Digital Signal and Image Processing

Written Assignment #2

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2022/3/20 DSIP Problem Set 2

Question 01.

Solution: (a) $\tilde{X}_3[k] = \sum_{n=0}^{3N-1} \tilde{x}[n] W_{3N}^{kn} = \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{kn} + \sum_{n=N}^{2N-1} \tilde{x}[n] W_{3N}^{kn} + \sum_{n=2N}^{3N-1} \tilde{x}[n] W_{3N}^{kn}$

$$= \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{kn} + \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{k(n+N)} + \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{k(n+2N)}$$

$$= \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{kn} (1 + W_{3N}^{kN} + W_{3N}^{2kN})$$

$$= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k'=0}^{N-1} \tilde{X}[k'] W_N^{-k'n} \cdot W_{3N}^{kn} (1 + W_{3N}^{kN} + W_{3N}^{2kN})$$

$$= \frac{1}{N} (1 + W_{3N}^{kN} + W_{3N}^{2kN}) \sum_{n=0}^{N-1} W_{3N}^{kn} \sum_{\ell=0}^{N-1} \tilde{X}[\ell] W_N^{-\ell n}$$

$$= \frac{1}{N} (1 + W_3^{k} + W_3^{2k}) \sum_{n=0}^{N-1} W_{3N}^{kn} \sum_{\ell=0}^{N-1} \tilde{X}[\ell] W_N^{-\ell n}$$

CASE 01. When $k = 3m+1$ or $3m+2$ $(m \in \mathbb{Z})$. $1 + W_3^k + W_3^{2k} = 0$

$\therefore \tilde{X}_3[k] = 0$

CASE 02. When $k = 3m$. $(m \in \mathbb{Z})$

$$\tilde{X}_3[k] = \frac{3}{N} \sum_{n=0}^{N-1} W_{3N}^{kn} \sum_{\ell=0}^{N-1} \tilde{X}[\ell] W_N^{-\ell n}$$

$$= \frac{3}{N} \sum_{\ell=0}^{N-1} \tilde{X}[\ell] \sum_{n=0}^{N-1} W_N^{(m-\ell)n}$$

We know $\sum_{n=0}^{N-1} W_N^{(m-\ell)n} = \begin{cases} 0 & \text{if. } m-\ell \neq 0 \\ N & \text{if. } m-\ell = 0 \end{cases}$

$$\therefore \tilde{X}_3[k] = \frac{3}{N} \cdot N \tilde{X}[m] = 3 \tilde{X}[\tfrac{k}{3}]$$

Thus, $\tilde{X}_3[k] = \begin{cases} 3\tilde{X}[\frac{k}{3}] & \text{if } 3 \mid k \\ 0 & \text{if } 3 \nmid k \end{cases}$ $\square$

(b) Solution: $\tilde{X}[0] = \tilde{x}[0]\cdot W_2^0 + \tilde{x}[1]\cdot W_2^0 = 1+2 = 3.$

$$\tilde{X}[1] = \tilde{x}[0]\cdot W_2^0 + \tilde{x}[1]\cdot W_2^1 = 1-2 = -1.$$

$$\tilde{X}_3[0] = \sum_{n=0}^{5} \tilde{x}[n]\, W_6^0 = 1+2+1+2+1+2 = 9$$

$$\tilde{X}_3[1] = \sum_{n=0}^{5} \tilde{x}[n]\, W_6^n = \left(e^0 + e^{-\frac{2}{6}\pi} + e^{-\frac{4}{6}\pi}\right) + 2\cdot\left(e^{-\frac{\pi}{6}} + e^{-\frac{3}{6}\pi} + e^{-\frac{5}{6}\pi}\right) = 0$$

$$\tilde{X}_3[2] = \sum_{n=0}^{5} \tilde{x}[n]\, W_6^{2n} = \sum_{n=0}^{5} \tilde{x}[n]\, W_3^n = 3\left(e^0 + e^{-\frac{\pi}{3}} + e^{-\frac{2}{3}\pi}\right) = 0$$

$$\tilde{X}_3[3] = \sum_{n=0}^{5} \tilde{x}[n]\, W_6^{3n} = \sum_{n=0}^{5} \tilde{x}[n]\, W_2^n = 1-2+1-2+1-2 = -3$$

$$\tilde{X}_3[4] = \sum_{n=0}^{5} \tilde{x}[n]\, W_6^{4n} = \sum_{n=0}^{5} \tilde{x}[n]\, W_3^{2n} = 3\left(e^0 + e^{-\frac{2}{3}\pi} + e^{-\frac{\pi}{3}}\right) = 0$$

$$\tilde{X}_3[5] = \sum_{n=0}^{5} \tilde{x}[n]\, W_6^{5n} = \sum_{n=0}^{5} \tilde{x}[n]\, W_6^{-n} = \left(e^0 + e^{-\frac{4}{6}\pi} + e^{-\frac{2}{6}\pi}\right) + 2\cdot\left(e^{-\frac{\pi}{6}} + e^{-\frac{3}{6}\pi} + e^{-\frac{5}{6}\pi}\right)$$

$$= 0.$$

Since $\forall n.\ \tilde{X}[n] = \tilde{X}[n+2].\ \tilde{X}_3[n] = \tilde{X}_3[n+6]$

Thus $\tilde{X}[n] = \begin{cases} 3 & n = 2k \\ -1 & n = 2k+1 \end{cases} \quad (k \in \mathbb{Z})$

$$\tilde{X}_3[n] = \begin{cases} 9 & n = 6k \\ -3 & n = 6k+3 \\ 0 & \text{otherwise} \end{cases} \quad (k \in \mathbb{Z})$$

Meanwhile, Let $\tilde{Y}[n] = \begin{cases} 3\tilde{X}[\frac{n}{3}] & n = 3k \\ 0 & \text{otherwise} \end{cases} \quad (k \in \mathbb{Z})$

We know $\tilde{Y}[n] = \begin{cases} 9 & n = 6k \\ -3 & n = 6k+3 \\ 0 & \text{otherwise} \end{cases} = \tilde{X}_3[n]$

Therefore, our result in (a) is verified. $\square$

Question 02.

(a) Solution: $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-nj\omega} = \sum_{n=0}^{+\infty} \alpha^n e^{-nj\omega} = \frac{1}{1-\alpha e^{-j\omega}}$ $\square$

(b) Solution: $\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, W_N^{+kn} = \sum_{n=0}^{N-1} \sum_{r=-\infty}^{+\infty} x[n+rN]\, W_N^{kn} = \sum_{n=0}^{N-1} \sum_{r=0}^{+\infty} x[n+rN]\, W_N^{kn}$

$$= \sum_{n=0}^{N-1} \sum_{r=0}^{+\infty} \alpha^{n+rN}\, W_N^{kn} = \sum_{n=0}^{N-1} W_N^{kn} \cdot \frac{\alpha^n}{1-\alpha^N}$$

$$= \frac{1}{1-\alpha^N} \sum_{n=0}^{N-1} \left(\alpha \cdot e^{-\frac{2j\pi k}{N}}\right)^n$$

$$= \frac{1}{1-\alpha^N} \cdot \frac{1-\alpha^N \cdot e^{-\frac{2j\pi N}{N}}}{1-\alpha e^{-\frac{2jk\pi}{N}}}$$

$$= \frac{1}{1-\alpha e^{\frac{-2jk\pi}{N}}}$$ $\square$

(c) Solution: Compare the results in (a) and (b). We know

$\tilde{X}[k]$ is the result of sampling on $X(e^{j\omega})$,

where the sampling interval $\omega_s = \frac{2\pi}{N}$. $\square$