Exercise Sheet 7

Discrete Mathematics, 2020.10.13

1. Show that for any sets A and B

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$$

2. Show that $A \times \bigcup B = \bigcup\{A \times X | X \in B\}$

3. Here is a proof of $C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$. Show its corresponding first order logic proof.

   Proof.

   For any x,

$$\begin{array}{ll} x \in C \cap (A \cup B) & \iff \\ x \in C \wedge (x \in A \vee x \in B) & \iff \\ (x \in C \wedge x \in A) \vee (x \in C \wedge x \in B) & \iff \\ x \in (C \cap A) \cup (C \cap B) & \end{array}$$

   Qed.

4. Use the language of ZF to state empty set's uniqueness and prove your statement. **Note**: you are only allow to use the predicates $\in$ and $=$ in your statement in this task.

5. We define a set A to be "$\in$-well-ordered" if for every nonempty subset x of A, x must have a $\in$-least element y. In other words,

$$A \text{ is } \in\text{-well-ordered} \quad \leftrightarrow \quad \forall x(x \subseteq A \wedge \neg x = \emptyset \rightarrow \exists y(y \in x \wedge \forall z(z \in x \rightarrow y \in z \vee y = z)))$$

   Prove that:

   a) 0 is $\in$-well-ordered

   b) if n is $\in$-well-ordered, then $n \cup \{n\}$ is $\in$-well-ordered.

6. Prove that there exists at most one "smallest inductive set", i.e. if

   - Inductive(u)
   - $\forall x(\text{Inductive}(x) \rightarrow u \subseteq x)$
   - Inductive(v)
   - $\forall x(\text{Inductive}(x) \rightarrow v \subseteq x)$

   then $u = v$.

7. Suppose u and v are two inductive sets. Prove that $u \cap v$ is also inductive.

8. Suppose u is an inductive set.

   a) Prove that $\{x \in u \mid \forall v(v \subseteq u \wedge \text{Inductive}(v) \rightarrow x \in v)\}$ is also inductive.

   b) Prove that $\{x \in u \mid \forall v(v \subseteq u \wedge \text{Inductive}(v) \rightarrow x \in v)\}$ is the smallest inductive subset of u.

9. Prove there exists at least one "smallest inductive set". **Hint**: you can use the conclusions above.

10. **Note: in this task, you will prove the induction principle of natural numbers.** Suppose X is a set such that

- $0 \in X$,
- for any $n \in \mathbb{N}$, $n \in X$ implies $n \cup \{n\} \in X$.

Prove that

a) $\mathbb{N} \cap X$ is an inductive set.

b) for any $n \in \mathbb{N}$, $n \in X$ always holds.