[Solution of Homework 4] Martingale and Stopping Time

Doob's martingale inequality

Let $\{X_t\}_{t\geq 0}$ be a martingale with respect to itself where $X_t \geq 0$ for every t. Prove that for every $n \in \mathbb{N}$,

$$\mathbf{Pr}\left[\max_{0\leq t\leq n} X_t \geq \alpha\right] \leq \frac{\mathbf{E}\left[X_0\right]}{\alpha}.$$

▶ Hint

Proof. We define a stopping time τ when the first element that is greater that α occurs, otherwise set $\tau = n$. Formally,

$$\tau = \begin{cases} n & \max_{0\leq t\leq n} X_t < \alpha \\ \arg\min_{t\leq n}\{X_t \geq \alpha\} & o.w. \end{cases}$$

By definition of τ, we have

$$\mathbf{Pr}\left[\max_{0\leq t\leq n} X_t \geq \alpha\right] = \mathbf{Pr}\left[X_\tau \geq \alpha\right]$$

τ is bounded, so we apply Optional Stopping Theorem to obtain that $E[X_\tau] = E[X_0]$. Therefore, by Markov's Inequality,

$$\mathbf{Pr}\left[\max_{0\leq t\leq n} X_t \geq \alpha\right] = \mathbf{Pr}\left[X_\tau \geq \alpha\right] \leq \frac{\mathbf{E}\left[X_\tau\right]}{\alpha} = \frac{\mathbf{E}\left[X_0\right]}{\alpha}$$

Biased one-dimensional random walk

We study the biased random walk in this exercise. Let $X_t = \sum_{i=1}^{t} Z_i$ where each $Z_i \in \{-1, 1\}$ is independent, and satisfies $\mathbf{Pr}\left[Z_i = -1\right] = p \in (0, 1)$.

- Define $S_t = \sum_{i=1}^{t}(Z_i + 2p - 1)$. Show that $\{S_t\}_{t\geq 0}$ is a martingale.
  Proof.

$$
\begin{aligned}
&\mathbf{E}\left[S_t \mid Z_1, Z_2, \ldots, Z_{t-1}\right] \\
&= \mathbf{E}\left[Z_t \mid Z_1, Z_2, \ldots, Z_{t-1}\right] + (2p-1) + S_{t-1} \\
&= \mathbf{E}\left[Z_t\right] + (2p-1) + S_{t-1} \\
&= 1 - p - p + (2p-1) + S_{t-1} \\
&= S_{t-1}.
\end{aligned}
$$

So $\{S_t\}_{t\ge 0}$ is a martingale.

- Define $P_t = \left(\frac{p}{1-p}\right)^{X_t}$. Show that $\{P_t\}_{t\ge 0}$ is a martingale.
  Proof.

$$
\begin{aligned}
&\mathbf{E}\left[P_t \mid Z_1, Z_2, \ldots, Z_{t-1}\right] \\
&= P_{t-1}\mathbf{E}\left[\left(\frac{p}{1-p}\right)^{Z_t} \,\middle|\, Z_1, Z_2, \ldots, Z_{t-1}\right] \\
&= P_{t-1}\mathbf{E}\left[\left(\frac{p}{1-p}\right)^{Z_t}\right] \\
&= P_{t-1}\left(\frac{p}{1-p}(1-p) + \frac{1-p}{p}p\right) \\
&= P_{t-1}
\end{aligned}
$$

- Suppose the walk stops either when $X_t = -a$ or $X_t = b$ for some $a, b > 0$. Let τ be the stopping time. Compute $\mathbf{E}[\tau]$.
  Proof.
  When $p = \frac{1}{2}$, we've showed that $\mathbf{E}[\tau] = ab$, so we suppose $p \neq \frac{1}{2}$ in the following proof.
  Consider a time period of length $T = a + b$. In each period of time, the walk stops with probability at least $p^{a+b} + (1-p)^{a+b}$. If we divide the time into consecutive periods in this manner, in expected finite time, we can meet some period with the event happened. Therefore, $\mathbf{E}[\tau] < \infty$. And

$$
\begin{aligned}
|P_t - P_{t-1}| &= \left(\frac{p}{1-p}\right)^{X_t} + \left(\frac{p}{1-p}\right)^{X_{t-1}} \\
&< 2\max\left(\left(\frac{p}{1-p}\right)^{-a}, \left(\frac{p}{1-p}\right)^{b}\right),
\end{aligned}
$$

saying that $|P_t - P_{t-1}|$ is bounded by constant. So we apply OST and obtain that

$$
\mathbf{Pr}\left[X_\tau = -a\right]\left(\frac{p}{1-p}\right)^{-a} + \mathbf{Pr}\left[X_\tau = b\right]\left(\frac{p}{1-p}\right)^{b} = \mathbf{E}\left[P_\tau\right] = \mathbf{E}\left[P_0\right] = 1.
$$

Solving this equation, we get $\mathbf{Pr}\left[X_\tau = -a\right] = \frac{1-\left(\frac{p}{1-p}\right)^b}{\left(\frac{p}{1-p}\right)^{-a}-\left(\frac{p}{1-p}\right)^b}$. Since $|S_t - S_{t-1}| = |Z_t + 2p - 1| < 2$, applying OST, it follows that

$$\mathbf{Pr}\left[X_\tau = -a\right](-a) + \mathbf{Pr}\left[X_\tau = b\right]b + \mathbf{E}\left[\tau\right](2p-1) = \mathbf{E}\left[S_\tau\right] = \mathbf{E}\left[S_0\right] = 0.$$

So when $p \neq \frac{1}{2}$,

$$\mathbf{E}\left[\tau\right] = \frac{1-\left(\frac{p}{1-p}\right)^b}{\left(\frac{p}{1-p}\right)^{-a}-\left(\frac{p}{1-p}\right)^b}\frac{a+b}{2p-1} - \frac{b}{2p-1}.$$

Longest common subsequence

A *subsequence* of a string s is any string that can be obtained from s by removing a few characters (not necessarily continuous). Consider two uniformly random strings $x, y \in \{0,1\}^n$. Let X denote the length of their *longest common subsequence*.

- Show that there exist two constants $\frac{1}{2} < c_1 < c_2 < 1$ such that $c_1 n < \mathbf{E}\left[X\right] < c_2 n$ for sufficiently n.
  Proof.
  Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. For the lower bound, assuming that the length of common subsequence of (x_{2k-1}, x_{2k}) and (y_{2k-1}, y_{2k}) is l_k, we have

$$\mathbf{E}\left[X\right] \geq \mathbf{E}\left[\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} l_k\right] = \left(\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}2\right)+\frac{1}{2}\left(\frac{3}{4}+\frac{1}{4}2\right)\right)\left\lfloor\frac{n}{2}\right\rfloor = \frac{9}{8}\left\lfloor\frac{n}{2}\right\rfloor.$$

  So we could take $c_1 = \frac{9}{16}$.
  For the upper bound,

$$\mathbf{E}\left[X\right] \leq \mathbf{Pr}\left[X \geq c_2 n\right]n + \mathbf{Pr}\left[x < c_2 n\right]c_2 n = (\mathbf{Pr}\left[X \geq c_2 n\right](1-c_2) + c_2)n.$$

  And by Stirling's approximation,

$$\mathbf{Pr}\left[X \geq c_2 n\right] \leq \frac{\binom{n}{c_2 n}^2}{2^{c_2 n}} \sim \frac{1}{2\pi(1-c_2)c_2 n}\left(\frac{1}{(\sqrt{2}c_2)^{c_2}(1-c_2)^{1-c_2}}\right)^{2n},$$

  for suffiently large n. We take $c_2 = 0.91$ so that $\mathbf{Pr}\left[X \geq c_2 n\right]$ is $o(1)$ and then $\mathbf{E}\left[X\right] \leq c_2 n$ when n is sufficiently large.

- Prove that X is concentrated around $\mathbf{E}\left[X\right]$.
  Proof.
  We could regard X as a function of $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$.

$$X = f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n).$$

And obviously f is 1-Lipschitz function since changing exactly one character in x, y only add or delete at most one character in the longest common subsequences. Therefore, by Mcdarmid's Inequality,

$$\mathbf{Pr}\left[X - \mathbf{E}\left[X\right] \geq t\right] \leq 2e^{\frac{-t^2}{n}},$$

which means that X is concentrated around $\mathbf{E}\left[X\right]$.