Discrete Mathematics Exercise 13

Qiu Yihang, 2020/11/07

1. ***Solution:***

Graph	Directed/Undirected Edges	Multiple Edges or not	Loops or not	Type
3	Undirected	No	No	Simple Graph
5	Undirected	Yes	Yes	Pseudograph
7	Directed	No	Yes	Directed Graph
9	Directed	Yes	Yes	Directed Multigraph

2. ***Solution:***

Graph 1) has 6 vertices and 6 edges.

In graph 1), $\deg(a) = 2, \deg(b) = 4, \deg(c) = 1, \deg(d) = 0, \deg(e) = 2, \deg(f) = 3.$

Graph 2) has 5 vertices and 13 edges.

In graph 2), $\deg(a) = 6, \deg(b) = 6, \deg(c) = 6, \deg(d) = 5, \deg(e) = 3.$

3. ***Solution:***

The graph has 4 vertices and 8 edges.

$\deg^-(a) = 2, \deg^-(b) = 3, \deg^-(c) = 2, \deg^-(d) = 1.$

$\deg^+(a) = 2, \deg^+(b) = 4, \deg^+(c) = 1, \deg^+(d) = 1.$

4. ***Solution:***

21) There exists a partition $(\{a, b, c, d\}, \{e\})$ such that when you assign red to a, b, c, d and blue to e, no two adjacent vertices are assigned the same color.

Thus, **graph 21)** is bipartite.

24) There exists a partition $(\{a, \mathrm{c}, \mathrm{e}\}, \{\mathrm{b}, \mathrm{d}, \mathrm{f}\})$ such that when you assign red to $\mathrm{a}, \mathrm{c}, \mathrm{e}$ and blue to $b, \mathrm{d}, \mathrm{f}$, no two adjacent vertices are assigned the same color.

Thus, **graph 24)** is bipartite.

5. ***Proof:***

We can construct a graph G, in which vertices represent men and women and two vertices u, v are adjacent if and only if u is willing to marry v.

Then every vertex in G is adjacent with exactly k vertices since every man is willing to marry exactly k of the women and every woman is willing to marry exactly k of the men.

Since a man is willing to marry a woman if and only if she is willing to marry him, we know every edge in G is undirected.

Thus, G is an undirected graph.

Obviously, (in the exercise) a man is not willing to marry another man and a woman is not willing to marry another woman, **i.e.** exists a partition $\{M, W\}$ in which M includes all vertices representing men and W includes all vertices representing women and when you assign red to all vertices in M and blue to all vertices in W, no two adjacent vertices are assigned the same color.

Thus, G is a bipartite graph and $\{M, W\}$ is a bipartition.

Firstly, we prove $|M| = |W|$.

Let $n = |M|$.

Since G is a bipartite graph with a bipartition $\{M, W\}$ and every vertex in G is incident with k edges, we know there exist nk edges connecting vertices in M and vertices in W.

Considering every vertex in G is incident with k edges and every vertex from W is only adjacent to vertices from M, we know $|W| = nk/k = n = |M|$.

Now, we prove that for all $A \subseteq M$, $|A| \leq |\mathcal{N}(A)|$ holds.

We prove it by contradiction.

Suppose exists some $X \subseteq M$ **s.t.** $|X| > |\mathcal{N}(X)|$.

We can find an A **s.t.** $\forall X \subseteq M \left((|X| > |\mathcal{N}(X)|) \to A \subseteq X\right)$. (※)

Since $\mathcal{N}(u) \subseteq \mathcal{N}(A)$, $|\mathcal{N}(A)| \geq |\mathcal{N}(u)| = k$.

Thus, $|A| > |\mathcal{N}(A)| \geq k$, **i.e.** $|A| \geq k + 1$.

From (※) we know for any $u \in A$, $|A\backslash\{u\}| \leq |\mathcal{N}(A\backslash\{u\})|$, **i.e.** $|A| - 1 \leq |\mathcal{N}(A\backslash\{u\})|$.

Since $\mathcal{N}(A\backslash\{u\}) \subseteq \mathcal{N}(A)$, $|\mathcal{N}(A)| \geq |\mathcal{N}(A\backslash\{u\})| \geq |A| - 1$.

Meanwhile, $|A| > |\mathcal{N}(A)|$.

Thus, $|\mathcal{N}(A)| = |A| - 1$.

Then there exist $|A|k$ edges connecting vertices from A and vertices from $\mathcal{N}(A)$.

Considering every vertex in $\mathcal{N}(A)$ is adjacent to at most k vertices in A, by pigeonhole principle, since $|A|k - |\mathcal{N}(A)|k = k > 0$, there exist at least one vertex with more than k adjacent vertices. **Contradiction.**

Thus, for all $A \subseteq M$, $|A| \leq |\mathcal{N}(A)|$ holds.

By Hall's Theorem, we know there exist a complete matching $\mathcal{M}$ from M to W.

Knowing $|M| = |W|$, $\mathcal{M}$ is also a complete matching from W to M.

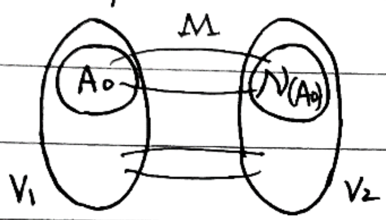
Therefore, there exists a matching that everyone is matched with someone that they are willing to marry.

(打字太累，改手写了) **QED**

6. Proof: Let $a = \max\limits_{a \in A_1} \operatorname{def}(A)$ and $\operatorname{def}(A_0) = a$, i.e. $|A_0| - |\mathcal{N}(A_0)| = a$.

Let $n(A)$ represents the number of [redacted] vertices of A $(A \subseteq V_1)$ that are endpoints of a matching M of G.

First, we prove that $n(V_1) \leq |V_1| - a = |V_1| - |A_0| + |\mathcal{N}(A_0)|$.



Obviously, $n(A_0) \leq \min\{|A_0|, |\mathcal{N}(A_0)|\}$

$n(V_1 \backslash A_0) \leq |V_1 \backslash A_0| = |V_1| - |A_0|$

(Since it is a matching)

Thus, $n(V_1) = n(A_0) + n(V_1 \backslash A_0) \leq |\mathcal{N}(A_0)| + |V_1| - |A_0|$.

Then we prove there exists a matching M_0 s.t. $n(V_1) = |V_1| - |A_0| + |\mathcal{N}(A_0)|$

We add a new vertices to V_2 and connect all of them to all vertices in V_1, and we get a new graph H, which is still a bipartite graph with a bipartition (V_1, V_2').

Thus, for any $A \subseteq V_1$, $|\mathcal{N}_H(A)| = |\mathcal{N}_G(A)| + a = |\mathcal{N}_G(A)| + |A_0| - |\mathcal{N}_G(A_0)|$

$\because |A| - |\mathcal{N}_G(A)| \leq |A_0| - |\mathcal{N}_G(A_0)|$ $\therefore |\mathcal{N}_H(A)| \geq |A|$.

By Hall's Marriage Theorem, there exists a complete matching M from V_1 to V_2'.

Now we prove all the new vertices we added, i.e. the vertices from $V_2' \setminus V_2$, are endpoints of the matching M by contradiction.

If exists $u \in V_2' \setminus V_2$ and u is not an endpoint of the matching M.

Then the matching M is also a complete matching from V_1 to $V_2' \setminus \{u\}$.

~~By Hall's Marriage Theorem, we know for any $A \subseteq V_1$, $|\mathcal{N}_H(A)|$~~

Let H with the vertex u and its incident edges deleted be H'.

By Hall's Marriage Theorem, we know for any $A \subseteq V_1$, $|\mathcal{N}_{H'}(A)| \geq |A|$.

Nevertheless, for $A_0 \subseteq V_1$, $|\mathcal{N}_{H'}(A_0)| = |\mathcal{N}_H(A_0)| - 1 = |\mathcal{N}_G(A_0)| + |A_0| - |\mathcal{N}_G(A_0)| - 1$

$= |A_0| - 1 < |A_0|$. $\boxed{\text{Contradiction!}}$

Thus, $\mathbf{Max}\, n(V_1) = |V_1| - |A_0| + |\mathcal{N}(A_0)| = |V_1| - \mathbf{Max}_{A \subseteq V_1} \operatorname{def}(A)$.

In other words, the maximal number of vertices of V_1 that are endpoints of a matching of G equals $|V_1| - \mathbf{Max}_{A \subseteq V_1} \operatorname{def}(A)$.

QED