Discrete Mathematics Exercise 20

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1. ***Solution:***

Let U be the set containing all 270 students involved in the survey.

Let A be the set containing students in U who like brussels sprouts.

Let B be the set containing students in U who like broccoli.

Let C be the set containing students in U who like cauliflower.

Then $|U| - |A \cup B \cup C|$ is the number of students who do not like any of these vegetables in U.

By Inclusion-exclusion, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 64 + 94 + 58 - 26 - 28 - 22 + 14 = 154$. Thereby, $|U| - |A \cup B \cup C| = 270 - 154 = 116$.

Thus, there are 116 of 270 students who do not like any of these vegetables.

2. ***Proof:***

To prove $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{l=1}^{n} \sum_{1 \le k_1 < \ldots < k_l \le n} (-1)^{l+1} \left|\bigcap_{j=1}^{l} A_{k_j}\right|$ by induction of the number of finite sets involved. Let it be n.

BASE STEP 01. $n = 1$. $|A| = |A|$.

BASE STEP 02. $n = 2$. $|A \cup B| = |A| + |B| - |A \cap B|$.

I.H. Suppose when $1 \le n \le m$, $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{l=1}^{n} \sum_{1 \le k_1 < \ldots < k_l \le n} (-1)^{l+1} \left|\bigcap_{j=1}^{l} A_{k_j}\right|$.

INDUCTION STEP.

When $n = m + 1$,

$$\left|\bigcup_{i=1}^{m+1} A_i\right| = \left|\left(\bigcup_{i=1}^{m} A_i\right) \cup A_{m+1}\right| = \left|\bigcup_{i=1}^{m} A_i\right| + |A_{m+1}| - \left|\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{m+1}\right|$$

$$= \sum_{l=1}^{m} \sum_{1 \le k_1 < \ldots < k_l \le m} (-1)^{l+1} \left|\bigcap_{j=1}^{l} A_{k_j}\right| + |A_{m+1}| - \left|\bigcup_{i=1}^{n} (A_i \cap A_{m+1})\right|$$

$$= \sum_{l=1}^{m} \sum_{1 \le k_1 < \ldots < k_l \le m} (-1)^{l+1} \left|\bigcap_{j=1}^{l} A_{k_j}\right| + |A_{m+1}| - \sum_{l=1}^{m} \sum_{1 \le k_1 < \ldots < k_l \le m} (-1)^{l+1} \left|\bigcap_{j=1}^{l} \left(A_{k_j} \cap A_{m+1}\right)\right|$$

$$= \sum_{l=1}^{m} \sum_{1 \le k_1 < \ldots < k_l \le m} (-1)^{l+1} \left|\bigcap_{j=1}^{l} A_{k_j}\right| + \sum_{l=1}^{m+1} \sum_{1 \le k_1 < \ldots < k_l = m+1} (-1)^{l+1} \left|\bigcap_{j=1}^{l} A_{k_j}\right|$$

$$= \sum_{l=1}^{m+1} \sum_{1 \le k_1 < \ldots < k_l \le m+1} (-1)^{l+1} \left|\bigcap_{j=1}^{l} A_{k_j}\right|$$

Thus, $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{l=1}^{n} \sum_{1 \le k_1 < \ldots < k_l \le n} (-1)^{l+1} \left|\bigcap_{j=1}^{l} A_{k_j}\right|$ still holds.

QED

3. ***Solution:***

The number of derangements of a set with 7 elements is $7!\left(\sum_{i=0}^{n}(-1)^i \cdot \frac{1}{i!}\right) = 1854.$

4. ***Solution:***

Label each student using a number from $\{1, 2, 3, \dots, n\}$.

Let $f(x)$ be the seat that student x should take in the first class and $g(x)$ be the seat that student x should take in the second class.

Consider the remaining occasions.

Let $A = \{(f, g) | \exists x \left(f(x) = g(x)\right)\}, A_i = \{(f, g) | f(i) = g(i)\}$.

Thus, the number of ways that these seats can be assigned with no student assigned the same seat for both classes is $(n!)^2 - |A|$. Also, $A = \bigcup_{i=1}^{n} A_i$.

For any l, $1 \le l \le n$, $\left|\bigcap_{j=1}^{l} A_{k_j}\right| = n!\,(n-l)!$

$$|A| = \left|\bigcup_{i=1}^{n} A_i\right| = \sum_{l=1}^{n} \sum_{1\le k_1<\dots<k_l\le n} (-1)^{l+1} \left|\bigcap_{j=1}^{l} A_{k_j}\right| = \sum_{l=1}^{n} \sum_{1\le k_1<\dots<k_l\le n} (-1)^{l+1}\, n!\,(n-l)!$$

$$= \sum_{l=1}^{n} (-1)^{l+1} \binom{l}{n} n!\,(n-l)!$$

Thus, the number of ways that these seats can be assigned with no student assigned the same seat for both classes is $(n!)^2 - |A| = (n!)^2 - \sum_{l=1}^{n}(-1)^{l+1}\binom{l}{n} n!\,(n-l)!$

$$= n! \sum_{l=0}^{n} (-1)^{l} \binom{l}{n} (n-l)!$$

$$= (n!)^2 \sum_{l=0}^{n} (-1)^{l} \frac{1}{l!}$$

5. ***Solution:***

Consider the remaining occasions.

Let $f: \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be any way digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are arranged.

Let $A = \{f | \exists x, x \text{ is even}, f(x) = x\}$. Let $A_i = \{f | f(i) = i\}$. Obviously $A = \bigcup_{i=0}^{4} A_{2i}$.

For any l, $1 \le l \le 5$, $\left|\bigcap_{j=1}^{l} A_{k_j}\right| = (10 - l)!$

$$|A| = \left|\bigcup_{i=0}^{4} A_{2i}\right| = \sum_{l=1}^{5} \sum_{0\le k_1<\dots<k_l\le 4} (-1)^{l+1} \left|\bigcap_{j=1}^{l} A_{k_j}\right| = \sum_{l=1}^{5} \sum_{0\le k_1<\dots<k_l\le 4} (-1)^{l+1}(10-l)!$$

$$= \sum_{l=1}^{5} (-1)^{l+1} \binom{l}{5} (10-l)! = 1458120.$$

Thus, the number of ways that the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are arranged with no even digit is in its original position is $10! - |A| = 2170680$.