



Foundation Calculus and Mathematical Techniques (CELEN037)

Problem Sheet 6

Topics: Indefinite Integrals

Topic 1: Integrals of the form $\int \sin^m x \cos^n x \, dx$

1. Evaluate the following integrals

(i) $\int \frac{\sin^3 x}{\cos^4 x} \, dx$

(ii) $\int \cot^3 x \cdot \sin x \, dx$

(iii) $\int \sin^2 x \cdot \cos^2 x \, dx$

(iv) $\int \sin^4 x \cdot \cos^5 x \, dx$

(v) $\int \sin^4 x \, dx$

Topic 2: Integrals of the form $\int \frac{f'(x)}{f(x)} \, dx$

2. Evaluate the following integrals

(i) $\int \frac{2 + 2 \cos 3x}{3x + \sin 3x} \, dx$

(ii) $\int \frac{1 - \tan x}{1 + \tan x} \, dx$

(iii) $\int \frac{\ln x + 1}{x \ln x} \, dx$

(iv) $\int \frac{3}{\cos^{-1} x \cdot \sqrt{4 - 4x^2}} \, dx$

(v) $\int \frac{4 \sin^3 x - 3 \sin x}{4 \cos^4 x - 3 \cos x} \, dx$

Topic 3: Integration after Completing the Square in the denominator

3. Evaluate the following integrals

(i) $\int \frac{1}{x^2 - 8x + 12} \, dx$

(ii) $\int \frac{1}{7 - 12x - 9x^2} \, dx$

(iii) $\int \frac{1}{\sqrt{8x - x^2}} \, dx$

(iv) $\int \frac{1}{\sqrt{25x^2 - 20x + 14}} \, dx$

(v) $\int \frac{1}{x^2 + 2px + q} \, dx$ where $p, q \in \mathbb{R}$ and $p^2 - q < 0$

(vi) $\int \frac{1}{x^2 + 2px + q} \, dx$ where $p, q \in \mathbb{R}$ and $p^2 - q > 0$

Topic 4: The method of t -substitution

4. Evaluate the following integrals

(i) $\int \frac{1}{3 + 2 \sin x} dx$

(ii) $\int \frac{1}{3 - \cos x} dx$

(iii) $\int \frac{\sec x}{\tan x - \sec x + 1} dx$

(iv) $\int \frac{1}{4 \sin x + 6 \cos x + 3} dx$

(v) $\int \frac{\sin x + \cos x}{2 \cos 2x} dx$

(vi) $\int \frac{1}{a + b \sin x} dx$ when $a^2 < b^2$

(vii) $\int \frac{1}{a + b \cos x} dx$ when $a^2 > b^2$

Topic 5: Integrals of the form $\int \frac{1}{a \cos^2 x + b \sin^2 x + c} dx$

5. Evaluate $\int \frac{1}{a \cos^2 x + b \sin^2 x + c} dx$ when a, b, c satisfy the following given conditions:

(i) $b + c = 0$

(ii) $b + c \neq 0$ and $(a + c)(b + c) > 0$

(iii) $b + c \neq 0$ and $a + c = 0$

(iv) $b + c \neq 0$ and $(a + c)(b + c) < 0$

Answers

1.

(i) $\frac{1}{3 \cos^3 x} - \frac{1}{\cos x} + C$

(ii) $-\frac{1}{\sin x} - \sin x + C$

(iii) $\frac{4x - \sin 4x}{32} + C$

(iv) $\frac{\sin^5 x}{5} - \frac{2 \sin^7 x}{7} + \frac{\sin^9 x}{9} + C$

(v) $\frac{x - \sin 2x}{4} + \frac{4x + \sin 4x}{32} + C$

2.

(i) $\frac{2}{3} \cdot \ln |3x + \sin 3x| + C$

(ii) $\ln |\sin x + \cos x| + C$

(iii) $\ln |x \ln x| + C$

(iv) $-\frac{3}{2} \cdot \ln |\cos^{-1} x| + C$

(v) $\frac{1}{3} \cdot \ln |\cos 3x| + C$

3.

(i) $\frac{1}{4} \cdot \ln \left| \frac{x-6}{x-2} \right| + C$

(ii) $\frac{1}{6\sqrt{11}} \cdot \ln \left| \frac{3x+2+\sqrt{11}}{3x+2-\sqrt{11}} \right| + C$

(iii) $\sin^{-1} \left(\frac{x-4}{4} \right) + C$

(iv) $\frac{1}{5} \cdot \ln \left| 5x-2+\sqrt{25x^2-20x+14} \right| + C$

(v) $\frac{1}{\sqrt{q-p^2}} \cdot \tan^{-1} \left(\frac{x+p}{\sqrt{q-p^2}} \right) + C$

(vi) $\frac{1}{2\sqrt{p^2-q}} \cdot \ln \left| \frac{x+p-\sqrt{p^2-q}}{x+p+\sqrt{p^2-q}} \right| + C$

4.

$$(i) \quad \frac{2}{\sqrt{5}} \cdot \tan^{-1} \left[\frac{3}{\sqrt{5}} \tan \left(\frac{x}{2} \right) + \frac{2}{\sqrt{5}} \right] + C$$

$$(ii) \quad \frac{\sqrt{2}}{2} \cdot \tan^{-1} \left[\sqrt{2} \tan \left(\frac{x}{2} \right) \right] + C$$

$$(iii) \quad \ln \left| \frac{\tan \left(\frac{x}{2} \right)}{\tan \left(\frac{x}{2} \right) - 1} \right| + C$$

$$(iv) \quad \frac{1}{\sqrt{43}} \cdot \ln \left| \frac{3 \tan \left(\frac{x}{2} \right) - 4 + \sqrt{43}}{3 \tan \left(\frac{x}{2} \right) - 4 - \sqrt{43}} \right| + C$$

$$(v) \quad \frac{1}{2\sqrt{2}} \cdot \ln \left| \frac{\tan \left(\frac{x}{2} \right) + 1 + \sqrt{2}}{\tan \left(\frac{x}{2} \right) + 1 - \sqrt{2}} \right| + C$$

$$(vi) \quad \frac{1}{a} \cdot \sqrt{\frac{a^2}{b^2 - a^2}} \cdot \ln \left| \frac{\tan \left(\frac{x}{2} \right) + \frac{b}{a} - \sqrt{\frac{b^2}{a^2} - 1}}{\tan \left(\frac{x}{2} \right) + \frac{b}{a} + \sqrt{\frac{b^2}{a^2} - 1}} \right| + C$$

$$(vii) \quad \frac{2}{a+b} \cdot \sqrt{\frac{a+b}{a-b}} \cdot \tan^{-1} \left[\sqrt{\frac{a-b}{a+b}} \cdot \tan \left(\frac{x}{2} \right) \right] + C$$

5.

$$(i) \quad \frac{\tan x}{a+c} + C$$

$$(ii) \quad \frac{1}{b+c} \cdot \sqrt{\frac{b+c}{a+c}} \cdot \tan^{-1} \left(\sqrt{\frac{b+c}{a+c}} \cdot \tan x \right) + C$$

$$(iii) \quad -\frac{\cot x}{b+c} + C$$

$$(iv) \quad \frac{1}{2(b+c)} \cdot \sqrt{\frac{b+c}{-(a+c)}} \cdot \ln \left| \frac{\tan x - \sqrt{\frac{-(a+c)}{b+c}}}{\tan x + \sqrt{\frac{-(a+c)}{b+c}}} \right| + C$$