Probability Theory and Mathematical Statistics 概率统计

Homework 1122-1125

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5-8. 解：记每户每日用电量为 X_i. 则 $X_i \sim U[0,20]$ $E(X_i)=10$. $D(X_i)=\frac{20^2}{12}=\frac{100}{3}$

由中心极限定理，$\sum_{i=1}^{n} X_i \xrightarrow[n\to+\infty]{\text{依分布收敛}} N(10n, \frac{100}{3}n)$

(1) $\mu = nE(X_i)=75000$. 因n很大，近似认为 $\sum_{i=1}^{7500} X_i - 75000 \sim N(0, \frac{750000}{3})$

$P(\sum_{i=1}^{7500} X_i - 75000 \geq 1000) = 1-\Phi\left(\frac{1000}{\sqrt{750000/3}}\right) = 1-\Phi\left(\frac{1000}{\sqrt{2500}}\right) = 1-\Phi(2) \doteq 1-0.9772$

$= 0.0228$ □

(2) 设每天至少供应电量为 x kW·h.

则 $P(\sum_{i=1}^{7500} X_i \leq x) \geq 99.9\%$ 即 $\Phi\left(\frac{x-75000}{\sqrt{750000/3}}\right) \geq 0.999$.

查表知 $\frac{x-75000}{500} \geq 3.08$. $\Rightarrow x \geq 76540$ $\therefore x_{min}=76540$ □

5-9. (2) 解：记第i周销售量为 X_i. 记 $Y_n=\sum_{i=1}^{n} X_i$. 由中心极限定理 $Y_n \xrightarrow[n\to+\infty]{\text{依分布收敛}} N(n, 0.4n)$

(在9(1)中已求得 $E(X_i)=1$ $D(X_i)=0.4$)

则 $Y_{52} \overset{\text{近似}}{\sim} N(52, 20.8)$

$P(42 \leq Y_{52} \leq 62) = P(-10 \leq Y_{52}-52 \leq 10) = \Phi\left(\frac{10}{\sqrt{20.8}}\right) - \Phi\left(-\frac{10}{\sqrt{20.8}}\right) = 2\Phi\left(\frac{10}{\sqrt{20.8}}\right)-1$

$\doteq 2\Phi(2.193)-1 = 0.9714$ □

5-10. 证明：记第i次试验"成功的次数"为 X_i. 即 $X_i = \begin{cases} 1 & \text{成功} \\ 0 & \text{不成功} \end{cases}$

有 $E(X_i)=p$. $D(X_i)=p(1-p)$. $\xi_n=\sum_{i=1}^{n} X_i$

由 De Moivre-Laplace 中心极限定理知 $\xi_n \overset{\text{近似}}{\sim} N(np, np(1-p))$

$\therefore \frac{\xi_n}{n} \overset{\text{近似}}{\sim} N(p, \frac{p(1-p)}{n})$

$\therefore P\left(\left|\frac{\xi_n}{n}-p\right|<\varepsilon\right) = P\left(-\varepsilon < \frac{\xi_n}{n}-p < \varepsilon\right) \approx \Phi\left(\varepsilon\sqrt{\frac{n}{p(1-p)}}\right) - \Phi\left(-\varepsilon\sqrt{\frac{n}{p(1-p)}}\right)$

$= 2\Phi\left(\varepsilon\sqrt{\frac{n}{p(1-p)}}\right) - 1$. 得证 □

5-11. (3) 解：记第i位顾客消费额 X_i． $Y_n=\sum_{i=1}^{n}X_i$ 由11(2)知 $E(X_i)=30$． $D(X_i)=150$．

由中心极限定理． $Y_n \xrightarrow[n\to+\infty]{\text{以分布收敛}} N(30n,150n)$

$\therefore Y_{20}-6000 \overset{\text{近似}}{\sim} N(0,30000)$

$P(5800\le Y_{20}\le 6200)= P(-200\le Y_{20}-6000\le 200)\approx \Phi\left(\frac{200}{\sqrt{30000}}\right)-\Phi\left(-\frac{200}{\sqrt{30000}}\right)$

$=2\Phi\left(\frac{200}{\sqrt{30000}}\right)-1\approx 2\Phi(1.15)-1=0.7498$ □

6-1. 解：(1) 总体是 所有计算机专业本科生就业后的薪酬情况 □

(2) 样本是 被调查的所在地区近三年200名毕业生(计算机专业本科生)现在的月薪情况 □

(3) 样本容量是200. □

6-5. 解：认为是简单样本.

(1) $P(\max\{X_1,X_2,X_3\}<5)=P(\{X_1<5\}\cap\{X_2<5\}\cap\{X_3<5\})$

$P(X_i<5)=\Phi\left(\frac{5-2}{\sqrt{9}}\right)=\Phi(1.0)=0.8413$

$\therefore P(\max\{X_1,X_2,X_3\}<5)=\Phi^3(1.0)=0.8413^3=0.5955$ □

(2) $P(\{-2.5<X_1<3.5\}\cup\{2<X_3<6.5\})$

$=P(-2.5<X_1<3.5)+P(2<X_3<6.5)-P(-2.5<X_1<3.5)P(2<X_3<6.5)$

$=\Phi\left(\frac{3.5-2}{\sqrt{9}}\right)-\Phi\left(\frac{-2.5-2}{\sqrt{9}}\right)+\Phi\left(\frac{6.5-2}{\sqrt{9}}\right)-\Phi\left(\frac{2-2}{\sqrt{9}}\right)-\left(\Phi\left(\frac{3.5-2}{\sqrt{9}}\right)-\Phi\left(\frac{-2.5-2}{\sqrt{9}}\right)\right)\left(\Phi\left(\frac{6.5-2}{\sqrt{9}}\right)-\Phi\left(\frac{2-2}{\sqrt{9}}\right)\right)$

$=\Phi(0.5)-\Phi(-1.5)+\Phi(1.5)-\Phi(0)-(\Phi(0.5)-\Phi(-1.5))(\Phi(1.5)-\Phi(0))$

$=\Phi(0.5)+2\Phi(1.5)-\frac{3}{2}-(\Phi(0.5)+\Phi(1.5)-1)(\Phi(1.5)-\frac{1}{2})$

$=0.7873$ □

(3) $E(X_1^2X_2^2X_3^2)=E(X_1^2)E(X_2^2)E(X_3^2)=(E(X_1)^2+D(X_1))(E(X_2)^2+D(X_2))(E(X_3)^2+D(X_3))$

$=(2^2+9)^3=2197$ □

(4). $E(X_1X_2X_3)=E(X_1)E(X_2)E(X_3)=2^3=8$

$D(X_1X_2X_3)=E(X_1^2X_2^2X_3^2)-E(X_1X_2X_3)^2=2133$ □

$D(2X_1-3X_2-X_3)=4D(X_1)+9D(X_2)+D(X_3)=14D(X)=14\times 9=126$ □

补充题. 解: $X_{(1)}=\min\{X_1,\cdots X_n\}$. $P(X_{(1)}=2)=P(\{X_1=2\}\cap\cdots\cap\{X_n=2\})=P(X=2)^n=\frac{1}{2^n}$.

$P(X_{(1)}=0)=P(\{X_1=0\}\cup\cdots\cup\{X_n=0\})=1-P(\{X_1\neq 0\}\cap\cdots\cap\{X_n\neq 0\})$

$=1-(1-P(X=0))^n=1-\frac{3^n}{4^n}$.

$P(X_{(1)}=1)=1-P(X_{(1)}=0)-P(X_{(1)}=2)=\frac{3^n}{4^n}-\frac{1}{2^n}$.

$X_{(n)}=\max\{X_1,\cdots X_n\}$. $P(X_{(n)}=0)=P(\{X_1=0\}\cap\cdots\cap\{X_n=0\})=P(X=0)^n=\frac{1}{4^n}$.

$P(X_{(n)}=2)=1-P(X_{(n)}\neq 2)=1-P(\{X_1\neq 2\}\cap\cdots\cap\{X_n\neq 2\})=1-(1-P(X=2))^n=1-\frac{1}{2^n}$.

$P(X_{(n)}=1)=1-P(X_{(n)}=0)-P(X_{(n)}=2)=\frac{1}{2^n}-\frac{1}{4^n}$.

综上, $X_{(1)}$ 的分布列如下.

$X_{(1)}$	0	1	2
P	$1-\frac{3^n}{4^n}$	$\frac{3^n}{4^n}-\frac{1}{2^n}$	$\frac{1}{2^n}$

$X_{(n)}$ 的分布列如下.

$X_{(n)}$	0	1	2
P	$\frac{1}{4^n}$	$\frac{1}{2^n}-\frac{1}{4^n}$	$1-\frac{1}{2^n}$

□

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6-2. 解: (1), (2), (3), (7) 是统计量. □

6-6. 解: $E(\bar{X})=E(X)=\mu$. $D(\bar{X})=\frac{D(X)}{n}=\frac{36}{n}$. $\therefore \bar{X}\sim N(\mu,\frac{36}{n})$

$P(|\bar{X}-\mu|<1)=2\Phi\left(\frac{1}{\sqrt{36/n}}\right)-1=2\Phi\left(\frac{\sqrt{n}}{6}\right)-1\geqslant 0.95$

$\Rightarrow \Phi\left(\frac{\sqrt{n}}{6}\right)\geqslant 0.975$ 查表知 $\frac{\sqrt{n}}{6}\geqslant 1.96$ $\Rightarrow \sqrt{n}\geqslant 11.76$ $\Rightarrow n\geqslant 11.76^2\approx 138.3$

$n\in\mathbb{N}^*$. $\therefore n_{min}=139$ □

6-7. 解: $E(\bar{X})=E(X)=\mu$. $D(\bar{X})=\frac{D(X)}{n}=\frac{\sigma^2}{n}$ $E(\bar{X}^2)=D(\bar{X})+E^2(\bar{X})=\frac{\sigma^2}{n}+\mu^2$

$D(X_i-\bar{X})=E((X_i-\bar{X})^2)=\frac{n-1}{n}E(S^2)=\frac{n-1}{n}\sigma^2=D(X_j-\bar{X})$

$i\neq j$ 时. $cov(X_i-\bar{X}, X_j-\bar{X})=cov(X_i,X_j)-cov(X_i,\bar{X})-cov(\bar{X},X_j)+cov(\bar{X},\bar{X})$

$$=0-cov(X_i,\frac{X_i}{n})-cov(X_j/n,X_j)+D(\bar{X})$$

$$=0-\frac{1}{n}D(X_i)-\frac{1}{n}D(X_j)+D(\bar{X})$$

$$=0-\frac{1}{n}\sigma^2-\frac{1}{n}\sigma^2+\frac{\sigma^2}{n}=-\frac{\sigma^2}{n}$$

$$\rho(X_i-\bar{X}, X_j-\bar{X})=\frac{cov(X_i-\bar{X}, X_j-\bar{X})}{\sqrt{D(X_i-\bar{X})\,D(X_j-\bar{X})}}=\frac{-\frac{\sigma^2}{n}}{\frac{n-1}{n}\sigma^2}=-\frac{1}{n-1}$$

$i=j$ 时显然 $\rho=1$. 综上. $\rho(X_i-\bar{X}, X_j-\bar{X})=\begin{cases}-\frac{1}{n-1}, & i\neq j\\ 1, & i=j\end{cases}$ □.

补充题. 解: 显然有 $X_i>\theta$ $(\forall i\in\{1,2,\cdots,n\})$

$P(X_i\geq y)=\int_y^{+\infty}\lambda e^{-\lambda(x-\theta)}dx=e^{-\lambda(y-\theta)}$

$P(Y_n>\theta+\varepsilon)=P(X_1\geq\theta+\varepsilon, X_2>\theta+\varepsilon,\cdots X_n>\theta+\varepsilon)$

$$=e^{-n\lambda\varepsilon}$$

当 $n\to\infty$ 时. $P(Y_n>\theta+\varepsilon)=e^{-n\lambda\varepsilon}\to 0$ 即 $\lim_{n\to+\infty}(P(|Y_n-\theta|>\varepsilon))=0$

$\therefore Y_n\xrightarrow[n\to+\infty]{P}\theta$. □