Exercise Sheet 3

Discrete Mathematics, 2020.9.25

1. a) Prove that $p \to (q \to p)$ is a tautology.
   b) Prove that $(p \to q \to r) \to (p \to q) \to (p \to r)$ is a tautology.
   c) Prove that $p \to q \to r \equiv (p \wedge q) \to r$.
   Here, $\to$ is right associative, i.e. $\phi \to \psi \to \chi$ means $\phi \to (\psi \to \chi)$.

2. (P35, Ex.22-24, [R])
   a) Show that $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ are logically equivalent.
   b) Show that $(p \to r) \wedge (q \to r)$ and $(p \vee q) \to r$ are logically equivalent.
   c) Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent.

3. (P35, Ex.31, [R]) Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

4. (P35, Ex.32, [R]) Show that $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$ are not logically equivalent.

5. Consider the compound proposition $\phi = p \to (q \oplus r)$ where p, q, r are propositional variables.

   (a) Find a compound proposition ψ in CNF such that $\phi \equiv \psi$.

   (b) Use the algorithm that we learned in class to construct a compound proposition ψ in CNF such that ϕ is satisfiable if and only if ψ is satisfiable.

6. (P36, Ex.52, [R]) In this exercise we will show that $\{|\}$ is a functionally complete collection of logical operators. (**Note**: $p|q$ means p NAND q. The proposition p NAND q is true when either p or q, or both, are false; and it is false when both p and q are true. The operators $|$ is called the Sheffer stroke after H. M. Sheffer)

   a) Show that $\phi|\phi$ is logically equivalent to $\neg\phi$.

   b) Show that $(\phi|\psi)|(\phi|\psi)$ is logically equivalent to $\phi \wedge \psi$.

   c) Show that $\{|\}$ is a functionally complete collection of logical operators based on the results above.

7. Consider potential process of determining whether

   $$(\neg p_1 \vee p_2) \wedge (\neg p_1 \vee p_3 \vee p_5) \wedge (\neg p_2 \vee p_4) \wedge (\neg p_3 \vee \neg p_4) \wedge (p_1 \vee p_5 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (p_2 \vee \neg p_3) \wedge (p_6 \vee \neg p_5)$$

   is SAT or UNSAT.

   a) Calculus UnitPro($\mathcal{J}_1$) where $\mathcal{J}_1 = [p_1 \mapsto \mathbf{T}, p_3 \mapsto \mathbf{F}]$.

   b) Calculus UnitPro($\mathcal{J}_2$) where $\mathcal{J}_2 = [p_3 \mapsto \mathbf{F}]$.