# Exercise Sheet 5

Discrete Mathematics, 2020.9.29

1. Consider the first order language with symbol set $S = \{R\}$ in which $R$ represents a binary predicate.

   a) Let $\mathcal{J}_1$ be an $S$-interpretation such that
      - the domain in $\mathcal{J}_1$ is $\mathbb{N}$,
      - $\mathcal{J}_1(R, a, b) = \mathbf{T}$ if and only if $a < b$.

      Prove that $[\![\forall x \exists y R(x, y)]\!]_{\mathcal{J}_1} = \mathbf{T}$

   b) Let $\mathcal{J}_2$ be an $S$-interpretation such that
      - the domain in $\mathcal{J}_2$ is $\mathbb{N}$,
      - $\mathcal{J}_2(R, a, b) = \mathbf{T}$ if and only if $a > b$.
      - $\mathcal{J}_2(x) = 0$

      Prove that $[\![\exists y R(x, y)]\!]_{\mathcal{J}_2} = \mathbf{F}$

   c) Let $\mathcal{J}_3$ be an $S$-interpretation such that
      - the domain in $\mathcal{J}_3$ is $\mathbb{N}$,
      - $\mathcal{J}_3(R, a, b) = \mathbf{T}$ if and only if $a > b$.

      Prove that $[\![\forall x \exists y R(x, y)]\!]_{\mathcal{J}_3} = \mathbf{F}$

   d) Prove that $[\![\forall x \forall y (R(x, y) \to \exists z (R(x, z) \wedge R(z, y))]\!]_{\mathcal{J}_1} = \mathbf{F}$.

   e) Let $\mathcal{J}_4$ be an $S$-interpretation such that
      - the domain in $\mathcal{J}_4$ is $\mathbb{Q}$ (rational numbers, 有理数集),
      - $\mathcal{J}_4(R, a, b) = \mathbf{T}$ if and only if $a < b$.

      Prove that $[\![\forall x \forall y (R(x, y) \to \exists z (R(x, z) \wedge R(z, y))]\!]_{\mathcal{J}_4} = \mathbf{T}$.

2. Consider the first order language with symbol set $S = \{f, R\}$ in which $f$ represents a binary function and $R$ represents a binary predicate.

   a) Let $\mathcal{J}_1$ be an $S$-interpretation such that
      - the domain in $\mathcal{J}_1$ is $\mathbb{N}$,
      - $\mathcal{J}_1(f, a, b) = a + b$,
      - $\mathcal{J}_1(R, a, b) = \mathbf{T}$ if and only if $a = b$.

      Prove that $[\![\forall x \forall y R(f(x, y), f(y, x))]\!]_{\mathcal{J}_1} = \mathbf{T}$.

   b) Let $\mathcal{J}_2$ be an $S$-interpretation such that
      - the domain in $\mathcal{J}_2$ is $\mathbb{N}$,
      - $\mathcal{J}_2(f, a, b) = a * b$,
      - $\mathcal{J}_2(R, a, b) = \mathbf{T}$ if and only if $a = b$.

      Prove that $[\![\forall x \forall y R(f(x, y), f(y, x))]\!]_{\mathcal{J}_2} = \mathbf{T}$.

   c) Let $\mathcal{J}_3$ be an $S$-interpretation such that
      - the domain in $\mathcal{J}_3$ is $\{\mathbf{T}, \mathbf{F}\}$,
      - $\mathcal{J}_3(f, a, b) = [\![\wedge]\!] (a, b)$,
      - $\mathcal{J}_3(R, a, b) = \mathbf{T}$ if and only if $a = b$.

      Prove that $[\![\forall x \forall y R(f(x, y), f(y, x))]\!]_{\mathcal{J}_3} = \mathbf{T}$.

d) Prove that $\forall x \forall y R(f(x,y), f(y,x))$ is not valid.

3. (P56, Ex.43, [R]) Consider the first order language with symbol set $S = \{P, Q\}$ in which $P$ and $Q$ represent two unary predicates. Determine whether $\forall x(P(x) \to Q(x))$ and $\forall x P(x) \to \forall x Q(x)$ are logically equivalent. Justify your answer.

4. Is $\neg \forall x(\phi \to \psi)$ logically equivalent to $\exists x(\phi \wedge \neg \psi)$? You do not need to give a formal proof, but try to explain the intuition behind your answer.

5. (P67, Ex.32, [R]) Consider the first order language with symbol set $S = \{P, Q, T\}$ in which $P$, $Q$ represent two unary predicates and $T$ represents a ternary predicate. Express the negations of each of these statements so that all negation symbols immediately precede predicates.
a) $\exists z \forall y \forall x T(x, y, z)$
b) $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$
c) $\exists x \exists y (P(x, y) \leftrightarrow Q(y, x))$
d) $\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y))$

6. a) Prove that if $\phi \vDash \psi$ then $\forall x \phi \vDash \forall x \psi$.
   b) Prove that if $\Phi, \phi \vDash \psi$ and $x$ does not freely occur in $\Phi$ then $\Phi, \forall x \phi \vDash \forall x \psi$.
   c) Demonstrate an example in which
      - $\Phi, \phi \vDash \psi$
      - $x$ does freely occur in $\Phi$
      - $\Phi, \forall x \phi \nvDash \forall x \psi$.