Probability Theory and Mathematical Statistics 概率统计

Homework 1115

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5-5. 解: $E(X_n)=0.5(-\sqrt{\ln n})+0.5\sqrt{\ln n}=0 \quad \Rightarrow \frac{1}{n}\sum_{i=1}^{n}E(X_i)=0$

$E(X_n^2)=0.5\ln n+0.5\ln n=\ln n.$ $\quad D(X_n)=E(X_n^2)-E^2(X_n)=\ln n$

由Chebyshev不等式, $P\left(\left|\frac{1}{n}\sum_{i=1}^{n}X_i-\frac{1}{n}\sum_{i=1}^{n}E(X_i)\right|\geq\varepsilon\right)\leq\frac{D(\frac{1}{n}\sum_{i=1}^{n}X_i)}{\varepsilon^2}=\frac{\sum_{i=1}^{n}D(X_i)}{n^2\varepsilon^2}$ (∵独立)

$=\frac{\sum_{i=1}^{n}\ln n}{n^2\varepsilon^2}$

即 $P\left(\left|\frac{1}{n}\sum_{i=1}^{n}X_i-\frac{1}{n}\sum_{i=1}^{n}E(X_i)\right|<\varepsilon\right)=1-\frac{\sum_{i=1}^{n}\ln n}{n^2\varepsilon^2}$. 当$n\to\infty$时. 该式$\to 1$.

即 $\lim_{n\to\infty}P\left(\left|\frac{1}{n}\sum_{i=1}^{n}X_i-\frac{1}{n}\sum_{i=1}^{n}E(X_i)\right|<\varepsilon\right)=1$. , $\{X_n\}$服从大数定律 □

5-7. 证明: 记 $A_1=X_1,\ A_2=X_3,\ A_3=X_5,\ \cdots \quad B_1=X_2,\ B_2=X_4,\ B_3=X_6,\ \cdots$

则 $A_1,\cdots A_m\cdots$ 相互独立且同分布; $B_1,\cdots B_m\cdots$ 相互独立且同分布.

由Khintchine大数定律知: $\lim_{m\to\infty}P\left(\left|\frac{1}{m}\sum_{i=1}^{m}A_i-\mu\right|<\varepsilon\right)=1=\lim_{m\to\infty}P\left(\left|\frac{1}{m}\sum_{i=1}^{m}B_i-\mu\right|<\varepsilon\right)$

∵ [redacted] $\left|\frac{1}{n}\sum_{i=1}^{n}X_i-\mu\right|=\frac{1}{2}\left|\frac{2}{n}\sum_{i=1}^{\frac{n}{2}}A_i-\mu+\frac{2}{n}\sum_{i=1}^{\frac{n}{2}}B_i-\mu\right|$

∴ $\left|\frac{2}{n}\sum_{i=1}^{\frac{n}{2}}A_i-\mu\right|<\varepsilon$ 且 $\left|\frac{2}{n}\sum_{i=1}^{\frac{n}{2}}B_i-\mu\right|<\varepsilon$ 时. 有 $\left|\frac{1}{n}\sum_{i=1}^{n}X_i-\mu\right|<\frac{1}{2}(\varepsilon+\varepsilon)=\varepsilon$

∴ $\lim_{n\to\infty}P\left(\left|\frac{1}{n}\sum_{i=1}^{n}X_i-\mu\right|<\varepsilon\right)=\frac{1}{2}\left(\lim_{n\to\infty}P\left(\left|\frac{1}{n}\sum_{i=1}^{n}A_i-\mu\right|<\varepsilon\right)+\lim_{n\to\infty}P\left(\left|\frac{1}{n}\sum_{i=1}^{n}B_i-\mu\right|<\varepsilon\right)\right)$

$=\frac{1}{2}(1+1)=1.$ 得证 □

[redacted]

$D(X)=(0-1)^2\times0.2+(1-1)^2\times0.6+(2-1)^2\times0.2=0.4$

5-9. 解: $E(X)=0\times0.2+1\times0.6+2\times0.2=1$. 记一年累积销售量为$Y$.

则 $Y=\sum_{i=1}^{52}X_i$, 其中X_i相互独立且同分布, 均服从与X相同的分布. $\quad E(X_i)=1$

∴ $E(Y)=\sum_{i=1}^{52}E(X_i)=52.$ [redacted] $D(Y)=\sum_{i=1}^{52}D(X_i)=20.8$

(1) 取$\varepsilon=10$. 由Chebyshev不等式知 $P(42<Y<62)=P(|Y-E(Y)|<10)\leq1-\frac{D(Y)}{\varepsilon^2}$

$=0.792$ □

5.11. 解：(1) $\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{60} k(30-|x-30|)\,dx = \frac{1}{2}\cdot 60\cdot 30\cdot k = 1 \Rightarrow k = \frac{1}{900}$ □

(2) 由对称性：$E(X)=30$. $D(X) = 2\int_0^{30} \frac{1}{900}(30-x)x^2\,dx = 150$

记第i位顾客消费额 X_i. X_i 独立同分布，与 X 分布相同，记每天营业额为 Y 元。

$Y = \sum_{i=1}^{200} X_i$. $E(Y) = \sum_{i=1}^{200} E(X_i) = 6000$. $D(Y) = \sum_{i=1}^{200} D(X_i) = 30000$

由 Chebyshev 不等式，

$P(5800 \le Y \le 6200) = P(|Y-E(Y)| \le 200)$

$\ge 1 - \frac{30000}{200^2} = \frac{1}{4}$ □

补充1. 证明：构造函数 $f(X) = \begin{cases} 1, & |X| \ge \varepsilon \\ 0, & |X| < \varepsilon \end{cases}$ 令 $Y = f(X)\cdot|X|^k$ 为随机变量.

显然有 $|f(X)\cdot X^k| \le |X|^k \Rightarrow E(f(X)\cdot|X|^k) \le E(|X|^k)$

而 $P(|X|\ge\varepsilon)\cdot\varepsilon^k \le E(f(X)\cdot|X|^k) \le E(|X|^k)$ $(\because \varepsilon>0)$

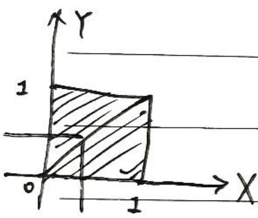
$\Rightarrow P(|X|\ge\varepsilon) \le \frac{E(|X|^k)}{\varepsilon^k}$ $(\because \varepsilon>0)$ □

补充2. 解：$\{X_i\}$ 独立同分布，故 $\{X_i^2\}$ 独立同分布. $E(X_i^2)$ 存在.

由 Khintchine 大数定律知 $\lim_{n\to+\infty}\left(\left|\frac{1}{n}\sum_{i=1}^n X_i^2 - E(X_i^2)\right| \ge \varepsilon\right) = 0$

$\therefore \frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow[n\to+\infty]{P} E(X_i^2) = \int_0^{+\infty} \frac{1}{2}x^2 e^{-\frac{x}{2}}\,dx = D(X_i) + E(X_i)^2 = \frac{1}{0.5^2} + \left(\frac{1}{0.5}\right)^2 = 8$ □

补充3. 解：$E(\max\{X,Y\}) = 2\int_0^1 x\,dx\int_0^x dy = \frac{2}{3}$ $E((\max\{X,Y\})^2) = 2\int_0^1 x^2\,dx\int_0^x dy = \frac{1}{2}$



由 Chebyshev 不等式， $D(\max\{X,Y\}) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$

$P\left(|\max\{X,Y\} - E(\max\{X,Y\})| < \frac{1}{3}\right) > \frac{D(\max\{X,Y\})}{(1/3)^2} = \frac{1}{2}$ □

真值计算：$|\max\{X,Y\} - \frac{2}{3}| < \frac{1}{3} \Rightarrow \frac{1}{3} < \max\{X,Y\} < 1$.

$P\left(|\max\{X,Y\} - E(\max\{X,Y\})| < \frac{1}{3}\right) = \frac{1^2 - (\frac{1}{3})^2}{1^2} = \frac{8}{9}$ □

由此可见估计值与真值有较大差距.

补充4. 解: X_i 独立同分布 $\Rightarrow$ $\sin X_i^2$ 独立同分布.

$E(\sin X_i^2) = \int_1^2 1\cdot \sin x^2 \, dx \doteq 0.4945$ 存在

由 Khinchine 大数定律: $\lim\limits_{n\to\infty} \left(\left| \frac{1}{n}\sum\limits_{i=1}^{n} \sin X_i^2 - E(\sin X_i^2) \right| \geq \varepsilon \right) = 0$.

$\therefore \frac{1}{n}\sum\limits_{i=1}^{n} \sin X_i^2 \xrightarrow[n\to+\infty]{P} \int_1^2 \sin x^2 \, dx$ □

该结果可以用于 $\int_1^2 \sin x^2 \, dx$ 具体数值计算. □