Exercise Sheet 9

Discrete Mathematics, 2020.10.23

1. ([R], Page 617, Exercise 57) Consider the equivalence relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x - y \text{ is an integer } \}$.
   a) What is the equivalence class of 1 for this equivalence relation?
   b) What is the equivalence class of 1/2 for this equivalence relation?

2. ([R], Page 617, Exercise 54) Suppose that R_1 and R_2 are equivalence relations on a set A. Let P_1 and P_2 be the partitions that correspond to R_1 and R_2, respectively. Show that $R_1 \subseteq R_2$ if and only if P_1 is a refinement of P_2.
   Refinement:A partition P_1 is called a refinement of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2.

3. ([R], Page 583, Exercise 58) Let R be a symmetric relation. Show that R^n is symmetric for all positive integers n.

4. Prove that for any relations R, S, we have that

$$R \circ S \subseteq S \Leftrightarrow (\bigcup_{n=1}^{\infty} R^n) \circ S \subseteq S \ .$$

5. For any binary relation $R \subseteq A \times A$, we call R a **preorder** on A if R is reflexive on A and transitive, and we call R a **partial order** on A if R is reflexive on A, antisymmetric and transitive.

   a) Suppose $R = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid \text{there is no integer } n \text{ such that } b < n \leq a\}$. Prove that R is a preorder on $\mathbb{R}$ but R is not a partial order on $\mathbb{R}$.
   b) Suppose $R \subseteq A \times A$ is a preorder. Prove that $R^+ = R$ (where R^+ represents the transitive closure of R).
   c) Suppose $R \subseteq A \times A$ is a preorder. Prove that $R \cap R^{-1}$ is an equivalence relation on A.
   d) Suppose $R \subseteq A \times A$ is a preorder. Let $B = \{[a]_{R \cap R^{-1}} \mid a \in A\}$ and $S = \{([a], [b]) \mid aRb\} \subseteq B \times B$. Prove that S is a partial order on B.