Mathematical Logic Homework 05

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1 Provability in Sentential Logic

1.1 $(A \wedge B) \vee (\neg A \vee \neg B)$ is Provable

Proof. We can construct the following proof tree.

$$\dfrac{\{A\vee\neg A\} \quad \dfrac{\{B\vee\neg B\} \quad \dfrac{\dfrac{[A]\quad[B]}{A\wedge B}\wedge\text{-I}}{(A\wedge B)\vee(\neg A\vee\neg B)}\vee\text{-I1} \quad \dfrac{\dfrac{[\neg B]}{\neg A\vee\neg B}\vee\text{-I2}}{(A\wedge B)\vee(\neg A\vee\neg B)}\vee\text{-I2}}{(A\wedge B)\vee(\neg A\vee\neg B)}\vee\text{-E} \quad \dfrac{\dfrac{[\neg A]}{\neg A\vee\neg B}\vee\text{-I1}}{(A\wedge B)\vee(\neg A\vee\neg B)}\vee\text{-I2}}{(A\wedge B)\vee(\neg A\vee\neg B)}\vee\text{-E}$$

Thus, exists a proof tree of $(A \wedge B) \vee (\neg A \vee \neg B)$ without any undischarged assumptions.

i.e. $\vdash (A \wedge B) \vee (\neg A \vee \neg B)$,

i.e. $(A \wedge B) \vee (\neg A \vee \neg B)$ is provable. ■

1.2 $(A \wedge B) \vee (\neg A \wedge \neg B)$ is Not Provable

Proof. The truth table of $(A \wedge B) \vee (\neg A \wedge \neg B)$ is as follows.

A	B	$(A \wedge B) \vee (\neg A \wedge \neg B)$
True	True	True
True	False	False
False	True	False
False	False	True

Therefore, $\nvDash (A \wedge B) \vee (\neg A \wedge \neg B)$.

By **Soundness Thm.**, for any wff α, $\vdash \alpha \implies \vDash \alpha$, i.e. $\nvDash \alpha \implies \nvdash \alpha$.

Therefore, $(A \wedge B) \vee (\neg A \wedge \neg B)$ is not provable. ■

2 Translation into wffs in First-Order Logic

2.1 There is No Such A Set that Every Set is Its Member

Solution. There is no such a set that every set is its member.

($\neg$ there is such a set that every set is its member).

($\neg$ $\exists$ x such that every set is its member).

($\neg$ $\exists$ x $\forall$ y, y is a member of x).

$\underline{\boldsymbol{(\neg\ \exists x\ \forall y \quad y \in x)}}$. ■

2.2 Problem 2.2

Solution. Every farmer who owns a donkey needs hay, and every farmer who owns a donkey beats it.

(Every farmer who owns a donkey needs hay $\wedge$ every farmer who owns a donkey beats it)

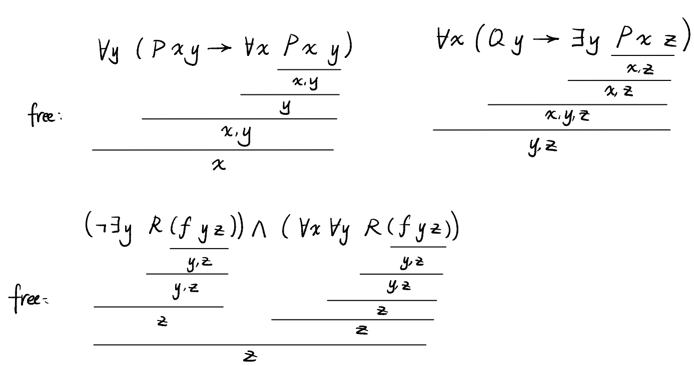
($\forall x$ (x is a farmer and owns a donkey $\rightarrow$ x needs hay) $\wedge$ $\forall x$ (x is a farmer and owns a donkey $\rightarrow$ x beats the donkey))

($\forall x$ (($F\ x \wedge \exists y$, y is a donkey and x owns y) $\rightarrow H\ x$)) $\wedge$ ($\forall x$ (($F\ x \wedge \exists y$, y is a donkey and x owns y) $\rightarrow$ x beats y))

$\underline{(\forall x\ ((F\ x \wedge \exists y\ (D\ y\ \wedge\ O\ x\ y)) \rightarrow H\ x) \wedge (\forall x\ (F\ x \wedge \exists y\ (D\ y\ \wedge\ O\ x\ y)) \rightarrow B\ x\ y))}$ ■

3 Variables Occurring Free

Solution.



Thus, variables occurring free in each wff are as follows.

wff	variables occurring free
$\forall y\ (P\ x\ y \rightarrow \forall x\ P\ x\ y)$	x
$\forall x(Qx \rightarrow \exists y\ P\ x\ z)$	y, z
$(\neg\exists y\ R(f\ y\ z)) \wedge (\forall x \forall y\ R(f\ y\ z))$	z

■

4 Problem 4

4.1 $\models_{\mathfrak{N}} \exists v_0,\ v_0 \dot{+} v_0 \doteq v_1[s]$

Solution. There exists an assignment $s(v_0|1)$ s.t. $\overline{s(v_0|1)}(v_0) + \overline{s(v_0|1)}(v_0) = 1 + 1 = \overline{s(v_0|1)}(v_1) = 2$,

i.e. $\models_{\mathfrak{A}} v_0 \dot{+} v_0 \doteq v_1[s(v_0|1)]$.

Thus, $\models_{\mathfrak{N}} \exists v_0,\ v_0 \dot{+} v_0 \doteq v_1[s]$. ■

4.2 $\not\models_{\mathfrak{N}} \exists v_0,\ v_0 \dot{\times} v_0 \doteq v_1[s]$

Solution. Assume $\models_{\mathfrak{N}} \exists v_0,\ v_0 \dot{\times} v_0 \doteq v_1[s]$.

Then Exists an assignment $s(v_0|a)$ s.t. $\overline{s(v_0|a)}(v_0) \times \overline{s(v_0|a)}(v_0) = \overline{s(v_0|a)}(v_1) = a \times a = 2$,

i.e. $a = \sqrt{2} \notin |\mathfrak{N}| = \mathbb{N}$. <u>**Contradiction.**</u>

Thus, $\not\models_{\mathfrak{N}} \exists v_0,\ v_0 \dot{\times} v_0 \doteq v_1[s]$. ■

4.3 $\models_{\mathfrak{N}} \forall v_0 \exists v_1\ v_0 \doteq v_1[s]$

Solution. For any $a \in |\mathfrak{N}| = \mathbb{N}$, exists an assignment $s(v_0|a)(v_1|b)$ where $b = a$

s.t. $\overline{s(v_0|a)(v_1|b)}(v_0) = \overline{s(v_0|a)(v_1|b)}(v_1) = a.$

i.e. for any $a \in |\mathfrak{N}| = \mathbb{N}$, exists $b = a$ s.t. $\models_{\mathfrak{A}} v_0 \doteq v_1[s(v_0|a)(v_1|b)]$.

i.e. for any $a \in |\mathfrak{N}| = \mathbb{N}$, $\models_{\mathfrak{A}} \exists v_1\ v_0 \doteq v_1[s(v_0|a)]$.

Thus, $\models_{\mathfrak{N}} \forall v_0 \exists v_1\ v_0 \doteq v_1[s]$. ■

4.4 $\models_{\mathfrak{N}} \forall v_0 \forall v_1\ v_0 \dot{+} \dot{1} \dot{<} v_1 \to \exists v_2\ v_0 \dot{<} v_2 \land v_2 \dot{<} v_1[s]$

Solution. For any $a, b \in |\mathfrak{N}| = \mathbb{N}$,

CASE 1. When $a + 1 < b$. In this case, $\models_{\mathfrak{A}} v_0 \dot{+} \dot{1} \dot{<} v_1[s(v_0|a)(v_1|b)]$.

There exists an assignment $\hat{s} = s(v_0|a)(v_1|b)(v_2|c)$ where $c = a + 1$

s.t. $\models_{\mathfrak{A}} v_0 \dot{<} v_2 \land v_2 \dot{<} v_1[\hat{s}]$.

CASE 2. When $a + 1 \geq b$. In this case, $\not\models_{\mathfrak{A}} v_0 \dot{+} \dot{1} \dot{<} v_1[s(v_0|a)(v_1|b)]$.

Thus, for any $a, b \in |\mathfrak{N}| = \mathbb{N}$, we have

$$\models_{\mathfrak{A}} v_0 \dot{+} \dot{1} \dot{<} v_1[s(v_0|a)(v_1|b)] \Longrightarrow \models_{\mathfrak{A}} \exists v_2\ v_0 \dot{<} v_2 \land v_2 \dot{<} v_1[s(v_0|a)(v_1|b)]$$

i.e. for any $a, b \in |\mathfrak{N}| = \mathbb{N}$, $\models_{\mathfrak{A}} v_0 \dot{+} \dot{1} \dot{<} v_1 \to \exists v_2\ v_0 \dot{<} v_2 \land v_2 \dot{<} v_1[s(v_0|a)(v_1|b)]$.

Therefore, $\models_{\mathfrak{N}} \forall v_0 \forall v_1\ v_0 \dot{+} \dot{1} \dot{<} v_1 \to \exists v_2\ v_0 \dot{<} v_2 \land v_2 \dot{<} v_1[s]$. ■

4.5 $\not\models_{\mathfrak{N}} \forall v_0 \forall v_1 \; v_0 \dot{<} v_2 \land v_2 \dot{<} v_1 [s]$

Solution. Exists $a = 80 \in |\mathfrak{N}| = \mathbb{N}, \; b = 1 \in |\mathfrak{N}| = \mathbb{N}$

s.t. $\overline{s(v_0|a)(v_1|b)}(v_0) = 80 \geq \overline{s(v_0|a)(v_1|b)}(v_2) = 4$ and

$\overline{s(v_0|a)(v_1|b)}(v_2) = 4 \geq \overline{s(v_0|a)(v_1|b)}(v_1) = 1$.

i.e. $\not\models_{\mathfrak{A}} v_0 \dot{<} v_2 \land v_2 \dot{<} v_1 [s(v_0|a)(v_1|b)]$.

Thus, $\not\models_{\mathfrak{N}} \forall v_0 \forall v_1 \; v_0 \dot{<} v_2 \land v_2 \dot{<} v_1 [s]$. ■

5 $\models_{\mathfrak{A}} (\alpha \to \forall x \; \alpha) [s]$ If x Does Not Occur Free In α

Proof. Recall the following theorem.

Thm. *Let $\mathfrak{A}$ be a structure for $\mathbb{L}$, s_1 and s_2 be two assignments for $\mathfrak{A}$ and ϕ be a wff for $\mathbb{L}$. If $s_1(y) = s_2(y)$ for every y that occurs free in ϕ, then*

$$\models_{\mathfrak{A}} \phi[s_1] \iff \models_{\mathfrak{A}} \phi[s_2]$$ □

The proof of the proposition that $\models_{\mathfrak{A}} (\alpha \to \forall x \; \alpha) [s]$ if x does not occur free in α is as follows.

Since x does not occur free in α, we know for any $a \in |\mathfrak{A}|$, $s(x|a)(y) = s(y)$ for any variable y occurring free in α (since $y \neq x$).

Thus, by **Theorem**, we have $\models_{\mathfrak{A}} \alpha[s] \iff \models_{\mathfrak{A}} \alpha[s(x|a)]$ for any $a \in |\mathfrak{A}|$.

Thus, when $\models_{\mathfrak{A}} \alpha[s]$, we have $\models_{\mathfrak{A}} \alpha[s(x|a)]$ for any $a \in |\mathfrak{A}|$.

i.e. When $\models_{\mathfrak{A}} \alpha[s]$, we have $\models_{\mathfrak{A}} \forall x \; \alpha[s]$.

Therefore, $\models_{\mathfrak{A}} (\alpha \to \forall x \; \alpha) [s]$. ■

6 Sufficient and Necessary Condition for Monoid

Solution. The sentence σ should be

$$(\forall x \; (x \mathbin{\dot{\circ}} \dot{e} \doteq x \;\land\; \dot{e} \mathbin{\dot{\circ}} x \doteq x) \;\land\; \forall x \forall y \forall z \;\; (x \mathbin{\dot{\circ}} y) \mathbin{\dot{\circ}} z \doteq x \mathbin{\dot{\circ}} (y \mathbin{\dot{\circ}} z))$$

Now we prove that for any structure $\mathfrak{A}$, $|\mathfrak{A}|$ is a monoid with $\dot{e}^{\mathfrak{A}}$ as the identity and $\dot{\circ}^{\mathfrak{A}}$ as the associative operator **iff.** $\models_{\mathfrak{A}} \sigma$.

Sufficiency. Suppose $\models_{\mathfrak{A}} \sigma$.

Then $\models_{\mathfrak{A}} \forall x \; (x \mathbin{\dot{\circ}} \dot{e} \doteq x \;\land\; \dot{e} \mathbin{\dot{\circ}} x \doteq x)$ and $\models_{\mathfrak{A}} \forall x \forall y \forall z \;\; (x \mathbin{\dot{\circ}} y) \mathbin{\dot{\circ}} z \doteq x \mathbin{\dot{\circ}} (y \mathbin{\dot{\circ}} z)$.

i.e. for any $a \in |\mathfrak{A}|$, $a \mathbin{\dot{\circ}} e = e \mathbin{\dot{\circ}} a = a$.

For any $a, b, c, \in |\mathfrak{A}|, (a \mathbin{\dot{\circ}} b) \mathbin{\dot{\circ}} c \doteq a \mathbin{\dot{\circ}} (b \mathbin{\dot{\circ}} c)$.

Thus, $|\mathfrak{A}|$ is a monoid with $\dot{e}^{\mathfrak{A}}$ as the identity and $\dot{\circ}^{\mathfrak{A}}$ as the associative operator. □

Necessity. Assume $|\mathfrak{A}|$ is a monoid with $\dot{e}^{\mathfrak{A}}$ as the identity and $\dot{\circ}^{\mathfrak{A}}$ as the associative operator.

Then for any $a \in |\mathfrak{A}|$, $a \mathbin{\dot{\circ}} e = e \mathbin{\dot{\circ}} a = a$.

For any $a, b, c \in |\mathfrak{A}|$, $(a \mathbin{\dot{\circ}} b) \mathbin{\dot{\circ}} c \doteq a \mathbin{\dot{\circ}} (b \mathbin{\dot{\circ}} c)$.

Since $\overline{s(w|d)}(w) = d$ for any assignment s for $\mathfrak{A}$ and any $d \in |\mathfrak{A}|$, we know

For any assignment s for $\mathfrak{A}$ and any $a \in |\mathfrak{A}|$,

$$\vDash_{\mathfrak{A}} (x \mathbin{\dot{\circ}} \dot{e} \doteq x \ \wedge\ \dot{e} \mathbin{\dot{\circ}} x \doteq x)[s(x|a)].$$

For any assignment s for $\mathfrak{A}$ and any $a, b, c \in |\mathfrak{A}|$,

$$\vDash_{\mathfrak{A}} (x \mathbin{\dot{\circ}} y) \mathbin{\dot{\circ}} z \doteq x \mathbin{\dot{\circ}} (y \mathbin{\dot{\circ}} z)\,[s(x|a)(y|b)(z|c)].$$

i.e. for any assignment s for $\mathfrak{A}$,

$$\vDash_{\mathfrak{A}} \forall x\, (x \mathbin{\dot{\circ}} \dot{e} \doteq x \wedge \dot{e} \mathbin{\dot{\circ}} x \doteq x)[s] \text{ and } \vDash_{\mathfrak{A}} \forall x \forall y \forall z\ (x \mathbin{\dot{\circ}} y) \mathbin{\dot{\circ}} z \doteq x \mathbin{\dot{\circ}} (y \mathbin{\dot{\circ}} z)\,[s].$$

i.e. for any assignment s for $\mathfrak{A}$,

$$\vDash_{\mathfrak{A}} \forall x\, (x \mathbin{\dot{\circ}} \dot{e} \doteq x \ \wedge\ \dot{e} \mathbin{\dot{\circ}} x \doteq x) \wedge \vDash_{\mathfrak{A}} \forall x \forall y \forall z\ (x \mathbin{\dot{\circ}} y) \mathbin{\dot{\circ}} z \doteq x \mathbin{\dot{\circ}} (y \mathbin{\dot{\circ}} z)\,[s].$$

i.e. $\vDash_{\mathfrak{A}} \forall x\, (x \mathbin{\dot{\circ}} \dot{e} \doteq x \ \wedge\ \dot{e} \mathbin{\dot{\circ}} x \doteq x) \wedge \vDash_{\mathfrak{A}} \forall x \forall y \forall z\ (x \mathbin{\dot{\circ}} y) \mathbin{\dot{\circ}} z \doteq x \mathbin{\dot{\circ}} (y \mathbin{\dot{\circ}} z)$.

i.e. $\vDash_{\mathfrak{A}} \sigma$. $\square$

In conclusion, for any structure $\mathfrak{A}$, $|\mathfrak{A}|$ is a monoid with $\dot{e}^{\mathfrak{A}}$ as the identity and $\dot{\circ}^{\mathfrak{A}}$ as the associative operator **iff.** $\vDash_{\mathfrak{A}} \sigma$. $\blacksquare$