2

1 (a) $y = e^{-x}$

$\therefore \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{e^{-(x+h)} - e^{-x}}{h}$ — 1.

$= \lim_{h \to 0} e^{-x}\left[\frac{e^{-h} - 1}{h}\right]$

$= -e^{-x} \lim_{h \to 0} \left(\frac{e^{-h} - 1}{-h}\right)$

$= -e^{-x} \cdot (1)$

$= -e^{-x}.$ — 1 Mark (if above steps are ok)

(b) (i) $y = \sin\left(\cos\left(e^{2x+3}\right)\right)$

~~1 Mark~~

Let $v = e^{2x+3} \Rightarrow \frac{dv}{dx} = 2e^{2x+3}$

$u = \cos v \Rightarrow \frac{du}{dv} = -\sin v$

1 Mark

$\therefore y = \sin(\cos(v)) = \sin u \Rightarrow \frac{dy}{du} = \cos u.$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

$= \cos u \cdot (-\sin v) \cdot 2e^{2x+3}$

$= -2e^{2x+3} \cos\left(\cos\left(e^{2x+3}\right)\right) \cdot \sin\left(e^{2x+3}\right)$

2 Mark if stated method is NOT used & answer is correct

1

Only final answer 1 mark (ie no steps)

ii) $y = c_1 e^{mx} + c_2 e^{-mx}$

$\Rightarrow \dfrac{dy}{dx} = m \cdot c_1 e^{mx} - c_2 m e^{-mx}$

$\Rightarrow \dfrac{d^2y}{dx^2} = c_1 \cdot m^2 e^{mx} + c_2 \cdot m^2 e^{-mx}$ — 1 Mark

$= m^2 \left(c_1 e^{mx} + c_2 e^{-mx} \right)$

$= m^2 y$

$\therefore \dfrac{d^2y}{dx^2} - m^2 y = 0.$ — 1 Mark

c) $x = y\sqrt{1-y^2}$ — 1 Mark

$\therefore \dfrac{dx}{dy} = y \cdot \dfrac{1}{\cancel{2}\sqrt{1-y^2}} (-\cancel{2}y) + \sqrt{1-y^2} \cdot (1)$

$\Rightarrow \dfrac{dx}{dy} = \dfrac{-y^2 + 1 - y^2}{\sqrt{1-y^2}} = \dfrac{1-2y^2}{\sqrt{1-y^2}}.$ 1 Mark Simplification

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\left(\dfrac{dx}{dy}\right)} = \dfrac{\sqrt{1-y^2}}{1-2y^2}.$ — 1 Mark

Deduct 1 mark if Correct method not applied.

2 (a) $x = a\cos\theta \Rightarrow \dfrac{dx}{d\theta} = -a\sin\theta$

$y = b\sin\theta \Rightarrow \dfrac{dy}{d\theta} = b\cos\theta$

i) $\therefore \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{b\cos\theta}{-a\sin\theta} = -\dfrac{b}{a}\cot\theta$ (1) Mark

ii) $\therefore$ Gradient at $\theta = \frac{\pi}{4} = \left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = -\dfrac{b}{a}.(1) = -\dfrac{b}{a}.$

$= m.$

1 Mark

iii) Eqn. of tgt line is

$y - y_1 = m(x - x_1)$

Now $y_1 = y\big|_{\text{at } \theta=\frac{\pi}{4}} = b\sin\frac{\pi}{4} = \dfrac{b}{\sqrt{2}}$

1 Mark

& $x_1 = x\big|_{\text{at } \theta=\frac{\pi}{4}} = a\cos\frac{\pi}{4} = \dfrac{a}{\sqrt{2}}$

$\therefore$ Tgt line is $y - \dfrac{b}{\sqrt{2}} = -\dfrac{b}{a}\left(x - \dfrac{a}{\sqrt{2}}\right)$

$\Rightarrow ay - \dfrac{ab}{\sqrt{2}} = -bx + \dfrac{ab}{\sqrt{2}}$

$\Rightarrow bx + ay = \sqrt{2}\,ab$

1 Mark

OR $\dfrac{x}{a} + \dfrac{y}{b} = \sqrt{2}.$

iv) Normal line is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$\Rightarrow y - \frac{b}{\sqrt{2}} = \frac{a}{b}\left(x - \frac{a}{\sqrt{2}}\right)$$

$$\Rightarrow by - \frac{b^2}{\sqrt{2}} = ax - \frac{a^2}{\sqrt{2}}$$

$$\Rightarrow ax - by = \frac{a^2 - b^2}{\sqrt{2}}.$$

1 Mark for answer or A.E.F.

b) $y = (\ln x)^x$

$$\Rightarrow \ln y = x \ln(\ln x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{\ln x} \cdot \frac{1}{x} + \ln(\ln x)$$ 1 Mark

1 Mark

$$\Rightarrow \frac{dy}{dx} = (\ln x)^x \left[\frac{1}{\ln x} + \ln(\ln x)\right]$$

c) $x^3 + x^2y + xy^2 + y^3 = 4xy$

$$\Rightarrow 3x^2 + x^2\frac{dy}{dx} + y \cdot 2x + x \cdot 2y\frac{dy}{dx} + y^2 + 3y^2\frac{dy}{dx}$$

1 Mark

$$= 4x \cdot \frac{dy}{dx} + 4y.$$ 1 Mark.

$$\Rightarrow \frac{dy}{dx}\left(x^2 + 2xy + 3y^2 - 4x\right) = 4y - 3x^2 - 2xy - y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{4y - 3x^2 - 2xy - y^2}{x^2 + 2xy + 3y^2 - 4x}.$$ 1 Mark

3 a i) $f(x) = x^3 + x^2 - 8x - 15$

$\therefore \underline{f'(x) = 3x^2 + 2x - 8}$ 1 Mark

For st. points, $f'(x) = 0 \Rightarrow 3x^2 + 6x - 4x - 8 = 0$

$\Rightarrow (3x-4)(x+2) = 0$

$\Rightarrow x = -2$ or $\frac{4}{3}$.

OR $(-2,-3)$ and $(\frac{4}{3}, -21.5)$ 1 Mark

ii) $f''(x) = 6x + 2$

$\therefore f''(x)\Big|_{x=-2} < 0 \Rightarrow$ OR $(-2,-3)$ $x = -2$ is a point of max. value

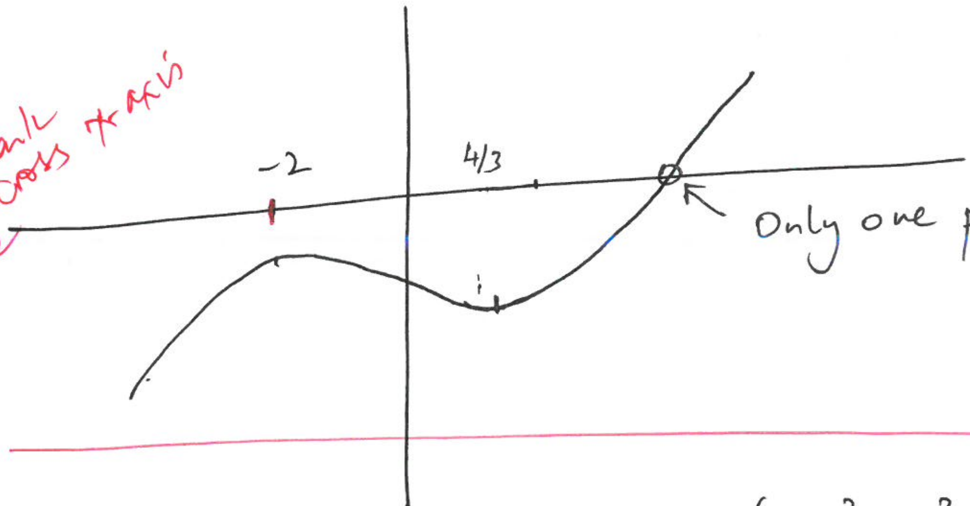
& $f(-2) = -3$. 1 Mark

& $f''(x)\Big|_{x=\frac{4}{3}} = 6\left(\frac{4}{3}\right) + 2 > 0 \Rightarrow$ OR $(\frac{4}{3}, -21.5)$ $x = \frac{4}{3}$ is a point of min value.

$f\left(\frac{4}{3}\right) \approx -21.5$ 1 Mark

iii)

1 Mark Curve must cross the axis exactly once



Only one positive root.

iv) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \left(\dfrac{x_n^3 + x_n^2 - 8x_n - 15}{3x_n^2 + 2x_n - 8}\right)$

Steps are imp.

$$= \frac{3x_n^3+2x_n^2-8x_n-x_n^3+x_n^2+8x_n+15}{3x_n^2+2x_n-8}$$

$$\therefore x_{n+1} = \frac{2x_n^3+x_n^2+15}{3x_n^2+2x_n-8}.$$

1 mark

1 mark any no. $> \frac{4}{3}$

Let $x_0 = 1.5$ because $x = \frac{4}{3}$ is a point of min value so choosing $x > \frac{4}{3}$.

n	x_n
0	13.7142857
1	9.1864585
2	6.2603886
3	4.4629054
4	3.5053270
5	3.1620075
6	3.1155048
7	3.1146794
8	3.1146791
9	3.1146791

Must be 7 digits after decimal

$\therefore$ Positive root correct to 6 dp. is

$x^* = 3.114679.$

1 mark for 6 dp

b) Given $\frac{dh}{dt} = 3$ cm/sec ; to find $\left.\frac{dv}{dt}\right|_{r=5}$.

$$V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dv}{dt} = \frac{1}{3}\pi . r^2 \frac{dh}{dt}$$

$$\therefore \left.\frac{dv}{dt}\right|_{r=5} = \frac{1}{3}\pi (5)^2 . (3) = 25\pi \text{ cm}^3/\text{sec}.$$

1 mark

4 a) i) $\displaystyle\int \frac{1+x+x^2}{x^2}\,dx$

$$= \int \left(\frac{1}{x^2} + \frac{1}{x} + 1\right) dx$$

$$= -\frac{1}{x} + \ln|x| + x + C$$ ① Mark

ii) $\displaystyle\int \left(e^{2x} + \sin x \cdot \sec^2 x\right) dx$

$$= \int e^{2x}\,dx + \int \sec x \cdot \tan x\,dx$$

$$= \frac{e^{2x}}{2} + \sec x + C$$ ① Mark

iii) $\displaystyle\int \frac{1}{x\left[1+(\ln x)^2\right]}\,dx$

Let $\ln x = t \Rightarrow \frac{1}{x}\,dx = dt$ 1 Mark

$$\therefore I = \int \frac{dt}{1+t^2}$$

$$= \tan^{-1}(t) + C$$

$$= \tan^{-1}(\ln x) + C$$ 1 Mark

(b) (i) $\int \sin 7x \cos 3x \, dx$

$$= \frac{1}{2} \int \left[\sin(7x+3x) + \sin(7x-3x) \right] dx$$

$$= \frac{1}{2} \int (\sin 10x + \sin 4x) \, dx$$ 1 Mark

$$= \frac{1}{2}\left(\frac{-\cos 10x}{10}\right) + \frac{1}{2}\left(\frac{-\cos 4x}{4}\right) + C$$

$$= -\frac{1}{20} \cos 10x - \frac{1}{\cancel{4}\,8} \cos 4x + C.$$ 1 Mark

ii) $\int \sin^7 x \cos^3 x \, dx$

$$= \int \sin^7 x \cdot \cos^2 x \cdot \cos x \, dx$$

Let ~~cos~~ $\sin x = t \Rightarrow \cos x \, dx = dt$ 1 Mark

& $\cos^2 x = 1 - \sin^2 x = 1 - t^2$

$$\therefore I = \int t^7 (1-t^2) \, dt$$

$$= \frac{t^8}{8} - \frac{t^{10}}{10} + C$$

$$= \frac{1}{8} (\sin x)^8 - \frac{1}{10} (\sin x)^{10} + C$$ 1 Mark

c) $$I = \int \frac{\overset{f'(x)}{2x+3}}{\sqrt{\underset{f(x)}{1+3x+x^2}}}\,dx$$

Let $1+3x+x^2 = t^2$

$\Rightarrow (3+2x)\,dx = 2t\,dt$

1 mark for substitution or showing use of formula

$$\therefore I = \int \frac{2\cancel{t}\,dt}{\cancel{\sqrt{t^2}}}$$

$= 2t + C$

$= 2\sqrt{1+3x+x^2} + C$ 1 mark

5 (a)

$$\frac{8}{(x-3)(3x-1)} = \frac{A}{x-3} + \frac{B}{3x-1}$$

$$\Rightarrow \quad A(3x-1) + B(x-3) = 8$$

Put $x=3 \Rightarrow A(8)=8 \Rightarrow \boxed{A=1}$ 1 Mark

& $x=\frac{1}{3} \Rightarrow A(0)+B\left(\frac{1}{3}-3\right)=8 \Rightarrow B\left(-\frac{8}{3}\right)=8$

$\Rightarrow \boxed{B=-3}$ 1 Mark

$$\therefore \quad f(x) = \frac{8}{(x-3)(3x-1)} = \frac{1}{x-3} - \frac{3}{3x-1}$$

$$\therefore \int f(x)\,dx = \int \frac{1}{(x-3)}\,dx - 3\int \frac{1}{3x-1}\,dx$$

$$= \ln|x-3| - \cancel{3}\frac{\ln|3x-1|}{\cancel{3}} + C$$

1 Mark

$$\therefore \int_1^2 f(x)\,dx = \left[\ln\left|\frac{x-3}{3x-1}\right|\right]_1^2$$

$$= \ln\left|\frac{-1}{5}\right| - \ln\left|\frac{-2}{2}\right|$$

$$= \ln\left|\frac{1}{5}\right| - \ln|1|$$

$$= -\ln 5.$$

1 Mark

b i) $I = \int e^x \sin x \, dx$

Let $u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow du = \cos x \, dx$

$dv = e^x dx \Rightarrow v = e^x$

1 Mark

$\therefore I = uv - \int v \, du = e^x \sin x - \int \underset{dv}{e^x} \underset{u}{\cos x} \, dx$

1 Mark

Let $u = \cos x \Rightarrow du = -\sin x \, dx$

$dv = e^x dx \Rightarrow v = e^x$

$\therefore I = e^x \sin x - \left[e^x \cos x - \int e^x (-\sin x) \, dx \right]$

$I = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

$\therefore I = e^x (\sin x - \cos x) - I$

$\therefore 2I = e^x (\sin x - \cos x)$

$\therefore I = \int e^x \sin x \, dx = \frac{e^x}{2} (\sin x - \cos x) + C.$

1 Mark

b) ii) $$\text{Area} = \int_0^{\pi/2} e^x \sin x \, dx$$

$$= \left[\frac{e^x}{2} (\sin x - \cos x) \right]_0^{\pi/2}$$

$$= \frac{e^{\pi/2}}{2}(1-0) - \frac{e^0}{2}(1-0)$$

$$= \frac{1}{2}\left[e^{\pi/2} + 1 \right].$$ 1 mark

c) $$\text{Volume} = \pi \int_0^{\pi/4} (\tan x)^2 \, dx$$

$$= \pi \int_0^{\pi/4} \tan^2 x \, dx$$

$$= \pi \int_0^{\pi/4} (\sec^2 x - 1) \, dx$$ 1 mark

$$= \pi \left[\tan x - x \right]_0^{\pi/4}$$

$$= \pi \left[1 - \frac{\pi}{4} - 0 + 0 \right]$$

$$= \pi - \frac{\pi^2}{4}$$ 1 mark

12

6 (a) $$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $\cos x = t \Rightarrow -\sin x\,dx = dt$

$\Rightarrow \sin x\,dx = -dt$

1 mark

x	0	$\pi/2$
t	1	0

$$= \int_1^0 \frac{-dt}{1+t^2} = \int_0^1 \frac{1}{1+t^2}\,dt$$

$$= \left[\tan^{-1} t\right]_0^1$$

$$= \frac{\pi}{4}.$$

1 mark

b) i) $$\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$$

$$= \int \frac{1}{\sqrt{x^2+2x+1+4}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+1)^2+2^2}}\,dx$$

1 for completing the square

$$= \cancel{\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C}$$

$$= \ln\left|(x+1)+\sqrt{x^2+2x+5}\right| + C$$

1 Mark

13

b) ii)

let $\tan(x/2) = t$

$$\Rightarrow \quad dx = \frac{2\,dt}{1+t^2}$$

1 Mark

$$\sin x = \frac{2t}{1+t^2} \quad \& \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\therefore I = \int \frac{\frac{2\,dt}{1+t^2}}{2\left(\frac{1-t^2}{1+t^2}\right) + \left(\frac{2t}{1+t^2}\right) + 3}$$

$$= \int \frac{2\,dt}{2(1-t^2) + 2t + 3(1+t^2)}$$

$$= \int \frac{2\,dt}{2 - 2t^2 + 2t + 3 + 3t^2}$$

$$= \int \frac{2\,dt}{t^2 + 2t + 5}$$

1 Mark

$$= \cancel{2}\left[\frac{1}{\cancel{2}} \tan^{-1}\left(\frac{t+1}{2}\right)\right] + C$$

$$= \tan^{-1}\left(\frac{\tan(x/2) + 1}{2}\right) + C.$$

1 Mark

c) $a = \frac{\pi}{4}$ $b = \frac{\pi}{2}$ and $n = 4$

$\therefore h = \frac{b-a}{n} = \frac{\pi/2 - \pi/4}{4} = \frac{\pi}{16}$. 1 Mark

2 Marks

1 Mark

x	$\pi/4$	$5\pi/16$	$6\pi/16$	$7\pi/16$	$8\pi/16 = \pi/2$
$f(x)$	0.9003163	0.8469280	0.7842133	0.7135855	0.6366198
f_n	f_0	f_1	f_2	f_3	$f_4 = f_n$

Digits 6 or 7 no problem.

By Simpson's rule,

$$\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \approx \frac{\pi/16}{3}\left[f_0 + 4(f_1 + f_3) + 2 \cdot f_2 + f_4\right]$$

$$\approx \underline{0.611787} \quad (6\,dp).$$

1 Mark

must be 6 dp

7 (a)

i) $\operatorname{cosec} y \, dx + \cos^2 x \, dy = 0$

$\Rightarrow \operatorname{cosec} y \, dx = -\cos^2 x \, dy$ 1 Mark for S. of Vari.

$\Rightarrow \int \sec^2 x \, dx = \int -\sin y \, dy$

$\Rightarrow \tan x = \cos y + C$ 1 Mark for G. Sol^n.

Now, $y(\pi/4) = 0 \Rightarrow$ When $x = \frac{\pi}{4}$; $y = 0$

$\Rightarrow \tan\frac{\pi}{4} = \cos 0 + C \Rightarrow C = 0$.

$\therefore \tan x = \cos y$ is particular solution. 1 Mark.

1 Mark for S. of V. either (i) or (ii)

ii) $\frac{dy}{dx} = \frac{y \cdot \cos x}{1 + \sin x}$

$\Rightarrow \int \frac{1}{y} \, dy = \int \frac{\cos x}{1 + \sin x} \, dx$

$\Rightarrow \ln|y| = \ln|1 + \sin x| + \ln C$

$\Rightarrow y = C(1 + \sin x)$ 1 Mark

Now $y(0) = 1 \Rightarrow$ When $x = 0$, $y = 1$

$\therefore 1 = C(1 + 0) \Rightarrow C = 1$

$\therefore y = 1 + \sin x$ is particular sol^n. 1 Mark.

b) $y = a\cos^{-1}x + b$

$$\Rightarrow \frac{dy}{dx} = \frac{-a}{\sqrt{1-x^2}} + 0$$

$$\Rightarrow \frac{d^2y}{dx^2} = -a \cdot \frac{1}{\cancel{2}}(1-x^2)^{-3/2} \cdot (-\cancel{2}x)$$

$$= +ax(1-x^2)^{-3/2}$$

1 Mark.

$$LHS = (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}$$

$$= (1-x^2)\left[+ax(1-x^2)^{-3/2}\right] - x \cdot \left(\frac{a}{\sqrt{1-x^2}}\right)$$

$$= +ax \cdot (1-x^2)^{-1/2} - \frac{ax}{\sqrt{1-x^2}}$$

$$= \frac{+ax}{\sqrt{1-x^2}} - \frac{ax}{\sqrt{1-x^2}}$$

$= 0 = RHS$. (proved).

1 Mark

OR

$$y = a\cos^{-1}x + b \Rightarrow \frac{dy}{dx} = \frac{-a}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{dy}{dx} = a$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2$$

(17)

Diff $\Rightarrow (1-x^2) \cdot \cancel{2}\cancel{\frac{dy}{dx}} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{\cancel{2}} \cdot (-\cancel{2}x) = 0.$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0.$$

c) i)
$$\frac{dm}{dt} = km$$

$$\Rightarrow \int \frac{dm}{m} = \int k\,dt$$

$$\Rightarrow \ln m = kt + C \quad \text{——} (1)$$

1 Mark.

~~⇒~~ Now at $t=0$, $m=m_0$.

$$\Rightarrow \ln m_0 = C$$

$$\therefore (1) \Rightarrow \ln m = kt + \ln m_0$$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = kt$$

$$\Rightarrow \frac{m}{m_0} = e^{kt} \Rightarrow \underline{m = m_0 e^{kt}}$$

1 Mark

ii) When $t=8$ & $k=1.5$

$$m = m_0 \cdot e^{8(1.5)} = \underline{m_0 \cdot e^{12}}$$

or

$$\approx \underline{162754.79\ m_0}$$

1 Mark.

8 a) i)

$$f(x) = \ln(1+x) \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$$

$$f^{iv}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{iv}(0) = -6$$

2 Marks

2 Mark

ii) $\therefore$ Maclaurin's series expansion is

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$= 0 + x \cdot (+1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

1 Mark

iii)

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

1 Mark

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$- \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \right]$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

1 Mark

b) $f(x) = \dfrac{1}{1-x} \Rightarrow f(0) = 1$

$\Rightarrow f'(x) = \dfrac{+1}{(1-x)^2} \Rightarrow f'(0) = 1$

$f''(x) = \dfrac{2}{(1-x)^3} \Rightarrow f''(0) = 2$

1 Mark

$f'''(x) = \dfrac{6}{(1-x)^4} \Rightarrow f'''(0) = 6$

$f^{iv}(x) = \dfrac{24}{(1-x)^5} \Rightarrow f^{iv}(0) = 24$

1 Mark

$\therefore \dfrac{1}{1-x} = f(0) + x \cdot f'(0) + \dfrac{x^2}{2!} f''(0) + \dfrac{x^3}{3!} f'''(0) + \cdots$

$= 1 + x + \dfrac{x^2}{2} \cdot (2) + \dfrac{x^3}{6} (6) + \cdots$

$= 1 + x + x^2 + x^3 + \cdots$

1 Mark for proving