[Homework 5] Poisson Process & Poisson Approximation (Due: May 25, 2022)

Problem 1

Customers arrive according to a Poisson process of rate λ per hour. Joe does not want to stay until the store closes at $T = 10$ p.m., so he decides to close up when the first customer after time $T - s$ arrives. He wants to leave early but he does not want to lose any business so he is happy if he leaves before T and no one arrives after.

- What is the probability he achieves his goal?
- What is the optimal value of s and the corresponding success probability? (That is, the value s maximizing the success probability)

Problem 2

- Assume $X \sim \texttt{Poisson}(\lambda)$ for some integer $\lambda \geq 1$. Prove that for any $k = 0, 1, \ldots, \lambda - 1$, it holds that $\mathbf{Pr}\left[X = \lambda + k\right] \geq \mathbf{Pr}\left[X = \lambda - k - 1\right]$. Use this to conclude that $\mathbf{Pr}\left[X \geq \lambda\right] \geq \frac{1}{2}$.
- Recall the setting of Corollary 4 in Lecture 10. Prove that if $\mathbf{E}\left[f(X_1, \ldots, X_n)\right]$ is monotonically increasing in m, then

$$\mathbf{E}\left[f(X_1, \ldots, X_n)\right] \leq 2 \cdot \mathbf{E}\left[f(Y_1, \ldots, Y_n)\right].$$

- Recall the birthday problem in Lecture 2 and assume notations there. Now suppose we would like to estimate the probability of the event "*there exists four students who share the same birthday*". Assume there are 50 students in the class ($n = 50$ and $m = 365$). Use Poisson approximation to show that the probabilty is at most 1%.