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Foundation Calculus & Mathematical Techniques CELEN037

Weekly Worksheet-3 **Topics:** Parametric Differentiation, Maclaurin's series, Equations of tangent and normal lines, Newton-Raphson Method, Increasing and Decreasing functions.

Type 1: Parametric Differentiation

1. Use the method of parametric differentiation to find $\dfrac{dy}{dx}$.

   (i) $x = a\cos\theta$ and $y = b\sin\theta\ ;\ \theta \in [\,0, 2\pi\,]$. Also find $\left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{4}}$.

   (ii) $x = 3\sin\theta - \cos\theta$ and $y = 3\cos\theta - \sin\theta$. Also find $\left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{4}}$.

   (iii) $x = a\,(t - \sin t)$ and $y = a\,(1 - \cos t)$ (a is constant). Also find $\left.\dfrac{dy}{dx}\right|_{t=\frac{\pi}{2}}$.

   (iv) $x = \cos^{-1}(\sin\theta)$ and $y = \sin^{-1}(\cos\theta)$; $\dfrac{\pi}{2} < \theta < \pi$.

   (v) $x = v_0\, t\, \cos\alpha$ and $y = v_0\, t\, \sin\alpha - \dfrac{1}{2}\, g\, t^2$ (v_0, α and g are constant).

   (vi) $x = \dfrac{2t}{1+t^2}$ and $y = \dfrac{1-t^2}{1+t^2}$. **Hint**: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2}$.

Type 2: Maclaurin's series

2. Obtain the Maclaurin's series expansions of the following functions:

   (i) $f(x) = \cos x$.

   (ii) $f(x) = \dfrac{1}{1-x}$; $x \neq 1$.

   (iii) $f(x) = \sqrt{1-x}$; $x < 1$.

   (iv) $f(x) = \sin x + \cos x$.

   (v) $f(x) = \tan x$ **Note**: Differentiate up to the fifth order.

   (vi) Obtain Maclaurin's series expansion of $f(x) = e^x$.

   Also show that $\dfrac{1}{2}\left(e^x - e^{-x}\right) = \displaystyle\sum_{1}^{\infty} \dfrac{x^{2k-1}}{(2k-1)\,!}$.

   (vii) Obtain Maclaurin's series expansion of $f(x) = \ln(1+x)$.

   Hence show that $\ln\left(\dfrac{1+x}{1-x}\right) = 2\displaystyle\sum_{1}^{\infty} \dfrac{x^{2k-1}}{(2k-1)}$.

   $(viii)$ Given $f(x) = x\, e^{-x}$, show that $f(x) = x - x^2 + \dfrac{x^3}{2} - \dfrac{x^4}{6} + \dfrac{x^5}{24} - \dfrac{x^6}{120} + \cdots\cdots$

Type 3: Equations of tangent and normal lines

3. Obtain the equations of tangent and normal lines to the following curves at given points:

   (i) $y = \dfrac{1}{x}$ at $(-1, -1)$.

   (ii) $y = x^2 - 2$ at $(1, -1)$.

   (iii) $e^x\, y^2 + x^2 = 9$ at $(0, 3)$.

Type 4: Newton-Raphson Method $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \qquad (n = 0, 1, 2, ...)}$

4. Use the Newton-Raphson method to approximate the root of:

   (i) $\sin x = x - 1,$ correct to 5 d.p, starting with $x_0 = 1$.

   (ii) $2\sin x - x = 0,$ correct to 6 d.p, starting with $x_0 = 2$.

   (iii) $2\cos x - x^2 = 0,$ correct to 6 d.p, starting with $x_0 = 1$.

   (iv) $x^4 - x^2 = 1,$ correct to 6 d.p, starting with $x_0 = -1.5$.

   (v) $x^4 - \sin x = 1,$ correct to 7 d.p, starting with $x_0 = 1.5$.

   (vi) $x^3 - 2x - 5 = 0,$ correct to 8 d.p, starting with $x_0 = 2$.

   (vii) $x^3 - 2x - 5 = 0,$ correct to 8 d.p, starting with $x_0 = 2$.

   $(viii)$ Use the Newton-Raphson method to approximate the value of $\sqrt[3]{2}$, correct to 8 d.p., by starting with $x_0 = 1$.

Type 5: Increasing and Decreasing functions

5. (i) Show that $f(x) = \tan x$ is increasing $\forall\, x \in D_f$.

   (ii) Show that $f(x) = \cos x$ is increasing in the fourth quadrant.

   (iii) Show that $f(x) = e^{-2x} + 1$ is always decreasing.

   (iv) Show that $f(x) = x^2 + 2x - 3$ is decreasing when $x < -1$.

   (v) Given $f(x) = x^3 - 5x + 1$. Determine the intervals in which the function f is increasing and the intervals in which it is decreasing.

   (vi) Given $f(x) = 2x^4 + 3x^3 - 9x^2 + 7$. Determine the intervals in which the function f is increasing and the intervals in which it is decreasing. Also draw a rough sketch for the curve $y = f(x)$.