Lecture 3

Topics covered in this lecture session

1. Trigonometric functions.
2. More about Trigonometric functions.
3. Solving Trigonometric equations.
4. Formulae for addition, factor and multi-angle.

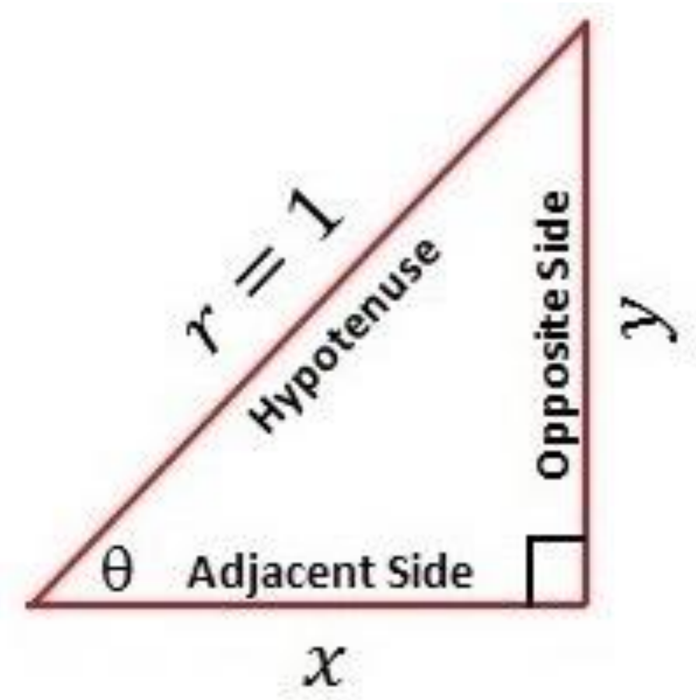
Trigonometric functions

$$\cos\theta = \frac{\text{Adjacent Side}}{\text{Hypotenuse}} = \frac{x}{r} = \frac{x}{1} = x$$

$$\sin\theta = \frac{\text{Opposite Side}}{\text{Hypotenuse}} = \frac{y}{r} = \frac{y}{1} = y$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad ; \quad \cos\theta \neq 0$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad ; \quad \sin\theta \neq 0$$



$$\sec\theta = \frac{1}{\cos\theta} \quad ; \quad \cos\theta \neq 0$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad ; \quad \sin\theta \neq 0$$

Trigonometric identities

The basic trigonometric identities are:

$$\cos^2\theta + \sin^2\theta = 1 \qquad (1)$$

$$1 + \tan^2\theta = \sec^2\theta \quad ; \quad \cos\theta \neq 0$$

obtained by dividing (1) by $\cos^2\theta$

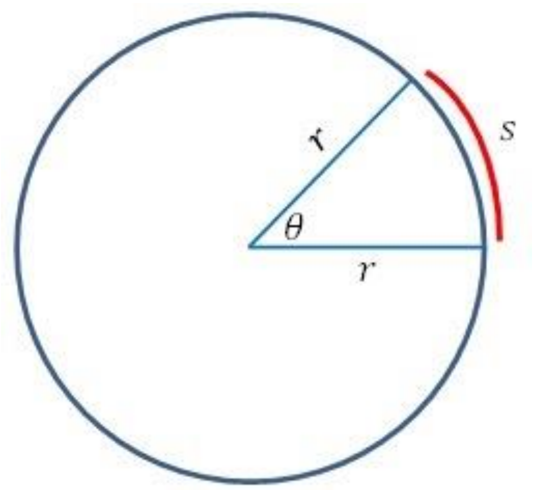
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta \quad ; \quad \sin\theta \neq 0$$

obtained by dividing (1) by $\sin^2\theta$

Conversion (degree $\longleftrightarrow$ radians)

By definition, the length of the enclosed arc (s) is equal to the radius (r) multiplied by the magnitude of the angle (θ) in radians.

$$s = r\,\theta \quad \Rightarrow \quad \theta = \frac{s}{r}$$



$\therefore$ For one complete revolution (360^o), the magnitude in radians is

$$360^\circ = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi$$

$$\Rightarrow \boxed{\pi = 180^\circ}$$

An important relation to convert degrees to radians and vice-versa.

Conversion (degree $\longleftrightarrow$ radians)

$$\boxed{\text{angle in radians} = \text{angle in degrees} \times \left(\frac{\pi}{180^\circ}\right)}$$

$$\boxed{\text{angle in degrees} = \text{angle in radians} \times \left(\frac{180^\circ}{\pi}\right)}$$

$$45^\circ = 45^\circ \times \left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{4} \text{ radians}$$

$$270^\circ = 270^\circ \times \left(\frac{\pi}{180^\circ}\right) = \frac{3\pi}{2} \text{ radians}$$

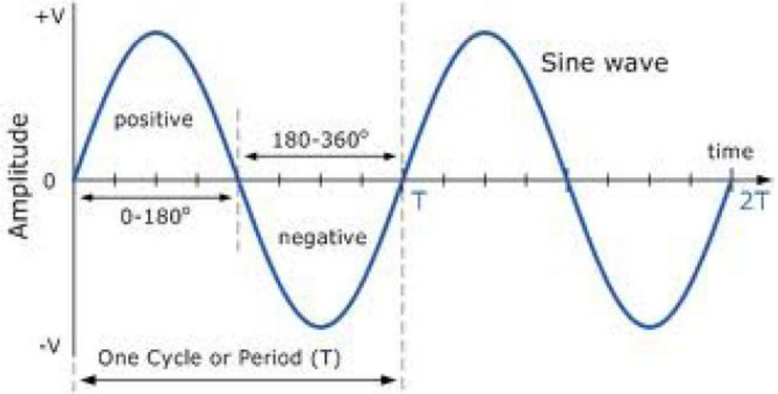
$$\frac{\pi}{6} \text{ radians} = \left(\frac{180^\circ}{\pi}\right) \times \frac{\pi}{6} = 30^\circ$$

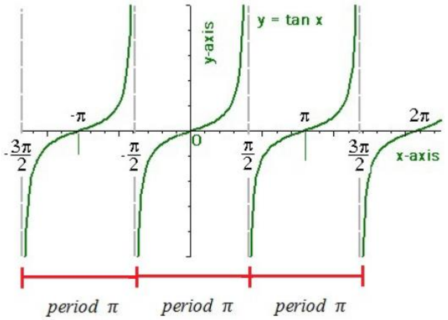
$$\frac{5\pi}{12} \text{ radians} = \left(\frac{180^\circ}{\pi}\right) \times \frac{5\pi}{12} = 75^\circ$$

Periodic functions

If $f(x+p)=f(x)$, the function f is called periodic and p is defined as its period. The smallest positive value of p is called the Principal period of f.

Trigonometric function	Principal Period
cos	2π
sin	
sec	
cosec	
tan	π
cot	





Periods of Trigonometric functions

where T_1 is the trig function sin, cos, sec or cosec.

Principal period of $a\,T_1(bx+c)+d$ is $\frac{2\pi}{|b|}$

e.g. principal period of : $2\cos\,(4x-5)+6 \;=\; \frac{2\pi}{4}$

---

where T_2 is the trig function tan or cot.

Principal period of $a\,T_2(bx+c)+d$ is $\frac{\pi}{|b|}$

e.g. principal period of : $3\tan\,(4-5x)+2=\frac{\pi}{|(-5)|}\;=\;\frac{\pi}{5}$

Q1

Find the Principal Period of $\operatorname{cosec}\left(\frac{1}{3}x - \pi\right)$

A $\dfrac{3\pi}{2}$

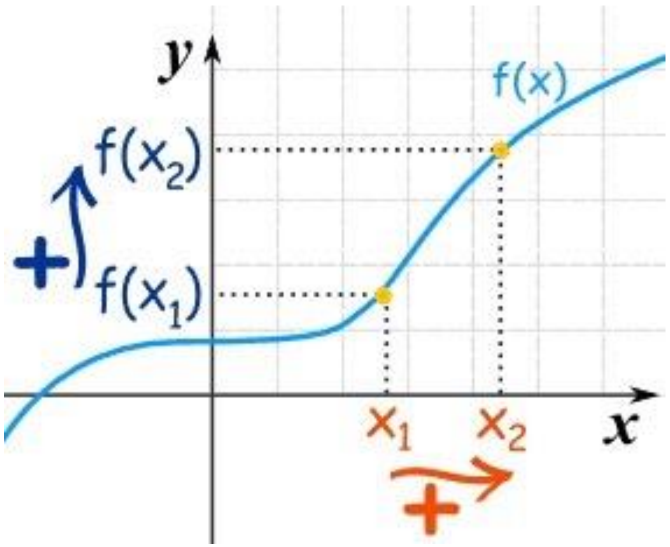
B 4π

$\boxed{\text{C} \quad 6\pi}$

Increasing and Decreasing functions

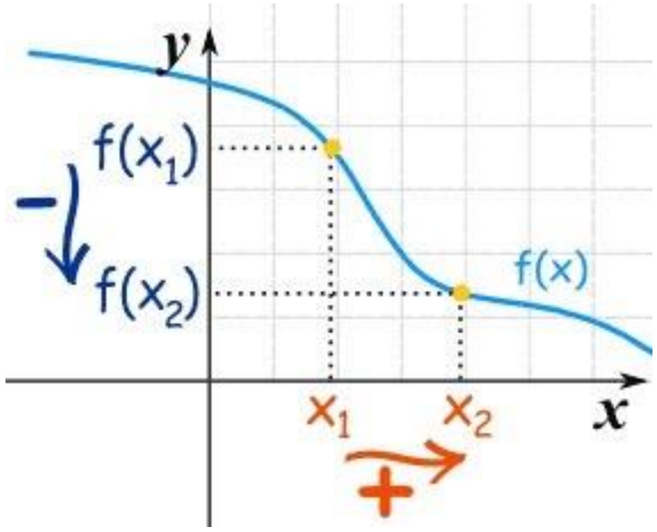
If $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$,

then the function f is said to be an increasing (↑) function.



If $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$,

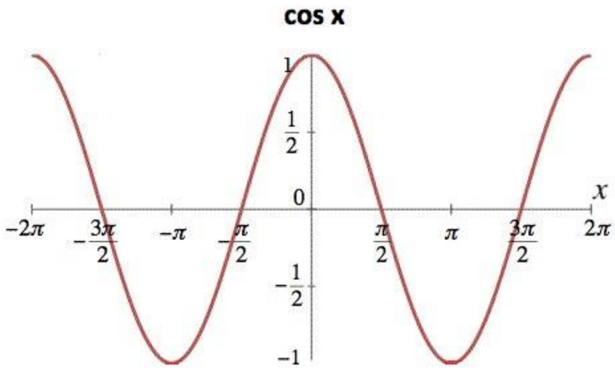
then the function f is said to be a decreasing (↓) function.

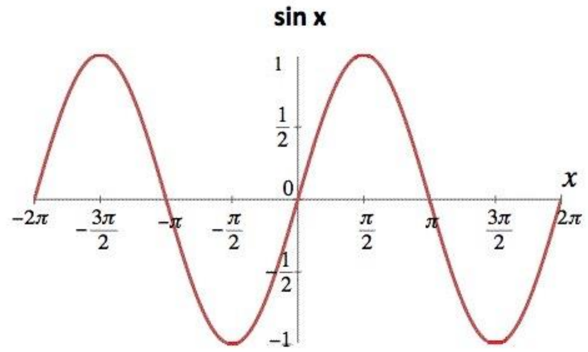


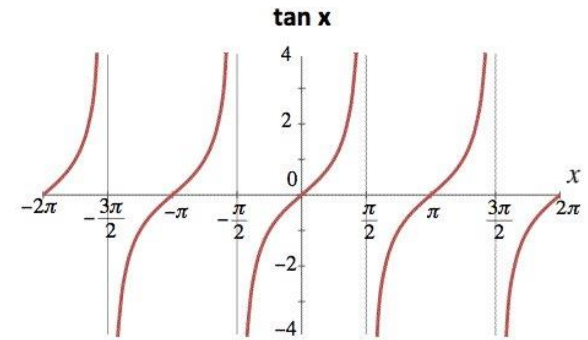
Increasing and Decreasing functions

Quadrant	1	2	3	4
cos	↓	↓	↑	↑
sin	↑	↓	↓	↑
tan	↑	↑	↑	↑

Quadrant	1	2	3	4
sec	↑	↑	↓	↓
cosec	↓	↑	↑	↓
cot	↓	↓	↓	↓





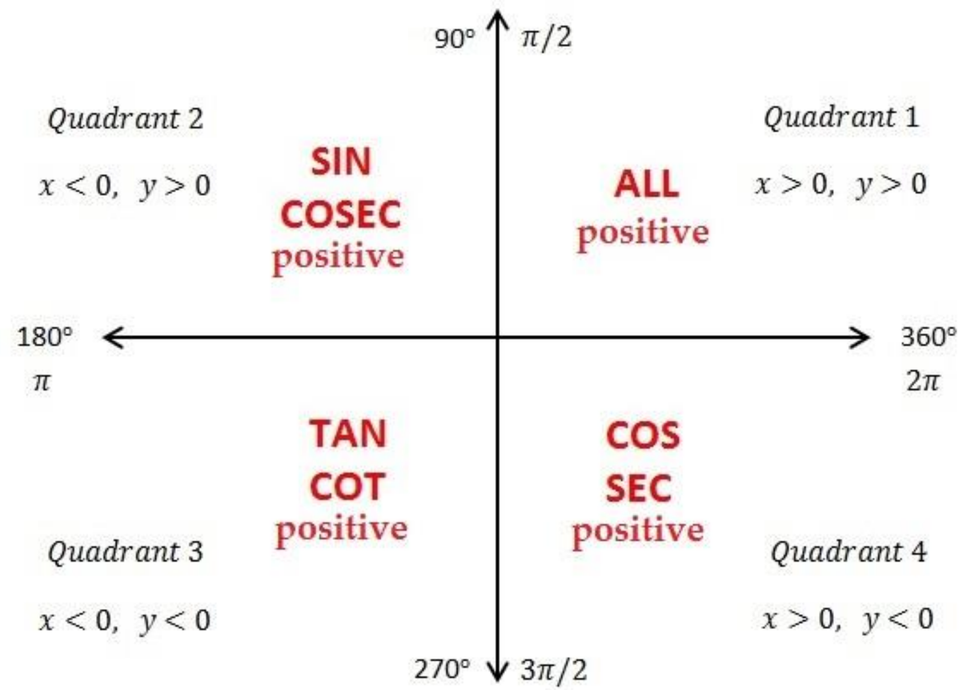


Signs of Trigonometric functions in the quadrants

$$\cos\theta = \frac{x}{r}$$

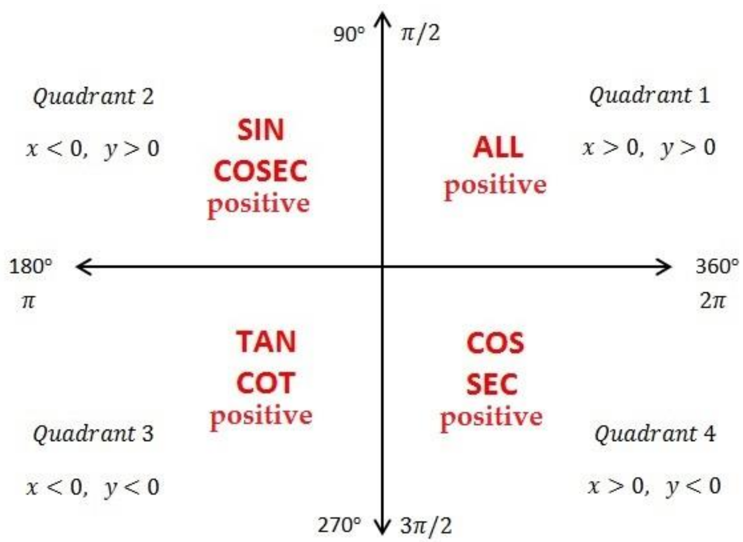
$$\sin\theta = \frac{y}{r}$$

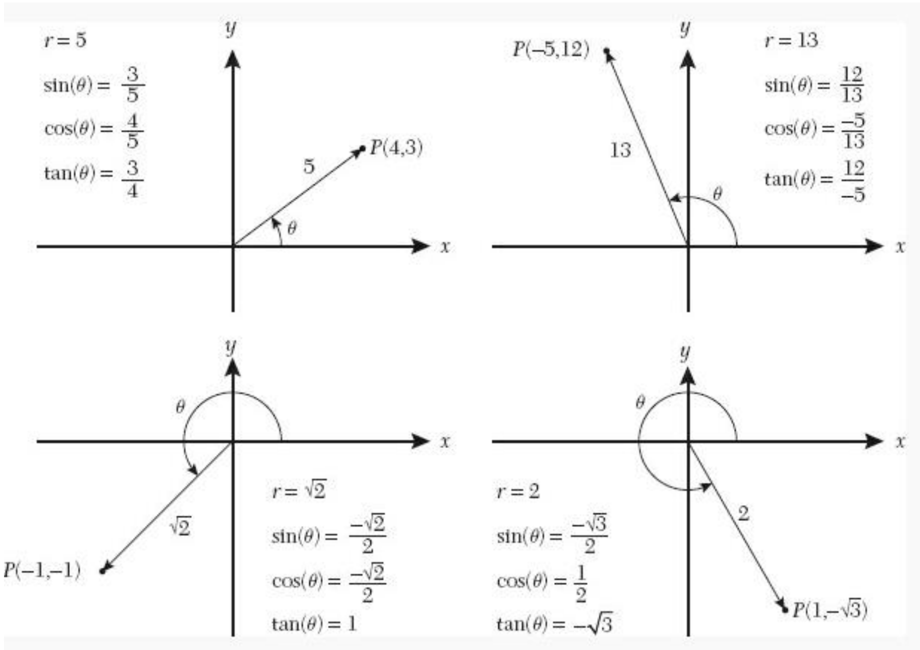
$$\tan\theta = \frac{y}{x}$$



For unit circle ($r = 1$) $\cos\theta = x$ & $\sin\theta = y$

Signs of Trigonometric functions in the quadrants

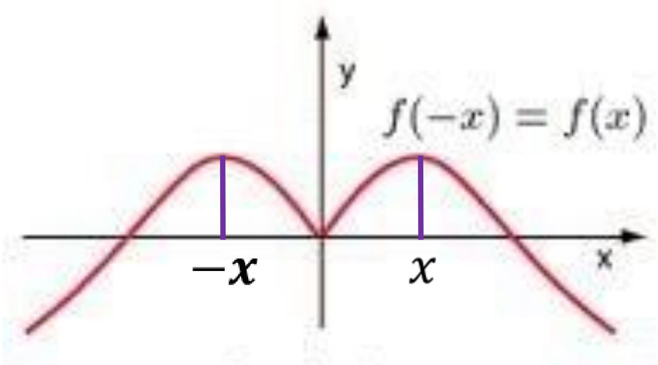




Example If $\tan\theta = \dfrac{-3}{4}$; $\dfrac{3\pi}{2} \le \theta \le 2\pi$, find $\cos\theta$ and $\sin\theta$.

Even and Odd Trigonometric functions

The function f is said to be an even function if $f(-x) = f(x)$

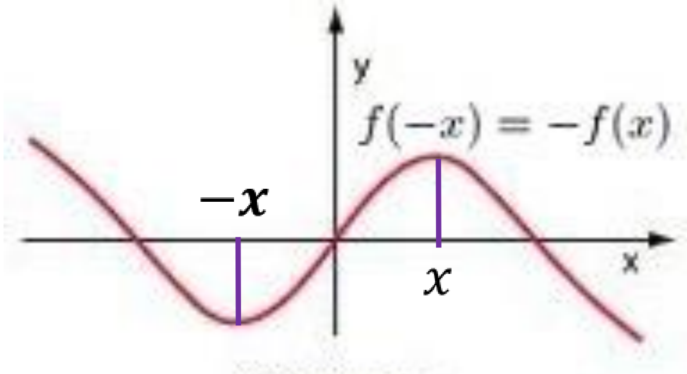


Even function

e.g. $\cos(-\theta) = \cos\theta$

$\Rightarrow$ $\cos$ is an even function.

The function f is said to be an odd function if $f(-x) = -f(x)$



Odd function

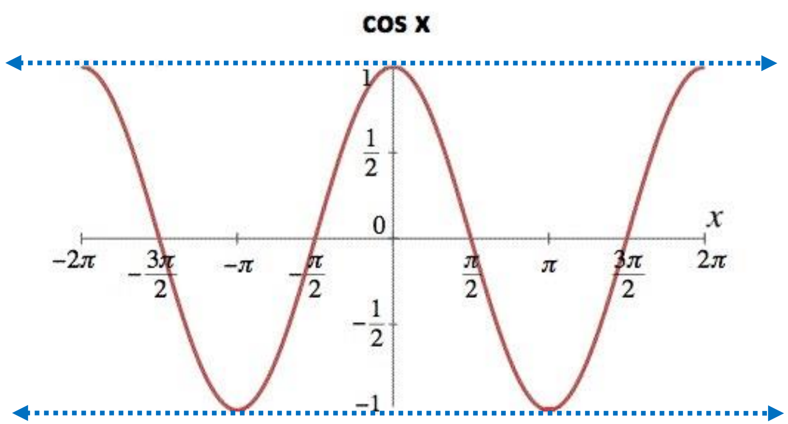
e.g. $\sin(-\theta) = -\sin\theta$

$\Rightarrow$ $\sin$ is an odd function.

Range of Trigonometric functions

From the graph of cosine function, it is clear that

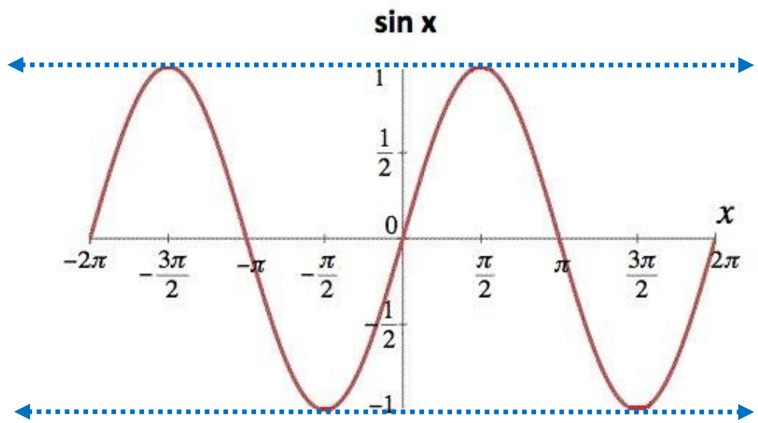
$$-1 \leq \cos\theta \leq 1$$



$\therefore$ Range of $\cos$ function is $[-1, 1]$

Similarly,

$$-1 \leq \sin\theta \leq 1$$

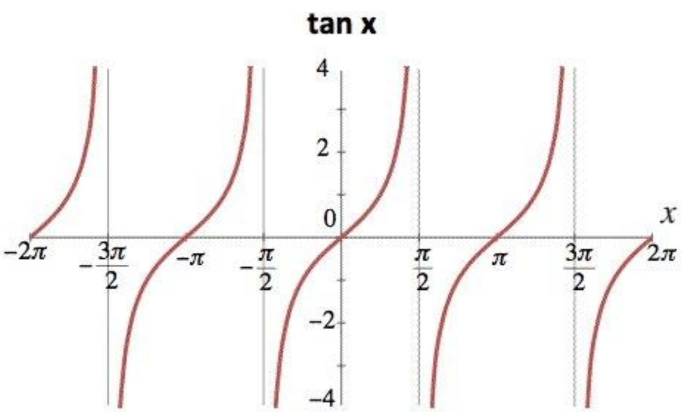


$\therefore$ Range of $\sin$ function is $[-1, 1]$

Range of Trigonometric functions

Also, $\tan\theta \in \mathbb{R}$

$\cot\theta \in \mathbb{R}$

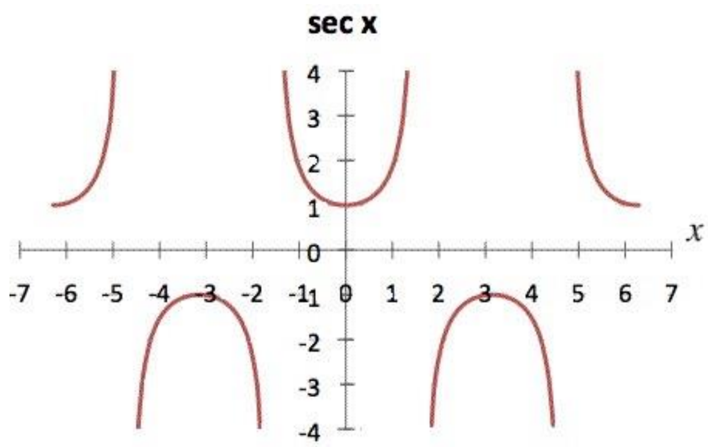


$\therefore$ Range of $\tan$ function is $\mathbb{R}$.

Range of $\cot$ function is $\mathbb{R}$.

And, $\sec\theta \le -1$ **or** $\sec\theta \ge 1$

$\mathrm{cosec}\,\theta \le -1$ **or** $\mathrm{cosec}\,\theta \ge 1$



$\therefore$ Range of $\sec$ function is $\mathbb{R} - (-1, 1)$.

Range of cosec function is $\mathbb{R} - (-1, 1)$.

Range of Trigonometric functions

Trigonometric function	Range
sin and cos	$[-1, 1]$
sec and cosec	$\mathbb{R} - (-1, 1)$
tan and cot	$\mathbb{R}$

Q2

Find the Range of $y = 3\cos x + 2$

A $[-1, 1]$

B $\boxed{[-1, 5]}$

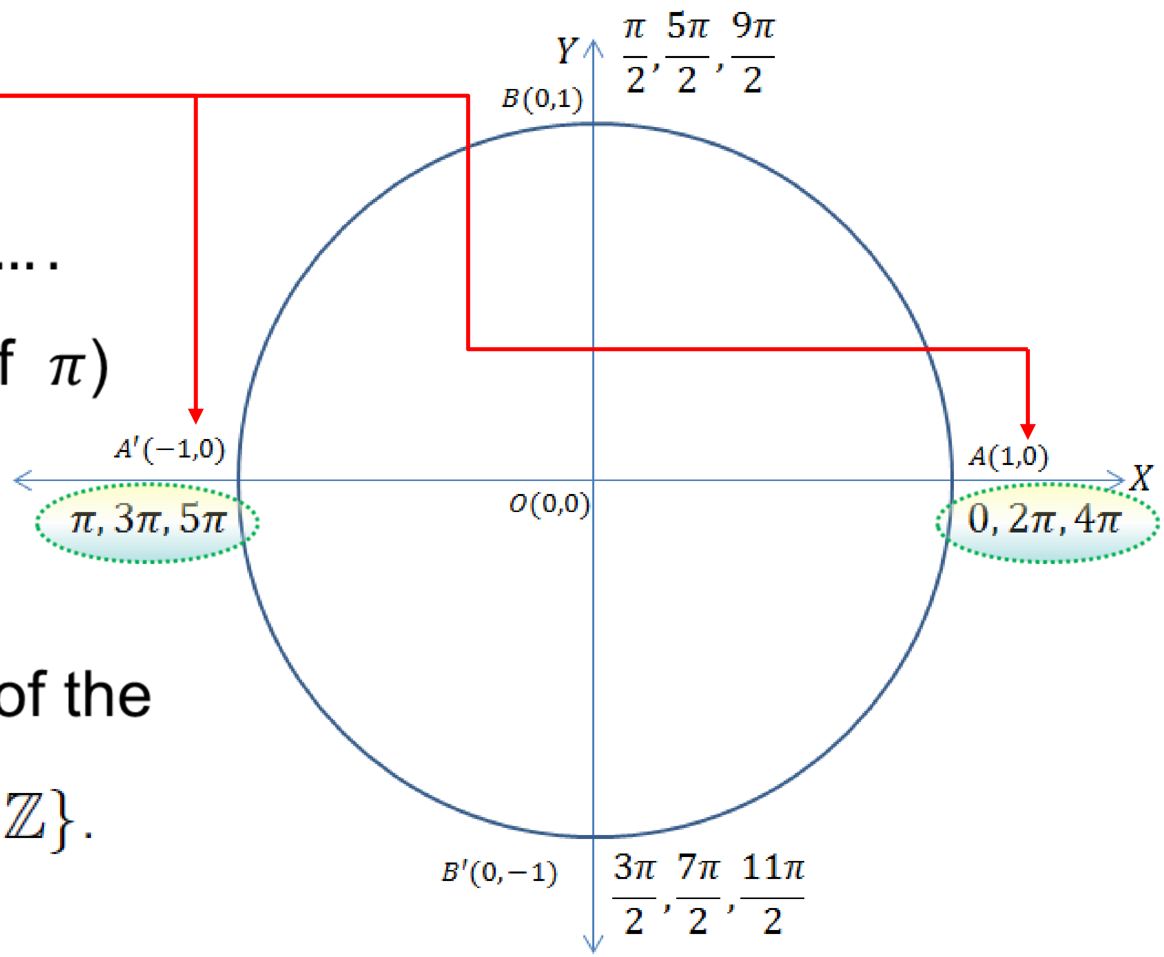
C $[-3, 2]$

Sets of Zeros of Trigonometric functions

Consider $\sin\theta = 0$

Then, $\theta = 0\,, \pi, 2\pi, 3\pi, \ldots.$

(i.e. multiples of π)



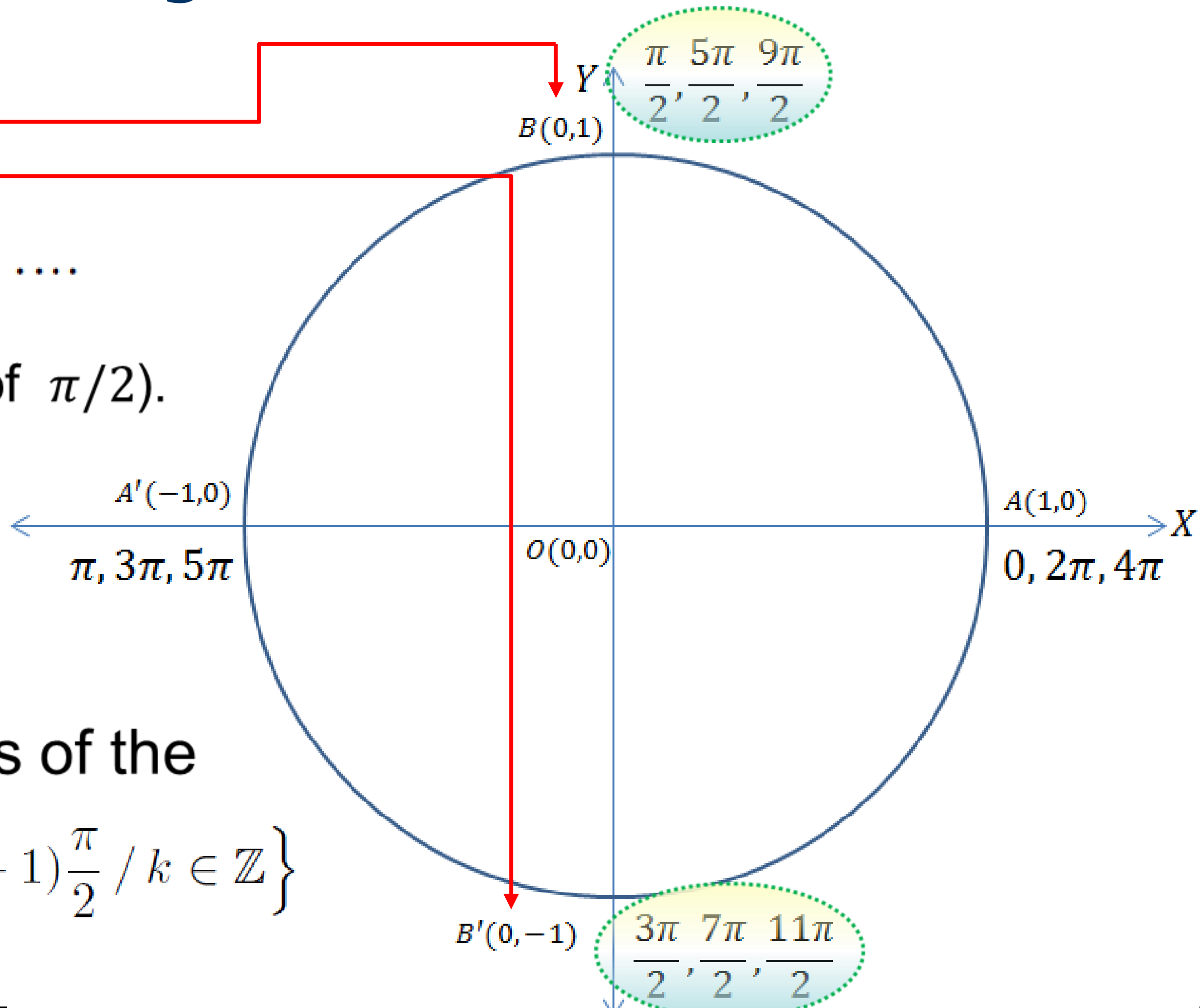
Thus, the set of zeros of the sin function $\{k\pi \,/\, k \in \mathbb{Z}\}$.

Sets of Zeros of Trigonometric functions

Similarly, $\cos\theta = 0$

$$\Rightarrow \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

(i.e. odd multiples of $\pi/2$).



Thus, the set of zeros of the $\cos$ function is $\left\{(2k+1)\frac{\pi}{2} \,/\, k \in \mathbb{Z}\right\}$

Note...

Function	Domain	Range	Set of zeros	Period
cos	$\mathbb{R}$	$[-1,1]$	$\left\{(2k+1)\frac{\pi}{2} \,/\, k\in\mathbb{Z}\right\}$	2π
sin	$\mathbb{R}$	$[-1,1]$	$\{k\pi \,/\, k\in\mathbb{Z}\}$	2π
tan	$\mathbb{R}-\left\{(2k+1)\frac{\pi}{2} \,/\, k\in\mathbb{Z}\right\}$	$\mathbb{R}$	$\{k\pi \,/\, k\in\mathbb{Z}\}$	π
sec	$\mathbb{R}-\left\{(2k+1)\frac{\pi}{2} \,/\, k\in\mathbb{Z}\right\}$	$\mathbb{R}-(-1,1)$	ϕ	2π
cosec	$\mathbb{R}-\{k\pi \,/\, k\in\mathbb{Z}\}$	$\mathbb{R}-(-1,1)$	ϕ	2π
cot	$\mathbb{R}-\{k\pi \,/\, k\in\mathbb{Z}\}$	$\mathbb{R}$	$\left\{(2k+1)\frac{\pi}{2} \,/\, k\in\mathbb{Z}\right\}$	π

Solving Trigonometric equations

A trigonometric equation is an equation containing one or more trigonometric functions of the variable, say θ.

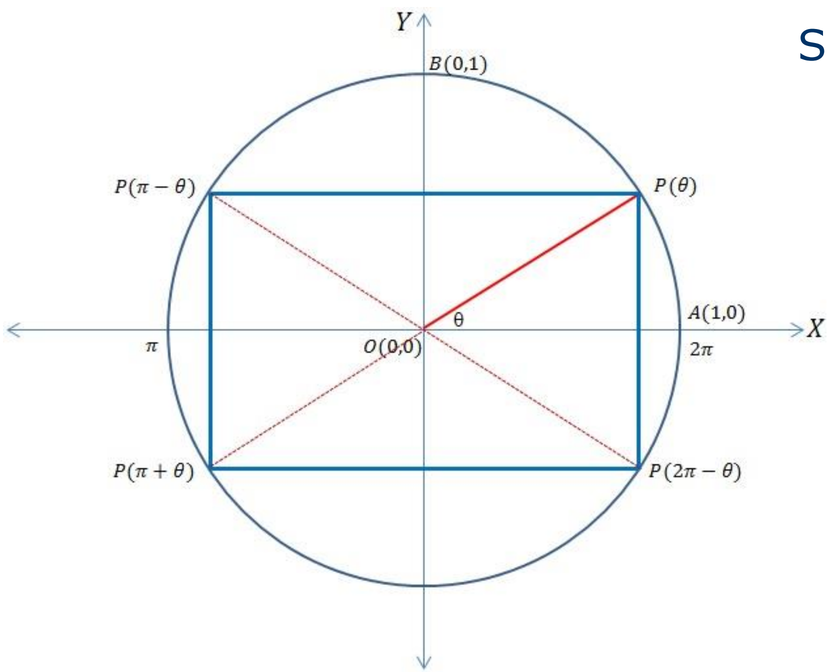
Solving for θ means finding the values of θ (in given interval) which makes the trigonometric equation true.

e.g. The solution of $\cos\theta = \frac{1}{2}$ in $\left(0, \frac{\pi}{2}\right)$ is $\frac{\pi}{3}$ radian

whereas its solution in $(0, 2\pi)$ is

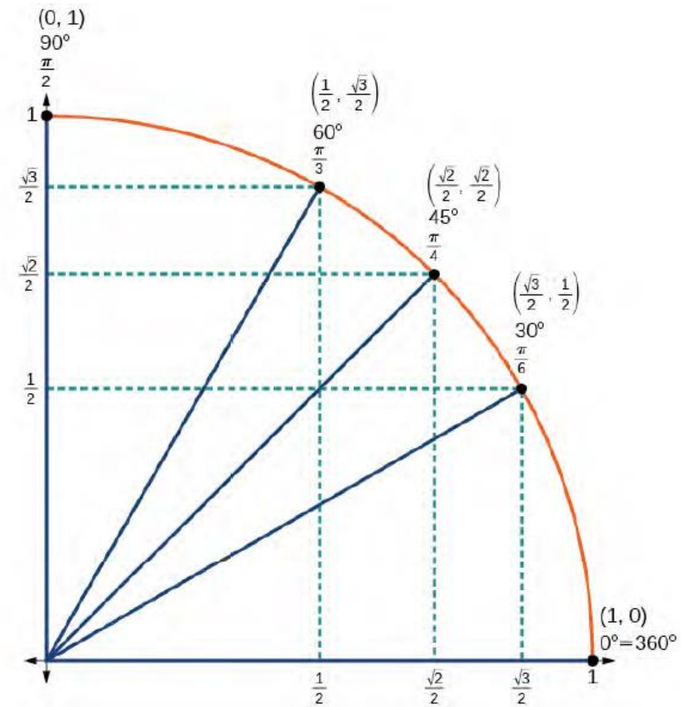
$$\frac{\pi}{3} \quad \text{or} \quad \left(2\pi - \frac{\pi}{3}\right) = \frac{5\pi}{3} \text{ radians.}$$

Solving Trigonometric equations (Review of angular measure)



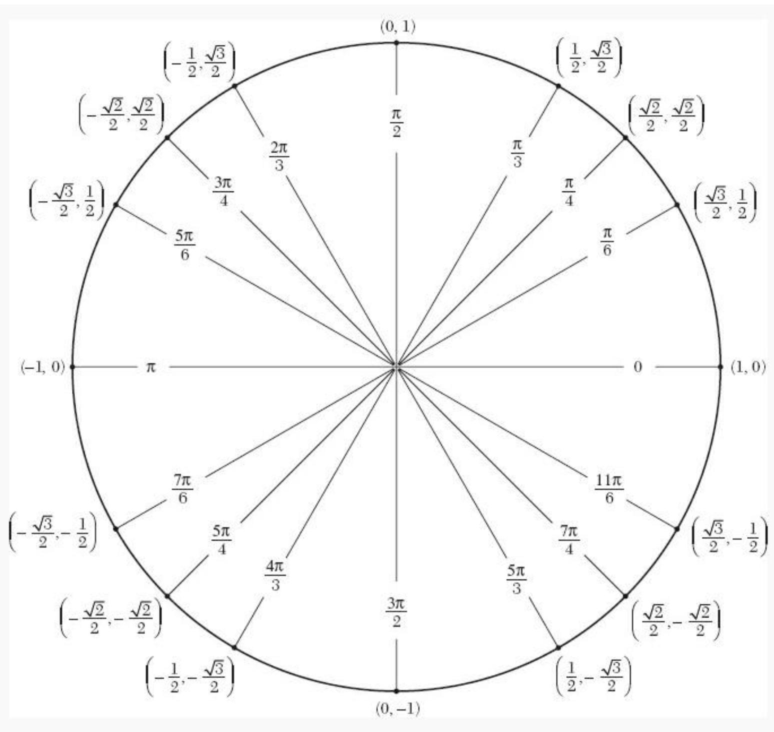
Reference Angle $\theta \in (0, 2\pi)$

Sine and Cosine of Special Angles of $\theta \in \left[0, \frac{\pi}{2}\right]$



Coordinate of Point P on the unit circle is $(\sin\theta, \cos\theta)$

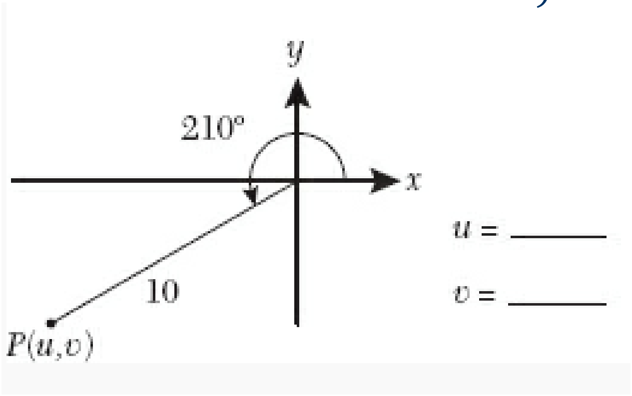
Solving Trigonometric equations (Review of angular measure)



Special angles shown for the unit circle

Q3

Find u and v from the figure below (note circle has a radius of 10)



A 5 and $\frac{3}{2}$

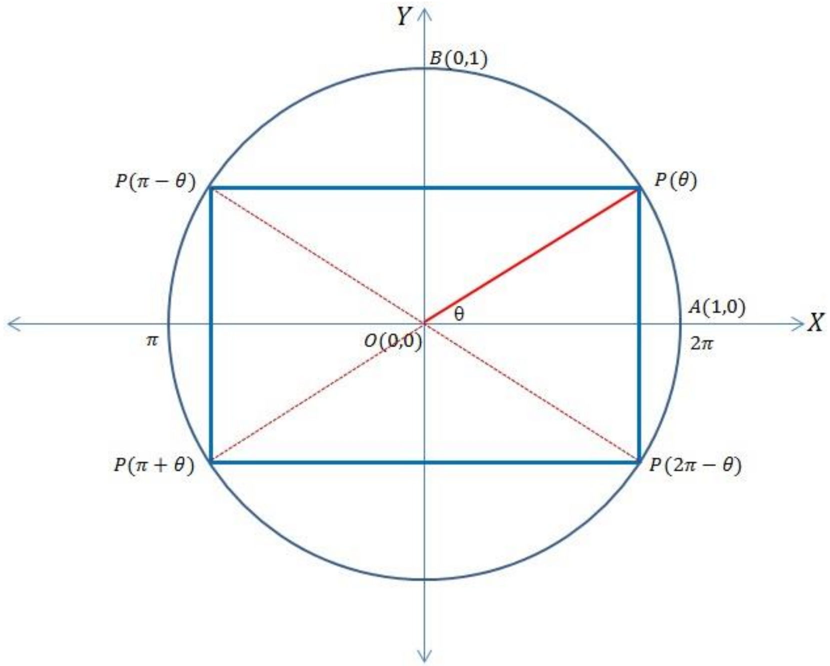
B $-10\sqrt{2}$ and $\frac{10\sqrt{3}}{2}$

$\boxed{\text{C} \quad -5, -5\sqrt{3}}$

Worked Examples

1. Solve: $\tan^2\theta - 2\sec\theta + 1 = 0 \quad ; \quad 0 \le \theta \le \pi.$

2. Solve: $2\cot^2\theta = 7\operatorname{cosec}\theta - 8 \quad ; \quad 0 < \theta < 180°.$



Addition and factor formulae

Note: $x(A+B) = xA + xB$, but $\sin(A+B) \neq \sin A + \sin B$.

$$\boxed{\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
\sin(A-B) &= \sin A \cos B - \cos A \sin B \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
\cos(A-B) &= \cos A \cos B + \sin A \sin B \\
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
\tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B}
\end{aligned}}$$

Addition and factor formulae

Example

$$\sin 75^\circ = \sin\left(45^\circ + 30^\circ\right)$$

$$\boxed{\sin(A+B) = \sin A \cos B + \cos A \sin B}$$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Addition and Factor formulae

$$\sin(A+B) = \sin A\cos B + \cancel{\cos A\sin B}$$

$$\sin(A-B) = \sin A\cos B - \cancel{\cos A\sin B}$$

Adding

$$\boxed{\sin(A+B) + \sin(A-B) = 2\sin A\cos B}$$

$$\sin(A+B) = \cancel{\sin A\cos B} + \cos A\sin B$$

$$-\sin(A-B) = -\cancel{\sin A\cos B} + \cos A\sin B$$

Subtracting

$$\boxed{\sin(A+B) - \sin(A-B) = 2\cos A\sin B}$$

Similarly, it can be proved that

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$$

$$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$$

Addition and Factor formulae

Writing $A+B=C$ and $A-B=D$ $\Rightarrow$ $A=\dfrac{C+D}{2}$ and $B=\dfrac{C-D}{2}$

$$\begin{aligned}
\sin(A+B)+\sin(A-B) &= 2\sin A\cos B\\
\sin(A+B)-\sin(A-B) &= 2\cos A\sin B\\
\cos(A+B)+\cos(A-B) &= 2\cos A\cos B\\
\cos(A+B)-\cos(A-B) &= -2\sin A\sin B
\end{aligned}$$

$$\begin{aligned}
\sin C+\sin D &= 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\\
\sin C-\sin D &= 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\\
\cos C+\cos D &= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\\
\cos C-\cos D &= -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)
\end{aligned}$$

Example

Prove that $\sin 50^\circ+\sin 10^\circ=\sin 70^\circ$

Multi-angle formulae

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\boxed{\begin{aligned} &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\boxed{\begin{aligned} \sin^2\theta &= \frac{1}{2}(1-\cos 2\theta) \\ \cos^2\theta &= \frac{1}{2}(1+\cos 2\theta) \end{aligned}}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

Worked Example

Prove that $\dfrac{\sin 3\theta}{1+2\cos 2\theta} = \sin\theta$. Hence deduce the value of $\sin 15^o$.

With $t = \tan\left(\dfrac{\theta}{2}\right)$,

useful formulae in Calculus

$\sin\theta = \dfrac{2t}{1+t^2}$	$\cos\theta = \dfrac{1-t^2}{1+t^2}$	$\tan\theta = \dfrac{2t}{1-t^2}$

Further Reading (click on links)

Foundation Algebra by P. Gajjar.

Chapter 5, and Chapter 6 (Sections 6.1 to 6.7)

Foundations of Mathematics by P. Brown.

Chapter 4 (Sections 4.1 to 4.12)

THANKS FOR YOUR ATTENTION