Stochastic Process Homework 02

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Mar.07 - Mar.22, 2022

0 Reference

This homework is completed with the help of discussions with Sun Yilin and Ji Yikun. The completion of **2.3** is based on the notes of AI2613 last year.

1 Optimal Coupling

Proof. Since

$$D_{\mathrm{TV}}(\mu,\nu)=\frac{1}{2}\sum_{i\in\Omega}|\mu(i)-\nu(i)|=\sum_{i\in\Omega,\nu(i)\geq\mu(i)}(\nu(i)-\mu(i)), \tag{1}$$

it is natural to maximize $\omega(i,i)$, i.e. set $\omega(i,i)=\min\{\mu(i),\nu(i)\}$.

Considering $\mu(i)-\min\{\mu(i),\nu(i)\}=\max\{0,\mu(i)-\nu(i)\},\nu(j)-\min\{\mu(j),\nu(j)\}=\max\{0,\nu(j)-\mu(j)\}$, we construct a feasible coupling ω as follows.

$$\omega(i,j)=\begin{cases}\min\{\mu(i),\nu(i)\}, & i=j\\ C\max\{0,\mu(i)-\nu(i)\}\max\{0,\nu(j)-\mu(j)\}, & i\neq j\end{cases}$$

Now we solve C. We know

$$\begin{aligned}\mu(i)&=\sum_{j\in\Omega}\omega(i,j)\\&=\sum_{j\neq i\in\Omega}C\max\{0,\mu(i)-\nu(i)\}\max\{0,\nu(j)-\mu(j)\}+\min\{\mu(i),\nu(i)\}\\&=\min\{\mu(i),\nu(i)\}+C\max\{0,\mu(i)-\nu(i)\}\sum_{j\neq i\in\Omega}\max\{0,\nu(j)-\mu(j)\}\end{aligned}$$

CASE 01. When $\min(\mu(i),\nu(i))=\mu(i)$, obvious the equation above holds.

CASE 02. When $\min(\mu(i),\nu(i))=\nu(i)$,

by (1), we have

$$
\begin{aligned}
\mu(i) &= \nu(i) + C\left(\mu(i)-\nu(i)\right) \sum_{j \neq i \in \Omega, \nu(j) \geq \mu(j)} \left(\nu(j)-\mu(j)\right) \\
&= \nu(i) + C\left(\mu(i)-\nu(i)\right) D_{\mathrm{TV}}\left\{\mu, \nu\right\}
\end{aligned}
$$

Thus,

$$
C = \frac{1}{D_{\mathrm{TV}}\left(\mu, \nu\right)}
$$

i.e.

$$
\omega(i,j) = \begin{cases} \min\left\{\mu(i), \nu(i)\right\}, & i = j \\ \frac{1}{D_{\mathrm{TV}}(\mu,\nu)} \max\left\{0, \mu(i)-\nu(i)\right\} \max\left\{0, \nu(j)-\mu(j)\right\}, & i \neq j \end{cases}
$$

By the symmetry of μ and ν, i and j in the equation, we know

$$
\nu(j) = \sum_{i \in \Omega} \omega(i,j)
$$

also holds. Thus, ω is a valid coupling.

Now we prove $\mathbf{Pr}_{(X,Y)\sim\omega}\left[X \neq Y\right] = D_{\mathrm{TV}}\left(\mu, \nu\right)$.

$$
\begin{aligned}
\mathbf{Pr}_{(X,Y)\sim\omega}\left[X = Y\right] &= \sum_{i \in \Omega} \omega_(i,i) \\
&= \sum_{i \in \Omega} \min\left\{\mu(i), \nu(i)\right\} \\
&= \sum_{i \in \Omega, \mu(i) \geq \nu(i)} \nu(i) + \sum_{i \in \Omega, \mu(i) < \nu(i)} \mu(i) \\
&= \sum_{i \in \Omega} \nu(i) + \sum_{i \in \Omega, \nu(i) \geq \mu(i)} \left(\mu(i)-\nu(i)\right) \\
&= 1 - \sum_{i \in \Omega, \nu(i) \geq \mu(i)} \left(\nu(i)-\mu(i)\right) \\
&= 1 - D_{\mathrm{TV}}\left(\mu, \nu\right) && \text{(by (1))}
\end{aligned}
$$

i.e.,

$$
\mathbf{Pr}_{(X,Y)\sim\omega}\left[X \neq Y\right] = 1 - \mathbf{Pr}_{(X,Y)\sim\omega}\left[X = Y\right] = D_{\mathrm{TV}}\left(\mu, \nu\right).
$$

Thus, there exists a coupling

$$
\omega(i,j) = \begin{cases} \min\left\{\mu(i), \nu(i)\right\}, & i = j \\ \frac{1}{D_{\mathrm{TV}}(\mu,\nu)} \max\left\{0, \mu(i)-\nu(i)\right\} \max\left\{0, \nu(j)-\mu(j)\right\}, & i \neq j \end{cases}
$$

s.t. $\mathbf{Pr}_{(X,Y)\sim\omega}\left[X \neq Y\right] = D_{\mathrm{TV}}\left(\mu, \nu\right)$. ■

2 Stochastic Dominance

2.1 Binomial Distribution Case

Proof. First we prove the sufficiency.

Assume $p \geq q$. Let $X \sim \texttt{Binom}(n,p), Y \sim \texttt{Binom}(n,q)$.

For any $a \in \Omega$,

CASE 01. When $a \geq n+1$, $\mathbf{Pr}\left[X \geq a\right] = \mathbf{Pr}\left[Y \geq a\right] = 0$.

CASE 02. When $a \leq -1$, $\mathbf{Pr}\left[X \geq a\right] = \mathbf{Pr}\left[Y \geq a\right] = 1$.

CASE 03. When $1 \leq a \leq n$, we have

$$\mathbf{Pr}\left[X \geq a\right] = \sum_{i=a}^{n} \mathbf{Pr}\left[X = i\right] = \sum_{i=a}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$
$$\mathbf{Pr}\left[Y \geq a\right] = \sum_{i=a}^{n} \mathbf{Pr}\left[Y = i\right] = \sum_{i=a}^{n} \binom{n}{i} q^i (1-q)^{n-i}$$

Consider $f(x) = \sum_{i=a}^{n} \binom{n}{i} x^i (1-x)^{n-i}$. When $x \in [0,1]$,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x} &= nx^{n-1} + \sum_{i=a}^{n-1} \binom{n}{i} \left(ix^{i-1}(1-x)^{n-i} - (n-i)x^i(1-x)^{n-i-1}\right) \\ &= nx^{n-1} + \sum_{i=a}^{n-1} \binom{n}{i} \left((i-nx)x^{i-1}(1-x)^{n-i-1}\right) \geq 0\end{aligned}$$

i.e. $f(x)$ is monotonously increasing on $[0,1]$.

Since $p, q \in [0,1]$, $p \geq q$, we know $f(p) \geq f(q)$, i.e.

$$\mathbf{Pr}\left[X \geq a\right] \geq \mathbf{Pr}\left[Y \geq a\right].$$

Thus, for any $a \in \Omega$, $\mathbf{Pr}\left[X \geq a\right] \geq \mathbf{Pr}\left[Y \geq a\right]$, i.e. $\texttt{Binom}(n,p) \succeq \texttt{Binom}(n,q)$.

Now we prove the necessity.

Assume $\texttt{Binom}(n,p) \succeq \texttt{Binom}(n,q)$. Let $X \sim \texttt{Binom}(n,p), Y \sim \texttt{Binom}(n,q)$.

Then we have

$$\begin{aligned}\mathbf{Pr}\left[X \geq n\right] \geq \mathbf{Pr}\left[Y \geq n\right] &\iff \mathbf{Pr}\left[X = n\right] \geq \mathbf{Pr}\left[Y = n\right] \\ &\iff p^n \geq q^n \\ &\iff p \geq q.\end{aligned}$$

In conclusion, for any $p, q \in [0,1]$, $\texttt{Binom}(n,p) \succeq \texttt{Binom}(n,q)$ **iff.** $p \geq q$. ∎

2.2 Monotone Coupling

Proof. First we prove the sufficiency.

Assume exists a monotone coupling ω of μ and ν. Then we know

$$1 = \mathbf{Pr}_{(X,Y)\sim\omega}\left[X \geq Y\right] = \sum_{i\in\Omega}\sum_{j\leq i}\omega(i,j) = 1 = \sum_{i\in\Omega}\sum_{j\in\Omega}\omega(i,j)$$

Thus, when $i < j,\ \omega(i,j) = 0$.

$$\begin{aligned}
\forall a \in \Omega, \quad \mathbf{Pr}_{X\sim\mu}\left[X \geq a\right] &= \mathbf{Pr}_{(X,Y)\sim\omega}\left[X \geq a\right] = \sum_{i\in\Omega, i\geq a}\mathbf{Pr}_{(X,Y)\sim\omega}\left[X = i\right] \\
&= \sum_{i\in\Omega, i\geq a}\sum_{j\in\Omega}\omega(i,j) \ = \sum_{j\in\Omega}\sum_{i\in\Omega, i\geq a}\omega(i,j) \\
&\geq \sum_{j\in\Omega, j\geq a}\sum_{i\in\Omega, i\geq j}\omega(i,j) \ = \sum_{j\in\Omega, j\geq a}\sum_{i\in\Omega, i\geq j}\omega(i,j) + 0 \\
&= \sum_{j\in\Omega, j\geq a}\sum_{i\in\Omega, i\geq j}\omega(i,j) \ + \sum_{j\in\Omega, j\geq a}\sum_{i\in\Omega, i< j}\omega(i,j) \\
&= \sum_{j\in\Omega, j\geq a}\sum_{i\in\Omega}\omega(i,j) \ = \sum_{j\in\Omega, j\geq a}\mathbf{Pr}_{(X,Y)\sim\omega}\left[Y = j\right] \\
&= \mathbf{Pr}_{(X,Y)\sim\omega}\left[Y \geq a\right] \\
&= \mathbf{Pr}_{Y\sim\nu}\left[Y \geq a\right].
\end{aligned}$$

i.e. $\quad \mu \succeq \nu$.

Now we prove the necessity, i.e. to construct a coupling ω s.t. $\mathbf{Pr}_{(X,Y)\sim\omega}\left[X \geq Y\right] = 1$.

Assume $\mu \succeq \nu$. We construct ω as follows.

First we set $\omega(i,j) = 0$ for $i < j$.

We have $\mu(n) = \mathbf{Pr}_{X\sim\mu}\left[X \geq n\right] \geq \mathbf{Pr}_{Y\sim\nu}\left[Y \geq n\right] = \nu(n)$. Set $\omega(n,n) = \nu(n)$.

For the remaining part, we determine $\omega(i,j)$ in the following order

$$\begin{aligned}
&\omega(n, n-1), \omega(n-1, n-1), \\
&\omega(n, n-2), \omega(n-1, n-2), \omega(n-2, n-2), \\
&..., \\
&\omega(n, 1), \omega(n-1, 1), ..., \omega(1, 1)
\end{aligned}$$

by the following method.

$$\omega(i,j) = \min\left\{\mu(i) - \sum_{k=j+1}^{n}\omega(i,k), \quad \nu(j) - \sum_{k=i+1}^{n}\omega(k,j)\right\}$$

We prove ω is a valid coupling as follows. Obvious $\mu(i) = \sum_{j\in\Omega}\omega(i,j), \nu(j) = \sum_{j\in\Omega}\omega(i,j)$, which is maintained and guaranteed by how we determined $\omega(i,j)$.

Meanwhile, $\omega(i,j) \geq 0$, (since $\sum_{k=j+1}^{n}\omega(i,k) \leq \mu(i), \sum_{k=i+1}^{n}\omega(k,j) \leq \nu(j)$).

Thus, ω is a valid coupling s.t. $\mathbf{Pr}_{(X,Y)\sim\omega}[X \geq Y] = 1$.

In conclusion, $\mu \succeq \nu$ **iff.** exists a coupling ω s.t. $\mathbf{Pr}_{(X,Y)\sim\omega}[X \geq Y] = 1$. ■

2.3 Erdős–Rényi Model Random Graph

- *The completion of this problem is based on the notes of* **AI2613** *last year.*

Proof. We can generate $G \sim \mathcal{G}(n,p)$ and $H \sim \mathcal{G}(n,q)$ simultaneously, where $p, q \in [0,1],\ p \geq q$.

For each pair of vertices (u,v), we independently pick $r_{\{u,v\}} \sim \texttt{Uniform}(0,1)$. We determine whether graph G and H has edge $\{u,v\}$ as follows.

$$\begin{cases} \text{both } G \text{ and } H \text{ have edge } \{u,v\}\,, & r_{\{u,v\}} \in [0,q] \\ \text{only } G \text{ has edge } \{u,v\}\,, & r_{\{u,v\}} \in (q,p] \\ \text{neither has edge } \{u,v\}\,, & r_{\{u,v\}} \in (p,1] \end{cases}$$

Obvious H is always a subgraph of G. If H is connected, G is for sure connected.

Thus, $\mathbf{Pr}[G \text{ is connected}] \geq \mathbf{Pr}[H \text{ is connected}]$.

Therefore, for any $p, q \in [0,1]$ s.t. $p \geq q$,

$$\mathbf{Pr}_{G\sim\mathcal{G}(n,p)}[G \text{ is connected}] \geq \mathbf{Pr}_{H\sim\mathcal{G}(n,q)}[H \text{ is connected}]\,.$$

Qed. ■

3 Total Variation Distance is Non-Increasing

Proof. By **Coupling Lemma**, we know

$$\Delta(t) \leq \mathbf{Pr}_{(X,Y)\sim\omega_t}\left[X \neq Y\right],$$

where ω_t is a coupling of μ_t and π.

We construct a coupling ω_t as follows, where $(X_t, Y_t) \sim \omega_t, (X_{t+1}, Y_{t+1}) \sim \omega_{t+1}$.

$$\begin{cases} X_{t+1} = X_t = Y_t = Y_{t+1}, & \text{if } X_t = Y_t \\ X_{t+1} \sim \mu_t,\ Y_{t+1} \sim \pi, & \text{if } X_t \neq Y_t \end{cases}$$

Then we have

$$\begin{aligned} \Delta(t+1) &\leq \mathbf{Pr}_{(X,Y)\sim\omega_{t+1}}\left[X \neq Y\right] \\ &= \mathbf{Pr}\left[X_{t+1} \neq Y_{t+1}\right] \\ &\leq \mathbf{Pr}\left[X_t \neq Y_t\right] \\ &= \Delta(t). \end{aligned}$$

Qed. ■