Discrete Mathematics Exercise 6

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1.

a) ***Proof:*** For any $a \in \mathbb{N}$, there exists a $b = a + 1 \in \mathbb{N}$ such that $a < b$,

namely for any $a \in \mathbb{N}$, $[\![\mathcal{R}(x,y)]\!]_{\mathcal{J}_{1[x\mapsto a, y\mapsto a+1]}} = \mathbf{T}$.

namely for any $a \in \mathbb{N}$, $[\![\exists y \mathcal{R}(x,y)]\!]_{\mathcal{J}_{1[x\mapsto a]}} = \mathbf{T}$.

In other words, $[\![\forall x \exists y \mathcal{R}(x,y)]\!]_{\mathcal{J}_1} = \mathbf{T}$.

QED

b) ***Proof:*** Since $\mathcal{J}_2(x) = 0$, for any $b \in \mathbb{N}$, we know $b \nless 0$, thus $[\![\mathcal{R}(x,y)]\!]_{\mathcal{J}_{2[y\mapsto b]}} = \mathbf{F}$.

namely there does not exist a $b \in \mathbb{N}$ such that $[\![\mathcal{R}(x,y)]\!]_{\mathcal{J}_{2[y\mapsto b]}} = \mathbf{T}$.

In other words, $[\![\exists y \mathcal{R}(x,y)]\!]_{\mathcal{J}_2} = \mathbf{F}$.

QED

c) ***Proof:*** Let $\mathcal{J}_3(x) = 0$. For any $b \in \mathbb{N}$, we know $b \nless 0$, thus $[\![\mathcal{R}(x,y)]\!]_{\mathcal{J}_{3[x\mapsto 0, y\mapsto b]}} = \mathbf{F}$.

namely there exist an $a = 0$, $[\![\exists y \mathcal{R}(x,y)]\!]_{\mathcal{J}_{3[x\mapsto a]}} = \mathbf{F}$.

In other words, $[\![\forall x \exists y \mathcal{R}(x,y)]\!]_{\mathcal{J}_3} = \mathbf{F}$.

QED

d) ***Proof:*** There exists $a = 0 \in \mathbb{N}$, $b = 1 \in \mathbb{N}$ such that $[\![\mathcal{R}(x,y)]\!]_{\mathcal{J}_{1[x\mapsto a, y\mapsto b]}} = \mathbf{T}$.

For $\mathcal{J}_1(z)$, there exists two cases:

(1) $\mathcal{J}_1(z) = 0$. In this case, $[\![\mathcal{R}(x,z)]\!]_{\mathcal{J}_{1[x\mapsto a, z\mapsto 0]}} = \mathbf{F}$.

So $\left[\!\!\left[\big(\mathcal{R}(x,z) \land \mathcal{R}(z,y)\big)\right]\!\!\right]_{\mathcal{J}_{1[x\mapsto a, y\mapsto b, z\mapsto 0]}} = \mathbf{F}$.

(2) $\mathcal{J}_1(z) = d \geq 1, d \in \mathbb{N}$. In this case, $[\![\mathcal{R}(z,y)]\!]_{\mathcal{J}_{1[x\mapsto a, z\mapsto d]}} = \mathbf{F}$.

So $\left[\!\!\left[\big(\mathcal{R}(x,z) \land \mathcal{R}(z,y)\big)\right]\!\!\right]_{\mathcal{J}_{1[x\mapsto a, y\mapsto b, z\mapsto d]}} = \mathbf{F}$.

Therefore, for any $\mathcal{J}_1(z) = d \in \mathbb{N}$, $\left[\!\!\left[\big(\mathcal{R}(x,z) \land \mathcal{R}(z,y)\big)\right]\!\!\right]_{\mathcal{J}_{1[x\mapsto a, y\mapsto b, z\mapsto d]}} = \mathbf{F}$,

namely $\left[\!\!\left[\exists z\big(\mathcal{R}(x,z) \land \mathcal{R}(z,y)\big)\right]\!\!\right]_{\mathcal{J}_{1[x\mapsto a, y\mapsto b]}} = \mathbf{F}$.

Thus, there exists an S-Interpretation $\mathcal{J}_1$ where $\mathcal{J}_1(x) = 0$, $\mathcal{J}_1(y) = 1$ such that

$$\left[\!\!\left[\mathcal{R}(x,y) \rightarrow \exists z\big(\mathcal{R}(x,z) \land \mathcal{R}(z,y)\big)\right]\!\!\right]_{\mathcal{J}_{1[x\mapsto 0, y\mapsto 1]}} = \mathbf{F}.$$

In other words, $\left[\!\left[\forall x \forall y \left(\mathcal{R}(x,y) \to \exists z\big(\mathcal{R}(x,z) \wedge \mathcal{R}(z,y)\big)\right)\right]\!\right]_{\mathcal{J}_1} = \mathbf{F}.$

QED

e)

Proof: For any $a, b \in \mathbb{Q}, a < b$, there exists a $c = \dfrac{a+b}{2} \in \mathbb{Q}$ ***s.t.*** $a < c$ and $c < b$,

namely $[\![\mathcal{R}(x,y) \to \exists z\big(\mathcal{R}(x,z) \wedge \mathcal{R}(z,y)\big)]\!]_{\mathcal{J}_{4[x\mapsto a, y\mapsto b]}} = \mathbf{T}.$

In other words, $\left[\!\left[\forall x \forall y \left(\mathcal{R}(x,y) \to \exists z\big(\mathcal{R}(x,z) \wedge \mathcal{R}(z,y)\big)\right)\right]\!\right]_{\mathcal{J}_4} = \mathbf{T}.$ ***QED***

2.

a) ***Proof:*** For any $a, b \in \mathbb{N}, a + b = b + a$, namely $f(a,b) = f(b,a)$.

Thus, for any $a, b \in \mathbb{N}$, $[\![\mathcal{R}\big(f(x,y), f(y,x)\big)]\!]_{\mathcal{J}_{1[x\mapsto a, y\mapsto b]}} = \mathbf{T}$

In other words, $[\![\forall x \forall y \mathcal{R}\big(f(x,y), f(y,x)\big)]\!]_{\mathcal{J}_1} = \mathbf{T}.$ ***QED***

b) ***Proof:*** For any $a, b \in \mathbb{N}, a * b = b * a$, namely $f(a,b) = f(b,a)$.

Thus, for any $a, b \in \mathbb{N}$, $[\![\mathcal{R}\big(f(x,y), f(y,x)\big)]\!]_{\mathcal{J}_{2[x\mapsto a, y\mapsto b]}} = \mathbf{T}$

In other words, $[\![\forall x \forall y \mathcal{R}\big(f(x,y), f(y,x)\big)]\!]_{\mathcal{J}_2} = \mathbf{T}.$ ***QED***

c) ***Proof:*** For any $a, b \in \{\mathbf{T}, \mathbf{F}\}, a \wedge b = b \wedge a$, namely $f(a,b) = f(b,a)$.

Thus, for any $a, b \in \{\mathbf{T}, \mathbf{F}\}$, $[\![\mathcal{R}\big(f(x,y), f(y,x)\big)]\!]_{\mathcal{J}_{3[x\mapsto a, y\mapsto b]}} = \mathbf{T}$

In other words, $[\![\forall x \forall y \mathcal{R}\big(f(x,y), f(y,x)\big)]\!]_{\mathcal{J}_3} = \mathbf{T}.$ ***QED***

d) ***Proof:*** There exists an S-Interpretation $\mathcal{J}_4$ such that

- The domain of $\mathcal{J}_4$ is $\mathbb{R}$.
- $\mathcal{J}_4(f, x, y) = x - y$.
- $\mathcal{J}_4(\mathcal{R}, a, b) = \mathbf{T}$ if and only if $a = b$.

There exists $a = 0, b = 1$ such that $a - b \neq b - a$,

namely $[\![\mathcal{R}\big(f(x,y), f(y,x)\big)]\!]_{\mathcal{J}_{4[x\mapsto a, y\mapsto b]}} = \mathbf{F}.$

Thus, $[\![\forall x \forall y \mathcal{R}\big(f(x,y), f(y,x)\big)]\!]_{\mathcal{J}_4} = \mathbf{F}.$

In other words, $\forall x \forall y \mathcal{R}\big(f(x,y), f(y,x)\big)$ is not valid. ***QED***

3. ***Solution:***

First, we prove that the proposition if $\Phi \vDash \psi$, then $\Phi \vDash \forall x \psi$ is false.

Let $\Phi = \{\phi\}$, $\phi = P(x)$, and $\psi = Q(x)$.

There exists an S-Interpretation $\mathcal{J}$ such that

- The domain is $\mathbb{N}$.

- $\mathcal{J}(P(x)) = \mathbf{T}$ if and only if $x > 1$.
- $\mathcal{J}(Q(x)) = \mathbf{T}$ if and only if $x > 0$.

It's plain to see that $\Phi \vDash \psi$, $[\![\forall x\psi]\!]_{\mathcal{J}} = \mathbf{F}$ since exists $-1 \in \mathbb{N}$, $[\![\psi]\!]_{\mathcal{J}_{[x\mapsto -1]}} = \mathbf{F}$.

However, $\Phi \nvDash \forall x\psi$ since there exists an S-Interpretation $\mathcal{J}_{[x\mapsto 1]}$ such that $[\![\phi]\!]_{\mathcal{J}_{[x\mapsto 1]}} = \mathbf{T}$, $[\![\forall x\psi]\!]_{\mathcal{J}_{[x\mapsto 1]}} = \mathbf{F}$.

Thus, the proposition if $\Phi \vDash \psi$, then $\Phi \vDash \forall x\psi$ is false.

Therefore, $\phi \vdash \psi$ in the FOL can't imply $\phi \vDash \psi$.

In other words, the first order logic with this proof rule is impossible to be sound.

4. ***Solution:*** ***a***) 0,1,4,9,16,25,36,49,64,81.

b) $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$

5. ***Solution:***

a) Incorrect. $\{\emptyset\}$ only has one member, namely $\emptyset$. In fact, $\{\emptyset\} \subseteq \{\emptyset\}$.

b) Correct. $\{\{\emptyset\}\}$ is a set with one member $\{\emptyset\}$, so $\{\emptyset\} \in \{\{\emptyset\}\}$.

c) Incorrect. Let $A = \{1\}$. In this case, $\mathcal{P}(A) = \{\emptyset, \{1\}\}$. The subsets of $\mathcal{P}(A)$ are $\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}$. A is not among them, so $A \nsubseteq \mathcal{P}(A)$.

d) Correct. A is absolutely a subset of A itself, so A is for sure a member of $\mathcal{P}(A)$, namely $A \in \mathcal{P}(A)$.

6. ***a***)***Proof:*** Let $\mathcal{J}$ be an S-Interpretation. The domain is A.

When $[\![\forall x\phi]\!]_{\mathcal{J}} = \mathbf{T}$, for any $a \in A$, $[\![\phi]\!]_{\mathcal{J}_{[x\mapsto a]}} = \mathbf{T}$, from $\phi \vDash \psi$ we know $[\![\psi]\!]_{\mathcal{J}_{[x\mapsto a]}} = \mathbf{T}$.

Therefore, for any $a \in A$, $[\![\psi]\!]_{\mathcal{J}_{[x\mapsto a]}} = \mathbf{T}$, namely $[\![\forall x\psi]\!]_{\mathcal{J}} = \mathbf{T}$.

In other words, $\forall x\phi \vDash \forall x\psi$. ***QED***

b)***Proof:*** We prove it by contradiction.

Assume $\Phi, \phi \vDash \psi$ and $\Phi, \forall x\phi \nvDash \forall x\psi$.

Then there exists an S-Interpretation $\mathcal{J}$ where the domain is A such that $[\![\forall x\phi]\!]_{\mathcal{J}} = \mathbf{T}$, $[\![\forall x\psi]\!]_{\mathcal{J}} = \mathbf{F}$, namely $[\![\exists x\neg\psi]\!]_{\mathcal{J}} = \mathbf{T}$, and for any $a \in A$ and $\varphi \in \Phi$, $[\![\varphi]\!]_{\mathcal{J}_{[x\mapsto a]}} = \mathbf{T}$ (because x does not freely occur in Φ).

So there exists an $a_0 \in A$ such that $[\![\neg\psi]\!]_{\mathcal{J}_{[x\mapsto a_0]}} = \mathbf{T}, [\![\psi]\!]_{\mathcal{J}_{[x\mapsto a_0]}} = \mathbf{F}.$

Since $[\![\forall x\phi]\!]_{\mathcal{J}} = \mathbf{T}$, for any $a \in A$, $[\![\phi]\!]_{\mathcal{J}_{[x\mapsto a]}} = \mathbf{T}.$

Thus, exists an $a_0 \in A$ such that for any $\varphi \in \Phi$, $[\![\varphi]\!]_{\mathcal{J}_{[x\mapsto a_0]}} = \mathbf{T}$ and $[\![\phi]\!]_{\mathcal{J}_{[x\mapsto a_0]}} = \mathbf{T}$, $[\![\psi]\!]_{\mathcal{J}_{[x\mapsto a_0]}} = \mathbf{F}$, which is a ***contradiction*** since $\Phi, \phi \vDash \psi$.

In other words, $\Phi, \forall x\phi \vDash \forall x\psi$.

QED

c) Solution: Let $\Phi = \{\chi\}$, $\chi = P(x)$, $\phi = Q(x)$ and $\psi = T(x)$.

There exists an S-Interpretation $\mathcal{J}$ such that

- The domain is $\mathbb{N}$.
- $\mathcal{J}(P(x)) = \mathbf{T}$ if and only if $x \geq 1$.
- $\mathcal{J}(Q(x)) = \mathbf{T}$ if and only if x is a natural number.
- $\mathcal{J}(T(x)) = \mathbf{T}$ if and only if $x > 0$.

It's plain to see that $\Phi, \phi \vDash \psi$, $[\![\forall x\phi]\!]_{\mathcal{J}} = \mathbf{T}$ since for any $a \in \mathbb{N}$, $[\![\phi]\!]_{\mathcal{J}_{[x\mapsto a]}} = \mathbf{T}.$

However, $[\![\forall x\psi]\!]_{\mathcal{J}} = \mathbf{F}$ because there exists $a = 0$ such that $a \ngtr 0$, namely $[\![\psi]\!]_{\mathcal{J}_{[x\mapsto 0]}} = \mathbf{F}.$

So there exists an S-Interpretation $\mathcal{J}_{[x\mapsto 0]}$ such that $[\![\chi]\!]_{\mathcal{J}_{[x\mapsto 0]}} = \mathbf{T}$, $[\![\forall x\phi]\!]_{\mathcal{J}_{[x\mapsto 0]}} = \mathbf{T}$, $[\![\forall x\psi]\!]_{\mathcal{J}_{[x\mapsto 0]}} = \mathbf{F}.$

Thus, $\Phi, \forall x\phi \nvDash \forall x\psi$.

7. *Solution:* A = {1}, B = {1,{1}} is a feasible solution.