[Homework 1] Review of Probability Theory

Probability Space of Tossing Coins

Let us construct the probability space of tossing an infinite sequence of independent fair coins. Let $\Omega = \{0,1\}^{\mathbb{N}}$. We can write each $\omega \in \Omega$ as an infinite sequence $\omega = (\omega_1, \omega_2, \dots)$ where $\omega_i \in \{0,1\}$.

1. Let $n \in \mathbb{N}$. For every $s = (s_1, \dots, s_n) \in \{0,1\}^n$, let

   $$C_s = \{\omega \in \Omega \mid \omega_1 = s_1, \dots, \omega_n = s_n\}.$$

   Prove that for every $n \in \mathbb{N}$, the collection $\{C_s\}_{s \in \{0,1\}^n}$ forms a partition of Ω.

2. Let $\mathcal{F}_n$ be the σ-algebra generated by $\{C_s\}_{s \in \{0,1\}^n}$ (that is, the minimal σ-algebra containing sets in $\{C_s\}_{s \in \{0,1\}^n}$). Note that $\mathcal{F}_n$ is called the σ-algebra of *tossing n coins*. Prove that there exists a bijection between $\mathcal{F}_n$ and $2^{\{0,1\}^n}$.

3. Prove that $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \dots$ is increasing. The collection $\{\mathcal{F}_n\}_{n \geq 1}$ is called a *filtration*.

4. Let $\mathcal{F}_\infty = \bigcup_{n \geq 1} \mathcal{F}_n$. Prove that $\mathcal{F}_\infty$ is an algebra[1] (not necessarily a σ-algebra) and $\mathcal{F}_\infty \neq 2^\Omega$.

5. Let $\mathcal{B}(\Omega) \triangleq \sigma(\mathcal{F}_\infty)$ be the minimal σ-algebra containing $\mathcal{F}_\infty$. Prove that for any $\omega \in \Omega$, it holds that $\{\omega\} \in \mathcal{B}(\Omega) \setminus \mathcal{F}_\infty$.

6. Prove that for every $A \in \mathcal{F}_\infty$, there exist some $n \in \mathbb{N}$ and $s_1, \dots, s_k \in \{0,1\}^n$ such that $A = C_{s_1} \cup \dots \cup C_{s_k}$. Although the choice of n might not be unique, prove that the value $\frac{k}{2^n}$ only depends on A.

7. Prove that there exists a unique probability measure $P : \mathcal{B}(\Omega) \to [0,1]$ satisfying for every $A \in \mathcal{F}_\infty$, $P(A) = \frac{k}{2^n}$ where k and n are defined in the last question.

Then $(\Omega, \mathcal{B}(\Omega), P)$ is our probability space for tossing coins, and it is isomorphic to the Lebesgue measure on $[0,1]$.

8. Formalize $X \sim \mathtt{Geom}(1/2)$ in this probability space.

Conditional Expectation

1. Let X be a random variable and $f : \mathbb{R} \to \mathbb{R}$ be a measurable function (that is, for every borel set $A \in \mathscr{R}$, $f^{-1}(A) \in \mathscr{R}$). We usually use $f(X)$ to denote the random variable: $\omega \in \Omega \mapsto f(X(\omega)) \in \mathbb{R}$. Prove that $f(X)$ is $\sigma(X)$-measurable.
2. Let Y, Y' be two random variables such that $\sigma(Y) = \sigma(Y')$. Prove that $\mathbf{E}[X \mid Y] = \mathbf{E}[X \mid Y']$.
3. The fact you just proved should convince you that the conditional expectation $\mathbf{E}[X \mid Y]$ only depends on the σ-algebra $\sigma(Y)$ (but not the value of Y). Let Ω be the set of outcomes and $X : \Omega \to \mathbb{R}$ be a random variable. Let $\mathcal{F}$ be a σ-algebra on Ω. Can you define the notation $\mathbf{E}[X \mid \mathcal{F}]$?
4. (The coarser always wins) Let $\mathcal{F}_1, \mathcal{F}_2$ be two σ-algebra such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $X : \Omega \to \mathbb{R}$ be a random variable. Prove that $\mathbf{E}[\mathbf{E}[X \mid \mathcal{F}_1] \mid \mathcal{F}_2] = \mathbf{E}[\mathbf{E}[X \mid \mathcal{F}_2] \mid \mathcal{F}_1] = \mathbf{E}[X \mid \mathcal{F}_1]$.

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1. A set $\mathcal{F}$ is an algebra if for every $A, B \in \mathcal{F}$, it holds $A^c \in \mathcal{F}$ and $A \cup B \in \mathcal{F}$. ↩