Discrete Mathematics Exercise 8

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1. ***Solution:***

For any $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Q}$. Thus, $\forall x \in \mathbb{R}\,(x\mathcal{R}x)$, **i.e.** $\mathcal{R}$ is reflexive.

For any $x, y \in \mathbb{R}, x - y \in \mathbb{Q} \to y - x = -(x - y) \in \mathbb{Q}$.

Thus, $\forall x, y \in \mathbb{R}\ (x\mathcal{R}y \to y\mathcal{R}x)$, **i.e.** $\mathcal{R}$ is symmetric.

Exists $x = 2, y = 3 \in \mathbb{R}$ **s.t.** $x\mathcal{R}y \wedge y\mathcal{R}x$ but $x \neq y$. Thus, $\mathcal{R}$ is not antisymmetric.

For any $x, y, z \in \mathbb{R}, x - y \in \mathbb{Q} \wedge y - z \in \mathbb{Q} \to x - z = (x - y) + (y - z) \in \mathbb{Q}$.

Thus, $\forall x, y, z \in \mathbb{R}\ (x\mathcal{R}y \wedge y\mathcal{R}z \to x\mathcal{R}z)$, **i.e.** $\mathcal{R}$ is transitive.

In conclusion, $\mathcal{R}$ is reflexive, symmetric and transitive, but is not antisymmetric.

2. b) ***Solution:***

Exists $x = 0 \in \mathbb{Z}$ **s.t.** $x^2 = 0 \ngeq 1$, **i.e.** $\neg x\mathcal{R}x$. Thus, $\mathcal{R}$ is not reflexive.

For any $x, y \in \mathbb{Z}$, $xy \geq 1 \to yx \geq 1$. Thus, $\forall x, y \in \mathbb{Z}\,(x\mathcal{R}y \to y\mathcal{R}x)$, **i.e.** $\mathcal{R}$ is symmetric.

Exists $x = 1, y = 2 \in \mathbb{Z}$ **s.t.** $x\mathcal{R}y \wedge y\mathcal{R}x$ but $x \neq y$. Thus, $\mathcal{R}$ is not antisymmetric.

Exists $x = 0.5, y = 4, z = 0.3 \in \mathbb{Z}$ **s.t.** $x\mathcal{R}y \wedge y\mathcal{R}z$ but $\neg x\mathcal{R}z$. Thus, $\mathcal{R}$ is not transitive.

In conclusion, $\mathcal{R}$ is symmetric but is not reflexive or antisymmetric or transitive.

f) ***Solution:***

For any $x \in \mathbb{Z}$, x and x are for sure both negative or both nonnegative. Thus, $\forall x \in \mathbb{Z}\,(x\mathcal{R}x)$, **i.e.** $\mathcal{R}$ is reflexive.

For any $x, y \in \mathbb{Z}$, that x and y are both negative or both nonnegative implies y and x are both negative or both nonnegative. Thus, $\forall x, y \in \mathbb{Z}\,(x\mathcal{R}y \to y\mathcal{R}x)$, **i.e.** $\mathcal{R}$ is symmetric.

Exists $\forall x = 1, y = 2 \in \mathbb{Z}$ **s.t.** $x\mathcal{R}y \wedge y\mathcal{R}x$ but $x \neq y$. Thus, $\mathcal{R}$ is not antisymmetric.

For any $x, y, z \in \mathbb{R}$, that x and y are both negative or both nonnegative and that y and z are both negative or both nonnegative implies x and z are both negative or both nonnegative.

Thus, $\forall x, y, z \in \mathbb{Z}\,(x\mathcal{R}y \wedge y\mathcal{R}z \to x\mathcal{R}z)$, **i.e.** $\mathcal{R}$ is transitive.

In conclusion, $\mathcal{R}$ is reflexive, symmetric and transitive but is not antisymmetric.

3. ***Solution:***

a) $\mathcal{R}_1 \circ \mathcal{R}_1 = \{(a, b) \in \mathbb{R}^2 |\, a > b\}$.

For any $(a, b) \in \mathcal{R}_1 \circ \mathcal{R}_1$, $\exists c \in \mathbb{R}\,(a, c), (c, b) \in \mathcal{R}_1$ **i.e.** $(a > c) \wedge (c > b)$, **i.e.** $a > b$.

Thus, $\mathcal{R}_1 \circ \mathcal{R}_1 = \{(a, b) \in \mathbb{R}^2 |\, a > b\}$.

b) $\mathcal{R}_1 \circ \mathcal{R}_2 = \{(a, b) \in \mathbb{R}^2 |\, a > b\}$

For any $(a, b) \in \mathcal{R}_1 \circ \mathcal{R}_2$, $\exists c \in \mathbb{R}\,(a, c) \in \mathcal{R}_2, (c, b) \in \mathcal{R}_1$ **i.e.** $(a \geq c) \wedge (c > b)$, **i.e.** $a > b$.

Thus, $\mathcal{R}_1 \circ \mathcal{R}_2 = \{(a, b) \in \mathbb{R}^2 |\, a > b\}$.

c) $\mathcal{R}_1 \circ \mathcal{R}_3 = \mathbb{R}^2$

For any $(a,b) \in \mathcal{R}_1 \circ \mathcal{R}_3$, $\exists c \in \mathbb{R}\ (a,c) \in \mathcal{R}_3, (c,b) \in \mathcal{R}_1$ **i.e.** $(a < c) \wedge (c > b)$.

Thus, $\mathcal{R}_1 \circ \mathcal{R}_3 = \mathbb{R}^2$.

e) $\mathcal{R}_1 \circ \mathcal{R}_5 = \{(a,b) \in \mathbb{R}^2 |\ a > b\}$

For any $(a,b) \in \mathcal{R}_1 \circ \mathcal{R}_5$, $\exists c \in \mathbb{R}\ (a,c) \in \mathcal{R}_5, (c,b) \in \mathcal{R}_1$ **i.e.** $(a = c) \wedge (c > b)$, **i.e.** $a > b$.

Thus, $\mathcal{R}_1 \circ \mathcal{R}_5 = \{(a,b) \in \mathbb{R}^2 |\ a > b\}$.

f) $\mathcal{R}_1 \circ \mathcal{R}_6 = \mathbb{R}^2$

For any $(a,b) \in \mathcal{R}_1 \circ \mathcal{R}_6$, $\exists c \in \mathbb{R}\ (a,c) \in \mathcal{R}_6, (c,b) \in \mathcal{R}_1$ **i.e.** $(a \neq c) \wedge (c > b)$.

Thus, $\mathcal{R}_1 \circ \mathcal{R}_6 = \mathbb{R}^2$.

g) $\mathcal{R}_2 \circ \mathcal{R}_3 = \mathbb{R}^2$

For any $(a,b) \in \mathcal{R}_2 \circ \mathcal{R}_3$, $\exists c \in \mathbb{R}\ (a,c) \in \mathcal{R}_3, (c,b) \in \mathcal{R}_2$ **i.e.** $(a < c) \wedge (c \geq b)$.

Thus, $\mathcal{R}_2 \circ \mathcal{R}_3 = \mathbb{R}^2$.

h) $\mathcal{R}_3 \circ \mathcal{R}_3 = \{(a,b) \in \mathbb{R}^2 |\ a < b\}$

For any $(a,b) \in \mathcal{R}_3 \circ \mathcal{R}_3$, $\exists c \in \mathbb{R}\ (a,c) \in \mathcal{R}_3, (c,b) \in \mathcal{R}_3$ **i.e.** $(a < c) \wedge (c < b)$, **i.e.** $a < b$.

Thus, $\mathcal{R}_3 \circ \mathcal{R}_3 = \{(a,b) \in \mathbb{R}^2 |\ a < b\}$.

4. ***Proof:***

To prove the composition operator is associative, we only need to prove that

$$\forall x \forall y \left((x,y) \in (\mathcal{R}_3 \circ \mathcal{R}_2) \circ \mathcal{R}_1 \leftrightarrow (x,y) \in \mathcal{R}_3 \circ (\mathcal{R}_2 \circ \mathcal{R}_1)\right).$$

$(a,b) \in (\mathcal{R}_3 \circ \mathcal{R}_2) \circ \mathcal{R}_1$ **iff.** $\exists c\big((a,c) \in \mathcal{R}_1 \wedge (c,b) \in \mathcal{R}_3 \circ \mathcal{R}_2\big)$

iff. $\exists c \exists d\ \big((a,c) \in \mathcal{R}_1 \wedge (c,d) \in \mathcal{R}_2 \wedge (d,b) \in \mathcal{R}_3\big)$

iff. $\exists d\big((a,d) \in \mathcal{R}_2 \circ \mathcal{R}_1 \wedge (d,b) \in \mathcal{R}_3\big)$

iff. $(a,b) \in \mathcal{R}_3 \circ (\mathcal{R}_2 \circ \mathcal{R}_1)$

Thus, the composition operator is associative over relations.

QED

5. ***Proof:***

To prove that *a relation* $\mathcal{R}$ *on a set* $\mathbb{A}$ *is transitive* **iff.** $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$, we just need to prove that *a relation* $\mathcal{R}$ *on a set* $\mathbb{A}$ *is not transitive* **iff.** $\mathcal{R} \circ \mathcal{R} \nsubseteq \mathcal{R}$.

a relation $\mathcal{R}$ *on a set* $\mathbb{A}$ *is not transitive*

iff. $\exists x,y,z \in \mathbb{A}\ \neg(x\mathcal{R}y \wedge y\mathcal{R}z \rightarrow x\mathcal{R}z)$

iff. $\exists x,y,z \in \mathbb{A}\ \neg(\neg(x\mathcal{R}y \wedge y\mathcal{R}z) \vee x\mathcal{R}z)$

iff. $\exists x,y,z \in \mathbb{A}\ (x\mathcal{R}y \wedge y\mathcal{R}z) \wedge \neg x\mathcal{R}z$

iff. $\exists x,z \in \mathbb{A}\ \big(\exists y \in \mathbb{A}\ (\ (x,y) \in \mathcal{R} \wedge (y,z) \in \mathcal{R} \wedge (x,z) \notin \mathcal{R})\big)$

iff. $\exists x,z \in \mathbb{A}\ \big((x,z) \in \mathcal{R} \circ \mathcal{R} \wedge (x,z) \notin \mathcal{R}\big)$

iff. $\mathcal{R} \circ \mathcal{R} \nsubseteq \mathcal{R}$

QED

6. ***Proof:***

Since a relation $\mathcal{R}$ on a set $\mathbb{A}$ is antisymmetric **iff.** $\forall x,y \in \mathbb{A}\ (x\mathcal{R}y \wedge y\mathcal{R}x \to x=y)$ and $\mathcal{R} \cap \mathcal{R}^{-1} \subseteq I_{\mathbb{A}}$ **iff.** $\forall x,y \in \mathbb{A}\ \big((x,y) \in \mathcal{R} \wedge (x,y) \in \mathcal{R}^{-1} \to\ (x,y) \in I_{\mathbb{A}}\big)$, we just need to prove $\forall x,y \in \mathbb{A}$ $(x\mathcal{R}y \wedge y\mathcal{R}x \to x=y)$ **iff.** $\forall x,y \in \mathbb{A}\ \big((x,y) \in \mathcal{R} \wedge (x,y) \in \mathcal{R}^{-1} \to (x,y) \in I_{\mathbb{A}}\big)$.

$\forall x,y \in \mathbb{A}\ \ \big((x,y) \in \mathcal{R} \wedge (x,y) \in \mathcal{R}^{-1} \to (x,y) \in I_{\mathbb{A}}\big)$
iff. $\forall x,y \in \mathbb{A}\ \ \big((x,y) \in \mathcal{R} \wedge (x,y) \in \mathcal{R}^{-1} \to x=y\big)$
iff. $\forall x,y \in \mathbb{A}\ \ \big((x,y) \in \mathcal{R} \wedge (y,x) \in \mathcal{R} \to x=y\big)$
iff. $\forall x,y \in \mathbb{A}\ \ (x\mathcal{R}y \wedge y\mathcal{R}x \to x=y)$

QED

7. ***Disproof:***

$\mathcal{R}_1 \cup \mathcal{R}_2$ might not be an equivalence relation on $\mathbb{A}$.
For example:
$\mathbb{A} = \mathbb{N},\ \mathcal{R}_1 = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \text{ and } b \text{ are congruent modulo } 7\}$,
$\mathcal{R}_2 = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \text{ and } b \text{ are congruent modulo } 2\}$.
Since $9\mathcal{R}_1 2$ and $2\mathcal{R}_2 4$, $(9,2),(2,4) \in \mathcal{R}_1 \cup \mathcal{R}_2$. However, $(9,4) \notin \mathcal{R}_1 \cup \mathcal{R}_2$.
Therefore, $\mathcal{R}_1 \cup \mathcal{R}_2$ is not transitive, and is thereby not an equivalence relation on $\mathbb{A}$.

Thus, when $\mathcal{R}_1$ and $\mathcal{R}_2$ are equivalence relations on $\mathbb{A}$, $\mathcal{R}_1 \cup \mathcal{R}_2$ might not be an equivalence relation on $\mathbb{A}$.

QED

8. ***Proof:***

Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are equivalence relations,

$$(\forall a \in \mathbb{A}\ (\ (a,a) \in \mathcal{R}_1)\ ) \wedge (\forall a \in \mathbb{A}\ (\ (a,a) \in \mathcal{R}_2)\ ) \Rightarrow \forall a \in \mathbb{A}\ \big((a,a) \in \mathcal{R}_1 \wedge (a,a) \in \mathcal{R}_2\big)$$
$$\Rightarrow \forall a \in \mathbb{A}\ \big((a,a) \in \mathcal{R}_1 \cap \mathcal{R}_2\big).$$

Thus, $\mathcal{R}_1 \cap \mathcal{R}_2$ is reflexive.

Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are equivalence relations,

$$\big(\forall a,b \in \mathbb{A}\ (\ (a,b) \in \mathcal{R}_1 \to (b,a) \in \mathcal{R}_1\ )\big) \wedge\ \big(\forall a,b \in \mathbb{A}\ (\ (a,b) \in \mathcal{R}_2 \to (b,a) \in \mathcal{R}_2\ )\big)$$
$$\Rightarrow \forall a,b \in \mathbb{A}\ \big((a,b) \in \mathcal{R}_1 \to (b,a) \in \mathcal{R}_1\big) \wedge \big((a,b) \in \mathcal{R}_2 \to (b,a) \in \mathcal{R}_2\big)$$
$$\Rightarrow \forall a,b \in \mathbb{A}\ \Big(\ \big((a,b) \in \mathcal{R}_1 \wedge (a,b) \in \mathcal{R}_2\big) \to \big((b,a) \in \mathcal{R}_1 \wedge (b,a) \in \mathcal{R}_2\big)\ \Big)$$
$$\Rightarrow \forall a,b \in \mathbb{A}\ ((a,b) \in \mathcal{R}_1 \cap \mathcal{R}_2 \to (b,a) \in \mathcal{R}_1 \cap \mathcal{R}_2).$$

Thus, $\mathcal{R}_1 \cap \mathcal{R}_2$ is symmetric.

Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are equivalence relations,

$$\big(\forall a,b,c \in \mathbb{A}\ (a\mathcal{R}_1\mathrm{b} \wedge b\mathcal{R}_1\mathrm{c} \to a\mathcal{R}_1 c)\big) \wedge \big(\forall a,b,c \in \mathbb{A}\ (a\mathcal{R}_2\mathrm{b} \wedge b\mathcal{R}_2\mathrm{c} \to a\mathcal{R}_2 c)\big)$$
$$\Rightarrow \forall a,b,c \in \mathbb{A}\ \big((a\mathcal{R}_1\mathrm{b} \wedge b\mathcal{R}_1\mathrm{c} \to a\mathcal{R}_1 c) \wedge (a\mathcal{R}_2\mathrm{b} \wedge b\mathcal{R}_2\mathrm{c} \to a\mathcal{R}_2 c)\big)$$
$$\Rightarrow \forall a,b,c \in \mathbb{A}\ \big((a\mathcal{R}_1\mathrm{b} \wedge b\mathcal{R}_1\mathrm{c} \wedge a\mathcal{R}_2\mathrm{b} \wedge b\mathcal{R}_2\mathrm{c}) \to (a\mathcal{R}_1 c \wedge a\mathcal{R}_2 c)\big)$$
$$\Rightarrow \forall a,b,c \in \mathbb{A}\ \big((a,b) \in \mathcal{R}_1 \wedge (a,b) \in \mathcal{R}_2 \wedge (b,c) \in \mathcal{R}_1 \wedge (b,c) \in \mathcal{R}_2\big) \to \big((a,c) \in \mathcal{R}_1 \wedge (a,c) \in \mathcal{R}_2\big)$$
$$\Rightarrow \forall a,b,c \in \mathbb{A}\ \big((a,b) \in \mathcal{R}_1 \cap \mathcal{R}_2 \wedge (b,c) \in \mathcal{R}_1 \cap \mathcal{R}_2\big) \to \big((a,c) \in \mathcal{R}_1 \cap \mathcal{R}_2\big)$$

Thus, $\mathcal{R}_1 \cap \mathcal{R}_2$ is transitive.

Since $\mathcal{R}_1 \cap \mathcal{R}_2$ is reflexive, symmetric and transitive, $\mathcal{R}_1 \cap \mathcal{R}_2$ is an equivalence relation on $\mathbb{A}$.

QED