Discrete Mathematics Exercise 3

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1. ***a***)

Proof: The truth value table of $p \to (q \to p)$ is as follows.

p	q	$p \to (q \to p)$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**T**

For any truth assignment $\mathcal{J}$, $[\![p \to (q \to p)]\!]_{\mathcal{J}} = \boldsymbol{T}$. In other words, $p \to (q \to p)$ is a tautology.

QED

b)

Proof: The truth value table of $(p \to q \to r) \to (p \to q) \to (p \to r)$ is as follows.

p	q	r	$(p \to q \to r) \to (p \to q) \to (p \to r)$
T	**T**	**T**	**T**
T	**T**	**F**	**T**
T	**F**	**T**	**T**
T	**F**	**F**	**T**
F	**T**	**T**	**T**
F	**T**	**F**	**T**
F	**F**	**T**	**T**
F	**F**	**F**	**T**

For any truth assignment $\mathcal{J}$, $[\![(p \to q \to r) \to (p \to q) \to (p \to r)]\!]_{\mathcal{J}} = \boldsymbol{T}$. In other words, $(p \to q \to r) \to (p \to q) \to (p \to r)$ is a tautology.

QED

c)

Proof: The truth value table of $p \to q \to r \equiv (p \wedge q) \to r$ is as follows.

p	q	r	$p \to q \to r$	$(p \wedge q) \to r$
T	**T**	**T**	**T**	**T**
T	**T**	**F**	**F**	**F**
T	**F**	**T**	**T**	**T**
T	**F**	**F**	**T**	**T**
F	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**
F	**F**	**F**	**T**	**T**

For any truth assignment $\mathcal{J}$, $[\![p \to q \to r]\!]_{\mathcal{J}} = [\![(p \wedge q) \to r]\!]_{\mathcal{J}}$.

In other words, $p \to q \to r \equiv (p \wedge q) \to r$.

QED

2. ***a***)

Proof: The truth table of $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ is as follows.

p	q	r	$(p \to q) \wedge (p \to r)$	$p \to (q \wedge r)$
T	**T**	**T**	**T**	**T**
T	**T**	**F**	**F**	**F**
T	**F**	**T**	**F**	**F**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**
F	**F**	**F**	**T**	**T**

For any assignment $\mathcal{J}$, $[\![(p \to q) \wedge (p \to r)]\!]_{\mathcal{J}} = [\![p \to (q \wedge r)]\!]_{\mathcal{J}}$.
In other words, $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$.

QED

b)

Proof: The truth table of $(p \to r) \wedge (q \to r)$ and $(p \vee q) \to r$ is as follows.

p	q	r	$(p \to r) \wedge (q \to r)$	$(p \vee q) \to r$
T	**T**	**T**	**T**	**T**
T	**T**	**F**	**F**	**F**
T	**F**	**T**	**T**	**T**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**T**
F	**T**	**F**	**F**	**F**
F	**F**	**T**	**T**	**T**
F	**F**	**F**	**T**	**T**

For any assignment $\mathcal{J}$, $[\![(p \to r) \wedge (q \to r)]\!]_{\mathcal{J}} = [\![(p \vee q) \to r]\!]_{\mathcal{J}}$.
In other words, $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$.

QED

c)

Proof: The truth table of $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ is as follows.

p	q	r	$(p \to q) \vee (p \to r)$	$p \to (q \vee r)$
T	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**
T	**F**	**T**	**T**	**T**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**
F	**F**	**F**	**T**	**T**

For any assignment $\mathcal{J}$, $[\![(p \to q) \vee (p \to r)]\!]_{\mathcal{J}} = [\![p \to (q \vee r)]\!]_{\mathcal{J}}$.
In other words, $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$.

QED

3. *Proof:*

When $[\![p]\!]_{\mathcal{J}} = \boldsymbol{F}, [\![q]\!]_{\mathcal{J}} = \boldsymbol{T}, [\![r]\!]_{\mathcal{J}} = \boldsymbol{F}$, $[\![(p \to q) \to r]\!]_{\mathcal{J}} = \boldsymbol{F}$ while $[\![p \to (q \to r)]\!]_{\mathcal{J}} = \boldsymbol{T}$.
So $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

QED

4. *Proof:*

When $[\![p]\!]_{\mathcal{J}} = \boldsymbol{T}, [\![q]\!]_{\mathcal{J}} = \boldsymbol{F}, [\![r]\!]_{\mathcal{J}} = \boldsymbol{F}$, $[\![(p \wedge q) \to r]\!]_{\mathcal{J}} = \boldsymbol{T}$ while $[\![(p \to r) \wedge (q \to r)]\!]_{\mathcal{J}} = \boldsymbol{F}$.
So $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$ are not logically equivalent.

QED

5. (a) *Solution:*

The truth value table of $\phi = p \to (q \oplus r)$ is as follows.

p	q	r	$p \to (q \oplus r)$	ψ
T	**T**	**T**	**F**	**F**
T	**T**	**F**	**T**	**T**
T	**F**	**T**	**T**	**T**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**
F	**F**	**F**	**T**	**T**

We can construct a proposition $\chi = (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r)$ in DNF, which is logically equivalent to $\neg\phi$.

We can construct a proposition $\psi = \neg\chi = (\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r)$ in CNF, which is logically equivalent to ϕ. From the truth value table of ϕ and ψ, it's obvious to see that $\phi \equiv \psi$.

Thus, $\psi = (\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r)$ is a feasible solution.

(b) *Solution:*

$$\phi = p \to (q \oplus r)$$

$$\Longrightarrow \Big((q \oplus r) \leftrightarrow p_1\Big) \wedge \Big((p \to p_1) \leftrightarrow p_2\Big) \wedge p_2$$

We can list the truth value table of $(q \oplus r) \leftrightarrow p_1$ and $(p \to p_1) \leftrightarrow p_2$ as follows:

q	r	p_1	$(q \oplus r) \leftrightarrow p_1$		p	p_1	p_2	$(p \to p_1) \leftrightarrow p_2$
T	**T**	**T**	**F**		**T**	**T**	**T**	**T**
T	**T**	**F**	**T**		**T**	**T**	**F**	**F**
T	**F**	**T**	**T**		**T**	**F**	**T**	**F**
T	**F**	**F**	**F**		**T**	**F**	**F**	**T**
F	**T**	**T**	**T**		**F**	**T**	**T**	**T**
F	**T**	**F**	**F**		**F**	**T**	**F**	**F**
F	**F**	**T**	**F**		**F**	**F**	**T**	**T**
F	**F**	**F**	**T**		**F**	**F**	**F**	**F**

Then we can build disjunctive clauses that are logically equivalent to $(q \oplus r) \leftrightarrow p_1$ and $(p \to p_1) \leftrightarrow p_2$ respectively:

$$(q \oplus r) \leftrightarrow p_1 \equiv (\neg q \vee \neg r \vee \neg p_1) \wedge (\neg q \vee r \vee p_1) \wedge (q \vee \neg r \vee p_1) \wedge (q \vee r \vee \neg p_1),$$
$$(p \rightarrow p_1) \leftrightarrow p_2 \equiv (\neg p \vee \neg p_1 \vee p_2) \wedge (\neg p \vee p_1 \vee \neg p_2) \wedge (p \vee \neg p_1 \vee p_2) \wedge (p \vee p_1 \vee p_2).$$

Thus, we can construct a proposition

$$\psi = (\neg q \vee \neg r \vee \neg p_1) \wedge (\neg q \vee r \vee p_1) \wedge (q \vee \neg r \vee p_1) \wedge (q \vee r \vee \neg p_1) \wedge$$
$$(\neg p \vee \neg p_1 \vee p_2) \wedge (\neg p \vee p_1 \vee \neg p_2) \wedge (p \vee \neg p_1 \vee p_2) \wedge (p \vee p_1 \vee p_2) \wedge p_2 \ ,$$

which is in CNF such that ϕ is satisfiable if and only if ψ is satisfiable.

6. ***a***)

Proof: The truth value table of $\phi \downarrow \phi$ and $\neg\phi$ is as follows.

ϕ	$\phi \downarrow \phi$	$\neg\phi$
T	**F**	**F**
F	**T**	**T**

For any truth assignment $\mathcal{J}$, $[\![\phi \downarrow \phi]\!]_{\mathcal{J}} = [\![\neg\phi]\!]_{\mathcal{J}}$. In other words, $\phi \downarrow \phi \equiv \neg\phi$.

QED

b)

Proof: The truth value table of $(\phi \downarrow \psi) \downarrow (\psi \downarrow \phi)$ and $\phi \wedge \psi$ is as follows.

ϕ	ψ	$(\phi \downarrow \psi) \downarrow (\psi \downarrow \phi)$	$\phi \wedge \psi$
T	**T**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**F**	**F**
F	**F**	**F**	**F**

For any truth assignment $\mathcal{J}$, $[\![(\phi \downarrow \psi) \downarrow (\psi \downarrow \phi)]\!]_{\mathcal{J}} = [\![\phi \wedge \psi]\!]_{\mathcal{J}}$.

In other words, $(\phi \downarrow \psi) \downarrow (\psi \downarrow \phi) \equiv \phi \wedge \psi$.

QED

c)

Proof:

It's a theorem that $\{\neg, \wedge\}$ is functionally complete.

According to ***a***) and ***b***), we can replace $\neg\phi$ with $\phi \downarrow \phi$ and replace $\phi \wedge \psi$ with $(\phi \downarrow \psi) \downarrow (\psi \downarrow \phi)$.

Thus, for any set of propositional variables $\mathbf{\Sigma}$ and any $\boldsymbol{f}$, which is a mapping from $\mathbf{\Sigma}$'s truth assignments to truth values, there exists a compound proposition ϕ that involves only "$\downarrow$" such that $[\![\phi]\!]_{\mathcal{J}} = \boldsymbol{f}(\mathcal{J})$ for every $\mathcal{J}$.

In other words, $\{\downarrow\}$ is functionally complete.

QED

7. ***a***)

Solution: Given that $\mathcal{J}_1 = [p_1 \mapsto \boldsymbol{T}, p_3 \mapsto \boldsymbol{F}]$.

Let $(\neg p_1 \vee p_2) \wedge (\neg p_1 \vee p_3 \vee p_5) \wedge (\neg p_2 \vee p_4) \wedge (\neg p_3 \vee \neg p_4) \wedge (p_1 \vee p_5 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (p_2 \vee \neg p_3) \wedge (p_6 \vee \neg p_5)$ to be $\boldsymbol{\phi}$.

Since we want ϕ to be ***True***, we need every disjunctive clause in ϕ to be ***True***.

In other words, we need

$$[\![\neg p_1 \vee p_2]\!]_{\mathcal{J}_1} = \boldsymbol{T} \quad ①, \ [\![\neg p_1 \vee p_3 \vee p_5]\!]_{\mathcal{J}_1} = \boldsymbol{T} \quad ②, \ [\![\neg p_2 \vee p_4]\!]_{\mathcal{J}_1} = \boldsymbol{T}, \quad ③$$

$$[\![\neg p_3 \vee \neg p_4]\!]_{\mathcal{J}_1} = \boldsymbol{T} \quad ④, \ [\![p_1 \vee p_5 \vee \neg p_2]\!]_{\mathcal{J}_1} = \boldsymbol{T} \quad ⑤, \ [\![p_2 \vee p_3]\!]_{\mathcal{J}_1} = \boldsymbol{T} \quad ⑥,$$

$$[\![p_2 \vee \neg p_3]\!]_{\mathcal{J}_1} = \boldsymbol{T} \quad ⑦, \ [\![p_6 \vee \neg p_5]\!]_{\mathcal{J}_1} = \boldsymbol{T} \qquad ⑧$$

Under the truth assignment $\mathcal{J}_1$, ④⑤⑦ is already satisfied.

Since $[\![\neg p_1]\!]_{\mathcal{J}_1} = \boldsymbol{F}$, we know from ① that we need $p_2 \mapsto \boldsymbol{T}$. From ② we know $p_5 \mapsto \boldsymbol{T}$. Therefore, from ④ we know that we need $p_4 \mapsto \boldsymbol{T}$. Similarly, we could figure out that $p_6 \mapsto \boldsymbol{T}$.

Thus, $\mathrm{Unit}Pro(\mathcal{J}_1) = [p_1 \mapsto \boldsymbol{T}, p_2 \mapsto \boldsymbol{T}, p_3 \mapsto \boldsymbol{F}, p_4 \mapsto \boldsymbol{T}, p_5 \mapsto \boldsymbol{T}, p_6 \mapsto \boldsymbol{T}]$.

b)

Solution: Given that $\mathcal{J}_2 = [p_3 \mapsto \boldsymbol{F}]$.

Let $(\neg p_1 \vee p_2) \wedge (\neg p_1 \vee p_3 \vee p_5) \wedge (\neg p_2 \vee p_4) \wedge (\neg p_3 \vee \neg p_4) \wedge (p_1 \vee p_5 \vee \neg p_2) \wedge (p_2 \vee p_3) \wedge (p_2 \vee \neg p_3) \wedge (p_6 \vee \neg p_5)$ to be ϕ.

Since we want ϕ to be ***True***, we need every disjunctive clause in ϕ to be ***True***.

In other words, we need

$$[\![\neg p_1 \vee p_2]\!]_{\mathcal{J}_2} = \boldsymbol{T} \quad ①, \ [\![\neg p_1 \vee p_3 \vee p_5]\!]_{\mathcal{J}_2} = \boldsymbol{T} \quad ②, \ [\![\neg p_2 \vee p_4]\!]_{\mathcal{J}_2} = \boldsymbol{T}, \quad ③$$

$$[\![\neg p_3 \vee \neg p_4]\!]_{\mathcal{J}_2} = \boldsymbol{T} \quad ④, \ [\![p_1 \vee p_5 \vee \neg p_2]\!]_{\mathcal{J}_2} = \boldsymbol{T} \quad ⑤, \ [\![p_2 \vee p_3]\!]_{\mathcal{J}_2} = \boldsymbol{T} \quad ⑥,$$

$$[\![p_2 \vee \neg p_3]\!]_{\mathcal{J}_2} = \boldsymbol{T} \quad ⑦, \ [\![p_6 \vee \neg p_5]\!]_{\mathcal{J}_2} = \boldsymbol{T} \qquad ⑧$$

Under the truth assignment $\mathcal{J}_2$, ④⑦ is already satisfied.

Since $[\![p_3]\!]_{\mathcal{J}_2} = \boldsymbol{F}$, from ⑥ we know that we need $p_2 \mapsto \boldsymbol{T}$. Then $[\![\neg p_2]\!]_{\mathcal{J}_2} = \boldsymbol{F}$, we know that we need $p_4 \mapsto \boldsymbol{T}$.

Thus, $\mathrm{Unit}Pro(\mathcal{J}_2) = [p_2 \mapsto \boldsymbol{T}, p_3 \mapsto \boldsymbol{F}, p_4 \mapsto \boldsymbol{T}]$.