# Linear and Convex Optimization Homework 08

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**0. Preparation**

Complete `newton.py` and `ista.py`. The completed code (with `newton`, `damped_newton`, `soft_th` and `ista` function) is enclosed in the zip file.

**1.(a)** ***Solution:***

$$\nabla f = (e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} - e^{-x_1-0.1}, 3e^{x_1+3x_2-0.1} - 3e^{x_1-3x_2-0.1})$$

$$\nabla^2 f = \begin{pmatrix} e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1} & 3e^{x_1+3x_2-0.1} - 3e^{x_1-3x_2-0.1} \\ 3e^{x_1+3x_2-0.1} - 3e^{x_1-3x_2-0.1} & 9e^{x_1+3x_2-0.1} + 9e^{x_1-3x_2-0.1} \end{pmatrix}$$

I made some adjustments to the given `p1.py` and `utils.py`. The code is enclosed in the zip file.

Use Newton's Method to solve the problem numerically.

The solution and the number of iterations is given below.

```
In [1]: runfile('D:/Textbooks/2021-2022-1/Linear and Convex
Optimization/hw8/p1.py', wdir='D:/Textbooks/2021-2022-1/Linear and
Convex Optimization/hw8')

Newton's method with initial point = [-1.5  1. ]
  number of iterations: 5
  solution: [-3.46573590e-01 -1.41533711e-10]
  value: 2.5592666966582156
```

Fig.01. Results of Program 1(a)

The visualization of the trajectory of $x_k$ and the change of error $f(x_k) - f(x^*)$ are as follows.

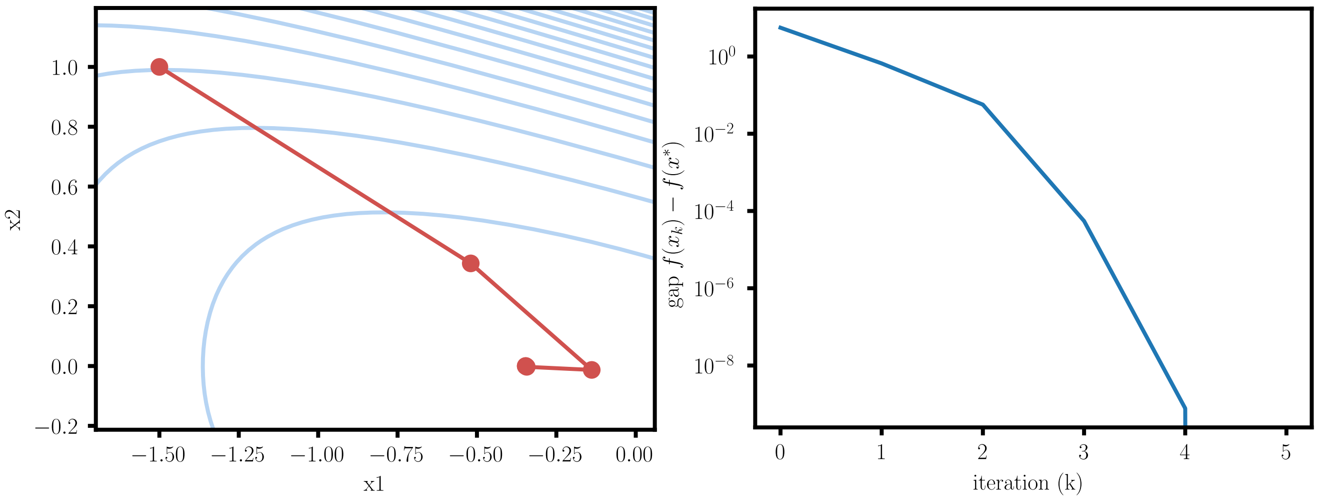


Fig.02. Trajectory of $x_k$ Produced by Newton's Method and Change of Gap $f(x_k) - f(x^*)$

**(c)** ***Solution:***

Set initial point to $(1.5,1)$. Use Newton's Method to solve the problem numerically.

The solution and the number of iterations is as follows.

```
Newton's method with initial point = [1.5 1. ]
  number of iterations: 8
  solution: [-3.46573573e-01  1.13424622e-08]
  value: 2.5592666966582165
```

Fig.03. Results of Program 1(c)

The visualization of the trajectory of $x_k$ and the change of error $f(x_k) - f(x^\star)$ are as follows.

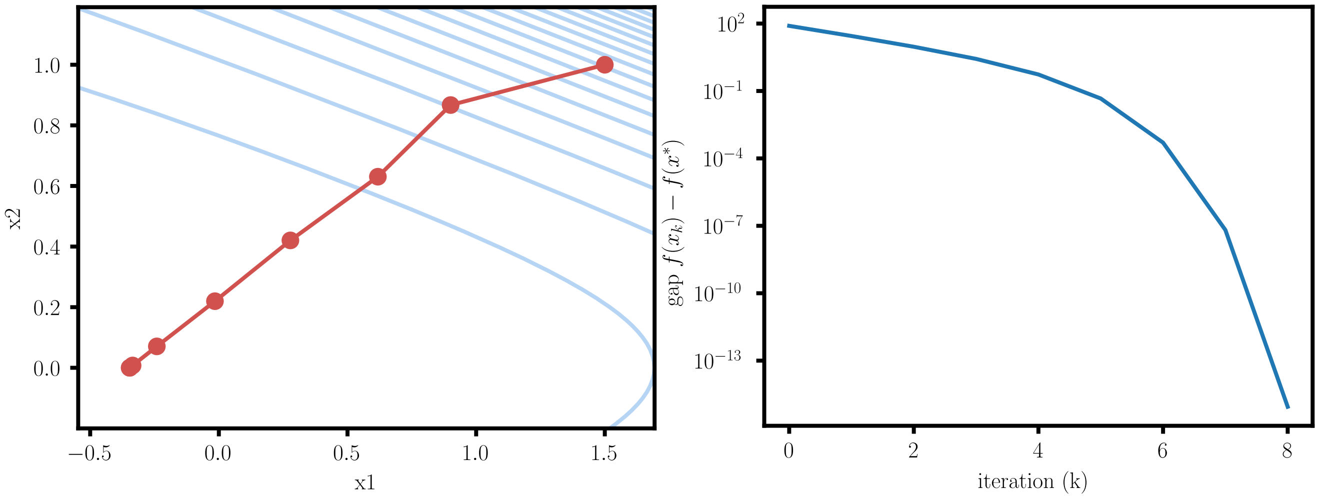


Fig.04. Trajectory of $x_k$ Produced by Newton's Method and Change of Gap $f(x_k) - f(x^\star)$

**2.(a)** ***Proof:***

$$\nabla^2 f(\boldsymbol{w}) = \frac{\partial \nabla f(\boldsymbol{w})}{\partial \boldsymbol{w}} = \sum_{i=1}^{m} \frac{\partial \nabla f(\boldsymbol{w})}{\partial\left(1 - \sigma(y_i \boldsymbol{x}_i^T \boldsymbol{w})\right)} \frac{\partial\left(1 - \sigma(y_i \boldsymbol{x}_i^T \boldsymbol{w})\right)}{\partial(y_i \boldsymbol{x}_i^T \boldsymbol{w})} \frac{\partial(y_i \boldsymbol{x}_i^T \boldsymbol{w})}{\partial \boldsymbol{w}}$$

$$= \sum_{i=1}^{m} (-y_i \boldsymbol{x}_i)\left(-\sigma'(y_i \boldsymbol{x}_i^T \boldsymbol{w})\right) y_i \boldsymbol{x}_i^T = \sum_{i=1}^{m} y_i^2 \sigma'(y_i \boldsymbol{x}_i^T \boldsymbol{w}) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

$$= \sum_{i=1}^{m} \sigma'(y_i \boldsymbol{x}_i^T \boldsymbol{w}) \boldsymbol{x}_i \boldsymbol{x}_i^T$$

(Since $|y_i| = 1 \Rightarrow y_i^2 = 1$)

*Qed.* ■

**(b)** ***Solution:***

Use Damped Newton's Method to solve the problem numerically.

The solution, the number of outer iterations and the total number of iterations in the inner loop is as follows.

```
In [2]: runfile('D:/Textbooks/2021-2022-1/Linear and Convex Optimization/hw8/
p2.py', wdir='D:/Textbooks/2021-2022-1/Linear and Convex Optimization/hw8')

Damped Newton's method
  number of iterations in outer loop: 8
  total number of iterations in inner loop: 9
  solution: [-1.46915219  4.43702925 -4.36570087]
  value: 2.8766828251471
```

Fig.05. Results of Program 2(b)

The visualization of the change of step size and the change of error $f(x_k) - f(x^\star)$ are as follows.

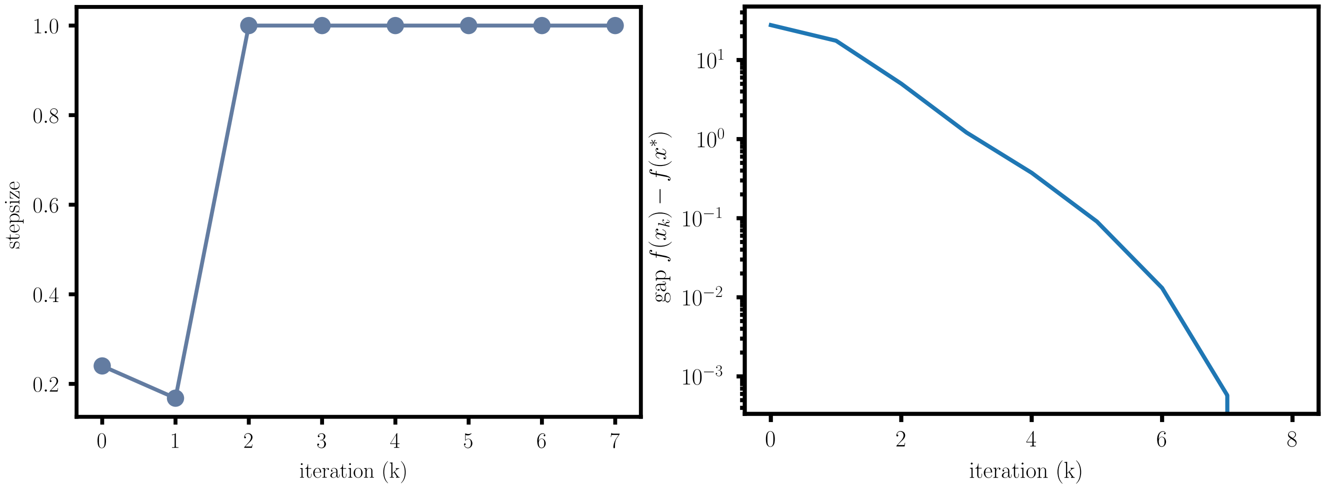


Fig.06. Change of Step Size and Change of Gap $f(x_k) - f(x^\star)$

**(c)** ***Proof:***

Use pure Newton's Method to solve the problem numerically with initial point $w_0 = (1,1,0)$.

The program cannot work out a solution and raises an exception of "singular matrix", telling that it failed to calculate the inverse of a certain matrix.

The output of the first three $w_k$ produced by pure Newton's Method and corresponding $f(w_k)$ is as follows.

```
[array([1., 1., 0.]), array([-28.48343542,  -4.83721275,  44.85294118]), array([
34330.73081749,   5658.96521501, -54499.71921383])]
[30.54235587171705, 142.6264703473908, inf]
```

Fig.07. The First Three $w_k$ Produced by Newton's Method and Corresponding $f(w_k)$

Obviously, pure Newton's Method <u>**diverges**</u> in this case. ■

**3.(a)** ***Solution:***

The Newton step can be written as

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \left(\nabla^2 f(\boldsymbol{x}_k)\right)^{-1} \nabla f(\boldsymbol{x}_k)$$

Given that $f(x) = (x-a)^4$,

$$\nabla f(x) = 4(x-a)^3, \nabla^2 f(x) = 12(x-a)^2.$$

Thus, the Newton step can be written explicitly as

$$x_{k+1} = x_k - \frac{4(x_k-a)^3}{12(x_k-a)^2} = x_k - \frac{x_k - a}{3} = \frac{2x_k + a}{3}$$

■

**(b)** ***Proof:***

$$y_{k+1} = |x_{k+1} - a| = \left|\frac{2x_k + a}{3} - a\right| = \left|\frac{2x_k - 2a}{3}\right| = \frac{2}{3}|x_k - a| = \frac{2}{3}y_k$$

*Qed.* ■

**(c)** ***Solution:***

From (b) we know $|x_{k+1} - a| = \frac{2}{3}|x_k - a|$.

$$|x_k - a| < \varepsilon \Leftrightarrow \left(\frac{2}{3}\right)^k |x_0 - a| < \varepsilon \Leftrightarrow k > O(\log\varepsilon)$$

Thus, $|x_k - a|$ decays to zero exponentially. ■

**4.(a)** ***Solution:***

I made some adjustments to the given p4.py. The code is enclosed in the zip file.

The solution and the number of iterations is as follows.

```
In [1]: runfile('D:/Textbooks/2021-2022-1/Linear and Convex
Optimization/hw8/p4.py', wdir='D:/Textbooks/2021-2022-1/Linear and
Convex Optimization/hw8')

lambda =  2
number of iterations: 169
solution: [1.00000000e+00 9.24600449e-09]
value: 6.5
```

Fig.08. Results of Program 4(a)

The visualization of the trajectory of $x_k$ and the change of error $f(x_k) - f(x^\star)$ are as follows.

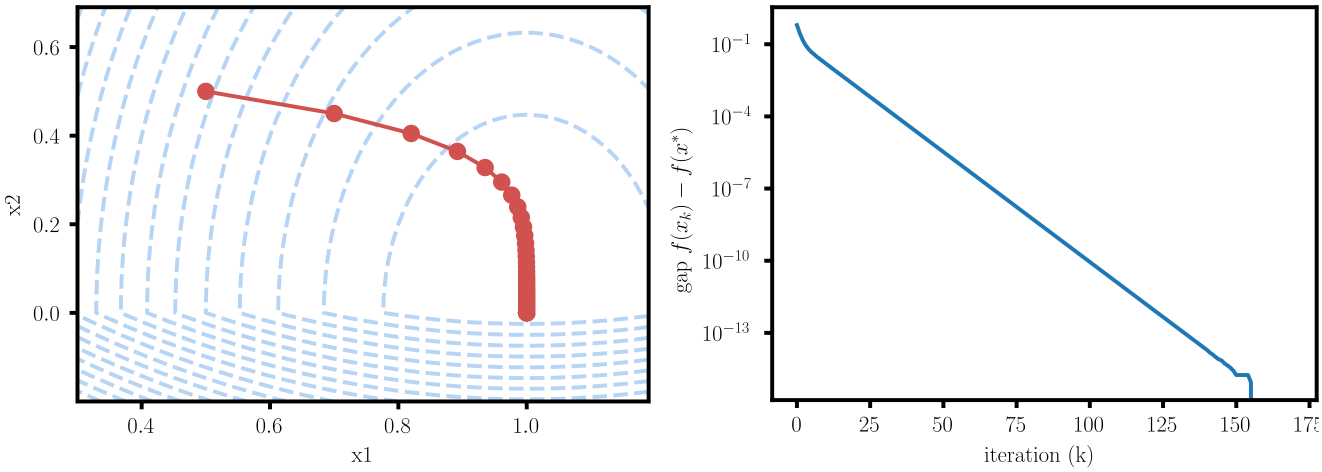


Fig.09. Trajectory of $x_k$ Produced by ISTA and Change of Gap $f(x_k) - f(x^\star)$

**(b)** ***Solution:***

Set $\lambda = 1$. The solution and the number of iterations is as follows.

**<u>There are no zeros in $w^\star$.</u>** ■

```
lambda =  1
number of iterations: 169
solution: [1.25       0.99999999]
value: 4.875
```

Fig.10. Results of Program 4(b)

The visualization of the trajectory of $x_k$ and the change of error $f(x_k) - f(x^\star)$ are as follows.

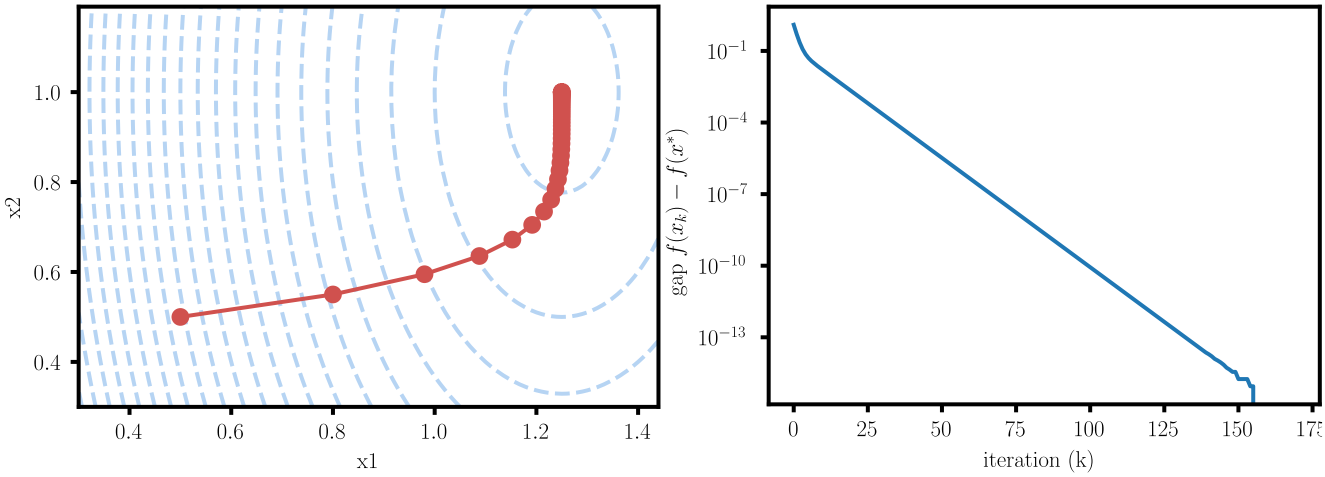


Fig.11. Trajectory of $x_k$ Produced by ISTA and Change of Gap $f(x_k) - f(x^\star)$

**(c)** ***Solution:***

Set $\lambda = 6$. The solution and the number of iterations is as follows.

Ignoring numerical errors, the solution should be $(0,0)$, **i.e.** **<u>there are two zeros in $w^\star$.</u>** ■

```
lambda =  6
number of iterations: 38
solution: [1.85659632e-09 0.00000000e+00]
value: 8.500000000000002
```

Fig.12. Results of Program 4(a)

The visualization of the trajectory of $x_k$ and the change of error $f(x_k) - f(x^\star)$ are as follows.

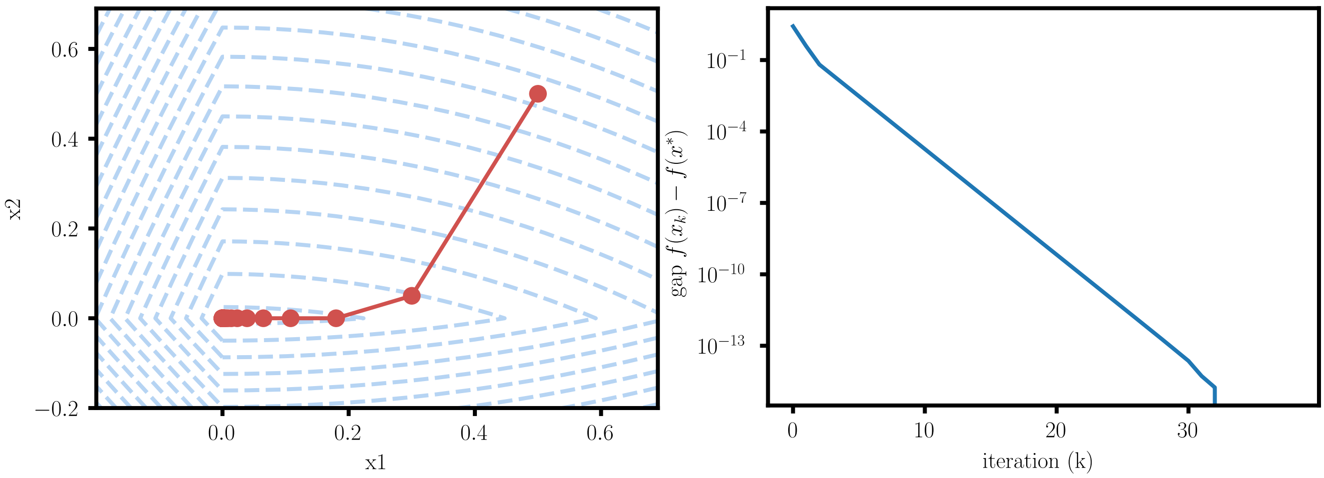


Fig.13. Trajectory of $x_k$ Produced by ISTA and Change of Gap $f(x_k) - f(x^\star)$