The University of Nottingham Ningbo China

Centre for English Language Education

SEMESTER TWO 2020-2021

FOUNDATION CALCULUS & MATHEMATICAL TECHNIQUES

Time allowed 1 hour 30 Minutes

Candidates may complete the front cover of their answer book and sign their attendance card but must NOT write anything else until the start of the examination period is announced.

This paper contains EIGHT questions which carry equal marks.

Attempt all questions.

An indication is given of the weighting of each subsection of a question by means of a figure enclosed by square brackets, eg. [5], immediately following that subsection.

Only CELE approved calculator (with university logo) is allowed during this exam.

Dictionaries are not allowed with one exception.Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

Do not turn examination paper over until instructed to do so.

ADDITIONAL MATERIAL: *Formula Sheet (attached to the back of the question paper).*

INFORMATION FOR INVIGILATORS:

1. *Please give a 15 minutes warning before the end of the exam.*
2. *Please collect Answer Booklets, Question Papers and Formula Sheet at the end of the exam.*

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1. (a) Given $y = \dfrac{x}{x+1}$.

   (i) Find $\dfrac{dy}{dx}$ by using the first principles.

   (ii) Find k if $\left.\dfrac{dy}{dx}\right|_{\left(1,\frac{1}{2}\right)} + k\,x = 0.$ [3]

   (b) Given $y = \dfrac{\sin x}{1+\cos x}$, use the quotient rule for differentiation to show that

   $$\frac{dy}{dx} = \frac{1}{1+\cos x}.$$ [3]

   (c) (i) Given that the curve $y = x^2 + e^x$ passes through the point $(0, 1)$, find $\left.\dfrac{d^2y}{dx^2}\right|_{x=0}$.

   (ii) Given $y = \cos^{-1}(\sin x)$; $\dfrac{\pi}{2} < x < \pi$, find $\dfrac{dy}{dx}$.

   (iii) Given $g(x) = f(x) \cdot \sec x$ where $f(\pi/4) = 0$ and $f'(\pi/4) = 2$. Find $g'(\pi/4)$.

   [4]

2. (a) Given $y = x + \cos(xy)$.

   (i) Use the method of implicit differentiation to find the gradient of the curve at the point $P(0, 1)$.

   (i) Hence find the equation of the tangent line at the point P.

   (ii) Show that the equation of the normal line at the point P is $x + y - 1 = 0$. [4]

   (b) Given $f(x) = 4\,x^3 - x^2 - 2\,x + 1$.

   (i) Find the stationary points of f.

   (ii) Use the second derivative test to classify the stationary points of f obtained in 2(b)(i) as the points of maximum or minimum values.

   (iii) Sketch the graph of $y = f(x)$. [5]

   (c) Find the values of x for which the function $f(x) = x^2 - 3\ln x$; $x \in \mathbb{R}^+$ is increasing.

   [1]

3. (a) Given $x = (\theta - \pi) \cdot \cos\theta$ and $y = 3\,\theta \cdot \sin\theta$, use the method of parametric differentiation to find $\left.\dfrac{dy}{dx}\right|_{\theta=\pi/2}$. [3]

(b) Given that the equation $f(x) = x^4 - \cos x = 0$ has a real root in the interval $(0, 1)$.

Use the Newton-Raphson formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f\,'(x_n)}$; $n = 0, 1, 2, 3, \ldots$

to approximate this root by considering the zeroth approximation, $x_0 = 0.5$.

Give your answers correct to 5 decimal places (d.p.), and tabulate all values obtained in your calculations. [5]

(c) The area and circumference of a circle of radius r are given by πr^2 and $2\pi r$ respectively. Given that the area of the circle is decreasing at a rate of 0.5 cm^2/sec, find the rate at which its circumference is decreasing when the radius is 4 cm. [2]

4. (a) (i) Given $f(x) = \cos 2x$, obtain the Maclaurin's series expansion of f up to the terms with x^6.

(ii) Use the relationship $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ to write the Maclaurin's series expansion of $\sin^2 x$, up to the terms with x^6. [6]

(b) (i) Use the Maclaurin's series expansion to show that for $|\,x\,| \le 1$,

$$\sqrt{1-x} \;\approx\; 1 - \frac{x}{2} - \frac{x^2}{8}.$$

(ii) Use the substitution $x = \dfrac{1}{50}$ in the expansion given in 4(b)(i) to approximate $\sqrt{2}$.

Give your answer correct to 3 d.p. [4]

5. (a) Evaluate the following integrals:

$$(i)\ \int \left[\frac{x^3 - x^2}{x - 1} + \frac{1}{\sqrt{36 - x^2}} \right] dx$$

$$(ii)\ \int \left(\frac{e^{3x} + 1}{e^x} \right)\ dx.$$ [4]

(b) Evaluate $\displaystyle\int \frac{\cot x}{100 - \big[\ \ln(\sin x)\ \big]^2}\ dx$ by using the substitution $\ln(\sin x) = t$. [2]

(c) (i) Evaluate $\displaystyle\int \sin^9 x \cdot \cos^5 x\ \ dx$ by using appropriate substitution.

(ii) Given $\displaystyle\int f(ax + b)\ \ dx = \frac{1}{a}\ F(ax + b) + C$, where F is the antiderivative of f,

evaluate $\displaystyle\int \cos 9x \cdot \sin 5x\ \ dx.$ [4]

6. (a) Use the method of t-substitution to evaluate $\displaystyle\int \frac{1}{2\sin x - \cos x}\ dx.$ [3]

(b) (i) Evaluate $\displaystyle\int \frac{e^{2x} - 1}{e^{2x} + 1}\ dx.$

(ii) Evaluate $\displaystyle\int \frac{1}{\cos^2 x - 2}\ dx.$ [3]

(c) Given $f(x) = \dfrac{x}{(x + 1)(x + 2)}$.

(i) Express $f(x)$ as a sum of partial fractions.

(ii) Hence evaluate $\displaystyle\int f(x)\ dx.$

(iii) Also show that $\displaystyle\int_0^1 f(x)\ dx = \ln\left(\frac{9}{8}\right)$. [4]

7. (a) (i) Evaluate $\displaystyle\int_0^{\pi/4} \tan^7 x \cdot \sec^2 x \;\, dx$ by using the substitution $\tan x = t$.

(ii) Given $\displaystyle\int_2^k \frac{1}{|\,x\,|\sqrt{x^2-4}} \;\, dx = \frac{\pi}{6}$, find the constant k. [3]

(b) Given $\displaystyle\int_0^{\pi} p\,x\,\cos x \; dx = 1$, use the method of integration by parts for definite integrals to find the value of the constant p. [3]

(c) Consider the region R bounded by the curve $f(x) = e^{x/2}$, lines $x = -1$, $x = 0$, and the x-axis.

(i) Find the area of the region R.

(ii) Calculate the volume of the solid of revolution when the region R defined above is rotated around the x-axis. [4]

8. (a) (i) Solve the variable-separable form ordinary differential equation: $\displaystyle\frac{dy}{dx} = \frac{1+\sin x}{1-\cos y}$.

(ii) Solve the initial value problem (IVP): $\displaystyle\frac{dy}{dx} = \frac{y \cdot \tan^2 x}{1+y^2} \quad ; \quad y(0) \;=\; 1.$ [5]

(b) Show that $y = \displaystyle\frac{\sin x + a}{x} - \cos x$ satisfies the first order ordinary differential equation

$x\,\displaystyle\frac{dy}{dx} + y = x \cdot \sin x$, where a is an arbitrary constant. [2]

(c) The differential equation model for the growth of population $(P(t))$ is defined by

$$\frac{dP}{dt} = k\,P, \qquad \text{where } k > 0 \text{ is constant, and } t \text{ is time.} \tag{8.1}$$

(i) Show that the general solution of (8.1) is given by

$$P = P_0 \cdot e^{kt}, \qquad \text{where } P_0 = P(0) \equiv \text{initial population.}$$

(ii) If the population doubles in 20 years, use the result in 8(c)(i) to estimate the population when $t = 50$ years. [3]

Formula Sheet for Foundation Calculus (CELEN037)

- **Differentiation: Useful results**

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \qquad \textbf{and} \qquad \lim_{h\to 0} \cos h = 1$$

$$\lim_{m\to 0} \ (1+m)^{1/m} = e$$

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1 \quad \textbf{and} \quad \lim_{h\to 0} \frac{a^h - 1}{h} = \log_e a$$

If $u = f(x)$, $v = g(x)$, and $w = h(x)$ are differentiable functions of x, then,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(u \cdot v) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

$$\frac{d}{dx}(u \cdot v \cdot w) = u\,v \cdot \frac{dw}{dx} + v\,w \cdot \frac{du}{dx} + u\,w \cdot \frac{dv}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

$$\frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} \qquad \textbf{and} \qquad \frac{dy}{dx} = \frac{1}{\left(\dfrac{dx}{dy}\right)}$$

- **Derivatives of standard functions**

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\,\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\,\mathrm{cosec}^2 x$$

$$\frac{d}{dx}(\mathrm{cosec} x) = -\,\mathrm{cosec}\, x\ \cot x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x\ \log_e a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-\,1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

- **Integration**

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\,\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \mathrm{cosec}^2 x\,dx = -\,\cot x + C$$

$$\int \sec x\ \tan x\,dx = \sec x + C$$

$$\int \mathrm{cosec}\, x\ \cot x\,dx = -\,\mathrm{cosec}\, x + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a > 0)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\ \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{|x|\,\sqrt{x^2-a^2}} = \frac{1}{a}\ \sec^{-1}\left|\frac{x}{a}\right| + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\,\left|x + \sqrt{x^2-a^2}\right| + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\,\left|x + \sqrt{x^2+a^2}\right| + C$$

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ \ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ \ln\left|\frac{x+a}{x-a}\right| + C$$

- **Some useful results**

$$\int [\, f(x)\,]^n \; f'(x)\, dx = \frac{1}{n+1} \; [\, f(x)\,]^{n+1} + C$$

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + C$$

- **Integration by Parts**

If u and v are continuous functions of x, then

$$\int u\, dv = u\, v - \int v\, du$$

- **Numerical Integration**

Trapezium Rule:

$$\int_a^b f(x)\;\, dx \approx \frac{h}{2}\; [f_0 + 2\; (f_1 + f_2 + ... + f_{n-1}) + f_n]$$

Simpson's Rule:

$$\int_a^b f(x)\;\, dx \approx \frac{h}{3}\; [\, f_0 + 4\, f_1 + 2\, f_2 + 4\, f_3 + 2\, f_4 + ... + 2\, f_{n-2} + 4\, f_{n-1} + f_n\,]$$

where $h = \dfrac{b-a}{n}$.

- **Maclaurin's series**

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!}\, f''(0) + \frac{x^3}{3!}\, f'''(0) + ... + \frac{x^n}{n!}\, f^{(n)}(0) + ...$$

- **Trigonometry**

$$\begin{cases} \cos^2\theta + \sin^2\theta = 1 \\ \tan^2\theta + 1 = \sec^2\theta \\ \cot^2\theta + 1 = \operatorname{cosec}^2\theta \end{cases}$$

$$\begin{cases} \sin(A \pm B) = \sin A\cos B \pm \cos A \sin B \\ \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \\ \tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \end{cases}$$

$$\begin{cases} 2\sin A\cos B = \sin(A+B) + \sin(A-B) \\ 2\cos A \sin B = \sin(A+B) - \sin(A-B) \\ 2\cos A\cos B = \cos(A+B) + \cos(A-B) \\ -2\sin A \sin B = \cos(A+B) - \cos(A-B) \end{cases}$$

$$\begin{cases} \sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right) \\ \sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right) \\ \cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right) \\ \cos C - \cos D = -2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right) \end{cases}$$

$$\begin{cases} \sin 2\theta = 2\sin\theta\cos\theta \quad ; \quad \sin\theta = 2\sin\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{\theta}{2}\right) \\ \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \\ \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} \end{cases}$$

$$\begin{cases} \sin^2\theta = \dfrac{1}{2}(1-\cos 2\theta) \quad ; \quad 1 - \cos\theta = 2\sin^2\left(\dfrac{\theta}{2}\right) \\ \cos^2\theta = \dfrac{1}{2}(1+\cos 2\theta) \quad ; \quad 1 + \cos\theta = 2\cos^2\left(\dfrac{\theta}{2}\right) \end{cases}$$

$$\begin{cases} \sin 3\theta = 3\sin\theta - 4\sin^3\theta \\ \cos 3\theta = 4\cos^3\theta - 3\cos\theta \end{cases}$$

$$\begin{cases} \sin\theta = \dfrac{2t}{1+t^2} \\ \cos\theta = \dfrac{1-t^2}{1+t^2} \qquad \text{where } t = \tan\left(\dfrac{\theta}{2}\right) \\ \tan\theta = \dfrac{2t}{1-t^2} \end{cases}$$