[Homework 4] Martingale (Due: May 8, 2022)

Problem 1 (Doob's martingale inequality)

Let $\{X_t\}_{t\geq 0}$ be a martingale with respect to itself where $X_t \geq 0$ for every t. Prove that for every $n \in \mathbb{N}$,

$$\mathbf{Pr}\left[\max_{0\leq t\leq n} X_t \geq \alpha\right] \leq \frac{\mathbf{E}\left[X_0\right]}{\alpha}.$$

▼ Hint

Consider the stopping time $\tau = \arg\min_{t\leq n}\{X_t \geq \alpha\}$ or $\tau = n$ if $X_t < \alpha$ for all $0 \leq t \leq n$.

Problem 2 (Biased one-dimensional random walk)

We study the biased random walk in this exercise. Let $X_t = \sum_{i=1}^{t} Z_i$ where each $Z_i \in \{-1, 1\}$ is independent, and satisfies $\mathbf{Pr}\left[Z_i = -1\right] = p \in (0,1)$.

- Define $S_t = \sum_{i=1}^{t}(Z_i + 2p - 1)$. Show that $\{S_t\}_{t\geq 0}$ is a martingale.
- Define $P_t = \left(\frac{p}{1-p}\right)^{X_t}$. Show that $\{P_t\}_{t\geq 0}$ is a martingale.
- Suppose the walk stops either when $X_t = -a$ or $X_t = b$ for some $a, b > 0$. Let τ be the stopping time. Compute $\mathbf{E}\left[\tau\right]$.

Problem 3 (Longest common subsequence)

A *subsequence* of a string s is any string that can be obtained from s by removing a few characters (not necessarily continuous). Consider two uniformly random strings $x, y \in \{0,1\}^n$. Let X denote the length of their *longest common subsequence*.

- Show that there exist two constants $\frac{1}{2} < c_1 < c_2 < 1$ such that $c_1 n < \mathbf{E}\left[X\right] < c_2 n$ for sufficiently n.
- Prove that X is well-concentrated around $\mathbf{E}\left[X\right]$ using tools developed in the class.

▼ Hint

To find c_2, you can try to estimate the following probabilty: there exist $S, T \subset [n]$ such that (1) $|S| = |T|$ and both two sets are *large*; and (2) $x_S = y_T$ where x_S and y_T are the restrictions of x and y on S and T respectively.

> 【选做题，联动AI2615】设计一个$O(n^2)$的动态规划算法计算x与y的最长公共子序列