Discrete Mathematics Exercise 16

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Lemma. In the rooted tree G with the root r,

if u is an ancestor of v in G and let the unique simple path from r to v be: $r, e_1, x_1, \dots, e_k, x_k = u, e_{k+1}, x_{k+1}, \dots e_n, x_n = v$, then $r, e_1, x_1, \dots, e_k, x_k = u$ is the unique simple path from r to u.

Proof: We know from the definition that $r, e_1, x_1, \dots, e_k, x_k = u$ is a simple path from r to u.

Since G is a tree, we know the path is the unique simple path from r to u.

Qed.

1. ***Proof:***

Suppose the root of the rooted tree G is r. Let $\mathcal{L}(u) = \text{the level of the vertex } u \text{ in } G$.

For any $(u, v) \in \mathcal{R}_1$, u is v's ancestor in G, **i.e.** the unique simple path from r to v includes u and $u \neq v$. Let the path be: $r, e_1, x_1, \dots, e_k, x_k = u, e_{k+1}, x_{k+1}, \dots e_n, x_n = v\ (k+1 \leq n)$.

Therefore, $r, e_1, x_1, \dots, e_k, x_k = u$ is the unique simple path from r to u.

Meanwhile, $\mathcal{L}(v) = n$, $\mathcal{L}(u) = k$. Thus, $\mathcal{L}(u) + 1 \leq \mathcal{L}(v)$, **i.e.** $\mathcal{L}(u) < \mathcal{L}(v)$.

In other words, $u'\text{s level}$ is strictly smaller than $v'\text{s level}$ in G.

Thus, $(u, v) \in \mathcal{R}_2$.

Therefore, $\mathcal{R}_1 \subseteq \mathcal{R}_2$. **QED**

2. ***Proof:***

Suppose the root of the rooted tree G is r.

Since u, v are ancestors of w, the unique simple path from r to w includes u, v.

Let the path be: $r, e_1, x_1, \dots e_n, x_n = w$

Suppose $x_j = u$, $x_k = v$.

Then $r, e_1, x_1, \dots e_j, x_j = u$ is the unique simple path from r to u.

Then $r, e_1, x_1, \dots e_k, x_k = v$ is the unique simple path from r to v.

CASE 1. $j = k$. Then $x_k = u = v$.

CASE 2. $j < k$. Then the unique simple path from r to v includes $x_j = u$. Since $u = x_j \neq x_k = v$ (otherwise the path is not a simple path), u is an ancestor of v.

CASE 3. $j > k$. Then the unique simple path from r to u includes $x_k = v$. Since $u = x_j \neq x_k = v$ (otherwise the path is not a simple path), v is an ancestor of u.

Therefore, either **(i)** u is v's ancestor, **(ii)** v is u's ancestor, or **(iii)** $u = v$.

QED

3. ***Proof:***

Suppose the root of the rooted tree G is r.

There exists a unique simple path from r to v: $r, e_1, x_1, \dots e_n, x_n = v$. Since v is a descendant of u, u is included in the path. Let $x_k = u$. Then $r, e_1, x_1, \dots e_i, x_i$ is the unique simple path from

r to x_i $(1 \leq i \leq n)$. (*)

Let $i = k$. Then $x_n = v, e_n, \dots x_{k+1}, e_{k+1}, x_k = u$ is a simple path and is the unique simple path from v to u (Otherwise, there exists a simple circuit. Contradiction).

For $k \leq i \leq n,$ from (*) we know $r, e_1, x_1, \dots e_i, x_i$ is the unique simple path from r to x_i, which includes $x_k = u$.

CASE 1. $x_i = x_k = u$.

CASE 2. $x_i \neq u$. Then u is x_i's ancestor.

Thus, the unique simple path from v to u only passes through u's descendants and u.

QED

4. ***Proof:***

Proof by induction.

Let $\mathcal{I}(G) = \text{the number of internal vertices in } G$. Let $L(G) = the\ number\ of\ leaves\ in\ G$.

BASE STEP 1. When G has no internal vertex,

since G is a rooted tree, there is at least one vertex in G, **i.e.** the root vertex, which is not an internal vertex and is thus a leaf.

Therefore, G has more leaves than internal vertices.

BASE STEP 2. When G has only one internal vertex,

the only one internal vertex is the root (otherwise it is the case in base step 1).

Since every internal vertex has more than two children, thus there are at least two leaves.

Therefore, G has more leaves than internal vertices.

INDUCTIVE STEP.

Suppose the statement holds when G has k internal vertices.

When G has $k+1$ vertices, we can always find an internal vertex whose children are all leaves (otherwise we can keep choosing an internal child of the chosen vertex instead). Let it be u. Suppose u has n children, $n \geq 2$.

Let G^* be G with all children of u removed. Then G^* is a tree with k internal vertices and all internal vertices of G^* have more than two children.

Thus, $\mathcal{I}(G^*) < L(G^*)$.

Meanwhile, $\mathcal{I}(G) = \mathcal{I}(G^*) + 1$ (the one added is u), $L(G) = L(G^*) - 1 + n$ (u is a leaf in G^* but is not a leaf in G).

Thus, $\mathcal{I}(G) = \mathcal{I}(G^*) + 1 = \mathcal{I}(G^*) + 2 - 1 < L(G^*) + n - 1 = L(G)$.

Therefore, G has more leaves than internal vertices.

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QED

Lemma ([R][43]) Let G be a connected graph. Show that if T is a spanning tree of T constructed using depth-first search, then an edge of G not in T must be a back edge, that is, it must connect a vertex to one of its ancestors or one of its descendants in T.

5. ***Proof:***

We can construct a spanning tree of $G = (V, E)$ using depth-first search process and name it $T = (V, E')$.

CASE 1. $k = 1$.

Then G only contains one vertex. Otherwise, G is not connected.

In this case, G has at most 0 edges, $0 = n(k-1)$, **i.e.** the maximum number can be reached.

CASE 2. $k \geq 2$.

Since G does not contain a simple path of length k, **i.e.** G does not contain any paths of length more than $(k-1)$, **i.e.** the height of T is at most $(k-1)$.

Therefore, for any $u \in V$, for any $e \in E$ which is incident with u, either $e \in E'$ or $e \in E \backslash E'$.

From **[R][43]** we know any $e \in E \backslash E'$, e connects u to one of u's ancestors or one of u's descendants in T. Since the height of T is at most $(k-1)$, there are at most $(k-2)$ vertices that are either u's ancestors or u's descendants. Therefore, there are at most $(k-2)$ edges in $E \backslash E'$ that are incident with u. (since G is a simple graph)

Thus, $E \backslash E'$ contains at most $n(k-2)$ edges. Meanwhile, E' contains at most $(n-1)$ edges.

Thus, E contains at most $n(k-2) + n - 1 = n(k-1) - 1$ edges.

Since $n(k-1) - 1 < n(k-1)$, G contains at most $n(k-1)$ edges and the maximum number cannot be reached.

In conclusion, G contains at most $n(k-1)$ edges, which can be reached only when $k = 1$.

QED