Digital Signal and Image Processing

Written Assignment #1

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2022/3/7 DSIP Problem Set 1

Question 01.

(a) Solution: $\|x-y\|^2 = \langle x-y, x-y\rangle$

$$= \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle$$

$$= \int_{-\infty}^{+\infty} x^2(t)\,dt - 2\int_{-\infty}^{+\infty} x(t)y(t)\,dt + \int_{-\infty}^{+\infty} y^2(t)\,dt$$

$$= \int_{-\infty}^{+\infty} x^2(t)\,dt - 2\int_0^T c\,x(t)\,dt + \int_0^T c^2\,dt$$

$$= \int_{-\infty}^{+\infty} x^2(t)\,dt - 2c\int_0^T x(t)\,dt + c^2 T$$

$$= T\cdot c^2 - \left(2\int_0^T x(t)\,dt\right)\cdot c + \int_{-\infty}^{+\infty} x^2(t)\,dt$$

$\min\limits_{c} \|x-y\|^2$. Let $f(c) = \|x-y\|^2$.

$$f'(c) = 2cT - 2\int_0^T x(t)\,dt = 0 \Rightarrow c = \frac{\int_0^T x(t)\,dt}{T}. \quad \square$$

Therefore, $P_V(x)(t) = \begin{cases} \frac{1}{T}\int_0^T x(\tau)\,d\tau, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}$ $\square$

(b) Solution: We can find $e(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$.

For any $y(t) \in V$, $y(t) = \begin{cases} c & 0 \le t < T \\ 0 & \text{otherwise} \end{cases} = c\cdot e(t)$. (By definition).

Thus, $V = \mathrm{span}(e)$, where $e(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise.} \end{cases}$

$$\langle x(t), e(t)\rangle = \int_{-\infty}^{+\infty} x(t)e(t)\,dt = \int_0^T x(t)\,dt.$$

By the definition of projection, $\begin{cases} y(t) \in \mathrm{span}(e) \\ x(t) - y(t) \perp e(t) \end{cases}$ Let $y(t) = ce(t)$.

$$\langle x-y, e\rangle = 0 \Leftrightarrow \langle x-ce, e\rangle = 0 \Leftrightarrow \langle x,e\rangle - c\langle e,e\rangle = 0$$

$$\therefore\ c = \frac{\langle x, e\rangle}{\langle e, e\rangle} = \frac{\int_0^T x(t)\,dt}{\int_0^T e^2(t)\,dt} = \frac{1}{T}\int_0^T x(t)\,dt$$

i.e. $$y(t) = P_V(x)(t) = \begin{cases} \frac{1}{T}\int_0^T x(t)\,dt & 0 \le t < T. \\ 0 & \text{otherwise} \end{cases}$$ □

(c) Proof: From (a) and (b), we know both methods eventually yields

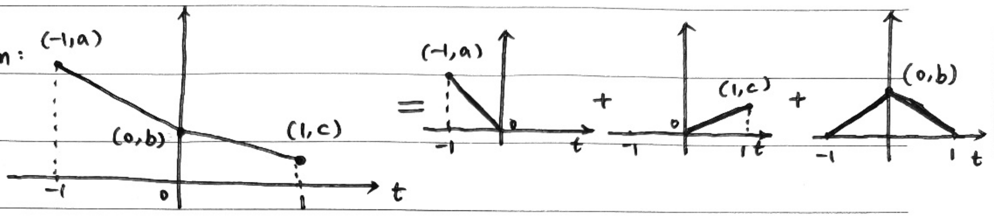
$$y(t) = \begin{cases} \frac{1}{T}\int_0^T x(t)\,dt & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

I personally prefer the former one since it is more direct. □

Question 02.

(a) Solution:

Proof:



Obvious, $\forall f \in V$. $$f = a\varphi_{-1} + b\varphi_0 + c\varphi_1$$
$$= f(-1)\varphi_{-1} + f(0)\varphi_0 + f(1)\varphi_1$$ □

Another Proof: $\forall f \in V$. $$f(t) = \begin{cases} f(0) + [f(0) - f(-1)]t. & -1 \le t \le 0 \\ f(0) + [f(1) - f(0)]t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Obvious. $$f(t) = f(0)\varphi_0(t) + [-(f(0) - f(-1)) + f(0)]\varphi_{-1}(t)$$
$$+ [f(1) - f(0) + f(0)]\varphi_1(t)$$
$$= f(0)\varphi_0 + f(-1)\varphi_{-1} + f(1)\varphi_1.$$ □

(b) Solution: $$\langle \varphi_{-1}, \varphi_0\rangle = \int_{-\infty}^{+\infty} \varphi_{-1}(t)\varphi_0(t)\,dt = \int_{-1}^{0} -t\cdot(t+1)\,dt = \frac{1}{6}$$
$$\langle \varphi_0, \varphi_1\rangle = \int_{-\infty}^{+\infty} \varphi_0(t)\varphi_1(t)\,dt = \int_0^1 t(-t+1)\,dt = \frac{1}{6}$$
$$\langle \varphi_{-1}, \varphi_1\rangle = \int_{-\infty}^{+\infty} \varphi_{-1}(t)\varphi_1(t)\,dt = 0.$$

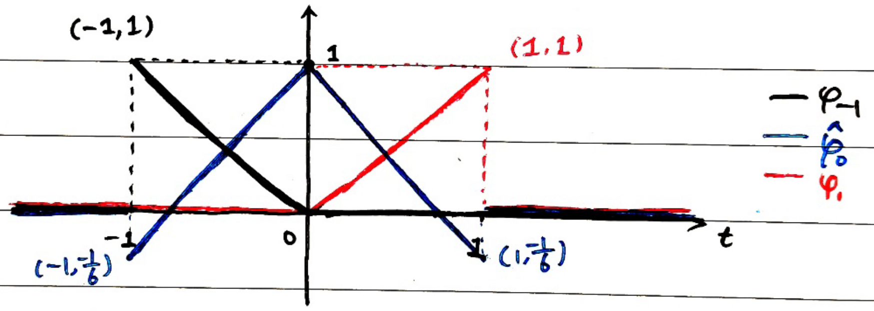
Thus, φ_{-1} and φ_1 are orthogonal. □

(c) Solution: By Kindergarten Formula, we have

$$Pr_{V_1}(\varphi_0) = \langle \varphi_0, \varphi_{-1} \rangle \varphi_{-1} + \langle \varphi_0, \varphi_1 \rangle \varphi_1$$
$$= \frac{1}{6}\varphi_{-1} + \frac{1}{6}\varphi_1 .$$

$$\hat{\varphi}_0 = \varphi_0 - Pr_{V_1}(\varphi_0) = \varphi_0 - \frac{1}{6}\varphi_{-1} - \frac{1}{6}\varphi_1$$

The visualization of $\{\varphi_{-1}, \hat{\varphi}_0, \varphi_1\}$ is as follows.



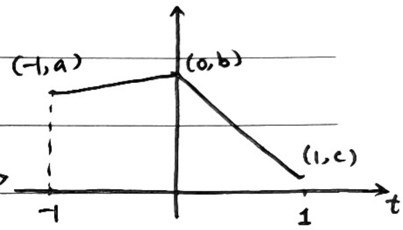
Since $\varphi_0 = \hat{\varphi}_0 + \frac{1}{6}\varphi_1 + \frac{1}{6}\varphi_{-1}$, it's obvious that

$$V = \text{span}\{\varphi_0, \varphi_{-1}, \varphi_1\} = \text{span}\{\varphi_{-1}, \hat{\varphi}_0, \varphi_1\}.$$

i.e. all functions in V can be represented as the linear combination of the family $\{\varphi_{-1}, \hat{\varphi}_0, \varphi_1\}$. $\square$

(d) Solution: Suppose $f(t) = a\varphi_{-1}(t) + b\varphi_0(t) + c\varphi_1(t)$.

By (a), we can plot $f(t)$ as follows. ($f(-1)=a$, $f(0)=b$, $f(1)=c$)



By definition, we know

$$f = \underset{f}{\arg\min} \|f-g\| = \underset{f}{\arg\min} \|f-g\|^2.$$

$$\|f-g\|^2 = \langle f-g, f-g \rangle = \langle f,f \rangle - 2\langle f,g \rangle + \langle g,g \rangle$$
$$= \langle a\varphi_{-1} + b\varphi_0 + c\varphi_1, a\varphi_{-1} + b\varphi_0 + c\varphi_1 \rangle - 2\langle f,g \rangle + \langle g,g \rangle$$
$$= a^2\langle \varphi_{-1}, \varphi_{-1} \rangle + b^2\langle \varphi_0, \varphi_0 \rangle + c^2\langle \varphi_1, \varphi_1 \rangle + 2ab\langle \varphi_{-1}, \varphi_0 \rangle + 2bc\langle \varphi_0, \varphi_1 \rangle$$
$$+ 2ac\langle \varphi_{-1}, \varphi_1 \rangle - 2\langle f, g \rangle + \langle g, g \rangle$$
$$= a^2\int_{-1}^{0} t^2 dt + b^2\int_{-1}^{1} \varphi_0^2 \, dt + c^2\int_0^1 t^2 dt + \frac{ab}{3} + \frac{bc}{3} + 0$$
$$-2\int_{-1}^{\frac{2}{3}} f(t)\,dt + \int_{-1}^{\frac{2}{3}} 1 \, dt$$
$$= \frac{a^2}{3} + \frac{2}{3}b^2 + \frac{c^2}{3} + \frac{ab}{3} + \frac{bc}{3} - 2\left(\frac{1}{2}\cdot(a+b)\cdot 1 + \frac{1}{2}\cdot\left(b + \frac{2}{3}c + \frac{1}{3}b\right)\cdot\frac{2}{3}\right) + \frac{5}{3}$$
$$= \frac{a^2}{3} + \frac{2}{3}b^2 + \frac{c^2}{3} + \frac{ab}{3} + \frac{bc}{3} - a - \frac{17}{9}b - \frac{4}{9}c + \frac{5}{3}$$

Let $h(a,b,c) = \|f-g\|^2$.

$$
\begin{cases}
\dfrac{\partial h}{\partial a} = \dfrac{2}{3}a + \dfrac{b}{3} - 1 = 0 \\[2mm]
\dfrac{\partial h}{\partial b} = \dfrac{4}{3}b + \dfrac{a}{3} + \dfrac{c}{3} - \dfrac{17}{9} = 0 \\[2mm]
\dfrac{\partial h}{\partial c} = \dfrac{2}{3}c + \dfrac{b}{3} - \dfrac{4}{9} = 0
\end{cases}
\Rightarrow
\begin{cases}
a = \dfrac{11}{12} \\[2mm]
b = \dfrac{7}{6} \\[2mm]
c = \dfrac{1}{12}
\end{cases}
$$

Thus, $f(t) = \frac{11}{12}\varphi_{-1}(t) + \frac{7}{6}\varphi_0(t) + \frac{1}{12}\varphi_1(t)$.

The visualization of $g(t)$ and $f(t)$ is as follows.