Linear and Convex Optimization Homework 10

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1. ***Solution:***

The Lagrangian function is

$$\begin{aligned}\mathcal{L}(\boldsymbol{x},\boldsymbol{\mu}) &= x_1^2+x_2^2+\mu_1((x_1-1)^2+(x_2-1)^2-1)+\mu_2((x_1-1)^2+{x_2}^2-1)\\ &= (1+\mu_1+\mu_2)x_1^2+(1+\mu_1+\mu_2)x_2^2-2(\mu_1+\mu_2)x_1-2\mu_1x_2+\mu_1.\end{aligned}$$

Thus, the KKT condition is

$$\begin{cases}\nabla\mathcal{L}_{x_1}(\boldsymbol{x},\boldsymbol{\mu}) = (2+2\mu_1+2\mu_2)x_1-2(\mu_1+\mu_2)=0\\ \nabla\mathcal{L}_{x_2}(\boldsymbol{x},\boldsymbol{\mu}) = (2+2\mu_1+2\mu_2)x_2-2\mu_1=0\\ \mu_1\geq 0,\ \ \mu_2\geq 0\\ \mu_1((x_1-1)^2+(x_2-1)^2-1)=0\\ \mu_2((x_1-1)^2+{x_2}^2-1)=0\\ (x_1-1)^2+(x_2-1)^2-1\leq 0\\ (x_1-1)^2+{x_2}^2-1\leq 0\end{cases}$$

■

Now we solve the KKT condition.

CASE 1. When $g_1(\boldsymbol{x})$ is inactive. Then $\mu_1=0\Rightarrow x_2=0\Rightarrow x_1=1\Rightarrow 2=0.$

Contradiction. Discard.

CASE 2. $g_1(\boldsymbol{x})$ and $g_2(\boldsymbol{x})$ are both active. Then $\mu_1>0,\mu_2>0\Rightarrow\begin{cases}x_1=\frac{\mu_1+\mu_2}{\mu_1+\mu_2+1}\\ x_2=\frac{\mu_1}{\mu_1+\mu_2+1}\end{cases}.$

Plug into complementary slackness equation, we have

$$\begin{cases}x_1=\dfrac{\sqrt{3}}{2}+1\\ x_2=\dfrac{1}{2}\end{cases}\Rightarrow\frac{1}{\mu_1+\mu_2+1}=-\frac{\sqrt{3}}{2}<0$$

It violates $\mu_1>0,\mu_2>0$. Discard.

CASE 3. $g_1(\boldsymbol{x})$ is active while $g_2(\boldsymbol{x})$ is inactive. Then $\mu_1>0,\mu_2=0.$

Thus, $\begin{cases}x_1=\frac{\mu_1}{\mu_1+1}=1-\frac{\sqrt{2}}{2}\\ x_2=\frac{\mu_1}{\mu_1+1}=1-\frac{\sqrt{2}}{2}\end{cases}\Rightarrow\begin{cases}x_1=1-\frac{\sqrt{2}}{2}\\ x_2=1-\frac{\sqrt{2}}{2}\\ \mu_1=\sqrt{2}\\ \mu_2=0\end{cases}.$

(The other solution is discarded since it violates $\mu_1\geq 0$.)

In this case, $f(\boldsymbol{x})=3-2\sqrt{2}$. Minimum.

In conclusion, the minimum point is $\left(1-\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{2}\right)^T.$

The corresponding Lagrangian multiplier is $\begin{cases}\mu_1=\sqrt{2}\\ \mu_2=0\end{cases}.$ ■

2.(a) ***Solution:***

$$\begin{cases}(x_1-1)^2+(x_2-1)^2\le 1\\(x_1-1)^2+(x_2+1)^2\le 1\end{cases}\Rightarrow\begin{cases}x_1=1\\x_2=0\end{cases}$$

Thus, the feasible set is $X=\{(1,0)^T\}$.

The sketch of the feasible set (shown in red) and the level set of the objective function (shown in light blue) is as follows.

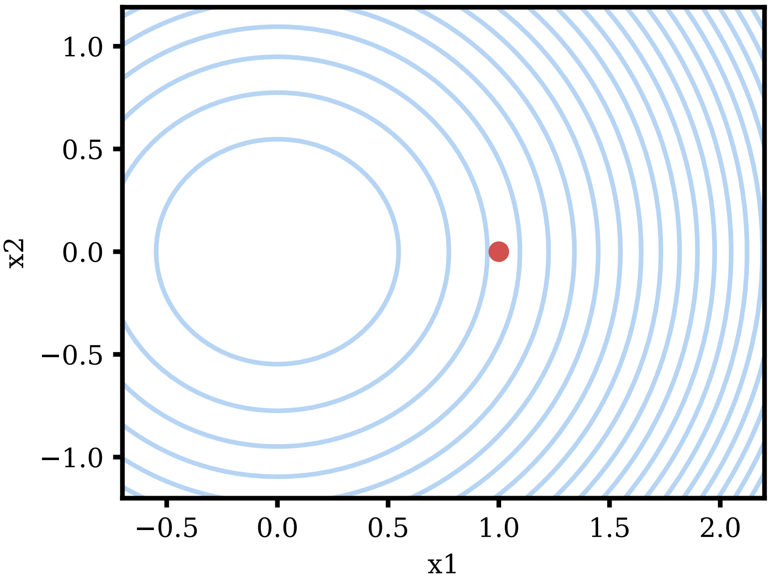

Fig.01. The Sketch of the Feasible Set and the Level Set of the Objective Function

Obviously, the optimal point $\boldsymbol{x}^\star=(1,0)^T$ and the optimal value $f(\boldsymbol{x}^\star)=1$. ■

(b) ***Solution:***

The Lagrangian function is

$$\begin{aligned}\mathcal{L}(\boldsymbol{x},\boldsymbol{\mu})&=x_1^2+x_2^2+\mu_1((x_1-1)^2+(x_2-1)^2-1)+\mu_2((x_1-1)^2+(x_2+1)^2-1)\\&=(1+\mu_1+\mu_2)x_1^2+(1+\mu_1+\mu_2)x_2^2-2(\mu_1+\mu_2)x_1-2(\mu_1-\mu_2)x_2+\mu_1+\mu_2.\end{aligned}$$

Thus, the KKT conditions are as follows.

$$\begin{cases}\nabla\mathcal{L}_{x_1}(\boldsymbol{x},\boldsymbol{\mu})=(2+2\mu_1+2\mu_2)x_1-2(\mu_1+\mu_2)=0\\\nabla\mathcal{L}_{x_2}(\boldsymbol{x},\boldsymbol{\mu})=(2+2\mu_1+2\mu_2)x_2-2(\mu_1-\mu_2)=0\\\mu_1\ge 0,\ \ \mu_2\ge 0\\\mu_1((x_1-1)^2+(x_2-1)^2-1)=0\\\mu_1((x_1-1)^2+(x_2+1)^2-1)=0\\(x_1-1)^2+(x_2-1)^2-1\le 0\\(x_1-1)^2+(x_2+1)^2-1\le 0\end{cases}$$

We already know the optimal point is $(1,0)^T$. Plug it into the KKT condition and we get $2=0$, **i.e.** the KKT condition can never be satisfied. Therefore, there exist **NO** such Lagrange multipliers that satisfy the KKT condition. ■

Let $g_1(\boldsymbol{x})=(x_1-1)^2+(x_2-1)^2-1, g_2(\boldsymbol{x})=(x_1-1)^2+(x_2+1)^2-1$.

$$\nabla g_1(\boldsymbol{x})=(2x_1-2,2x_2-2),\nabla g_2(\boldsymbol{x})=(2x_1-2,2x_2+2).$$

Considering the gradient of constraints, $\nabla g_1(\boldsymbol{x}^\star) = (0, -2), \nabla g_2(\boldsymbol{x}^\star) = (0,2)$, are not linearly independent, we know $\boldsymbol{x}^\star$ **is not a regular point.** ■

3. ***Solution:***

Lagrangian function is

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\mu}) = \left(x_1 - \frac{9}{4}\right)^2 + (x_2 - 2)^2 - \mu_1(-x_1^2 + x_2) + \mu_2(x_1 + x_2 - 6) - \mu_3 x_1 - \mu_4 x_2$$

Consider the case of $\boldsymbol{x}^{(1)}$. It violates the constraint $-x_1^2 + x_2 \geq 0$.

Thus, $\boldsymbol{x}^{(1)}$ is not feasible and is of course **not an optimal solution** to the problem. ■

Consider the case of $\boldsymbol{x}^{(2)}$. Only the constraint $x_1 \geq 0$ is active. Then $\mu_1 = \mu_2 = \mu_4 = 0$.

$$\nabla \mathcal{L}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\mu}) = \left(2x_1 - \frac{9}{2} - \mu_3, 2x_2 - 4\right) = \mathbf{0} \Rightarrow \mu_3 = -\frac{9}{2},$$

which violates $\mu_3 \geq 0$.

Thus, $\boldsymbol{x}^{(2)}$ is **not an optimal solution** to the problem. ■

Consider the case of $\boldsymbol{x}^{(3)}$. Only the constraint $-x_1^2 + x_2 \geq 0$ is active.

Then $\mu_2 = \mu_3 = \mu_4 = 0$.

$$\nabla \mathcal{L}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\mu}) = \left(2x_1 - \frac{9}{2} + 2\mu_1 x_1, 2x_2 - 4 - \mu_1\right) = \mathbf{0} \Rightarrow \mu_1 = \frac{1}{2}.$$

Therefore, $\boldsymbol{x}^{(3)}$ is **an optimal solution** to the problem.

The optimal value is

$$\left(x_1^{(3)} - \frac{9}{4}\right)^2 + \left(x_2^{(3)} - 2\right)^2 = \frac{5}{8}.$$

The corresponding Lagrange multiplier is $(\mu_1, \mu_2, \mu_3, \mu_4) = \left(\frac{1}{2}, 0,0,0\right)$. ■

4.(a) ***Proof:***

The Lagrangian function is

$$\mathcal{L}(\boldsymbol{x}, \lambda, \boldsymbol{\mu}) = \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{z}\|^2 + \lambda \boldsymbol{y}^T \boldsymbol{x} - \boldsymbol{\mu}^T \boldsymbol{x}$$

The KKT condition is

$$\begin{cases} \nabla \mathcal{L}_{x_i}(\boldsymbol{x}, \lambda, \boldsymbol{\mu}) = x_i - z_i + \lambda y_i - \mu_i = 0 \\ \boldsymbol{y}^T \boldsymbol{x} = \mathbf{0} \\ \boldsymbol{\mu} \geq \mathbf{0} \\ \boldsymbol{x} \geq \mathbf{0} \\ \boldsymbol{\mu}\boldsymbol{x} = \mathbf{0} \end{cases}$$

Let the optimal solution be $\boldsymbol{x}^\star = (x_1^\star, x_2^\star, \dots, x_n^\star)$.

By the assumption, since all constraints are affine, even critical optimal points satisfy KKT conditions, **i.e.** all optimal points satisfy KKT conditions. Thus, $\boldsymbol{x}^\star$ satisfy KKT condition.

We know $x_i^\star = z_i - \lambda y_i + \mu_i$.

CASE 01. $x_i^\star > 0$. Then $\mu_i^\star = 0$. Thus, $x_i^\star = z_i - \lambda y_i > 0$.

CASE 02. $x_i^\star = 0$. Then $\mu_i^\star > 0$. Thus, $z_i - \lambda y_i = -\mu_i^\star < 0$, **i.e.** $(z_i - \lambda y_i)^+ = 0$.

In conclusion, $x_i^\star = (z_i - \lambda y_i)^+$.

Plug $x_i^\star$ into $\boldsymbol{y}^T\boldsymbol{x} = \boldsymbol{0}$, we know $\sum_{i=1}^n y_i(z_i - \lambda y_i)^+ = 0$.

Therefore, exists a λ **s.t.** the solution $\boldsymbol{x}^\star$ takes the form

$$x_i^\star = (z_i - \lambda y_i)^+, \qquad i = 1,2,\dots,n$$

where λ satisfies

$$\sum_{i=1}^n y_i(z_i - \lambda y_i)^+ = 0.$$

Qed. ■

(b) ***Solution:***

The code is enclosed in the submission with the name "`p4.py`".

The result is as follows.

```
In [26]: runfile('D:/Textbooks/2021-2022-1/Linear and Convex Optimization/
p4.py', wdir='D:/Textbooks/2021-2022-1/Linear and Convex Optimization')
[0.33333333 1.33333333 1.66666667]
0.6666666666666666
```

Fig.02. The Output

Thus, the solution is $\left(\frac{1}{3}, \frac{4}{3}, \frac{5}{3}\right)^T$ with the corresponding Lagrange multiplier be $\frac{2}{3}$. ■