Probability Theory and Mathematical Statistics 概率统计

Homework 1108-1111

邱一航 520030910155

11/08 周一

4-19. 解: $f_X(x)=\frac{1}{\sqrt{2\pi}\cdot\sqrt{2}}\, e^{-\frac{(x-1)^2}{2\cdot 2}}$ $\quad\therefore X\sim N(1,2)$

$f_Y(y)=\frac{1}{\sqrt{2\pi}\cdot 1}\cdot e^{-\frac{(y-2)^2}{2\cdot 1}}$ $\quad\therefore Y\sim N(2,1)$

Z 为 $2X-Y+8$ 的线性函数，故 Z 服从正态分布. $E(Z)=2E(X)-E(Y)+8=8$

$D(Z)=D(2X)+D(Y)=4D(X)+D(Y)=9$ □

4-22. (1) 证明: $P(X\leq a)=0 \Rightarrow E(X)\geq a$

$P(X>b)=1-P(X\leq b)=0 \Rightarrow E(X)\leq b$

$\therefore a\leq E(X)\leq b.$ □

(2) 证明: 记 $Y=\frac{X-a}{b-a}$. 则 Y 在 $[0,1]$ 内分布.

$Y\leq 1 \Rightarrow Y^2\leq Y \Rightarrow E(Y^2)\leq E(Y).$ $\quad\therefore E(Y^2)-(E(Y))^2\leq E(Y)-(E(Y))^2\leq \frac{1}{4}$

$D(X)=(b-a)^2D(Y)=(b-a)^2\left(E(Y^2)-(E(Y))^2\right)\leq \frac{1}{4}(b-a)^2$ □

4-24. 解: $X\sim U(1,3) \Rightarrow E(X)=2.$ $\quad E(X^2)=\int_1^3 x^2 f_X(x)\,dx=\frac{13}{3}.$

$Y\sim N(0,1) \Rightarrow E(Y)=0.$ $\quad E(Y^2)=[E(Y)]^2+D(Y)=1$

X,Y 相互独立 $\Rightarrow E(XY)=E(X)E(Y)=0$ $\quad E(X^2Y^2)=E(X^2)E(Y^2)$

$D(XY)=E(X^2Y^2)-[E(XY)]^2=E(X^2)E(Y^2)-0=\frac{13}{3}.$ □

4-26. 解: $f_X(x)=\int_{-\infty}^{+\infty} f(x,y)\,dy=\int_0^{+\infty} e^{-(x+y)}\,dy=e^{-x}.$ $\quad f_Y(y)=e^{-y}$

$E(X)=\int_{-\infty}^{+\infty} x f(x)\,dx=\int_0^{+\infty} x e^{-x}\,dx=1$ $\quad E(Y)=\int_0^{+\infty} y e^{-y}\,dy=1$

$E(X^2)=\int_{-\infty}^{+\infty} x^2 f(x)\,dx=\int_0^{+\infty} x^2 e^{-x}\,dx=2$ $\quad E(Y^2)=2$

$\therefore D(X)=E(X^2)-[E(X)]^2=2-1=1.$ $\quad D(Y)=E(Y^2)-[E(Y)]^2=2-1=1.$

[redacted] $E(XY)=\int_0^{+\infty} xe^{-x}\,dx\int_0^{+\infty} ye^{-y}\,dy=1.$ $\therefore \operatorname{cov}(X,Y)=E(XY)-E(X)E(Y)=0$

$\Rightarrow \rho_{XY}=0.$ $\quad C=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

综上: $E(X)=1$, $E(Y)=1$, $D(X)=1$, $D(Y)=1$. $\operatorname{cov}(X,Y)=0$.

$\rho_{XY}=0.$ $\quad C=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ □

4-27. [redacted] 证明: $P(X=-1)=\frac{3}{8}$ $\quad P(X=0)=\frac{1}{4}.$ $\quad P(X=1)=\frac{3}{8}$

$P(Y=-1)=\frac{3}{8}$ $\quad P(Y=0)=\frac{1}{4}$ $\quad P(Y=1)=\frac{3}{8}$

验证知 $P(X=-1,Y=-1)=\frac{1}{8}\neq P(X=-1)P(Y=-1)=\frac{9}{64}$

$\therefore$ X, Y 不相互独立. □

$E(X)=0.$ $E(Y)=0.$ (对称性可知) $\quad D(X)=E(X^2)=2\cdot\frac{3}{8}\cdot 1=\frac{3}{4}$

$D(Y)=E(Y^2)=\frac{3}{4}$ $\quad E(XY)=1\cdot\frac{1}{8}+1\cdot\frac{1}{8}-1\cdot\frac{1}{8}-1\cdot\frac{1}{8}=0$

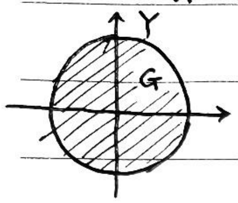
$\operatorname{cov}(X,Y)=E(XY)-E(X)E(Y)=0$ $\Rightarrow \rho_{XY}=0.$ $\therefore$ X, Y 不相关 □

4-28 证明: 若X与Y不相关, 则 $\operatorname{cov}(X,Y)=E(XY)-E(X)E(Y)=0$

$E(X)=P(A),\ E(Y)=P(B)$ $\quad E(XY)=P(AB)$

$\Rightarrow P(A)P(B)=P(AB)$ $\Rightarrow$ A, B 相互独立 $\Rightarrow$ X, Y 相互独立 □

4-30. 证明: $E(X)=0.$ ~~E(Y)=0 (对~~ $E(Y)=0$ (对称性可知)



$\cancel{f_X(x)=\int_{-\infty}^{+\infty} f(x,y)\,dy=2\sqrt{1-x^2},\quad f_Y(y)=2\sqrt{1-y^2}}$

$\cancel{D(X)=E(X^2)=\int_{-1}^{1} 2x^2\sqrt{1-x^2}\,dx}$

$\operatorname{cov}(x,y)=\iint_G (x-0)(y-0)\,f(x,y)\,dx\,dy=0$ (对称性)

$\Rightarrow \rho_{XY}=0.$ 即 X, Y 不相关 □

$f_X(x)=\int_{-\infty}^{+\infty} f(x,y)\,dy=2\sqrt{1-x^2}.$ $\quad f_Y(y)=2\sqrt{1-y^2}$

$P(-\frac{\sqrt{3}}{2}\le X\le\frac{\sqrt{3}}{2},\ Y\ge\frac{1}{2})=\frac{\pi}{3}-\frac{1}{2}\cdot\sqrt{3}\cdot\frac{1}{2}=\frac{\pi}{3}-\frac{\sqrt{3}}{4}$

$\neq P(-\frac{\sqrt{3}}{2}\le X\le\frac{\sqrt{3}}{2})\,P(Y\ge\frac{1}{2})=(\frac{2\pi}{3}+\frac{\sqrt{3}}{2})(\frac{\pi}{3}-\frac{\sqrt{3}}{4})$ $\therefore$ X, Y 不相互独立 □

补充1. 已知X, Y相互独立且服从$N(0,2)$. 求$D(|X-Y|)$.

解: 记 $Z=X-Y$. 为X, Y线性函数. $E(Z)=E(X)-E(Y)=0$.

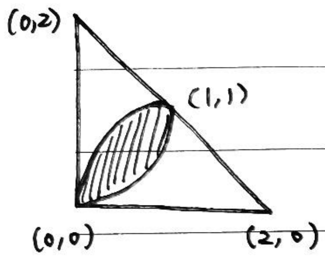
$\because$ X, Y相互独立 $\therefore$ $D(Z)=D(X)+D(Y)=4$ $\therefore$ $Z\sim N(0,4)$

$$D(|Z|)=E(|Z|^2)-[E(|Z|)]^2=E(Z^2)-(E(|Z|))^2=D(Z)-(E(|Z|))^2$$

$$=4-\left(2\int_0^{+\infty}x\frac{1}{\sqrt{2\pi}\cdot 2}e^{-\frac{x^2}{8}}dx\right)^2=4-\left(\frac{4}{\sqrt{2\pi}}\int_0^{+\infty}e^{-\left(\frac{x^2}{8}\right)}d\left(\frac{x^2}{8}\right)\right)^2$$

$$=4-\left(\frac{2\sqrt{2}}{\sqrt{\pi}}\right)^2=4-\frac{8}{\pi}$$ □

补充2. 解: 记投一个质点落在$y=x^2$与$y=\sqrt{x}$围成区域的概率为p.



$$S_{阴}=1-2\left(\int_0^1 x^2dx\right)=\frac{1}{3}$$

$$p=\frac{S_{阴}}{S}=\frac{1/3}{\frac{1}{2}\cdot 2\cdot 2}=\frac{1}{6}.$$

记落在该区域内的质点数为X. 则 $X\sim B(10,p)$

即 $p_x=C_{10}^x p^x(1-p)^{10-x}$

$$E(X)=\sum_{i=0}^{10}xC_{10}^x p^x(1-p)^{10-x}=\frac{5}{3}$$ □

$$E(X^2)=\sum_{i=0}^{10}x^2C_{10}^x p^x(1-p)^{10-x}=\frac{25}{6}\Rightarrow D(X)=E(X^2)-[E(X)]^2=\frac{25}{18}$$ □

11/01 周四

29. 解: Z是X, Y的线性函数. 故Z也服从正态分布.

$$E(Z)=E\left(\frac{X}{3}+\frac{Y}{2}\right)=\frac{1}{3}E(X)+\frac{1}{2}E(Y)=\frac{1}{3}$$ □

$$D(Z)=D\left(\frac{X}{3}\right)+D\left(\frac{Y}{2}\right)+2Cov\left(\frac{X}{3},\frac{Y}{2}\right)=\frac{1}{9}D(X)+\frac{1}{4}D(Y)+2\rho_{XY}\cdot\sqrt{\frac{1}{9}D(X)\cdot\frac{1}{4}D(Y)}$$

$$=3$$ □

考虑到 $D(Y)=D\left(2Z-\frac{2}{3}X\right)=D(2Z)+D\left(\frac{2}{3}X\right)+2Cov\left(2Z,\frac{2}{3}X\right)$

$$=4D(Z)+\frac{4}{9}D(X)+2\rho_{XZ}\sqrt{4D(Z)\cdot\frac{4}{9}D(X)}$$

$\Rightarrow \rho_{XZ}=0.$ □

31. 解：由题知：$D(X)=1.\quad D(Y)=5.\quad Cov(X,Y)=2.=cov(Y,X)$

$$D(U)=D(X-2Y)=D(X)+4D(Y)-4cov(X,Y)=13$$

$$D(V)=D(2X-Y)=4D(X)+D(Y)-4cov(X,Y)=1$$

$$cov(U,V)=cov(X-2Y,2X-Y)=2D(X)-cov(X,Y)-4cov(Y,X)+2D(Y)$$

$$=2$$

$$\rho_{UV}=\frac{cov(U,V)}{\sqrt{D(U)D(V)}}=\frac{2}{13}\sqrt{13}$$ □

补充1. 证明：$E\left((tX-Y)^2\right)=E\left(t^2X^2-2tXY+Y^2\right)=t^2E(X^2)-2tE(XY)+E(Y^2)\geq 0$

$(\because (tX-Y)^2\geq 0,\ \forall X.\ \forall Y)$ 即不等式 $t^2E(X^2)-2tE(XY)+E(Y^2)\geq 0$ 恒有解（对 $t\in R$）

$\therefore \Delta=4E^2(XY)-4E(X^2)E(Y^2)\leq 0 \Rightarrow E^2(XY)\leq E(X^2)E(Y^2)$ □

（取等当且仅当 $\exists t_0$ s.t. $(t_0X-Y)^2=0$ 即 $Y=t_0X$ 即 $P(Y=t_0X)=1$.） □

补充2. 解：记 $X>Y$ 的可能性为 p. $P(X=Y)\doteq 0$.

$$p=P(X<Y)=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\cdot 3}e^{-\frac{(x-2)^2}{18}}dx\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\cdot 2}e^{-\frac{(y-2)^2}{8}}dy$$

$$=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\cdot 2}e^{-\frac{(y-2)^2}{8}}dy\int_{y-2}^{+\infty}\frac{1}{\sqrt{2\pi}\cdot 3}e^{-\frac{u^2}{18}}du$$

$$=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\cdot 2}e^{-\frac{(y-2)^2}{8}}dy\int_{-\infty}^{2-y}\frac{1}{\sqrt{2\pi}\cdot 3}e^{-\frac{u^2}{18}}du$$

$$=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\cdot 2}e^{-\frac{(-v)^2}{8}}dv\int_{-\infty}^{-v}\frac{1}{\sqrt{2\pi}\cdot 3}e^{-\frac{u^2}{18}}du$$

$$=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\cdot 2}e^{-\frac{v^2}{8}}dv\int_{-\infty}^{v}\frac{1}{\sqrt{2\pi}\cdot 3}e^{-\frac{u^2}{18}}du$$

$$=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\cdot 2}e^{-\frac{(y-2)^2}{8}}dy\int_{-\infty}^{y}\frac{1}{\sqrt{2\pi}\cdot 3}e^{-\frac{(x-2)^2}{18}}dx=P(Y<X)$$

$\therefore p=\frac{1}{2}$

几何分布.

$P(Z_1=k) = C_{k-1}^{0}\, p(1-p)^{k-1} = p(1-p)^{k-1}$. $\therefore E(Z_1)=\frac{1}{p}=2$ □

$D(Z_1)=\frac{1-p}{p^2}=2$ □

$Z_2 \sim B(10,p)$ $\therefore E(Z_2)=np=5$. $D(Z_2)=np(1-p)=\frac{5}{2}$ □□

补充3. 解:(1) $f_X(x)=\begin{cases} 1\cdot e^{-1\cdot x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$ $Z=XY$. 由Y的概率分布与$f_X(x)$知,

$\begin{cases} Z>0 \Leftrightarrow Y=1 \\ Z<0 \Leftrightarrow Y=-1 \end{cases}$

$\therefore z>0$ 时. $f_Z(z)=(1-p)f_X(z)$; $z<0$时. $f_Z(z)=p f_X(-z)$

$$\therefore f_Z(z)=\begin{cases} (1-p)e^{-z}, & z>0 \\ 0, & z=0 \\ pe^{+z}, & z<0 \end{cases}$$ □

(2) $E(Z)=\int_{-\infty}^{0} zpe^{+z}\,dz + \int_0^{+\infty} +z(1-p)e^{-z}\,dz$

$=\int_0^{+\infty} -zpe^{-z}\,dz + (1-p)\int_0^{+\infty} ze^{-z}\,dz$

$=-p(1)+(1-p)\cdot 1 = 1$.

$E(Z^2)=\int_{-\infty}^{0} z^2pe^{z}\,dz + \int_0^{+\infty} z^2(1-p)e^{-z}\,dz$

$=-\int_0^{+\infty} z^2pe^{-z}dz + (1-p)\int_0^{+\infty} z^2e^{-z}dz = -2p+2(1-p)$

$=2-4p$. $\Rightarrow$ $D(Z)=2-4p-1=1-4p$.

$E(X)=1$. $E(Y)=-p+(1-p)=1-2p$. $D(Y)=4p(1-p)$

$D(X)=1$.

$E(XZ)=\iint_{\infty} f(x,z)xz\,dxdz = \int_0^{+\infty} e^{-x}(-p\cdot x^2+(1-p)x^2)\,dx$

$=\int_0^{+\infty} x^2e^{-x}(1-2p)\,dx = (1-2p)E(X^2) = 2(1-2p)=2-4p$

$\mathrm{cov}(X,Z)=E(XZ)-E(X)E(Z) = 2-4p-1 = 1-4p = 0 \Leftrightarrow p=\frac{1}{4}$ □

(3) $P(X<1)=\int_0^1 e^{-x}\,dx = 1-e^{-1}$. $P(Z<1)=p+(1-p)(1-e^{-1})$

$=1-(1-p)e^{-1}$

$P(X<1, Z<1) = P(X<1)=1-e^{-1} \neq P(X<1)P(Z<1)$

□

$\therefore X,Z$不相互独立