Probability Theory and Mathematical Statistics **概率统计**

Homework 1206-1209

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7-2. 解: $E(X) = \int_{-\infty}^{\infty} f(x;\theta)\, x\, dx = \int_1^{\theta} \frac{2\theta^2}{(\theta^2-1)x^2}\, dx = +\frac{2\theta^2}{\theta^2-1}\left(1-\frac{1}{\theta}\right)$

$= \frac{2\theta(\theta-1)}{\theta^2-1} = \frac{2\theta}{\theta+1}$

矩法估计. 则 $\frac{2\hat{\theta}}{\hat{\theta}+1} = \bar{X} \Rightarrow \hat{\theta} = \frac{\bar{X}}{2-\bar{X}}$. 为矩估计量 □

$\hat{\theta} = \frac{\bar{x}}{2-\bar{x}}$ 为矩估计值 □

7-3. 解: $E(X) = \lambda$.

矩法估计 $E(X) \approx \bar{X} = \frac{1\cdot 20+2\cdot 10+3\cdot 2+4\cdot 1}{17+20+10+2+1} = 1$ 即 $\hat{\lambda}_1 = 1$

(矩估计值) □

最大似然函数 $L = (P(X=0))^{17}(P(X=1))^{20}(P(X=2))^{10}(P(X=3))^{2}(P(X=4))^{1}$

$\because \ln\frac{\lambda^k e^{-\lambda}}{k!} = k\ln\lambda - \lambda - \ln k!$

$\therefore \ln L = 17(-\lambda) + 20\ln\lambda - 20\lambda + 20\ln\lambda - 10\lambda - \ln 2 + 6\ln\lambda - 2\lambda - \ln 6$

$+ 4\ln\lambda - \lambda - \ln 24 = 50\ln\lambda - 50\lambda - \ln 2 - \ln 6 - \ln 24$

$\frac{\partial \ln L}{\partial\lambda} = 0 \Rightarrow \frac{50}{\lambda} - 50 = 1 \Rightarrow \hat{\lambda}_2 = 1$ 为最大似然估计值 □

7-4. 解: 矩法估计. $E(X) = 2\theta(1-\theta) + 2\theta^2 + 3(1-2\theta) = 3-4\theta \approx \bar{X}$.

样本 $\bar{x} = \frac{3+1+3+0+3+1+2+3}{8} = 2 \Rightarrow \hat{\theta}_1 = \frac{1}{4}$ 矩估计值 □

最大似然估计.

最大似然函数 $L(\theta) = (\theta^2)^1\cdot(2\theta(1-\theta))^2\cdot(\theta^2)^1\cdot(1-2\theta)^4$

$$\ln L = 2\ln\theta + 2\ln 2 + 2\ln\theta + 2\ln(1-\theta) + 2\ln\theta + 4\ln(1-2\theta) = 6\ln\theta + 2\ln(1-\theta) + 4\ln(1-2\theta) + 2\ln 2$$

$$\frac{\partial \ln L}{\partial\theta} = 0 \Rightarrow \frac{6}{\theta} - \frac{2}{1-\theta} - \frac{8}{1-2\theta} = 0$$

$$\Rightarrow 12\theta^2 - 14\theta + 3 = 0 \quad \Rightarrow \theta_1 = \frac{7+\sqrt{13}}{12} \qquad \theta_2 = \frac{7-\sqrt{13}}{12}$$

(代入知是L最小，舍)

$\therefore \hat{\theta}_2 = \frac{7-\sqrt{13}}{12}$ 最大似然估计值 □

7-6. 解：(3) $E(X) = \int_0^1 x \cdot \sqrt{\theta}\, x^{\sqrt{\theta}-1}\, dx = \int_0^1 \sqrt{\theta}\, x^{\sqrt{\theta}}\, dx = \frac{\sqrt{\theta}}{\sqrt{\theta}+1}$

$\therefore$ 矩估计量 $\hat{\theta}_1 = \left(\frac{\bar{X}}{1-\bar{X}}\right)^2 = \left(\frac{1}{1-\sum_{i=1}^n X_i} - 1\right)^2$ □

最大似然函数 $L(\theta) = \prod_{i=1}^n \sqrt{\theta}\, X_i^{\sqrt{\theta}-1}$

$\ln L = \sum_{i=1}^n \left(\frac{1}{2}\ln\theta + (\sqrt{\theta}-1)\ln X_i\right) \Rightarrow \frac{\partial \ln L}{\partial\theta} = \frac{n}{2\theta} + \left(\sum_{i=1}^n \ln X_i\right) \cdot \frac{1}{2}\frac{1}{\sqrt{\theta}} = 0$

$\Rightarrow$ 最大似然估计量 $\hat{\theta}_2 = \left(\frac{n}{\sum_{i=1}^n \ln X_i}\right)^2$ □

(5) 解：矩法估计 $E(X) = \int_\mu^{+\infty} \frac{x}{\theta} e^{-\frac{x-\mu}{\theta}}\, dx = \int_0^{+\infty} \frac{x+\mu}{\theta} e^{-\frac{x}{\theta}}\, dx$

$= \theta + \frac{\mu}{\theta}\int_0^{+\infty} e^{-\frac{x}{\theta}}\, dx = \theta + \mu \approx \bar{X}$

$E(X^2) = \int_\mu^{+\infty} \frac{x^2}{\theta} e^{-\frac{x-\mu}{\theta}}\, dx = \int_0^{+\infty} \frac{(x+\mu)^2}{\theta} e^{-\frac{x}{\theta}}\, dx = \int_0^{+\infty} \frac{x^2+2\mu x+\mu^2}{\theta} e^{-\frac{x}{\theta}}\, dx$

$= (\theta^2+\theta^2) + 2\mu\theta + \mu^2 = 2\theta^2 + 2\mu\theta + \mu^2 \approx D(X) + \bar{X}^2$

$\Rightarrow \theta^2 = (2\theta^2 + 2\theta\mu + \mu^2) - (\theta+\mu)^2 \approx D(X) = \sum_{i=1}^n (X_i - \bar{X})^2$

$\therefore \hat{\theta}_1 = \sqrt{\sum_{i=1}^n (X_i - \bar{X})^2}$ 为矩估计量 □

$\hat{\mu}_1 = \bar{X} - \sqrt{\sum_{i=1}^n (X_i - \bar{X})^2}$

最大似然估计. $L(\theta) = \begin{cases} 0 & (\mu > X_{(1)}) \\ \prod_{i=1}^n \frac{1}{\theta} e^{-\frac{X_i-\mu}{\theta}} & (\mu \le X_{(1)}) \end{cases}$

$\ln L(\theta) = \sum_{i=1}^n \left(-\ln\theta - \frac{X_i-\mu}{\theta}\right) \Rightarrow \frac{\partial \ln L}{\partial\theta} = -\frac{n}{\theta} + \left(\sum_{i=1}^n X_i\right)\frac{1}{\theta^2} - \frac{\mu}{\theta^2} = 0$ (*)

$\frac{\partial \ln L}{\partial\mu} = \frac{n}{\theta} > 0$ (μ越大 L越大)

实际上，$\mu > X_{(1)}$. $L(\theta)=0$. ∴ 当 $\hat{\mu}_2 = X_{(1)}$ 时 L 最大

代入(*)得 $\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i - X_{(1)}$

综上：$\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i - X_{(1)}$. $\hat{\mu}_2 = X_{(1)}$ 为最大似然估计量 □

补充1. 解：

$$f_X(x) = \begin{cases} \theta, & 0 \le x < 1 \\ 1-\theta, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

矩法估计. $E(X) = \int_0^1 \theta x\,dx + \int_1^2 (1-\theta)x\,dx = \frac{1}{2}\theta + \frac{3}{2}(1-\theta) = \frac{3}{2} - \theta \approx \bar{X}$

矩估计值：$\hat{\theta}_1 = \frac{3}{2} - \bar{X} = \frac{3}{2} - \frac{1}{5}(0.4+0.6+1.2+1.8+0.7) = 0.56$ □

最大似然估计. $L(\theta) = \theta\cdot\theta\cdot(1-\theta)\cdot(1-\theta)\cdot\theta = \theta^3(1-\theta)^2$

$\ln L(\theta) = 3\ln\theta + 2\ln(1-\theta)$ $\frac{\partial \ln L}{\partial\theta} = \frac{3}{\theta} - \frac{2}{1-\theta} = 0$

最大似然估计值 $\hat{\theta}_2 = \frac{3}{5}$ □

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7-9. 解：$E\left(c\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2\right) = E\left(cX_1^2 + 2c\sum_{i=2}^{n-1}X_i^2 + cX_n^2 - 2c\sum_{i=1}^{n-1}X_iX_{i+1}\right)$

$= cE(X_1^2) + 2c\sum_{i=2}^{n-1}E(X_i^2) + cE(X_n^2) - 2c\sum_{i=1}^{n-1}E(X_i)E(X_{i+1})$

(∵ X_i, X_{i+1} 相互独立)

$= 2(n-1)c\,E(X^2) - 2c(n-1)E^2(X)$

$= 2c(n-1)D(X) = 2c(n-1)\sigma^2$

要使 $c\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2$ 为 σ^2 的无偏估计量. 则 $2c(n-1)=1$ $\Rightarrow c = \frac{1}{2n-2}$ □

7-10. 解：记总体为 X. 则 $E(X) = D(X) = \lambda$.

$E\left(\sum_{i=1}^{n-1}X_iX_{i+1}\right) = \sum_{i=1}^{n-1}E(X_i)E(X_{i+1})$ (∵ X_i, X_{i+1} 相互独立)

$= (n-1)E^2(X) = (n-1)\lambda^2$

∴ λ^2 的无偏估计量是 $\frac{1}{n-1}\sum_{i=1}^{n-1}X_iX_{i+1}$ □

7-13. 解：(1) 矩法估计 $E(X)=\int_{\theta}^{+\infty}2xe^{-2(x-\theta)}dx=\int_{0}^{+\infty}2(x+\theta)e^{-2x}dx$

$=\frac{1}{2}+\theta\approx\bar{X}\quad\Rightarrow$ 矩估计量 $\hat{\theta}_1=\bar{X}-\frac{1}{2}$ □

最大似然估计 $L(\theta)=\begin{cases}\prod_{i=1}^{n}2e^{-2(X_i-\theta)}, & \theta\leq X_{(1)}\\ 0, & \theta>X_{(1)}\end{cases}$

$\ln L(\theta)=\sum_{i=1}^{n}(\ln 2-2(X_i-\theta))=n\ln 2-2\sum_{i=1}^{n}X_i+2n\theta\quad(\theta\leq X_{(1)})$

$\Rightarrow\frac{\partial}{\partial\theta}\ln L(\theta)=2n>0\quad\therefore\theta$越大，$L$越大.

而 $\theta>X_{(1)}$ 时 $L(\theta)=0\quad\therefore\theta_{max}=X_{(1)}$ 即最大似然估计量 $\hat{\theta}_2=X_{(1)}$ □

(2) $E(\hat{\theta}_1)=E(\bar{X})-\frac{1}{2}=E(X)-\frac{1}{2}=\frac{1}{2}+\theta-\frac{1}{2}$

$\therefore\hat{\theta}_1$是无偏估计量 □

$E(\hat{\theta}_2)=E(X_{(1)})$ 记 $Z=X_{(1)}$

$P(Z\leq z)=1-P(Z>z)=1-P(X_1>z,\cdots X_n>z)$

$=1-[P(X>z)]^n=1-\left(2\int_{z}^{+\infty}e^{-2(x-\theta)}dx\right)^n=1-e^{-2n(z-\theta)}$

$\Rightarrow f_Z(z)=\begin{cases}2ne^{-2n(z-\theta)} & x>\theta\\ 0 & x\leq\theta\end{cases}$

$E(Z)=\int_{\theta}^{+\infty}2nze^{-2n(z-\theta)}dz=\int_{0}^{+\infty}2n(z+\theta)e^{-2nz}dz$

$=\frac{1}{2n}+\theta\quad\therefore\hat{\theta}_2$ 不是无偏估计量

修正 $\hat{\theta}_2$ 为无偏估计：$\hat{\theta}_2'=\hat{\theta}_2-\frac{1}{2n}=X_{(1)}-\frac{1}{2n}$ □

(3) $D(\hat{\theta}_1)=D(\bar{X})=D\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)=\frac{n}{n^2}D(X)=\frac{1}{n}D(X)$

$=\frac{1}{n}D(X-\theta)=\frac{1}{n}D(\underbrace{E(2)}_{\text{指数分布}})=\frac{1}{4n}.$

$D(\hat{\theta}_2')=D(Z)=D(Z-\theta)=D(\underbrace{E(2n)}_{\text{指数分布}})=\frac{1}{4n^2}$

$\because n\geq1,\quad\therefore D(\hat{\theta}_1)\geq D(\hat{\theta}_2')\quad\therefore\hat{\theta}_2'$ 比 $\hat{\theta}_1$ 更有效 □

7-14. 解：$S_1^2=\frac{\sigma^2}{n}\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$ 其中：$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2\sim\chi^2(n)$

$E(S_1^2)=\frac{\sigma^2}{n}\cdot n=\sigma^2$ $D(S_1^2)=\left(\frac{\sigma^2}{n}\right)^2\cdot 2n=\frac{2\sigma^4}{n}$

$S_2^2=\frac{\sigma^2}{n-1}\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2$ 其中：$\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2\sim\chi^2(n-1)$

$E(S_2^2)=\frac{\sigma^2}{n-1}\cdot(n-1)=\sigma^2$ $D(S_2^2)=\left(\frac{\sigma^2}{n-1}\right)^2\cdot 2(n-1)=\frac{2\sigma^4}{n-1}$

$\therefore$ S_1^2, S_2^2 是 σ^2 的无偏统计量 且 $D(S_1^2)<D(S_2^2)$ $\therefore$ S_1^2 更有效 □

7-16. (1) 证明：$E(\bar{X})=E(X)=\mu$. $D(\bar{X})=\frac{1}{n}D(X)=\frac{\sigma^2}{n}$ $\therefore$ $\bar{X}$ 是 μ 的无偏估计量

由 Rao-Cramer 不等式知，对 μ 的任意无偏估计量 $\hat{\mu}$，有：

$$D(\hat{\mu})\geqslant I(\mu)=\frac{1}{nE\left[\left(\frac{\partial \ln f(X;\mu)}{\partial \mu}\right)^2\right]}$$

其中 $E\left(\left(\frac{\partial \ln f(X;\mu)}{\partial\mu}\right)^2\right)=E\left(\left(-\frac{2\mu}{2\sigma^2}+\frac{2X}{2\sigma^2}\right)^2\right)=E\left(\frac{(X-\mu)^2}{\sigma^4}\right)=\frac{1}{\sigma^2}E\left(\left(\frac{X-\mu}{\sigma}\right)^2\right)$

$=\frac{1}{\sigma^2}E(\chi^2(1))=\frac{1}{\sigma^2}$

$\therefore$ $D(\hat{\mu})\geqslant\frac{\sigma^2}{n}=D(\bar{X})\Rightarrow$ $\bar{X}$ 是 μ 的有效估计量 □

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(2) 证明：$E(S^2)=\sigma^2$. 即 S^2 是 σ^2 的无偏估计量

$$D(S^2)=\frac{(\sigma^2)^2}{(n-1)^2}D\left(\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2\right)=\frac{\sigma^4}{(n-1)^2}D(\chi^2(n-1))=\frac{2\sigma^4}{n-1}$$

由 Rao-Cramer 不等式，对 σ^2 的任意无偏估计量 $\hat{\sigma}^2$，有：

$$D(\hat{\sigma}^2)\geqslant I(\sigma^2)=\frac{1}{nE\left[\left(\frac{\partial \ln f(X;\sigma^2)}{\partial\sigma^2}\right)^2\right]}$$

$$\ln f(X;\sigma^2)=-\frac{1}{2}\ln 2\pi-\frac{1}{2}\ln\sigma^2-\frac{(X_i-\mu)^2}{2\sigma^2}$$

$$\frac{\partial \ln f(X;\sigma^2)}{\partial\sigma^2}=-\frac{1}{2\sigma^2}+\Big(\quad (X_i-\mu)^2\Big)\cdot\frac{1}{2\sigma^4}$$

$$E\left[\left(\frac{\partial \ln f(X;\sigma^2)}{\partial \sigma^2}\right)^2\right] = \frac{1}{4\sigma^4} - \frac{1}{2\sigma^6}E\left((X_i-\mu)^2\right) + \frac{1}{4\sigma^8}E\left((X_i-\mu)^4\right)$$

$$= \frac{1}{4\sigma^4} - \frac{1}{2\sigma^4}E\left(\left(\frac{X_i-\mu}{\sigma}\right)^2\right) + \frac{1}{4\sigma^4}\left(\left(\frac{X_i-\mu}{\sigma}\right)^4\right)$$

$$= \frac{1}{4\sigma^4} - \frac{1}{2\sigma^4}E\left(\chi^2(1)\right) + \frac{1}{4\sigma^4}\left(E^2\left(\chi^2(1)\right) + D\left(\chi^2(1)\right)\right)$$

$$= \frac{1}{4\sigma^4} - \frac{1}{2\sigma^4} + \frac{3}{4\sigma^4} = \frac{1}{2\sigma^4}.$$

$\therefore I(\sigma^2) = \frac{2\sigma^4}{n} < D(s^2)$

$\therefore$ S^2 不是 σ^2 的有效估计量. □

补充题1.

解 (1) $E(X) = \frac{\theta}{2} \approx \bar{X}$. 矩估计量 $\hat{\theta}_1 = 2\bar{X}$. □

最大似然函数 $L(\theta) = \begin{cases} 0 & (\theta < X_{(n)}) \\ \frac{1}{\theta^n} & (\theta \geq X_{(n)}) \end{cases}$

$\frac{\partial L(\theta)}{\partial \theta} = -n\theta^{-n+1} < 0$. $\therefore$ θ 越小. L 越大.

而 $\theta < X_{(n)}$ 时 $L(\theta) = 0$ $\therefore$ $\theta_{min} = X_{(n)}$ 即 最大似然估计值 $\hat{\theta}_2 = X_{(n)}$ □

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(2) $E(\hat{\theta}_1) = E(2\bar{X}) = 2E(\bar{X}) = 2E(X) = 2\cdot\frac{\theta}{2} = \theta$, $\hat{\theta}_1$是无偏估计量. □

记 $Z = X_{(n)}$. 则 $F_Z(z) = P(Z \leq z) = P(X_1 \leq z, \cdots X_n \leq z)$

$= [P(X \leq z)]^n = \begin{cases} \left(\frac{z}{\theta}\right)^n. & 0 \leq z \leq \theta. \\ 0. & \text{otherwise} \end{cases}$

$\Rightarrow f_Z(z) = \begin{cases} n\cdot\frac{z^{n-1}}{\theta^n} & 0 \leq z \leq \theta \\ 0 & \text{otherwise} \end{cases}$

$E(\hat{\theta}_2) = E(Z) = \int_0^\theta nz\cdot\frac{z^{n-1}}{\theta^n}\,dz = \int_0^\theta \frac{nz^n}{\theta^n}\,dz = \frac{n}{n+1}\theta$

$\therefore$ $\hat{\theta}_2$ 不是无偏估计量. 无偏修正 $\hat{\theta}_2' = \frac{n+1}{n}\hat{\theta}_2 = \frac{n+1}{n}X_{(n)}$ □

补充题2.

$X_i-\bar{X}$ 服从正态分布.

解: (1) 记 $Z=|X_i-\bar{X}|$. $E(X_i-\bar{X})=0$. $D(X_i-\bar{X})=D(X_i)+D(X_i)+2\operatorname{cov}(X_i,\bar{X})$

$=\sigma^2+\sigma^2+2\cdot\frac{1}{n^2}\operatorname{cov}(X_i,X_i)=\sigma^2+\sigma^2+\frac{2}{n^2}\sigma^2=2\sigma^2+\frac{2}{n^2}\sigma^2$, $\therefore X_i-\bar{X}\sim N(0,2(1+\frac{1}{n^2})\sigma^2)$

$z\geq 0$时: $F_Z(z)=P(Z\leq z)=P(-z\leq X_i-\bar{X}\leq z)=\int_{-z}^{z}f_{X_i-\bar{X}}(x)\,dx$

$$=\int_{-z}^{z}\frac{1}{\sqrt{2\pi}\sqrt{2+\frac{2}{n^2}}\sigma}e^{-\frac{x^2}{4\sigma^2+\frac{4}{n^2}\sigma^2}}dx$$

$$f_Z(z)=\frac{d}{dz}F_Z(z)=f_{X_i-\bar{X}}(z)+f_{X_i-\bar{X}}(-z)=2\cdot\frac{1}{\sqrt{2\pi}\sqrt{2+\frac{2}{n^2}}\sigma}e^{-\frac{z^2}{4\sigma^2+\frac{4}{n^2}\sigma^2}}$$

$$=\frac{1}{\sqrt{\pi(1+\frac{1}{n^2})}\sigma}e^{-\frac{z^2}{4(1+\frac{1}{n^2})\sigma^2}}\quad(z\geq 0)$$

$z<0$时. $F_Z(z)=0$. $f_Z(z)=0$

$$\text{综上,}\ f_Z(z)=\begin{cases}\frac{1}{\sqrt{\pi(1+\frac{1}{n^2})}\sigma}e^{-\frac{z^2}{4(1+\frac{1}{n^2})\sigma^2}} & (z\geq 0)\\ 0 & (\text{otherwise})\end{cases}$$ □

(2) $$E(Z)=\int_0^{+\infty}zf_Z(z)\,dz=\frac{1}{2}\int_0^{+\infty}\frac{1}{\sqrt{\pi(1+\frac{1}{n^2})}\sigma}e^{-\frac{z^2}{4(1+\frac{1}{n^2})\sigma^2}}d(z^2)$$

$$=\frac{1}{2\sqrt{\pi(1+\frac{1}{n^2})}\sigma}\cdot 4(1+\frac{1}{n^2})\sigma^2=2\sqrt{\frac{n^2+1}{\pi n^2}}\sigma$$

要使 $k\sum_{i=1}^{n}|X_i-\bar{X}|$ 为无偏估计量. 则 $E\left(k\sum_{i=1}^{n}|X_i-\bar{X}|\right)=2k\sqrt{\frac{n^2+1}{\pi n^2}}\sigma=\sigma$

$\therefore k=\frac{1}{2}\sqrt{\frac{\pi n}{n^2+1}}$ □