Probability Theory and Mathematical Statistics 概率统计

Homework 1129

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6-9. 解：(1) $X_1, \cdots X_{10} \sim N(0, 0.25) \Rightarrow 2X_1, \cdots 2X_{10} \sim N(0,1)$

则 $\sum_{i=1}^{10}(2X_i)^2 = 4\sum_{i=1}^{10} X_i^2 \sim \chi^2(10)$

$\therefore P(\sum_{i=1}^{10} X_i^2 \geq 4) = P(4\sum_{i=1}^{10} X_i^2 \geq 16) = P(\chi^2(10) \geq 16) = 0.1$ （查表可知） □

(2) $\sum_{i=1}^{10}(X_i - \bar{X})^2 \cdot \frac{1}{0.25} \sim \chi^2(9)$.

$\therefore P(\sum_{i=1}^{10}(X_i - \bar{X})^2 \geq 4.23) = P(4\sum_{i=1}^{10}(X_i - \bar{X})^2 \geq 16.92) = P(\chi^2(9) \geq 16.92)$

查表可知 $P(\sum_{i=1}^{10}(X_i - \bar{X})^2 \geq 4.23) = P(\chi^2(9) \geq 16.92) = 0.05$ □

6-10. 解：(1) 要使 $aX_1^2 + b(X_2+X_3+X_4)^2 + c\sum_{i=5}^{9}(X_i - Y)^2$ 服从 χ^2 分布，

则 $\sqrt{a}X_1 \sim N(0,1)$. $\sqrt{b}(X_2+X_3+X_4) \sim N(0,1)$. $\sqrt{c}X_i \sim N(0,1)$

$\because X_i \sim N(0,4) \quad \therefore a = \frac{1}{4}. \quad b = \frac{1}{12}. \quad c = \frac{1}{4}$ □

(2) 要使 $d\frac{X_1+X_2}{\sqrt{X_3^2+X_4^2+X_5^2}}$ 服从 t 分布. 则 $d\frac{X_1+X_2}{\sqrt{X_3^2+X_4^2+X_5^2}} = \frac{Z}{\sqrt{W/n}}$.

其中 $Z \sim N(0,1)$. $W \sim \chi^2(n)$. 记 $Z = d_1(X_1+X_2)$. $W = d_2(X_3^2+X_4^2+X_5^2)$

显然 $W \sim \chi^2(n)$ 时 $n=3$. $\therefore d = \frac{d_1}{\sqrt{d_2/3}}$.

$\because X_i \sim N(0,4)$. $d_1 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{4}} = \frac{1}{2\sqrt{2}}$ $\quad d_2 = \frac{1}{4}$

$\therefore d = \frac{\frac{1}{2\sqrt{2}}}{\frac{1}{\sqrt{12}}} = \frac{\sqrt{6}}{2}$ 自由度 $n=3$ □

6-11. 解：$\because X_{n+1} - \bar{X} \sim N(0, \frac{n+1}{n}\sigma^2)$ $\quad \frac{n-1}{\sigma^2}S^2 \sim \chi^2(n-1)$. $\Rightarrow \frac{X_{n+1}-\bar{X}}{\sqrt{\frac{n+1}{n}}\sigma} \sim N(0,1)$

$\therefore Y_1 = \frac{X_{n+1}-\bar{X}}{S}\sqrt{\frac{n}{n+1}} = \frac{\frac{X_{n+1}-\bar{X}}{\sqrt{\frac{n+1}{n}}\sigma}}{\sqrt{\frac{(n-1)S^2}{\sigma^2(n-1)}}} \triangleq \frac{Y}{\sqrt{\frac{Z}{n-1}}}$ 其中 $Y = \frac{X_{n+1}-\bar{X}}{\sqrt{\frac{n+1}{n}}\sigma} \sim N(0,1)$

$Z = \frac{(n-1)}{\sigma^2}S^2 \sim \chi^2(n-1)$

Y, Z 相互独立

$\therefore Y_1 \sim t(n-1)$. □

$$Y_2 = \frac{(X_{n+1}-\bar{X})^2 / (\frac{n+1}{n}\sigma^2)}{(n-1)S^2/(\sigma^2(n-1))} \triangleq \frac{Y^2/1}{Z/n-1}$$. 其中 $Y\sim N(0,1)$. $\Rightarrow Y^2\sim\chi^2(1)$

$Z\sim\chi^2(n-1)$. 两者相互独立

$\therefore Y_2 \sim F(1, n-1)$ □

6-13. 解: (1) $Y = \frac{(X_1+X_2)^2}{(X_1-X_2)^2} = \frac{\frac{1}{2}(X_1+X_2)^2/1}{\frac{1}{2}(X_1-X_2)^2/1}$ $\because \frac{1}{\sqrt{2}}(X_1+X_2)$. $\frac{1}{\sqrt{2}}(X_1-X_2)\sim N(0,1)$

$\therefore \frac{1}{2}(X_1+X_2)^2$. $\frac{1}{2}(X_1-X_2)^2\sim\chi^2(1)$. 两者独立

$\therefore Y\sim F(1,1)$ □

(2) $Z = \frac{X_1^2/1}{\sum_{i=2}^{n} X_i^2/(n-1)}$ $\because X_1\sim N(0,1)$. $X_i\sim N(0,1)$

$\therefore X_1^2\sim\chi^2(1)$. $\sum_{i=2}^{n} X_i^2\sim\chi^2(n-1)$. 两者相互独立

$\therefore Z\sim F(1, n-1)$ □

补充题1. 解: $Y = \sum_{i=1}^{n}(X_i+X_{n+i}-2\bar{X})^2 = (2+\frac{4}{n})\sigma^2\sum_{i=1}^{n}\left(\frac{X_i+X_{n+i}-2\bar{X}}{\sqrt{2\sigma^2+\frac{4}{n}\sigma^2}}\right)^2$

$\because X_i+X_{n+i}-2\bar{X}\sim N(0, \sigma^2+\sigma^2+\frac{4}{n}\sigma^2)$ $\therefore \frac{X_i+X_{n+i}-2\bar{X}}{\sqrt{2\sigma^2+\frac{4}{n}\sigma^2}}\sim N(0,1)$

$\therefore \sum_{i=1}^{n}\left(\frac{X_i+X_{n+i}-2\bar{X}}{\sqrt{2\sigma^2+\frac{4}{n}\sigma^2}}\right)^2\sim\chi^2(n)$

$E(Y) = (2+\frac{4}{n})\sigma^2 E(\chi^2(n)) = (2n+4)\sigma^2$ □

补充题2. 解: (1) $X_1-3\sim N(0,6)$. $X_2-3\sim N(0,6)$ $\therefore \frac{(X_1-3)^2}{6}\sim\chi^2(1)$. $\frac{(X_2-3)^2}{6}\sim\chi^2(1)$.

两者相互独立 当 $k_1=1$ 时. $k_1\frac{(X_1-3)^2}{(X_2-3)^2} = \frac{[(X_1-3)^2/6]/1}{[(X_2-3)^2/6]/1}\sim F(1,1)$ □□

(2) $k_2\frac{Y_1}{Y_2} = k_2\frac{\frac{n-1}{\cancel{n}}S_{1\sim n}^2}{\frac{n-1}{\cancel{n}}S_{n+1\sim 2n}^2} = k_2\frac{\frac{n-1}{6}S_{1\sim n}^2}{\frac{n-1}{6}S_{n+1\sim 2n}^2}$ $\because \frac{n-1}{6}S_{1\sim n}^2\sim\chi^2(n-1)$.

$\frac{n-1}{6}S_{n+1\sim 2n}^2\sim\chi^2(n-1)$ 两者相互独立

$\therefore k_2\frac{Y_1}{Y_2}$ 在 $k_2=1$ 时. $k_2\frac{Y_1}{Y_2} = \frac{\frac{n-1}{6}S_{1\sim n}^2}{\frac{n-1}{6}S_{n+1\sim 2n}^2}\sim F(n-1, n-1)$ □□

(3) $k_3\frac{Z_1}{Y_2} = k_3\frac{\sum_{i=1}^{n}(X_i-3)^2}{\sum_{i=n+1}^{2n}(X_i-\bar{X}_2)^2} = k_3\frac{\frac{1}{6}\sum_{i=1}^{n}(X_i-3)^2}{\frac{1}{6}\sum_{i=n+1}^{2n}(X_i-\bar{X}_2)^2}$ $\because \frac{X_i-3}{\sqrt{6}}\sim N(0,1)$

$\therefore \frac{1}{6}\sum_{i=1}^{n}(X_i-3)^2 \sim \chi^2(n)$

另一方面，$\frac{1}{6}\sum_{i=n+1}^{2n}(X_i-\bar{X}_2)^2 \sim \chi^2(n-1)$. 且分子、分母两者相互独立

$\therefore k_3=1$ 时，$k_3\frac{Z_1}{Y_2} \sim F(n,n-1)$ □

(4) $k_4\frac{\bar{X}_1-3}{\sqrt{Y_2}} = k_4\frac{\frac{\sqrt{n}}{\sqrt{6}}(\bar{X}_1-3)}{\sqrt{\frac{\frac{1}{6}\sum_{i=n+1}^{2n}(X_i-\bar{X}_2)^2}{n-1}\cdot\frac{n-1}{\cancel{n}}\cancel{n}}} = k_4\sqrt{\frac{1}{n-1}}\cdot\frac{\frac{\sqrt{n}}{\sqrt{6}}(\bar{X}_1-3)}{\sqrt{\frac{\frac{1}{6}\sum_{i=n+1}^{2n}(X_i-\bar{X}_2)^2}{n-1}}}$

$\because \frac{1}{\sqrt{6}}(\bar{X}_1-3)\sim N(0,1)$, $\frac{1}{6}\sum_{i=n+1}^{2n}(X_i-\bar{X}_2)^2 \sim \chi^2(n-1)$ 两者相互独立

当 $k_4=\sqrt{n-1}$ 时. $k_4\frac{\bar{X}_1-3}{\sqrt{Y_2}} \sim t(n-1)$. □

(5) $k\frac{X_1-3}{\sqrt{Y_2}} = k\sqrt{\frac{n}{n-1}}\cdot\frac{\frac{1}{\sqrt{6}}(X_1-3)}{\sqrt{\frac{\frac{1}{6}\sum_{i=n+1}^{2n}(X_i-\bar{X}_2)^2}{n-1}}}$ 此时 $\frac{1}{\sqrt{6}}(X_1-3)\sim N(0,1)$

$\frac{1}{6}\sum_{i=n+1}^{2n}(X_i-\bar{X}_2)^2 \sim \chi^2(n-1)$ 相互独立

$\therefore k=\sqrt{\frac{n-1}{n}}$ 时. $k\frac{X_1-3}{\sqrt{Y_2}} \sim t(n-1)$ □