Discrete Mathematics Exercise 15

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1. ***Solution:*** **a), b), d), f)** are trees.

2. ***Proof:***

We prove there are no simple circuits in G by contradiction.

Assume there is a simple circuit in G.

First, since G is connected, there are at least $(n-1)$ edges in G. (because the addition of an edge will decrease at most one connected component and if there are no edges in a graph with n vertices, there are n connected components.)

Thus, if you remove any edge from G, the new graph has only $(n-2)$ edges and is definitely not connected.

If there is a simple circuit in G, if you remove an edge involved in the simple circuit, the new graph is still connected. **Contradiction.**

Thus, there are no simple circuits in G, **i.e.** G is a connected graph with no simple circuits.

In other words, G is a tree.

QED

3. ***Proof:***

We prove G is connected by induction.

First, we prove that if G has no simple circuits, there are at most $(n-1)$ edges in G and when there are $(n-1)$ edges in G without simple circuits, G is connected.

BASE STEP. $n=1$. Obvious. $n=2$. Obvious.

INDUCTION STEP.

If $n=k$, there are at most $(k-1)$ edges in G if G has no simple circuits and when there are $(n-1)$ edges in G without simple circuits, G is connected.

When $n=k+1$, we can at most add an edge between the new vertex v and a vertex u in G. If we add two edges, the second edge is either from a vertex in G to another in G or from v to a vertex x (not u) in G.

In the former case, there is already a simple path between the two vertices. Thus, the addition of the new edge generates a circuit in the new graph.

In the latter case, there is a simple path from u to x. Thus, the addition of the first edge ensures there is a simple path from v to x. The addition of the second edge (from v to x) generates a circuit in the new graph.

Thus, when $n=k+1$, there are at most $(n-1)$ edges in G if G has no simple circuits and when there are $(n-1)$ edges in G without simple circuits, G is connected.

Thus, G is connected, **i.e.** G is a connected graph with no simple circuits.

In other words, G is a tree.

QED