Assignment VI for AI2615 (Spring 2022)

May 28, 2022

Due: Saturday, June 11, 2022.

Choose *any three* of the following questions (excluding the last one). Each question carries $\frac{100}{3}$ points. (You are encouraged to solve as many the remaining questions as possible.)

Problem 1

Given an undirected graph $G = (V, E)$ with $n = |V|$, decide if G contains a clique with size exactly $n/2$. Prove that this problem is NP-complete.

Problem 2

Consider the decision version of *Knapsack*. Given a set of n items with weights $w_1, \ldots, w_n \in \mathbb{Z}^+$ and values $v_1, \ldots, v_n \in \mathbb{Z}^+$, a capacity constraint $C \in \mathbb{Z}^+$, and a positive integer $V \in \mathbb{Z}^+$, decide if there exists a subset of items with total weight at most C and total value at least V. Prove that this decision version of Knapsack is NP-complete.

Problem 3

Given two undirected graphs G and H, decide if H is a subgraph of G. Prove that this problem is NP-complete.

Problem 4

Given an undirected graph $G = (V, E)$ and an integer k, decide if G has a spanning tree with maximum degree at most k. Prove that this problem is NP-complete.

Problem 5

Given a ground set $U = \{1, \ldots, n\}$, a collection of its subsets $\mathcal{S} = \{S_1, \ldots, S_m\}$, and a positive integer k, the *set cover* problem asks if we can find a subcollection $\mathcal{T} \subseteq \mathcal{S}$ such that $\bigcup_{S \in \mathcal{T}} S = U$ and $|\mathcal{T}| = k$. Prove that set cover is NP-complete.

Problem 6

Given a collection of integers (can be negative), decide if there is a subcollection with sum exactly 0. Prove that this problem is NP-complete.

Problem 7

In an undirected graph $G = (V, E)$, each vertex can be colored either black or white. After an initial color configuration, a vertex will become black if all its neighbors are black, and the updates go on and on until no more update is possible. (Notice that once a vertex is black, it will be black forever.) Now, you are given an initial configuration where all vertices are white, and you need to change k vertices from white to black such that all vertices will eventually become black after updates. Prove that it is NP-complete to decide if this is possible.

Problem 8

Suppose we want to allocate n items $S = \{1, \dots, n\}$ to two agents. The two agents may have different values for each item. Let $u_1, u_2, \dots, u_n$ be agent 1's values for those n items, and $v_1, v_2, \dots, v_n$ be agent 2's values for those n items. An allocation is a partition (A, B) for S, where A is the set of items allocated to agent 1 and B is the set of items allocated to agent 2. An allocation (A, B) is *envy-free* if, based on each agent's valuation, (s)he believes the set (s)he receives is (weakly) more valuable than the set received by the other agent. Formally, (A, B) is envy-free if

$$\sum_{i \in A} u_i \geq \sum_{j \in B} u_j \qquad \text{agent 1 thinks } A \text{ is more valuable}$$

and

$$\sum_{i \in B} v_i \geq \sum_{j \in A} v_j \qquad \text{agent 2 thinks } B \text{ is more valuable.}$$

Prove that deciding if an envy-free allocation exists is NP-complete.

Problem 9

Given a ground set $U = \{1, \dots, n\}$ and a collection of its subsets $\mathcal{S} = \{S_1, \dots, S_m\}$, the *exact cover* problem asks if we can find a subcollection $\mathcal{T} \subseteq \mathcal{S}$ such that $\bigcup_{S \in \mathcal{T}} S = U$ and $S_i \cap S_j = \emptyset$ for any $S_i, S_j \in \mathcal{T}$. Prove that exact cover is NP-complete.

Problem 10

How long does it take you to finish the assignment (including thinking and discussion)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Please write down their names here.