Stochastic Process Homework 04

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Apr.30 - May.8, 2022

0 Reference and Notations

In the following sections, we use the following notations.

Notaion	Meaning
$\overline{X_{1,n}}$	$X_1, X_2, ...X_n$

Table 1: Notations.

This homework is completed with the help of discussions with **Ji Yikun**.

1 Doob's Martingale Inequality

Proof. For any given $n \in \mathbb{N}$,

Let $X_\tau = \max\limits_{0 \le t \le n} X_t \ge \alpha$. By **Markov's Inequality**, we have

$$\mathbf{Pr}\left[X_\tau \ge \alpha\right] \le \frac{\mathbb{E}\left[X_\tau\right]}{\alpha}.$$

Consider the stopping time τ.

$$\tau = \begin{cases} t, & \exists\ t \in [0, n] \text{ s.t. } X_t \ge \alpha \\ n, & \forall\ 0 \le t \le n,\ X_t < \alpha \end{cases}$$

Thus, $\mathbf{Pr}\left[\tau \le n\right] = 1$.

Since $\{X_t\}$ is a martingale w.r.t. $\{X_t\}$ and $\exists\ N = n \in \mathbb{N}$ s.t. $\mathbf{Pr}\left[\tau \le N\right] = 1$ for the stopping time τ, by **Optional Stopping Time Theorem**, we have $\mathbb{E}\left[X_\tau\right] = \mathbb{E}\left[X_0\right].$

Therefore,

$$\mathbf{Pr}\left[X_\tau \ge \alpha\right] \le \frac{\mathbb{E}\left[X_0\right]}{\alpha}.$$

Qed. ■

2 Biased One-dimensional Random Walk

2.1 $\{S_t\}_{t\geq 0}$ is a Martingale

Proof. We have

$$\begin{aligned}\mathbb{E}\left[S_{t+1} \mid \overline{Z_{1,t}}\right] &= \mathbb{E}\left[S_t + Z_{t+1} + 2p - 1 \mid \overline{Z_{1,t}}\right] = S_t + 2p - 1 + \mathbb{E}\left[Z_{t+1} \mid \overline{Z_{1,t}}\right] \\ &= S_t + 2p - 1 + (-1)\cdot p + 1\cdot(1-p) = S_t.\end{aligned}$$

Thus, $\{S_t\}_{t\geq 0}$ is a martingale. ∎

2.2 $\{P_t\}_{t\geq 0}$ is a Martingale

Proof. We have

$$\begin{aligned}\mathbb{E}\left[P_{t+1} \mid \overline{Z_{1,t}}\right] &= \mathbb{E}\left[P_t\left(\frac{p}{1-p}\right)^{Z_{t+1}} \mid \overline{Z_{1,t}}\right] = P_t\cdot\mathbb{E}\left[\left(\frac{p}{1-p}\right)^{Z_{t+1}} \mid \overline{Z_{1,t}}\right] \\ &= P_t\cdot\left(p\cdot\frac{1-p}{p} + (1-p)\cdot\frac{p}{1-p}\right) = P_t.\end{aligned}$$

Thus, $\{P_t\}_{t\geq 0}$ is a martingale. ∎

2.3 Average Number of Steps, i.e. $\mathbb{E}[\tau]$

Proof. Define $p_a \triangleq \mathbf{Pr}[X_\tau = a], p_b \triangleq \mathbf{Pr}[X_\tau = b] = 1 - p_a$, $q = \max(p, 1-p)$.

For any $N \in \mathbb{N}$, we have

$$\mathbf{Pr}[\tau \leq N(a+b)] \geq \sum_{k=0}^{N}\mathbf{Pr}[\tau = k(a+b)] \geq \sum_{k=0}^{N} q^{k(a+b)} = \frac{1-q^{(N+1)(a+b)}}{1-q^{a+b}}$$

Thus, for any $t \in \mathbb{N}$, let $m = \left\lfloor \frac{t}{a+b} \right\rfloor (a+b) \leq t$.

$$\mathbf{Pr}[\tau > t] = 1 - \mathbf{Pr}[\tau \leq t] \leq 1 - \mathbf{Pr}[\tau \leq m] \leq q^{a+b}\frac{1-q^m}{1-q^{a+b}} \to 0 \text{ (as } t\to\infty)$$

$$t\cdot\mathbf{Pr}[\tau = t] = t(1 - \mathbf{Pr}[\tau \leq m]) \leq tq^{a+b}\frac{1-q^m}{1-q^{a+b}} \to 0 \text{ (as } t\to\infty)$$

Thus, we know $\mathbf{Pr}[\tau < \infty] = 1,\ \mathbb{E}[\tau] < \infty$.

CASE 01. $p \neq \frac{1}{2}$.

We have already shown that $\{P_t\}_{t\geq 0}$ is a martingale. Obvious for all $t \leq \tau$, $|P_t| \leq 1$. Meanwhile, we have $\mathbf{Pr}\left[\tau < \infty\right] = 1$. By **Optional Stopping Time Theorem**, we know

$$\mathbb{E}\left[P_t\right] = \mathbb{E}\left[P_1\right] \Longleftrightarrow p_a\left(\frac{p}{1-p}\right)^{-a} + p_b\left(\frac{p}{1-p}\right)^{b} = p \cdot \frac{1-p}{p} + (1-p)\cdot\frac{p}{1-p} = 1.$$

This yields that

$$p_a = \frac{1-\left(\frac{p}{1-p}\right)^b}{\left(\frac{p}{1-p}\right)^{-a} - \left(\frac{p}{1-p}\right)^{b}}, \quad p_b = \frac{\left(\frac{p}{1-p}\right)^{-a} - 1}{\left(\frac{p}{1-p}\right)^{-a} - \left(\frac{p}{1-p}\right)^{b}}.$$

Meanwhile, we have already proved that $\{S_t\}_{t\geq 0}$ is a martingale.

Also, $\forall t \leq \tau$, $\mathbb{E}\left[|S_{t+1} - S_t| \mid \overline{Z_{1,t}}\right] = \mathbb{E}\left[2p - 1 + Z_{t+1} \mid \overline{Z_{1,t}}\right] = 2p - 1 + \mathbb{E}\left[Z_{t+1} \mid \overline{Z_{1,t}}\right] < 2p$.

Moreover, $\mathbb{E}\left[\tau\right] < \infty$.

Therefore, by **Optional Stopping Time Theorem**, we have

$$\begin{aligned}\mathbb{E}\left[S_\tau\right] = \mathbb{E}\left[S_1\right] = 0 \Longleftrightarrow \mathbb{E}\left[S_\tau\right] = \mathbb{E}\left[\sum_{i=1}^{\tau}(Z_i + 2p - 1)\right] &= \mathbb{E}\left[(2p-1)\tau + \sum_{i=1}^{\tau} Z_i\right] \\ &= (2p-1)\mathbb{E}\left[\tau\right] + p_a \cdot a + p_b \cdot (-b) = 0.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\left[\tau\right] = \frac{ap_a - bp_b}{2p-1} &= \frac{a + b - a\left(\frac{p}{1-p}\right)^b - b\left(\frac{p}{1-p}\right)^{-a}}{(2p-1)\left[\left(\frac{p}{1-p}\right)^{-a} - \left(\frac{p}{1-p}\right)^{b}\right]} \\ &= \frac{(a+b)(1-p)^b p^a - ap^{a+b} - b(1-p)^{a+b}}{(2p-1)[(1-p)^{a+b} - p^{a+b}]}.\end{aligned}$$

CASE 02. $p = \frac{1}{2}$.

Since $\mathbb{E}\left[X_{t+1} \mid \overline{Z_{1,t}}\right] = \mathbb{E}\left[X_t + Z_{t+1} \mid \overline{Z_{1,t}}\right] = \mathbb{E}\left[X_t \mid \overline{Z_{1,t}}\right] + \mathbb{E}\left[Z_{t+1} \mid \overline{Z_{1,t}}\right] = X_t + 0 = X_t$, we know $\{X_t\}_{t\geq 0}$ is a martingale.

Meanwhile, $\mathbf{Pr}\left[\tau < \infty\right] = 1$; $\forall\ t \leq \tau,\ |X_t| \leq \max(a, b)$.

By **Optional Stopping Time Theorem**, we have $\mathbb{E}\left[X_\tau\right] = \mathbb{E}\left[X_1\right] = 0$, i.e. $ap_a + bp_b = 0$.

This yields that

$$p_a = \frac{b}{a+b}, \quad p_b = \frac{a}{a+b}.$$

Construct $Y_t = X_t^2 - t$. Since

$$\begin{aligned}\mathbb{E}\left[Y_{t+1} \mid \overline{Z_{1,t}}\right] &= \mathbb{E}\left[X_{t+1}^2 - (t+1) \mid \overline{Z_{1,t}}\right] = \mathbb{E}\left[(X_t + Z_{t+1})^2 - (t+1) \mid \overline{Z_{1,t}}\right] \\ &= \mathbb{E}\left[X_t^2 + 2X_tZ_{t+1} + Z_{t+1}^2 - (t+1) \mid \overline{Z_{1,t}}\right] \\ &= \mathbb{E}\left[X_t^2 \mid \overline{Z_{1,t}}\right] + 2\mathbb{E}\left[X_tZ_{t+1} \mid \overline{Z_{1,t}}\right] + \mathbb{E}\left[Z_{t+1}^2 \mid \overline{Z_{1,t}}\right] - (t+1) \\ &= X_t^2 + 2X_t\mathbb{E}\left[Z_{t+1} \mid \overline{Z_{1,t}}\right] + \mathbb{E}\left[Z_{t+1}^2 \mid \overline{Z_{1,t}}\right] - (t+1) \\ &= X_t^2 + 0 + 1 - (t+1) = X_t^2 - t = Y_t,\end{aligned}$$

we know $\{Y_t\}_{t\geq 0}$ is a martingale.

Also, we have

$$\begin{aligned}\mathbb{E}\left[|Y_{t+1}-Y_t| \mid \overline{Z_{1,t}}\right] &= \mathbb{E}\left[2X_tZ_{t+1}+Z_{t+1}^2-1 \mid \overline{Z_{1,t}}\right] \\ &= X_t\cdot\mathbb{E}\left[Z_{t+1} \mid \overline{Z_{1,t}}\right]+\mathbb{E}\left[Z_{t+1}^2 \mid \overline{Z_{1,t}}\right]-1 \\ &= 0+1-1=0.\end{aligned}$$

Moreover, $\mathbb{E}\left[\tau\right]<\infty$.

By **Optional Stopping Time Theorem**, we have

$$\begin{aligned}\mathbb{E}\left[Y_\tau\right]=\mathbb{E}\left[Y_1\right]=0 &\Longleftrightarrow \mathbb{E}\left[X_\tau^2-\tau\right]=\mathbb{E}\left[X_\tau^2\right]-\mathbb{E}\left[\tau\right]=0 \\ &\Longleftrightarrow \mathbb{E}\left[\tau\right]=\mathbb{E}\left[X_\tau^2\right]=p_a\cdot a^2+p_b\cdot b^2=\frac{a^2b+b^2a}{a+b}=ab.\end{aligned}$$

In conclusion,

$$\mathbb{E}\left[\tau\right]=\begin{cases}\frac{(a+b)(1-p)^bp^a-ap^{a+b}-b(1-p)^{a+b}}{(2p-1)[(1-p)^{a+b}-p^{a+b}]}, & p\neq\frac{1}{2}\\ ab, & p=\frac{1}{2}\end{cases}$$

■

3 Longest Common Subsequence

Notation: *In this section, we define*

- $X_{(i,j)}$ as the length of longest common subsequence of $x[i:j]$ and y.
- $X_{(i,j),(k,l)}$ as the length of longest common subsequence of $x[i:j]$ and $y[k:l]$.

3.1 Range of $\mathbb{E}\left[\boldsymbol{X}\right]$

Proof. First we prove the existence of c_1.

CASE 01. When $n=2$, all possible cases are as follows.

x	y	X	x	y	X	x	y	X	x	y	X
00	00	2	00	01	1	00	10	1	00	11	0
01	00	1	01	01	2	01	10	1	01	11	1
10	00	1	10	01	1	10	10	2	10	11	1
11	00	0	11	01	1	11	10	1	11	11	2

Table 2: All Possible Cases.

Thus,

$$\mathbb{E}[X] = \frac{2(2+1+1+0+1+2+1+1)}{2^2 \cdot 2^2} = \frac{9}{8} > c_1 \cdot 2.$$

Therefore, we have $c_1 < \frac{9}{16}$.

Let $c_1^* \triangleq \frac{9}{16}$.

CASE 02. When $n = 3$,

x	y	X	x	y	X	x	y	X	x	y	X
000	000	3	000	001	2	000	010	2	000	011	2
000	100	2	000	101	1	000	110	1	000	111	0
001	000	2	001	001	3	001	010	2	001	011	2
001	100	2	001	101	2	001	110	1	001	111	1

Table 3: Some Typical Cases.

Thus, we know

$$\begin{aligned}\mathbb{E}[X] &= \frac{2\cdot(3+2+2+2+2+1+1+0)+6\cdot(2+3+2+2+2+2+1+1)}{2^3 \cdot 2^3} \\ &= \frac{29}{16} > \frac{9}{16} \cdot 3 = c_1^* \cdot 3 \geq c_1 \cdot 3.\end{aligned}$$

The inequality already holds.

CASE 03. When $n \geq 4$, we can divide x and y into smaller pieces with length 2 or 3.

When n is even, we have

$$\mathbb{E}[X] \geq \sum_{k=1}^{n/2} \mathbb{E}\left[X_{(2k-1,2k),(2k-1,2k)}\right] = \sum_{k=1}^{n/2} c_1^* \cdot 2 = c_1^* \cdot n.$$

When n is odd, we have

$$\begin{aligned}\mathbb{E}[X] &\geq \sum_{k=1}^{(n-3)/2} \mathbb{E}\left[X_{(2k-1,2k),(2k-1,2k)}\right] + \mathbb{E}\left[X_{(n-2,n),(n-2,n)}\right] \\ &> \sum_{k=1}^{(n-3)/2} c_1^* \cdot 2 + c_1^* \cdot 3 = c_1^* \cdot n.\end{aligned}$$

In conclusion, for any $n \geq 2, n \in \mathbb{N}$, $\mathbb{E}[X] \geq \frac{9}{16} n$.

In other words, there exists $c_1 \in (\frac{1}{2}, \frac{9}{16})$ s.t.

$$\frac{1}{2} < c_1 < 1 \text{ while } \mathbb{E}[X] \geq \frac{9}{16} n > c_1 n \text{ holds for sufficiently } n.$$

(For example, $c_1 = 17/32$ is a feasible constant.) $\square$

Now we prove the existence of c_2.

Inspired by the hint, consider $X_{(i,j),(k,l)}$ when $j-i+1$ and $l-k+1$ are large enough.

Let $j-i+1 = l-k+1 = m,\ x' = x[i:j],\ y' = y[k:l]$. We have

$$\begin{aligned}\mathbf{Pr}\left[X_{(i,j),(k,l)} \geq t\right] &= \mathbf{Pr}\left[\text{exists } S,T \subset [m], |S| = |T| = t \text{ s.t. } x'_S = y'_T\right] \\ &\leq \frac{2^t\binom{m}{t}\binom{m}{t}\cdot 2^{m-t}\cdot 2^{m-t}}{2^m\cdot 2^m} = \frac{1}{2^t}\binom{m}{t}^2\end{aligned}$$

(Since the RHS might count the same sequence more than once.)

Since m is large enough, by **Stirling's Formula**, we know

$$\begin{aligned}\mathbf{Pr}\left[X_{(i,j),(k,l)} \geq t\right] &\leq \frac{1}{2^t}\left(\frac{m!}{t!(m-t)!}\right)^2 \\ &\approx 2^{-t}\left(\frac{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m}{\sqrt{2\pi t}\left(\frac{t}{e}\right)^t\sqrt{2\pi(m-t)}\left(\frac{m-t}{e}\right)^{m-t}}\right)^2 \\ &= \frac{1}{\pi}\frac{m^{2m+1}}{2^{t+1}\cdot t^{2t+1}\cdot (m-t)^{2m-2t+1}}\end{aligned}$$

We know $X = X_{(1,n),(1,n)}$. Let $t = \mu n$. This yields

$$\begin{aligned}\mathbf{Pr}\left[X \geq \mu n\right] &\leq \frac{1}{\pi}\frac{n^{2n+1}}{2^{\mu n+1}(\mu n)^{2\mu n+1}\left((1-\mu)n\right)^{2n-2\mu n+1}} \\ &= \frac{1}{2\pi\mu(1-\mu)}\frac{1}{n}\left(\frac{1}{2^{\mu/2}\mu^\mu(1-\mu)^{1-\mu}}\right)^{2n} \to 0 \qquad (\text{as } n\to\infty)\end{aligned}$$

i.e. $\mathbf{Pr}\left[X < \mu n\right] = 1 - \mathbf{Pr}\left[X \geq \mu n\right] \to 1 \qquad (\text{as } n \to \infty)$

When $\mu \geq 0.91$, we have $\frac{1}{2^{\mu/2}\mu^\mu(1-\mu)^{1-\mu}} < 1$, i.e.

$$n\cdot\mathbf{Pr}\left[X \geq \mu n\right] \leq \frac{1}{2\pi\mu(1-\mu)}\cdot\left(\frac{1}{2^{\mu/2}\mu^\mu(1-\mu)^{1-\mu}}\right)^{2n} \to 0 \qquad (\text{as } n\to\infty)$$

Let $\mu < c_2 < 1$. Then

$$\begin{aligned}\mathbb{E}\left[X\right] &= \mathbb{E}\left[X|X<\mu n\right]\cdot\mathbf{Pr}\left[X<\mu n\right] + \mathbb{E}\left[X|X\geq\mu n\right]\cdot\mathbf{Pr}\left[X\geq\mu n\right] \\ &\leq \mu n\cdot\mathbf{Pr}\left[X<\mu n\right] + \sum_{\mu\leq k\leq 1} kn\cdot\mathbf{Pr}\left[X = kn\right] \\ &< \mu n\cdot\mathbf{Pr}\left[X<\mu n\right] + n\cdot\mathbf{Pr}\left[X\geq\mu n\right] = \mu n + (1-\mu)n\cdot\mathbf{Pr}\left[X\geq\mu n\right] \\ &\leq \mu n + \frac{1}{2\pi\mu}\cdot\left(\frac{1}{2^{\mu/2}\mu^\mu(1-\mu)^{1-\mu}}\right)^{2n} \to \mu n \qquad (\text{as } n\to\infty) \\ &< c_2 n. \qquad\qquad (\text{for sufficiently } n)\end{aligned}$$

In other words, c_2 exists.

(For example, set $\mu = 0.96$. Then $c_2 = 0.99$ is a reasonable constant for $n \geq 3$.) □

In conclusion, exist constants c_1, c_2 s.t. for sufficiently n,

$$\frac{1}{2} < c_1 < c_2 < 1,\ c_1 n < \mathbb{E}\left[X\right] < c_2 n. \qquad \blacksquare$$

3.2 $\boldsymbol{X}$ is Well-Concentrated around $\mathbb{E}[\boldsymbol{X}]$

Proof. We can construct a function $f(\boldsymbol{z}) \triangleq f(x_1, x_2, ..., x_n, y_1, y_2, ...y_n) \triangleq X$.

Obvious $f(\boldsymbol{z}) - f(\boldsymbol{z'}) \leq \|\boldsymbol{z} - \boldsymbol{z'}\|_1 = \sum_{i=1}^{n} |x_i - x_i'| + \sum_{i=1}^{n} |y_i - y_i'|$.

Thus, f is 1-*Lipschitz*.

By **McDiarmid's Inequality**, since f is 1-*Lipschitz* and $x_1, x_2, ...x_n, y_1, y_2, ...y_n$ are obviously independent to each other, we have

$$\begin{aligned} &\mathbf{Pr}\left[\Big| f(x_1, x_2, ...x_n, y_1, y_2, ...y_n) - \mathbb{E}\left[f(x_1, x_2, ...x_n, y_1, y_2, ...y_n)\right] \Big| \geq t\right] \leq 2e^{-\frac{2t^2}{n}} \\ \iff &\mathbf{Pr}\left[\big|X - \mathbb{E}\left[X\right]\big| \geq t\right] \leq 2e^{-\frac{2t^2}{n}} \end{aligned}$$

i.e. X is well-concentrated around $\mathbb{E}[X]$. ■

3.3 (Optional) Dynamic Programming for LCS

Solution. Let $f(i, j)$ be the length of LCS between $x[1:i]$ and $y[1:j]$.

State Transition Equation.

$$f(i,j) = \begin{cases} f(i-1, j-1) + 1, & x[i] = y[j] \\ \max(f(i-1, j), f(i, j-1)), & x[i] \neq y[j] \end{cases}$$

Boundaries. $f(\cdot, 0) = 0,\ f(0, \cdot) = 0$.

The final result. $f(n, n)$. ■