Discrete Mathematics Exercise 2

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1. ***a***)

Proof:

It's obvious that the truth value of $[\![\phi]\!]_{\mathcal{J}}$ is either ***True*** or ***False*** under any truth assignment $\mathcal{J}$.

When $[\![\phi]\!]_{\mathcal{J}} = \boldsymbol{T}$, since $\phi \vDash \psi$, we know that $[\![\psi]\!]_{\mathcal{J}} = \boldsymbol{T}$. It's plain to see that $[\![\phi \wedge \psi]\!]_{\mathcal{J}} = [\![\boldsymbol{T} \wedge \boldsymbol{T}]\!]_{\mathcal{J}} = \boldsymbol{T} = [\![\phi]\!]_{\mathcal{J}}$ and $[\![\phi \vee \psi]\!]_{\mathcal{J}} = [\![\boldsymbol{T} \vee \boldsymbol{T}]\!]_{\mathcal{J}} = \boldsymbol{T} = [\![\psi]\!]_{\mathcal{J}}$.

When $[\![\phi]\!]_{\mathcal{J}} = \boldsymbol{F}$, it's plain to see that $[\![\phi \wedge \psi]\!]_{\mathcal{J}} = [\![\boldsymbol{F} \wedge \psi]\!]_{\mathcal{J}} = \boldsymbol{F} = [\![\phi]\!]_{\mathcal{J}}$ and $[\![\phi \vee \psi]\!]_{\mathcal{J}} = [\![\boldsymbol{T} \vee \psi]\!]_{\mathcal{J}} = [\![\psi]\!]_{\mathcal{J}}$.

In short, for any truth assignment $\mathcal{J}$, the truth value of $\phi \wedge \psi$ is consistent with that of ϕ, while the truth value of $\phi \vee \psi$ is consistent with that of ψ.

In other words, if $\phi \vDash \psi$, then $\phi \wedge \psi \equiv \phi$ and $\phi \vee \psi \equiv \psi$.

QED

b)

Proof:

We already know that $\phi \vDash \phi \vee \psi$. According to the conclusion in a), $\phi \wedge (\phi \vee \psi) \equiv \phi$, which is the Absorption Law – 1.

Similarly, from the statement $\phi \wedge \psi \vDash \phi$, we could reach the conclusion that $(\phi \wedge \psi) \vee \phi \equiv \phi$. Then by the Communitive Law of disjunction, $\phi \vee (\phi \wedge \psi) \equiv (\phi \wedge \psi) \vee \phi \equiv \phi$, which is the Absorption Law – 2.

QED

2. ***a***)

Proof: The truth table of $\neg(p \oplus q)$ and $(\neg p) \oplus q$ is as follows.

p	q	$\neg(p \oplus q)$	$(\neg p) \oplus q$
T	**T**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**F**	**F**
F	**F**	**T**	**T**

For any assignment $\mathcal{J}$, $[\![\neg(p \oplus q)]\!]_{\mathcal{J}} = [\![(\neg p) \oplus q]\!]_{\mathcal{J}}$. In other words, $\neg(p \oplus q) \equiv (\neg p) \oplus q$.

QED

b)

Proof: The truth table of $p \oplus (\neg p) \oplus q$ and $\neg q$ is as follows.

p	q	$p \oplus (\neg p) \oplus q$	$\neg q$
T	**T**	**F**	**F**
T	**F**	**T**	**T**
F	**T**	**F**	**F**
F	**F**	**T**	**T**

For any assignment $\mathcal{J}$, $[\![p \oplus (\neg p) \oplus q]\!]_{\mathcal{J}} = [\![\neg q]\!]_{\mathcal{J}}$. In other words, $p \oplus (\neg p) \oplus q \equiv \neg q$.

QED

3. *Solution:*

The truth table of $\neg(p \wedge \neg(q \oplus r))$ is as follows.

p	q	r	$\neg(p \wedge \neg(q \oplus r))$
T	**T**	**T**	**F**
T	**T**	**F**	**T**
T	**F**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**T**	**F**	**T**
F	**F**	**T**	**T**
F	**F**	**F**	**T**

We notice that when $[\![p]\!]_{\mathcal{J}} = \boldsymbol{F}$, $[\![\neg(p \wedge \neg(q \oplus r))]\!]_{\mathcal{J}} = \boldsymbol{T}$. We also notice that when $[\![p]\!]_{\mathcal{J}} = \boldsymbol{T}, [\![q]\!]_{\mathcal{J}} = \boldsymbol{T}, [\![r]\!]_{\mathcal{J}} = \boldsymbol{F}$ or $[\![p]\!]_{\mathcal{J}} = \boldsymbol{T}, [\![q]\!]_{\mathcal{J}} = \boldsymbol{F}, [\![r]\!]_{\mathcal{J}} = \boldsymbol{T}$, $[\![\neg(p \wedge \neg(q \oplus r))]\!]_{\mathcal{J}} = \boldsymbol{T}$.

Thus, we could construct a disjunctive normal form $\psi = (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p)$ such that $\phi \vDash \psi$.

Now we prove that $\psi \vDash \phi$.

Suppose there exists a truth assignment $\mathcal{J}$ such that $[\![\psi]\!]_{\mathcal{J}} = \boldsymbol{T}$. There are three possible cases:

1) $[\![\neg p]\!]_{\mathcal{J}} = \boldsymbol{T}$, namely $[\![p]\!]_{\mathcal{J}} = \boldsymbol{F}$. In this case, we know from the truth value table that $[\![\phi]\!]_{\mathcal{J}} = \boldsymbol{T}$.

2) $[\![p \wedge q \wedge \neg r]\!]_{\mathcal{J}} = \boldsymbol{T}$, namely $[\![p]\!]_{\mathcal{J}} = \boldsymbol{T}, [\![q]\!]_{\mathcal{J}} = \boldsymbol{T}, [\![r]\!]_{\mathcal{J}} = \boldsymbol{F}$. In this case, we know from the truth value table that $[\![\phi]\!]_{\mathcal{J}} = \boldsymbol{T}$.

3) $[\![p \wedge \neg q \wedge r]\!]_{\mathcal{J}} = \boldsymbol{T}$, namely $[\![p]\!]_{\mathcal{J}} = \boldsymbol{T}, [\![q]\!]_{\mathcal{J}} = \boldsymbol{F}, [\![r]\!]_{\mathcal{J}} = \boldsymbol{T}$. In this case, we know from the truth value table that $[\![\phi]\!]_{\mathcal{J}} = \boldsymbol{T}$.

So as long as $[\![\psi]\!]_{\mathcal{J}} = \boldsymbol{T}$, $[\![\phi]\!]_{\mathcal{J}} = \boldsymbol{T}$. In other words, $\psi \vDash \phi$.

Therefore, $\psi \equiv \phi$.

So $\psi = (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p)$ is a disjunctive normal form of ϕ.