Discrete Mathematics Exercise 5

Qiu Yihang, 2020/10/03

1.

a) ***Proof:*** For any given $x \in \mathbb{N}$, there exists a $y = x + 1 \in \mathbb{N}$ such that $x < y$.

In other words, $[\![\forall x \exists y \mathcal{R}(x, y)]\!]_{\mathcal{J}_1} = \mathbf{T}$. ***QED***

b) ***Proof:*** Since $\mathcal{J}_2(x) = 0$, $[\![\mathcal{R}(x, y)]\!]_{\mathcal{J}_2} = [\![\varphi]\!]_{\mathcal{J}_2}$ $(\varphi = 0 > y)$. For any $y \in \mathbb{N}$, $y \geq 0$.

In other words, $[\![\exists y \mathcal{R}(x, y)]\!]_{\mathcal{J}_2} = \mathbf{F}$. ***QED***

c) ***Proof:*** There exists an $x = 0 \in \mathbb{N}$ such that for any $y \in \mathbb{N}, y \geq x$, namely $\nexists y \in \mathbb{N}, x < y$.

In other words, $[\![\forall x \exists y \mathcal{R}(x, y)]\!]_{\mathcal{J}_3} = \mathbf{F}$. ***QED***

d) ***Proof:*** There exists $x = 0 \in \mathbb{N}$, $y = 1 \in \mathbb{N}$ such that $[\![\mathcal{R}(x, y)]\!]_{\mathcal{J}_1} = \mathbf{T}$.

For any $z \in \mathbb{N}$, there exists two cases:

(1) $z = 0$. In this case, $[\![\mathcal{R}(x, z)]\!]_{\mathcal{J}_1} = \mathbf{F}$, so $\left[\!\left[\left(\mathcal{R}(x, z) \wedge \mathcal{R}(z, y)\right)\right]\!\right]_{\mathcal{J}_1} = \mathbf{F}$.

(2) $z \geq 1, z \in \mathbb{N}$. In this case, $[\![\mathcal{R}(z, y)]\!]_{\mathcal{J}_1} = \mathbf{F}$, so $\left[\!\left[\left(\mathcal{R}(x, z) \wedge \mathcal{R}(z, y)\right)\right]\!\right]_{\mathcal{J}_1} = \mathbf{F}$.

Therefore, for any $z \in \mathbb{N}$, $\left[\!\left[\left(\mathcal{R}(x, z) \wedge \mathcal{R}(z, y)\right)\right]\!\right]_{\mathcal{J}_1} = \mathbf{F}$,

namely $\left[\!\left[\exists z\left(\mathcal{R}(x, z) \wedge \mathcal{R}(z, y)\right)\right]\!\right]_{\mathcal{J}_1} = \mathbf{F}$.

Thus, there exists an S-Interpretation $\mathcal{J}_1$ where $\mathcal{J}_1(x) = 0$, $\mathcal{J}_1(y) = 1$ such that

$$\left[\!\left[\mathcal{R}(x, y) \to \exists z\left(\mathcal{R}(x, z) \wedge \mathcal{R}(z, y)\right)\right]\!\right]_{\mathcal{J}_1} = [\![\mathbf{T} \to \mathbf{F}]\!]_{\mathcal{J}_1} = \mathbf{F}.$$

In other words, $\left[\!\left[\forall x \forall y\left(\mathcal{R}(x, y) \to \exists z\left(\mathcal{R}(x, z) \wedge \mathcal{R}(z, y)\right)\right)\right]\!\right]_{\mathcal{J}_1} = \mathbf{F}$.

QED

e)

Proof: For any given $x, y \in \mathbb{Q}, x < y$, there exists a $z = \frac{x+y}{2} \in \mathbb{Q}$ ***s.t.*** $x < z$ and $z < y$.

In other words, $\left[\!\left[\forall x \forall y\left(\mathcal{R}(x, y) \to \exists z\left(\mathcal{R}(x, z) \wedge \mathcal{R}(z, y)\right)\right)\right]\!\right]_{\mathcal{J}_4} = \mathbf{T}$. ***QED***

2.

a) ***Proof:*** For any given $a, b \in \mathbb{N}, a + b = b + a$, namely $f(a, b) = f(b, a)$.

In other words, $\left[\!\left[\forall x \forall y \mathcal{R}\left(f(x, y), f(y, x)\right)\right]\!\right]_{\mathcal{J}_1} = \mathbf{T}$. ***QED***

b) ***Proof:*** For any given $a, b \in \mathbb{N}, a * b = b * a$, namely $f(a, b) = f(b, a)$.

In other words, $\left[\!\left[\forall x \forall y \mathcal{R}\left(f(x, y), f(y, x)\right)\right]\!\right]_{\mathcal{J}_2} = \mathbf{T}$. ***QED***

c) ***Proof:*** For any given $a, b \in \{\mathbf{T}, \mathbf{F}\}, a \wedge b = b \wedge a$, namely $f(a, b) = f(b, a)$.

In other words, $\left[\!\left[\forall x \forall y \mathcal{R}\left(f(x, y), f(y, x)\right)\right]\!\right]_{\mathcal{J}_3} = \mathbf{T}$. ***QED***

d) ***Proof:*** There exists an S-Interpretation $\mathcal{J}_4$ such that

- The domain of $\mathcal{J}_4$ is $\mathbb{R}$.
- $\mathcal{J}_4(f, x, y) = x - y$.
- $\mathcal{J}_4(\mathcal{R}, a, b) = \mathbf{T}$ if and only if $a = b$.

There exists $x = 0, y = 1$ such that $x - y \neq y - x$,

namely $[\![\forall x \forall y \mathcal{R}(f(x, y), f(y, x))]\!]_{\mathcal{J}_4} = \mathbf{F}$.

In other words, $\forall x \forall y \mathcal{R}(f(x, y), f(y, x))$ is not valid. ***QED***

3. ***Solution:***

There exists an S-Interpretation $\mathcal{J}$ such that

- The domain of $\mathcal{J}$ is $\mathbb{N}$.
- $\boldsymbol{P}(x)$ mean whether x can be divided by 3.
- $\boldsymbol{Q}(x)$ mean whether x is a negative number.

Under this S-Interpretation $\mathcal{J}$, $[\![\forall x(P(x) \to Q(x))]\!]_{\mathcal{J}} = \mathbf{F}$ while $[\![\forall x P(x)]\!]_{\mathcal{J}} = \mathbf{F}$ and $[\![\forall x Q(x)]\!]_{\mathcal{J}} = \mathbf{F}$, namely $[\![\forall x P(x) \to \forall x Q(x)]\!]_{\mathcal{J}} = \mathbf{T}$.

In this case, $[\![\forall x(P(x) \to Q(x))]\!]_{\mathcal{J}} \neq [\![\forall x P(x) \to \forall x Q(x)]\!]_{\mathcal{J}}$.

In other words, $\forall x(P(x) \to Q(x))$ and $\forall x P(x) \to \forall x Q(x)$ are not logically equivalent.

4. ***Solution:***

$\neg \forall x(\phi \to \psi) \equiv \exists x\,(\neg(\phi \to \psi)) \equiv \exists x\,(\neg(\neg\phi \vee \psi)) \equiv \exists x\,(\phi \wedge \neg\psi)$.

So $\neg \forall x(\phi \to \psi)$ and $\exists x\,(\phi \wedge \neg\psi)$ are logically equivalent.

5. ***a***) ***Solution:*** $\forall z \exists y \exists x \neg T(x, y, z)$

b) ***Solution:*** $(\forall x \forall y \neg P(x, y)) \vee (\exists x \exists y \neg Q(x, y))$

c) ***Solution:*** $\forall x \forall y\left((P(x, y) \wedge \neg Q(y, x)) \vee (\neg P(x, y) \wedge Q(y, x))\right)$

d) ***Solution:*** $\exists y \forall x \forall z(\neg T(x, y, z) \wedge \neg Q(x, y))$

6. ***a***) ***Proof:*** When $[\![\forall x \phi]\!]_{\mathcal{J}} = \mathbf{T}$, for any x, $[\![\phi]\!]_{\mathcal{J}} = \mathbf{T}$.

Since $\phi \vDash \psi$, we know that for any x, $[\![\psi]\!]_{\mathcal{J}} = \mathbf{T}$. Thus, $[\![\forall x \psi]\!]_{\mathcal{J}} = \mathbf{T}$.

In other words, $\forall x \phi \vDash \forall x \psi$. ***QED***

b) ***Proof:*** When $[\![\forall x \phi]\!]_{\mathcal{J}} = \mathbf{T}$ and $[\![\Phi]\!]_{\mathcal{J}} = \mathbf{T}$ (x does not freely occur in Φ), for any x, $[\![\phi]\!]_{\mathcal{J}} = \mathbf{T}, [\![\Phi]\!]_{\mathcal{J}} = \mathbf{T}$.

Since $\Phi, \phi \vDash \psi$, we know that for any x, $[\![\psi]\!]_{\mathcal{J}} = \mathbf{T}$. Thus, $[\![\forall x \psi]\!]_{\mathcal{J}} = \mathbf{T}$.

In other words, $\Phi, \forall x \phi \vDash \forall x \psi$.

QED

c*) *Solution: Let $\mathcal{J}$ be an S-Interpretation where the domain is $\mathbb{N}$.
Let $\Phi = \{\chi\}, \chi = P(x)$, which means $x \geq 1$.
Let $\phi = Q(x)$, which means x is a natural number. Thus, $[\![\forall x \phi]\!]_{\mathcal{J}} = \mathbf{T}$.
Let $\psi = T(x)$, which means $x > 0$.
It's plain to see that $\Phi, \phi \vDash \psi$.
However, $[\![\forall x \psi]\!]_{\mathcal{J}} = \mathbf{F}$ because there exists $x = 0$ such that $x \ngtr 0$.
Thus, $\Phi, \forall x \phi \nvDash \forall x \psi$.