Stochastic Process Homework 05

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0 Reference

This time, I finish the homework on my own.

1 Get Off Work Earlier

1.1 Probability that Joe Achieves the Goal

Solution. Let $N(t)$ be the number of customers arrive between $T-s$ and $T-s+t$.

Obvious $N(t)$ is a Poisson Process.

Suppose the first customer after $T-s$ arrives at $T-s+\tau_1$. Then $\tau_1 \sim \texttt{Exp}(\lambda)$.

Suppose the second customer after $T-s$ arrives at $T-s+\tau_1+\tau_2$. Then $\tau_2 \sim \texttt{Exp}(\lambda)$.

Therefore, we have

$$\begin{aligned}
\mathbf{Pr}\left[\text{Joe achieves his goal}\right] &= \mathbf{Pr}\left[0 \le \tau_1 \le s \wedge \tau_1+\tau_2 > s\right] \\
&= \int_0^s \lambda e^{-\lambda t} \cdot \mathbf{Pr}\left[\tau_2 > s-t\right] \mathrm{d}t \\
&= \lambda \int_0^s e^{-\lambda t} e^{-\lambda(s-t)} \mathrm{d}t = \lambda \int_0^s e^{-\lambda s} \mathrm{d}t \\
&= \lambda s e^{-\lambda s}
\end{aligned}$$

Thus, the probability that Joe achieves his goal is $\underline{\boldsymbol{\lambda s e^{-\lambda s}}}$. ■

1.2 Optimal Value of s

Solution. Let $f(s) = \lambda s e^{-\lambda s}$.

$$f'(s) = \lambda\left(1-\lambda s\right) e^{-\lambda s} = 0 \implies s^* = \frac{1}{\lambda},\ f(s^*) = e^{-1}.$$

Thus, the optimal value of s is $\underline{\boldsymbol{\lambda^{-1}}}$ and the corresponding probability is $\underline{\boldsymbol{e^{-1}}}$. ■

2 Poisson Process

2.1 $\mathbf{Pr}\left[X \geq \lambda\right] \geq \frac{1}{2}$

Proof. Since $X \sim \texttt{Poisson}(\lambda)$, for $k = 0, 1, 2, ...\lambda - 1$,

$$\begin{aligned}
\mathbf{Pr}\left[X = \lambda + k\right] &= \frac{\lambda^{\lambda+k}}{(\lambda+k)!}e^{-\lambda} = \frac{\lambda^{2k+1}}{\prod_{i=-k}^{k}(\lambda+i)} \frac{\lambda^{\lambda-k-1}}{(\lambda-k-1)!} \\
&= \frac{\lambda^2}{(\lambda-k)(\lambda+k)} \cdot \frac{\lambda^2}{(\lambda-k+1)(\lambda+k-1)} \cdot ... \frac{\lambda^2}{(\lambda-1)(\lambda+1)} \cdot \frac{\lambda^{\lambda-k-1} \cdot e^{-\lambda}}{(\lambda-k-1)!} \\
&= \frac{\lambda^2}{\lambda^2 - k^2} \cdot \frac{\lambda^2}{\lambda^2 - (k-1)^2} \cdot ... \frac{\lambda^2}{\lambda^2 - 1} \cdot \mathbf{Pr}\left[X = \lambda - k - 1\right] \\
&\geq \mathbf{Pr}\left[X = \lambda - k - 1\right].
\end{aligned}$$

■

Then we have

$$\begin{aligned}
2\mathbf{Pr}\left[X \geq \lambda\right] &= \mathbf{Pr}\left[X \geq 2\lambda\right] + \sum_{k=0}^{\lambda-1} \mathbf{Pr}\left[X = \lambda + k\right] + \mathbf{Pr}\left[X \geq \lambda\right] \\
&\geq \mathbf{Pr}\left[X \geq 2\lambda\right] + \sum_{k=0}^{\lambda} \mathbf{Pr}\left[X = k\right] + \mathbf{Pr}\left[X \geq \lambda\right] = \mathbf{Pr}\left[X \geq 2\lambda\right] + 1 \geq 1.
\end{aligned}$$

$$\iff \quad \mathbf{Pr}\left[X \geq \lambda\right] \geq \frac{1}{2}.$$

■

2.2 $\mathbb{E}\left[f(X_1, X_2, ...X_n)\right] \leq 2 \cdot \mathbb{E}\left[f(Y_1, Y_2, ...Y_n)\right]$

Proof. Since $Y_i \sim \texttt{Poisson}(\frac{m}{n})$, we know $\sum_{i=1}^{n} Y_i \sim \texttt{Poisson}(m)$.

$$\begin{aligned}
\mathbb{E}\left[f(Y_1, Y_2, ...Y_n)\right] &= \sum_{k=0}^{\infty} \mathbb{E}\left[f(Y_1, Y_2, ...Y_n) \middle| \sum_{i=1}^{n} Y_i = k\right] \mathbf{Pr}\left[\sum_{i=1}^{n} Y_i = k\right] \\
&\geq \sum_{k=m}^{\infty} \mathbb{E}\left[f(Y_1, Y_2, ...Y_n) \middle| \sum_{i=1}^{n} Y_i = k\right] \mathbf{Pr}\left[\sum_{i=1}^{n} Y_i = k\right] \\
&= \sum_{k=m}^{\infty} \mathbb{E}\left[f(X_1, X_2, ...X_n) \middle| \mathbb{E}\left[X_i\right] = \frac{k}{n}\right] \mathbf{Pr}\left[\sum_{i=1}^{n} Y_i = k\right] \\
&\geq \sum_{k=m}^{\infty} \mathbb{E}\left[f(X_1, X_2, ...X_n) \middle| \mathbb{E}\left[X_i\right] = \frac{m}{n}\right] \mathbf{Pr}\left[\sum_{i=1}^{n} Y_i = k\right] \\
&\qquad \text{(Since } \mathbb{E}\left[f(X_1, X_2, ...X_n)\right] \text{ is monotonously increasing in } m\text{)} \\
&= \mathbb{E}\left[f(X_1, X_2, ...X_n)\right] \mathbf{Pr}\left[\sum_{i=1}^{n} Y_i \geq m\right] \\
&\geq \frac{1}{2}\mathbb{E}\left[f(X_1, X_2, ...X_n)\right] \qquad \text{(By } \mathbf{2.1}\text{)}
\end{aligned}$$

Thus, $\mathbb{E}\left[f(X_1, X_2, ...X_n)\right] \leq 2 \cdot \mathbb{E}\left[f(Y_1, Y_2, ...Y_n)\right]$. ■

2.3 Poisson Approximation of Birthday Problem

Proof. $n = 365, m = 50$.

Let X_i be the number of students whose birthday is the i-th day of a year ($i = 1, 2, ..., 365$).

Then we know

$$\sum_{i=1}^{365} X_i = m = 50,\ X_i \sim \texttt{Binom}\left(50, \frac{1}{365}\right).$$

Let $f(X_1, X_2, ...X_n) = \mathbb{1}[\exists i \text{ s.t. } X_i \geq 4]$. Then $\mathbb{E}[f(X_1, X_2, ...X_n)] = \mathbf{Pr}[\exists i \text{ s.t. } X_i \geq 4]$ is the probability of the event *"there exists four students who share the same birthday"*.

Poisson Approximation.

Construct i.i.d. $Y_i \sim \texttt{Poisson}\left(\frac{50}{365}\right)$ $(i = 1, 2, ...365)$ conditioned on $\sum_{i=1}^{365} Y_i = 50$.

Let $\lambda \triangleq \frac{50}{365}$. Then we have

$$\mathbf{Pr}[\exists i \text{ s.t. } Y_i \geq 4] = 1 - \mathbf{Pr}[\forall i, Y_i \leq 3] = 1 - \left[\left(\frac{1}{0!} + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!}\right) e^{-\lambda}\right]^{365}.$$

Meanwhile, it is trivial that $\mathbb{E}[f(X_1, X_2, ...X_{365})] = \mathbf{Pr}[\exists i \text{ s.t. } X_i \geq 4]$ is monotonously increasing in m. (The more students there are, the more likely that $\mathbb{E}[X_i]$ are larger, which leads to higher probability that exists four students who share the same birthday).

By **2.2**, we know

$$\begin{aligned}
\mathbf{Pr}[\exists i \text{ s.t. } X_i \geq 4] &= \mathbb{E}[f(X_1, X_2, ...X_n)] \\
&\leq 2 \cdot \mathbb{E}[f(Y_1, Y_2, ...Y_n)] = 2 \cdot \mathbf{Pr}[\exists i \text{ s.t. } Y_i \geq 4] \\
&= 2 - 2\left[\frac{6 + 6\lambda + 3\lambda^2 + \lambda^3}{6} e^{-\lambda}\right]^{365} \\
&\approx 0.9578\% < 1\%
\end{aligned}$$

Thus, the probability that *exists four students who share the same birthday* is at most 1%. ∎