Discrete Mathematics Exercise 7

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1. ***Proof:***

Firstly, we prove that $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

For any $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$, we know from the definition of power set and union that $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$, **i.e.** $x \subseteq A$ and $x \subseteq B$.

It's plain to see that $x \subseteq A \cap B$, **i.e.** $x \in \mathcal{P}(A \cap B)$.

Thus, for any $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$, $x \in \mathcal{P}(A \cap B)$.

In other words, $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Now we prove $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

For any $x \in \mathcal{P}(A \cap B)$, we know from the definition of power set and union that $x \subseteq A \cap B$, **i.e.** $x \subseteq A$ and $x \subseteq B$. Thus, $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$, **i.e.** $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Therefore, for any $x \in \mathcal{P}(A \cap B)$, $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

In other words, $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

Since $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$ and $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$,

we know $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

QED

2. ***Proof:***

Firstly, we prove that $A \times \bigcup B \subseteq \bigcup\{A \times X \mid X \in B\}$.

For any $z \in A \times \bigcup B$, $\exists a \in A, b \in \bigcup B$ such that $z = (a, b)$.

From $b \in \bigcup B$, we know $\exists X \in B$ such that $b \in X$, **i.e.** $z \in A \times X$.

Therefore, for any $z \in A \times \bigcup B$, $z \in \bigcup\{A \times X \mid X \in B\}$.

In other words, $A \times \bigcup B \subseteq \bigcup\{A \times X \mid X \in B\}$.

Now we prove $A \times \bigcup B \supseteq \bigcup\{A \times X \mid X \in B\}$.

For any $z \in \bigcup\{A \times X \mid X \in B\}$, $\exists a \in A, X \in B, b \in X$ such that $z = (a, b)$.

From $X \in B, b \in X$, we know $b \in \bigcup B$.

Therefore, for any $z \in \bigcup\{A \times X \mid X \in B\}$, $z \in A \times \bigcup B$.

In other words, $A \times \bigcup B \supseteq \bigcup\{A \times X \mid X \in B\}$.

Since $A \times \bigcup B \subseteq \bigcup\{A \times X \mid X \in B\}$ and $A \times \bigcup B \supseteq \bigcup\{A \times X \mid X \in B\}$,

we know that $A \times \bigcup B = \bigcup\{A \times X \mid X \in B\}$.

QED

3. ***Proof:***

Enter a new proof context, introduce an arbitrary x.

$\mathrm{ZF} \vdash x \in C \cap (A \cup B) \leftrightarrow x \in C \wedge (x \in A \vee x \in B)$.

$\mathrm{ZF} \vdash x \in C \cap (A \cup B) \leftrightarrow (x \in C \wedge x \in A) \vee (x \in C \wedge x \in B)$.

$\mathrm{ZF} \vdash x \in C \cap (A \cup B) \leftrightarrow x \in (C \cap A) \cup (C \cap B)$.

Exit the proof context.

Using its conclusion, we can prove $\mathrm{ZF} \vdash \forall x\big(x \in C \cap (A \cup B) \leftrightarrow x \in (C \cap A) \cup (C \cap B)\big)$.

Then we can prove $\mathrm{ZF} \vdash C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$.

QED

4. ***Proof:***

Assume there exists two empty sets $\emptyset_1$ and $\emptyset_2$.

$\mathrm{ZF} \vdash \forall x(\neg x \in \emptyset_1) \wedge \forall x(\neg x \in \emptyset_2) \rightarrow \forall x(\neg x \in \emptyset_1 \wedge \neg x \in \emptyset_2)$.

$\mathrm{ZF} \vdash \forall x(\neg x \in \emptyset_1) \wedge \forall x(\neg x \in \emptyset_2) \rightarrow \forall x\big((\neg x \in \emptyset_1 \wedge \neg x \in \emptyset_2) \vee (x \in \emptyset_1 \wedge x \in \emptyset_2)\big)$.

$\mathrm{ZF} \vdash \forall x(\neg x \in \emptyset_1) \wedge \forall x(\neg x \in \emptyset_2) \rightarrow \forall x(x \in \emptyset_1 \leftrightarrow x \in \emptyset_2)$.

Then using the axiom of extensionality,

$\mathrm{ZF} \vdash \forall x(\neg x \in \emptyset_1) \wedge \forall x(\neg x \in \emptyset_2) \rightarrow \emptyset_1 = \emptyset_2$.

QED

5. ***a*)*****Proof:*** Since $\forall x\big(\neg(x \subseteq \emptyset \wedge \neg x = \emptyset)\big)$,

we know $\forall x\Big(x \subseteq \emptyset \wedge \neg x = \emptyset \rightarrow \exists y\big(y \in x \wedge \forall z(z \in x \rightarrow y \in z \vee y = z)\big)\Big)$,

$0 = \{\} = \emptyset$ is $\in$-well-ordered. ***QED***

b*)**Proof:***

When n is $\in$-well-ordered,

$$\forall x\Big(x \subseteq n \wedge \neg x = \emptyset \rightarrow \exists y\big(y \in x \wedge \forall z(z \in x \rightarrow y \in z \vee y = z)\big)\Big). \quad ①$$

For any $x \subseteq n \cup \{n\} \wedge \neg x = \emptyset$, there exists two cases:

1) $x \subseteq n \wedge \neg x = \emptyset$

From ① we know $\exists y\big(y \in x \wedge \forall z(z \in x \rightarrow y \in z \vee y = z)\big)$.

2) $\exists x', x' \subseteq n \wedge \neg x' = \emptyset$ such that $x = x' \cup \{n\}$.

From ① we know $\exists y\big(y \in x' \wedge \forall z(z \in x' \rightarrow y \in z \vee y = z)\big)$.

Since $y \in x'$, $y \in x$.

For any z, $z \in x \Rightarrow \big(\exists w\,(w \in x') \wedge (w \in z) \wedge (y \in w \vee y = w)\big) \Rightarrow y \in z \vee y = z$.

Thus, $\exists y\big(y \in x \wedge \forall z(z \in x \rightarrow y \in z \vee y = z)\big)$.

In conclusion, for any $x \subseteq n \cup \{n\} \wedge \neg x = \emptyset \rightarrow \exists y\big(y \in x \wedge \forall z(z \in x \rightarrow y \in z \vee y = z)\big)$.

Therefore, $\forall x\Big(x \subseteq n \cup \{n\} \wedge \neg x = \emptyset \rightarrow \exists y\big(y \in x \wedge \forall z(z \in x \rightarrow y \in z \vee y = z)\big)\Big)$.

In other words, when n is $\in$-well-ordered, $n \cup \{n\}$ is $\in$-well-ordered.

QED

6. ***Proof:***

Since $\forall x\,(\textbf{Inductive}(x) \rightarrow u \subseteq x)$ and $\textbf{Inductive}(v)$, $u \subseteq v$.

Similarly, since $\forall x\,(\textbf{Inductive}(x) \rightarrow v \subseteq x)$ and $\textbf{Inductive}(u)$, $v \subseteq u$.

From $u \subseteq v$ and $v \subseteq u$, we know $u = v$.

QED

7. ***Proof:***

Since u and v are two inductive sets,

$$\emptyset \in u \wedge \forall x\,(x \in u \rightarrow x \cup \{x\} \in u) \text{ and } \emptyset \in v \wedge \forall x\,(x \in v \rightarrow x \cup \{x\} \in v).$$

$\emptyset \in u$ and $\emptyset \in v \Rightarrow \emptyset \in u \cap v$.

For any $x \in u \cap v$,

$x \in u \cap v \Leftrightarrow x \in u \wedge x \in v \Rightarrow x \cup \{x\} \in u \wedge x \cup \{x\} \in v \Leftrightarrow x \cup \{x\} \in u \cap v$.

Thus, $\emptyset \in u \cap v \wedge \forall x\,(x \in u \cap v \rightarrow x \cup \{x\} \in u \cap v)$.

In other words, $u \cap v$ is an inductive set.

QED

8. ***a*)*Proof:***

Let $\{x \in u \mid \forall v(v \subseteq u \wedge \mathbf{Inductive}(v) \rightarrow x \in v)\}$ be X.

According to the definition of inductive sets, we know $\forall v\,(\mathbf{Inductive}(v) \rightarrow \emptyset \in v)$.

Therefore, $\forall v\,(v \subseteq u \wedge \mathbf{Inductive}(v) \rightarrow \emptyset \in v)$, **i.e.** $\emptyset \in X$.

Now we prove for any $y \in X, y \cup \{y\} \in X$.

Since $y \in X$, **i.e.** $\forall v\,(v \subseteq u \wedge \mathbf{Inductive}(v) \rightarrow y \in v)$, also considering that $\forall v(t \in v \wedge \mathbf{Inductive}(v)) \rightarrow (t \cup \{t\} \in w)$, we know $\forall v\,(v \subseteq u \wedge \mathbf{Inductive}(v) \rightarrow y \cup \{y\} \in v)$.

In other words, $y \cup \{y\} \in X$.

Since $(\emptyset \in X) \wedge \forall y(y \in X \rightarrow y \cup \{y\} \in X)$, X is an inductive set.

QED

b*)*Proof:

Let $\{x \in u \mid \forall v(v \subseteq u \wedge \mathbf{Inductive}(v) \rightarrow x \in v)\}$ be X.

We prove X is the smallest inductive subset of u by contradiction.

Assume the smallest inductive subset of u is Y and $X \neq Y$.

We know from the definition of X that $\forall x(x \in X \rightarrow x \in Y)$, **i.e.** $X \subseteq Y$.

Since $X \neq Y$, X is a proper subset of Y and is therefore a proper subset of u. Given that $\mathbf{Inductive}(X)$, X is an inductive proper subset of Y and u, which is contrast to the definition of Y (because Y is the smallest inductive subset of u).

Thus, X is the smallest inductive subset of u.

QED

9. ***Proof:***

From the conclusion of 7, we know that

$$\mathbf{Inductive}(u_1) \wedge \mathbf{Inductive}(u_2) \rightarrow \mathbf{Inductive}(u_1 \cap u_2). \qquad ①$$

We can construct a set X such that $\forall x\,(\mathbf{Inductive}(x) \rightarrow x \in X)$.

Thus, there exists $A = \bigcap\{x \in X \mid \mathbf{Inductive}(x)\}$.

Using ①, we know that A is an inductive set.

Then we prove A is the smallest inductive set of all inductive sets.

For any $u \in X$, $\exists y = \{x \in u \mid \forall v(v \subseteq u \wedge \textbf{Inductive}(v) \to x \in v)\}$.
From the conclusion of 8, we know that $\textbf{Inductive}(y)$ holds, **i.e.** $y \in X$, $A \subseteq y$.
Now we prove $A = y$ by contradiction.
Assume $A \neq y$.
Then A is a proper set of y.
From the conclusion of 8 we know y is the smallest inductive subset of u.
Given $\textbf{Inductive}(A) \wedge A \subseteq y \wedge A \neq y$,
we know that $\exists A\, \exists x\ (x \in y) \wedge \neg(A \subseteq u \wedge \textbf{Inductive}(A) \to x \in A)$,
which is a ***contradiction*** since $y = \{x \in u \mid \forall v(v \subseteq u \wedge \textbf{Inductive}(v) \to x \in v)\}$.
So, $A = y$, **i.e.** for any $u \in X$, A is the smallest inductive subset of u.

Thus, $\forall u\ (u \in X \to A \subseteq u)$.
Therefore, A is the smallest inductive set of any inductive set u.
In other words, there exists at least one "smallest inductive set".

QED

10.*a*) ***Proof:***

Since $0 \in X$ and $0 \in \mathbb{N}$ (according to the definition of $\mathbb{N}$), $0 \in \mathbb{N} \cap X$.
For any $y \in \mathbb{N} \cap X$,
$y \in \mathbb{N} \cap X \Leftrightarrow y \in \mathbb{N} \wedge y \in X \Rightarrow y \cup \{y\} \in \mathbb{N} \wedge y \cup \{y\} \in X \Leftrightarrow y \cup \{y\} \in \mathbb{N} \cap X$.
(since for any $y \in \mathbb{N}$, $y \in X$ implies $y \cup \{y\} \in X$)

In other words, $\mathbb{N} \cap X$ is inductive.

QED

b*)** ***Proof:

For any $n \in \mathbb{N}$, $n = 0 \vee \exists x\ (x \in \mathbb{N} \wedge n = x \cup \{x\})$.
Thus, to prove that for any $n \in \mathbb{N}$, $n \in X$ holds, we only need to prove that $0 \in \mathbb{N}$ and for any $x \in \mathbb{N}$, $x \cup \{x\} \in X$ (since $x \cup \{x\} \in \mathbb{N}$ holds).
Firstly, we know that $0 \in \mathbb{N} \cap X$. Thus, for $0 \in \mathbb{N}$, $0 \in X$.
Assume $x \in \mathbb{N}$ and $x \in X$ holds.
Since for any $n \in \mathbb{N}$, $n \in X$ implies $n \cup \{n\} \in X$, we know $x \cup \{x\} \in X$.
Therefore, for any $x \in \mathbb{N}$, $x \cup \{x\} \in X$.

Thereby, for any $n \in \mathbb{N}$, $n \in X$ always holds.

QED