Algorithm Homework 06

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June 2022

1 Problem 01 - Clique with Half Size

Proof. First we prove that the problem is *NP*.

We prove that the problem is polynomial-time verifiable.

Consider the algorithm $V : \{0,1\}^* \times \{0,1\}^* \to \{0,1\},\ (x,y) \mapsto V(x,y)$. x is a $0,1$-sequence representing a graph $G = (V,E)$ while y is a $0,1$-sequence representing a clique V'. Obvious $|y| = |x|^{O(1)}$. $V(x,y)$ checks whether $|V'| = \frac{n}{2} = \frac{|V|}{2}$ and whether V' is a clique on the graph represented by x. Obvious the checking process takes $O(|E|)$ time, i.e. V terminates in $|x|^{O(1)}$ time.

Thus, V is a verifier. Therefore, the problem is polynomial-time verifiable, i.e. NP. $\square$

Let the *exact k-clique* problem on G be whether G contain a clique with size exactly k. Now we prove *exact k-clique* $\leq_K$ *clique with half size*.

We convert the *exact k-clique* problem on G into a *clique with half size* problem on G' as follows.

Let the clique on G be $\mathcal{V}_G$. Let the clique on G' be $\mathcal{V}$.

CASE 01. $k = \frac{|V|}{2}$. Construct $G'_k = (V'_k, E'_k) = G = (V,E)$.

Obvious the solutions of the two problems are exactly the same.

CASE 02. $k < \frac{|V|}{2}$.

First we construct a complete graph $G_c = (V_c, E_c)$ with $|V_c| = |V| - 2k$,

(i.e. $E_c = \{(u,v) \mid \forall u, v \in V_c,\ u \neq v\}$.)

Then we construct $G'_k = (V'_k, E'_k)$, where $V'_k = V \cup V_c$ and $E'_k = E \cup E_c \cup (V \times V'_k)$.

Obvious $\mathcal{V}$ must contain V_c and $\mathcal{V}_G = \mathcal{V} \setminus V_c$, i.e. $|\mathcal{V}_G| = |\mathcal{V}| - |V_c|$.

Thus, $|\mathcal{V}_G| = k \Longleftrightarrow |\mathcal{V}| = |\mathcal{V}_G| + |V_c| = k + (|V| - 2k) = |V| - k = \frac{|V'_k|}{2}$.

Therefore, *exact k-clique* on G' and *clique with half size* G has the same solution.

CASE 03. $k > \frac{|V|}{2}$.

First we construct a graph $G_n = (V_n, E_n)$, where $|V_n| = 2k - |V|, E_n = \varnothing$.

Then we construct $G'_k = (V'_k, E'_k)$, where $V'_k = V \cup V_n$ and $E'_k = E \cup E_n = E$.

Obvious $\mathcal{V}$ cannot contain V_c and $\mathcal{V}_G = \mathcal{V}$, i.e. $|\mathcal{V}_G| = |\mathcal{V}|$.

Thus, $|\mathcal{V}_G| = k \Longleftrightarrow |\mathcal{V}| = k = \frac{2k}{2} = \frac{|V|+2k-|V|}{2} = \frac{|V_k'|}{2}$.

Therefore, *exact k-clique* on G' and *clique with half size* G has the same solution.

Through the process above, we can convert a *k-clique* problem on G into a *clique with half size* problem on G'. Thus,

$$\textit{exact k-clique} \ \leq_K \ \textit{clique with half size}.$$

Moreover, *k-clique* $\leq_K$ *exact k-clique.*

For any *k-clique* problem, we can solve a series of *exact k-clique* problems, i.e. *exact k-clique*, *exact* $(k+1)$-*clique*, ... *exact* $|V|$-*clique*. Solving these $|V|-k+1$ problems, we can decide the solution of *k-clique*.

Since the number of the series of *exact k-clique* problem is polynomial, we have

$$\textit{k-clique} \ \leq_K \ \textit{exact k-clique} \implies \textit{k-clique} \ \leq_K \ \textit{clique with half size}.$$

Meanwhile, we know $k-clique$ problem is NP-complete.

Therefore, the problem is NP-complete. ■

2 Problem 02 - (C, V)-Knapsack

Proof. First we prove the (C, V)-*Knapsack* problem is NP.

Consider the algorithm $V : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}, (x, y) \mapsto V(x, y)$. x is a $0,1$-sequence representing $n, w_1, w_2, ...w_n, v_1, v_2, ...v_n, C, V$, y is a 0,1-sequence representing a subset of the n items, i.e. $\mathcal{I} \subset \{1, 2, ...n\}$. $V(x, y)$ checks whether y is a valid arrangement with total value of items at least V, i.e. to check whether $\sum_{i\in\mathcal{I}} w_i \leq C$ and $\sum_{i\in\mathcal{I}} v_i \geq V$. Obvious this takes at most $O(n)$ time, i.e. $V(x, y)$ terminates in $|x|^{O(1)}$ time.

Thus, V is a verifier. Then the problem is polynomial-time verifiable, i.e. NP. □

Now we prove *Subset Sum* $\leq_K$ (C, V)-*Knapsack.*

For any *Subset Sum* problem, i.e. given $n, a_1, a_2, ...a_n$ and W, decide whether exists $\mathcal{I} \in [n]$ s.t. $\sum_{i\in\mathcal{I}} a_i = W$, we can convert it into the (C, V)-*Knapsack* problems as follows.

Given n. Given $w_i = a_i, v_i = a_i$ for any $i \in [n]$. Determine whether exists a subset of items with total weight at most $C = W$ and total value at least $V = W$. The (W, W)-*Knapscak* problem returning 1 means $\exists\, \mathcal{I} \subset [n]$ s.t. $\sum_{i\in\mathcal{I}} a_i \leq W, \sum_{i\in\mathcal{I}} a_i \geq W \implies \sum_{i\in\mathcal{I}} a_i = W$.

Thus, the solutions for the two problems above are exactly the same. Therefore,

$$\textit{Subset Sum} \ \leq_K \ (C, V)\textit{-Knapsack}.$$

Meanwhile, *Subset Sum* is NP-complete.

Thus, (C, V)-*Knapsack* is also NP-complete. ■

3 Problem 3 - Subgraph Problem

Proof. First we prove that the *subgraph problem* is NP.

Consider the algorithm $V : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}, (x,y) \mapsto V(x,y)$. x is a $0,1$-sequence representing $G = (V_G, E_G), H = (V_H, E_H)$, y is a 0,1-sequence representing a mapping from V_H to V_G, noting which vertex in H is corresponding to which vertex in G. $V(x,y)$ checks that under y, whether H is a subgraph of G, i.e. to check whether $\{y(u), y(v)\} \in E$ **iff.** $\{u,v\} \in E$. Obvious this process takes at most $O(|E|^2)$, i.e. $V(x,y)$ terminates in $|x|^{O(1)}$ time.

Thus, V is a verifier. Therefore, the problem is polynomial-time verifiable, i.e. NP. □

Now we prove that *exact k-clique* $\leq_K$ *subgraph problem.*

We can convert the *exact k-clique* problem on G into a *subgraph problem* on G as follows.

For any two vertices in a clique, exists an edge between them on the original graph. Then we know the complete graph of vertices in the clique is a subgraph of the original graph.

Thus, we can construct a complete graph $\mathcal{G}$ with k vertices. Determine whether $\mathcal{G}$ is a subgraph of G. When $\mathcal{G}$ is a subgraph of G, we know exists at least k vertices on G which can induce a clique on G, i.e. exists a clique on G with size $\geq k$. Otherwise, there does not exist any clique on G with size $\geq k$.

Thus, the solutions of the two problems are exactly the same, i.e.

$$\textit{k-clique} \ \leq_K \ \textit{subgraph problem}.$$

Meanwhile, *k-clique* is NP-Complete.

Therefore, *subgraph problem* is NP-complete. ■

4 Rating and Feedback

The completion of the homework takes me one day, about 15 hours in total (including thinkings on problem 4-9 without writing a formal proof). Still, writing a formal solution is the most time-consuming part.

The ratings of each problem is as follows.

Problem	Rating
1	3
2	2
3	2

Table 1: Ratings.

This time I finish all problems on my own.