Discrete Mathematics Exercise 9

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1. ***Solution:***

a) The equivalence class of 1 for this equivalence relation is $\mathbb{Z}$.

b) The equivalence class of 1/2 for this equivalence relation is $\{x \mid x = k + {}^{1}/_{2}, k \in \mathbb{Z}\}$.

2. ***Proof:***

First, we prove the "if" part.

Since P_1 correspond to $\mathcal{R}_1$, for any $p_1 \in P_1$, exists $a \in \mathbb{A}$ **s.t.** $p_1 = [a]_{\mathcal{R}_1}$.

When P_1 is a refinement of P_2, $\forall p_1\,(\,(p_1 \in P_1) \wedge (\,\exists p_2(\,(p_2 \in P_2) \wedge (p_1 \subseteq p_2)\,)\,)\,)$

$$\Longrightarrow \forall a\,(\,(a \in \mathbb{A}) \wedge (\,\exists p_2(\,(p_2 \in P_2) \wedge ([a]_{\mathcal{R}_1} \subseteq p_2)\,)\,)\,).$$

For any $a \in \mathbb{A}$, $\exists b(\,(b \in \mathbb{A}) \wedge ([a]_{\mathcal{R}_1} \subseteq [b]_{\mathcal{R}_2})\,)$ **(**because P_2 correspond to $\mathcal{R}_2$**)**.

Since $a \in [a]_{\mathcal{R}_1}$, $[b]_{\mathcal{R}_2} = [a]_{\mathcal{R}_2}$.

Thus, for any $a \in \mathbb{A}$, $[a]_{\mathcal{R}_1} \subseteq [a]_{\mathcal{R}_2}$, **i.e.** $\forall a \forall x\,((a \in \mathbb{A}) \wedge (x \in \mathbb{A}) \wedge (a\mathcal{R}_1 x \rightarrow a\mathcal{R}_2 x))$.

In other words, $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

Now we prove the "only if" part.

When $\mathcal{R}_1 \subseteq \mathcal{R}_2$, $\forall a \forall x\,((a \in \mathbb{A}) \wedge (x \in \mathbb{A}) \wedge (a\mathcal{R}_1 x \rightarrow a\mathcal{R}_2 x))$,

i.e. $\forall a\,(a \in \mathbb{A}) \wedge ([a]_{\mathcal{R}_1} \subseteq [a]_{\mathcal{R}_2})$.

Since P_1 correspond to $\mathcal{R}_1$, for any $p_1 \in P_1$, exists $a \in \mathbb{A}$ **s.t.** $p_1 = [a]_{\mathcal{R}_1}$.

Thus, for any $p_1 \in P_1$, exists $a \in \mathbb{A}$ **s.t.** $p_1 = [a]_{\mathcal{R}_1}$ and $p_2 = [a]_{\mathcal{R}_2}$ **s.t.** $p_2 \in P_2, p_1 \subseteq p_2$.

In other words, P_1 is a refinement of P_2.

In conclusion, $\mathcal{R}_1 \subseteq \mathcal{R}_2$ **iff.** P_1 is a refinement of P_2.

QED

3. ***Proof:***

Let $\mathcal{R} \subseteq \mathbb{A} \times \mathbb{A}$.

Lemma. For any $n \in \mathbb{N}^+$, $\mathcal{R}^n \circ \mathcal{R} = \mathcal{R} \circ \mathcal{R}^n$.

Proof.

1) $n = 1$. $\mathcal{R} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{R}$ obviously holds.

2) $n \geq 2$.

Since composition is associative, we know $\mathcal{R}^n \circ \mathcal{R} = \mathcal{R}^{n\text{-}1} \circ \mathcal{R} \circ \mathcal{R} = \mathcal{R}^{n\text{-}1} \circ (\mathcal{R} \circ \mathcal{R}) = \mathcal{R}^{n\text{-}1} \circ \mathcal{R}^2 = \cdots = \mathcal{R}^2 \circ \mathcal{R}^{n-1} = \mathcal{R} \circ \mathcal{R} \circ \mathcal{R}^{n-1} = \mathcal{R} \circ (\mathcal{R} \circ \mathcal{R}^{n-1}) = \mathcal{R} \circ \mathcal{R}^n$.

Qed.

Now we prove for all positive integers n, $\mathcal{R}^n$ is a symmetric relation.

1) $n = 1$. $\mathcal{R}^n = \mathcal{R}^1 = \mathcal{R}$ is a symmetric relation.

2) $n \geq 2$.

IH. When $n = k$ $(k \in \mathbb{N}^+)$, $\mathcal{R}^n$ is a symmetric relation.

When $n = k + 1$, $\mathcal{R}^n = \mathcal{R}^{k+1} = \mathcal{R}^k \circ \mathcal{R}$.

For any $(a, b) \in \mathcal{R}^{k+1}$, exists $c \in \mathbb{A}$ **s.t.** $a\mathcal{R}c, c(\mathcal{R}^k)b$.

Since both $\mathcal{R}^k$ and $\mathcal{R}$ are symmetric relations, $c\mathcal{R}a, b(\mathcal{R}^k)c$.
Thus, $(b,a) \in \mathcal{R} \circ \mathcal{R}^k$.
According to Lemma, we know $\mathcal{R} \circ \mathcal{R}^k = \mathcal{R}^k \circ \mathcal{R} = \mathcal{R}^{k+1}$. Thus, $(b,a) \in \mathcal{R}^{k+1}$.
In other words, $\mathcal{R}^{k+1}$ is a symmetric relation, **i.e.** $\mathcal{R}^n$ is a symmetric relation.

Therefore, for all positive integers n, $\mathcal{R}^n$ is a symmetric relation.

QED

4. ***Proof:***

Let $\mathcal{R}, \mathcal{S} \subseteq \mathbb{A} \times \mathbb{A}$.
Lemma 1. $\mathcal{R} \circ \mathcal{S} \subseteq \mathcal{S} \implies \forall n(n \in \mathbb{N}^+) \land (\mathcal{R}^n \circ \mathcal{S} \subseteq \mathcal{S})$.
Proof.

1) $n = 1$. $\mathcal{R} \circ \mathcal{S} \subseteq \mathcal{S}$ obviously holds.
2) $n \geq 2$.

IH. When $n = k$ $(k \in \mathbb{N}^+)$, $\mathcal{R}^k \circ \mathcal{S} \subseteq \mathcal{S}$, **i.e.** for any $(a,b) \in \mathcal{R}^k \circ \mathcal{S}$, $(a,b) \in \mathcal{S}$.
When $n = k+1$, since $\mathcal{R}^{k+1} \circ \mathcal{S} = \mathcal{R}^k \circ \mathcal{R} \circ \mathcal{S} = \mathcal{R} \circ \mathcal{R}^k \circ \mathcal{S}$, (by **3. Lemma**)
we know for any $(a,b) \in \mathcal{R}^{k+1} \circ \mathcal{S}$, exists $c \in \mathbb{A}$ **s.t.** $a(\mathcal{R}^k \circ \mathcal{S})c, c\mathcal{R}b$.
Thus, $a\mathcal{S}c, c\mathcal{S}b$. So $a\mathcal{S}b$.
In other words, $\mathcal{R}^{k+1} \circ \mathcal{S} \subseteq \mathcal{S}$, **i.e.** $\mathcal{R}^n \circ \mathcal{S} \subseteq \mathcal{S}$.

Qed.

Lemma 2. $\left(\bigcup_{n=1}^{\infty} \mathcal{R}^n\right) \circ \mathcal{S} = \bigcup_{n=1}^{\infty} (\mathcal{R}^n \circ \mathcal{S})$.

Proof.

First we prove $\left(\bigcup_{n=1}^{\infty} \mathcal{R}^n\right) \circ \mathcal{S} \subseteq \bigcup_{n=1}^{\infty} (\mathcal{R}^n \circ \mathcal{S})$.

For any $(a,b) \in \left(\bigcup_{n=1}^{\infty} \mathcal{R}^n\right) \circ \mathcal{S}$, exists $c \in \mathbb{A}$ **s.t.** $a\mathcal{S}c, c\left(\bigcup_{n=1}^{\infty} \mathcal{R}^n\right)b$.
Thus, exists positive integer n **s.t.** $(c,b) \in \mathcal{R}^n$, **i.e.** $c(\mathcal{R}^n)b$.
Thus, $(a,b) \in \mathcal{R}^n \circ \mathcal{S}$.

Thus, $(a,b) \in \bigcup_{n=1}^{\infty} (\mathcal{R}^n \circ \mathcal{S})$.

Then we prove $\left(\bigcup_{n=1}^{\infty} \mathcal{R}^n\right) \circ \mathcal{S} \supseteq \bigcup_{n=1}^{\infty} (\mathcal{R}^n \circ \mathcal{S})$.

For any $(a,b) \in \bigcup_{n=1}^{\infty} (\mathcal{R}^n \circ \mathcal{S})$, exists positive integer n **s.t.** $(a,b) \in \mathcal{R}^n \circ \mathcal{S}$.

Thus, exists $c \in \mathbb{A}$ **s.t.** $a\mathcal{S}c, c(\mathcal{R}^n)b$.

Thus, $a\mathcal{S}c, (c,b) \in \left(\bigcup_{n=1}^{\infty} \mathcal{R}^n\right)$.

Thus, $(a,b) \in \left(\bigcup_{n=1}^{\infty} \mathcal{R}^n\right) \circ \mathcal{S}$.

Qed.

Firstly, we prove "$\implies$" holds.

From **Lemma 1**, we know for all positive integers $n, \mathcal{R}^n \circ \mathcal{S} \subseteq \mathcal{S}$.

From **Lemma 2**, we know for any $(a,b) \in \left(\bigcup_{n=1}^{\infty} \mathcal{R}^n\right) \circ \mathcal{S}$, $(a,b) \in \bigcup_{n=1}^{\infty} (\mathcal{R}^n \circ \mathcal{S})$.

Thus, exists positive integer n **s.t.** $(a,b) \in \mathcal{R}^n \circ \mathcal{S}$.

Thus, $(a,b) \in \mathcal{S}$.

So $\left(\bigcup_{n=1}^{\infty} \mathcal{R}^n\right) \circ \mathcal{S} \subseteq \mathcal{S}$.

Now we prove "$\Leftarrow$" holds by contradiction.

If $\mathcal{R} \circ \mathcal{S} \nsubseteq \mathcal{S}$, then exists $(a,b) \in \mathcal{R} \circ \mathcal{S}$ **s.t.** $(a,b) \notin \mathcal{S}$.

Thus, $(a,b) \in \bigcup_{n=1}^{\infty} (\mathcal{R}^n \circ \mathcal{S})$. (By **Lemma 2**)

Since $\left(\bigcup_{n=1}^{\infty} \mathcal{R}^n\right) \circ \mathcal{S} \subseteq \mathcal{S}$, **i.e.** $\bigcup_{n=1}^{\infty} (\mathcal{R}^n \circ \mathcal{S}) \subseteq \mathcal{S}$ (By **Lemma 2**), $(a,b) \in \mathcal{S}$.

Contradiction.

Therefore, $\mathcal{R} \circ \mathcal{S} \subseteq \mathcal{S}$.

In conclusion, $\mathcal{R} \circ \mathcal{S} \subseteq \mathcal{S} \Leftrightarrow \left(\bigcup_{n=1}^{\infty} \mathcal{R}^n\right) \circ \mathcal{S} \subseteq \mathcal{S}$.

QED

5. a) ***Proof:***

1) It's obvious that there's no integer n **s.t.** $a < n \leq a$, **i.e.** $a\mathcal{R}a$. Thus, $\mathcal{R}$ is reflexive.

2) When $a\mathcal{R}b \wedge b\mathcal{R}c$, we prove $a\mathcal{R}c$ by contradiction.

If $\neg a\mathcal{R}c$, then exists an integer n **s.t.** $c < n \leq a$.

Therefore, either $c < n \leq b$ or $b < n \leq a$, **i.e.** $\neg a\mathcal{R}b \vee \neg b\mathcal{R}c$. ***Contradiction.***

Thus, $a\mathcal{R}b \wedge b\mathcal{R}c \rightarrow a\mathcal{R}c$, **i.e.** $\mathcal{R}$ is transitive.

3) Exists $a = 1/2, b = 0$ **s.t.** $a\mathcal{R}b$ and $b\mathcal{R}a$. Thus, $\mathcal{R}$ is not antisymmetric.

Since $\mathcal{R}$ is reflexive and transitive but is not antisymmetric, $\mathcal{R}$ is a preorder on $\mathbb{R}$ but is not a partial order on $\mathbb{R}$.

QED

b) ***Proof:***

Since $\mathcal{R}$ is a preorder, $\mathcal{R}$ is transitive.

In this case, $\mathcal{R} \subseteq T \rightarrow \mathcal{R} \subseteq T$ obviously holds. Thus, $\forall T\ (\mathcal{R} \subseteq T \wedge \textbf{Transitive}(T) \rightarrow \mathcal{R} \subseteq T)$.

Because $\mathcal{R} \subseteq \mathcal{R}$, $\mathcal{R}$ is transitive and $\forall T\ (\mathcal{R} \subseteq T \wedge \textbf{Transitive}(T) \rightarrow \mathcal{R} \subseteq T)$, $\mathcal{R}$ is the transitive closure of itself, **i.e.** $\mathcal{R} = \mathcal{R}^+$.

QED

c) ***Proof:***

Since $\mathcal{R}$ is a preorder, $\mathcal{R}$ is reflexive and transitive.

1) Since for any $a \in \mathbb{A}$, $a\mathcal{R}a$, we know $a(\mathcal{R}^{-1})a$.

Thus, $\forall a(a \in \mathbb{A}) \wedge a(\mathcal{R} \cap \mathcal{R}^{-1})a$, **i.e.** $\mathcal{R} \cap \mathcal{R}^{-1}$ is reflexive.

2) For any $a(\mathcal{R} \cap \mathcal{R}^{-1})b$, $a\mathcal{R}b \wedge a(\mathcal{R}^{-1})b$, **i.e.** $a\mathcal{R}b \wedge b\mathcal{R}a$.

Thus, $a\mathcal{R}b \wedge b\mathcal{R}a \wedge a(\mathcal{R}^{-1})b \wedge b(\mathcal{R}^{-1})a$.

Thus, $b(\mathcal{R} \cap \mathcal{R}^{-1})a$.

So, for any $a(\mathcal{R} \cap \mathcal{R}^{-1})b$, $b(\mathcal{R} \cap \mathcal{R}^{-1})a$. In other words, $\mathcal{R} \cap \mathcal{R}^{-1}$ is symmetric.

3) For any $a(\mathcal{R} \cap \mathcal{R}^{-1})b$ and $b(\mathcal{R} \cap \mathcal{R}^{-1})c$, $a\mathcal{R}b \wedge a(\mathcal{R}^{-1})b \wedge b\mathcal{R}c \wedge b(\mathcal{R}^{-1})c$.
Since both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are transitive, we know $a\mathcal{R}c \wedge a(\mathcal{R}^{-1})c$.
Thus, $a(\mathcal{R} \cap \mathcal{R}^{-1})c$. In other words, $\mathcal{R} \cap \mathcal{R}^{-1}$ is transitive.

Therefore, $\mathcal{R} \cap \mathcal{R}^{-1}$ is reflexive, symmetric and transitive.
In other words, $\mathcal{R} \cap \mathcal{R}^{-1}$ is an equivalence relation on $\mathbb{A}$.

QED

d) ***Proof:***

Since $\mathcal{R}$ is a preorder, $\mathcal{R}$ is reflexive and transitive.

1) For any $b \in \mathbb{B}$, exists $a \in \mathbb{A}$ **s.t.** $b = [a]$.
Since $\mathcal{R}$ is reflexive, $a\mathcal{R}a$, we know $([a],[a]) \in \mathcal{S}$, **i.e.** $(b,b) \in \mathcal{S}$.
In other words, $\mathcal{S}$ is reflexive on $\mathbb{B}$.
2) For any $x, y \in \mathbb{B}$, exists $a, b \in \mathbb{A}$ **s.t.** $x = [a], y = [b]$.
When $(x,y) \in \mathcal{S} \wedge (y,x) \in \mathcal{S}$, **i.e.** $([a],[b]) \in \mathcal{S} \wedge ([b],[a]) \in \mathcal{S}$,
we know from the definition that $a(\mathcal{R} \cap \mathcal{R}^{-1})b \wedge b(\mathcal{R} \cap \mathcal{R}^{-1})a$, **i.e.** $[a] = [b]$, **i.e.** $x = y$.
Thus, for any $x, y \in \mathbb{B}$, $x\mathcal{S}y \wedge y\mathcal{S}x \to x = y$.
In other words, $\mathcal{S}$ is antisymmetric on $\mathbb{B}$.
3) For any $x, y, z \in \mathbb{B}$, exists $a, b, c \in \mathbb{A}$ **s.t.** $x = [a], y = [b], z = [c]$.
When $x\mathcal{S}y \wedge y\mathcal{S}z$, **i.e.** $([a],[b]) \in \mathcal{S} \wedge ([b],[c]) \in \mathcal{S}$,
we know from the definition that $a(\mathcal{R} \cap \mathcal{R}^{-1})b \wedge b(\mathcal{R} \cap \mathcal{R}^{-1})c$.
Thus, $a(\mathcal{R} \cap \mathcal{R}^{-1})c$. (We have proved that $\mathcal{R} \cap \mathcal{R}^{-1}$ is transitive in **c).**)
Thus, for any $x, y, z \in \mathbb{B}$, $x\mathcal{S}y \wedge y\mathcal{S}z \to x\mathcal{S}z$.
In other words, $\mathcal{S}$ is transitive on $\mathbb{B}$.

Therefore, $\mathcal{S}$ is reflexive, antisymmetric and transitive.
In other words, $\mathcal{S}$ is a partial order on $\mathbb{B}$.

QED