Discrete Mathematics Exercise 14

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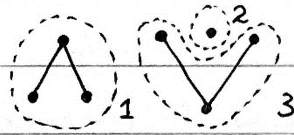
1. ***Solution:***

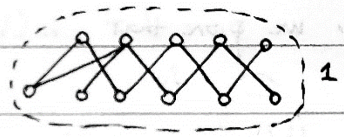
Path: **a), b)**.
Simple path: **a).**
Circuits: **b)**.
The length of Path a) is **4** and the length of path b) is **4**.

2. ***Solution:***

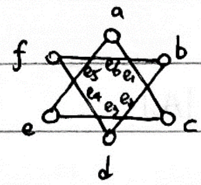
3) There are 3 connected components.



4) There is only 1 connected component.



5) There are 2 connected components: (V_1, E_1), (V_2, E_2),



in which $V_1 = \{a, c, e\}$, $E_1 = \{e_1, e_3, e_5\}$,
$V_2 = \{b, d, f\}$, $E_2 = \{e_2, e_4, e_6\}$.

3. ***Solution:***

Strongly connected components of graph **a)** are G_1, G_2, G_3, in which G_1, G_2, G_3 are the subgraphs induced by $\{a, b, e\}, \{c\}, \{d\}$ respectively.

Strongly connected components of graph **b)** are G_1, G_2, G_3, G_4, in which G_1, G_2, G_3, G_4 are the subgraphs induced by $\{c, d, e\}, \{a\}, \{b\}, \{f\}$ respectively.

Strongly connected components of graph **c)** are G_1, G_2, in which G_1, G_2 are the subgraphs induced by $\{a, b, c, d, f, g, h, i\}, \{e\}$ respectively.

4. ***Proof:***

First, we prove the "only if" part.

When the simple path G is bipartite with a bipartition (V_1, V_2), a path starting from a vertex in V_1 with an odd-number length ends at a vertex in V_2, and a path starting from a vertex in V_2 with an odd-number length ends at a vertex in V_1.

Since a circuit is firstly a path and it starts from and ends at the same vertex, G has no circuits with an odd number of edges.

Then we prove the "if" part. Let $G = \{V, E\}$.

For $u, v \in V$ and u is connected to v, let $d(u, v) =$ the least length of the path from u to v. Since there are no circuits in G with an odd number of edges, **i.e.** the length of all circuits in G is even. Therefore, if $d(u, v)$ is even, all paths from u to v is even. The same works for the case when $d(u, v)$ is odd.

Let $V_0^* = V$.

Pick $u_0 \in V_0^*$.

Let $V_{10} = \{v \in V_0^* \mid u_0 \text{ and } v \text{ is connected and } d(u_0, v) \text{ is odd}\}$.

Let $V_{20} = \{v \in V_0^* \mid u_0 \text{ and } v \text{ is connected and } d(u_0, v) \text{ is even}\}$.

Let $V_1^* = V_0^* \backslash (V_{10} \cup V_{20})$.

Pick $u_1 \in V_1^*$.

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Pick $u_n \in V_n^*$.

Let $V_{1n} = \{v \in V_n^* \mid u_n \text{ and } v \text{ is connected and } d(u_n, v) \text{ is odd}\}$.

Let $V_{2n} = \{v \in V_n^* \mid u_n \text{ and } v \text{ is connected and } d(u_n, v) \text{ is even}\}$.

Let $V_{n+1}^* = V_n^* \backslash (V_{1n} \cup V_{2n})$.

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Let $V_1 = \bigcup_{i=0}^{\infty} V_{1i}, V_2 = \bigcup_{i=0}^{\infty} V_{2i}$.

Now we prove (V_1, V_2) is a bipartition by contradiction.

Assume there exists $u, v \in V_1$ **s.t.** u is adjacent to v. Thus, u is connected to v. We can find $w \in \bigcup_{i=0}^{\infty}\{u_i\}$ **s.t.** w is connected to u, **i.e.** w is connected to v. From the definition of V_1, we know $d(w, u)$ and $d(w, v)$ is odd, **i.e.** all paths from w to v is odd. On the other hand, exists a path $w, x_1, x_2, \dots x_m, u$, whose length is odd. Then there exists a path with an even length from w to v, **i.e.** $w, x_1, x_2, \dots x_m, u, v$, whose length is even. **Contradiction.**

Thus, a simple graph G is bipartite if and only if it has no circuits with an odd number of edges.

QED

5. ***Proof:***

Since exists a path from u to v: u, e_0, v, u and v are connected in graph G'.

Thus, $[u]_{\text{Conn}(G')} = [v]_{\text{Conn}(G')}$.

For any $x, y \in [u]_{\text{Conn}(G)}$, obviously x and y are still connected in graph G'.

For any $x, y \in [v]_{\text{Conn}(G)}$, obviously x and y are still connected in graph G'.

For any $x \in [u]_{\text{Conn}(G)}, y \in [v]_{\text{Conn}(G)}$, there exist a path from x to u: $x, e_1, x_1, e_2, \dots, e_n, u$ and a path from y to v: $y, e_{n+1}, x_{n+1}, e_{n+2}, \dots, e_m, v$. Since G' is undirected, there exists a path from x to y: $x, e_1, x_1, e_2, \dots, e_n, u, e_0, v, e_m, x_m, e_{m-1}, \dots, e_{n+1}, y$, **i.e.** x and y are connected in graph G'.

Let $x = u$, we know $[u]_{\text{Conn}(G)} \subseteq [u]_{\text{Conn}(G')}$ and $[v]_{\text{Conn}(G)} \subseteq [u]_{\text{Conn}(G')}$.

Thus, $[u]_{\text{Conn}(G)} \cup [v]_{\text{Conn}(G)} \subseteq [u]_{\text{Conn}(G')}$.

Now we prove $[u]_{\text{Conn}(G')} \subseteq [u]_{\text{Conn}(G)} \cup [v]_{\text{Conn}(G)}$ by contradiction.

If exists $x \in [u]_{\text{Conn}(G')}$ and $x \notin [u]_{\text{Conn}(G)} \cup [v]_{\text{Conn}(G)}$.

1) $x = u$. Obviously $u \in [u]_{\text{Conn}(G)} \cup [v]_{\text{Conn}(G)}$. **Contradiction.**

2) $x \neq u$. Thus, exists **a simple path** from x to u: $x, e_1, x_1, e_2, \dots, e_n, u$. Since $x \notin [u]_{\text{Conn}(G)} \cup [v]_{\text{Conn}(G)}$ and $G' = (V, E \cup \{e_0\})$, we know e_0 is definitely included in the path.

Thus, v is included in the path, **i.e.** x is connected to v in graph G.

Thus, $x \in [v]_{\text{Conn}(G)} \subseteq [u]_{\text{Conn}(G)} \cup [v]_{\text{Conn}(G)}$. **Contradiction.**

Thus, $[u]_{\text{Conn}(G')} \subseteq [u]_{\text{Conn}(G)} \cup [v]_{\text{Conn}(G)}$.

Therefore, $[u]_{\mathrm{Conn}(G')} = [v]_{\mathrm{Conn}(G')} = [u]_{\mathrm{Conn}(G)} \cup [v]_{\mathrm{Conn}(G)}$.

QED

6. ***Proof:***

0) Obviously V is a finite set and $I \subseteq \mathcal{P}(V)$.

1) By the definition of indepent sets of vertices, pick any $u \in V$, let $v^* = u$ and $k = 0$, we know that $\emptyset \in I$.

2) For any $A \subseteq B \subseteq V$, let $A = \{a_1, a_2, \dots, a_s\}, B = \{b_1, b_2, \dots, b_k\}$ $(s < k)$.

If $B \subseteq I$, we know there exist $v^* \in V$ **s.t.** B is an independent set whose source vertex is v^*.

In other words, exist k pairwise-disjoint paths $\rho_1, \rho_2, \dots, \rho_k$ **s.t.** ρ_i connects v^* and b_i.

Since for any $a_i \in A$, $a_i \in B$, we pick these ρ_j **s.t.** $b_j = a_i \in A$ and let $\rho'_i = \rho_j$. Then we get s pairwise-disjoint paths $\rho'_1, \rho'_2, \dots, \rho'_s$ **s.t.** ρ'_i connects v^* and a_i.

Thus, $A \subseteq I$.

3) For any $A, B \in I, |A| < |B|$, let $A = \{a_1, a_2, \dots, a_s\}, B = \{b_1, b_2, \dots, b_k\}$ $(s < k)$.

We prove there exists $x \in B \backslash A$ **s.t.** $A \cup \{x\} \in I$ by contradiction.

Assume for any $x \in B \backslash A$, no matter which $v \in V \backslash A \cup \{x\}$ we pick as source vertex v^*, there exists no pair of pair-wise disjoint paths $\rho_1, \rho_2, \dots, \rho_s, \rho_{s+1}$ **s.t.** ρ_i connects v^* and a_i $(1 \leq i \leq s)$ and ρ_{s+1} connects v^* and x.

In other words, for any $v \in V \backslash A \cup \{x\}$ we pick as sourse vertex v^*, any path from v^* to x and any path of v^* to a certain a_i $(1 \leq i \leq s)$ have common edges. Therefore, the first edge in any path from v^* to x and any path from $v*$ to a_i must be the same. In this case, we call there is a conflict between x and a_i.

For any designated source vertex v^*, we know for any $x, y \in B \backslash A$, if there are conflicts between x and a_i and between y and a_i, $x = y$. (Otherwise, we know there must be at least one common edge, **i.e.** the first edge in the path, in the path from v^* to x and v^* to y. Thus, if $x \in B \backslash A$, $y \notin B \backslash A$. **Contradiction.**)

Let $C = A \cap B$. Obviously there is no vertex $x \in B \backslash A$ **s.t.** there is a conflict between x and a $y \in C$. Otherwise, $x \notin B$.

Therefore, for any designated source vertex v^*, there are at most $(s - |C|)$ vertices in $B \backslash A$ **s.t.** there are conflicts between the vertex and a $y \in A \backslash C$ since $|A \backslash C| = s - |C|$.

That is to say, there are at least $(k - s)$ vertices in $B \backslash A$ which would not cause conflicts with any $y \in A \backslash C$, **i.e.** exists a designated $v^* \in V \backslash A \cup \{x\}$ and there exist pair-wise disjoint paths $\rho_1, \rho_2, \dots, \rho_s, \rho_{s+1}$ **s.t.** ρ_i connects v^* and a_i $(1 \leq i \leq s)$ and ρ_{s+1} connects v^* and the vertex. $(k - s \geq 1)$ **Contradiction.**

Thus, exist at least one vertex $x \in B \backslash A$, exists a desginated $v^* \in V \backslash A \cup \{x\}$ **s.t.** there exist $(s + 1)$ pair-wise disjoint paths $\rho_1, \rho_2, \dots, \rho_s, \rho_{s+1}$ **s.t.** ρ_i connects v^* and a_i $(1 \leq i \leq s)$ and ρ_{s+1} connects v^* and x.

In other words, exist at least one vertex $x \in B \backslash A$ **s.t.** $A \cup \{x\} \in I$.

Thus, (V, I) is a finite matroid.

QED