# [Homework 2]: Finite Markov Chains, Coupling

## Problem 1 (Optimal Coupling)

Let $\Omega$ be a finite state space and $\mu, \nu$ be two distributions over $\Omega$. Prove that there exists a coupling $\omega$ of $\mu$ and $\nu$ such that

$$\mathbf{Pr}_{(X,Y)\sim\omega}\left[X \neq Y\right] = D_{\mathrm{TV}}(\mu, \nu).$$

You need to explicitly describe how $\omega$ is constructed.

## Problem 2 (Stochastic Dominance)

Let $\Omega \subseteq \mathbb{Z}$ be a finite set of integers. Let $\mu$ and $\nu$ be two distributions over $\Omega$. We say $\mu$ is *stochastic dominance* over $\nu$ if for $X \sim \mu$, $Y \sim \nu$ and any $a \in \Omega$,

$$\mathbf{Pr}\left[X \geq a\right] \geq \mathbf{Pr}\left[Y \geq a\right].$$

We write $\mu \succeq \nu$.

- Consider the binomial distirbution $\mathtt{Binom}(n, p)$ where $X \sim \mathtt{Binom}(n, p)$ satisfies for any $a = 0, 1, \ldots, n$, $\mathbf{Pr}\left[X = a\right] = \binom{n}{a} \cdot p^a \cdot (1-p)^{n-a}$. Prove that for any $p, q \in [0, 1]$, $\mathrm{Binom}(n, p) \succeq \mathrm{Binom}(n, q)$ if and only if $p \geq q$.
- A coupling $\omega$ of $\mu$ and $\nu$ is *monotone* if $\mathbf{Pr}_{(X,Y)\sim\omega}\left[X \geq Y\right] = 1$. Prove that $\mu \succeq \nu$ if and only if a monotone coupling of $\mu$ and $\nu$ exists.
- Consider the Erdős–Rényi (https://en.wikipedia.org/wiki/Erd%C5%91s%E2%80%93R%C3%A9nyi_model) model $\mathcal{G}(n, p)$ for random graph. In this model, each $G \sim \mathcal{G}(n, p)$ is a simple undirected random graph with $n$ vertices where each $\{i, j\} \in \binom{[n]}{2}$ is present with probability $p$ independently. Prove that for any $p, q \in [0, 1]$ satisfying $p \geq q$, it holds that $\mathbf{Pr}_{G\sim\mathcal{G}(n,p)}\left[G \text{ is connected}\right] \geq \mathbf{Pr}_{H\sim\mathcal{G}(n,q)}\left[H \text{ is connected}\right]$.

## Problem 3 (Total Variation Distance is Non-Increasing)

Let $P$ be the transition matrix of an irreducible and aperiodic Markov chain with state space $\Omega$. Let $\pi$ be its stationary distribution. Let $\mu_0$ be an arbitrary distribution on $\Omega$ and $\mu_t^{\mathtt{T}} = \mu_0^{\mathtt{T}} P^t$ for every $t \geq 0$. For every $t \geq 0$, let $\Delta(t) = D_{\mathtt{TV}}(\mu_t, \pi)$ be the total variation distance between $\mu_t$ and $\pi$. Prove that $\Delta(t+1) \leq \Delta(t)$ for every $t \geq 0$.