Exercise Sheet 6

Discrete Mathematics, 2020.10.9

1. Consider the first order language with symbol set $S = \{R\}$ in which R represents a binary predicate.

   a) Let $\mathcal{J}_1$ be an S-interpretation such that
      - the domain in $\mathcal{J}_1$ is $\mathbb{N}$,
      - $\mathcal{J}_1(R, a, b) = \mathbf{T}$ if and only if $a < b$.

      Prove that $[\![\forall x \exists y R(x, y)]\!]_{\mathcal{J}_1} = \mathbf{T}$

   b) Let $\mathcal{J}_2$ be an S-interpretation such that
      - the domain in $\mathcal{J}_2$ is $\mathbb{N}$,
      - $\mathcal{J}_2(R, a, b) = \mathbf{T}$ if and only if $a > b$.
      - $\mathcal{J}_2(x) = 0$

      Prove that $[\![\exists y R(x, y)]\!]_{\mathcal{J}_2} = \mathbf{F}$

   c) Let $\mathcal{J}_3$ be an S-interpretation such that
      - the domain in $\mathcal{J}_3$ is $\mathbb{N}$,
      - $\mathcal{J}_3(R, a, b) = \mathbf{T}$ if and only if $a > b$.

      Prove that $[\![\forall x \exists y R(x, y)]\!]_{\mathcal{J}_3} = \mathbf{F}$

   d) Prove that $[\![\forall x \forall y (R(x, y) \to \exists z (R(x, z) \wedge R(z, y)))]\!]_{\mathcal{J}_1} = \mathbf{F}$.

   e) Let $\mathcal{J}_4$ be an S-interpretation such that
      - the domain in $\mathcal{J}_4$ is $\mathbb{Q}$ (rational numbers, 有理数集),
      - $\mathcal{J}_4(R, a, b) = \mathbf{T}$ if and only if $a < b$.

      Prove that $[\![\forall x \forall y (R(x, y) \to \exists z (R(x, z) \wedge R(z, y)))]\!]_{\mathcal{J}_4} = \mathbf{T}$.

2. Consider the first order language with symbol set $S = \{f, R\}$ in which f represents a binary function and R represents a binary predicate.

   a) Let $\mathcal{J}_1$ be an S-interpretation such that
      - the domain in $\mathcal{J}_1$ is $\mathbb{N}$,
      - $\mathcal{J}_1(f, a, b) = a + b$,
      - $\mathcal{J}_1(R, a, b) = \mathbf{T}$ if and only if $a = b$.

      Prove that $[\![\forall x \forall y R(f(x, y), f(y, x))]\!]_{\mathcal{J}_1} = \mathbf{T}$.

   b) Let $\mathcal{J}_2$ be an S-interpretation such that
      - the domain in $\mathcal{J}_2$ is $\mathbb{N}$,
      - $\mathcal{J}_2(f, a, b) = a * b$,
      - $\mathcal{J}_2(R, a, b) = \mathbf{T}$ if and only if $a = b$.

      Prove that $[\![\forall x \forall y R(f(x, y), f(y, x))]\!]_{\mathcal{J}_2} = \mathbf{T}$.

   c) Let $\mathcal{J}_3$ be an S-interpretation such that
      - the domain in $\mathcal{J}_3$ is $\{\mathbf{T}, \mathbf{F}\}$,
      - $\mathcal{J}_3(f, a, b) = [\![\wedge]\!]\,(a, b)$,
      - $\mathcal{J}_3(R, a, b) = \mathbf{T}$ if and only if $a = b$.

      Prove that $[\![\forall x \forall y R(f(x, y), f(y, x))]\!]_{\mathcal{J}_3} = \mathbf{T}$.

   d) Prove that $\forall x \forall y R(f(x, y), f(y, x))$ is not valid.

3. If a first order logic have the following proof rule, is it possible to be sound? Why?

   - If $\Phi \vdash \psi$, then $\Phi \vdash \forall x \psi$

4. a) ([R], Page 125, Exercise 1(c)) List the members of this set:

$$\{x|x \text{ is the square of an integer and } x < 100\}$$

   b) List all subsets of $\{\emptyset, \{\emptyset\}\}$.

5. Are the following statements correct? Explain why (informally).

   a) ([R], Page 125, Exercise 10(c)) $\{\emptyset\} \in \{\emptyset\}$
   b) ([R], Page 125, Exercise 10(d)) $\{\emptyset\} \in \{\{\emptyset\}\}$
   c) For any set A, $A \subseteq \mathcal{P}(A)$.
   d) For any set A, $A \in \mathcal{P}(A)$.

6. a) Prove that if $\phi \vDash \psi$ then $\forall x \phi \vDash \forall x \psi$.
   b) Prove that if $\Phi, \phi \vDash \psi$ and x does not freely occur in Φ then $\Phi, \forall x \phi \vDash \forall x \psi$.
   c) Demonstrate an example in which
      - $\Phi, \phi \vDash \psi$
      - x does freely occur in Φ
      - $\Phi, \forall x \phi \nvDash \forall x \psi$.

7. ([R], Page 126, Exercise 18) Find two sets A and B such that $A \in B$ and $A \subseteq B$