Category theory notes

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1 Definition of a category

Definition 1 *A category $\mathcal{C}$ has objects a and arrows $f : \mathsf{a} \to \mathsf{b}$ between objects. $\mathcal{C}_0$ denotes the objects of catogory $\mathcal{C}$ and $\mathcal{C}_1$ denotes its arrows. For an arrow $f : \mathsf{a} \to \mathsf{b}$ the object a is called its domain and b its codomain: also denoted $\mathsf{a} = dom(f)$ and $\mathsf{b} = cod(f)$. A category $\mathcal{C}$ must adhere to the following **axioms**:*

- *if f and g are two functions such that $cod(f) = dom(g)$, then the composition gf (or $g.f$) is a function from $dom(f)$ to $cod(g)$:*

  $$(\mathsf{a} \xrightarrow{f} \mathsf{b} \xrightarrow{g} \mathsf{c}) \quad \mapsto \quad (\mathsf{a} \xrightarrow{gf} \mathsf{c})$$

- *composition of arrows is **associative**: i.e. given $f : \mathsf{a} \to \mathsf{b}$, $g : \mathsf{b} \to \mathsf{c}$ and $h : \mathsf{c} \to \mathsf{d}$, then $(hg)f = h(gf) : \mathsf{a} \to \mathsf{d}$*

- *every element $\mathsf{a} \in \mathcal{C}_0$ has an identity arrow $id_{\mathsf{a}} : \mathsf{a} \to \mathsf{a}$ satisfying $id_{\mathsf{a}}.f = f \;\; \forall f \in \mathcal{C}_1$ with $cod(f) = \mathsf{a}$ and $g.id_{\mathsf{a}} = g \;\; \forall g \in \mathcal{C}_1$ with $dom(g) = \mathsf{a}$.*

Example 2 *A **preorder** is a set X together with a binary relation $\leq$ which is*

- *reflexive: i.e. $x \leq x \;\; \forall x \in X$*
- *transitive: i.e. $x \leq y, y \leq z \Rightarrow x \leq z \;\; \forall x, y, z \in X$*

any such a preorder can be seen as a category $\mathcal{C}$ with elements being the objects of X and a unique arrow $x \to y$ iff $x \leq y$.

Example 3 *A **monoid** is a set X with a binary operation, written like multiplication xy for $x, y \in X$ which is associative and has a unit element $e \in X$ such that $ex = xe = x \;\; \forall x \in X$. A monoid can be interpreted as a category with one object and one arrow x for every $x \in X$.*

Example 4 *Top, Grp, Mon, Rng, Grph, Pos are the categories with **topological spaces, groups, monoids, rings, graphs, posets** as objects and their respective homomorphisms (structure preserving functions) as arrows. Set is the category of sets with functions between sets as arrows.*

Definition 5 *A category $\mathcal{C}$ is called* ***small*** *iff both $\mathcal{C}_0$ and $\mathcal{C}_1$ are sets. The category is called* ***locally small*** *iff for any two objects $\mathsf{a}, \mathsf{b} \in \mathcal{C}$ the set of morphisms between a and b (denoted $Hom(\mathsf{a}, \mathsf{b})$) is a set.*

Definition 6 *Let $\mathcal{C}$ be a category, the* ***oposite category*** *is denoted $\mathcal{C}^{op}$. This category has the same objects, but arrows pointing the other way. I.e. if $f : \mathsf{a} \to \mathsf{b}$ in $\mathcal{C}_1$, then $\bar{f} : \mathsf{b} \to \mathsf{a}$ in $(\mathcal{C}^{op})_0$. Composition of arrows is defined as $\bar{f}\bar{g} = \bar{gf}$.*

Definition 7 *An arrow $f : \mathsf{a} \to \mathsf{b}$ is called a* ***monomorphism*** *if for any other object c and morphisms $g, h : \mathsf{c} \to \mathsf{a}$ $fg = fh$ implies $g = h$. The arrow is called* ***epimorphism*** *if for any object c and morphisms $g, h : \mathsf{b} \to \mathsf{c}$ $gf = hf$ implies $g = h$.*

Lemma 8 *Monomorphisms in Set correspond to injective functions, where epimorphisms in Set correspond to surjective functions.*

Lemma 9 *If gf is mono, then f is mono, so by duality: if fg is epi, then f is epi.*

Definition 10 *An epi $f : \mathsf{a} \to \mathsf{b}$ is called* ***split epi*** *if $\exists g : \mathsf{b} \to \mathsf{a}$ such that $fg = id_{\mathsf{b}}$. Dually a mono $f : \mathsf{a} \to \mathsf{b}$ is called* ***split mono*** *if $\exists g : \mathsf{b} \to \mathsf{a}$ such that $gf = id_{\mathsf{a}}$.*

Definition 11 *A morphism $f : \mathsf{a} \to \mathsf{b}$ is called an* ***isomorphism*** *if $\exists g : \mathsf{b} \to \mathsf{a}$ such that $fg = id_{\mathsf{b}}$ and and $gf = id_{\mathsf{a}}$. In this case the objects a and b are called* ***isomorphic****.*

Lemma 12

- *if two of f, g and fg are iso, then so is the third*
- *if f is epi and split mono, it is iso*
- *if f is split epi and mono, it is iso*

Definition 13 *An object $\mathsf{a} \in \mathcal{C}_0$ is called a* ***terminal object*** *if for any other object $\mathsf{b} \in \mathcal{C}_0$ there is a unique arrow $f : \mathsf{b} \to \mathsf{a}$. (E.g. singletons are terminal objects in Set) Similarly an object $\mathsf{a} \in \mathcal{C}_0$ is called a* ***initial object*** *if for ano other object $\mathsf{b} \in \mathcal{C}_0$ there is a unique arrow $f : \mathsf{a} \to \mathsf{b}$. (E.g. the empty set is the only initial object in Set.)*

Lemma 14 *Any two terminal objects are isomorphic. Same for any two initial objects.*

2 Functors and natural morphisms

Definition 15 *A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of operations $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ such that the following holds:*

- *for each $f : \mathsf{a} \to \mathsf{b}$ in $\mathcal{C}$: $F_1(f) : F_0(\mathsf{a}) \to F_0(\mathsf{b})$*

- *for each* $\mathsf{a} \xrightarrow{f} \mathsf{b} \xrightarrow{g} \mathsf{c}$ *:* $F_1(gf) = F_1(g)F_1(f)$.
- $F_0(id_{\mathsf{a}}) = id_{F_0(\mathsf{a})}$ *for each* $a \in \mathcal{C}$

Definition 16 *An endofunctor is a functor F from a category $\mathcal{C}$ to itself.*

Definition 17 *A functor $F : \mathcal{C} \to \mathcal{D}$ is called **covariant**, whereas a functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is called **contravariant**.*

Since functors are composable, the following definition can be made:

Definition 18 $\mathcal{C}\mathsf{at}$ *is the category whose objects are **small categories** and whose morphisms are functors between those categories.*

Now that functors are defined, its time to define natural morphisms:

Definition 19 *Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors of categories $\mathcal{C}$ and $\mathcal{D}$: a **natural morphism** $\alpha : F \Rightarrow G$ is a set of morphisms $\alpha_{\mathsf{a}} : F(\mathsf{a}) \to G(\mathsf{a})$ for any object $\mathsf{a} \in \mathcal{C}_0$ such that for any arrow $f : \mathsf{a} \to \mathsf{b}$ in $\mathcal{C}_1$ the following diagram in $\mathcal{D}$ commutes:*

$$\begin{tikzcd} F(\mathsf{a}) \arrow[r, "F(f)"] \arrow[d, "\alpha_{\mathsf{a}}"] & F(\mathsf{b}) \arrow[d, "\alpha_{\mathsf{b}}"] \\ G(\mathsf{a}) \arrow[r, "G(f)"] & G(\mathsf{b}) \end{tikzcd}$$

Definition 20 *A natural transformation $\alpha : F \Rightarrow G$ is called a **natural isomorphism** iff its components $\alpha_{\mathsf{a}} : \mathsf{a} \in \mathcal{C}$ are isomorphisms.*

Definition 21 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We can define the **Functor category** which is denoted as either $\mathcal{D}^{\mathcal{C}}$ or $[\mathcal{C}, \mathcal{D}]$ as follows:*

- *objects of this category are functors $F : \mathcal{C} \to \mathcal{D}$*
- *morphisms between two such functors F and G are natural transformations $\alpha : F \Rightarrow G$*

Definition 22 *Actually a category like* $\mathcal{C}\mathsf{at}$ *which has morphisms and natural transformations between morphisms is called a **2-category**.*

Definition 23 *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, the **product category** or **cartesian product of categories** $\mathcal{C} \times \mathcal{D}$ has objects (a, b) where $\mathsf{a} \in \mathcal{C}_0$ and $\mathsf{b} \in \mathcal{D}_0$ and morphisms $(f, b) : (\mathsf{a}, \mathsf{b}) \to (\mathsf{a}', \mathsf{b}')$ where $f : \mathsf{a} \to \mathsf{a}' \in \mathcal{C}_1$ and $g : \mathsf{b} \to \mathsf{b}' \in \mathcal{D}_1$.*

Definition 24 *A functor from any product category to another category is called a **bifunctor**.*

Definition 25 *A functor $F : \mathcal{C} \to \mathcal{D}$ is called **faithfull** if its induced function $F_1 : Hom(\mathsf{a}, \mathsf{b}) \to Hom(F(\mathsf{a}), F(\mathsf{b}))$ is injective for any two objects $\mathsf{a}, \mathsf{b} \in \mathcal{C}$. It's called **full** if this induced function is surjective for any two objects $\mathsf{a}, \mathsf{b} \in \mathcal{C}$.*

Definition 26 *A functor **reflects** a property if, whenever the image (of objects or arrows) has a property, then the origin has the property too.*

Lemma 27 *Faithfull functors reflect epis and monos*

Lemma 28 *Full and Faithfull functors reflect property of being initial or terminal object.*

Definition 29 *Suppose we have categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ together with functors:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] & \mathcal{E} \end{tikzcd}$$

If we have natural transformations $\alpha : F \Rightarrow F'$ and $\beta : G \Rightarrow G'$, then they cannot be trivially composed since their domains and codomains do not match. Writing out the inducing morphisms on objects, however, makes it easy to construct a morphism

$$\beta \circ \alpha : G \circ F \Rightarrow G' \circ F'$$

*and this composition is called the **horizontal composition** of natural transformations.*

3 limits and colimits

Definition 30 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. A **cone** for F exists of an object $\mathsf{d} \in \mathcal{D}_0$ and a natural transformation $\mu : \Delta_{\mathsf{d}} \Rightarrow F$. ($\Delta_{\mathsf{d}}$ is the constant functor which sends all objects to d and all arrows to id_{d}.) So μ is a family of morphisms $(\mu_{\mathsf{c}} : \mathsf{d} \to F(\mathsf{c}) | \mathsf{c} \in \mathcal{C}_0)$ such that for any morphism $f : \mathsf{c} \to \mathsf{c}'$ in $\mathcal{C}$ the following diagram commutes:*

$$\begin{tikzcd} & \mathsf{d} \arrow[dl, "\mu_{\mathsf{c}}"'] \arrow[dr, "\mu'_{\mathsf{c}}"] & \\ F(\mathsf{c}) \arrow[rr, "F(f)"] & & F(\mathsf{c}') \end{tikzcd}$$

*This cone is denoted as (d, μ) and d is called the **vertex** of the cone. In fact, cones of F form a category with morphisms $f : (\mathsf{d}, \mu) \to (\mathsf{d}', \mu')$ being a morphism $f : \mathsf{d} \to \mathsf{d}'$ in $\mathcal{C}_1$ such that $\mu'_{\mathsf{c}} g = \mu_{\mathsf{c}}$ for any $\mathsf{c} \in \mathcal{C}_0$. I.e. the following diagram commutes for any $\mathsf{c} \in \mathcal{C}_0$:*

$$\begin{tikzcd} \mathsf{d} \arrow[rr, "g"] \arrow[dr, "\mu_{\mathsf{c}}"] & & \mathsf{d}' \arrow[dl, "\mu'_{\mathsf{c}}"] \\ & F(\mathsf{c}) & \end{tikzcd}$$

So all cones over F form a category, which is denoted $Cone(F)$.

Note that the codomain of F in the previous definition is regularly called the **diagram of $\mathcal{C}$ in $\mathcal{D}$** and the category $\mathcal{C}$ is called the **index category** of this diagram.

Definition 31 *Let F be as in the previous definition. The terminal object in the category $Cone(F)$ is called the **limiting cone** for F or **limit** for the diagram F. It is also sometimes denoted **Lim**F. Note that I say **the** terminal object because terminal objects are unique up to unique isomorphism.*

Now its natural to define the co-definition for cones and limits:

Definition 32 *Let $F : \mathcal{E} \to \mathcal{C}$ be a functor. A **cocone** for this functor is a pair (ν, d) where d is an object in $\mathcal{C}$ and $\nu : F \Rightarrow \Delta_D$ is a natural transformation. So a morphism $f : (\nu, \mathsf{d}) \to (\nu', \mathsf{d}')$ in the category of cocones of F are morphisms $f : \mathsf{d} \to \mathsf{d}'$ such that for any $\mathsf{e} \in \mathcal{E}_0$ the following diagram commutes:*

$$\begin{tikzcd} & F(\mathsf{c}) & \\ \mathsf{d} \arrow[ur, "\nu_{\mathsf{c}}"] \arrow[rr, "f"] & & \mathsf{d}' \arrow[ul, "\mu'_{\mathsf{c}}"'] \end{tikzcd}$$

*The **colimit** for F is the initial object in the category of cocones over F.*

Now I will give a few examples of limits and colimits:

Example 33 *Let c, d be two objects of a category $\mathcal{C}$. These two objects can be seen as the image of the functor $F : \mathbf{2} \to \mathcal{C}$ where $\mathbf{2}$ is the **discrete category with two objects** (the category with two objects and no arrows except for the identity arrows). The functor F sends the one element of $\mathbf{2}$ to c and the other to d. The **product** of c and d is the limit of the category of cones over F. I.e. it is an object e together with two morphisms $p : \mathsf{e} \to \mathsf{c}$ and $q : \mathsf{e} \to \mathsf{d}$ such that for any other such object e' and morphisms p' and q' there is a unique morphism m that factorizes p' and q'. In diagrams this looks as follows:*

$$\begin{tikzcd} & \mathsf{e}' \arrow[dl, "p'"'] \arrow[d, dashed, "\exists! m"] \arrow[dr, "q'"] & \\ & \mathsf{e} \arrow[dl, "p"] \arrow[dr, "q"'] & \\ \mathsf{c} & & \mathsf{d} \end{tikzcd}$$

*The morphisms p and q are called the **projections** of the product. Its **coproduct** of c and d is the colimit of this functor F. (Write out the diagram yourself.)*

Example 34 *Let $\bar{\mathbf{2}}$ be the category $\begin{tikzcd} \mathsf{x} \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & \mathsf{y} \end{tikzcd}$. A functor $\bar{\mathbf{2}} \to \mathcal{C}$ is basically a parallel pair of morphisms $f, g : \mathsf{a} \to \mathsf{b}$ in $\mathcal{C}$. And a cone for this functor is basically a diagram*

$$\begin{tikzcd} & \mathsf{d} \arrow[dl, "\mu_{\mathsf{a}}"'] \arrow[dr, "\mu_{\mathsf{b}}"] & \\ \mathsf{a} \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & & \mathsf{b} \end{tikzcd}$$

*but $\mu_{\mathsf{b}} = f\mu_{\mathsf{a}} = g\mu_{\mathsf{a}}$ so this arrow is usually omitted, leaving the following diagram: $\begin{tikzcd} \mathsf{d} \arrow[r, "\mu_{\mathsf{a}}"] & \mathsf{a} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \mathsf{b} \end{tikzcd}$. The limit in this category of cones (I.e. the object d and morphism μ_{a}) is called the **equalizer** of the pair f and g. The definition of the **co equalizer** follows automatically.*

Lemma 35 *Every equalizer is a monomorpism.*

Covariantly: every coequalizer is epi.

Lemma 36 *let* $\begin{tikzcd} \mathsf{d} \arrow[r, "\mu_{\mathsf{a}}"] & \mathsf{a} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \mathsf{b} \end{tikzcd}$ *be an equalizer diagram. Then* μ_{a} *is an isomorphism iff* $f = g$*.*

Covariantly: a coequalizer is iso iff $f = g$*.*

Definition 37 *Every monomorphism* $f : \mathsf{a} \to \mathsf{b}$ *which is part of some equalizer diagram* $\begin{tikzcd} \mathsf{a} \arrow[r, "f"] & \mathsf{b} \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & \mathsf{c} \end{tikzcd}$ *is called* ***regular mono****. The definition of a* ***regular epi*** *follows automatically.*

Lemma 38 *If* f *is regular mono and epi, it is iso. Also every split mono is regular.*

Example 39 *A* ***pullback*** *is the limiting cone in the category of cones over a diagram like this:* $\begin{tikzcd} \mathsf{a} \arrow[r, "f"] & \mathsf{b} & \mathsf{c} \arrow[l, "g"] \end{tikzcd}$ *. Just to clarify, a cone for this diagram looks like this:*

$$\begin{tikzcd} & \mathsf{d} \arrow[dl, "p"] \arrow[dr, "q"] & \\ \mathsf{a} \arrow[dr, "f"] & & \mathsf{c} \arrow[dl, "g"] \\ & \mathsf{b} & \end{tikzcd}$$

. (The arrow from d *to* b *is omitted since it can be given as the composition* $gq = fp$*.) The co definitions of the pullback is called a* ***pushout****.*

Lemma 40 *In the pullback diagram given above, is* f *is mono, then* q *is too. Similarly if* f *is iso, then* q *is too.*

Co-Covariantly: let

$$\begin{tikzcd} & \mathsf{b} \arrow[dl, "f"] \arrow[dr, "g"] & \\ \mathsf{a} \arrow[dr, "p"] & & \mathsf{c} \arrow[dl, "q"] \\ & \mathsf{d} & \end{tikzcd}$$

be a pushout diagram, then f *epi implies that* q *is epi and* f *regular epi implies that* q *is regular epi.*

Lemma 41 *Given two commuting squares*

$$\begin{tikzcd} \mathsf{a} \arrow[r, "b"] \arrow[d, "a"] & \mathsf{b} \arrow[r, "c"] \arrow[d, "f"] & \mathsf{c} \arrow[d, "d"] \\ \mathsf{x} \arrow[r, "g"] & \mathsf{y} \arrow[r, "h"] & \mathsf{z} \end{tikzcd}$$

, then the following hold:

- *If both squares are pullbacks, then so is the composite. Covariantly: the composition of two pushouts is a pushout.*
- *If the right hand square and the composite are pullbacks, then so is the left hand square. Covariantly: if both the first square and the composition are pushouts, then so is the second square.*

Lemma 42 *$f : \mathsf{a} \to \mathsf{b}$ is mono iff*
$$\begin{tikzcd} \mathsf{a} \arrow[r, "id_{\mathsf{a}}"] \arrow[d, "id_{\mathsf{a}}"] & \mathsf{a} \arrow[d, "f"] \\ \mathsf{a} \arrow[r, "f"] & \mathsf{b} \end{tikzcd}$$
is a pullback diagram. Covariantly: f is epi iff
$$\begin{tikzcd} \mathsf{a} \arrow[r, "f"] \arrow[d, "f"] & \mathsf{a} \arrow[d, "id_{\mathsf{b}}"] \\ \mathsf{a} \arrow[r, "id_{\mathsf{b}}"] & \mathsf{b} \end{tikzcd}$$
is a pushout diagram.

4 yoneda

Definition 43 *A **profunctor** is a bifunctor which is contravariant in its first argument and covariant in its second and which codomain is Set. So a profunctor is a functor $F : \mathcal{C}^{op} \times \mathcal{D} \to Set$.*

Let $\mathcal{C}$ be a locally small category and $\mathsf{a} \in \mathcal{C}_0$. There is a functor $Hom(a, -) : \mathcal{C} \to Set$ defined on objects by $\mathsf{x} \mapsto Hom(\mathsf{a}, \mathsf{x})$ and which sends arrows $f : \mathsf{x} \to \mathsf{y}$ to the function $Hom(a, f) : Hom(\mathsf{a}, \mathsf{x}) \to Hom(\mathsf{a}, \mathsf{y})$ defined by $Hom(a, f)(h) := fh : \mathsf{a} \to \mathsf{y}$.

Similarly one can fix the second argument in the Hom set in order to get a contravariant functor $Hom(-, b) : \mathcal{C}^{op} \to Set$.

Combine these to functors in order to get a profunctor $Hom(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to Set$.

Definition 44 *Any functor $F : \mathcal{C} \to Set$ which is naturally isomorphic to the Hom functor $Hom(\mathsf{a}, -)$ for some object a is called **representable**.*

So this means that there are natural iso's $\alpha : Hom(\mathsf{a}, -) \to F$ and $\beta : F \to Hom(\mathsf{a}, -)$.

Naturality of α means that for all any arrow $f : \mathsf{x} \to \mathsf{y}$ the following diagram commutes:
$$Ff \circ \alpha_{\mathsf{x}} = \alpha_{\mathsf{y}} \circ Hom(\mathsf{a}, f)$$
.

(α_{x} is a function $Hom(\mathsf{a}, \mathsf{x}) \to F\mathsf{x}$.)

The following lemma is called the **yoneda lemma**:

Lemma 45 *Let $\mathcal{C}$ be any locally small category and furthermore let $\mathsf{a} \in \mathcal{C}_0$ and $F : \mathcal{C} \to Set$ be any functor such that $F\mathsf{a} \neq \emptyset$): There is an isomorphism (of sets)*
$$[\mathcal{C}, Set](Hom(\mathsf{a}, -), F) \cong Fa$$
(remember that $[\mathcal{C}, Set]$ is the category with functors $\mathcal{C} \to Set$ as objects and natural transformations as arrows.)

On one side, lets take any natural transformation $\alpha : Hom(\mathsf{a}, -) \to F$, then $\alpha_{\mathsf{a}} : Hom(\mathsf{a}, \mathsf{a}) \to Fa$ is a function of sets, which can be evaluated at

$id_{\mathsf{a}} \in Hom(\mathsf{a}, \mathsf{a})$ to get an element in Fa. This way we defined a function $[\mathcal{C}, Set](Hom(\mathsf{a}, -), F) \to Fa$.

Its inverse is a function $Fa \to [\mathcal{C}, Set](Hom(\mathsf{a}, -), F)$ which can be constructed as follows:

Take any object x in $\mathcal{C}$, some morphism $h : \mathsf{a} \to \mathsf{x}$ and a morphism $f : \mathsf{x} \to \mathsf{y}$. The naturality square for any natural morphism α given above becomes (pointwise) when acting on h:

$$\alpha_{\mathsf{y}}(Hom(\mathsf{a}, f)h) = (Ff)(\alpha_{\mathsf{x}} h)$$

and

$$Hom(\alpha, f)h = f \circ h$$

which leads to:

$$\alpha_{\mathsf{y}}(f \circ h) = (Ff)(\alpha_{\mathsf{x}} h)$$

So specializing $\mathsf{x} = \mathsf{a}$ and $h = id_{\mathsf{a}}$ this yields:

$$\alpha_{\mathsf{y}} f = (Ff)(\alpha_{\mathsf{a}} id_{\mathsf{a}})$$

The left hand side is the action of α_{y} on an arbitrary element $f \in Hom(\mathsf{a}, \mathsf{y})$ which it is completely determined by $\alpha_{\mathsf{a}} id_{\mathsf{a}}$. So a natural transformation α can be determined uniquely by choosing any value in F_{a} and setting this as the value for $\alpha_{\mathsf{a}} id_{\mathsf{a}}$.

Co-yoneda lemma states that

Lemma 46 $[\mathcal{C}^{op}, Set](Hom(-, \mathsf{a}), F) \cong F_{\mathsf{a}}$.

So using the Yoneda lemma and setting $F := Hom(\mathsf{b}, -)$ for any object b we get the following equality:

$$[\mathcal{C}, Set](Hom(\mathsf{a}, -), Hom(\mathsf{b}, -)) \cong Hom(\mathsf{b}, \mathsf{a})$$

Definition 47 *Define a function $h : \mathcal{C} \to Set^{\mathcal{C}^{op}}$ which sends a to the functor $h\mathsf{a} := Hom(\mathsf{a}, -) : \mathcal{C}^{op} \to Set$ which is an element of $Set^{\mathcal{C}}$. This function can be made into a functor be defining its action on a function $f : \mathsf{b} \to \mathsf{a}$. $hf \in [\mathcal{C}, Set](Hom(\mathsf{a}, -), Hom(\mathsf{b}, -))$ using the previously given isomorphism. So the function is actually a functor $h : \mathcal{C} \to Set^{\mathcal{C}^{op}}$ which is called the* ***Yoneda embedding****.*

Lemma 48 *The Yoneda embedding is full, faithfull and injective on objects.*

The use of the Yoneda lemma is often the following: Suppose we want to prove that two objects $\mathsf{a}, \mathsf{b} \in \mathcal{C}_0$ are isomorphic. It suffices to prove that for any object $\mathsf{x} \in \mathcal{C}_0$ there is a bijection $f_{\mathsf{x}} : Hom(\mathsf{x}, \mathsf{a}) \to Hom(\mathsf{x}, \mathsf{b})$ which is natural in x. Naturality in x means the for any morphism $g : \mathsf{x}' \to \mathsf{x}$ the following diagram commutes:

$$\begin{array}{ccc} Hom(\mathsf{x}, \mathsf{a}) & \xrightarrow{f_{\mathsf{x}}} & Hom(\mathsf{x}, \mathsf{b}) \\ \downarrow{\scriptstyle Hom(g, id_{\mathsf{a}})} & & \downarrow{\scriptstyle Hom(g, id_{\mathsf{b}})} \\ Hom(\mathsf{x}', \mathsf{a}) & \xrightarrow{f'_{\mathsf{x}}} & Hom(\mathsf{x}', \mathsf{b}) \end{array}$$

5 monads, adjunctions and T-algebras

Definition 49 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with functors $R : \mathcal{C} \to \mathcal{D}$ and $L : \mathcal{D} \to \mathcal{C}$. R is called **right adjoint** to the functor L (And conversely L **left adjoint** to R) if there are natural transformations*

$$\eta : I_{\mathcal{D}} \Rightarrow R \circ L$$

$$\epsilon : L \circ R \Rightarrow I_{\mathcal{C}}$$

*where $I_{\mathcal{D}}$ and $I_{\mathcal{C}}$ are the identity functors for the categories $\mathcal{D}$ and $\mathcal{C}$ respectively. The whole thing is called an **adjunction** and is usually denoted:*

$$L \dashv R$$

*In this definition η is called the **unit** and ϵ the **counit** of the adjunction.*

Note that on objects these natural transformations become moprhisms

$$\eta_{\mathsf{d}} : \mathsf{d} \to (R \circ L)(\mathsf{d})$$

$$\epsilon_{\mathsf{c}} : (L \circ R)(c) \to \mathsf{c}$$

for objects $\mathsf{d} \in \mathcal{D}^{op}$ and $\mathsf{c} \in \mathcal{C}^{op}$.

Lemma 50 *The unit and counit adhere to the following **triangular identities:***

$$\begin{tikzcd} L \arrow[r, "L\circ\eta"] \arrow[dr, equal] & L \circ R \circ L \arrow[d, "\epsilon\circ L"] \\ & L \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\eta\circ R"] \arrow[dr, equal] & R \circ L \circ R \arrow[d, "R\circ\epsilon"] \\ & R \end{tikzcd}$$

Here the compositions $L \circ \eta$, $\epsilon \circ L$, $\eta \circ R$ and $R \circ \epsilon$ are horizontal compositions or natural transformations. (Where $L = I_L : L \Rightarrow L$ is the identity natural transformation from l to L and R similar.)

Alternatively an adjunction can be described using hom sets:

Definition 51 *Let $R : \mathcal{C} \to \mathcal{D}$ and $L : \mathcal{D} \to \mathcal{C}$ be functors. L is left adjoint to R iff for objects $\mathsf{c} \in \mathcal{C}$ and $\mathsf{d} \in \mathcal{D}$ there is an isomorphism*

$$Hom(L\mathsf{d}, \mathsf{c}) \cong Hom(\mathsf{d}, R\mathsf{c})$$

which is natural in both c and d.

Lemma 52 *There is a **forgetfull functor** U from Mon (the category of monoids) to Set and conversely there is a **free functor** $F : Set \to Mon$ which maps a set to the monoid which is freely generated by the set. For this situation, F is left adjoint to U.*

The same goes for the forgetfull functor $U : Grp \to Set$ which maps a group to its underlying set and $F : Set \to Grp$ which maps a set to the group which is freely generated by this set.

Definition 53 *Let $\mathcal{C}$ be a category and $T : \mathcal{C} \to \mathcal{C}$ be any endofunctor. The **kleisli category** associated to $\mathcal{C}, T$ is denoted $\mathcal{C}_T$. It has the same objects as $\mathcal{C}$, but an arrow $\mathsf{a} \to \mathsf{b}$ in $\mathcal{C}_T$ is actually an arrow $\mathsf{a} \to T\mathsf{b}$ in $\mathcal{C}$. Arrows can be composed if T is a monad, as will be described shortly.*

Definition 54 *A **monad** is an endofunctor T together together with two natural transformations*

$$\mu : T^2 \Rightarrow T$$

and

$$\eta : I \Rightarrow T$$

which make the following two diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, "\mu\circ T"] \arrow[d, "T\circ\mu"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I \circ T \arrow[r, "\eta\circ T"] \arrow[dr] & T^2 \arrow[d, "\mu"] & T \circ I \arrow[l, "T\circ\eta"'] \arrow[dl] \\ & T & \end{tikzcd}$$

*Note that again $\mu \circ T$, $T \circ \mu$, $\eta \circ T$ and $T \circ \eta$ are horizontal compositions of natural transformations. μ is usually called **multiplication** and η **unit**.*

Lemma 55 *Let $\mathcal{C}$ be a category and (T, η, μ) a monad on $\mathcal{C}$, then the kleisli category $\mathcal{C}_T$ is defined and the composition of arrows are as follows: If $f : \mathsf{a} \to T\mathsf{b}, g : \mathsf{b} \to T\mathsf{c} \in \mathcal{C}_1$ (So $f : \mathsf{a} \to \mathsf{b}, g : \mathsf{b} \to \mathsf{c}$ in $(\mathcal{C}_T)_1$.) then $g \circ_T f = \mu \circ Tg \circ f$, is the kleisli composition of arrows. (Note that $g \circ_T f : a \to T\mathsf{c}$ in $\mathcal{C}_1$.)*

Ofcourse, its also possible to define a comonad:

Definition 56 *A **comonad** on a category $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with natural transformations*

$$\epsilon : T \Rightarrow I$$

and

$$\delta : T \Rightarrow T^2$$

*called **extract** and **duplicate** respectively. These natural transformations should make the following two diagrams commute:*

$$\begin{tikzcd} T \arrow[r, "\delta"] \arrow[d, "\delta"] & T^2 \arrow[d, "\delta\circ T"] \\ T^2 \arrow[r, "T\circ\delta"] & T^3 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I \circ T & T^2 \arrow[l, "\epsilon\circ T"'] \arrow[r, "T\circ\epsilon"] & T \circ I \\ & T \arrow[ul] \arrow[u, "\delta"] \arrow[ur] & \end{tikzcd}$$

The following theorem shows that every adjunction gives rise to a monad:

Theorem 57 *Let $L \dashv R$ be an adjunction. So we have natural transformations*

$$\eta : I_{\mathcal{D}} \Rightarrow R \circ L$$

and

$$\epsilon : L \circ R \Rightarrow I_{\mathcal{C}}$$

. *Then* $R \circ L$ *is a monad with unit* η *and multiplication* $\mu = R \circ \epsilon \circ L$.

Similarly $L \circ R$ *forms a comonad with* ϵ *as extract and* $\delta = L \circ \eta \circ R : (L \circ R)^2 \to L \circ R$.

Definition 58 *Let* $\mathcal{C}$ *be a category with endofunctor* T. *We will now define the* ***category of*** T***-algebras***. *Objects in this category are pairs* (a, f), *where* $\mathsf{a} \in \mathcal{C}_0$ *and* $f : F\mathsf{a} \to \mathsf{a}$ *in* $\mathcal{C}_1$. *Here* a *is often called the* ***carrier*** *and* f *is called the* ***evaluator function***. *A morphism* $\mu : (\mathsf{a}, f) \to (\mathsf{b}, g)$ *is simply a morphism* $\mu : \mathsf{a} \to \mathsf{b}$ *in* $\mathcal{C}_1$ *that makes the following diagram commute:*

$$\begin{tikzcd} T\mathsf{a} \arrow[r, "Tm"] \arrow[d, "f"] & T\mathsf{b} \arrow[d, "g"] \\ \mathsf{a} \arrow[r, "m"] & \mathsf{b} \end{tikzcd}$$

Any object in this category is called a T***-albegra***. *An initial object in this category is called the* ***initial algebra***.

Lemma 59 *Let* (i, j) *be an initial algebra, then the morphism* $j : T\mathsf{i} \to \mathsf{i}$ *is an isomorphism.*

Definition 60 *Let* $\mathcal{C}$ *be a category with endofunctor* T. *Objects in the* ***category of*** T***-coalgebras*** *are pairs* (a, f) *where* $\mathsf{a} \in \mathcal{C}_0$ *and* $f : \mathsf{a} \to T\mathsf{a}$. *And a morphism* $m : (\mathsf{a}, f) \to (\mathsf{b}, g)$ *in this category is a morphism* $m : \mathsf{a} \to \mathsf{b}$ *which makes the following diagram commute:*

$$\begin{tikzcd} T\mathsf{a} \arrow[r, "Tm"] & T\mathsf{b} \\ \mathsf{a} \arrow[r, "m"] \arrow[u, "f"] & \mathsf{b} \arrow[u, "g"'] \end{tikzcd}$$

Lemma 61 *The arrow* u *in a terminal objet* (t, u) *of a* T*-coalgebra is an isomorphism.*

Definition 62 *Let* $\mathcal{C}$ *be a category,* T *an endofunctor and* (a, f) *the initial* T*-algebra. (So* $\mathsf{a} \in \mathcal{C}_0$ *and* $f : T\mathsf{a} \to \mathsf{a}$ *is iso.) Let furthermore* (b, g) *be any other object in the category of* T*-algebras. Then, since* (a, f) *is the initial object, there is a unique arrow* $m : (\mathsf{a}, f) \to (\mathsf{b}, g)$ *in the category of* T*-algebras. So there is a unique* $m : \mathsf{a} \to \mathsf{b}$ *that makes the following square commute:*
$$\begin{tikzcd} T\mathsf{a} \arrow[r, "Tm"] \arrow[d, "f"] & T\mathsf{b} \arrow[d, "g"] \\ \mathsf{a} \arrow[r, "m"] & \mathsf{b} \end{tikzcd}.$$
This uniquely defined morphism m *is called the* ***catamorphism of*** g, *usually denoted* $\mathsf{cata}\, g$.

Note that, since f is iso, $\mathsf{cata}\, g$ can also be defined as the composition $\mathsf{cata}\, g = g \circ Tm \circ f^{-1}$.

So this equality actually gives a recursive definition of $m = \mathsf{cata}\, g$.

It's already been shown that every adjunction gives rise to a monad. What follows next is the opposite. I will show that every monad gives rise to a an adjunction.

Definition 63 *Let $\mathcal{C}$ be a category and T a monad. Since T is an endofunctor, we can define the category of T-algebras. This category is called the* ***Eilenberg-Moore category*** *and is usually denoted by $\mathcal{C}^T$.*

Theorem 64 *For such a T-algebra there is clearly a forgetful functor $U^T : \mathcal{C}^T \to \mathcal{C}$ which sends objects (a, f) to the carrier object a and morphisms of the T-algebra to the corresponding morphism in $\mathcal{C}_1$.*

As noted before a forgetful functor is left adjoint to a free functor $F^T : \mathcal{C} \to \mathcal{C}^T$. This free functor sends an object $\mathsf{a} \in \mathcal{C}_0$ to the object $(T\mathsf{a}, \mu_{\mathsf{a}})$, where μ is the multiplication of the monad T.

It even turns out that

$$U^T \circ F^T = T$$

as one might hope.