Probability Theory and Mathematical Statistics 概率统计

Homework 1018

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10/20

3-10. 解: $P(X=0)=1-0.6=0.4$ $P(X=1)=0.6$

$P(Y=1)=P(Y=1|X=0)P(X=0)+P(Y=1|X=1)P(X=1)=\frac{2}{5}$

$P(Y\neq 1)=1-P(Y=1)=\frac{3}{5}$

$P(X=0|Y=1)=\dfrac{P(Y=1|X=0)P(X=0)}{P(Y=1)}=\frac{1}{4}$ $P(X=1|Y=1)=1-\frac{1}{4}=\frac{3}{4}$

$P(X=0|Y\neq 1)=\dfrac{P(Y=2|X=0)P(X=0)+P(Y=3|X=0)P(X=0)}{P(Y\neq 1)}=\frac{1}{2}$ $P(X=1|Y\neq 1)=1-1/2=1/2$

$\therefore$ $\{Y=1\}$. $\{Y\neq 1\}$条件下X的条件分布律如下:

$X\|Y=1$	0	1
$P(X\|Y=1)$	$\frac{1}{4}$	$\frac{3}{4}$

$X\|Y\neq 1$	0	1
$P(X\|Y\neq 1)$	$\frac{1}{2}$	$\frac{1}{2}$

□

3-11. 解: $P(X_1+X_2+\cdots+X_n=r)=C_n^r p^r(1-p)^{n-r}$ (n重Bernoulli试验. 二项分布)

$P(X_i=1\cap X_1+X_2+\cdots+X_n=r)=C_{n-1}^{r-1}p^r(1-p)^{n-r}$ $\therefore P(X_i=1|X_1+\cdots+X_n=r)=\dfrac{C_{n-1}^{r-1}}{C_n^r}$

$P(X_i=0\cap X_1+X_2+\cdots+X_n=r)=C_{n-1}^{r}p^r(1-p)^{n-r}$ $\therefore P(X_i=0|X_1+\cdots+X_n=r)=\dfrac{C_{n-1}^{r}}{C_n^r}$

$X_i\|X_1+\cdots+X_n=r$	0	1
$P(X_i\|X_1+\cdots+X_n=r)$	$\frac{n-r}{n}$	$\frac{r}{n}$

□

3-14. 解: (1) ① $x>0$时.

$f_X(x)=\int_{-\infty}^{+\infty}f(x,y)\,dy=\int_0^{+\infty}\frac{x^3}{2}e^{-x(1+y)}\,dy$

$=\int_0^{+\infty}\frac{x^2}{2}e^{-x(1+y)}\,d[x(1+y)]=\frac{x^2}{2}e^{-x}$

② $x\leq 0$时. $f_X(x)=0$

$\therefore f_X(x)=\begin{cases}\frac{x^2}{2}e^{-x}, & x>0\\ 0, & x\leq 0\end{cases}$ □

(2) $x>0$ $\quad f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{x^3}{2}e^{-x(1+y)}}{\frac{x^2}{2}e^{-x}} = xe^{-xy}$

$\therefore f_{Y|X}(y|x) = \begin{cases} xe^{-xy}, & y>0 \\ 0, & y\leq 0 \end{cases} \quad (x>0)$ □

$f_{Y|X}(y|0.5) = \begin{cases} 0.5e^{-0.5y}, & y>0 \\ 0, & y\leq 0 \end{cases}$ □

(3) $P(Y\geq 1|X=0.5) = \int_1^{+\infty} f_{Y|X}(y|0.5)\,dy = \int_1^{+\infty}\frac{1}{2}e^{-\frac{y}{2}}\,dy = -\int_1^{+\infty} d(e^{-\frac{y}{2}})$

$= -0 + e^{-\frac{1}{2}} = e^{-0.5}$ □

3-15. 解：(1) ① $0<x<2$. $\quad f(x,y) = f_{Y|X}(y|x)\, f_X(x) = \frac{2+x}{6}\cdot f_{Y|X}(y|x)$

①' $f(x,y) = \frac{2+x}{6}\cdot\frac{1+xy}{1+x/2} = \frac{xy+1}{3} \quad (0<y<1)$

②' $f(x,y) = 0 \quad$ (otherwise)

② $x\leq 0$ 或 $x\geq 2$. $\quad f(x,y)=0$

综上：(X,Y)的联合概率密度为 $f(x,y) = \begin{cases} \frac{xy+1}{3} & (0<x<2,\ 0<y<1) \\ 0 & (\text{otherwise}) \end{cases}$ □

(2) $f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$

① $0<y<1$ 时. $\quad f_Y(y) = \int_0^2 \frac{xy+1}{3}\,dx = \frac{2}{3}y + \frac{2}{3}$

② 其它： $\quad f_Y(y) = 0$

$\therefore (X,Y)$ 关于 Y 的边缘概率密度为 $f_Y(y) = \begin{cases} \frac{2y+2}{3} & (0<y<1) \\ 0 & (\text{otherwise}) \end{cases}$ □

3-17. 解：(1)

Y \ X	-1	0	2	$P(Y=y)$
0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

由 $P(XY=0)=1$ 知 $P(X=-1, Y=1) = P(X=2, Y=1) = 0$

随后可求得联合分布律如左表.

(3) 要求 $0<y<1$.

① $0<x<2$. $f_{X|Y}(x|y)=\dfrac{f(x,y)}{f_Y(y)}=\dfrac{\frac{xy+1}{3}}{\frac{2y+2}{3}}=\dfrac{xy+1}{2y+2}$

② 其它. $f_{X|Y}(x|y)=0$

∴ 在$\{Y=y\}$条件下, X条件概率密度 $f_{X|Y}(x|y)=\begin{cases}\dfrac{xy+1}{2y+2} & (0<x<2)\\ 0 & (\text{otherwise})\end{cases}$

(要求 $0<y<1$) □

∴ 在$\{X=0\}$条件下Y的条件分布律如下

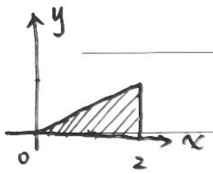
$Y\|X=0$	0	1
$P(Y\|X=0)$	0	1

□

(2) $P(X=-1, Y=0)=\frac{1}{4}\neq P(X=-1)P(Y=0)=\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$

∴ X与Y不相互独立 □

3-20. 解: (1) ① $0\leq x\leq 2$时 $f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy=\int_0^{\frac{x}{2}}\frac{3}{4}x\,dy=\frac{3}{8}x^2$



② $x<0$或$x>2$时 $f_X(x)=0$.

①′ $0\leq y\leq 1$时. $f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dy=\int_{2y}^{2}\frac{3}{4}x\,dx=\frac{3}{2}-\frac{3}{2}y^2$

②′ $y<0$或$y>1$时. $f_Y(y)=0$

综上: $f_X(x)=\begin{cases}\frac{3}{8}x^2, & 0\leq x\leq 2\\ 0, & \text{otherwise}\end{cases}$ $f_Y(y)=\begin{cases}\frac{3}{2}(1-y^2), & 0\leq y\leq 1\\ 0, & \text{otherwise}\end{cases}$ □□

(2) ∵ $f(x,y)=\frac{3}{4}x \neq f_X(x)f_Y(y)=\frac{3}{8}x^2\cdot\frac{3}{2}(1-y^2)$ ∴ X, Y不相互独立

• (如取 $x=1$, $y=\frac{1}{2}$: $\frac{3}{4}=f(1,\frac{1}{2})\neq f_X(1)f_Y(\frac{1}{2})=\frac{27}{64}$) □

补充1. 解: (1) ∵ $(X,Y)\sim N(2,4;2,9;0)$ ∴ $Y\sim N(2,9)$ $\mu_2=2$. $\sigma_2=3$.

$P(|Y|>1)=1-P(|Y|\leq 1)=1-P(-1\leq Y\leq 1)=1-F(1)+F(-1-0)$

$=1-\Phi\left(\frac{1-2}{3}\right)+\Phi\left(\frac{-1-2}{3}\right)=1-\Phi(-\frac{1}{3})+\Phi(-1)$

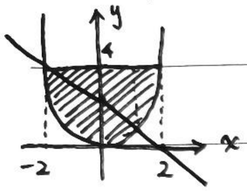
$=\Phi(\frac{1}{3})-\Phi(1)+1=0.6293-0.8413+1=0.7880$ □

(2) $f_{Y|X}(y|x=2) = \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left[y-\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right)\right]^2}$

$= \frac{1}{3\sqrt{2\pi}}\, e^{-\frac{1}{18}(y-2)^2}$ 即 $Y|X=2 \sim N(2,9)$

$\therefore P(Y<3|X=2) = \Phi\left(\frac{3-0-2}{3}\right) = \Phi\left(\frac{1}{3}\right) = 0.6293$ □

补充2. 解: (1) $S = \int_{-2}^{2}\int_{x^2}^{4} dy\,dx = \int_{-2}^{2} 4-x^2\,dx = 16-\frac{16}{3} = \frac{32}{3}$ 记围成区域为 G



$\because (X,Y) \sim U(y=x^2, y=4)$ $\therefore f(x,y) = \begin{cases} \frac{3}{32} & ((x,y)\in G) \\ 0 & ((x,y)\notin G) \end{cases}$

联立 $\begin{cases} y=x^2 \\ x+y=2 \end{cases} \Rightarrow x=y=1.$

$P(X+Y\geq 2) = \iint\limits_{X+Y\geq 2} f(x,y)\,dx\,dy = \frac{3}{32}\left(\int_{-2}^{1}\int_{2-x}^{4} dy\,dx + \int_{1}^{2}\int_{x^2}^{4} dy\,dx\right)$

$= \frac{3}{32}\left(\frac{9}{2}+\frac{5}{3}\right) = \frac{37}{64}$ □

(2) $P(X+Y\geq 2 \cap X\geq 1) = \iint\limits_{X+Y\geq 2,\ X>1} f(x,y)\,dx\,dy = \frac{3}{32}\int_{1}^{2}\int_{x^2}^{4} dy\,dx = \frac{3}{32}\cdot\frac{5}{3} = \frac{5}{32}$

$P(X>1) = \iint\limits_{X>1} f(x,y)\,dx\,dy = \frac{3}{32}\int_{1}^{2}\int_{x^2}^{4} dy\,dx$

$\therefore P(X+Y\geq 2 | X>1) = \frac{P(X+Y\geq 2 \cap X\geq 1)}{P(X>1)} = 1.$ □