Mathematical Logic Homework 04

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1 DNF and CNF Formalization

1.1 DNF Formalization

Solution. The truth table of $(A \leftrightarrow B) \leftrightarrow C$ is as follows.

$v(A)$	$v(B)$	$v(C)$	$\bar{v}\left((A \leftrightarrow B) \leftrightarrow C\right)$
True	True	True	True
True	True	False	False
True	False	True	False
True	False	False	True
False	True	True	False
False	True	False	True
False	False	True	True
False	False	False	False

Thus, the DNF of $(A \leftrightarrow B) \leftrightarrow C$ is

$$\underline{\boldsymbol{(A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge \neg C)}.} \qquad \blacksquare$$

1.2 CNF Formalization

Solution. We can derive the CNF of $(A \leftrightarrow B) \leftrightarrow C$ from its DNF.

$$\begin{aligned}
& (A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge \neg C) \\
\models\dashv\ & (A \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge \neg C)) \\
& \wedge (B \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge \neg C)) \\
& \wedge (C \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge \neg C))
\end{aligned}$$

$$
\begin{aligned}
\models\dashv\quad & \cdots \\
\models\dashv\quad & (A \lor A \lor \neg A \lor \neg A) \land (A \lor A \lor \neg A \lor \neg B) \land (A \lor A \lor \neg A \lor \neg C) \\
& \land (A \lor A \lor B \lor \neg A) \land (A \lor A \lor B \lor \neg B) \lor (A \lor A \lor B \lor \neg C) \\
& \land (A \lor A \lor C \lor \neg A) \land (A \lor A \lor C \lor \neg B) \land (A \lor A \lor C \lor \neg C) \\
& \land (A \lor \neg B \lor \neg A \lor \neg A) \land (A \lor \neg B \lor \neg A \lor \neg B) \land (A \lor \neg B \lor \neg A \lor \neg C) \\
& \land (A \lor \neg B \lor B \lor \neg A) \land (A \lor \neg B \lor B \lor \neg B) \lor (A \lor \neg B \lor B \lor \neg C) \\
& \land (A \lor \neg B \lor C \lor \neg A) \land (A \lor \neg B \lor C \lor \neg B) \land (A \lor \neg B \lor C \lor \neg C) \\
& \land (A \lor \neg C \lor \neg A \lor \neg A) \land (A \lor \neg C \lor \neg A \lor \neg B) \land (A \lor \neg C \lor \neg A \lor \neg C) \\
& \land (A \lor \neg C \lor B \lor \neg A) \land (A \lor \neg C \lor B \lor \neg B) \land (A \lor \neg C \lor B \lor \neg C) \\
& \land (A \lor \neg C \lor C \lor \neg A) \land (A \lor \neg C \lor C \lor \neg B) \land (A \lor \neg C \lor C \lor \neg C) \\
& \land (B \lor A \lor \neg A \lor \neg A) \land (B \lor A \lor \neg A \lor \neg B) \land (B \lor A \lor \neg A \lor \neg C) \\
& \land (B \lor A \lor B \lor \neg A) \land (B \lor A \lor B \lor \neg B) \lor (B \lor A \lor B \lor \neg C) \\
& \land (B \lor A \lor C \lor \neg A) \land (B \lor A \lor C \lor \neg B) \land (B \lor A \lor C \lor \neg C) \\
& \land (B \lor \neg B \lor \neg A \lor \neg A) \land (B \lor \neg B \lor \neg A \lor \neg B) \land (B \lor \neg B \lor \neg A \lor \neg C) \\
& \land (B \lor \neg B \lor B \lor \neg A) \land (B \lor \neg B \lor B \lor \neg B) \land (B \lor \neg B \lor B \lor \neg C) \\
& \land (B \lor \neg B \lor C \lor \neg A) \land (B \lor \neg B \lor C \lor \neg B) \land (B \lor \neg B \lor C \lor \neg C) \\
& \land (B \lor \neg C \lor \neg A \lor \neg A) \land (B \lor \neg C \lor \neg A \lor \neg B) \land (B \lor \neg C \lor \neg A \lor \neg C) \\
& \land (B \lor \neg C \lor B \lor \neg A) \land (B \lor \neg C \lor B \lor \neg B) \land (B \lor \neg C \lor B \lor \neg C) \\
& \land (B \lor \neg C \lor C \lor \neg A) \land (B \lor \neg C \lor C \lor \neg B) \land (B \lor \neg C \lor C \lor \neg C) \\
& \land (C \lor A \lor \neg A \lor \neg A) \land (C \lor A \lor \neg A \lor \neg B) \land (C \lor A \lor \neg A \lor \neg C) \\
& \land (C \lor A \lor B \lor \neg A) \land (C \lor A \lor B \lor \neg B) \lor (C \lor A \lor B \lor \neg C) \\
& \land (C \lor A \lor C \lor \neg A) \land (C \lor A \lor C \lor \neg B) \land (C \lor A \lor C \lor \neg C) \\
& \land (C \lor \neg B \lor \neg A \lor \neg A) \land (C \lor \neg B \lor \neg A \lor \neg B) \land (C \lor \neg B \lor \neg A \lor \neg C) \\
& \land (C \lor \neg B \lor B \lor \neg A) \land (C \lor \neg B \lor B \lor \neg B) \land (C \lor \neg B \lor B \lor \neg C) \\
& \land (C \lor \neg B \lor C \lor \neg A) \land (C \lor \neg B \lor C \lor \neg B) \land (C \lor \neg B \lor C \lor \neg C) \\
& \land (C \lor \neg C \lor \neg A \lor \neg A) \land (C \lor \neg C \lor \neg A \lor \neg B) \land (C \lor \neg C \lor \neg A \lor \neg C) \\
& \land (C \lor \neg C \lor B \lor \neg A) \land (C \lor \neg C \lor B \lor \neg B) \land (C \lor \neg C \lor B \lor \neg C) \\
& \land (C \lor \neg C \lor C \lor \neg A) \land (C \lor \neg C \lor C \lor \neg B) \land (C \lor \neg C \lor C \lor \neg C) \\
\models\dashv\quad & (A \lor B \lor C) \land (A \lor \neg B \lor \neg C) \land (\neg A \lor \neg B \lor C) \land (\neg A \lor B \lor \neg C)
\end{aligned}
$$

Thus, the CNF of $(A \leftrightarrow B) \leftrightarrow C$ is

$$\underline{\boldsymbol{(A \lor B \lor C) \land (A \lor \neg B \lor \neg C) \land (\neg A \lor \neg B \lor C) \land (\neg A \lor B \lor \neg C)}} \qquad \blacksquare$$

2 Problem 02

Proof. By Compactness Theorem, we know if Δ' is finitely satisfiable, Δ' is satisfiable.

Since $\Sigma \cup \{\neg\alpha\}$ is not satisfiable, we know $\Sigma \cup \{\neg\alpha\}$ is not finitely satisfiable.

i.e. exists a finite subset $\Delta^* \subset \Sigma \cup \{\neg\alpha\}$ s.t. Δ^* is not satisfiable.

CASE 01. $\neg\alpha \in \Delta^*$. Let $\Delta^* = \Delta \cup \{\neg\alpha\}$. Then $\Delta \subset \Sigma$.

We know $\Delta \cup \{\neg\alpha\}$ is not satisfiable $\iff \Delta \models \alpha$.

CASE 02. $\neg\alpha \notin \Delta^*$. Let $\Delta = \Delta^* \subset \Sigma$. Then Δ is not satisfiable. Thus, $\Delta \models \alpha$.

In conclusion, there exists some finite set Δ s.t. $\Delta \subset \Sigma$. ■

3 Semantic Consequences of Σ is Effectively Decidable

Proof. Since Σ is effectively enumerable, exists an algorithm $\mathcal{A}$ for enumerating members in Σ.

We can design an algorithm $\mathcal{B}$ as follows.

```
Algo. B
begin
    on Input α;
    for n = 1, 2, 3, ... do
        Run A until the n-th output appears;
        Let the outputs of A be σ_1, σ_2, ...σ_n ∈ Σ;
        Collect all sentence symbols appearing in these wff in the set S;
        Let V ≜ {f | f : S → {True, False}};
        not ← 0;
        yes ← 0;
        for v ∈ V do
            if v̄(¬α) = False then not ← not + 1;
            if v̄(α) = False then yes ← yes + 1;
        end
        if not == |V| then      Output:"YES";
        if yes == |V| then      Output:"NO";
    end
end
```

Let the set of semantic consequences of Σ be Γ.

Now we prove $\mathcal{B}$ is an algorithm for effectively determining membership in Γ.

First we prove the correctness.

If the algorithm returns "YES", then there exists a subset $\Delta = \{\sigma_1, \sigma_2, ...\sigma_n\} \subset \Sigma$ s.t. for all truth assignment v satisfying Δ, $\bar{v}(\neg\alpha) = \texttt{False} \implies \Delta \not\models \neg\alpha \implies \Sigma \not\models \neg\alpha$. Since for each wff α, either $\Sigma \models \alpha$ or $\Sigma \models \neg\alpha$, we know $\Sigma \models \alpha$, i.e. $\alpha \in \Gamma$.

If the algorithm returns "NO", then there exists a subset $\Delta = \{\sigma_1, \sigma_2, ...\sigma_n\} \subset \Sigma$ s.t. for all truth assignment v satisfying Δ, $\bar{v}(\alpha) = \texttt{False} \Longrightarrow \Delta \not\models \alpha \Longrightarrow \Sigma \not\models \alpha$, i.e. $\alpha \notin \Gamma$.

Thus, the result of $\mathcal{B}$ is correct. □

<u>Then we prove the algorithm will terminate within finite steps.</u>

In the loop of n, since S is finite, we know $\{f \mid f : S \to \{\texttt{True}, \texttt{False}\}\}$ is finite. Calculating $\bar{v}(\neg\alpha)$ and $\bar{v}(\alpha)$ can terminate within finite steps. Thus, each loop will terminate within finite steps.

If $\alpha \in \Gamma$, then exists a finite subset $\Delta \subset \Sigma$ s.t. $\Delta \models \alpha$. Obvious there must exist some k s.t. $\Delta \subset \{\sigma_1, \sigma_2 ... \sigma_k\}$. Then exists a v satisfying $\{\sigma_1, ...\sigma_k\}$ s.t. $\bar{v}(\alpha) = \texttt{True}$. Thus, $\mathcal{B}$ will terminate within k loops.

If $\alpha \notin \Gamma$, since either $\Sigma \models \alpha$ or $\Sigma \models \neg\alpha$, we know $\Sigma \models \neg\alpha \Longrightarrow \neg\alpha \in \Gamma$. Thus, $\mathcal{B}$ will still terminate within k loops.

Thus, $\mathcal{B}$ will return a result within finite steps. □

In conclusion, Γ is effectively decidable,

i.e. semantic consequences of Σ is effectively decidable. ■

4 Proof Trees

4.1 $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

Solution. The proof tree is as follows.

$$
\begin{prooftree}
\AxiomC{$[A \wedge (B \vee C)]$}
\RightLabel{$\wedge$-E}
\UnaryInfC{$B \vee C$}
\AxiomC{$[A \wedge (B \vee C)]$}
\RightLabel{$\wedge$-E1}
\UnaryInfC{A}
\AxiomC{$[B]$}
\RightLabel{$\wedge$-I}
\BinaryInfC{$A \wedge B$}
\RightLabel{$\vee$-I1}
\UnaryInfC{$(A \wedge B) \vee (A \wedge C)$}
\AxiomC{$[A \wedge (B \vee C)]$}
\RightLabel{$\wedge$-E1}
\UnaryInfC{A}
\AxiomC{$[C]$}
\RightLabel{$\wedge$-I}
\BinaryInfC{$A \wedge C$}
\RightLabel{$\vee$-I1}
\UnaryInfC{$(A \wedge B) \vee (A \wedge C)$}
\RightLabel{$\vee$-E}
\TrinaryInfC{$(A \wedge B) \vee (A \wedge C)$}
\RightLabel{$\to$-I}
\UnaryInfC{$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$}
\end{prooftree}
$$
■

4.2 $\neg(\neg A \vee \neg B) \to A \vee B$

Solution. The proof tree is as follows.

$$
\begin{prooftree}
\AxiomC{$\{B \vee \neg B\}$}
\AxiomC{$[B]$}
\RightLabel{$\vee$-I2}
\UnaryInfC{$A \vee B$}
\AxiomC{$[\neg B]$}
\RightLabel{$\vee$-I2}
\UnaryInfC{$\neg A \vee \neg B$}
\AxiomC{$[\neg(\neg A \vee \neg B)]$}
\RightLabel{$\neg$-E}
\BinaryInfC{$A \vee B$}
\RightLabel{$\vee$-E}
\TrinaryInfC{$A \vee B$}
\RightLabel{$\to$-I}
\UnaryInfC{$\neg(\neg A \vee \neg B) \to A \vee B$}
\end{prooftree}
$$
■

4.3 $((P \to Q) \to P) \to P$

Solution. The proof tree is as follows.

$$
\begin{prooftree}
\AxiomC{$\{P \vee \neg P\}$}
\AxiomC{$[P]$}
\AxiomC{$[P]$}
\AxiomC{$[\neg P]$}
\RightLabel{$\neg$-E}
\BinaryInfC{Q}
\RightLabel{$\to$-I}
\UnaryInfC{$P \to Q$}
\AxiomC{$[(P \to Q) \to P]$}
\RightLabel{$\to$-E}
\BinaryInfC{P}
\RightLabel{$\vee$-E}
\TrinaryInfC{P}
\RightLabel{$\to$-I}
\UnaryInfC{$((P \to Q) \to P) \to P$}
\end{prooftree}
$$

■

5 Part of the Proof of the Soundness of Natural Deduction

5.1 ¬-I Case

Proof. We have $\{\gamma_1, ...\gamma_n, \beta\} \vdash \delta \wedge \neg\delta$.

By **Inductive Hypothesis**, $\{\gamma_1, ...\gamma_n, \beta\} \vDash \delta \wedge \neg\delta$.

i.e. for any truth assignment v satisfying $\gamma_1, ...\gamma_n, \beta$, $\bar{v}(\delta \wedge \neg\delta) = \texttt{True}$. Thus, There exists no truth assignment v satisfies $\gamma_1, ...\gamma_n, \beta$.

Therefore, $\{\gamma_1, ...\gamma_n, \beta\}$ is not satisfiable.

Assume $\{\gamma_1, ...\gamma_n\} \nvDash \neg\beta$. For any truth assignment v satisfying $\{\gamma_1, ...\gamma_n\}$, $\bar{v}(\neg\beta) = \texttt{False}$, i.e. $\bar{v}(\beta) = \texttt{True}$, i.e. exists v satisfying $\{\gamma_1, ...\gamma_n\}$ s.t. v satisfies β, i.e. v satisfies $\{\gamma_1, ...\gamma_n, \beta\}$. **Contradiction.**

Thus, $\{\gamma_1, ...\gamma_n\} \vDash \neg\beta$.

Since $\{\gamma_1, ...\gamma_n\} \subset \Sigma$, we know $\Sigma \vDash \neg\beta$. ■

5.2 ¬-E Case

Proof. We know $\{\gamma_1, ...\gamma_n\} \vdash \beta$ and $\{\sigma_1, ...\sigma_m\} \vdash \neg\beta$.

By **Inductive Hypothesis**, $\{\gamma_1, ...\gamma_n\} \vDash \beta$ and $\{\sigma_1, ...\sigma_m\} \vDash \neg\beta$.

Since $\{\gamma_1, ...\gamma_n\} \subset \Sigma, \{\sigma_1, ...\sigma_m\} \subset \Sigma$, we know $\Sigma \vDash \beta$ and $\Sigma \vDash \neg\beta$.

Thus, for any truth assignment v satisfying Σ, $\bar{v}(\beta) = \bar{v}(\neg\beta) = \texttt{True}$.

Therefore, there exists no truth assignment v satisfying Σ, i.e. Σ is not satisfiable.

Then for any wff δ, $\Sigma \vDash \delta$.

Thus, $\Sigma \vDash \alpha$. ■