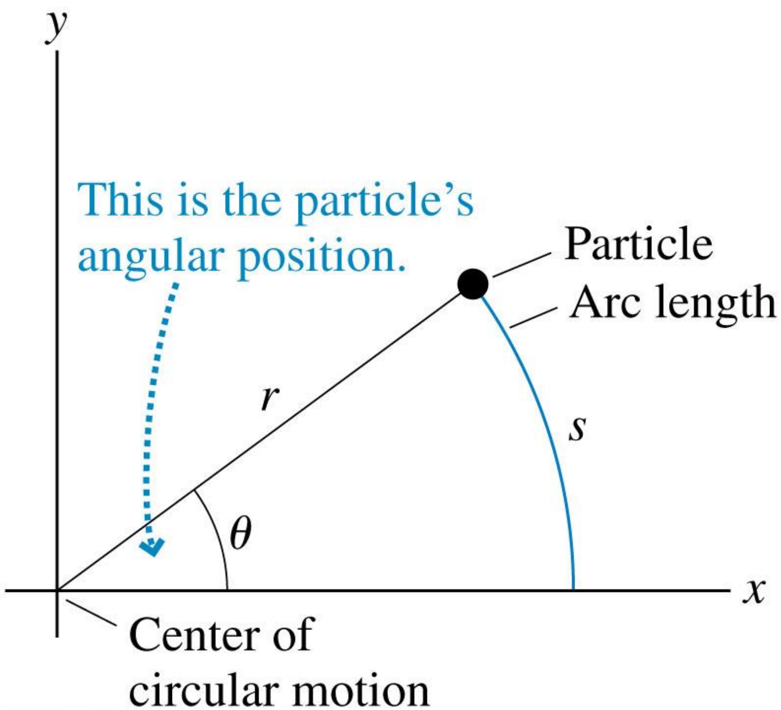
Uniform Circular Motion



- The relationship between the period, T, and the tangential speed, v, is:

$$v = \frac{1\ \text{circumference}}{1\ \text{period}} = \frac{2\pi r}{T}$$

Angular Position

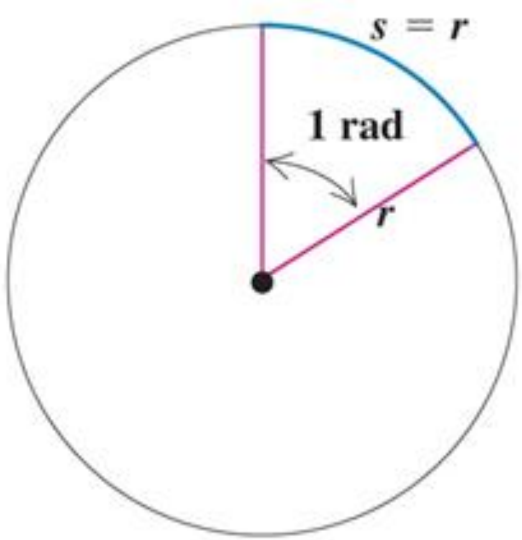


- Consider an object/particle at a distance r from the origin, at an angle θ from the positive x-axis.
- The angle may be measured in degrees, revolutions (rev) or **radians** (rad), that are related by:

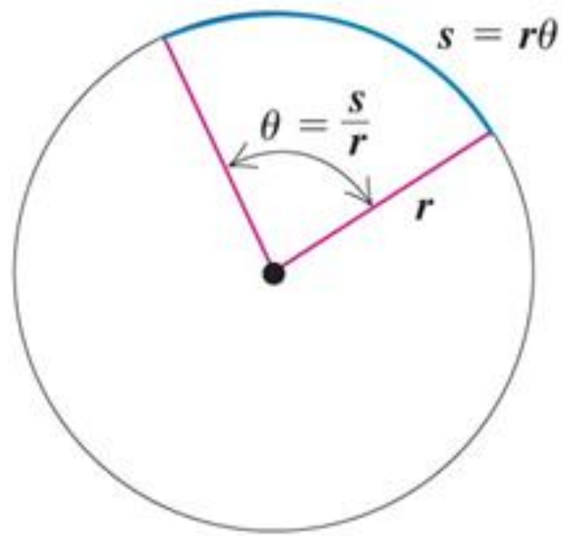
$$1\,rev = 360° = 2\pi\,rad$$

Radians

One radian is the angle at which the arc s has the same length as the radius r.

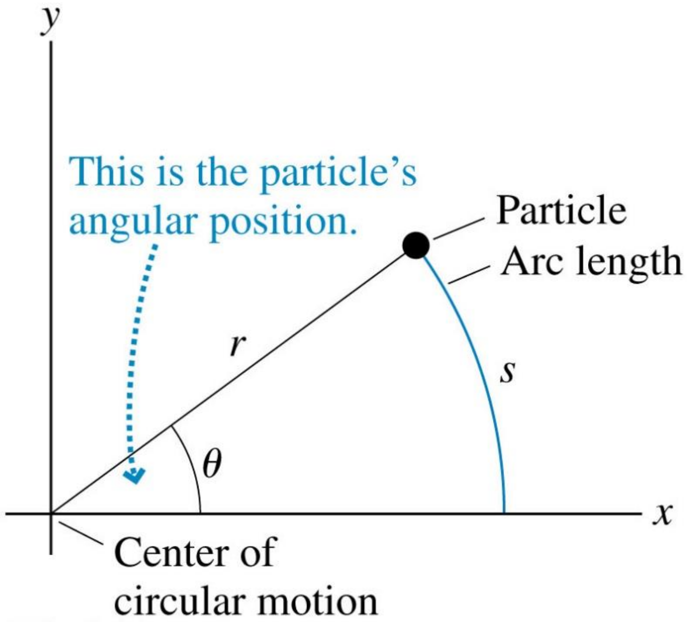


An angle θ in radians is the ratio of the arc length s to the radius r.



N.B. Because the length of the circumference of a circle is $2\pi r$, dividing this length by r, gives us 2π, which means that there are 2π radians in a circle.

Angular Position

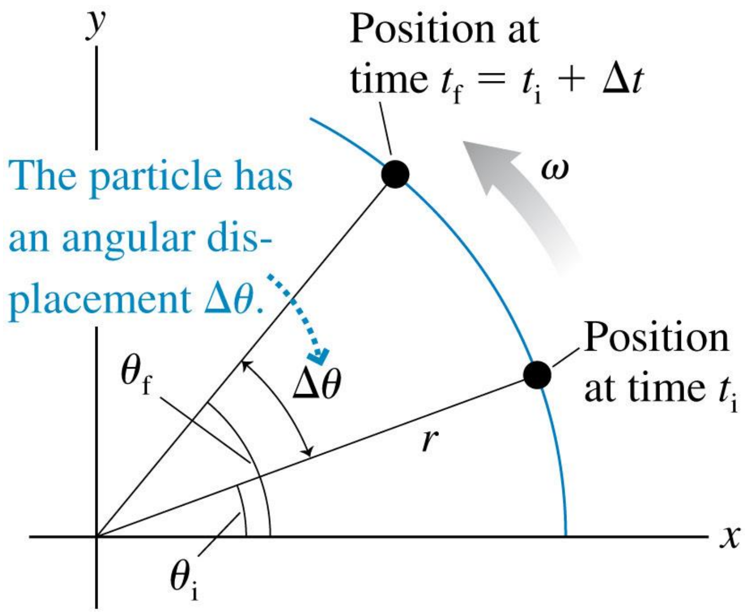


- Thus, if the angle is measured in radians, we can describe the angular position of the object, using the relationship below.

$$s = r\theta \qquad \text{(with } \theta \text{ in rad)}$$

- We can then go one step further, and describe the velocity of the object, referred to as **angular velocity**, because the object is moving in a circle; let's now consider this idea in more detail.

Angular Velocity

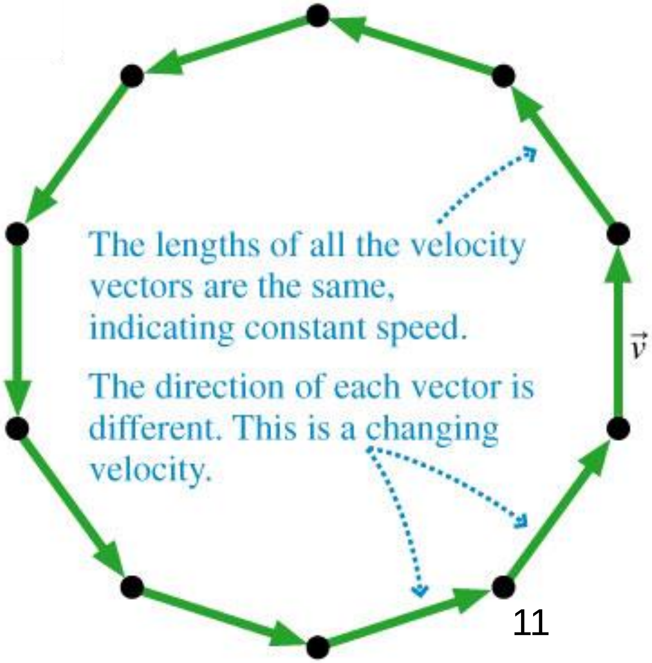


- A particle on a circular path moves through an **angular displacement** $\Delta\theta = \theta_f - \theta_i$ in a time interval $\Delta t = t_f - t_i$.
- Similar to linear motion, we can define average angular velocity, ω:
- The definition of **instantaneous angular velocity** is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{(angular velocity)}$$

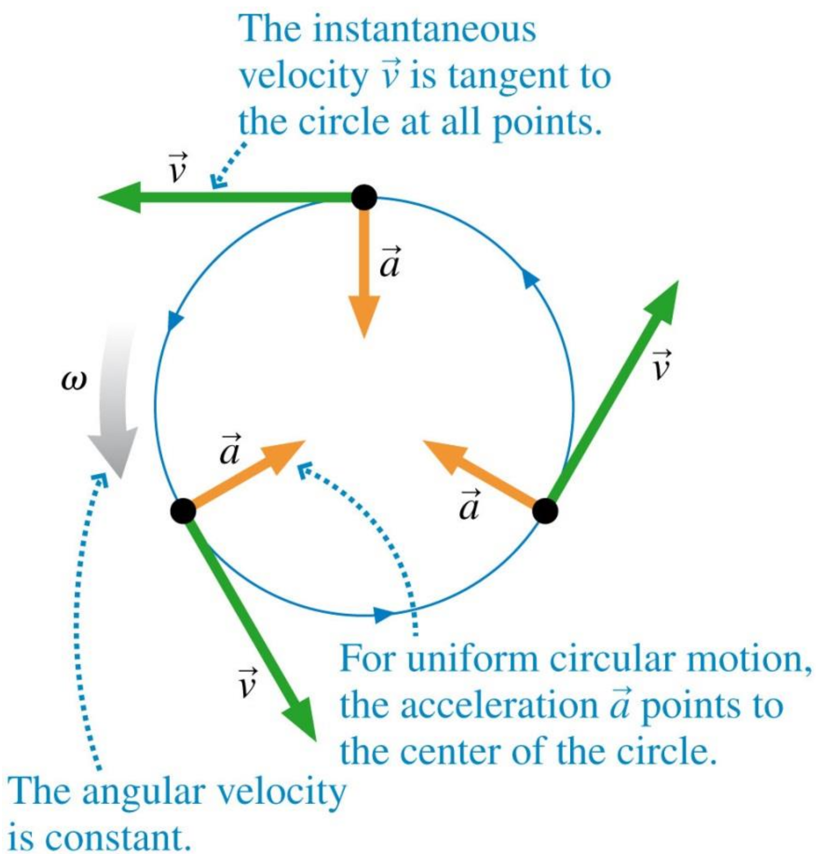
Uniform Circular Motion

- The figure to the right shows a motion diagram for a Ferris wheel.
- It has constant speed, but not constant velocity (because its direction is changing), so it is accelerating.
- We can choose a point on the wheel to track.
- For every pair of adjacent velocity vectors, we can subtract them to find the average acceleration near that point; we call this acceleration **centripetal acceleration**.

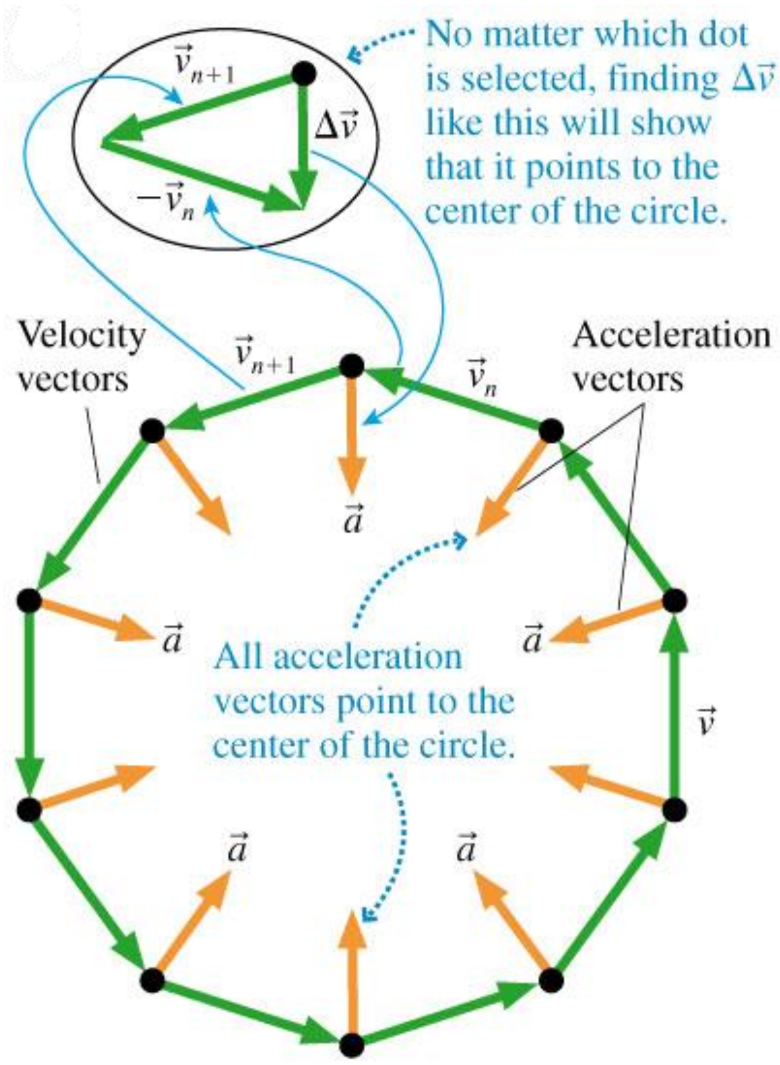


2. Centripetal Acceleration and Force

Centripetal Acceleration

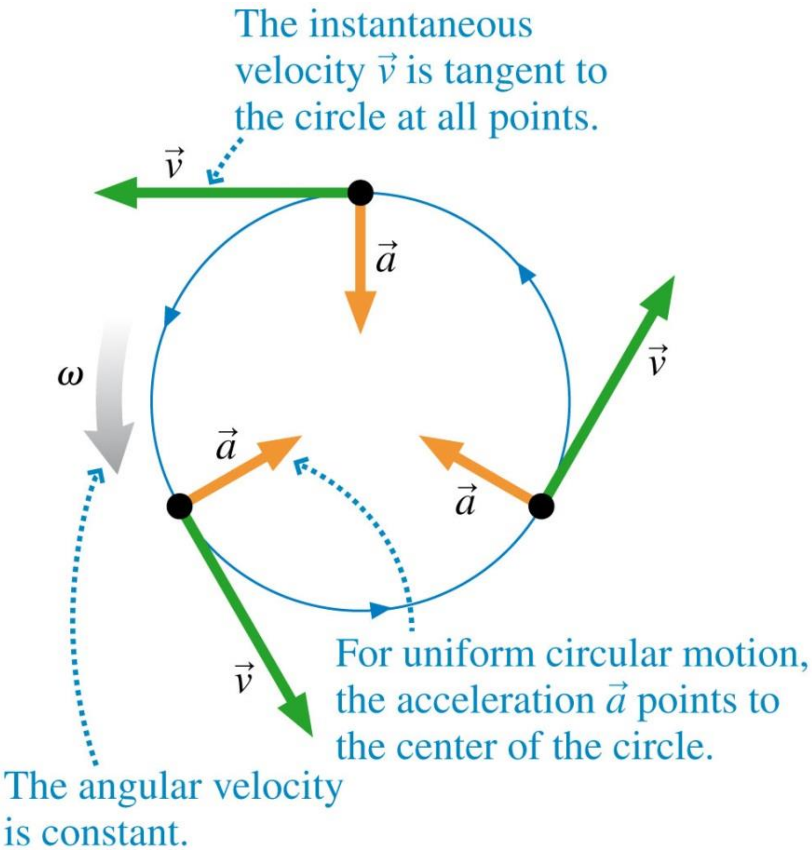


- In uniform circular motion, although the speed is constant, there is an **acceleration** because the **direction** of the velocity vector is always changing.
- The acceleration of uniform circular motion is called **centripetal acceleration**.



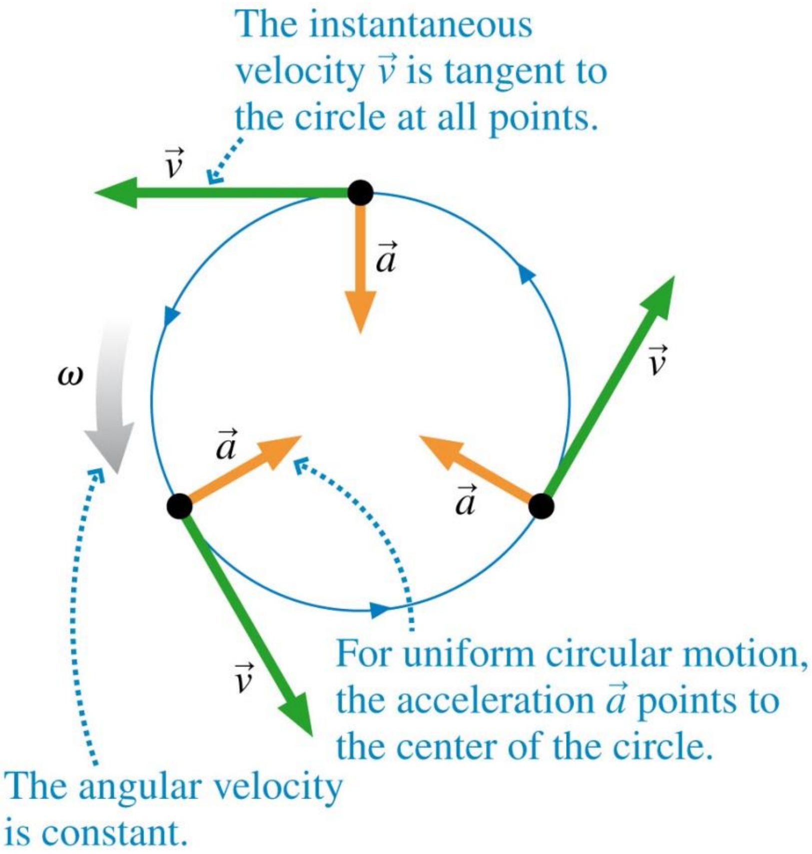
- At every point, the acceleration points toward the centre of the circle.
- This is an acceleration due to changing direction, not to changing speed.

Centripetal Acceleration



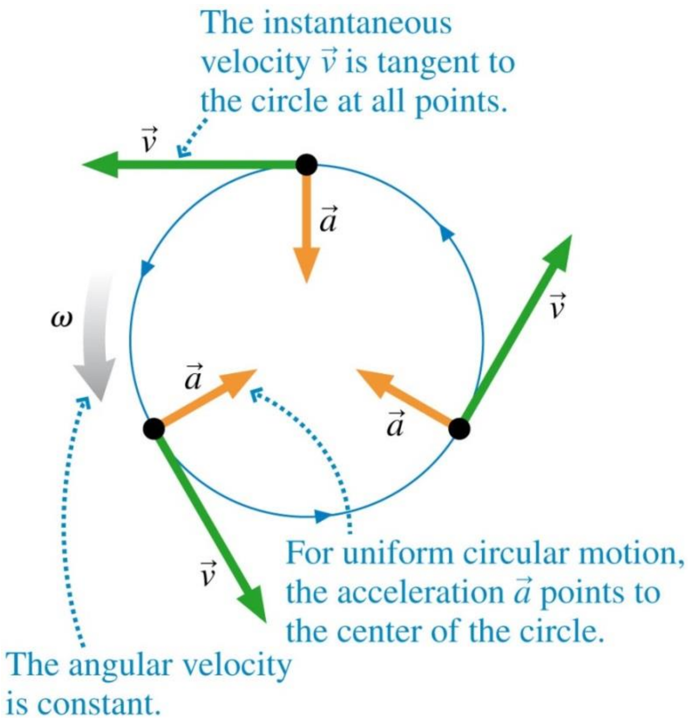
- The velocity vector $\vec{v}$ for uniform circular motion is always tangent to the circle.
- In other words, the velocity vector has only a tangential component, which we can call v_t.

Centripetal Acceleration



- The tangential velocity component v_t is the rate at which the particle moves around the circle, where s is the arc length measured from the positive x-axis.
- The arc length is $s = r\theta$

Centripetal Acceleration

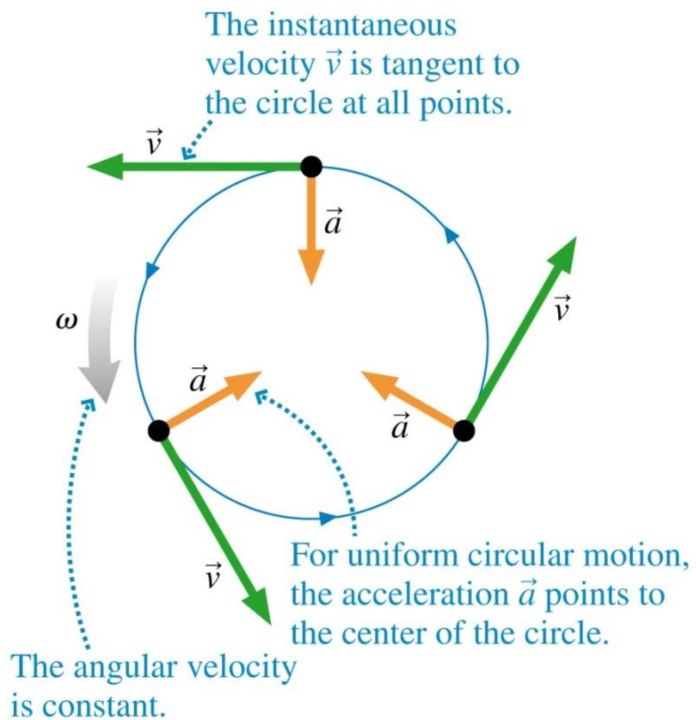


- The angular velocity is given by ω. Thus, the tangential velocity and the angular velocity are related by

$$v_t = \omega r \; (with \; \omega \; in \; \frac{rad}{s})$$

- The tangential velocity v_t is positive for counterclockwise (ccw) motion, while it is negative for clockwise motion (cw).

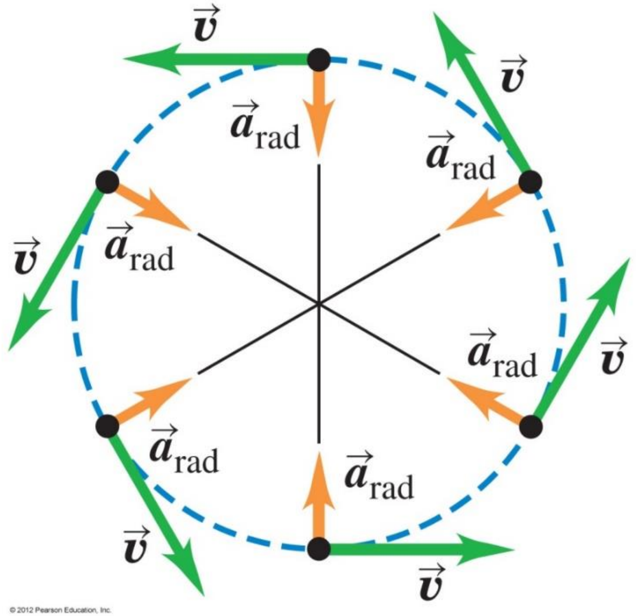
Centripetal Acceleration



$$\vec{a} = \left(\frac{v^2}{r}, \text{toward center of circle}\right) \qquad \text{(centripetal acceleration)}$$

Centripetal acceleration can be written in terms of angular velocity as: $a = \omega^2 r$.

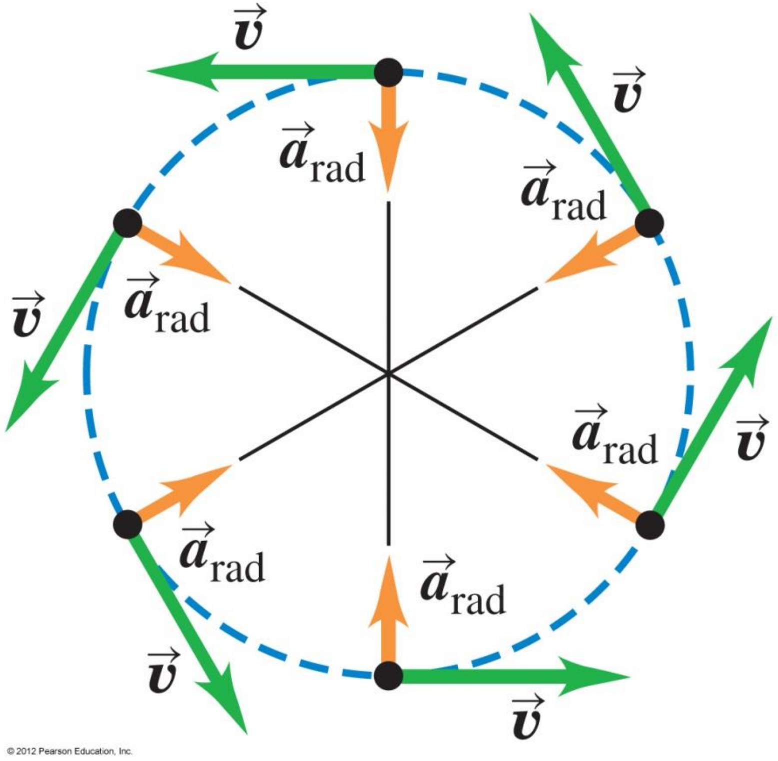
Centripetal Force



$$F_{net} = m\frac{v^2}{R}$$

- Building on the idea of centripetal acceleration, we can see that the object's acceleration toward the centre of the circle must be caused by a force, or several forces, such that their vector sum $\sum \vec{F}$ is a vector that is always directed toward the centre, with constant magnitude.

Centripetal Force

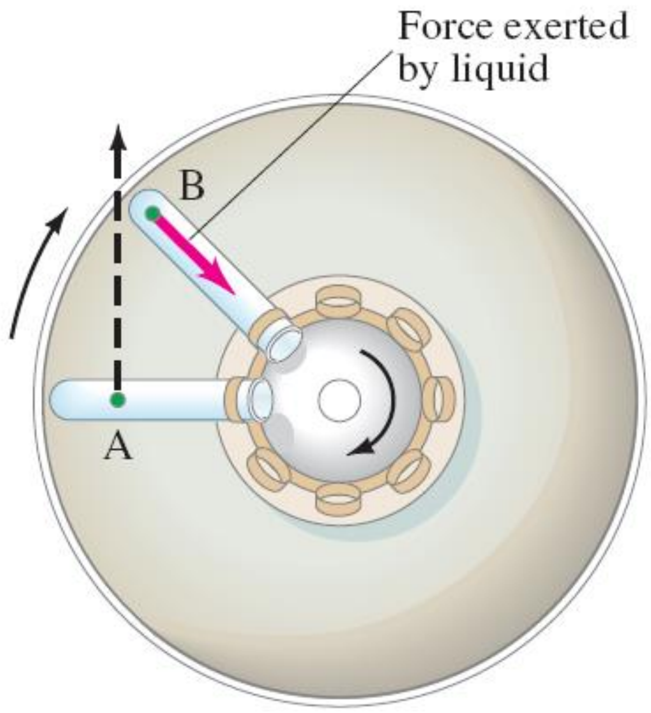


N.B.

When a particle moves in a circular path with constant speed (or uniform circular motion), there is no component of acceleration parallel to its instantaneous velocity; otherwise the particle's speed would change.

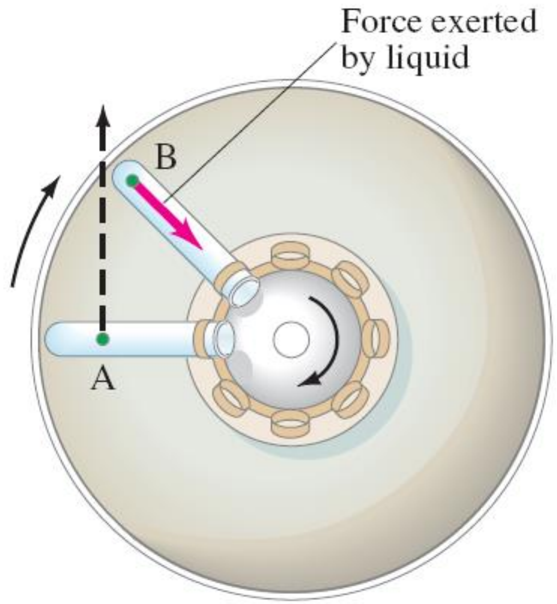
Let's now look at some **applications of circular motion**.

A Centrifuge



- Centrifuges are used to separate materials.
- Test tubes held in the centrifuge rotor are accelerated to very high rotational speeds.
- The small green dot represents a small particle, in a fluid-filled test tube.

A Centrifuge



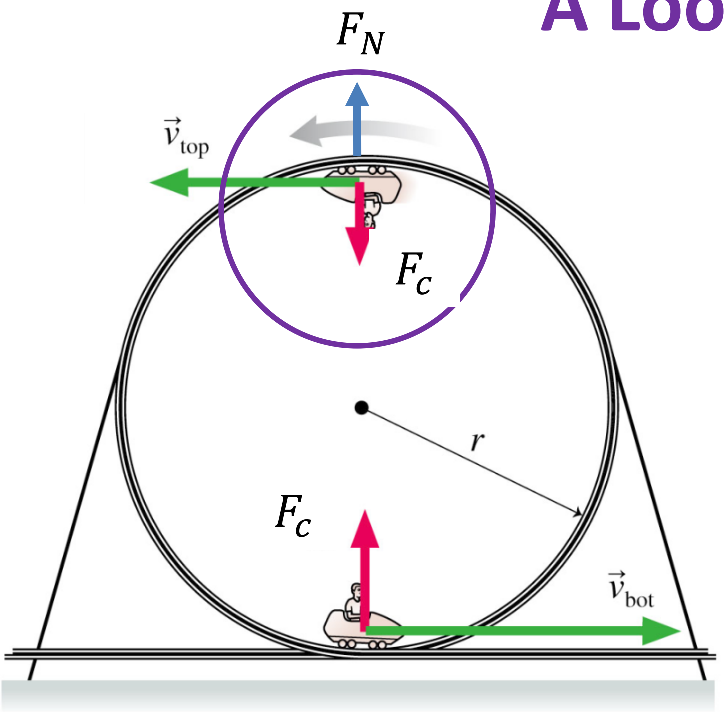
- At position A, the particle has a tendency to move in a straight line, but the fluid resists the motion of the particles, exerting a centripetal force that keeps the particles moving nearly in a circle.

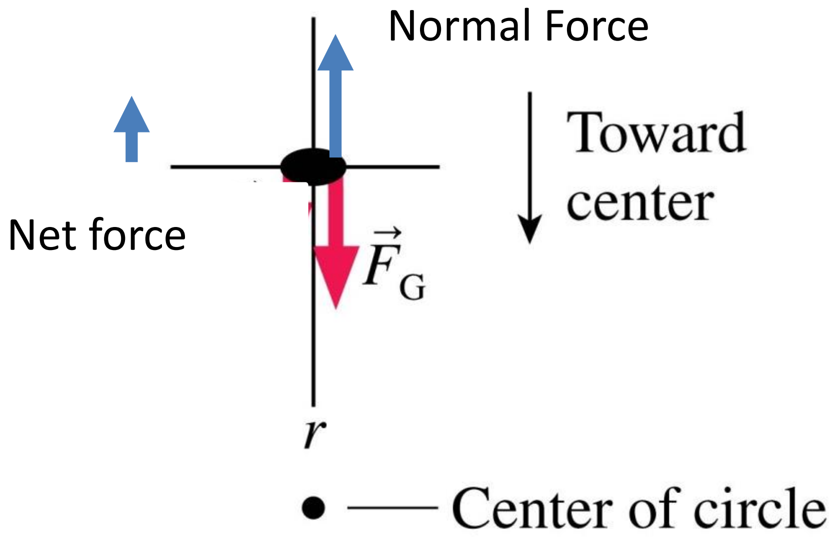
- The resistive force exerted by the fluid usually does not quite equal mv^2/r, and the particle moves slowly toward the bottom of the tube.
- Let's look at another interesting application, that of the **rollercoaster**.

A Rollercoaster

Q. Why doesn't the rollercoaster fall off the tracks when doing a loop-the-loop?

A Loop-the-Loop



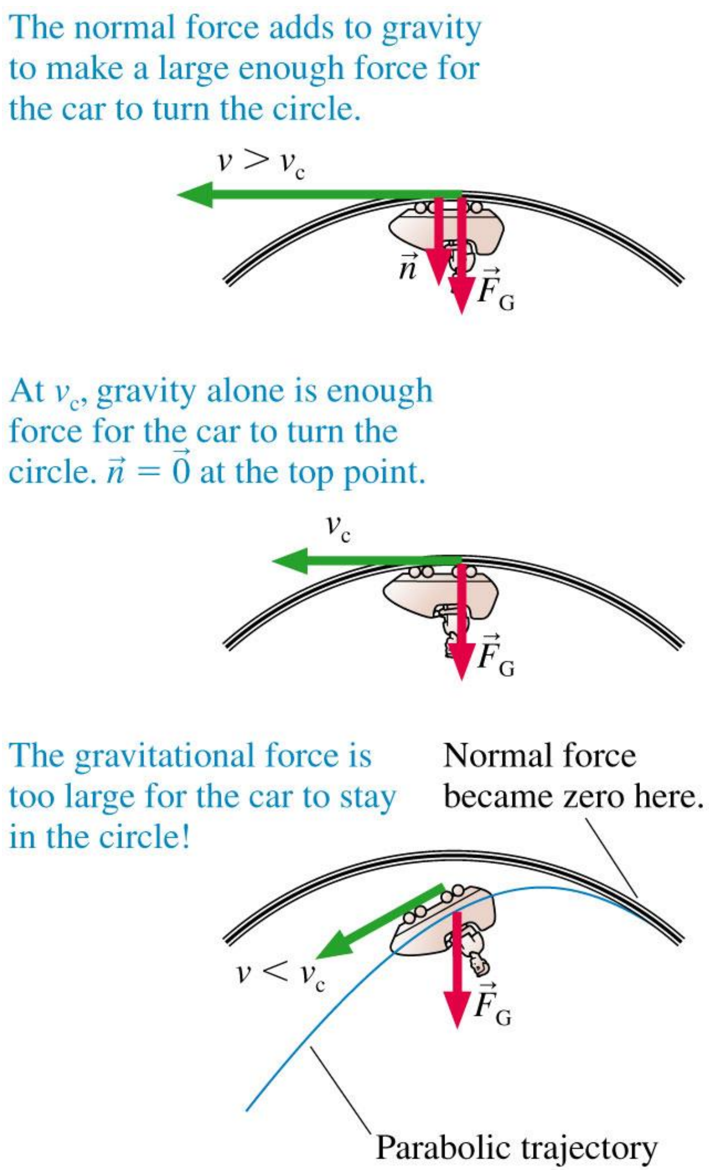


$$\Rightarrow\ n = \frac{m(v_{top})^2}{r} - mg$$

- As $\boldsymbol{v_{top}}$ decreases, there comes a point when $\boldsymbol{n}$ reaches zero.
- The speed at which $\boldsymbol{n = 0}$ is the minimum speed you must travel at to avoid falling off:

$$v_{minimum} = \sqrt{\frac{rmg}{m}} = \sqrt{rg}$$

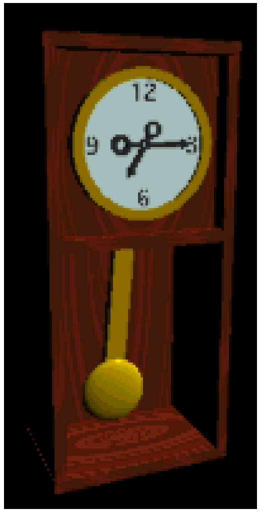
A Loop-the-Loop



- Obviously, when designing your rollercoaster, you should plan for it to travel a bit faster than this minimum speed.
- Health and safety is important ☺.

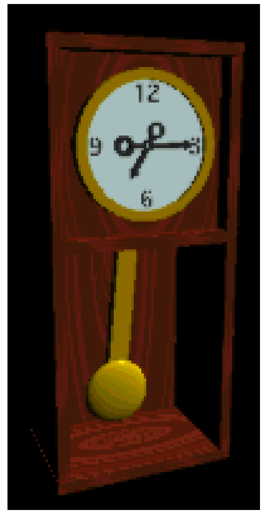
3. Periodic/Oscillatory Motion

Periodic/Oscillatory Motion



- Many everyday objects, such as springs and grandfather clocks, appear to move in a repeating pattern.
- We refer to such repeating movement as **periodic motion.**
- A mechanical system that undergoes periodic motion always has a stable **equilibrium position.**

Periodic/Oscillatory Motion

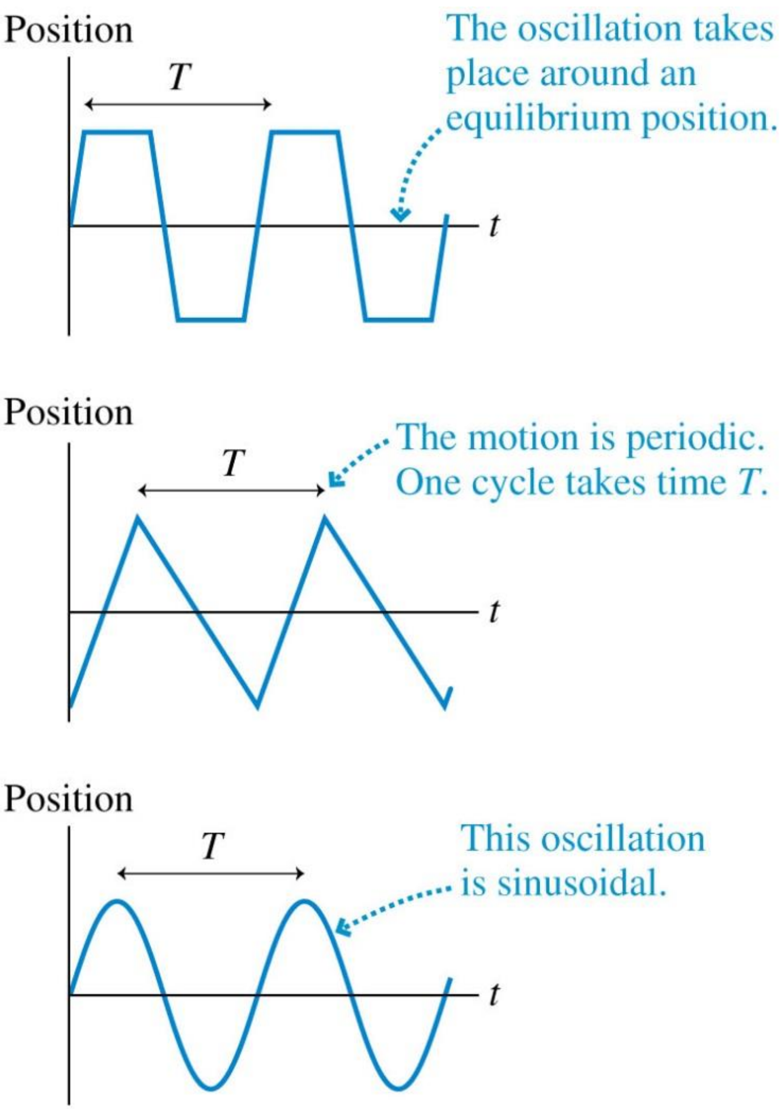


- When the object is moved away from its equilibrium/'average' position, a **force** comes into play that attempts to **return the object** to this position.

- However, by the time it reaches its equilibrium position, it has picked up **kinetic energy.**

- The object will slow down, and eventually stop. It will then start to **accelerate** back to its equilibrium position again.

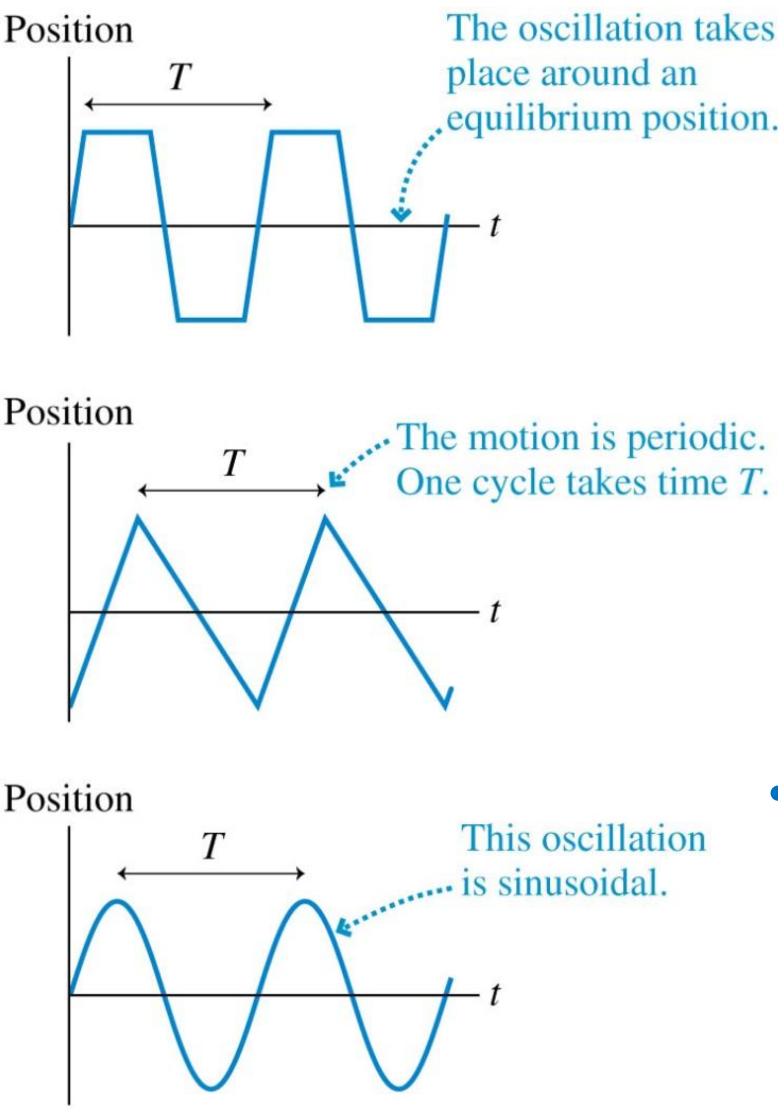
Periodic/Oscillatory Motion



- In general, objects that undergo any type of repetitive motion back-and-forth around an equilibrium position are called oscillators.

- The time to complete one full cycle, or one oscillation, is called the **period *T***.

Periodic/Oscillatory Motion



- The number of cycles per second is called the **frequency f**, measured in Hz:

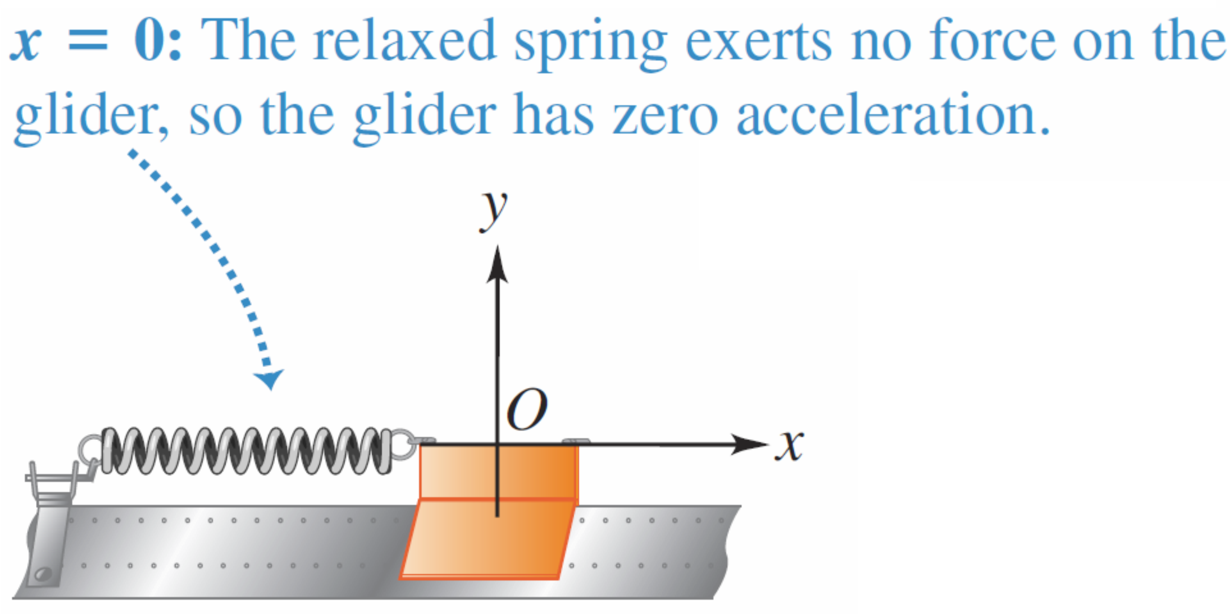
$$f = \frac{1}{T} \quad \text{or} \quad T = \frac{1}{f}$$

- $1\ \text{Hz} = 1\ \text{cycle per second} = 1\ \text{s}^{-1}$

- The simplest type of periodic motion is called **simple harmonic motion**, so let's have a look at it in more detail.

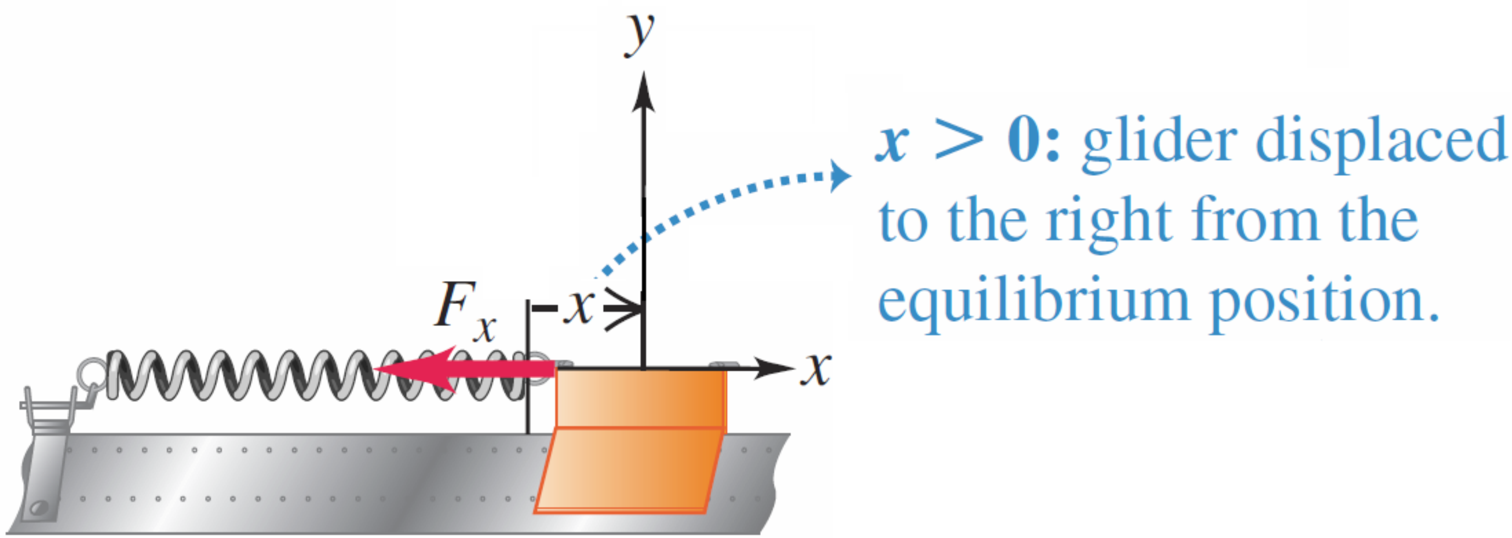
4. Simple Harmonic Motion

Simple Harmonic Motion



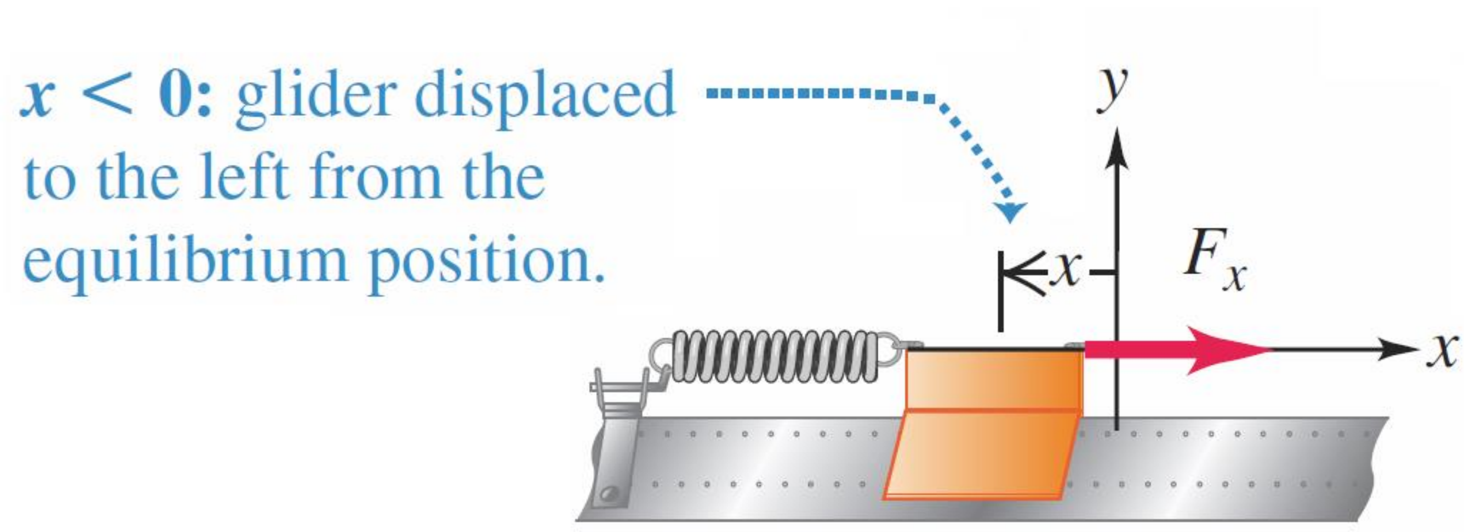
- Consider the example of an air track glider attached to a spring.
- We define the origin,O,as the equilibrium position of the system.
- We ignore friction, assuming it to be negligible.

Simple Harmonic Motion



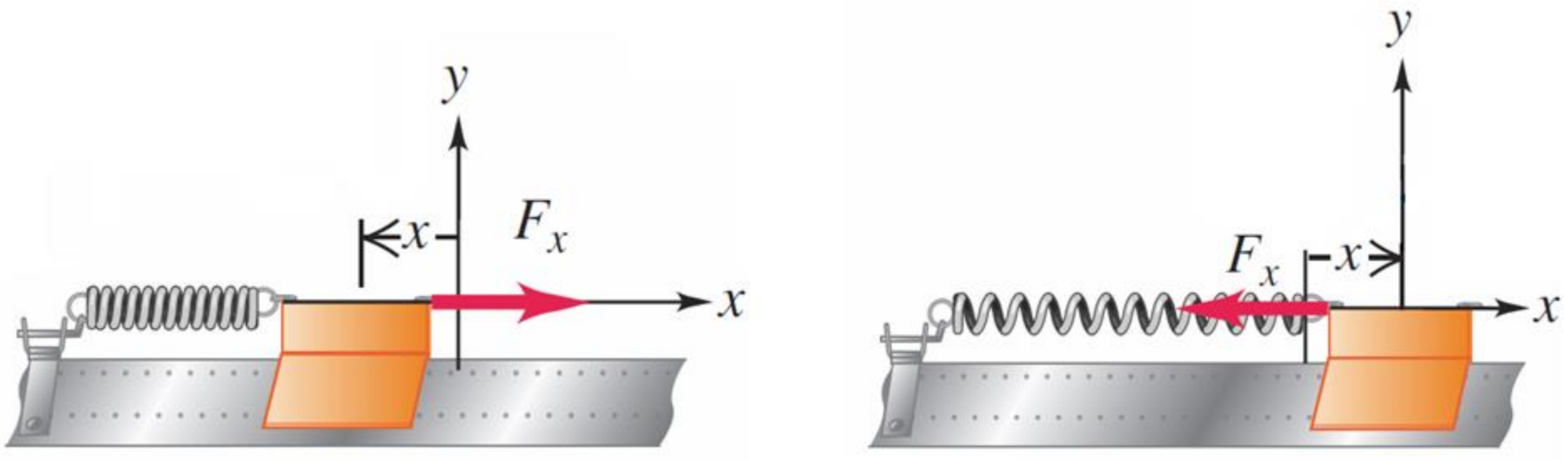
- When the glider is displaced to the right, a force will act to the left as the spring is stretched (**remember Hooke's law**).
- We call this force the **'restoring force'** because the force wants to restore the object/glider to its original/equilibrium position.
- Since our coordinate system defines $x > 0$ then $F_x < 0$.

Simple Harmonic Motion



- And conversely, when the glider is displaced to the left, a restoring force will act to the right as the spring is compressed.
- Since our coordinate system defines $x < 0$ then $F_x > 0$.

Simple Harmonic Motion

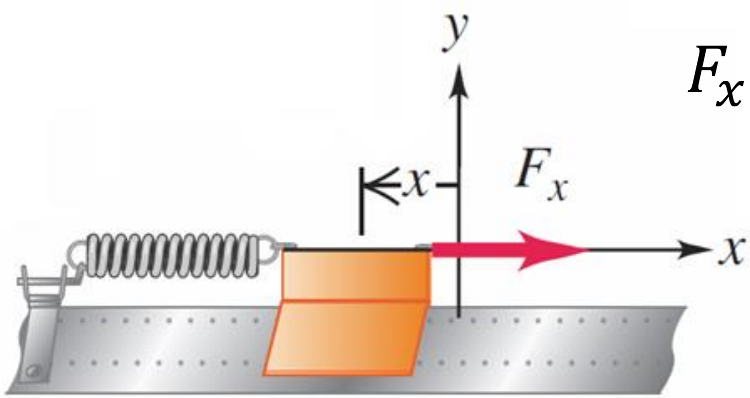


- When the system is displaced from equilibrium, the restoring force F_x will **always act in the opposite direction to the displacement** x.

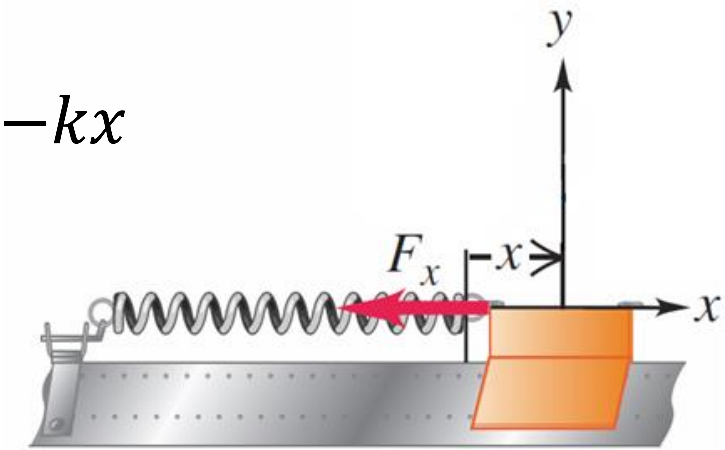
- The equation for the restoring force (**Hooke's law**) is therefore

$$F_x = -kx$$

Simple Harmonic Motion



$$F_x = -kx$$



- We can use our understanding of Newton’s 2nd law to obtain an equation for the acceleration of the glider/object:

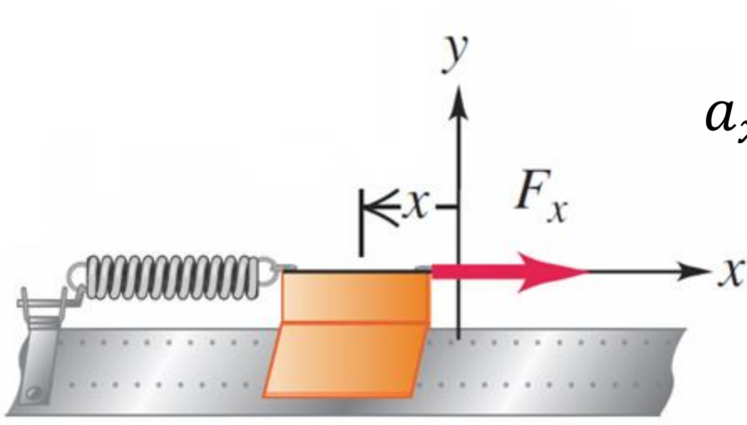
$$F_x = ma_x$$

$$ma_x = -kx$$

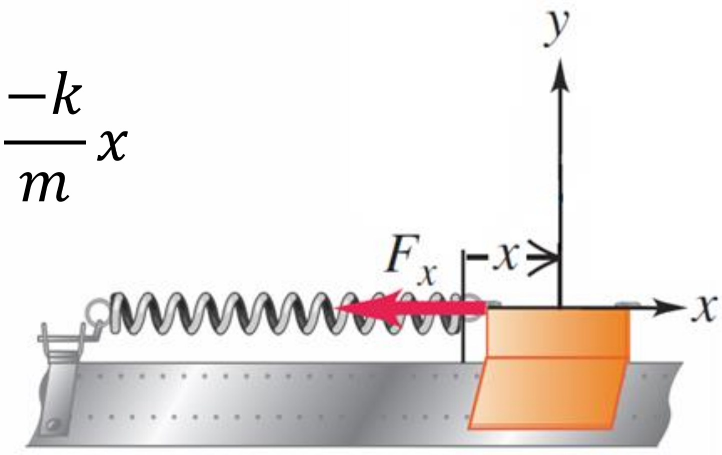
$$a_x = \frac{-k}{m}x$$

where $\frac{-k}{m}$ is a constant.

Simple Harmonic Motion

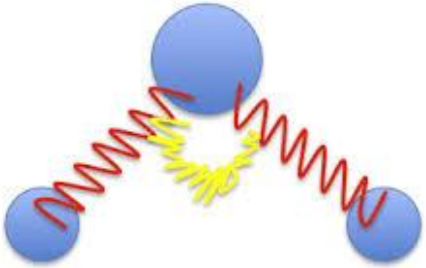


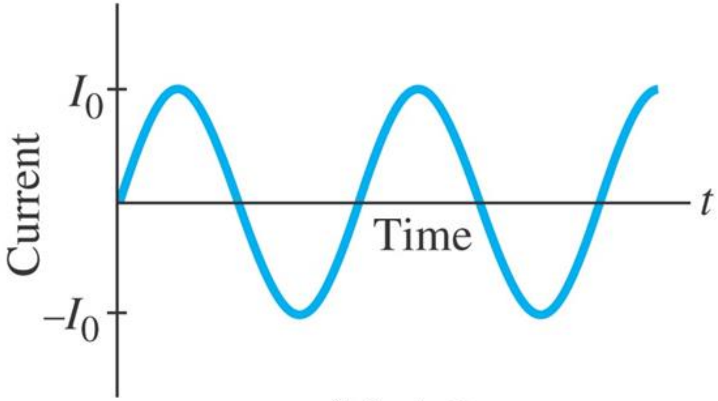
$$a_x = \frac{-k}{m} x$$



- Thus, the defining feature of an object which undergoes simple harmonic motion is that its **acceleration is directly proportional to its displacement from its equilibrium position**.
- Consequently, the restoring force acting on the object is directly proportional to the object's displacement from its equilibrium position.
- In more complex types of periodic motion, the restoring force may not be directly proportional to the displacement.

Simple Harmonic Motion





(b) AC

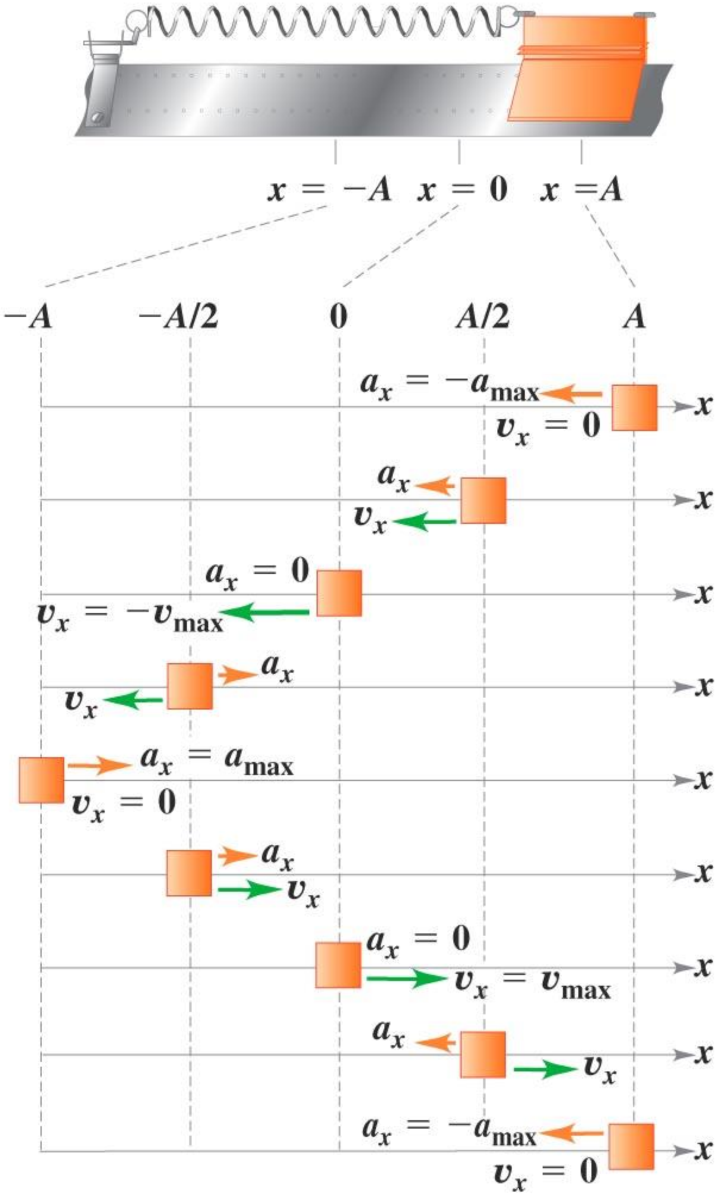
- However, many periodic motions are approximately simple harmonic if the **maximum displacement from the equilibrium position is small.**
- Examples of objects undergoing periodic motion that can be considered simple harmonic motion include the simple pendulum, molecular bonds, and, as we will see later, alternating electric current.
- Let's now look at some more quantities and equations that can be used to represent certain features of simple harmonic motion.

Features of Simple Harmonic Motion

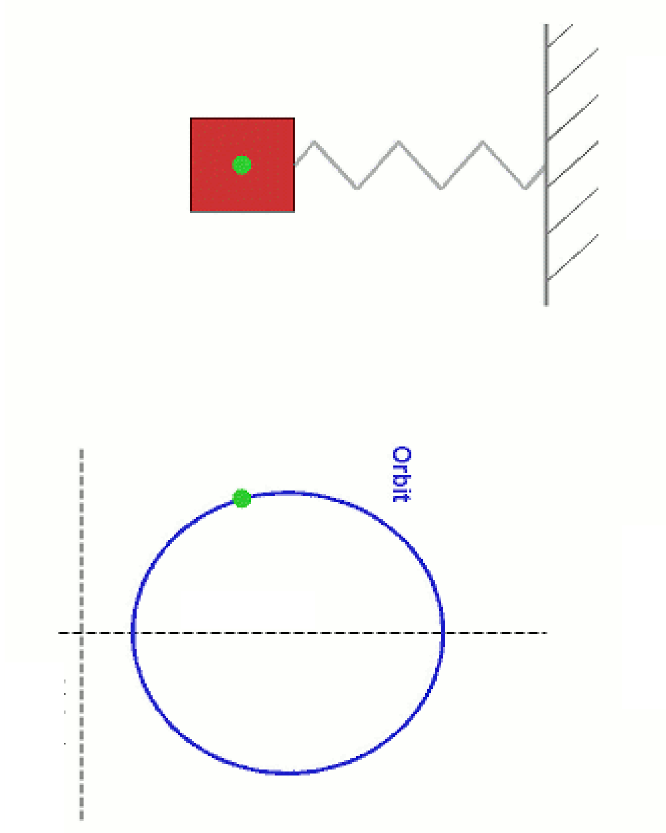
- The amplitude (A) is the **maximum magnitude** of the displacement.

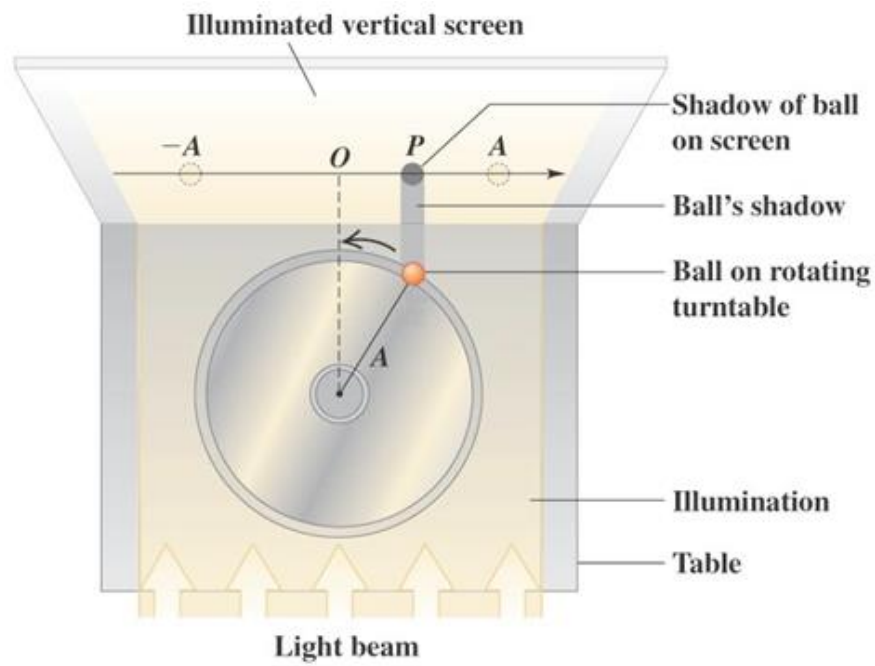
- Since simple harmonic motion is **symmetrical,** this means the overall motion is over a distance of $2A$.

- A **cycle (also known as an oscillation)** is one complete round trip. For example, from A to $-A$ and then back to A.



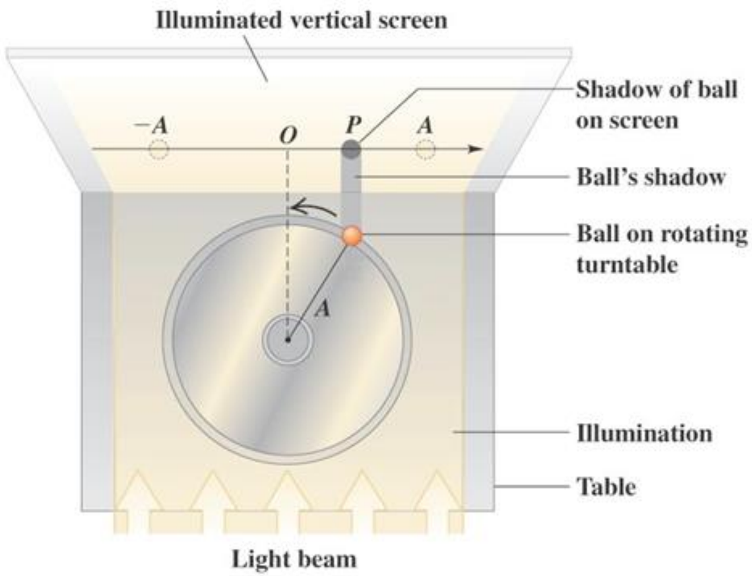
Features of Simple Harmonic Motion





- For a body undergoing simple harmonic motion, ***T*** is the **time/period** for one complete oscillation.
- This oscillation, when referenced to the uniform circular motion of a circle, (as in the **orange ball** on the rotating turntable) corresponds to one complete revolution of the circle.

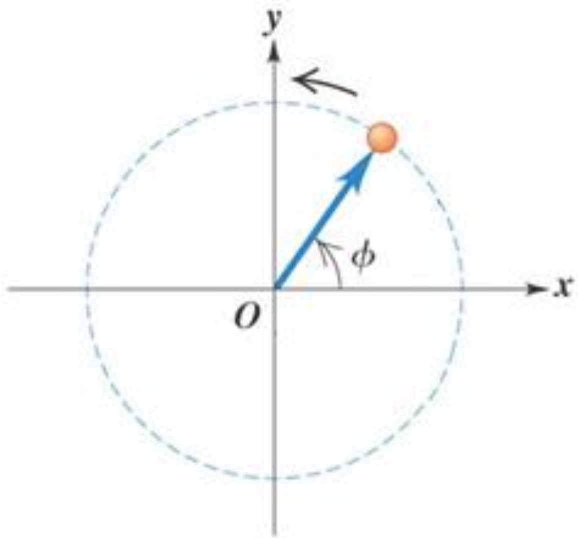
Features of Simple Harmonic Motion



$$\frac{2\pi \, radians}{T} = 2\pi \, radians \times f = 2\pi f = \omega$$

- If we divide the number of radians in a circle (2π) by the time it takes for one complete revolution of the circle ($\boldsymbol{T}$), we can construct a useful quantity called the **angular frequency ($\boldsymbol{\omega}$)**.
- It represents the rate of change of the angle, φ, with time, and is always measured in radians per second ($\mathrm{rads/s}$).

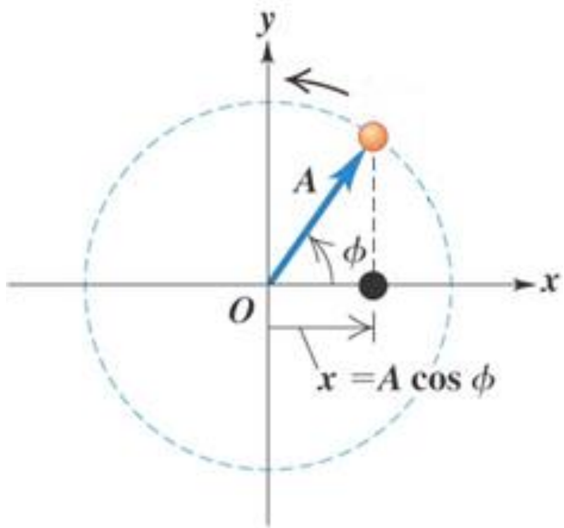
Features of Simple Harmonic Motion



- The **orange ball** moves at a constant speed, tracing an angle, φ, with φ measured in radians.
- ω is the rate of change of the angle, φ, with time:

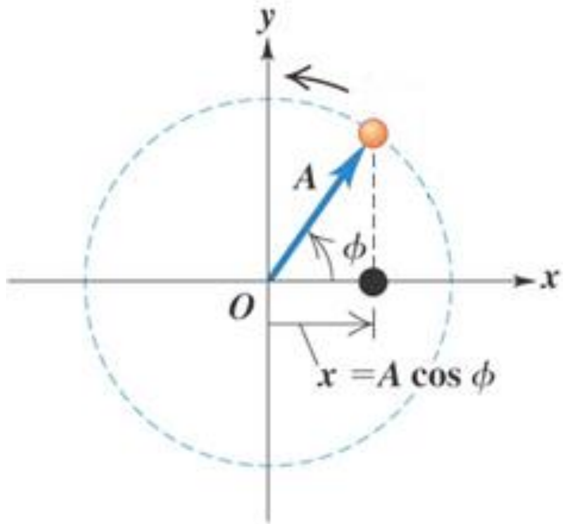
$$\omega = \frac{\Delta\varphi}{\Delta t} = 2\pi f$$

Features of Simple Harmonic Motion



- In the above picture, the **black ball** (and the function used to describe the position of this **black ball** on the x-axis) represents the position of the orange ball relative to the x-axis.
- x is the position of the **black ball** relative to the origin; A is the maximum displacement (or amplitude) of the ball relative to the origin; and φ is the angle that the ball makes relative to the origin.

Features of Simple Harmonic Motion



- Because ω is constant, φ increases uniformly with time; that is:

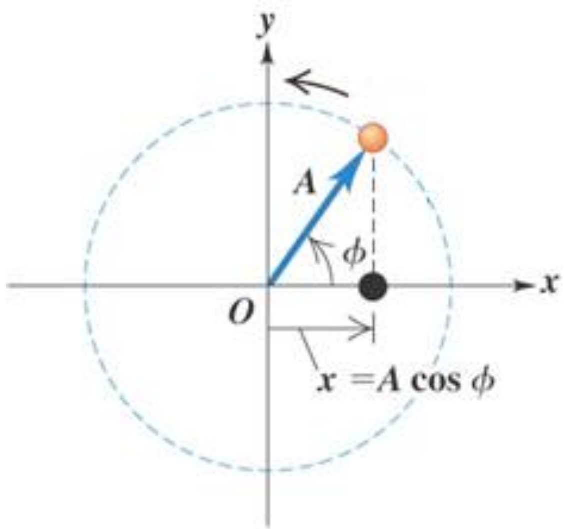
$$\varphi = \omega t$$

- Since $x = Acos\varphi$, then $x = Acos\omega t$
- We have defined $\omega = 2\pi f$, so we can also rewrite x as;

$$x = Acos2\pi ft$$

- This is the position of the black ball at time t.

Features of Simple Harmonic Motion



- From the position function for the ball, we can derive the equations for both its velocity and acceleration, using calculus:

$$x = Acos\omega t$$

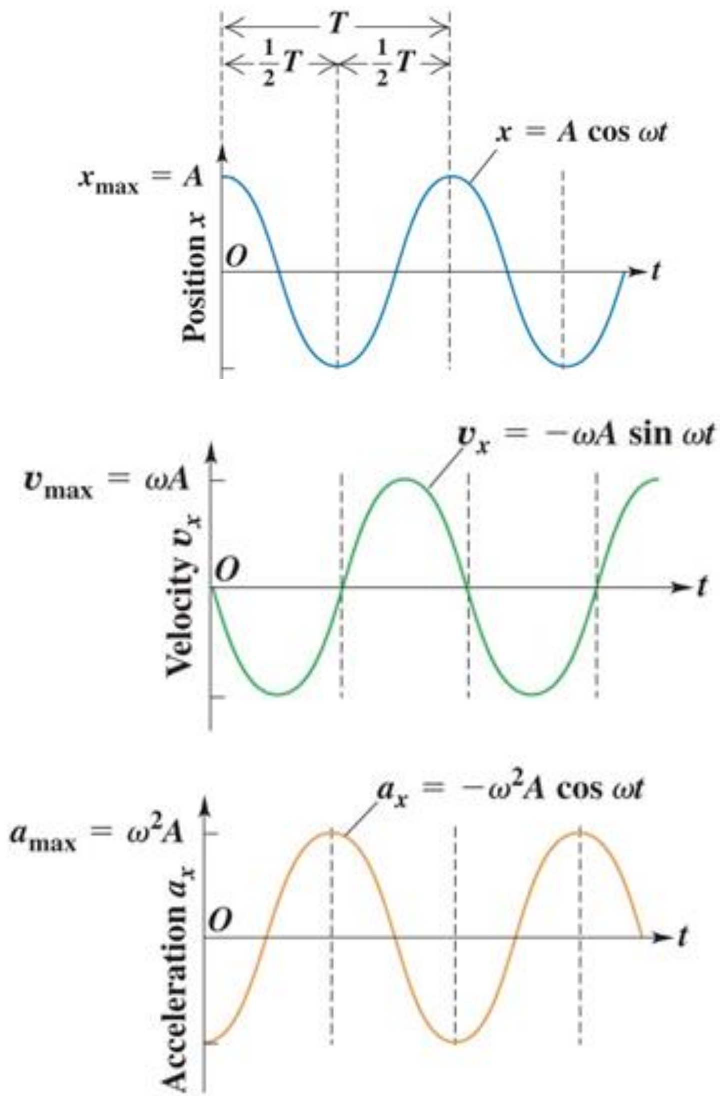
$$v = -\omega Asin\omega t$$

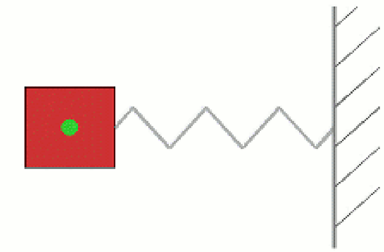
$$a = -(\omega)^2 Acos\omega t$$

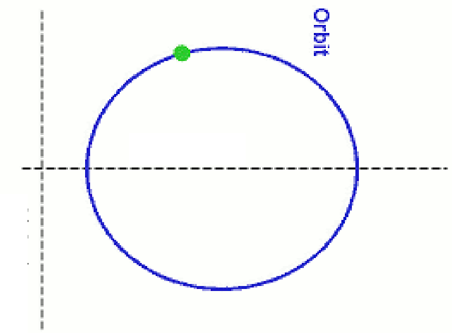
- By substituting in for x, we can also write acceleration as

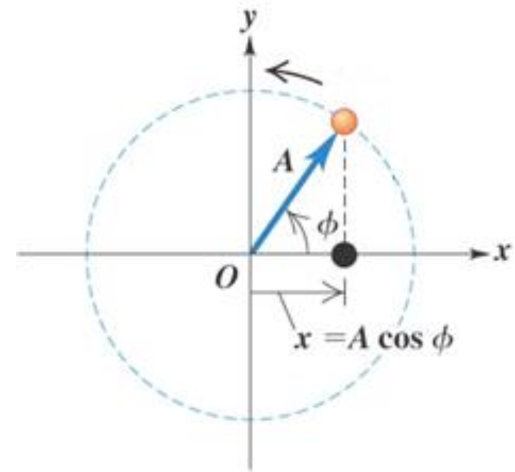
$$a = -(\omega)^2 x$$

Features of Simple Harmonic Motion

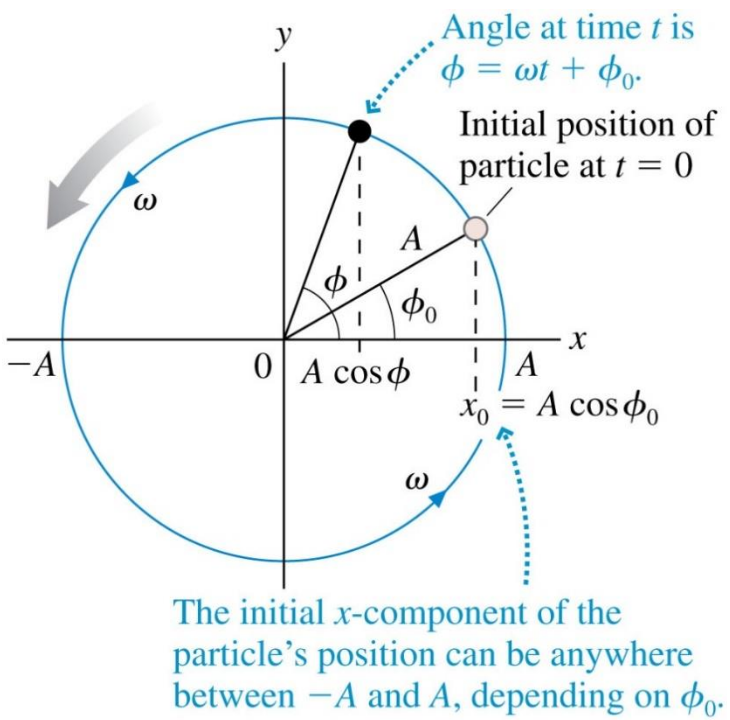








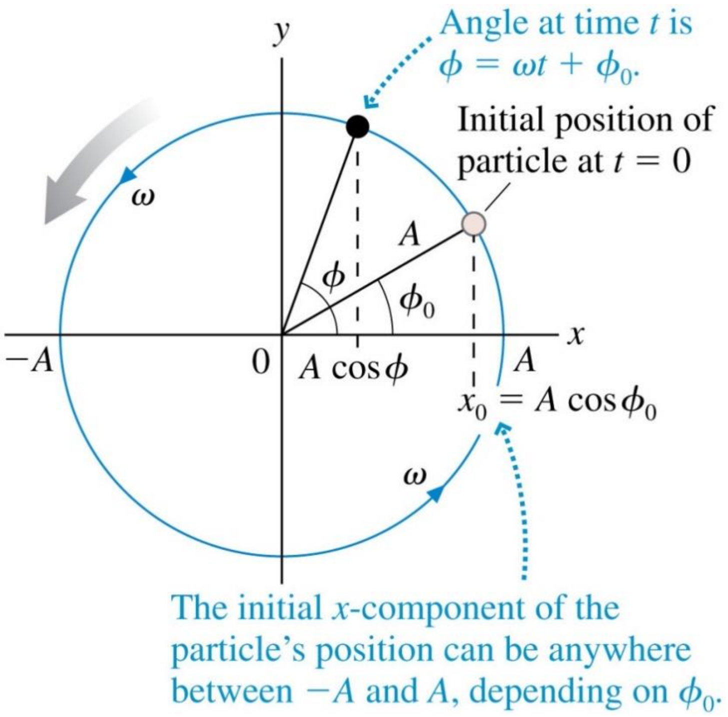
The Phase Constant



Q. What if an object in SHM is not initially at rest (equilibrium) at $x = A$ when $t = 0$?

A. Well, we can still use the cosine function, but we must add an extra term to it, called a **phase constant** $(\boldsymbol{\phi_0})$**,** measured in radians.

The Phase Constant

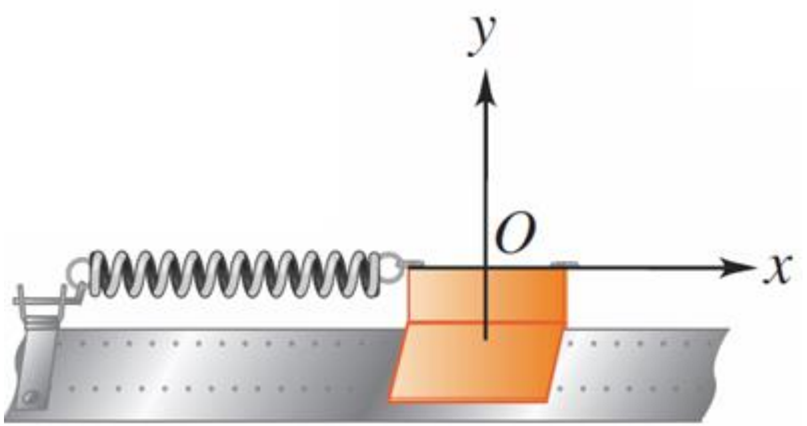


- In this case, the position function and velocity function are as follows:

$$x(t) = A\cos(\omega t + \phi_0)$$

$$v_x(t) = -\omega A \sin(\omega t + \phi_0) = -v_{max}\sin(\omega t + \phi_0)$$

Energy in Simple Harmonic Motion

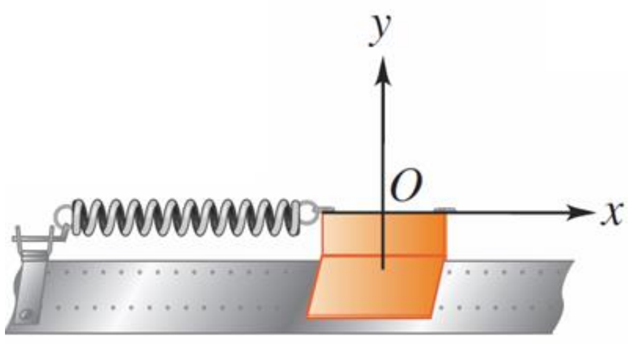


- Since we are ignoring friction, the **mechanical energy of the system is conserved**. Hence

$$E = \frac{1}{2}mv^2 + \frac{kx^2}{2} = constant$$

- This mechanical energy is linked to the displacement of the object's mass at a given time.
- When $x = A \; or \; x = -A$, the mass comes to rest, and all of the energy is in the form of **elastic potential energy**.

Energy in Simple Harmonic Motion



- When $x = A \; or \; x = -A$

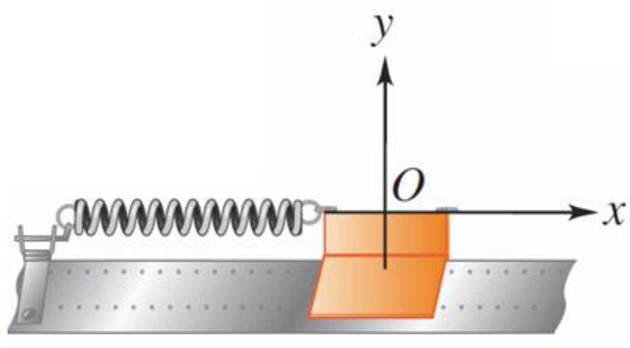
$$E = \frac{kA^2}{2} = constant$$

- Since **total energy is constant,** this must also equal the previous equation;

$$\frac{kA^2}{2} = \frac{1}{2}mv^2 + \frac{kx^2}{2}$$

- We can rearrange this to get an expression for the speed of the mass of the object at any given displacement.

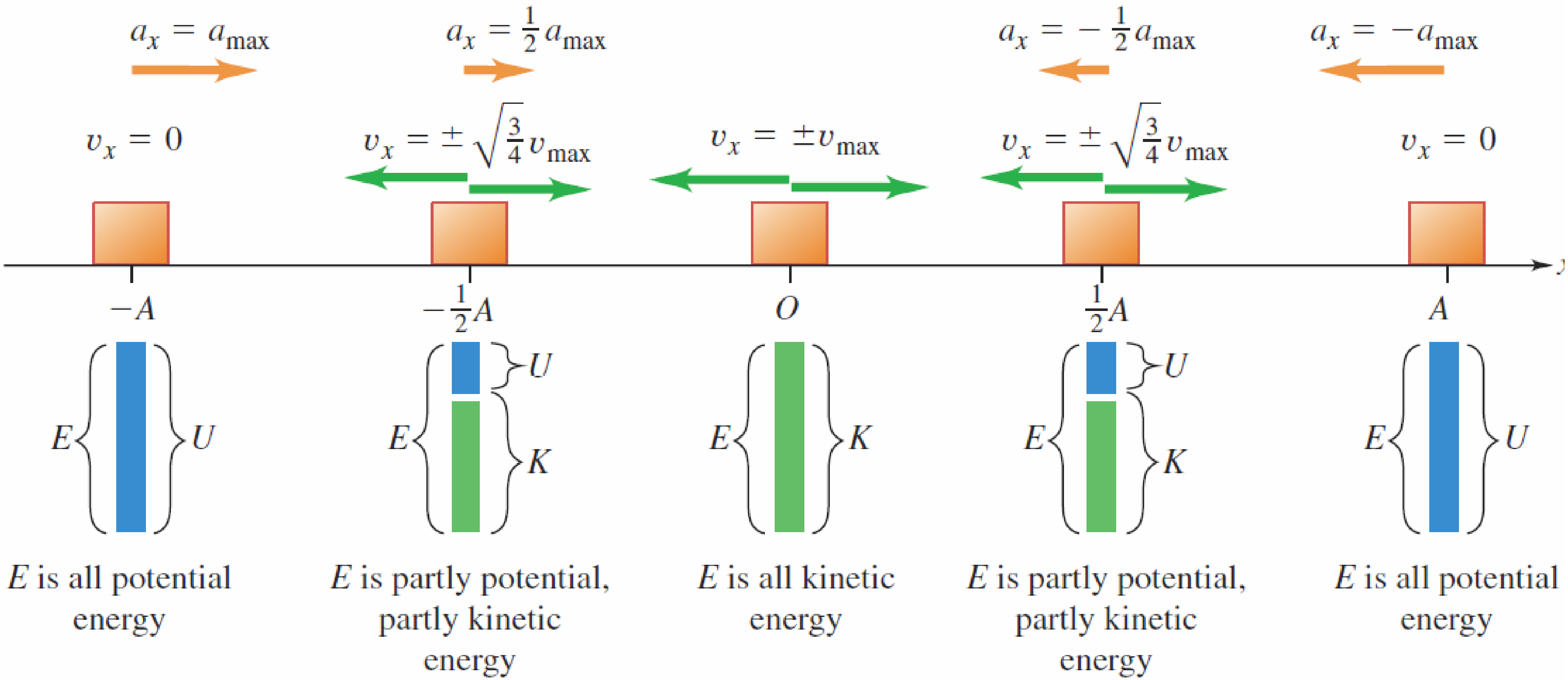
Energy in Simple Harmonic Motion



$$v_x = \pm\sqrt{\frac{k}{m}}A$$

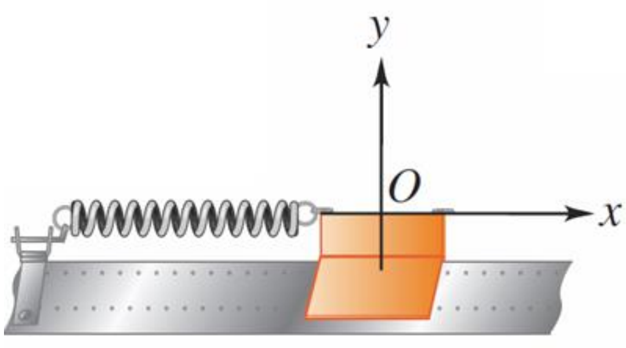
The **maximum speed** of the object occurs at the **equilibrium position**.

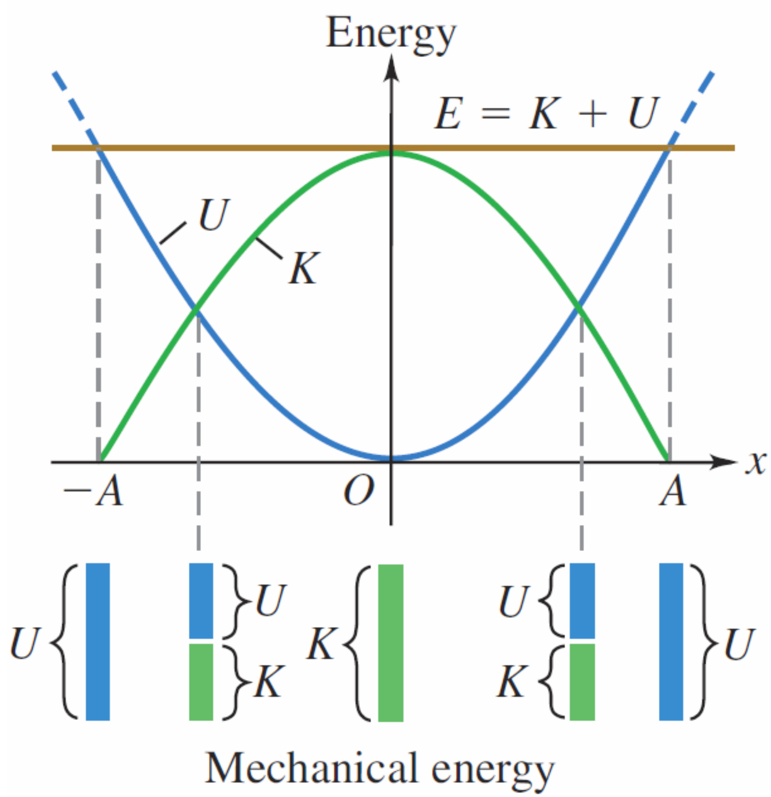
Energy in Simple Harmonic Motion



- As the object travels from A to $-A$, the energy is constantly changing from kinetic energy to elastic potential energy, and *vice versa*.

Energy in Simple Harmonic Motion





- The total energy, is always constant.
- This is true for all forms of SHM (where friction is ignored).

5. The Simple Pendulum

The Simple Pendulum

- The simple pendulum is a **mathematical model** based on a few simplifications/assumptions.

  - The pendulum is a **point mass.**
  - The pendulum is suspended from a **weightless string.**
  - The string is **inextensible.**
  - The pendulum is in a **uniform gravitational field.**

- With these in place, we can study the simple pendulum, and determine if its motion is simple harmonic.

The Simple Pendulum

- The path of the simple pendulum is an arc of a circle.

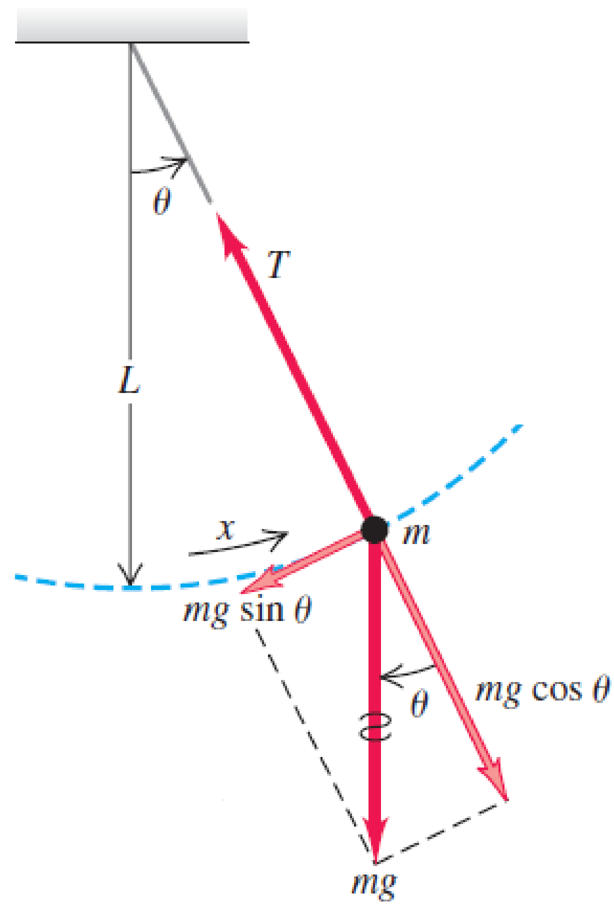
- So the displacement x measured along the arc is given by

$$x = L\theta$$

- For small angles of θ, the acceleration, a is proportional to θ, and hence x.

- If a pendulum performs SHM, then

$$a \propto -\theta$$



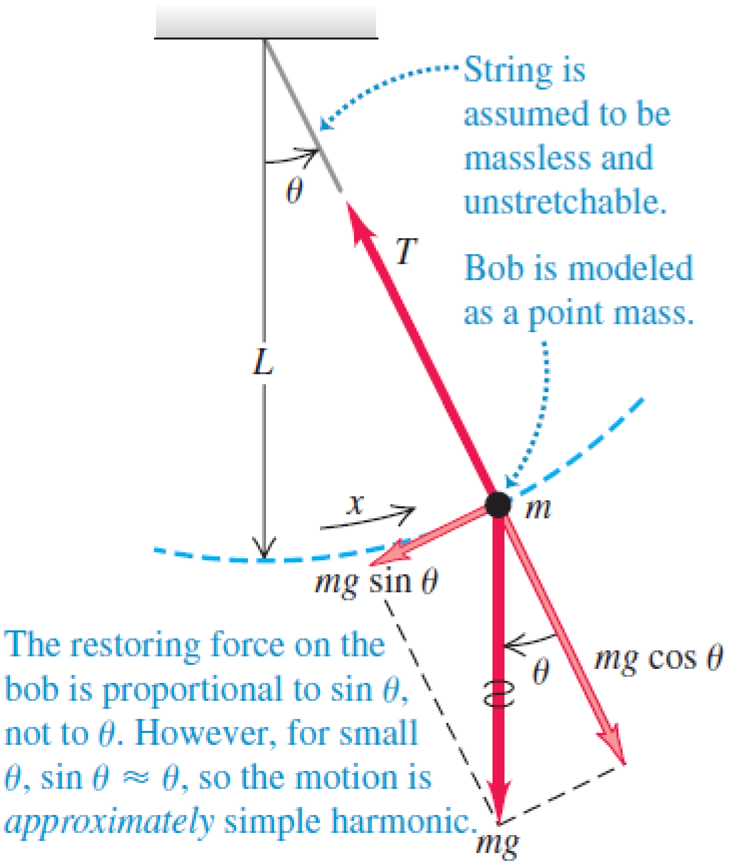
The Simple Pendulum

- Since $\sin\theta \sim \theta$ for small values

$$F = -mgsin\theta$$

$$ma = -mg\theta$$

$$a = -g\frac{x}{L}$$



The Simple Pendulum

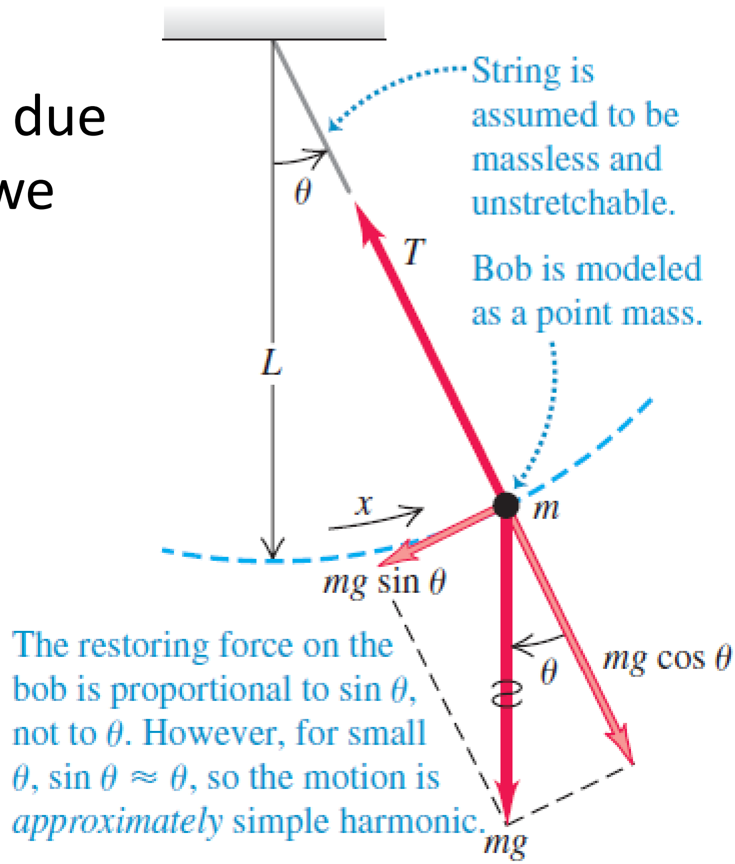
- We can then calculate the period (T) by comparing the equation for acceleration due to simple harmonic motion to the form we derived on the other slide:

$$a = -\frac{g}{L}x$$

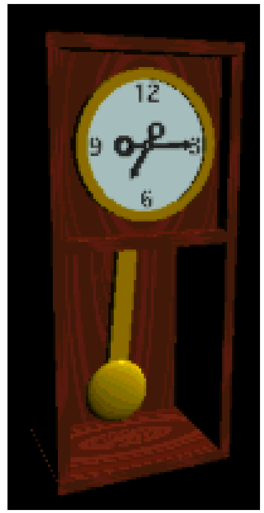
$$a = -(\omega)^2 x$$

- Therefore $\omega = \sqrt{\frac{g}{L}}$

$$\frac{2\pi}{T} = \sqrt{\frac{g}{L}} \quad \Rightarrow \quad \mathrm{T} = 2\pi\sqrt{\frac{L}{g}}$$



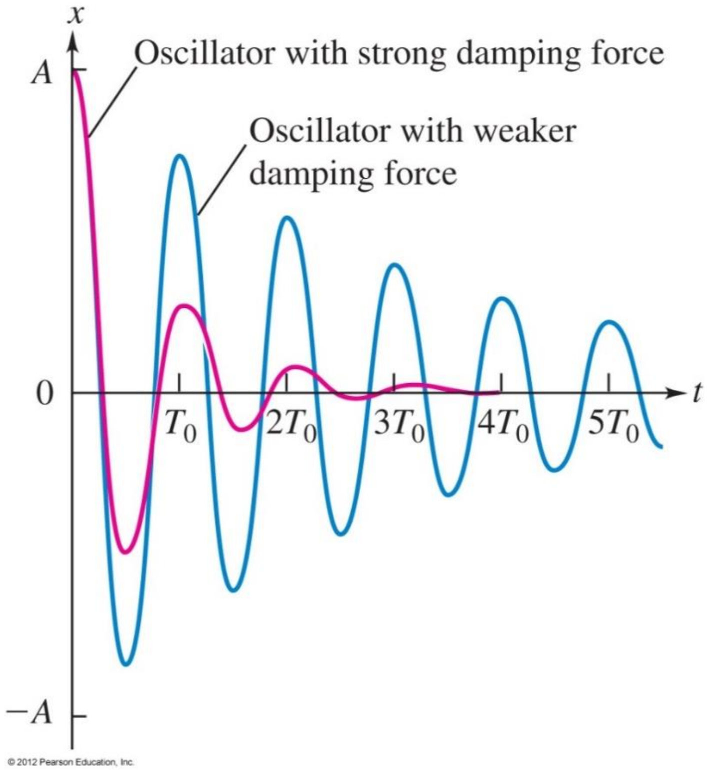
Simple Harmonic Motion thus Far



- The idealised oscillating systems (simple harmonic motion systems) that we've discussed so far are assumed to be frictionless, with no **non-conservative** forces.
- In other words, we've assumed that the total mechanical energy is constant, and such a system set into motion continues oscillating forever with no decrease in amplitude.
- The real world is different, however. Most oscillating systems undergo what we call **damping.**

Damping

(a) A real pendulum



- Real-world systems always have some friction (a non-conservative, or dissipative force, in other words), and oscillations do die-out with time unless we provide some means for replacing the mechanical energy lost to friction.
- In a situation such as this, we call the oscillations **damped oscillations**.

Summary of today's Lecture

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1. Circular Motion
2. Centripetal Force
3. Periodic/Oscillatory Motion.
4. Simple Harmonic Motion.
5. Representing Features of Simple Harmonic Motion
6. Energy in Simple Harmonic Motion.
7. The Simple Pendulum.

Lecture 8: Important Reading

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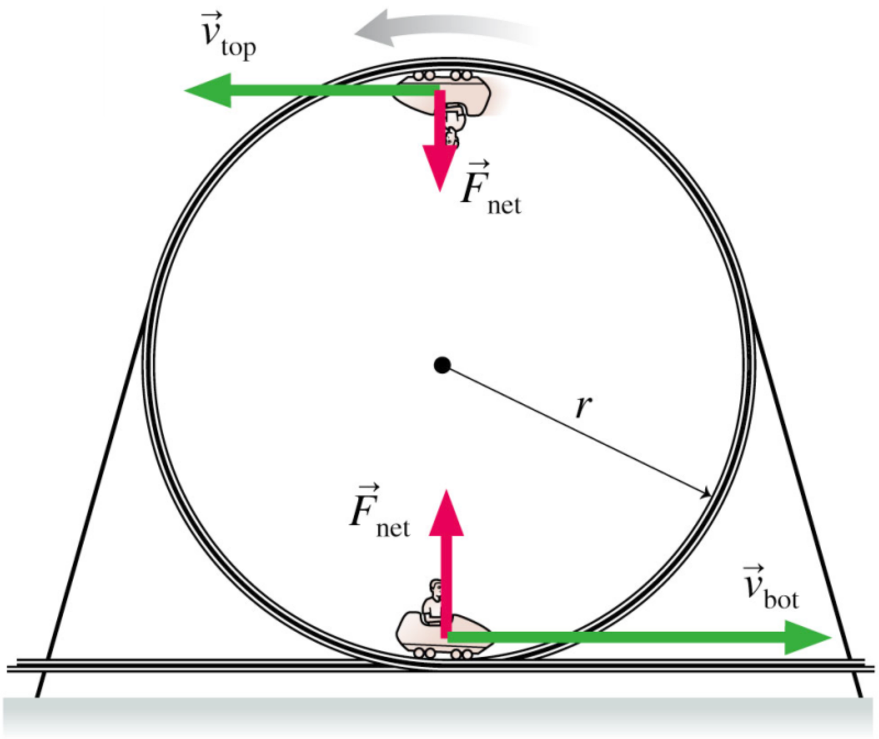
- **Ch. 5.2**, Uniform Circular Motion; p.141-144
- **Ch. 5.3**, Dynamics of Uniform Circular Motion; p.144-148
- **Ch. 14.1**, Oscillations of a Spring; p.428-429
- **Ch. 14.2**, Simple Harmonic Motion (SHM); p.430-435
- **Ch. 14.3**, Energy in the Simple Harmonic Oscillator; p.435-436
- **Ch. 14.4**, SHM Related to Uniform Circular Motion; p.437

Home Work

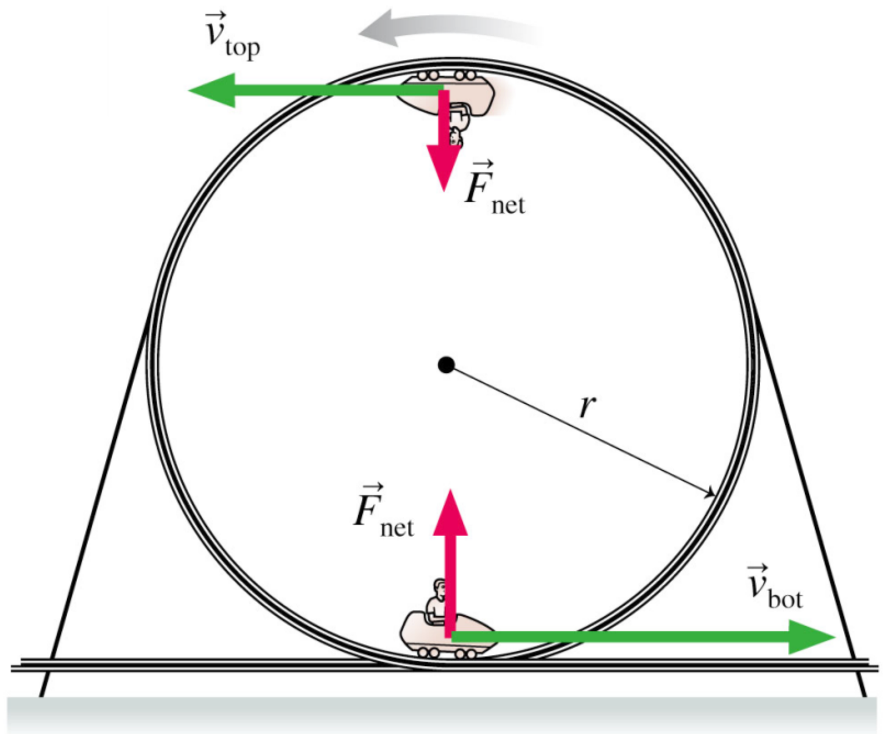
Do not forget to attempt the **Additional Problems** for this lecture before logging in to **Mastering Physics** to complete your assignments.

A Loop-the-Loop



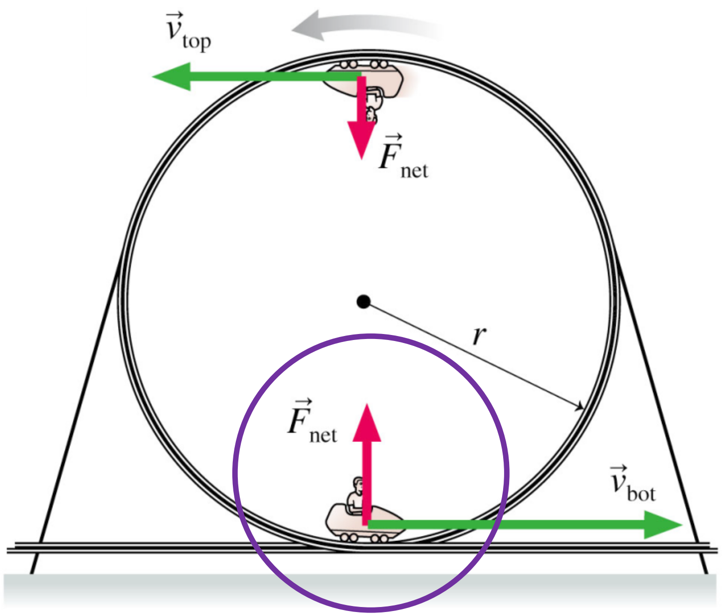
- The figure shows a roller-coaster going around a vertical loop-the-loop of radius r.
- Note that this is not uniform circular motion: the car slows down going up one side, and speeds up going down the other.

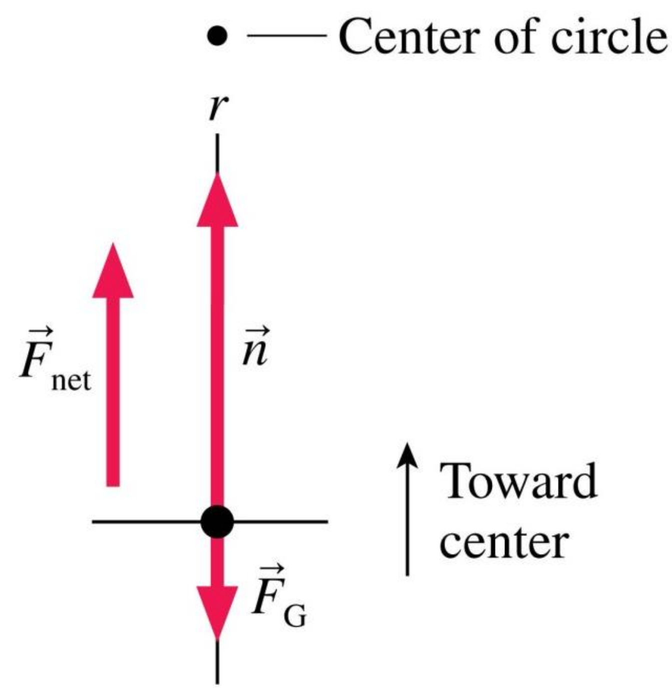
A Loop-the-Loop



- However, at the very top and very bottom points, only the car's direction is changing, so the acceleration is purely centripetal.
- Thus, because the car is moving in a circle at these points, there must be a net force toward the centre of the circle.

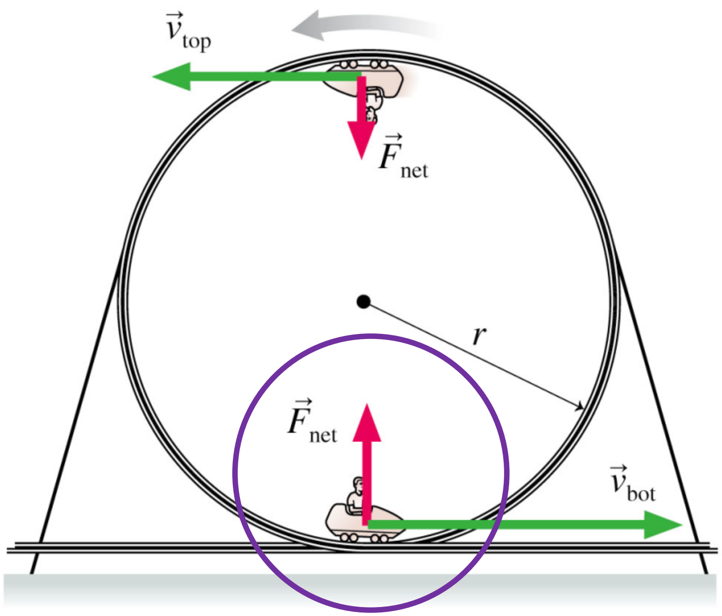
A Loop-the-Loop

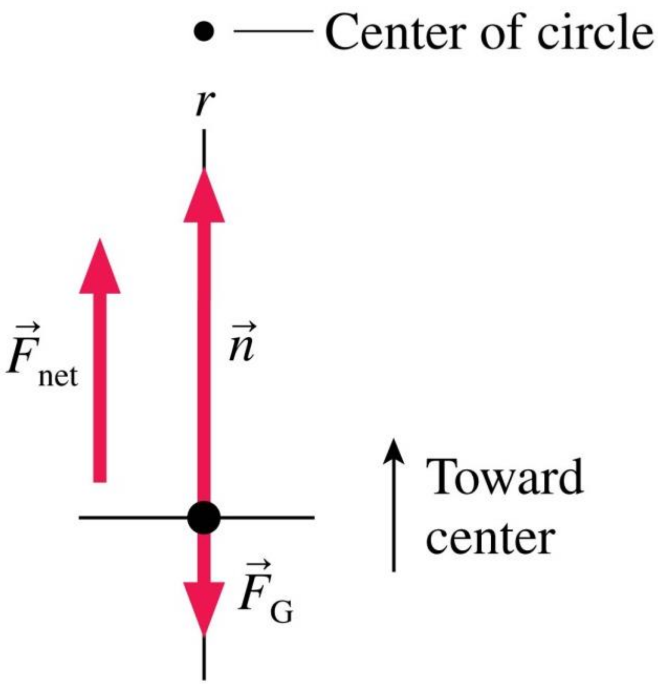




- The figure shows the roller-coaster free-body diagram at the bottom of the loop.
- Since the net force is toward the centre (upward at this point), $n > F_G$.

A Loop-the-Loop



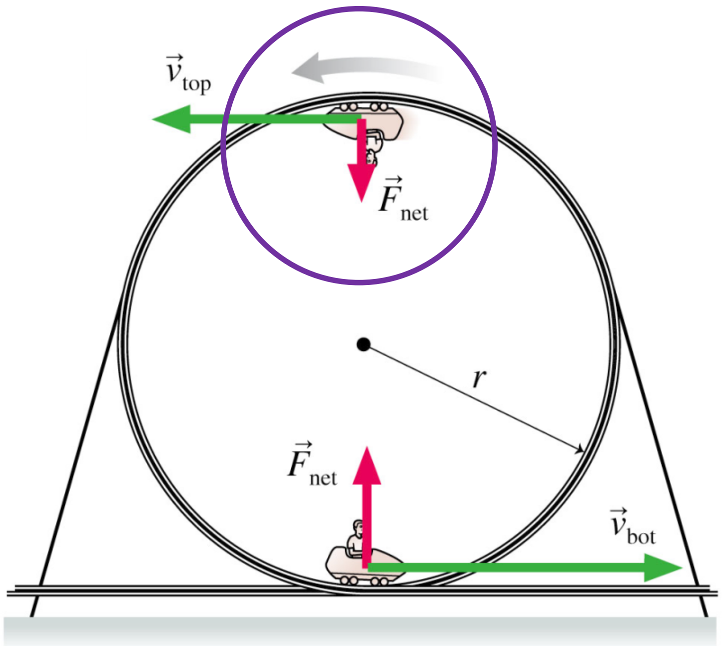


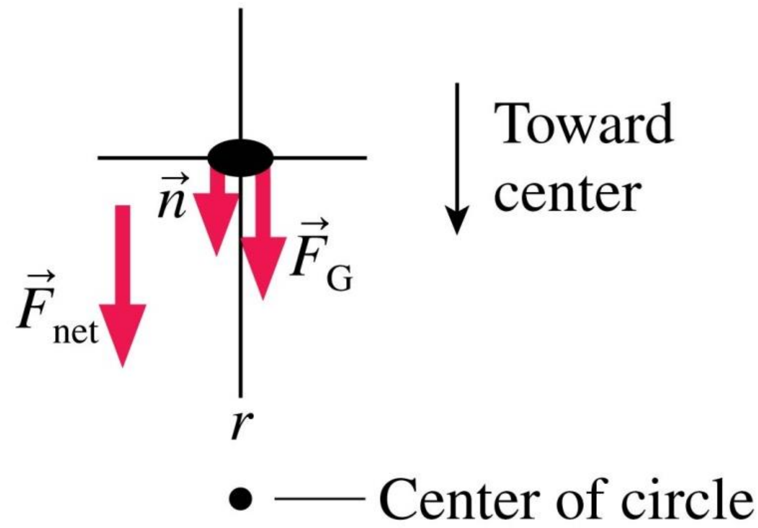
- This is why you 'feel slightly more heavy' at the bottom of the valley on a roller coaster.

$$\sum F_r = n_r + (F_G)_r = n - mg = ma_r = \frac{m(v_{bot})^2}{r}$$

$$\Rightarrow \; n = mg + \frac{m(v_{bot})^2}{r}$$

A Loop-the-Loop

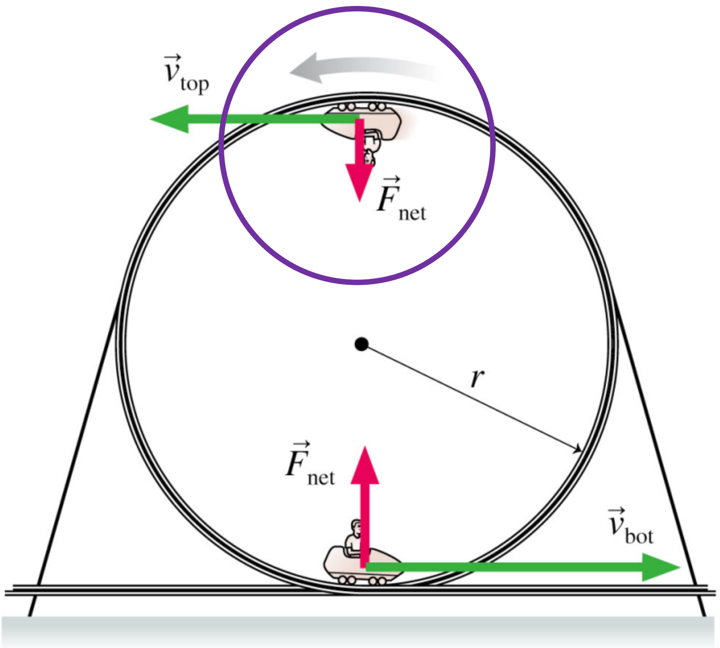


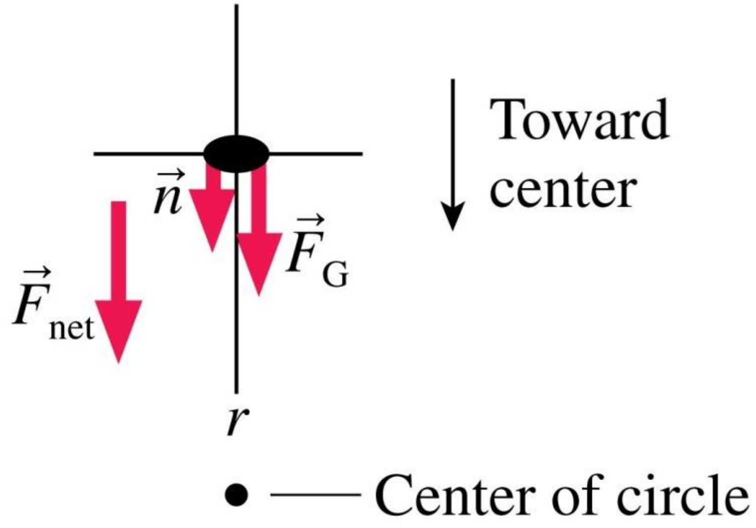


- The figure shows the roller-coaster free-body diagram at the top of the loop.
- Now, the normal force acts downward.
- The car is still moving in a circle, so the net force is also downward:

$$\sum F_r = n_r + (F_G)_r = n + mg = \frac{m(v_{top})^2}{r}$$

A Loop-the-Loop



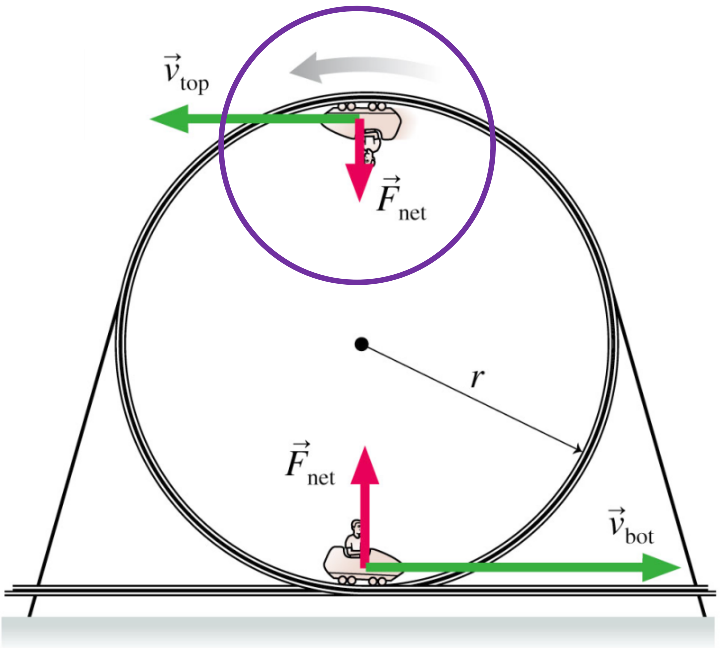


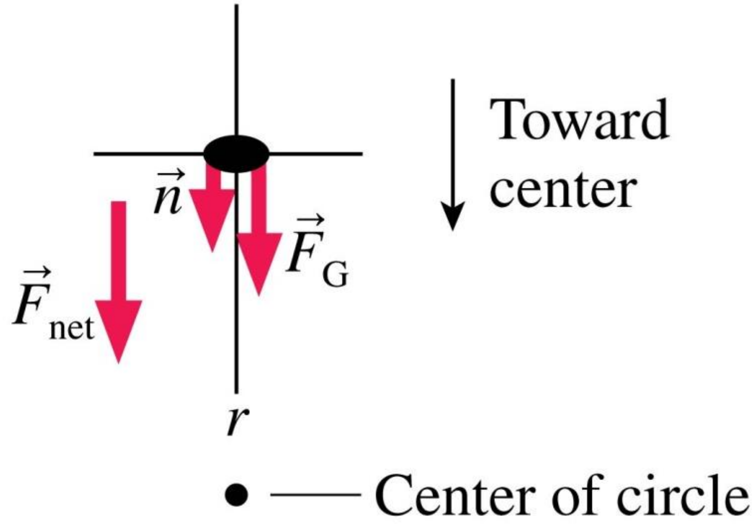
- This is why you 'feel slightly lighter' at the top of a loop-the-loop.

$$\sum F_r = n_r + (F_G)_r = n + mg = \frac{m(v_{top})^2}{r}$$

$$\Rightarrow \; n = \frac{m(v_{top})^2}{r} - mg$$

A Loop-the-Loop





$$\Rightarrow \; n = \frac{m(v_{top})^2}{r} - mg$$

- As v_{top} decreases, there comes a point when n reaches zero.
- The speed at which $n = 0$ is the minimum speed you must travel at to avoid falling off:

$$v_{minimum} = \sqrt{\frac{rmg}{m}} = \sqrt{rg}$$