Discrete Mathematics Exercise 12

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1. **a)** ***Solution:*** Yes.

b) ***Solution:*** No. f is not even a function from S into $\bigcup S$. (The domain of f is not S.)

c) ***Solution:*** Yes.

d) ***Solution:*** No. f is not even a function from $\mathcal{P}(\mathbb{N})$ into $\mathbb{N}$. (The domain of f is not $\mathcal{P}(\mathbb{N})$.)

2. ***Proof:***

First, we prove that $\bigcup\{ [a]_{\mathcal{R}} \mid a \in A\} = A$.

For any $y \in \bigcup\{ [a]_{\mathcal{R}} \mid a \in A\} = A$, exists $x \in \{ [a]_{\mathcal{R}} \mid a \in A\}$ **s.t.** $y \in x$.

Exists $a \in A$ **s.t.** $x = [a]_{\mathcal{R}}$. Since $y \in x,\ a\mathcal{R}y$.

Thus, $y \in A$. (since $\mathcal{R}$ is a relation on A)

Thus, $\bigcup\{ [a]_{\mathcal{R}} \mid a \in A\} \subseteq A$.

For any $y \in A$, $y \in [y]_{\mathcal{R}}$.

It's plain to see that $[y]_{\mathcal{R}} \in \{ [a]_{\mathcal{R}} \mid a \in A\}$. Thus, $y \in \bigcup\{ [a]_{\mathcal{R}} \mid a \in A\} = A$.

Thus, $A \subseteq \bigcup\{ [a]_{\mathcal{R}} \mid a \in A\}$.

Therefore, $\bigcup\{ [a]_{\mathcal{R}} \mid a \in A\} = A$.

Let $S = \{ [a]_{\mathcal{R}} \mid a \in A\}$.

By Axiom of Choices, we know there exists $f: S \to \bigcup S$ **s.t.** $f(X) \in X$ for any $X \in S$.

Now we prove f is an injection.

For any $X_1, X_2 \in S$, if $f(X_1) = f(X_2) = a$, then $a \in X_1$ and $a \in X_2$.

Thus, $X_1 = [a]_{\mathcal{R}}$ and $X_2 = [a]_{\mathcal{R}}$. Therefore, $X_1 = X_2$.

Thus, there exists an injection f from $\{ [a]_{\mathcal{R}} \mid a \in A\}$ into A.

QED