Mathematical Logic Homework 01

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0 Some Lemmas

Thm. The set X is *countable* **iff.** exists one-to-one mapping $f : X \to \mathbb{N}$. [*Already proved in class.*]

Thm. When $f : B \to C$ is surjective and $g : A \to B$ is bijective, $f \circ g : A \to C$ is surjective.

Proof. f is surjective $\Rightarrow$ For any $c \in C$, we can always find some $b \in B$ s.t. $f(b) = c$.

g is bijective $\Rightarrow$ g is surjective $\Rightarrow$ For any $b \in B$, we can always find some $a \in A$ s.t. $g(a) = b$.

Thus, for any $c \in C$, we can always find some $a \in A$ s.t. $f \circ g(a) = f(g(a)) = c$.

i.e. $f \circ g$ is surjective. ■

1 Question 01

1.1 Domain and Range of R

Solution. By the definition of *domain* and *range*, we know

$$\texttt{domain}(R) = \{1, 2, 3\},$$
$$\texttt{range}(R) = \{1.1, 3.2, 2.0\}$$
■

1.2 R is Not a Function

Solution. Since for $2 \in B$, exist 1.1 and 3.2 s.t. $\langle 2, 1.1\rangle \in R$ and $\langle 2, 3.2\rangle \in R$,

by the definition of *functions*, we know R is **not** a function. ■

2 Question 02

2.1 $f : \mathbb{N} \to A$ is Surjective $\Rightarrow$ A is Countable

Proof. When $f : \mathbb{N} \to A$ is surjective, for any $a \in A$, we know exists some $n \in \mathbb{N}$ s.t. $f(n) = a$.

Then we can construct a function $g : A \to \mathbb{N}$ as follows.

For any $a \in A$, we can pick a $n \in \mathbb{N}$ s.t. $f(n) = a$. Set $g(a) = n$.

Now we prove g is injective.

$g(x) = g(y) \Rightarrow\ f(g(x)) = f(g(y))$ (f is a function.) $\Rightarrow x = y$ (by the definition of g).

Thus, $g : A \to \mathbb{N}$ is a one-to-one mapping from A to $\mathbb{N}$.

Therefore, A is countable. ∎

2.2 $f : A \to \mathbb{N}$ is Surjective $\Rightarrow$ A is Infinite

Proof. We prove it by contradiction.

Assume A is finite. Then exists a bijective $g : A \to \{0, 1, 2, ..., |A| - 1\}$.

When $f : A \to \mathbb{N}$ is surjetcive, we know for any $n \in \mathbb{N}$, we can find some $a \in A$ s.t. $f(a) = n$.

Then $f \circ g^{-1} : \{0, 1, 2, ...|A| - 1\} \to \mathbb{N}$ is surjective, i.e. for any $n \in \mathbb{N}$, we can always find a $a \in \{0, 1, 2, ...|A| - 1\}$ s.t. $f \circ g^{-1}(a) = n$.

Since f and g are functions, i.e. $f \circ g^{-1} : A \to \mathbb{N}$ is a function, we know $\texttt{range}(f \circ g^{-1})$ is finite.

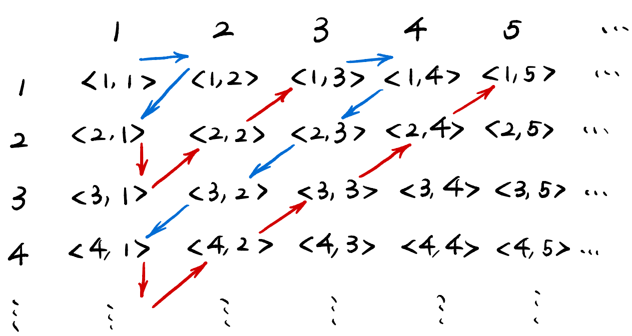
Therefore, exists $n \in \mathbb{N} \setminus \texttt{range}(f \circ g^{-1})$ s.t. for any $a \in \{0, 1, 2, ...|A| - 1\}, f \circ g^{-1}(a) \neq n$.

<u>**Contradiction!**</u>

Thus, A is not finite, i.e. A is infinite. ∎

3 Question 03

Proof. We can construct a listing without repetitions as follows.



Thus, $\mathbb{N} \times \mathbb{N}$ is enumerable. ∎