[Solution of Homework 1]

Probability Space of Tossing Coins

Let us construct the probability space of tossing an infinite sequence of independent fair coins. Let $\Omega = \{0,1\}^*$. We can write each $\omega \in \Omega$ as an infinite sequence $\omega = (\omega_1, \omega_2, \ldots)$ where $\omega_i \in \{0,1\}$.

1. Let $n \in \mathbb{N}$. For every $s = (s_1, \ldots, s_n) \in \{0,1\}^n$, let

$$C_s = \{\omega \in \Omega \mid \omega_1 = s_1, \ldots, \omega_n = s_n\}.$$

   Prove that for every $n \in \mathbb{N}$, the collection $\{C_s\}_{s \in \{0,1\}^n}$ forms a partition of Ω.
   Proof.
   For any $\omega \in \Omega$, there is exactly one $s = (\omega_1, \omega_2, \ldots, \omega_n) \in \{0,1\}^n$ such that $\omega \in C_s$. Therefore, $\cup_{s \in \{0,1\}^n} C_s = \Omega$ and $C_{s_1} \cap C_{s_2} = \varnothing$ for any $s_1 \neq s_2 \in \{0,1\}^n$, which is to say $\{C_s\}_{s \in \{0,1\}^n}$ forms a partition of Ω.

2. Let $\mathcal{F}_n$ be the σ-algebra generated by $\{C_s\}_{s \in \{0,1\}^n}$ (that is, the minimal σ-algebra containing sets in $\{C_s\}_{s \in \{0,1\}^n}$). Note that $\mathcal{F}_n$ is called the σ-algebra of *tossing n coins*. Prove that there exists a bijection between $\mathcal{F}_n$ and $2^{\{0,1\}^n}$.
   Proof.
   We construct a map $f : \mathcal{F}_n \to 2^{\{0,1\}^n}$ for any $A \in \mathcal{F}_n$:

$$f(A) = \cup_{\omega \in A} \{(\omega_1, \omega_2, \ldots, \omega_n)\}.$$

   For any $S = \{s^1, \ldots, s^k\} \in 2^{\{0,1\}^n}$, $f(\cup_{i=1}^k C_{s^i}) = S$. So f is surjective.
   Since both $\mathcal{F}_n$ and $2^{\{0,1\}^n}$ are of size 2^{2^n}, we can infer that f is a bijection between $\mathcal{F}_n$ and $2^{\{0,1\}^n}$.

3. Prove that $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \ldots$ is increasing. The collection $\{\mathcal{F}_n\}_{n \geq 1}$ is called a *filtration*.
   Proof.
   We will prove $\mathcal{F}_n \subsetneq \mathcal{F}_{n+1}$ for every n.
   Let $f : \mathcal{F}_n \to 2^{\{0,1\}^n}$ be the bijection defined in Problem 2. For any $A \in \mathcal{F}_n$, we write it as $\cup_{s \in f(A)} C_s$. For any $s = (s_1, s_2, \ldots, s_n) \in \{0,1\}^n$, we write $C_s = C_{(s_1, s_2, \ldots, s_n, 0)} \cup C_{(s_1, s_2, \ldots, s_n, 1)}$. Therefore, for any $A \in \mathcal{F}_n$,

$A = \cup_{s \in f(A)} \left(C_{(s_1,s_2,...,s_n,0)} \cup C_{(s_1,s_2,...,s_n,1)} \right) \in \mathcal{F}_{n+1}$, which implies $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. It is obvious that $\mathcal{F}_n \neq \mathcal{F}_{n+1}$ (For example, $C_{(s_1,s_2,...,s_{n+1})} \in \mathcal{F}_{n+1}$ but not in $\mathcal{F}_n$), so $\mathcal{F}_n \subsetneq \mathcal{F}_{n+1}$.

4. Let $\mathcal{F}_\infty = \bigcup_{n \geq 1} \mathcal{F}_n$. Prove that $\mathcal{F}_\infty$ is an algebra[1] (not necessarily a σ-algebra) and $\mathcal{F}_\infty \neq 2^\Omega$.
   Proof.
   For any $A \in \mathcal{F}_\infty$, there exists i such that $A \in \mathcal{F}_i$, so $A^c \in \mathcal{F}_i \subset \mathcal{F}_\infty$. For any $A, B \in \mathcal{F}_\infty$, there exist i, j such that $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$, so $A \cup B \in \mathcal{F}_{\max\{i,j\}} \subset \mathcal{F}_\infty$. Therefore, $\mathcal{F}_\infty$ is an algebra.
   2^Ω is not countable. $\mathcal{F}_n$ is countable for any n, so $\mathcal{F}_\infty$ is also countable. Therefore, $\mathcal{F}_\infty \neq 2^\Omega$. (In the following problem, we will show that any $\omega \in \Omega$, $\{\omega\} \in 2^\Omega \setminus \mathcal{F}_\infty$.)

5. Let $\mathcal{B}(\Omega) \triangleq \sigma(\mathcal{F}_\infty)$ be the minimal σ-algebra containing $\mathcal{F}_\infty$. Prove that for any $\omega \in \Omega$, it holds that $\{\omega\} \in \mathcal{B}(\Omega) \setminus \mathcal{F}_\infty$.
   Proof.
   For any $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, $\{\omega\} \in 2^\Omega$. However, there doesn't exist i such that $\{\omega\} \in \mathcal{F}_i$, hence $\{\omega\} \notin \mathcal{F}_\infty$.
   For any $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$ and n, $\omega \in C_{(\omega_1,\omega_2,...,\omega_n)}$. Therefore, $\{\omega\} = \cap_{n \geq 1} C_{(\omega_1,\omega_2,...,\omega_n)} \in \mathcal{B}(\Omega)$.

   (Notes that we use the union operation to define the σ-algebra, but for any $A_i \in \mathcal{F}, i \in \mathbb{N}$, we have $\overline{\cup_i \bar{A}_i} = \cap_i A_i \in \mathcal{F}$ because $\bar{A}_i \in \mathcal{F}$.)

6. Prove that for every $A \in \mathcal{F}_\infty$, there exist some $n \in \mathbb{N}$ and $s_1, \ldots, s_k \in \{0,1\}^n$ such that $A = C_{s_1} \cup \cdots \cup C_{s_k}$. Although the choice of n might not be unique, prove that the value $\frac{k}{2^n}$ only depends on A.
   Proof.
   There exists n such that $A \in \mathcal{F}_n$. Let $f_n : \mathcal{F}_n \to 2^{\{0,1\}^n}$ be the bijection defined in Problem 2. Let $f_n(A) = \{s^1, s^2, \ldots, s^{k_n}\}$. Then $A = C_{s^1} \cup \cdots \cup C_{s^{k_n}}$.
   Let n be the minimum index such that $A \in \mathcal{F}_n$. We can also find a set $S' = \{(s_1, s_2, \cdots, s_n, 0), (s_1, s_2, \cdots, s_n, 1) | s = (s_1, s_2, \ldots, s_n) \in f_n(A)\} \in 2^{\{0,1\}^{n+1}}$ such that $A = \cup_{s' \in S'} C_{s'}$ and $k_{n+1} := |S'| = 2k_n$. Therefore, $\frac{k_{n+1}}{2^{n+1}} = \frac{2k_n}{2^{n+1}} = \frac{k_n}{2^n}$. Applying this procedure inductively, we obtain that for any $i > n$, the value $\frac{k_i}{2^i} = \frac{k_n}{2^n}$, which is to say that $\frac{k}{2^n}$ only depends on A.

7. Prove that there exists a unique probability measure $P : \mathcal{B}(\Omega) \to [0,1]$ satisfying for every $A \in \mathcal{F}_\infty$, $P(A) = \frac{k}{2^n}$ where k and n are defined in the last question.
   Proof.
   We define a measure μ on F_∞ that $\mu(A) = \frac{k}{2^n}$ where k and n are defined in the last question:

   1. $\mu(C_s) = \frac{1}{2^n}$ for $s \in \{0,1\}^n$.

2. $\mu(A) = \sum_{i=1}^{k} P(C_{s^k}) = \frac{k}{2^n}$ for $A = \cup_{i=1}^{k} C_{s^k} \in \mathcal{F}_n$.
   For any disjoint sets $A_1, A_2, \cdots \in \mathcal{F}_\infty$ such that $\cup_{n\geq 1} A_n \in \mathcal{F}_\infty$, assuming $A_i = \cup_{s \in S_i} C_s$, we obtain that $A = \cup_{s\in S_i, i \in \mathbb{N}^+} C_s$.
   Therefore,

$$\mu(\cup_{n\geq 1} A_n) = \sum_{n\geq 1} \mu(A_n),$$

   and it is obvious that $\mu(\Omega) = \sum_{s\in 2^{\{0,1\}^n}} \mu(C_s) = 1$.
   And then we extend the measure μ on $\mathcal{F}_\infty$ to a measure on $\mathcal{B}(\Omega)$ by *Carathéodory Extension Theorem*. There exists a unique measure $P : \mathcal{B}(\Omega) \to [0, 1]$ such that $P(A) = \mu(A)$ for any $A \in \mathcal{F}_\infty$. Since $P(\Omega) = \mu(\Omega) = 1$, P is a probability measure.

Then $(\Omega, \mathcal{B}(\Omega), P)$ is our probability space for tossing coins, and it is isomorphic to the Lebesgue measure on $[0, 1]$.

8. Formalize $X \sim \texttt{Geom}(1/2)$ in this probability space.
   Solution.
   For any $\omega \in \Omega$, $X(\omega) := \min\{i \in \mathbb{N} | \omega_i = 1\}$.

Conditional Expectation

1. Let X be a random variable and $f : \mathbb{R} \to \mathbb{R}$ be a Borel function. We usually use $f(X)$ to denote the random variable: $\omega \in \Omega \mapsto f(X(\omega)) \in \mathbb{R}$. Prove that $f(X)$ is $\sigma(X)$-measurable.
   Proof.
   For any Borel set $B \subseteq \mathbb{R}$, $(f \circ X)^{-1}(B) = X^{-1}(f^{-1}(B)) \in \sigma(X)$.

2. Let Y, Y' be two random variables such that $\sigma(Y) = \sigma(Y')$. Prove that $\mathbf{E}[X \mid Y] = \mathbf{E}[X \mid Y']$.
   Proof.
   It suffices to show that $Y^{-1}(Y(\omega)) = Y'^{-1}(Y'(\omega))$. Since $\sigma(Y) = \sigma(Y')$, if there exists $\omega \in \Omega$ such that $Y^{-1}(Y(\omega)) \neq Y'^{-1}(Y'(\omega))$, $Y^{-1}(Y(\omega)) \cap Y'^{-1}(Y'(\omega)) \subseteq \sigma(Y)$, contradicting to the definition of $Y^{-1}(Y(\omega))$ and $Y'^{-1}(Y'(\omega))$. Therefore, $\mathbf{E}\left[X | Y^{-1}(Y(\omega))\right] = \mathbf{E}\left[X | Y'^{-1}(Y'(\omega))\right]$

3. The fact you just proved should convince you that the conditional expectation $\mathbf{E}[X \mid Y]$ only depends on the σ-algebra $\sigma(Y)$ (but not the value of Y). Let Ω be the set of outcomes and $X : \Omega \to \mathbb{R}$ be a random variable. Let $\mathcal{F}$ be a σ-algebra on Ω. Can you define the notation $\mathbf{E}[X \mid \mathcal{F}]$?
   Solution.
   Let $Y = \mathbf{E}[X \mid \mathcal{F}]$ such that $\int_A Y dP = \int_A X dP$ for any $A \in \mathcal{F}$ and Y is $\mathcal{F}$-

measurable.
If Ω is a countable set, we can explicitly give a definition of Y. First we need to prove the following lemma.

Lemma: any σ-algebra on a countable set has a unique partition.

- (Existence). For any $A \in \mathcal{F}$, if there exist a subset $A' \subset A$ and $A' \in \mathcal{F}$, we split A into $A \cap A'$ and $A \setminus A'$. Moerover, we repeat this procedure for $A \cap A'$ and $A \setminus A'$ respectively; otherwise A is a partition itself. Note that Ω is countable, therefore, the first step will proceed for at most countable times. According to the defintion of σ-algebra, we know the intersection of the set chain lies in $\mathcal{F}$ which claims the existence of the partition.
- (Uniqueness). There can only be one partition. Suppose there are two partition M_i and M_i', then there exists $M \in M_i$ and $M' \in M_i'$ such that $M \cap M \neq \varnothing$. It means that neither M_i nor M_i' is a partition.

Armed with above lemma, assume that the partition of $\mathcal{F}$ is formed by $\{M_i\}_{i>0}$. For any $\omega \in M_i$, $Y(\omega) = \mathbf{E}[X|\mathcal{F}](\omega) = \mathbf{E}[X|M_i] = \frac{\sum_{a \in Ran(X)} a \cdot P(X^{-1}(a) \cap M_i)}{P(M_i)}$.

4. (The coarser always wins) Let $\mathcal{F}_1, \mathcal{F}_2$ be two σ-algebra such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $X : \Omega \to \mathbb{R}$ be a random variable. Prove that $\mathbf{E}[\mathbf{E}[X \mid \mathcal{F}_1] \mid \mathcal{F}_2] = \mathbf{E}[\mathbf{E}[X \mid \mathcal{F}_2] \mid \mathcal{F}_1] = \mathbf{E}[X \mid \mathcal{F}_1]$.
Proof.
For $A \in \mathcal{F}_1 \subset \mathcal{F}_2$,

$$\int_A \mathbf{E}[X|\mathcal{F}_1]\, dP = \int_A X dP = \int_A \mathbf{E}[X|\mathcal{F}_2]\, dP.$$

According to the definition of conditional expectation, we know if X is $\mathcal{F}$-measurable, then $\mathbf{E}[X|\mathcal{F}] = X$.

Therefore, $\mathbf{E}[\mathbf{E}[X \mid \mathcal{F}_2] \mid \mathcal{F}_1] = \mathbf{E}[\mathbf{E}[X \mid \mathcal{F}_1] \mid \mathcal{F}_1] = \mathbf{E}[X \mid \mathcal{F}_1]$.
$\mathbf{E}[X \mid \mathcal{F}_1]$ is $\mathcal{F}_1$-measurable, therefore $\mathcal{F}_2$-measurble, so $\mathbf{E}[\mathbf{E}[X \mid \mathcal{F}_1] \mid \mathcal{F}_2] = \mathbf{E}[X \mid \mathcal{F}_1]$.

1. A set $\mathcal{F}$ is an algebra if for every $A, B \in \mathcal{F}$, it holds $A^c \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$. ↩