CELEN037 Seminar 6

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Topics

- Integrals of the Form $\int \sin^m x \cos^n x \; dx$
- Integrals of the Form $\int \frac{f'(x)}{f(x)} \; dx$
- Integration after Completing the Square in the Denominator
- The Method of t-Substitution
- Integrals of the Form $\int \frac{1}{a\cos^2 x + b\sin^2 x + c} \; dx$

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Notes

(i) If m and n are both odd, let $\cos x = t$ if $m < n$, and $\sin x = t$ if $m > n$

(ii) If m is odd and n is even, let $\cos x = t$

(iii) If m is even and n is odd, let $\sin x = t$

(iv) If m and n are both even, then transform the integrand using:

$$\cos^2 x = \frac{1+\cos 2x}{2} \quad \text{and} \quad \sin^2 x = \frac{1-\cos 2x}{2}$$

Example: Evaluate $\int \sin^3 x \cos^7 x \, dx$

Solution: Both m and n are odd.

$$\begin{aligned} I &= \int \sin^3 x \cos^7 x \, dx \\ &= \int \sin x \sin^2 x \cos^7 x \, dx \\ &= \int \sin x \,(1-\cos^2 x) \cos^7 x \, dx \\ &= \int \cos^7 x \sin x \, dx - \int \cos^9 x \sin x \, dx \end{aligned}$$

Let $\cos x = t \quad \Rightarrow \quad \sin x \, dx = -dt$

$$\begin{aligned} I &= -\int t^7 \, dt + \int t^9 \, dt \\ &= -\frac{t^8}{8} + \frac{t^{10}}{10} + C \\ &= -\frac{\cos^8 x}{8} + \frac{\cos^{10} x}{10} + C \end{aligned}$$

Practice Problems on Worksheet:

1: Q1(iii)

2: Q1(iv)

Answers:

1: $-\dfrac{\cos^3 x}{3} + \dfrac{\cos^5 x}{5} + C$

2: $\dfrac{\sin^3 x}{3} - \dfrac{2\sin^5 x}{5} + \dfrac{\sin^7 x}{7} + C$

Result

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

Example: Evaluate $\displaystyle\int \frac{x^2}{1+x^3}\,dx$

Solution:

Since

$$\frac{d}{dx}\left(1+x^3\right) = 3x^2$$

Hence

$$\int \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int \frac{3x^2}{1+x^3}\,dx$$

$$= \frac{1}{3}\ln\left|1+x^3\right| + C$$

Practice Problems on Worksheet:

1: Q2(vii)

2: Q2(viii)

3: Q2(ix)

4: Q2(x)

Answers:

1: $-\ln|\sec x - \tan x| + C$

2: $\ln|\operatorname{cosec} x - \cot x| + C$

3: $\ln|x| + x + C$

4: $\ln\left(e^x + e^{-x}\right) + C$

Practice Problems on Worksheet (Cont'd):

1: Q2(xi)

2: Q2(xii)

Answers:

1: $\ln\left(e^{x}+e^{-x}\right)+C$

2: $\ln\left|e^{x}-e^{-x}\right|+C$

Useful formulae

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C \qquad \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right|+C$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|+C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x+\sqrt{x^2+a^2}\right|+C$$

Example: Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+4x-5}}\,dx$

Solution:

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+4x-5}}dx &= \int \frac{1}{\sqrt{(x+2)^2-3^2}}\,dx \\ &= \ln\left|x+2+\sqrt{(x+2)^2-3^2}\right|+C\end{aligned}$$

Practice Problems on Worksheet:

1: Q3(iii)

2: Q3(iv)

3: Q3(ix)

4: Q3(x)

Answers:

1: $\dfrac{1}{\sqrt{5}}\tan^{-1}\left(\dfrac{x+2}{\sqrt{5}}\right)+C$

2: $\dfrac{1}{2\sqrt{13}}\ln\left|\dfrac{x+2+\sqrt{13}}{x+2-\sqrt{13}}\right|+C$

3: $\sin^{-1}\left(\dfrac{x-2}{3}\right)+C$

4: $\dfrac{1}{2}\ln\left(2x+1+\sqrt{(2x+1)^2+2}\right)+C$

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For integrals of the form:

$$\int \frac{1}{a + b\cos x + c\sin x}\,dx \quad \text{or} \quad \int \frac{1}{a + b\cos x}\,dx \quad \text{or} \quad \int \frac{1}{a + b\sin x}\,dx$$

t-substitution:

$$\text{Let } \tan\left(\frac{x}{2}\right) = t \;\Rightarrow\; dx = \frac{2\,dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{2t}{1-t^2}$$

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t-substitution:

Let $\tan\left(\frac{x}{2}\right) = t \Rightarrow dx = \dfrac{2dt}{1+t^2}, \quad \sin x = \dfrac{2t}{1+t^2}, \quad \cos x = \dfrac{1-t^2}{1+t^2}, \quad \tan x = \dfrac{2t}{1-t^2}$

Example: Evaluate $I = \displaystyle\int \frac{1}{1+2\cos x}\, dx$

Solution:

Let $\tan\left(\frac{x}{2}\right) = t$. Then

$$dx = \frac{2\ dt}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\Rightarrow \qquad I = \int \frac{1}{1+2\cdot\dfrac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$$

$$= \int \frac{2}{3-t^2}\, dt$$

$$= 2\int \frac{1}{\left(\sqrt{3}\right)^2 - t^2}\, dt$$

$$= 2\cdot\frac{1}{2\sqrt{3}} \ln\left|\frac{t+\sqrt{3}}{t-\sqrt{3}}\right| + C$$

$$= \frac{1}{\sqrt{3}} \ln\left|\frac{\tan\left(\frac{x}{2}\right)+\sqrt{3}}{\tan\left(\frac{x}{2}\right)-\sqrt{3}}\right| + C$$

Practice Problems on Worksheet:

1: Q4(iii)

2: Q4(iv)

3: Q4(v)

4: Q4(vi)

Answers:

1: $\frac{1}{\sqrt{5}} \ln \left| \frac{\tan\left(\frac{x}{2}\right) + \sqrt{5}}{\tan\left(\frac{x}{2}\right) - \sqrt{5}} \right| + C$

2: $\frac{1}{\sqrt{5}} \ln \left| \frac{\tan\left(\frac{x}{2}\right) - \frac{1}{\sqrt{5}}}{\tan\left(\frac{x}{2}\right) + \frac{1}{\sqrt{5}}} \right| + C$

3: $\frac{1}{\sqrt{15}} \ln \left| \frac{\tan\left(\frac{x}{2}\right) + \sqrt{\frac{5}{3}}}{\tan\left(\frac{x}{2}\right) - \sqrt{\frac{5}{3}}} \right| + C$

4: $\frac{2}{\sqrt{3}} \tan^{-1}\left(\sqrt{3}\tan\left(\frac{x}{2}\right)\right) + C$

Steps:

(i) Divide both the numerator and the denominator by $\cos^2 x$ and simplify

(ii) Substitute $\tan x$ by t $(\tan x = t)$, then $\sec^2 x\,dx = dt$

Example: Evaluate $\displaystyle\int \frac{1}{3\sin^2 x + 2}\,dx$

Solution:

$$I = \int \frac{1}{3\sin^2 x + 2}\,dx$$

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{3\sin^2 x + 2}{\cos^2 x}}\,dx$$

$$= \int \frac{\sec^2 x}{3\tan^2 x + 2\sec^2 x}\,dx$$

$$= \int \frac{\sec^2 x}{5\tan^2 x + 2}\,dx$$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\Rightarrow I = \int \frac{1}{5t^2 + 2}\,dt = \int \frac{1}{(\sqrt{5}t)^2 + (\sqrt{2})^2}\,dt$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{5}t}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{10}} \tan^{-1}\left(\frac{\sqrt{5}\tan x}{\sqrt{2}}\right) + C$$

Practice Problems on Worksheet:

1: Q5(v)

2: Q5(vi)

Answers:

1: $\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right) + C$

2: $\frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right) + C$

Office Hours

Day	Time	Venue
Tuesday	12:00 to 14:00	Trent 314a
	13:00 to 14:00	Trent 322
Thursday	16:30 to 17:30	TB 417
	17:00 to 18:00	IAMET 315
Friday	14:00 to 15:00	PB 330
	17:00 to 18:00	TB 417