Probability Theory and Mathematical Statistics 概率统计

Homework 1213

邱一航 520030910155

12/14

7-17. 解：(1) 枢轴量：$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$\because P\left(\left|\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right| < u_{0.025}\right) = 0.95 \Rightarrow -u_{0.025} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < u_{0.025} = 1.96$

$\Rightarrow -\frac{\sigma}{\sqrt{n}}u_{0.025}+\bar{X} < \mu < \frac{\sigma}{\sqrt{n}}u_{0.025}+\bar{X}$. 代入 $\bar{x} = \frac{1}{5}(x_1+\cdots+x_5) = 4.364$

得 μ 置信度为 95% 的置信区间是 (4.27, 4.46) □

单侧置信下限：$\because P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < u_{0.05}\right) = 0.95 \Rightarrow \mu > \bar{X} - \frac{\sigma}{\sqrt{n}}u_{0.05}$

取 $u_{0.05} \approx 1.645$. 得 $\mu > 4.285$ $\therefore \underline{\mu} = 4.285$ □

(2) 枢轴量：$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$

$\because P\left(\left|\frac{\bar{X}-\mu}{S/\sqrt{n}}\right| < t_{0.025}^{(n-1)}\right) = 0.95 \Rightarrow -\frac{S}{\sqrt{n}}t_{0.025}^{(n-1)} + \bar{X} < \mu < \frac{S}{\sqrt{n}}t_{0.025}^{(n-1)} + \bar{X}$

代入 $\bar{x} = 4.364$, $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2} = 0.054$. $t_{0.025}(4) = 2.776$ 得：

$4.30 < \mu < 4.43$. 即 μ 置信度 95° 的区间是 (4.30, 4.43) □

单侧置信上限：$P\left(\frac{\bar{X}-\mu}{S/\sqrt{n}} > t_{0.95}(n-1)\right) = 0.95$

$\therefore \mu < -\frac{S}{\sqrt{n}}t_{0.95}(n-1) + \bar{X} = \frac{S}{\sqrt{n}}t_{0.05}(n-1) + \bar{X} = 4.416$ □

7-18. 解：(1) 枢轴量：$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$ $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

$\because P\left(\left|\frac{\bar{X}-\mu}{S/\sqrt{n}}\right| < t_{0.025}(n-1)\right) = 0.95 \Rightarrow -\frac{S}{\sqrt{n}}t_{0.025}^{(n-1)} + \bar{X} < \mu < \bar{X} + \frac{S}{\sqrt{n}}t_{0.025}^{(n-1)}$

$P\left(\chi_{0.975}^2(n-1) < \frac{(n-1)S^2}{\sigma^2} < \chi_{0.025}^2(n-1)\right) = 0.95 \Rightarrow \frac{(n-1)S^2}{\chi_{0.025}^2(n-1)} < \sigma^2 < \frac{(n-1)S^2}{\chi_{0.975}^2(n-1)}$

代入 $\bar{x}=14.72$ $s^2=1.90648$. $t_{0.025}^{(29)}=2.045$. $\chi^{2\,(29)}_{0.025}=45.722$

$\chi^{2\,(29)}_{0.975}=16.047$ 得: $14.01<\mu<15.43$ □

$1.209<\sigma^2<3.445$ □

(2) $P\left(\frac{\bar{X}-\mu}{S/\sqrt{n}}<t_{0.05}^{(n-1)}\right)=0.95 \Rightarrow \underline{\mu}=-\frac{S}{\sqrt{n}}t_{0.05}^{(n-1)}+\bar{X}$

$P\left(\frac{\bar{X}-\mu}{S/\sqrt{n}}>t_{0.95}^{(n-1)}\right)=0.95 \Rightarrow \bar{\mu}=-\frac{S}{\sqrt{n}}t_{0.95}^{(n-1)}+\bar{X}=\frac{S}{\sqrt{n}}t_{0.05}^{(n-1)}+\bar{X}$

代入 $\bar{x}=14.72$, $s^2=1.90648$. $t_{0.05}^{(29)}=1.699$

∴ $\underline{\mu}=14.13$ $\bar{\mu}=15.31$ □

(3) $P\left(\frac{(n-1)S^2}{\sigma^2}<\chi^{2}_{0.05}(n-1)\right)=0.95 \Rightarrow \underline{\sigma^2}=\frac{(n-1)S^2}{\chi^2_{0.05}(n-1)}$

$P\left(\frac{(n-1)S^2}{\sigma^2}>\chi^{2}_{0.95}(n-1)\right)=0.95 \Rightarrow \overline{\sigma^2}=\frac{(n-1)S^2}{\chi^2_{0.95}(n-1)}$

代入 $\bar{x}=14.72$, $S^2=1.90648$ $\chi^{2}_{0.95}(n-1)=17.708$ $\chi^2_{0.05}(n-1)=42.557$ 得:

$\underline{\sigma^2}=1.299$ $\overline{\sigma^2}=3.122$ □

7-19. 解: (1) $E(X)=\int_{-\infty}^{+\infty}e^{y}\cdot\frac{1}{\sqrt{2\pi}\cdot 1}e^{-\frac{(y-\mu)^2}{2}}dy$

$=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2-(2\mu-2)y+\mu^2}{2}}dy=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-\mu+1)^2}{2}}\cdot e^{\frac{1}{2}+\mu}dy$

$=1\cdot e^{\mu+\frac{1}{2}}=e^{\mu+\frac{1}{2}}$ □

(2) 记随机变量 $Y=\ln X$ $Y_i=\ln X_i$ $Y\sim N(\mu,1)$

枢轴量: $\frac{\bar{Y}-\mu}{1/\sqrt{4}}=\frac{\bar{Y}-\mu}{1/2}$.

$P\left(\left|\frac{\bar{Y}-\mu}{1/2}\right|<u_{0.025}\right)=0.95 \Rightarrow \bar{Y}-\frac{1}{2}u_{0.025}<\mu<\bar{Y}+\frac{1}{2}u_{0.025}$

代入 $\bar{y}=\frac{1}{4}(\ln 0.50+\ln 1.25+\ln 0.80+\ln 2.00)=0$ $u_{0.025}=1.96$

∴ $-0.98<\mu<0.98$ 即 μ 置信度0.95的置信区间为 $(-0.98, 0.98)$ □

(3) $-0.98<\mu<0.98 \Rightarrow e^{-0.48}<e^{\mu+\frac{1}{2}}<e^{1.48}$

即 $E(X)$ 置信度0.95的置信区间为 $(e^{-0.48}, e^{1.48})$ □

7-20. 解: 枢轴量: $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$$P\left(\left|\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right| < u_{\frac{\alpha}{2}}\right) = 1-\alpha \quad\Rightarrow\quad \bar{X}-\frac{\sigma}{\sqrt{n}}u_{\frac{\alpha}{2}} < \mu < \bar{X}+\frac{\sigma}{\sqrt{n}}u_{\frac{\alpha}{2}}$$

其长度为 $2\frac{\sigma}{\sqrt{n}}u_{\frac{\alpha}{2}} \le l \quad\Rightarrow\quad n \ge \left(\frac{2\sigma}{l}u_{\frac{\alpha}{2}}\right)^2 = 4\frac{\sigma^2}{l^2}u_{\alpha/2}^2$ □

即容量至少要大于 $4\frac{\sigma^2}{l^2}u_{\alpha/2}^2$.

7-24. 解: (1) $F_Y(y) = P(Y\le y) = P\left(\frac{X_{(n)}}{\theta}\le y\right) = P(X_{(n)}\le \theta y)$

$= P(X_1\le\theta y, \cdots X_n\le\theta y) = [P(X\le\theta y)]^n$

当 $\theta y\in[0,\theta]$. $F_Y(y) = y^n$.

即 $y\in[0,1]$ $\Rightarrow$ $f_Y(y) = ny^{n-1}$

当 $y\notin[0,1]$ 时. $f_Y(y)=0$

$$\therefore\ f_Y(y) = \begin{cases} ny^{n-1}, & y\in[0,1] \\ 0 & y\notin[0,1] \end{cases}$$ □

记θ的置信度为 $(1-\alpha)$ 的单侧置信上限为 $\hat{\theta}_2(\alpha)$.

(2) 证明: 枢轴量取Y

$\because\ P(Y>\sqrt[n]{\alpha}) = 1-P(Y\le\sqrt[n]{\alpha}) = 1-(\sqrt[n]{\alpha})^n = 1-\alpha$

$\therefore\ \frac{X_{(n)}}{\hat{\theta}_2(\alpha)} = \sqrt[n]{\alpha} \Rightarrow \hat{\theta}_2(\alpha) = \frac{X_{(n)}}{\sqrt[n]{\alpha}}$ □

选做题:

7-21. 解: 取枢轴量 $\dfrac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}} \sim t(n+m-2)$

$$P\left(\left|\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}}\right| < t_{0.025}^{(n+m-2)}\right) = 0.95$$

$$\Rightarrow \bar{x}-\bar{y}-\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}\; t_{0.025}^{(n+m-2)} < \mu_1-\mu_2 < \bar{X}-\bar{Y}+\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\frac{(n-1)S_1^2+(m-1)S_2^2}{n+m-2}}\; t_{0.025}^{(n+m-2)}$$

代入 $\bar{x}=2.33$, $S_1^2=9$. $\bar{y}=0.75$. $S_2^2=4$. $n=21$. $m=11$. $t_{0.025}^{(n+m-2)}=2.042$ 得:

$$-0.48<\mu_1-\mu_2<3.64$$

∴ 两个总体均值差置信度为 0.95 的双侧置信区间为 $(-0.48, 3.64)$ □

7-22. 解: (1) 设甲城市家庭月消费支出的总体为 X, 乙为 Y. 则 $\bar{X}=\bar{x}_1$. $\bar{y}=\bar{x}_2$.

因为 $n, m>50$. 近似认为是正态分布. 取枢轴量 $\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{S_1^2/n+S_2^2/m}}\sim N(0,1)$

t分布

$$P\left(\left|\frac{(\bar{X}+\bar{Y})-(\mu_1+\mu_2)}{\sqrt{S_1^2/n+S_2^2/m}}\right|<u_{0.025}\right)=0.95$$

$$\Rightarrow \bar{X}-\bar{Y}-u_{0.025}\sqrt{\frac{S_1^2}{n}+\frac{S_2^2}{m}}<\mu_1-\mu_2<\bar{X}-\bar{Y}+u_{0.025}\sqrt{\frac{S_1^2}{n}+\frac{S_2^2}{m}}$$

代入 $\bar{x}=\bar{x}_1=3000$, $s_1=400$, $\bar{y}=\bar{x}_2=4200$. $s_2=500$. $n=61$, $m=121$ 得:

甲.乙两城市平均每户月消费支出间的差异置信度为 0.95 的置信区间为

$$(-1334.21,\ -1065.79)$$ □

(2) ∵ $\frac{(m-1)S_2^2}{\sigma_2^2}\sim\chi^2(m-1)$. $\frac{(n-1)S_1^2}{\sigma_1^2}\sim\chi^2(n-1)$

取枢轴量 $\frac{S_1^2}{S_2^2}\cdot\frac{\sigma_2^2}{\sigma_1^2}\sim F(n-1, m-1)$

$$P\left(F_{0.975}(n-1,m-1)<\frac{S_1^2}{S_2^2}\frac{\sigma_2^2}{\sigma_1^2}<F_{0.025}(n-1,m-1)\right)=0.95$$

$\Rightarrow \frac{\sigma_1^2}{\sigma_2^2}$ 置信度 0.95 的置信区间为 $\left(\frac{S_1^2}{S_2^2}\frac{1}{F_{0.025}(n-1,m-1)}<\frac{\sigma_1^2}{\sigma_2^2}<\frac{S_1^2}{S_2^2}\frac{1}{F_{0.975}(n-1,m-1)}\right)$

代入 $s_1=400$, $s_2=500$. $F_{0.025}(n-1,m-1)=1.53$. $F_{0.975}(n-1,m-1)=\frac{1}{1.53}=0.6536$

得: $\frac{\sigma_1^2}{\sigma_2^2}$ 置信度 0.95 的置信区间为: $(0.4183,\ 0.9792)$ □

7-23. 解：$n>50$. 由中心极限定理，近似认为 $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim N(0,1)$

(1) $\mu=\frac{1}{\lambda}$. 枢轴量：$\frac{\bar{X}-\mu}{S/\sqrt{n}} = \frac{\bar{X}-1/\lambda}{S/\sqrt{n}}$.

$P\left(\left|\frac{\bar{X}-\mu}{S/\sqrt{n}}\right| < u_{0.025}\right)=0.95 \Rightarrow \bar{X}-\frac{S}{\sqrt{n}}u_{0.025}<\mu<\bar{X}+\frac{S}{\sqrt{n}}u_{0.025}$

$$\frac{1}{\bar{X}+\frac{S}{\sqrt{n}}u_{0.025}}<\lambda<\frac{1}{\bar{X}-\frac{S}{\sqrt{n}}u_{0.025}}$$

代入 $\bar{x}=1034.17$, $s^2=495662.88$, $n=120$

$\therefore$ λ 置信度0.9的置信区间为 $(0.000862,\ 0.001101)$ □

μ 置信度0.9的置信区间为 $(908.20,\ 1160.14)$ □

(2) $P\left(\frac{\bar{X}-\mu}{S/\sqrt{n}}<u_{0.1}\right)=0.9 \Rightarrow \mu=\bar{X}+u_{0.1}\frac{S}{\sqrt{n}}$

代入值得：$\mu=951.91$ □