Category theory notes

November 25, 2021

1 Definition of a category

Definition 1 *A category $\mathcal{C}$ has objects $\mathfrak{a}$ and arrows $f : \mathfrak{a} \to \mathfrak{b}$ between objects. $\mathcal{C}^0$ denotes the objects of catogory $\mathcal{C}$ and $\mathcal{C}^1$ denotes its arrows. For an arrow $f : \mathfrak{a} \to \mathfrak{b}$ the object $\mathfrak{a}$ is called its domain and $\mathfrak{b}$ its codomain: also denoted $\mathfrak{a} = dom(f)$ and $\mathfrak{b} = cod(f)$. A category $\mathcal{C}$ must adhere to the following* axioms*:*

- *if f and g are two functions such that $cod(f) = dom(g)$, then the composition gf (or $g.f$) is a function from $dom(f)$ to $cod(g)$:*

  $$(\mathfrak{a} \xrightarrow{f} \mathfrak{b} \xrightarrow{g} \mathfrak{c}) \quad \mapsto \quad (\mathfrak{a} \xrightarrow{gf} \mathfrak{c})$$

- *composition of arrows is* associative*: i.e. given $f : \mathfrak{a} \to \mathfrak{b}$, $g : \mathfrak{b} \to \mathfrak{c}$ and $h : \mathfrak{c} \to \mathfrak{d}$, then $(hg)f = h(gf) : \mathfrak{a} \to \mathfrak{d}$*

- *every element $\mathfrak{a} \in \mathcal{C}^0$ has an identity arrow $id_{\mathfrak{a}} : \mathfrak{a} \to \mathfrak{a}$ satisfying $id_{\mathfrak{a}}.f = f \ \ \forall f \in \mathcal{C}^1$ with $cod(f) = \mathfrak{a}$ and $g.id_{\mathfrak{a}} = g \ \ \forall g \in \mathcal{C}^1$ with $dom(g) = \mathfrak{a}$.*

Example 2 *A* preorder *is a set X together with a binary relation $\leq$ which is*

- *reflexive: i.e. $x \leq x \ \ \forall x \in X$*
- *transitive: i.e. $x \leq y, y \leq z \Rightarrow x \leq z \ \ \forall x, y, z \in X$*

any such a preorder can be seen as a category $\mathcal{C}$ with elements being the objects of X and a unique arrow $x \to y$ iff $x \leq y$.

Example 3 *A* monoid *is a set X with a binary operation, written like multiplication xy for $x, y \in X$ which is associative and has a unit element $e \in X$ such that $ex = xe = x \ \ \forall x \in X$. A monoid can be interpreted as a category with one object and one arrow x for every $x \in X$.*

Example 4 *Top, Grp, Mon, Rng, Grph, Pos are the categories with* topological spaces, groups, monoids, rings, graphs, posets *as objects and their respective homomorphisms (structure preserving functions) as arrows. Set is the category of sets with functions between sets as arrows.*

Definition 5 *A category $\mathcal{C}$ is called* small *iff both $\mathcal{C}^0$ and $\mathcal{C}^1$ are sets. The category is called* locally small *iff for any two objects $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}$ the set of morphisms between $\mathfrak{a}$ and $\mathfrak{b}$ (denoted $Hom(\mathfrak{a}, \mathfrak{b})$) is a set.*

Definition 6 *Let $\mathcal{C}$ be a category, the* oposite category *is denoted $\mathcal{C}^{op}$. This category has the same objects, but arrows pointing the other way. I.e. if $f : \mathfrak{a} \to \mathfrak{b}$ in $\mathcal{C}^1$, then $\bar{f} : \mathfrak{b} \to \mathfrak{a}$ in $(\mathcal{C}^{op})^0$. Composition of arrows is defined as $\bar{f}\bar{g} = \bar{gf}$.*

Definition 7 *An arrow $f : \mathfrak{a} \to \mathfrak{b}$ is called a* monomorphism *if for any other object $\mathfrak{c}$ and morphisms $g, h : \mathfrak{c} \to \mathfrak{a}$ $fg = fh$ implies $g = h$. The arrow is called* epimorphism *if for any object $\mathfrak{c}$ and morphisms $g, h : \mathfrak{b} \to \mathfrak{c}$ $gf = hf$ implies $g = h$.*

Lemma 8 *Monomorphisms in Set correspond to injective functions, where epimorphisms in Set correspond to surjective functions.*

Lemma 9 *If gf is mono, then f is mono, so by duality: if fg is epi, then f is epi.*

Definition 10 *An epi $f : \mathfrak{a} \to \mathfrak{b}$ is called* split epi *if $\exists g : \mathfrak{b} \to \mathfrak{a}$ such that $fg = id_{\mathfrak{b}}$. Dually a mono $f : \mathfrak{a} \to \mathfrak{b}$ is called* split mono *if $\exists g : \mathfrak{b} \to \mathfrak{a}$ such that $gf = id_{\mathfrak{a}}$.*

Definition 11 *A morphism $f : \mathfrak{a} \to \mathfrak{b}$ is called an* isomorphism *if $\exists g : \mathfrak{b} \to \mathfrak{a}$ such that $fg = id_{\mathfrak{b}}$ and and $gf = id_{\mathfrak{a}}$. In this case the objects $\mathfrak{a}$ and $\mathfrak{b}$ are called* isomorphic.

Lemma 12

- *if two of f, g and fg are iso, then so is the third*
- *if f is epi and split mono, it is iso*
- *if f is split epi and mono, it is iso*

Definition 13 *An object $\mathfrak{a} \in \mathcal{C}^0$ is called a* terminal object *if for any other object $\mathfrak{b} \in \mathcal{C}^0$ there is a unique arrow $f : \mathfrak{b} \to \mathfrak{a}$. (E.g. singletons are terminal objects in Set) Similarly an object $\mathfrak{a} \in \mathcal{C}^0$ is called a* initial object *if for ano other object $\mathfrak{b} \in \mathcal{C}^0$ there is a unique arrow $f : \mathfrak{a} \to \mathfrak{b}$. (E.g. the empty set is the only initial object in Set.)*

Lemma 14 *Any two terminal objects are isomorphic. Same for any two initial objects.*

2 Functors and natural morphisms

Definition 15 *A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of operations $F_0 : \mathcal{C}^0 \to \mathcal{D}^0$ and $F_1 : \mathcal{C}^1 \to \mathcal{D}^1$ such that the following holds:*

- *for each $f : \mathfrak{a} \to \mathfrak{b}$ in $\mathcal{C}$: $F_1(f) : F_0(\mathfrak{a}) \to F_0(\mathfrak{b})$*

- *for each* $\mathfrak{a} \xrightarrow{f} \mathfrak{b} \xrightarrow{g} \mathfrak{c}$ *:* $F_1(gf) = F_1(g)F_1(f)$.

- $F_0(id_{\mathfrak{a}}) = id_{F_0(\mathfrak{a})}$ *for each* $a \in \mathcal{C}$

Definition 16 *An endofunctor is a functor F from a category $\mathcal{C}$ to itself.*

Definition 17 *A functor $F : \mathcal{C} \to \mathcal{D}$ is called* covariant*, whereas a functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is called* contravariant*.*

Since functors are composable, the following definition can be made:

Definition 18 $\mathcal{C}\mathfrak{at}$ *is the category whose objects are* small categories *and whose morphisms are functors between those categories.*

Now that functors are defined, its time to define natural morphisms:

Definition 19 *Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors of categories $\mathcal{C}$ and $\mathcal{D}$: a* natural morphism *$\alpha : F \Rightarrow G$ is a set of morphisms $\alpha_{\mathfrak{a}} : F(\mathfrak{a}) \to G(\mathfrak{a})$ for any object $\mathfrak{a} \in \mathcal{C}^0$ such that for any arrow $f : \mathfrak{a} \to \mathfrak{b}$ in $\mathcal{C}^1$ the following diagram in $\mathcal{D}$ commutes:*

$$\begin{tikzcd}
F(\mathfrak{a}) \arrow[r, "F(f)"] \arrow[d, "\alpha_{\mathfrak{a}}"] & F(\mathfrak{b}) \arrow[d, "\alpha_{\mathfrak{b}}"] \\
G(\mathfrak{a}) \arrow[r, "G(f)"] & G(\mathfrak{b})
\end{tikzcd}$$

Definition 20 *A natural transformation $\alpha : F \Rightarrow G$ is called a* natural isomorphism *iff its components $\alpha_{\mathfrak{a}}\mathfrak{a} \in \mathcal{C}$ are isomorphisms.*

Definition 21 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We can define the* Functor category *which is denoted as either $\mathcal{D}^{\mathcal{C}}$ or $[\mathcal{C}, \mathcal{D}]$ as follows:*

- *objects of this category are functors $F : \mathcal{C} \to \mathcal{D}$*

- *morphisms between two such functors F and G are natural transformations $\alpha : F \Rightarrow G$*

Definition 22 *Actually a category like $\mathcal{C}\mathfrak{at}$ which has morphisms and natural transformations between morphisms is called a* 2-category*.*

Definition 23 *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, the* product category *or* cartesian product of categories *$\mathcal{C} \times \mathcal{D}$ has objects $(\mathfrak{a}, \mathfrak{b})$ where $\mathfrak{a} \in \mathcal{C}^0$ and $\mathfrak{b} \in \mathcal{D}^0$ and morphisms $(f, b) : (\mathfrak{a}, \mathfrak{b}) \to (\mathfrak{a}', \mathfrak{b}')$ where $f : \mathfrak{a} \to \mathfrak{a}' \in \mathcal{C}^1$ and $g : \mathfrak{b} \to \mathfrak{b}' \in \mathcal{D}^1$.*

Definition 24 *A functor from any product category to another category is called a* bifunctor*.*

Definition 25 *A functor $F : \mathcal{C} \to \mathcal{D}$ is called* faithfull *if its induced function $F_1 : Hom(\mathfrak{a}, \mathfrak{b}) \to Hom(F(\mathfrak{a}), F(\mathfrak{b}))$ is injective for any two objects $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}$. It is called* full *if this induced function is surjective for any two objects $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}$.*

Definition 26 *A functor* reflects *a property if, whenever the image (of objects or arrows) has a property, then the origin has the property too.*

Lemma 27 *Faithfull functors reflect epis and monos*

Lemma 28 *Full and Faithfull functors reflect property of being initial or terminal object.*

3 limits and colimits

mention: definition of limits, initial/terminal objects, products and coproducts, equalizer and co equalizer, pullbacks and pushouts

4 yoneda

mention lemma and usual application of it

5 monads, adjunctions and T-algebras

mention relations between those, adjunctions for free and forgetfull