Mathematical Logic Homework 06

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1 $\models_{\mathfrak{A}} \forall v_2 Q v_1 v_2 [\![c^{\mathfrak{A}}]\!] \iff \models_{\mathfrak{A}} \forall v_3 Q c v_3$

Proof. We know

$$\begin{aligned}
\models_{\mathfrak{A}} \forall v_2 Q v_1 v_2 [\![c^{\mathfrak{A}}]\!] &\iff \text{for any } a \in |\mathfrak{A}|,\ \models_{\mathfrak{A}} Q v_1 v_2 [\![c^{\mathfrak{A}}, a]\!] \\
&\iff \text{for any } a \in |\mathfrak{A}|, \left(c^{\mathfrak{A}}, a\right) \in Q^{\mathfrak{A}}. \\
&\iff \text{for any } a \in |\mathfrak{A}|, \models_{\mathfrak{A}} Q c v_3 [\![a]\!]. \\
&\iff \models_{\mathfrak{A}} \forall v_3 Q c v_3.
\end{aligned}$$

■

2 Wffs Defining Relations in $\mathfrak{A}$

2.1 $\{0, 1\}$

Solution. We know

$$\begin{aligned}
a = 0 &\iff \text{for any } b \in |\mathfrak{A}|, a \times b = a \iff \text{for any } b \in |\mathfrak{A}|,\ \models_{\mathfrak{A}} a \mathbin{\dot\times} v_2 \doteq a\ [\![b]\!] \\
&\iff \models_{\mathfrak{A}} \forall v_2 \left(v_1 \mathbin{\dot\times} v_2 \doteq v_1\right) [\![a]\!] \iff \models_{\mathfrak{A}} \forall v_2 \left(v_1 \mathbin{\dot\times} v_2 \doteq v_1\right) [\![a]\!]
\end{aligned}$$

$$\begin{aligned}
a = 1 &\iff \text{for any } b \in |\mathfrak{A}|, a \times b = b \iff \text{for any } b \in |\mathfrak{A}|,\ \models_{\mathfrak{A}} a \mathbin{\dot\times} v_2 \doteq v_2\ [\![b]\!] \\
&\iff \models_{\mathfrak{A}} \forall v_2 \left(v_1 \mathbin{\dot\times} v_2 \doteq v_2\right) [\![a]\!] \iff \models_{\mathfrak{A}} \forall v_2 \left(v_1 \mathbin{\dot\times} v_2 \doteq v_2\right) [\![a]\!]
\end{aligned}$$

$$\begin{aligned}
a = 0 \text{ or } a = 1 &\iff \left(\models_{\mathfrak{A}} \forall v_2 \left(v_1 \mathbin{\dot\times} v_2 \doteq v_1\right) [\![a]\!]\right) \text{ or } \left(\models_{\mathfrak{A}} \forall v_2 \left(v_1 \mathbin{\dot\times} v_2 \doteq v_2\right) [\![a]\!]\right) \\
&\iff \models_{\mathfrak{A}} \left(\forall v_2 \left(v_1 \mathbin{\dot\times} v_2 \doteq v_1\right)\right) \vee \left(\forall v_2 \left(v_1 \mathbin{\dot\times} v_2 \doteq v_2\right)\right) [\![a]\!]
\end{aligned}$$

Thus, $\left(\forall \boldsymbol{v_2} \left(\boldsymbol{v_1} \mathbin{\dot\times} \boldsymbol{v_2} \doteq \boldsymbol{v_1}\right)\right) \vee \left(\forall \boldsymbol{v_2} \left(\boldsymbol{v_1} \mathbin{\dot\times} \boldsymbol{v_2} \doteq \boldsymbol{v_2}\right)\right)$ is a wff defining $\{0, 1\}$. ■

2.2 $\{2\}$

Solution. Let $\varphi_1(x) = \forall v_2 \left(x \dot{\times} v_2 \doteq v_2\right)$.

From **2.1**, we know $a = 1 \iff \vDash_{\mathfrak{A}} \phi_1(v_1)[\![a]\!]$, i.e. $\varphi_1(x)$ defines $\{1\}$.

$$
\begin{aligned}
\text{Then} \quad a = 2 &\iff \text{there is some } b \in |\mathfrak{A}|,\ a = b + b \text{ and } b = 1. \\
&\iff \text{there is some } b \in |\mathfrak{A}|,\ \vDash_{\mathfrak{A}} \left(a \doteq v_3 \dot{+} v_3 \wedge \varphi_1(v_3)\right) [\![b]\!]. \\
&\iff \vDash_{\mathfrak{A}} \exists v_3 \left(v_1 \doteq v_3 \dot{+} v_3 \wedge \varphi_1(v_3)\right) [\![a]\!].
\end{aligned}
$$

Thus, $\boldsymbol{\exists v_3 \left(v_1 \doteq v_3 \dot{+} v_3 \wedge \forall v_2 \left(v_3 \dot{\times} v_2 \doteq v_2\right)\right)}$ is a wff defining $\{2\}$. ■

2.3 $\{n \in \mathbb{N} \mid n$ is an even number$\}$

Solution. Let $\varphi_2(x) = \exists v_3 \left(x \doteq v_3 \dot{+} v_3 \wedge \varphi_1(v_3)\right)$. Then

$$
\begin{aligned}
a \text{ is an even number} &\iff \text{there is some } b \in |\mathfrak{A}| = \mathbb{N},\ a = b + b. \\
&\iff \text{there is some } b \in |\mathfrak{A}|,\ \vDash_{\mathfrak{A}} \left(a \doteq v_2 \dot{+} v_2\right) [\![b]\!]. \\
&\iff \vDash_{\mathfrak{A}} \exists v_2 \left(v_1 \doteq v_2 \dot{+} v_2\right) [\![a]\!].
\end{aligned}
$$

Thus, $\boldsymbol{\exists v_2 \left(v_1 \doteq v_2 \dot{+} v_2\right)}$ is a wff defining $\{n \in \mathbb{N} \mid n \text{ is an even number}\}$. ■

3 Homomorphism From $\mathfrak{N}_1$ and $\mathfrak{N}_2$

Proof. We can construct a function $h : \mathbb{N} \to \mathbb{N}$, $h(n) = 2^n$.

Now we prove that h is a homomorphism from $\mathfrak{N}_1$ to $\mathfrak{N}_2$.

0. Obvious h is a function from $|\mathfrak{N}_1| = \mathbb{N}$ to $|\mathfrak{N}_2| = \mathbb{N}$.

1. Since there is <u>**no**</u> predicate symbol in $\mathbb{L}$, it is definite that

for any n-ary predicate symbol R other than $\doteq$ and $a_1, ...a_n \in |\mathfrak{N}_1| = \mathbb{N}$,

$$(a_1, ...a_n) \in R^{\mathfrak{N}_1} \iff (h(a_1), ...h(a_n)) \in R^{\mathfrak{N}_2}.$$

2. There is only one function symbol in $\mathbb{L}$, i.e. $\dot{+}$.

For any $a, b \in |\mathfrak{N}_1| = \mathbb{N}$, we have

$$h\left(\dot{+}^{\mathfrak{N}_1}(a, b)\right) = h(a + b) = 2^{a+b} = 2^a \times 2^b = h(a) \times h(b) = \dot{+}^{\mathfrak{N}_2}\left(h(a), h(b)\right).$$

3. There is only one constant symbol in $\mathbb{L}$, i.e. $\dot{0}$.

We have $h\left(\dot{0}^{\mathfrak{N}_1}\right) = h(0) = 2^0 = 1 = \dot{0}^{\mathfrak{N}_2}$.

Therefore, $h : |\mathfrak{N}_1| \to |\mathfrak{N}_2|,\ n \mapsto 2^n$ satisifes the properties of homomorphisms,

i.e. there is a homomorphism from $\mathfrak{N}_1$ to $\mathfrak{N}_2$. ■