Probability Theory and Mathematical Statistics 概率统计

Homework 1011-1014

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2-41. 解：由题意可知 X 的分布列：

X	-2	-1	2
P	0.3	0.6	0.1

$\therefore Y=X^2-3$ 分布律为

Y	1	-2
	0.4	0.6

$Z=|X|$ 分布律为

Z	2	1
	0.4	0.6

□

2-42. 解：(1) $P(Y_1=-1)=P(X<0)=\frac{0-(-2)}{3-(-2)}=\frac{2}{5}$. $P(Y_1=1)=1-\frac{2}{5}=\frac{3}{5}$.

Y_1 的分布律为

Y	-1	1
	$\frac{2}{5}$	$\frac{3}{5}$

□

(2) $F_Y(y)=P(Y_2\le y)=P\left(\frac{X+1}{2}\le y\right)=P(X\le 2y-1)=F_X(2y-1)$

$$\therefore f_{Y_2}(y)=\frac{dF_Y(y)}{dy}=f_X(2y-1)\cdot 2$$

$$=\begin{cases}0, & y<-\frac{1}{2} \text{ 或 } y\ge 2\\ \frac{2}{5}, & -\frac{1}{2}\le y<2\end{cases}$$ □

43. 解：(1) ① $y>0$

$F_{Y_1}(y)=P(Y_1\le y)=P(X^2\le y)=P(-\sqrt{y}\le X\le\sqrt{y})=2\left(\Phi(\sqrt{y})-\frac{1}{2}\right)$

$$=2\Phi(\sqrt{y})-1=2\int_{-\infty}^{\sqrt{y}}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du-1$$

$$f_{Y_1}(y)=2\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{y}{2}}\cdot\frac{1}{2}\frac{1}{\sqrt{y}}=\frac{1}{\sqrt{2\pi y}}e^{-\frac{y}{2}}$$

② $y\le 0$ $f_{Y_1}(y)=0$

$$\therefore f_{Y_1}(y)=\begin{cases}\frac{1}{\sqrt{2\pi y}}e^{-\frac{y}{2}} & (y>0)\\ 0 & (y\le 0)\end{cases}$$ □

(2) 解: $Y_2 = e^{-X} \in (0, +\infty)$ $\therefore P(Y_2 \leq 0) = 0 \Rightarrow f_{Y_2}(y) = 0$

$y>0$ 时, $F_{Y_2}(y) = P(Y_2 \leq y) = P(e^{-X} \leq y) = P(X \geq -\ln y)$

$$= 1 - \Phi(-\ln y) = 1 - \int_{-\infty}^{-\ln y} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

$$f_{Y_2}(y) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln y)^2}{2}} = -\frac{1}{\sqrt{2\pi}} \cdot y^{-\frac{\ln y}{2}}$$

$$\therefore f_{Y_2}(y) = \begin{cases} -\frac{1}{\sqrt{2\pi}} y^{-\frac{\ln y}{2}} & (y>0) \\ 0 & (y \leq 0) \end{cases}$$ □

(3) 解: $Y_3 = X + |X| = \begin{cases} 2X & (X>0) \\ 0 & (X \leq 0) \end{cases}$ $\therefore Y_3 \in [0, +\infty)$

$y \geq 0$ 时, $F_{Y_3}(y) = P(2X \leq y) = P(X \leq \frac{y}{2}) = \Phi(\frac{y}{2})$

$y<0$ 时. $F_{Y_3}(y) = 0$

$$\therefore F_{Y_3}(y) = \begin{cases} \Phi(\frac{y}{2}) & (y \geq 0) \\ 0 & (y<0) \end{cases}$$ □

2-44. 解: $X \in \mathbb{R} \Rightarrow Y \in \mathbb{R}$

$$F_Y(y) = P(Y \leq y) = P(3-2X \leq y) = P(X \geq \frac{3-y}{2}) = 1 - P(X < \frac{3-y}{2})$$

$$= 1 - F(\frac{3-y}{2} - 0).$$ □

补充1. 解: 记误差为E. 由题意知 $E \sim U(-1, 1)$

(1) $P(|E| \leq 0.6) = P(E \leq 0.6) - P(E < -0.6) = \frac{1}{2} \cdot (0.6 - (-0.6+0)) = 0.6$

由题意. $Y \sim B(10, 0.6)$

$P(Y \geq 2) = 1 - P(Y=0) - P(Y=1) = 1 - (1-0.6)^{10} - C_{10}^1 \cdot 0.6 (1-0.6)^9$

$= 0.99832$ □

(2) 记测量n次（独立测量）中误差绝对值不超过0.6m的次数为Y_n.

则有 $Y_n \sim B(n, 0.6)$　　　$(n \geq 1)$

$P(Y_n \geq 1) = 1 - P(Y_n = 0) = 1 - (1-0.6)^n > 0.9 \Rightarrow 0.4^n < 0.1$

$\therefore n > \log_{0.4} 0.1$.　　n为自然数.　故 $n_{min} = 3$　　□

(3) $\because X \sim N(\mu, 0.25)$，$E = X - \mu$. $\Rightarrow E \sim N(0, 0.25)$　($\sigma^2 = 0.25$!)

此时 $P(|E| \leq 0.6) = 2\Phi\left(\frac{0.6}{0.25}\right) - 1 = 2\Phi(2.4) - 1 = 2 \times 0.9918 - 1$

$= 0.9836$

(i) $Y \sim B(20, 0.9836)$　（由题意是二项分布）

(ii) 记测量n次（独立测量）中误差绝对值不超过0.6m的次数为Y_n.

则 $Y_n \sim B(n, 0.9836)$　　　$(n \geq 1)$

$P(Y_n \geq 1) = 1 - P(Y_n = 0) = 1 - (1-0.9836)^n > 0.9 \Rightarrow 0.0164^n < 0.1$

$\therefore n > \log_{0.0164} 0.1$　　n为自然数　　故 $n_{min} = 1$　　□

开放1. 解：对各个地区的成绩分别计算平均值和方差，分别记为$\bar{x}_i$和σ_i^2.

对某地区考生成绩x（若在i地区）作以下标准化：$\tilde{x} = \frac{x - \bar{x}_i}{\sigma_i^2}$

这样得到的$\tilde{x}$间具有可比性（将每个地区的成绩都标准化为$N(0, 1)$分布）.

□

10/14 周四

3-1. 解：$P(X=0, Y=0) = P(X=1, Y=0) = P(X=0, Y=1) = P(X=3, Y=2) = 0$

$P(X=1, Y=1) = \frac{C_3^1 C_2^1}{C_7^4} = \frac{6}{35}$　　$P(X=2, Y=0) = \frac{C_3^2}{C_7^4} = \frac{3}{35}$

$P(X=0, Y=2) = \frac{C_2^2}{C_7^4} = 1/35$　　$P(X=2, Y=1) = \frac{C_3^2 C_2^1 C_2^1}{C_7^4} = \frac{12}{35}$

$P(X=3,\ Y=0) = \dfrac{C_3^3 C_2^1}{C_7^4} = \dfrac{2}{35}$ $\quad P(X=3,\ Y=1) = \dfrac{C_3^3 C_2^1}{C_7^4} = \dfrac{2}{35}$

$P(X=1,\ Y=2) = \dfrac{C_3^1 C_2^2 C_2^1}{C_7^4} = \dfrac{6}{35}$ $\quad P(X=2,\ Y=2) = \dfrac{C_3^2 C_2^2}{C_7^4} = \dfrac{3}{35}$

由此可得 (X,Y) 的联合分布律如下:

		X			
		0	1	2	3
Y	0	0	0	$\frac{3}{35}$	$\frac{2}{35}$
	1	0	$\frac{6}{35}$	$\frac{12}{35}$	$\frac{2}{35}$
	2	$\frac{1}{35}$	$\frac{6}{35}$	$\frac{3}{35}$	0

□

3-3. 解: (1) $P(X=i, Y=j) = P(X=i)\,P(Y=j \mid X=i) = \frac{1}{n} P(Y=j \mid X=i)$

$$= \begin{cases} \dfrac{1}{n(n-1)} & i \neq j \\ 0 & i = j \end{cases} \quad (i,j \in \{1,2,\cdots,n\})$$

□

(2) $n=3$ 时 (X,Y) 的联合分布律如下:

		X		
		1	2	3
Y	1	0	$\frac{1}{6}$	$\frac{1}{6}$
	2	$\frac{1}{6}$	0	$\frac{1}{6}$
	3	$\frac{1}{6}$	$\frac{1}{6}$	0

□

3-4. 解: $P(Y_1=0, Y_2=-1) = 0$. $\quad P(Y_1=1, Y_2=-1) = P(0 \leqslant X < 1) = \frac{1}{3}$.

$P(Y_1=2, Y_2=-1) = \frac{1}{3}$ $\quad P(Y_1=0, Y_2=1) = \frac{1}{3}$.

$P(Y_1=1, Y_2=1) = 0$ $\quad P(Y_1=2, Y_2=1) = 0$

(Y_1, Y_2) 联合分布律与边缘分布律如下:

p_{ij}		Y_1			$p_{i\cdot} := P(Y_2=y)$
		0	1	2	
Y_2	-1	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$
$p_{\cdot j}$ $:= P(Y_1=x)$		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

□

3-5. 解: (1) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dxdy = \int_0^{+\infty} dx \int_0^{+\infty} k e^{-2x-4y}\,dy$

$= \int_0^{+\infty} -\frac{k}{4}(0-e^{-2x})\,dx = \frac{k}{8} = 1 \quad \Rightarrow k=8$ □

(2) $P(0\le X\le 2, 0<Y\le 1) = \int_0^2\int_0^1 f(x,y)\,dxdy = \int_0^2\int_0^1 8\cdot e^{-2x-4y}\,dxdy$

$= \int_0^2$ [redacted] $2(e^{-2x}-e^{-2x-4})\,dx = 1-e^{-4}-e^{-4}+e^{-8} = 1-2\cdot e^{-4}+e^{-8}.$ □

(3) $P(X+Y<1) = \int_{-\infty}^{+\infty}\int_{-\infty}^{1-x} f(x,y)\,dxdy = \int_0^1\int_0^{1-x} 8\cdot e^{-2x-4y}\,dxdy$

$= \int_0^1 2(e^{-2x}+e^{2x-4})\,dx = 1+e^{-4}+e^{-2}+e^{-2}$

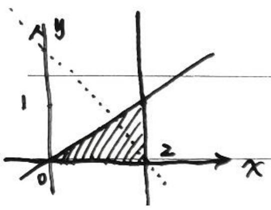
$= 1+2e^{-2}+e^{-4}$ □

3-6. 解: $P(X\ge 100, Y\ge 100) = F(+\infty,+\infty) - F(100,+\infty) - F(+\infty,100) + F(100,100)$

$= \cancel{1} - (\cancel{1}-\cancel{e^{-1}}) - (\cancel{1}-\cancel{e^{-1}}) + \cancel{1} - \cancel{2e^{-1}} + e^{-2}$

$= e^{-2}$ □

3-7. 解: (1) 由联合概率密度性质.



$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dxdy = \int_0^2 dx \int_0^{\frac{x}{2}} kx\,dy = \int_0^2 \frac{k}{2}x^2\,dx$

$= \frac{k}{6}x^3\Big|_0^2 = \frac{4}{3}k = 1 \quad \Rightarrow k=\frac{3}{4}$ □

(2) $x+y=2$ 与 $y=\frac{x}{2}$ 交点 $(\frac{4}{3},\frac{2}{3})$

$P(X+Y\le 2) = \iint\limits_{X+Y\le 2} f(x,y)\,dxdy = \int_0^{\frac{2}{3}} dy \int_{2y}^{2-y} \frac{3}{4}x\,dx$

$= \frac{3}{8}\int_0^{\frac{2}{3}}\left[(2-y)^2-(2y)^2\right]dy = \frac{3}{8}\int_0^{\frac{2}{3}}(-3y^2-4y+4)\,dy = \frac{5}{9}$ □

(3) $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$ ① $x<0$ 或 $x\ge 2$ 时. $f_X(x)=0$

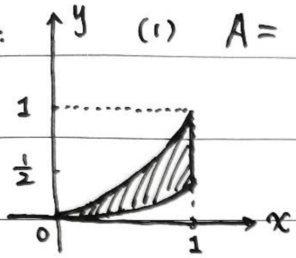
② $0\le x<2$ 时. $f_X(x) = \int_0^{\frac{x}{2}} \frac{3}{4}x\,dy = \frac{3}{8}x^2$

$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx$ ① $y<0$ 或 $y\ge 1$ 时 $f_Y(y)=0$

② $0\le y<1$ 时. $f_Y(y) = \int_{2y}^{2} \frac{3}{4}x\,dx = \frac{3}{2}-\frac{3}{2}y^2.$

综上: $f_X(x)=\begin{cases}\frac{3}{8}x^2, & 0\le x<2\\ 0 & \text{otherwise}\end{cases}$; $f_Y(y)=\begin{cases}\frac{3}{2}-\frac{3}{2}y^2, & 0\le y<1\\ 0, & \text{otherwise}\end{cases}$ □

3-9. 解:



(1) $A=\int_0^1 x^2-\frac{x^2}{2}\,dx=\int_0^1\frac{x^2}{2}\,dx=\frac{1}{6}$. (G的面积)

因为(X,Y)在G上服从均匀分布

$\therefore f(x,y)=\begin{cases}6, & (x,y)\in G\\ 0, & (x,y)\notin G.\end{cases}$ □

(2) $f_X(x)=\int_{-\infty}^{+\infty}f(x,y)\,dy$ ① $x<0$ 或 $x\ge 1$ 时. $f_X(x)=0$

② $0\le x<1$ 时. $f_X(x)=\int_{\frac{x^2}{2}}^{x^2}6\,dy=3x^2$

$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)\,dx$ ① $y<0$ 或 $y\ge 1$ 时. $f_Y(y)=0$.

② $0\le y<\frac{1}{2}$ 时. $f_Y(y)=\int_{\sqrt{y}}^{\sqrt{2y}}6\,dx=6(\sqrt{2y}-\sqrt{y})$

③ $\frac{1}{2}\le y<1$ 时. $f_Y(y)=\int_{\sqrt{y}}^{1}6\,dx=6-6\sqrt{y}$.

综上: $f_X(x)=\begin{cases}3x^2, & 0\le x\le 1\\ 0, & \text{otherwise}\end{cases}$; $f_Y(y)=\begin{cases}6(\sqrt{2y}-\sqrt{y}), & 0\le y<\frac{1}{2}\\ 6-6\sqrt{y}, & \frac{1}{2}\le y<1\\ 0, & \text{otherwise}\end{cases}$ □