Discrete Mathematics Exercise 10

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1. ***Proof:***

For any $(m,n) \in \mathbb{N} \times \mathbb{N}$, exists $x = 2^m(2n+1) - 1 \in \mathbb{N}$. Thus, f is an injection.

For any $x \in \mathbb{N}$,

1) x is even.

   Exists $m = 0, n = x/2 \in \mathbb{N}$ **s.t.** $x = 2^m(2n+1) - 1$.

2) x is odd. Therefore, $x + 1$ is even.

   Let $x + 1 = \prod_{i=1}^{s} {p_i}^{r_i}$, where all p_i are different prime integers and r_i are positive integers.

   Since $2 \mid x + 1$, let $p_1 = 2$, $m = r_1 \in \mathbb{N}$.

   Thus, for $i > 1$, p_i is odd.

   Thus, $\prod_{i=2}^{s} {p_i}^{r_i}$ is odd, let $\prod_{i=1}^{s} {p_i}^{r_i} = 2n+1$, then $n \in \mathbb{N}$.

Thus, for any $x \in \mathbb{N}$, exists $(m,n) \in \mathbb{N} \times \mathbb{N}$ **s.t.** $f(m,n) = x$. So f is a surjection.

Since f is both an injection and a surjection, f is a one-to-one correspondence between $\mathbb{N} \times \mathbb{N}$

QED

2. ***Solution:***

Let $C_0 = \{\, a \in [0,1] \mid \forall b \in B, a \neq G(b) \,\} = \{0,1\}, D_0 = \{\, F(a) \in (0,1) \mid a \in C_0 \,\} = \{1/3, 2/3\}$.

$C_1 = \{\, g(b) \in [0,1] \mid a \in D_0 \,\} = \{\, a \in [0,1] \backslash C_0 \mid \forall b \in B \backslash D_0, a \neq G(b) \,\} = \{1/3, 2/3\}$.

$D_1 = \{\, F(a) \in (0,1) \mid a \in C_1 \,\} = \{4/9, 5/9\}$.

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$$C_{n+1} = \{\, g(b) \in [0,1] \mid a \in D_n \,\} = \{\, a \in [0,1] \backslash \cup_{i=0}^{n} C_i \mid \forall b \in B \backslash \cup_{i=0}^{n} D_i \,, a \neq G(b) \,\}$$
$$= \left\{ \frac{3^{n+1}-1}{2 \cdot 3^{n+1}}, \frac{3^{n+1}+1}{2 \cdot 3^{n+1}} \right\}.$$

$$D_{n+1} = \{\, F(a) \in (0,1) \mid a \in C_{n+1} \,\} = \left\{ \frac{3^{n+2}-1}{2 \cdot 3^{n+2}}, \frac{3^{n+2}+1}{2 \cdot 3^{n+2}} \right\}.$$

Let $C = \cup_{i=0}^{\infty} C_i = \left\{ a \mid \left(a = \frac{3^n-1}{2 \cdot 3^n} \vee a = \frac{3^n+1}{2 \cdot 3^n} \right) \wedge n \in \mathbb{N} \right\}$,

$D = \cup_{i=0}^{\infty} D_i = \left\{ a \mid \left(a = \frac{3^n-1}{2 \cdot 3^n} \vee a = \frac{3^n+1}{2 \cdot 3^n} \right) \wedge n \in \mathbb{N}^+ \right\}$.

By Berstein's Theorem, we know $F'(x) = 1/3 + x/3$ is a bijection from C into D and $G'(x) = x$ is a bijection from $[0,1] \backslash C$ into $(0,1) \backslash D$.

We can construct a function $H(x) = \begin{cases} 1/3 + x/3, & x \in C \\ x, & x \notin C \end{cases}$ that is a bijection from $[0,1]$ into $(0,1)$.

3. a) ***Proof:*** There exists a bijection $F(x) = x$ from A into A. Thus, $A \approx A$. **QED**

b) ***Proof:***

Suppose $A \approx B$, then exists a bijection $F: A \to B$.

Thus, for any $a \in A$ exists only one $b \in B$ such that $F(a) = b$ and for any $b \in B$ exists only one

$a \in A$ such that $F(a) = b$. **(*)**

Let $F^{-1} = \{(a,b) \in A \times B \mid F(b) = a\}$.

Then for any $b \in B$, exists $a \in A$ **s.t.** $F(a) = b$, **i.e.** $b(F^{-1})a$.

For any $b \in B, a, a' \in A$, if $b(F^{-1})a \wedge b(F^{-1})a'$ by (*) we know $a = a'$. (Since $F(a) = F(a')$)

Thus, F^{-1} is a function.

By (*) we know for any $a \in A$ exists only one $b \in B$ such that $F^{-1}(b) = a$ and for any $b \in B$ exists only one $a \in A$ such that $F^{-1}(b) = a$, **i.e.** F^{-1} is a bijection.

Thus, exists a bijection $F^{-1}: B \to A$. Therefore, $B \approx A$.

QED

c) ***Proof:***

Suppose $A \approx B$ and $B \approx C$, then exists two bijections $F: A \to B$ and $G: B \to C$.

Therefore, F is both an injection and a surjection and so is G. Also, we know $G \circ F: A \to C$.

Since F is an injection and so is G, we know $G \circ F$ is an injection.

Since F is a surjection and so is G, we know $G \circ F$ is a surjection.

Thus, $G \circ F: A \to C$ is a bijection.

Therefore, $A \approx C$.

QED

4. a) ***Proof:***

For any $a \in A$, exists a $b = [a]_{\mathcal{R}} \in B$ **s.t.** $(a,b) \in F$.

For any $a \in A, b, b' \in B$, if aFb and aFb', then $b = b' = [a]_{\mathcal{R}}$.

Thus, F is a function from A into B.

QED

b) ***Proof:***

For any $b \in B$, exists an $a \in A$ **s.t.** $b = [a]_{\mathcal{R}}$, **i.e.** aFb, **i.e.** $F(a) = b$.

Thus, F is a surjection from A into B.

QED

c) ***Proof:***

For any $a, b \in A$, $(a,b) \in \mathcal{R}$ **iff.** $[a]_{\mathcal{R}} = [b]_{\mathcal{R}}$ **iff.** $F(a) = F(b)$.

Thus, $\mathcal{R} = \{\, (a,b) \mid F(a) = F(b) \,\}$.

QED

5. a) ***Proof:***

1) For any $a_1 \in A_1$, $a_2 \in A_2$,

$a_1 \mathcal{R}_1 a_1 \wedge a_2 \mathcal{R}_2 a_2$, **i.e.** $\big((a_1,a_2),(a_1,a_2)\big) \in (A_1 \times A_2) \times (A_1 \times A_2)$ **i.e.** $(a_1,a_2)\mathcal{R}(a_1,a_2)$.

Thus, $\mathcal{R}$ is reflexive.

2) For any $a_1, b_1 \in A_1$, $a_2, b_2 \in A_2$,

if $(a_1,a_2)\mathcal{R}(b_1,b_2)$, then $a_1\mathcal{R}_1 b_1 \wedge a_2 \mathcal{R}_2 b_2$, **i.e.** $a_2 \mathcal{R}_2 b_2 \wedge a_1 \mathcal{R}_1 b_1$.

Thus, $(b_1,b_2)\mathcal{R}(a_1,a_2)$.

In other words, $\mathcal{R}$ is symmetric.

3) For any $a_1, b_1, c_1 \in A_1$, $a_2, b_2, c_2 \in A_2$,

if $(a_1,a_2)\mathcal{R}(b_1,b_2)$ and $(b_1,b_2)\mathcal{R}(c_1,c_2)$, then $a_1\mathcal{R}_1 b_1 \wedge a_2\mathcal{R}_2 b_2 \wedge b_1 \mathcal{R}_1 c_1 \wedge b_2 \mathcal{R}_2 c_2$.

Thus, $a_1 \mathcal{R}_1 c_1 \wedge a_2 \mathcal{R}_2 c_2$. (Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are equivalence relations)

Thus, $(a_1, a_2)\mathcal{R}(c_1, c_2)$.
In other words, $\mathcal{R}$ is transitive.

Thus, $\mathcal{R}$ is an equivalence relation on $A_1 \times A_2$.

QED

b) *Proof:*

By definition we know $B_1 = \{ [a]_{\mathcal{R}_1} \mid a \in A_1 \}, B_2 = \{ [a]_{\mathcal{R}_2} \mid a \in A_2 \},$
$B = \{ [(a_1, a_2)]_{\mathcal{R}} \mid a_1 \in A_1, a_2 \in A_2 \}$.

Let $F = \left\{ \left(([a_1]_{\mathcal{R}_1}, [a_2]_{\mathcal{R}_2}), [(a_1, a_2)]_{\mathcal{R}} \right) \in (B_1 \times B_2) \times B \mid a_1 \in A_1, a_2 \in A_2 \right\}$.

First, we prove that F is a function.

1) For any $x \in B_1 \times B_2$, exists $a_1 \in A_1$ and $a_2 \in A_2$ **s.t.** $x = \left([a_1]_{\mathcal{R}_1}, [a_2]_{\mathcal{R}_2}\right)$.
   Thus, exists $y = [(a_1, a_2)]_{\mathcal{R}} \in B$ **s.t.** $x\ F\ y$.
2) For any $x \in B_1 \times B_2, y, y' \in B$, exists $a_1 \in A_1$ and $a_2 \in A_2$ **s.t.** $x = \left([a_1]_{\mathcal{R}_1}, [a_2]_{\mathcal{R}_2}\right)$
   if $x\ F\ y$ and $x\ F\ y'$, then $y = [(a_1, a_2)]_{\mathcal{R}} = y'$.

Thus, F is a function.

Then we prove F is an injection.
For any $x \in B_1 \times B_2$, exists $a_1 \in A_1$ and $a_2 \in A_2$ **s.t.** $x = \left([a_1]_{\mathcal{R}_1}, [a_2]_{\mathcal{R}_2}\right)$.
Then exists $y = [(a_1, a_2)]_{\mathcal{R}} \in B$ **s.t.** $x\ F\ y$.

Now we prove F is a surjection.
For any $x \in B$, exists $a_1 \in A_1$ and $a_2 \in A_2$ **s.t.** $y = [(a_1, a_2)]_{\mathcal{R}} \in B$.
Then exists $x = \left([a_1]_{\mathcal{R}_1}, [a_2]_{\mathcal{R}_2}\right) \in B_1 \times B_2$ **s.t.** $x\ F\ y$.

Thus, F is a bijection from $B_1 \times B_2$ into B.
In other words, $B_1 \times B_2 \approx B$.

QED