Stochastic Process Homework 01

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0 Reference

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1 Probability Space of Tossing Coins

1.1 $\{C_s\}_{s\in\{0,1\}^n}$ Forms a Partition

Proof. To prove $\{C_s\}_{s\in\{0,1\}^n}$ forms a partition of Ω, we just need to prove that for every $n \in \mathbb{N}$,

$$\bigcup \{C_s\}_{s\in\{0,1\}^n} = \Omega, \tag{1}$$

$$\forall c_s, c_{s'} \in \{C_s\}_{s\in\{0,1\}^n}, \quad c_s \cap c_{s'} = \varnothing. \tag{2}$$

Let $\mathbf{C}_n = \bigcup \{C_s\}_{s\in\{0,1\}^n}$.

First we prove (1), i.e. to prove

$$\mathbf{C}_n \subset \Omega, \quad \mathbf{C}_n \supset \Omega.$$

- For every $n \in \mathbb{N}$, since $\forall \omega \in \mathbf{C}_n,\ \exists s$ s.t. $\omega \in C_s \Rightarrow \omega \in \Omega$ (by the definition of C_s), we have $\mathbf{C}_n = \bigcup \{C_s\}_{s\in\{0,1\}^n} \subset \Omega$.
- For every $n \in \mathbb{N}$, since $\forall \omega \in \Omega, \exists s = (\omega_1, \omega_2, ...\omega_n)$ s.t. $\omega \in C_s \Rightarrow \omega \in \mathbf{C}_n$, we have $\mathbf{C}_n = \bigcup \{C_s\}_{s\in\{0,1\}^n} \supset \Omega$.

Thus, $\mathbf{C}_n = \bigcup \{C_s\}_{s\in\{0,1\}^n} = \Omega$.

Now we prove (2) by contradiction.

Assume exist s, s' s.t. $s \neq s', c_s \cap c_{s'} \neq \varnothing$, i.e. exists ω s.t. $\omega \in c_s, \omega \in c_{s'}$.

By the definition of $C_s, C_{s'}$, we have

$$\omega_1 = s_1 = s'_1, \omega_2 = s_2 = s'_2, ..., \omega_n = s_n = s'_n$$

i.e. $s = s'$. ***Contradiction***.

Thus, $\forall c_s, c_{s'} \in \{C_s\}_{s\in\{0,1\}^n}\,,\ c_s \cap c_{s'} = \varnothing$.

In conclusion, for every $n \in \mathbb{N}, \{C_s\}_{s\in\{0,1\}^n}$ forms a partition of Ω. ∎

1.2 Exists a Bijection Between $\mathcal{F}_n$ and $2^{\{0,1\}^n}$

Proof. We can construct an injective

$$f : \mathcal{F}_n \to 2^{\{0,1\}^n}.$$

$$S \mapsto S',$$

$$S' = \left\{ s \subset \{0,1\}^n \,\middle|\, \exists \omega \in S \text{ s.t. } \omega \in C_s \right\}$$

Meanwhile, we can also construct an injective

$$h : 2^{\{0,1\}^n} \to \mathcal{F}_n.$$

$$S' \mapsto S,$$

$$S = \left\{ \omega \in \Omega \,\middle|\, \exists s \in S' \text{ s.t. } \omega \in C_s \right\}.$$

By Cantor-Bernstein-Schroeder Theorem, there exists a bijection between $\mathcal{F}_n$ and $2^{\{0,1\}^n}$. ∎

1.3 $\{\mathcal{F}_n\}$ is Increasing

Proof. To prove $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq ...$ we just need to prove that

$$\forall n \in \mathbb{N},\ \mathcal{F}_n \subsetneq \mathcal{F}_{n+1}.$$

For every $S \in \mathcal{F}_n$, $\forall \omega \in S,\ \exists s \in \{0,1\}^n$ s.t. $\omega \in C_s$.

Let $s^{(0)} = (s,0) \in \{0,1\}^{(n+1)}$ and $s^{(1)} = (s,1) \in \{0,1\}^{(n+1)}$. We know either $\omega \in C_{s^{(0)}}$ or $\omega \in C_{s^{(1)}}$, i.e. $\omega \in C_{s^{(0)}} \cap C_{s^{(1)}}$.

Thus, for every $S \in \mathcal{F}_n$, $S \in \mathcal{F}_{n+1}$, i.e. $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

Now we prove $\mathcal{F}_n \neq \mathcal{F}_{n+1}$.

There exists $\hat{S} = \{(0,0,...,0,0)\} \subset \{0,1\}^{(n+1)}$, i.e. $C_{\hat{S}} \in \mathcal{F}_{n+1}$, while $C_{\hat{S}} \notin \mathcal{F}_n$.

Therefore,

$$\forall n \in \mathbb{N},\ \mathcal{F}_n \subsetneq \mathcal{F}_{n+1}.$$

i.e.

$$\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq ... \subsetneq \mathcal{F}_n \subsetneq \mathcal{F}_{n+1} \subsetneq ...$$ ∎

1.4 $\mathcal{F}_\infty$ is an Algebra While $\mathcal{F}_\infty \neq 2^\Omega$

Proof. First we prove $\mathcal{F}_\infty$ is an algebra.

For every $A \in \mathcal{F}_\infty = \bigcup_{n\geq 1} \mathcal{F}_n,\ \exists\ n \in \mathbb{N}, A \in \mathcal{F}_n$. Thus, $A^C = \Omega \backslash A \in \mathcal{F}_n \in \mathcal{F}_\infty$.

For every $A, B \in \mathcal{F}_\infty,\ \exists\ m, n \in \mathbb{N}$ s.t. $A \in \mathcal{F}_m, B \in \mathcal{F}_n$. Let $M = \max\{m, n\}$. From **1.3** we know $\mathcal{F}_m \subset \mathcal{F}_M$, $\mathcal{F}_n \subset \mathcal{F}_M$. Thus, $A \in \mathcal{F}_M, B \in \mathcal{F}_M$.

Since $\mathcal{F}_M$ is a σ-algebra (by its definition), we have $A \cup B \in \mathcal{F}_M \subset \mathcal{F}_\infty$.

In a nutshell, $\mathcal{F}_\infty$ is an algebra. ■

Now we prove $\mathcal{F}_\infty \neq 2^\Omega$.

Let $\mathbf{o} = (0, 0, ...) \in \{0, 1\}^{\mathbb{N}}$, $\mathbf{e} = (1, 1, ...) \in \{0, 1\}^{\mathbb{N}}$. There exists $\{\mathbf{o}, \mathbf{e}\} \subset \Omega$, i.e. $\{\mathbf{o}, \mathbf{e}\} \in 2^\Omega$, while $\forall n \in \mathbb{N}, \{\mathbf{o}, \mathbf{e}\} \notin \mathcal{F}_n$, i.e. $\{\mathbf{o}, \mathbf{e}\} \notin \mathcal{F}_\infty$.

Thus, $\mathcal{F}_\infty \neq 2^\Omega$. ■

1.5 $\{\omega\} \in \mathcal{B}(\Omega) \setminus \mathcal{F}_\infty$ for All $\omega \in \Omega$

Proof. First we prove $\forall \omega \in \Omega, \{\omega\} \in \mathcal{B}(\Omega)$ by contradiction.

Assume $\exists \omega \in \Omega, \{\omega\} \notin \mathcal{B}(\Omega)$.

There must exist $S \in \mathcal{B}(\Omega)$ s.t. $\omega \in S$ and $S \setminus \{\omega\} \neq \varnothing$. (For example, $C_{\{\omega_1\}}$ is a feasible S.) We can find a "smallest" set $\tilde{S}$ with such properties, i.e.

$$\forall S \in \mathcal{B}(\Omega) \text{ s.t. } \omega \in S, S \setminus \{\omega\} \neq \varnothing, \text{we have } \tilde{S} \subset S.$$

Obviously, $\forall \varphi \in \tilde{S} \setminus \{\omega\}$, we can find an $n \in \mathbb{N}$ s.t. $\omega_i = \varphi_i\ (i = 1, 2, ..., n-1)$ and $\omega_n \neq \varphi_n$. Then exists $C_{(\omega_1, \omega_2 ... \omega_n)} \in \mathcal{B}(\Omega)$ s.t. $\omega \in S, S \setminus \{\omega\} \neq \varnothing$.

Meanwhile, $\varphi \notin C_{(\omega_1, \omega_2 ... \omega_n)}$ while $\varphi \in \tilde{S}$. This yields $C_{(\omega_1, \omega_2 ... \omega_n)} \not\subset \tilde{S}$.

Contradiction to the definition of $\tilde{S}$.

Thus, $\forall \omega \in \Omega, \{\omega\} \in \mathcal{B}(\Omega)$.

Now we prove that $\forall \omega \in \Omega, \{\omega\} \notin \mathcal{F}_\infty$ by contradiction.

Assume $\exists \omega \in \Omega, \{\omega\} \in \mathcal{F}_\infty$. Then exists $n \in \mathbb{N}$ s.t. $\{\omega\} \in \mathcal{F}_n$.

Thus, exists $s_1, s_2, ...s_k \in \{0, 1\}^n$ s.t. $\bigcup_{i=1}^k C_{s_i} = \{\omega\}$.

Meanwhile, $\forall s_i$ s.t. $\omega \in s_i \in \{0, 1\}^n$, we have $\varphi = (\omega_1, \omega_2, ...\omega_n, 1 - \omega_{n+1}, \omega_{n+2}, ...) \in C_{s_i}$. Obvious $\varphi \neq \omega$. Thus, $\forall s_i \in \{0, 1\}^n, C_{s_i} \neq \{\omega\}$, i.e. $\bigcup_{i=1}^k C_{s_i} \neq \{\omega\}$.***Contradiction.***

Thus, $\forall \omega \in \Omega, \{\omega\} \notin \mathcal{F}_\infty$.

In conclusion, $\forall \omega \in \Omega, \omega \in \mathcal{B}(\Omega), \omega \notin \mathcal{F}_\infty \Longleftrightarrow \forall \omega \in \Omega, \{\omega\} \in \mathcal{B}(\Omega) \setminus \mathcal{F}_\infty$. ■

1.6 $\forall \boldsymbol{A} \in \boldsymbol{\mathcal{F}_\infty}, \exists\ \boldsymbol{n} \in \mathbb{N}, \boldsymbol{s_1, ..., s_k} \in \{\mathbf{0,1}\}^{\boldsymbol{n}}, \boldsymbol{A} = \bigcup_{\boldsymbol{i=1}}^{\boldsymbol{k}} \boldsymbol{C_{s_i}}$ with Unique $\frac{\boldsymbol{k}}{\mathbf{2}^{\boldsymbol{n}}}$

Proof. For every $A \in \mathcal{F}_\infty$, by the definition of $A \in \mathcal{F}_\infty$, we know $\exists\ n \in \mathbb{N}$ s.t. $A \in \mathcal{F}_n$.

Now we prove $\exists s_i \in \{0,1\}^n\ (i = 1, 2, ..., k), A = \bigcup_{i=1}^k C_{s_i}$ by contradiction.

Assume $\nexists\ s_i \in \{0,1\}^n\ (i = 1, 2, ..., k)$ s.t. $A = \bigcup_{i=1}^k C_{s_i}$.

Let $\mathbf{S}_S = \begin{cases} \bigcup_{s\in S} C_s, & S \neq \varnothing \\ \varnothing, & S = \varnothing \end{cases}$.

Since $\mathbf{S}_{\{0,1\}^n} = \bigcup_{s\in\{0,1\}^n} C_s = \Omega$, the only possible case is that $\forall S \subset \{0,1\}^n, S \neq \varnothing$, either $A^C \cap \mathbf{S}_s = \left\{\omega \in \mathbf{S}_S \middle|\ \omega \notin A\right\} \neq \varnothing$ or $A \cap \mathbf{S}_s^C = \left\{\omega \in A \middle|\ \omega \notin \mathbf{S}_S\right\} \neq \varnothing$.

By the definition of $\mathcal{F}_n$, we know $\mathcal{F}_n$ is a σ-algebra.

Therefore,

$$A^C \in \mathcal{F}_n.$$

$$\forall S \subset \{0,1\}^n,\ A^C \cup \mathbf{S}_S \in \mathcal{F}_n,$$

$$(A^C \cup \mathbf{S}_S)^C = A \cap \mathbf{S}_S^C \in \mathcal{F}_n.$$

Obviously, $\forall S' \subset \{0,1\}^n$ i.e. $S' \in 2^{\{0,1\}^n}$, $A \cap \mathbf{S}_S^C \neq \mathbf{S}_{S'}$. Otherwise, either $A \cap \mathbf{S}_S^C = \varnothing$, or $\exists S^\star = S \cup S'$ s.t. $A = \bigcup_{s\in S^\star} C_s$. ***Contracition.*** (Note that $S^\star$ is a finite set.)

Thus, $A \cap \mathbf{S}_S^C \in \mathcal{F}_n$ while $A \cap \mathbf{S}_S^C \notin \left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\}$.

Considering

$$\varnothing \in \left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\}, \Omega = \mathbf{S}_{\{0,1\}^n} \in \left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\},$$

$$\forall\ \mathbf{S}_X \in \left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\}, \mathbf{S}_X^C = \mathbf{S}_{\{0,1\}^n \setminus X} \in \left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\},$$

$$\forall\ \mathbf{S}_X, \mathbf{S}_Y \in \left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\}, \mathbf{S}_X \cup \mathbf{S}_Y = \mathbf{S}_{X\cup Y} \in \left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\},$$

we have

$$\left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\} \text{ is a } \sigma\text{−algebra}.$$

Meanwhile, it is obvious that $\left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\}$ contains sets in $\{C_s\}_{s\in\{0,1\}^n}$.

Since $\mathcal{F}_n$ is the minimal σ-algebra containing sets in $\{C_s\}_{s\in\{0,1\}^n}, \left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\} \subset \mathcal{F}_n$. Nevertheless, $A \cap \mathbf{S}_S^C \in \mathcal{F}_n$, $A \cap \mathbf{S}_S^C \notin \left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\} \Rightarrow \left\{\mathbf{S}_S \middle| S \subset \{0,1\}^n\right\} \not\subset \mathcal{F}_n$. ***Contradiction.***

Thus, for every $A \in \mathcal{F}_\infty, \exists\ n \in \mathbb{N}, \exists\ s_i \in \{0,1\}^n\ (i = 1, 2, ..., k)$ s.t. $\bigcup_{i=1}^k C_{s_i} = A$. ■

Now we prove for a fixed $A, \forall n \in \mathbb{N}, s_i \in \{0,1\}^n\ (i = 1, 2, ..., k)$ s.t. $A = \bigcup_{i=1}^k$, $k/2^n$ is unique.

From the analyses above, we can find a smallest $N_m \in \mathbb{N}$ s.t. $\exists s_i \in \{0,1\}^{N_m}\ (i = 1, 2, ..., n)$ s.t. $\bigcup_{i=1}^K s_i = A$.

We prove

$$\forall n \geq N_m, \exists s_i \in \{0,1\}^n\ (i = 1, 2, ..., n) \text{ s.t. } \bigcup_{i=1}^{k_n} s_i = A, \text{where } k_n/2^n = K/2^{N_m}$$

by induction.

BASE STEP. When $n = N_m$. Obvious.

INDUCTIVE HYPOTHESIS.

Assume when $n = N, \exists s_i \in \{0,1\}^n \; (i = 1, 2, ..., n)$ s.t. $\bigcup_{i=1}^{k_n} s_i = A$. Also, $k_n/2^n = K/2^{N_m}$.

INDUCTIVE STEP.

From **1.3** we know, $\mathcal{F}_n \subsetneq \mathcal{F}_{n+1}$. Thus, exist

$$s_i' = \begin{cases} (s_t, 0), & i = 2t-1, \\ (s_t, 1), & i = 2t \end{cases}.$$

s.t. $\bigcup_{i=1}^{k_{N+1}} s_i' = A$.

Obvious $k_{N+1} = 2k_N$. Thus,

$$\frac{k_{N+1}}{2^{N+1}} = \frac{2k_N}{2^{N+1}} = \frac{k_N}{2^N} = \frac{K}{2^{N_m}}.$$

Thus, for a fixed A, $\forall n \in \mathbb{N}, s_i \in \{0,1\}^n \; (i = 1, 2, ..., k)$ s.t. $A = \bigcup_{i=1}^{k}$, the value $k/2^n$ is unique.

Since the selection of n and k is related to A, we know $\frac{k}{2^n}$ depends on A. Considering for a fixed $A, k/2^n$ is unique, we know the value of $\frac{k}{2^n}$ ***only*** depends on A. ■

1.7 The Validity and Uniqueness of Probability Measure *P*

Proof. First we prove that P is a probability measure.

- For $\varnothing$, since $k = 0$ in this case, we have $P(\varnothing) = 0$.
- For Ω, set $n = 1, k = 2, s_1 = C_{(0)}, s_2 = C_{(1)}$. Thus, we have $P(\Omega) = 1$.
- For any $A \in \mathcal{F}_\infty$, we can find $n \in \mathbb{N}$ and $s_i \in \{0,1\}^n \; (i = 1, 2, ..., k)$ s.t. $\bigcup_{i=1}^{k} C_{s_i} = A$.

  Code sequences in $\{0,1\}^n \setminus \bigcup_{i=1}^{k} s_i$ as $s_1', ... s_{k'}'$.

  Since $(\bigcup_{i=1}^{k} C_{s_i}) \cup (\bigcup_{i=1}^{k'} C_{s_i'}) = \bigcup_{s \in \{0,1\}^n} C_s = \Omega$ and $\bigcup_{i=1}^{k} C_{s_i} \cap \bigcup_{i=1}^{k'} C_{s_i'} = \varnothing$,

  we know $A^C = \bigcup_{i=1}^{k'} C_{s_i'}$. Then $P(A^C) = k'/2^n$.

  Obvious $k' + k = \left| \{0,1\}^n \right| = 2^n$. Thus,

  $$P(A^C) = \frac{k'}{2^n} = \frac{2^n - k'}{2^n} = 1 - \frac{k}{2^n} = 1 - P(A).$$

- For a countable index set I, assume $A_i = \bigcup_{l=1}^{K_i} C_{s_{i,l}}$, where $s_{i,l} \in \{0,1\}^n$.

  When $\forall i \in I, A_i \in \mathcal{F}_\infty$ s.t. $\forall i \neq j \in I, A_i \cap A_j = \varnothing \Rightarrow (\bigcup_{l=1}^{K_i} C_{s_{i,l}}) \cap (\bigcup_{l=1}^{K_j} C_{s_{j,l}}) = \varnothing \Rightarrow$

  $$\exists S_{i,j} = \left\{ s_1', ..., s_{K_i+K_j}' \right\}, \; s_l' = \begin{cases} s_{i,l}, & l \leq K_i \\ s_{j,l-K_i}, & l > K_i \end{cases},$$

  $$|S_{i,j}| = K_i + K_j, \text{ s.t. } \bigcup_{s \in S_{i,j}} C_s = \bigcup_{l=1}^{K_i+K_j} C_{s_{i,l}} = A_i \cup A_j.$$

By induction, we know

$$\exists S = \left\{ s \in \{0,1\}^n \;\middle|\; \exists i \in I, 1 \leq l \leq K_i, s_{i,l} = s \right\}, |S| = \sum_{i \in I} K_i, \text{ s.t. } \bigcup_{i \in I} A_i = \bigcup_{s \in S} C_s.$$

Note S is a countable set.

Thus,

$$P(\bigcup_{i \in I} A_i) = \frac{|S|}{2^n} = \frac{\sum_{i \in I} K_i}{2^n} = \sum_{i \in I} \frac{K_i}{2^n} = \sum_{i \in I} P(A_i).$$

Therefore, P is a probability measure. ■

Now we prove P is the unique probability measure from $\mathcal{B}(\Omega)$ to $[0,1]$ by contradiction.

Consider measurable space $(\Omega, \mathcal{B}(\Omega))$ and measure P.

- Here we still use the notation in **1.1**.

  From **1.1** we know

  $$\exists A_i = \{C_s\}_{s \in \{0,1\}^n} \in \mathcal{B}(\Omega) \text{ s.t. } |P(A_i)| \leq 1 < \infty, \bigcup_{n \in \mathbb{N}} A_n = \Omega.$$

- Obvious $\{A_n\}_{n \in \mathbb{N}}$ satisfies $\forall i, j \in \mathbb{N}, i \neq j, A_i \cap A_j = \varnothing$. Thus, Ω can be covered with at most countably many measurable disjoint sets with finite measure.

- From **1.1**, **1.2**, and **1.3** we know

  $$\exists \mathcal{F}_1, \mathcal{F}_2, ... \in \mathcal{B}(\Omega) \text{ s.t. } \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq ... \mathcal{F}_n \subsetneq \mathcal{F}_{n+1} \subsetneq ..., \bigcup_{n \in \mathbb{N}} \mathcal{F}_n = \bigcup_{n \in \mathbb{N}} A_n = \Omega.$$

- Exists a strictly positive measurable

  $$f(\omega) = 1 \text{ for } \omega \in \Omega \text{ s.t. } \int_\Omega f(\omega) P(\mathrm{d}\omega) = 1 < \infty.$$

Thus, the measure space $(\Omega, \mathcal{B}(\Omega), P)$ is σ-finite.

By Carathéodory's Extension Theorem, we know P is unique on $(\Omega, \mathcal{B})$. ■

1.8 Formalization of Distribution $\texttt{Geom}(1/2)$ in $(\Omega, \mathcal{B}(\Omega), P)$

Solution. We can formalize Geometric Distribution in $(\Omega, \mathcal{B}(\Omega), P)$ as follows.

$$X : \Omega \to \mathbb{N}$$

$$\omega = (\omega_1, \omega_2, , ...) \mapsto n, \text{ where } \omega_n = 1 \text{ and } \omega_i = 0, \ i < n.$$

X given above is a formalization of $\texttt{Geom}(1/2)$ in the probability space $(\Omega, \mathcal{B}(\Omega), P)$. ■

2 Conditional Expectation

2.1 $f(X)$ is $\sigma(X)$-Measurable

Proof. By the definition of $\sigma(X)$, we know X is $\sigma(X)$-measurable, i.e. exists a Borel set $\mathcal{F}$ on $\mathbb{R}$ s.t.

$$X : (\Omega, \sigma(X)) \to (\mathbb{R}, \mathcal{F}); \quad \forall A \in \mathcal{F},\ X^{-1}(A) \in \sigma(X). \tag{3}$$

Now we consider $f(X) : (\Omega, \sigma(X)) \to (\mathbb{R}, \mathcal{F})$.

For every $B \in \mathcal{F}$, since f is a measurable function, $f^{-1}(B) \in \mathcal{F}$. Moreover, from (3) we know $X^{-1}(f^{-1}(B)) \in \sigma(X)$.

Thus, $\forall B \in \mathcal{F},\ X^{-1}(f^{-1}(B)) \in \sigma(X)$, i.e. $f(X) : (\Omega, \sigma(X)) \to (\mathbb{R}, \mathcal{F})$ is $\sigma(X)$-measurable. ∎

2.2 $\mathbf{E}[X|Y] = \mathbf{E}[X|Y']$

Proof. Let $f_Y(X) = \mathbf{E}[X|Y], f_{Y'}(X) = \mathbf{E}[X|Y']$.

By the definition of random variables, we know both $Y^{-1}(Y(\omega))$ and $(Y')^{-1}(Y'(\omega))$ exist.

Now we prove $\forall \omega \in \Omega,\ Y^{-1}(Y(\omega)) = (Y')^{-1}(Y'(\omega))$ by contradiction.

Let $A_Y(\omega) = Y^{-1}(Y(\omega)), A_{Y'}(\omega) = (Y')^{-1}(Y'(\omega))$. Assume $\exists \omega^\star$ s.t. $A_Y(\omega^\star) \neq A_{Y'}(\omega^\star)$.

Obvious $A_Y(\omega^\star) \in \sigma(Y), A_{Y'}(\omega^\star) \in \sigma(Y');\ \omega^\star \in A_Y(\omega^\star), \omega^\star \in A_{Y'}(\omega^\star)$. Since $\sigma(Y) = \sigma(Y')$, we know $A_Y^C(\omega^\star), A_{Y'}^C(\omega^\star) \in \sigma(Y), A_Y^C(\omega^\star) \cup A_{Y'}^C(\omega^\star) \in \sigma(Y)$, i.e. $A_Y(\omega^\star) \cap A_{Y'}(\omega^\star) = (A_Y^C(\omega^\star) \cup A_{Y'}^C(\omega^\star))^C \in \sigma(Y)$. Considering $A_Y(\omega^\star) \cap A_{Y'}(\omega^\star) \subsetneq A_Y(\omega)$, we know $\sigma(Y)$ is not the minimal σ-algebra on Y. ***Contradiction*** to the definition of $\sigma(Y)$.

Thus, $\forall \omega \in \Omega, Y^{-1}(Y(\omega)) = (Y')^{-1}(Y'(\omega))$.

Therefore,

$$\begin{aligned} \forall \omega \in \Omega,\ f_Y(\omega) &= \mathbf{E}[X|Y = Y(\omega)] = \mathbf{E}\left[X|Y^{-1}(Y(\omega))\right] = \mathbf{E}\left[(Y')^{-1}(Y'(\omega))\right] \\ &= \mathbf{E}[X|Y' = Y'(\omega)] = f_{Y'}(\omega). \end{aligned}$$

i.e.

$$\mathbf{E}[X|Y] = \mathbf{E}[X|Y'].$$ ∎

2.3 Definition of $\mathbf{E}[X|\mathcal{F}]$

Solution. For any given $\mathcal{F}$, we can find $\{A_i\}_{i \in I}$ s.t.

* $\forall i \in I, A_i \in \mathcal{F}$;
* $\forall s \in \mathcal{F}$, either $A_i \cap s = \varnothing$ or $A_i \cap s = A_i$;
* $\forall i, j \in I, A_i \cap A_j = \varnothing$.

Obvious, for every ω, exists exactly one i s.t. $\omega \in A_i$. Let $A_{\mathcal{F}}(\omega) = A_i$ s.t. $\omega \in A_i$.

We define $\mathbf{E}[X|\mathcal{F}]$ as follows.

$$\mathbf{E}[X|\mathcal{F}] : \Omega \to \mathbb{R},\ \mathbf{E}[X|\mathcal{F}](\omega) = \mathbf{E}[X|A_{\mathcal{F}}(\omega)]. \qquad \blacksquare$$

2.4 $\mathbf{E}\big[\ \mathbf{E}[\boldsymbol{X}|\boldsymbol{\mathcal{F}_1}] \mid \boldsymbol{\mathcal{F}_2}\ \big] = \mathbf{E}\big[\ \mathbf{E}[\boldsymbol{X}|\boldsymbol{\mathcal{F}_2}] \mid \boldsymbol{\mathcal{F}_1}\ \big] = \mathbf{E}[\boldsymbol{X}|\boldsymbol{\mathcal{F}_1}]$

Proof. Let $\mathbf{E}\big[\ \mathbf{E}[X|\mathcal{F}_1] \mid \mathcal{F}_2\ \big] = f_1,\ \mathbf{E}\big[\ \mathbf{E}[X|\mathcal{F}_2] \mid \mathcal{F}_1\ \big] = f_2,\ \mathbf{E}[X|\mathcal{F}_1] = g.\ f_1, f_2, g : \Omega \to \mathbb{R}$.

Let $\mathscr{A}_i = \{A_{\mathcal{F}_i}(\omega) | \forall \omega \in \Omega\}, i = 1, 2$.

First we prove $f_1 = g$, i.e. $\mathbf{E}\big[\ \mathbf{E}[X|\mathcal{F}_1] \mid \mathcal{F}_2\ \big] = \mathbf{E}[X|\mathcal{F}_1]$

For any $\omega \in \Omega$,

since $\mathcal{F}_1 \subset \mathcal{F}_2$, exists exactly one $A_{\mathcal{F}_1}(\omega) \in \mathscr{A}_1$ s.t. $A_{\mathcal{F}_1}(\omega) \supset A_{\mathcal{F}_2}(\omega)$, i.e.

$$\mathbf{Pr}\big[A_{\mathcal{F}_1}(\omega)|A_{\mathcal{F}_2}(\omega)\big] = 1; \qquad \forall S' \in \mathscr{A}_1 \setminus \{A_{\mathcal{F}_1}(\omega)\}, \mathbf{Pr}\big[S'|A_{\mathcal{F}_2}(\omega)\big] = 0.$$

Thus,

$$\begin{aligned}
f_1(\omega) &= \mathbf{E}\big[\ \mathbf{E}[X|\mathcal{F}_1] \mid A_{\mathcal{F}_2}(\omega)\ \big] \\
&= \sum_{x \in \mathbb{R}} x \cdot \mathbf{Pr}\big[\ \mathbf{E}[X|\mathcal{F}_1] = x \mid A_{\mathcal{F}_2}(\omega)\ \big] \\
&= \sum_{S \in \mathscr{A}_1} \mathbf{E}[X|S]\, \mathbf{Pr}\big[S|A_{\mathcal{F}_2}(\omega)\big] \\
&= \mathbf{E}[X|A_{\mathcal{F}_1}(\omega)] \\
&= g(\omega).
\end{aligned}$$

Now we prove $f_2 = g$, i.e. $\mathbf{E}\big[\ \mathbf{E}[X|\mathcal{F}_2] \mid \mathcal{F}_1\ \big] = \mathbf{E}[X|\mathcal{F}_1]$.

For any $\omega \in \Omega$,

since $\mathcal{F}_1 \subset \mathcal{F}_2$, we know exist $\boldsymbol{\mathcal{A}}(\omega) \triangleq \{A_{\mathcal{F}_2}(\omega')\}_{\omega' \in A_{\mathcal{F}_1}(\omega)}$ s.t.

$$\bigcup_{S \in \boldsymbol{\mathcal{A}}(\omega)} S = A_{\mathcal{F}_1}(\omega); \qquad \forall S, T \in \boldsymbol{\mathcal{A}}(\omega), S \cap T = \varnothing.$$

i.e.

$$\sum_{S \in \boldsymbol{\mathcal{A}}(\omega)} \mathbf{Pr}\big[S|A_{\mathcal{F}_1}(\omega)\big] = 1;$$

$$\forall S \in \boldsymbol{\mathcal{A}}(\omega), \mathbf{Pr}\big[S|A_{\mathcal{F}_1}(\omega)\big] > 0;$$

$$\forall S \in \mathscr{A}_2 \setminus \boldsymbol{\mathcal{A}}(\omega), \mathbf{Pr}\big[S|A_{\mathcal{F}_1}(\omega)\big] = 0.$$

Thus,

$$\begin{aligned}
f_2(\omega) &= \mathbf{E}\big[\ \mathbf{E}[X|\mathcal{F}_2] \mid A_{\mathcal{F}_1}(\omega)\ \big] \\
&= \sum_{x \in \mathbb{R}} x \cdot \mathbf{Pr}\big[\ \mathbf{E}[X|\mathcal{F}_2] = x \mid A_{\mathcal{F}_1}(\omega)\ \big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{S \in \mathscr{A}_2} \mathbf{E}\left[X|S\right] \mathbf{Pr}\big[S|A_{\mathcal{F}_1}(\omega)\big] \\
&= \sum_{S \in \boldsymbol{\mathcal{A}}(\omega)} \mathbf{E}\left[X|S\right] \mathbf{Pr}\big[S|A_{\mathcal{F}_1}(\omega)\big] \\
&= \mathbf{E}\left[X|A_{\mathcal{F}_1}(\omega)\right] \\
&= g(\omega).
\end{aligned}
$$

In conclusion, $f_1 = g = f_2$, i.e.

$$
\mathbf{E}\big[\ \mathbf{E}\left[X|\mathcal{F}_1\right] \mid \mathcal{F}_2\ \big] = \mathbf{E}\big[\ \mathbf{E}\left[X|\mathcal{F}_2\right] \mid \mathcal{F}_1\ \big] = \mathbf{E}\left[X|\mathcal{F}_1\right]. \qquad \blacksquare
$$