[Solution of Homework 5] Poisson Process & Poisson Approximation

Problem 1

Customers arrive according to a Poisson process of rate λ per hour. Joe does not want to stay until the store closes at $T = 10$ p.m., so he decides to close up when the first customer after time $T - s$ arrives. He wants to leave early but he does not want to lose any business so he is happy if he leaves before T and no one arrives after.

- What is the probability he achieves his goal?
- What is the optimal value of s and the corresponding success probability? (That is, the value s maximizing the success probability)

(a)

Solution
The goal of Joe is the event that there is only one customer arrived within time $T - s \sim T$ (the unit of variable s is hour). According to the fact that customers arrive according to a Poisson process of rate λ per hour, we have

$$\mathbf{Pr}\left[\text{Joe achieves his goal}\right] = s\lambda e^{-s\lambda}.$$

(b)

Solution
We can maximize $s\lambda e^{-s\lambda}$ when $s = \frac{1}{\lambda}$ where the maximum value is $\frac{1}{e}$.

Problem 2

- Assume $X \sim \texttt{Poisson}(\lambda)$ for some integer $\lambda \geq 1$. Prove that for any $k = 0, 1, \ldots, \lambda - 1$, it holds that $\mathbf{Pr}\left[X = \lambda + k\right] \geq \mathbf{Pr}\left[X = \lambda - k - 1\right]$. Use this to conclude that $\mathbf{Pr}\left[X \geq \lambda\right] \geq \frac{1}{2}$.

- Recall the setting of Corollary 4 in Lecture 10. Prove that if $\mathbf{E}\left[f(X_1,\ldots,X_n)\right]$ is monotonically increasing in m, then

$$\mathbf{E}\left[f(X_1,\ldots,X_n)\right] \le 2\cdot\mathbf{E}\left[f(Y_1,\ldots,Y_n)\right].$$

- Recall the birthday problem in Lecture 2 and assume notations there. Now suppose we would like to estimate the probability of the event "*there exists four students who share the same birthday*". Assume there are 50 students in the class ($n = 50$ and $m = 365$). Use Poisson approximation to show that the probabilty is at most 1%.

(a)

Solution
By definition, we have

$$\begin{aligned}\frac{\mathbf{Pr}\left[X=\lambda+k\right]}{\mathbf{Pr}\left[X=\lambda-k-1\right]} &= \frac{e^{-\lambda}\frac{\lambda^{\lambda+k}}{(\lambda+k)!}}{e^{-\lambda}\frac{\lambda^{\lambda-k-1}}{(\lambda-k-1)!}}\\ &= \frac{\lambda^{2k+1}}{(\lambda+k)(\lambda+k-1)\cdots(\lambda-k)}\\ &= \sum_{i=1}^{k}\frac{\lambda^2}{(\lambda+i)(\lambda-i)} \ge 1,\end{aligned}$$

which certifies

$$\mathbf{Pr}\left[X=\lambda+k\right] \ge \mathbf{Pr}\left[X=\lambda-k-1\right].$$

Then,

$$\begin{aligned}\mathbf{Pr}\left[X\ge\lambda\right] &= \frac{\sum_{t=\lambda}^{\infty}\mathbf{Pr}\left[X=t\right]}{\sum_{t=0}^{\infty}\mathbf{Pr}\left[X=t\right]}\\ &\ge \frac{\sum_{t=\lambda}^{2\lambda-1}\mathbf{Pr}\left[X=t\right]}{\sum_{t=0}^{2\lambda-1}\mathbf{Pr}\left[X=t\right]}\\ &= \frac{\sum_{t=\lambda}^{2\lambda-1}\mathbf{Pr}\left[X=t\right]}{\sum_{t=0}^{\lambda-1}\mathbf{Pr}\left[X=t\right]+\sum_{t=\lambda}^{2\lambda-1}\mathbf{Pr}\left[X=t\right]} \ge \frac{1}{2}.\end{aligned}$$

(b)

Solution

For convenience, we use $\mathbf{E}_m$ to denote the expectation according to the distribution in the m-balls-into-n-bins model.

$$\begin{aligned}
\mathbf{E}\left[f(Y_1,Y_2,\ldots,Y_n)\right] &= \sum_{k=0}^{\infty}\mathbf{E}\left[f(Y_1,Y_2,\ldots,Y_n)\;\middle|\;\sum_{i=1}^{n}Y_i=k\right]\mathbf{Pr}\left[\sum_{i=1}^{n}Y_i=k\right]\\
&\ge \sum_{k=m}^{\infty}\mathbf{E}\left[f(Y_1,Y_2,\ldots,Y_n)\;\middle|\;\sum_{i=1}^{n}Y_i=k\right]\mathbf{Pr}\left[\sum_{i=1}^{n}Y_i=k\right]\\
&= \sum_{k=m}^{\infty}\mathbf{E}_k\left[f(X_1,X_2,\ldots,X_n)\right]\mathbf{Pr}\left[\sum_{i=1}^{n}Y_i=k\right]\\
&\ge \mathbf{E}_m\left[f(X_1,X_2,\ldots,X_n)\right]\sum_{k=m}^{\infty}\mathbf{Pr}\left[\sum_{i=1}^{n}Y_i=k\right].
\end{aligned}$$

Note that $\sum_{i=1}^{n}Y_i\sim\mathrm{Possion}(m)$ and

$$\sum_{k=m}^{\infty}\mathbf{Pr}\left[\sum_{i=1}^{n}Y_i=k\right]\ge\frac{1}{2}.$$

Therefore,

$$\mathbf{E}\left[f(X_1,\ldots,X_n)\right]\le 2\cdot\mathbf{E}\left[f(Y_1,\ldots,Y_n)\right].$$

Solution

The birthday problem with n students is exactly the n-balls-into-m-bins model. For any $i \in [m]$, we define

$$X_i := \text{the number of students born in day } i.$$

Let $Y_i \sim \mathrm{Possion}(\frac{n}{m})$ and

$$f(X_1, \ldots, X_m) := \mathbf{1}[\max\{X_1, \ldots, X_m\} \geq 4]$$

which is monotone with respect to n. Then we have

$$\begin{aligned}
\mathbf{Pr}\left[\max\{X_1, \ldots, X_m\} \geq 4\right] &\leq 2 \cdot \mathbf{Pr}\left[\max\{Y_1, \ldots, Y_m\} \geq 4\right] \\
&= 2 \cdot (1 - \mathbf{Pr}\left[\max\{Y_1, \ldots, Y_m\} < 4\right]) \\
&= 2 \cdot (1 - \prod_{i=1}^{m} \mathbf{Pr}\left[Y_i < 4\right]) \\
&\leq 0.01.
\end{aligned}$$