Discrete Mathematics Exercise 11

Qiu Yihang, 2020/10/29

1. a) ***Solution:***

Let $A = \{x \in \mathbb{N} \mid x > 10\}$. We can construct a bijection $f: A \to \mathbb{N}$ where $f(x) = x - 11$.
Thus, A is a countably infinite set.

b) ***Solution:***

Let $B = \{x \in \mathbb{Z}^- \mid x \text{ is an odd negative integer}\}$.
We can construct a bijection $f: B \to \mathbb{N}$ where $f(x) = \lfloor (x+1)/2 \rfloor$.
Thus, B is a countably infinite set.

d) ***Solution:***

$\{x \in \mathbb{R} \mid 0 < x < 2\}$ is uncountable.

We can construct a bijection $f(x) = \begin{cases} 1/x(x-2) + 1 & 0 < x < 1 \\ 1/x(2-x) - 1 & 1 \le x < 2 \end{cases}$ from $(0,2)$ into $\mathbb{R}$.

Thus, $(0,2) \approx \mathbb{R}$.
Thus, $(0,2)$ is uncountable.

e) ***Solution:***

We can construct a bijection $f: A \times \mathbb{Z}^+ \to \mathbb{N}$ where $f(x,y) = \begin{cases} 2y-2 & x = 2 \\ 2y-1 & x = 3 \end{cases}$

Thus, $A \times \mathbb{Z}^+$ is countably infinite.

2. ***Proof:***

We can construct an injection f from $[0,1) \times [0,1)$ into $[0,1)$ **s.t.** when $x = 0.\overline{a_1 a_2 a_3 \dots a_n \dots\dots}$ and $y = 0.\overline{b_1 b_2 b_3 \dots b_n \dots\dots}$, $f(x,y) = 0.\overline{a_1 b_1 a_2 b_2 a_3 b_3 \dots a_n b_n \dots\dots}$.

(Specially, considering $0.\overline{a_1 a_2 a_3 \dots a_n 99999999 \dots\dots} = 0.\overline{a_1 a_2 a_3 \dots (a_n + 1) 00000000 \dots\dots}$, we made it a rule that we adopt the former way and abandon the latter one.)

We can also construct an injection $g(x) = (0, x)$ from $[0,1)$ into $[0,1) \times [0,1)$.

By Berstein's Theorem, we know there exists a bijection from $[0,1) \times [0,1)$ into $[0,1)$.

Thus, $[0,1) \times [0,1) \approx [0,1)$.

QED

3. ***Proof:***

$H(f)(b)(a) = f(a)(b)$ defines a function $H: (A \to (B \to C)) \to (B \to (A \to C))$.

First, we prove H is an injection.

For any $f_1, f_2 \in (A \to (B \to C))$,

$H(f_1) = H(f_2)$ **iff.** $\forall b \in B\, (H(f_1)(b) = H(f_2)(b))$
iff. $\forall b \in B\, \forall a \in A\, (H(f_1)(b)(a) = H(f_2)(b)(a))$
iff. $\forall b \in B\, \forall a \in A\, (f_1(a)(b) = f_2(a)(b))$ **iff.** $f_1 = f_2$.

Then we prove H is a surjection.

For any $h \in \big(B \to (A \to C)\big)$,

exists an $f \in \big(A \to (B \to C)\big)$ **s.t.** $\forall a \in A\ \forall b \in B\ \big(f(b)(a) = f(a)(b)\big)$.

Therefore, H is a bijection.

Thus, $\big(A \to (B \to C)\big) \approx \big(B \to (A \to C)\big)$.

QED

4. ***Proof:***

$H(f, g)(a) = \big(f(a), g(a)\big)$ defines a function $H: (A \to B) \times (A \to C) \to (A \to B \times C)$.

First, we prove H is an injection.

For any $(f_1, g_1), (f_2, g_2) \in (A \to B) \times (A \to C)$,

$H(f_1, g_1) = H(f_2, g_2)$ **iff.** $\forall a \in A\ \big(H(f_1, g_1)(a) = H(f_2, g_2)(a)\big)$

iff. $\forall a \in A\ \Big(\big(f_1(a), g_1(a)\big) = \big(f_2(a), g_2(a)\big)\Big)$

iff. $\forall a \in A\ \big(f_1(a) = f_2(a)\big) \wedge \big(g_1(a) = g_2(a)\big)$

iff. $(f_1 = f_2) \wedge (g_1 = g_2)$ **iff.** $(f_1, g_1) = (f_2, g_2)$.

Now we prove H is a surjection.

For any $h \in (A \to B \times C)$, exists a $(f, g) \in (A \to B) \times (A \to C)$ **s.t.** for any $a \in A$, $h(a) = (x, y)$, $f(a) = x, g(a) = y$.

Therefore, H is a bijection.

Thus, $(A \to B \times C) \approx (A \to B) \times (A \to C)$.

QED

5. ***Proof:***

Let A be the set of all functions from $\mathbb{R}$ into $\mathbb{R}$, **i.e.** $A = (\mathbb{R} \to \mathbb{R}) = \mathbb{R}^{\mathbb{R}}$.

Let B be the set of all binary relations on $\mathbb{R}$, **i.e.** $B = \mathcal{P}(\mathbb{R} \times \mathbb{R}) \approx 2^{\mathbb{R} \times \mathbb{R}}$.

Lemma. $\mathbb{N} \times \mathbb{R} \approx \mathbb{R} \times \mathbb{R}$.

Proof. We can construct an injection $f(x, y) = (x, y)$ from $\mathbb{N} \times \mathbb{R}$ into $\mathbb{R} \times \mathbb{R}$.

We can construct an injection g from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{N} \times \mathbb{R}$ **s.t.**

when $x = \overline{\dots a_k \dots a_3 a_2 a_1 . b_1 b_2 b_3 \dots b_n \dots \dots} \in \mathbb{R}$ and $y \in \mathbb{R}$,

let $a = \overline{a_1 b_1 a_2 b_2 a_3 b_3 \dots \dots a_n b_n \dots} \in \mathbb{N}$, $g(x, y) = (a, y)$.

(For those undefined a_n, let them be 0.)

By Berstein's Theorem, there exists a bijection from $\mathbb{N} \times \mathbb{R}$ into $\mathbb{R} \times \mathbb{R}$.

Thus, $\mathbb{N} \times \mathbb{R} \approx \mathbb{R} \times \mathbb{R}$.

Qed.

Thus, $\mathbb{R}^{\mathbb{R}} \approx (2^{\mathbb{N}})^{\mathbb{R}} \approx 2^{\mathbb{N} \times \mathbb{R}} \approx 2^{\mathbb{R} \times \mathbb{R}}$.

In other words, $A \approx B$.

QED