Stochastic Process Homework 03

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Apr.05 - Apr.16, 2022

0 Reference and Notations

In the following sections, we use the following notations.

Natotaion	Meaning
T_i	$\min\limits_{\tau} \{\tau > 0 \mid X_\tau = i\}$
$T_{i\to j}$	$\min\limits_{\tau} \{\tau > 0 \mid X_0 = i \wedge X_\tau = j\}$
N_i	$\sum_{t=0}^{\infty} \mathbb{1}[X_t = i]$
$\mathbf{Pr}_i[\cdot]$	$\mathbf{Pr}[\ \cdot \mid X_0 = i\ ]$
$\mathbb{E}_i[\cdot]$	$\mathbb{E}[\ \cdot \mid\ X_0 = i\ ]$

Table 1: Notations.

This homework is completed with the help of discussions with **Ji Yikun** and **Sun Yilin**.

1 FTMC for Countably Infinite Chains

1.1 [PR]+[A]+[I] $\Rightarrow$ [S]+[U]+[C] is A Generalization

Proof. To prove [PR]+[A]+[I] $\Rightarrow$ [S]+[U]+[C] implies that [F]+[A]+[I] $\Rightarrow$ [S]+[U]+[C],

we just to need to prove [F]+[A]+[I] implies [PR].

First we prove that [F]+[A]+[I] implies [Recurrence] by contradiction.

Assume exists a finite, aperiodic, and irreducable Markov Chain which is not recurrent. Then exists state i s.t. $\mathbf{Pr}_i[T_i < \infty] < 1$, i.e. $\mathbf{Pr}_i[N_i = \infty] < 1$. Then state i will never be visited after certain finite steps, which yields state i can not be reached from other states. **Contradiction** to the assumption that the Markov Chain is irreducable.

Now we prove [F]+[A]+[I]+[Recurrence] implies [PR] by contradiction.

Assume exsits a finite, aperiodic, irreducable, and recurrent Markov Chain s.t. there exists a state i, $\mathbb{E}_i[T_i] = \infty$.

Since the Markov Chain is finite and irreducable, we know

$$\forall j \in [n], \exists t > 0 \text{ s.t. } P^t(i,j) > 0.$$
$$\implies \mathbb{E}_i\left[T_j\right] \geq t \cdot P^t(i,j) + \left(1 - P^t(i,j)\right)\left(t + \mathbb{E}_i\left[T_i\right]\right) \geq \infty$$

Thus, $\forall j \in [n],\ \mathbb{E}_i\left[T_j\right] = \infty$.

This means $\forall t, \forall j \in [n], \lim_{t\to\infty} t \cdot \mathbf{Pr}_i\left[X_t = j \wedge j \text{ is never visited}\right] > 0$.

Meanwhile, since the Markov Chain is finite and irreducable, $\forall j \in [n], \exists \tau > 0,$ s.t. $\mathbf{Pr}_i\left[X_\tau = j\right]$ $= P^\tau(i,j) > 0$. Let $\beta = \mathbf{Pr}_i\left[X_\tau = j\right]$. Thus,

$$t \cdot \mathbf{Pr}_i\left[X_t = j \wedge j \text{ is never visited}\right] \leq t \cdot (1-\beta)^{\lceil \frac{t}{\tau} \rceil}\beta \to 0. \text{ (When } t \to \infty)$$

Contradiction.

Thus, [F]+[A]+[I] implies [PR].

If [PR]+[A]+[I] $\Rightarrow$ [S]+[U]+[C], since [F]+[A]+[I] implies [PR],

we know [F]+[A]+[I] $\Rightarrow$ [PR]+[A]+[I] $\Rightarrow$ [S]+[U]+[C].

In other words, [PR]+[A]+[I] $\Rightarrow$ [S]+[U]+[C] is a generalization. ■

1.2 $\mathbf{Pr}_{(i,j)}\left[T_{(k,k)} < \infty\right] = 1$ for Any i, j, k in the Given Markov Chain

Proof. Let $Q \in [0,1]^{\Omega\times\Omega}$ be the transition function of the Markov Chain. Then we have

$$Q\left((i,j),(i',j')\right) = P(i,i')P(j,j')$$

First we prove Q is also irreducable.

$\forall i,j,i',j' \in \Omega$, since P is irreducable, we know $\exists t_1, t_2,$ s.t. $P^{t_1}(i,i') > 0, P^{t_2}(j,j') > 0$. Then exists $t = t_1 \cdot t_2$ s.t. $Q^t\left((i,j),(i',j')\right) = P^t(i,i')P^t(j,j') > 0$. Thus, Q is irreducable.

Now we prove Q has a stationary distribution.

From **Lecture 5**, [I]+[PR] implies [S]+[U]. Thus, P has a unique stationary distribution π.

Define $\pi'(i,j) = \pi(i)\pi(j)$.

$$\begin{aligned}
\pi'(i,j) = \pi(i)\pi(j) &= \left(\sum_{i'} P(i',i)\pi(i)\right)\left(\sum_{j'} P(j',j)\pi(j)\right) \\
&= \sum_{i'}\sum_{j'} P(i',i)P(j',j)\pi(i)\pi(j) \\
&= \sum_{(i',j')} P(i',i)P(j',j)\pi(i)\pi(j) \\
&= \sum_{(i',j')} Q((i',j'),(i,j))\pi'(i,j)
\end{aligned}$$

Thus, we know $(\pi')^T = Q(\pi')^T$, i.e. π' is a stationary distribution of Q.

By **the Strong Law of Large Number for Markov Chain**, since Q is irreducable,

$$\forall i,j,i',j' \in \Omega,\ \mathbf{Pr}_{(i,j)}\left[\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}\mathbb{1}\left[X_t=(i',j')\right]=\frac{1}{\mathbb{E}_{(i',j')}\left[T_{(i',j')}\right]}\right]=1.$$

i.e.
$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}Q^t((i,j),(i',j')) &= \lim_{n\to\infty}\frac{1}{n}\sum_{t=1}\mathbb{E}_{(i,j)}\left[\mathbb{1}\left[X_t=(i',j')\right]\right]\\ &= \lim_{n\to\infty}\mathbb{E}_{(i,j)}\left[\frac{1}{n}\sum_{t=1}\mathbb{1}\left[X_t=(i',j')\right]\right]\\ &= \mathbb{E}_{(i,j)}\left[\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}\mathbb{1}\left[X_t=(i',j')\right]\right]\\ &\quad \text{(By \textbf{Bounded Convergence Theorem})}\\ &= \frac{1}{\mathbb{E}_{(i',j')}\left[T_{(i',j')}\right]}\end{aligned}$$

Set $(i,j)=(i',j')$, and we have

$$\forall i,j\in\Omega,\ \pi'(i,j)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}Q^t((i,j),(i,j))=\frac{1}{\mathbb{E}_{(i,j)}\left[T_{(i,j)}\right]}>0,$$

i.e. $\mathbf{Pr}_{(i,j)}\left[T_{(i,j)}<\infty\right]=1$, i.e. Q is positive recurrent.

Meanwhile, Q is irreducable. Thus, $\forall i,j,k\in\Omega,\ \mathbf{Pr}_{(i,j)}\left[T_{(k,k)}<\infty\right]=1.$ ■

1.3 FTMC for Countably Infinite Chains

Proof. In **Lecture 5**, we already proved [I]+[PR] implies [S]+[U].

Thus, irreducable, aperiodic, and positive recurrent Markov Chain P has a unique stationary distribution π. Now we just need to prove [PR]+[I]+[A] implies [C].

We construct a coupling ω as follows. Set $X_0\sim\mu_0, Y_0\sim\pi$. Let $X_t\sim\mu_t$.

$$(X,Y)\sim\omega,\ \begin{cases} X_{t+1}=X_t,\ Y_{t+1}=Y_t, & X_t=Y_t\\ X_t\to X_{t+1},\ Y_t\to Y_{t+1}\text{ randomly}, & X_t\neq Y_t\end{cases}$$

By **Coupling Lemma**, we know

$$D_{\mathrm{TV}}(\mu_t,\pi)\le\mathbf{Pr}_{(X,Y)\sim\omega}\left[X_t\neq Y_t\right].$$

Obvious $\mathbf{Pr}_{(X,Y)\sim\omega}\left[X_t\neq Y_t\right]\ge\mathbf{Pr}_{(X,Y)\sim\omega}\left[X_{t+1}\neq Y_{t+1}\right]$.

Since P is aperiodic, we know $\lim_{t\to\infty}\mathbf{Pr}_{(X,Y)\sim\omega}\left[X_t\neq Y_t\right]=0$.

Therefore, when $t\to\infty,\ \mu_t\to\pi$, i.e. Markov Chain P converges to π.

In conclusion, [PR]+[I]+[A] implies [S]+[U]+[C]. ■

2 A Randomized Algorithm for 3-SAT

In this section, we assume the same notations in the class.

2.1 The Probability of Correctness of the Given Algorithm

Solution. From **Lecture 6**, we know $\mathbb{E}\left[T_{Y_0\to n}\right] \le n^2$.

Thus,

$$1 - \mathbf{Pr}\left[\exists t \in [0, 2n^2] \text{ s.t. } Y_t = n\right] = \mathbf{Pr}\left[T_{Y_0\to n} > 2n^2\right] \le \frac{\mathbb{E}\left[T_{Y_0\to n}\right]}{2n^2} = \frac{1}{2}.$$

i.e. after $2n^2$ flipping operations, the correctness of the algorithm is at least 0.5.

Since we repeat it for 50 times,

we know the probability that the algorithm returns a faulty result is less than 0.5^{50}.

Thus, the probability of correctness of our algorithm is at least $\mathbf{1 - 0.5^{50}}$. ■

2.2 $\mathbf{Pr}\left[X_{t+1} = X_t + 1\right] \ge \frac{1}{3}$, $\mathbf{Pr}\left[X_{t+1} = X_t - 1\right] \le \frac{2}{3}$

Proof. Without loss of generality, suppose we choose the clause $x \vee y \vee z$ in round t.

We know assignment σ_t does not satisfy $x \vee y \vee z$, otherwise we would not choose the clause. Then $\sigma_t(x) = \sigma_t(y) = \sigma_t(z) =$ `False`. Let the satisfying assignment be σ. Possible cases are as follows.

$\sigma(x)$	$\sigma(y)$	$\sigma(z)$	$\mathbf{Pr}\left[X_{t+1} = X_t + 1\right]$	$\mathbf{Pr}\left[X_{t+1} = X_t - 1\right]$
`True`	`True`	`True`	1	0
`True`	`True`	`False`	2/3	1/3
`True`	`False`	`True`	2/3	1/3
`True`	`False`	`False`	1/3	2/3
`False`	`True`	`True`	2/3	1/3
`False`	`True`	`False`	1/3	2/3
`False`	`False`	`True`	1/3	2/3

Therefore,

$$\mathbf{Pr}\left[X_{t+1} = X_t + 1\right] \ge \frac{1}{3},\ \mathbf{Pr}\left[X_{t+1} = X_t - 1\right] \le \frac{2}{3}.$$

Qed. ■

2.3 $\Theta(2^n)$ Flipping Operations Are Needed to Ensure 0.99 Correctness

Proof. Similar to the proof of 2-SAT Random Algorithm, we design a random walk $\{Y_t\}_{t\leq 0}$ as follows.

Suppose we need to repeat the random flipping operations for at least C times to ensure the probability of the correctness of our algorithm is 0.99.

For $Y_t \notin \{0, n\}$,

$$Y_{t+1} = \begin{cases} Y_t + 1 & \text{w.p. } \frac{1}{3} \\ Y_t - 1 & \text{w.p. } \frac{2}{3} \end{cases}$$

For $Y_t = 0$, $Y_{t+1} = Y_t + 1$ w.p. 1.

For $Y_t = n$, $Y_{t+1} = Y_t - 1$ w.p. 1.

Then we have

$$\begin{aligned} \mathbf{Pr}\left[\text{the algorithm is correct.}\right] &\geq \mathbf{Pr}\Big[\exists t \in [0, C] \text{ s.t. } X_t = n\Big] \\ &\geq \mathbf{Pr}\Big[\exists t \in [0, C] \text{ s.t. } Y_t = n\Big] \\ &\qquad (\text{Since } \mathbf{Pr}\left[X_{t+1} = X_t + 1\right] \geq \mathbf{Pr}\left[Y_{t+1} = Y_t + 1\right]) \end{aligned}$$

Let $X_0 = Y_0 = i$.

Use $\mathscr{A}$ to denote the event that the first step is towards the right, i.e. $X_{t+1} = X_t + 1$.

Then

$$\begin{aligned} T_{k\to k+1} &= \mathbb{1}\left[\mathscr{A}\right] + \mathbb{1}[\bar{\mathscr{A}}](1 + T_{k-1\to k+1}) \\ &= \mathbb{1}\left[\mathscr{A}\right] + \mathbb{1}[\bar{\mathscr{A}}](1 + T_{k-1\to k} + T_{k\to k+1}) \\ \implies \qquad \mathbb{E}\left[T_{k\to k+1}\right] &= \frac{1}{3} + \frac{2}{3}(1 + \mathbb{E}\left[T_{k-1\to k}\right] + \mathbb{E}\left[T_{k\to k+1}\right]) \\ \implies \qquad \mathbb{E}\left[T_{k\to k+1}\right] &= 2\mathbb{E}\left[T_{k-1\to k}\right] + 3 \\ \implies \qquad \mathbb{E}\left[T_{k\to k+1}\right] + 3 &= 2\left(\mathbb{E}\left[T_{k-1\to k}\right] + 3\right). \end{aligned}$$

Since $\mathbb{E}\left[T_{0\to 1}\right] = 1$, we know $\mathbb{E}\left[T_{k\to k+1}\right] = 2^{k+2} - 3$.

Thus,

$$\begin{aligned} \mathbb{E}\left[T_{i\to n}\right] &= \mathbb{E}\left[\sum_{k=i}^{n-1} T_{k\to k+1}\right] = \sum_{k=i}^{n-1} \mathbb{E}\left[T_{k\to k+1}\right] \\ &= \sum_{k=i}^{n-1} \left(2^{k+2} - 3\right) = 2^{n+2} - 2^{i+2} - 3(n-i) \leq 2^{n+2} \end{aligned}$$

Therefore, we have

$$1 - \mathbf{Pr}\left[\exists t \in [0, C] \text{ s.t. } Y_t = n\right] = \mathbf{Pr}\left[T_{Y_0\to n} > C\right] \leq \frac{\mathbb{E}\left[T_{Y_0\to n}\right]}{C}.$$

We want to ensure the probability of correctness of the random algorithm is at least 0.99, i.e.

$$\frac{\mathbb{E}\left[T_{Y_0\to n}\right]}{C} = \frac{2^{n+2}}{C} \leq 0.01 \implies C \geq 400 \cdot 2^n \iff C = \Theta(2^n).$$

Qed. ■

2.4 Lower Bound for $\mathbf{Pr}\left[\exists t \in [1,3n] : X_t = n\right]$

Solution. Let $N_r = \sum_{t=0}^{3n-1} \mathbb{1}\left[X_{t+1} = X_t + 1\right], N_l = \sum_{t}^{3n-1} \mathbb{1}\left[X_{t+1} = X_t - 1\right]$.

Then N_r is the number of steps to the right and N_l is the number of steps to the left.

Start with $Y_0 = n - i$. The event $\exists t \in [1,3n]$ s.t. $X_t = n$ only occurs when $N_r - N_l = i$, i.e.

$$
\begin{aligned}
\mathbf{Pr}\left[\exists t \in [1,3n] : X_t = n\right] &= \mathbf{Pr}\left[N_r - N_l = i\right] \\
&\geq \binom{3i}{i}\left(\frac{1}{3}\right)^{2i}\left(\frac{2}{3}\right)^{i} \\
&\approx \frac{\sqrt{2\pi(3i)}\left(\frac{3i}{\mathrm{e}}\right)^{3i}}{\sqrt{2\pi(2i)}\left(\frac{2i}{\mathrm{e}}\right)^{2i}\sqrt{2\pi i}\left(\frac{i}{\mathrm{e}}\right)^{i}} \cdot \frac{2^i}{3^{3i}} \text{ (By Stirling Equation)} \\
&= \sqrt{\frac{3}{4\pi i}} \cdot \frac{1}{2^i}
\end{aligned}
$$

Thus, a good lower bound for $\mathbf{Pr}\left[\exists t \in [1,3n] : X_t = n\right]$ is

$$\sqrt{\frac{3}{4\pi i}} \cdot \frac{1}{2^i}$$

■

2.5 The Probability of Correctness of the Advanced Algorithm

Solution. From **2.4**, we know if we start with $X_0 = Y_0 = n - i$,

$$\mathbf{Pr}\left[\exists t \in [1,3n] : X_t = n\right] \geq \sqrt{\frac{3}{4\pi i}} \cdot \frac{1}{2^i}.$$

Use $\mathscr{S}$ to denote the event that the algorithm outputs a satisfying assignment.

Since σ_0 is uniform at random from all 2^n assignments, we have

$$
\begin{aligned}
\mathbf{Pr}\left[\mathscr{S}\right] &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^{n}\mathbf{Pr}\left[\exists t \in [1,3n] : X_0 = i \wedge X_t = n\right] \\
&\geq \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sqrt{\frac{3}{4\pi k}} \cdot \frac{1}{2^k} \geq \frac{1}{2^n}\sqrt{\frac{3}{4\pi n}}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^k} \\
&= \frac{1}{2^n}\sqrt{\frac{3}{4\pi n}}\left(1 + \frac{1}{2}\right)^{n} \\
&= \sqrt{\frac{3}{4\pi n}}\left(\frac{3}{4}\right)^{n}
\end{aligned}
$$

Thus, the probability that the algorithm outputs a satisfying assignment is at least

$$\sqrt{\frac{3}{4\pi n}}\left(\frac{3}{4}\right)^{n}$$

■

2.6 Advanced Algorithm Design

Solution. Inspired by **2.1**, we can make some adjustments to the original 3-SAT Random Algorithm.

We start with an assignment σ_0 which is uniform from all 2^n assignments of the variables.

We repeat the flipping operations for $3n$ times until a satisfying assignment is returned (and we output the assignment) or the number of epochs of repetitions have reached N.

End of the Advanced Algorithm. ■

Now we determine N.

From **2.5**, we know the algorithm returns a satisfying assignment w.p. $\sqrt{\frac{3}{4\pi n}}\left(\frac{3}{4}\right)^n$. Thus, the probability that the algorithm can not find a satisfying assignment after N repetitions is

$$\left(1-\sqrt{\frac{3}{4\pi n}}\left(\frac{3}{4}\right)^n\right)^N$$

Meanwhile, we want the probability of the correctness of our algorithm is at least 0.99, i.e.

$$\mathbf{Pr}\left[\text{The algorithm is correct.}\right] \geq 1-\left(1-\sqrt{\frac{3}{4\pi n}}\left(\frac{3}{4}\right)^n\right)^N \geq 0.99$$

$$\begin{aligned}
\implies \qquad N &\geq \log_{1-\sqrt{\frac{3}{4\pi n}}\left(\frac{3}{4}\right)^n} 0.01 = \frac{\log_{10} 0.01}{\log_{10}\left(1-\sqrt{\frac{3}{4\pi n}}\left(\frac{3}{4}\right)^n\right)} \\
&\approx \frac{-2}{-\sqrt{\frac{3}{4\pi n}}\left(\frac{3}{4}\right)^n \ln(10)} \\
&= \frac{2}{\ln(10)}\sqrt{\frac{4\pi}{3}}\cdot\sqrt{n}\left(\frac{4}{3}\right)^n \\
&= O\left(n^{1/2}\left(\frac{4}{3}\right)^n\right).
\end{aligned}$$

Choice of N. ■

The time complexity of our algorithm is at most $O(N\cdot 3n) = O(nN) = O\left(n^{3/2}\left(\frac{4}{3}\right)^n\right)$.

Therefore,

$$c = \frac{4}{3}.$$

■