Category theory notes

December 10, 2021

1 Definition of a category

Definition 1 *A category $\mathcal{C}$ has objects a and arrows $f : \mathsf{a} \to \mathsf{b}$ between objects. $\mathcal{C}_0$ denotes the objects of catogory $\mathcal{C}$ and $\mathcal{C}_1$ denotes its arrows. For an arrow $f : \mathsf{a} \to \mathsf{b}$ the object a is called its domain and b its codomain: also denoted $\mathsf{a} = dom(f)$ and $\mathsf{b} = cod(f)$. A category $\mathcal{C}$ must adhere to the following* axioms*:*

- *if f and g are two functions such that $cod(f) = dom(g)$, then the composition gf (or $g.f$) is a function from $dom(f)$ to $cod(g)$:*

  $$(\mathsf{a} \xrightarrow{f} \mathsf{b} \xrightarrow{g} \mathsf{c}) \quad \mapsto \quad (\mathsf{a} \xrightarrow{gf} \mathsf{c})$$

- *composition of arrows is* associative*: i.e. given $f : \mathsf{a} \to \mathsf{b}$, $g : \mathsf{b} \to \mathsf{c}$ and $h : \mathsf{c} \to \mathsf{d}$, then $(hg)f = h(gf) : \mathsf{a} \to \mathsf{d}$*

- *every element $\mathsf{a} \in \mathcal{C}_0$ has an identity arrow $id_\mathsf{a} : \mathsf{a} \to \mathsf{a}$ satisfying $id_\mathsf{a}.f = f \;\; \forall f \in \mathcal{C}_1$ with $cod(f) = \mathsf{a}$ and $g.id_\mathsf{a} = g \;\; \forall g \in \mathcal{C}_1$ with $dom(g) = \mathsf{a}$.*

Example 2 *A* preorder *is a set X together with a binary relation $\leq$ which is*

- *reflexive: i.e. $x \leq x \;\; \forall x \in X$*

- *transitive: i.e. $x \leq y, y \leq z \Rightarrow x \leq z \;\; \forall x, y, z \in X$*

any such a preorder can be seen as a category $\mathcal{C}$ with elements being the objects of X and a unique arrow $x \to y$ iff $x \leq y$.

Example 3 *A* monoid *is a set X with a binary operation, written like multiplication xy for $x, y \in X$ which is associative and has a unit element $e \in X$ such that $ex = xe = x \;\; \forall x \in X$. A monoid can be interpreted as a category with one object and one arrow x for every $x \in X$.*

Example 4 *Top, Grp, Mon, Rng, Grph, Pos are the categories with* topological spaces, groups, monoids, rings, graphs, posets *as objects and their respective homomorphisms (structure preserving functions) as arrows. Set is the category of sets with functions between sets as arrows.*

Definition 5 *A category $\mathcal{C}$ is called* small *iff both $\mathcal{C}_0$ and $\mathcal{C}_1$ are sets. The category is called* locally small *iff for any two objects $\mathsf{a}, \mathsf{b} \in \mathcal{C}$ the set of morphisms between a and b (denoted $Hom(\mathsf{a}, \mathsf{b})$) is a set.*

Definition 6 *Let $\mathcal{C}$ be a category, the* oposite category *is denoted $\mathcal{C}^{op}$. This category has the same objects, but arrows pointing the other way. I.e. if $f : \mathsf{a} \to \mathsf{b}$ in $\mathcal{C}_1$, then $\bar{f} : \mathsf{b} \to \mathsf{a}$ in $(\mathcal{C}^{op})_0$. Composition of arrows is defined as $\bar{f}\bar{g} = \bar{gf}$.*

Definition 7 *An arrow $f : \mathsf{a} \to \mathsf{b}$ is called a* monomorphism *if for any other object c and morphisms $g, h : \mathsf{c} \to \mathsf{a}$ $fg = fh$ implies $g = h$. The arrow is called* epimorphism *if for any object c and morphisms $g, h : \mathsf{b} \to \mathsf{c}$ $gf = hf$ implies $g = h$.*

Lemma 8 *Monomorphisms in Set correspond to injective functions, where epimorphisms in Set correspond to surjective functions.*

Lemma 9 *If gf is mono, then f is mono, so by duality: if fg is epi, then f is epi.*

Definition 10 *An epi $f : \mathsf{a} \to \mathsf{b}$ is called* split epi *if $\exists g : \mathsf{b} \to \mathsf{a}$ such that $fg = id_{\mathsf{b}}$. Dually a mono $f : \mathsf{a} \to \mathsf{b}$ is called* split mono *if $\exists g : \mathsf{b} \to \mathsf{a}$ such that $gf = id_{\mathsf{a}}$.*

Definition 11 *A morphism $f : \mathsf{a} \to \mathsf{b}$ is called an* isomorphism *if $\exists g : \mathsf{b} \to \mathsf{a}$ such that $fg = id_{\mathsf{b}}$ and and $gf = id_{\mathsf{a}}$. In this case the objects a and b are called* isomorphic.

Lemma 12

- *if two of f, g and fg are iso, then so is the third*
- *if f is epi and split mono, it is iso*
- *if f is split epi and mono, it is iso*

Definition 13 *An object $\mathsf{a} \in \mathcal{C}_0$ is called a* terminal object *if for any other object $\mathsf{b} \in \mathcal{C}_0$ there is a unique arrow $f : \mathsf{b} \to \mathsf{a}$. (E.g. singletons are terminal objects in Set) Similarly an object $\mathsf{a} \in \mathcal{C}_0$ is called a* initial object *if for ano other object $\mathsf{b} \in \mathcal{C}_0$ there is a unique arrow $f : \mathsf{a} \to \mathsf{b}$. (E.g. the empty set is the only initial object in Set.)*

Lemma 14 *Any two terminal objects are isomorphic. Same for any two initial objects.*

2 Functors and natural morphisms

Definition 15 *A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of operations $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ such that the following holds:*

- *for each $f : \mathsf{a} \to \mathsf{b}$ in $\mathcal{C}$: $F_1(f) : F_0(\mathsf{a}) \to F_0(\mathsf{b})$*

- *for each* $\mathsf{a} \xrightarrow{f} \mathsf{b} \xrightarrow{g} \mathsf{c}$ *:* $F_1(gf) = F_1(g)F_1(f)$.

- $F_0(id_{\mathsf{a}}) = id_{F_0(\mathsf{a})}$ *for each* $a \in \mathcal{C}$

Definition 16 *An endofunctor is a functor F from a category $\mathcal{C}$ to itself.*

Definition 17 *A functor* $F : \mathcal{C} \to \mathcal{D}$ *is called* covariant*, whereas a functor* $F : \mathcal{C}^{op} \to \mathcal{D}$ *is called* contravariant*.*

Since functors are composable, the following definition can be made:

Definition 18 $\mathcal{C}\mathsf{at}$ *is the category whose objects are* small categories *and whose morphisms are functors between those categories.*

Now that functors are defined, its time to define natural morphisms:

Definition 19 *Let* $F, G : \mathcal{C} \to \mathcal{D}$ *be two functors of categories* $\mathcal{C}$ *and* $\mathcal{D}$*: a* natural morphism $\alpha : F \Rightarrow G$ *is a set of morphisms* $\alpha_{\mathsf{a}} : F(\mathsf{a}) \to G(\mathsf{a})$ *for any object* $\mathsf{a} \in \mathcal{C}_0$ *such that for any arrow* $f : \mathsf{a} \to \mathsf{b}$ *in* $\mathcal{C}_1$ *the following diagram in* $\mathcal{D}$ *commutes:*

$$\begin{tikzcd} F(\mathsf{a}) \arrow[r, "F(f)"] \arrow[d, "\alpha_{\mathsf{a}}"] & F(\mathsf{b}) \arrow[d, "\alpha_{\mathsf{b}}"] \\ G(\mathsf{a}) \arrow[r, "G(f)"] & G(\mathsf{b}) \end{tikzcd}$$

Definition 20 *A natural transformation* $\alpha : F \Rightarrow G$ *is called a* natural isomorphism *iff its components* $\alpha_{\mathsf{a}}\mathsf{a} \in \mathcal{C}$ *are isomorphisms.*

Definition 21 *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. We can define the* Functor category *which is denoted as either* $\mathcal{D}^{\mathcal{C}}$ *or* $[\mathcal{C}, \mathcal{D}]$ *as follows:*

- *objects of this category are functors* $F : \mathcal{C} \to \mathcal{D}$

- *morphisms between two such functors F and G are natural transformations* $\alpha : F \Rightarrow G$

Definition 22 *Actually a category like* $\mathcal{C}\mathsf{at}$ *which has morphisms and natural transformations between morphisms is called a* 2-category*.*

Definition 23 *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two categories, the* product category *or* cartesian product of categories $\mathcal{C} \times \mathcal{D}$ *has objects* (a, b) *where* $\mathsf{a} \in \mathcal{C}_0$ *and* $\mathsf{b} \in \mathcal{D}_0$ *and morphisms* $(f, b) : (\mathsf{a}, \mathsf{b}) \to (\mathsf{a}', \mathsf{b}')$ *where* $f : \mathsf{a} \to \mathsf{a}' \in \mathcal{C}_1$ *and* $g : \mathsf{b} \to \mathsf{b}' \in \mathcal{D}_1$*.*

Definition 24 *A functor from any product category to another category is called a* bifunctor*.*

Definition 25 *A functor* $F : \mathcal{C} \to \mathcal{D}$ *is called* faithfull *if its induced function* $F_1 : Hom(\mathsf{a}, \mathsf{b}) \to Hom(F(\mathsf{a}), F(\mathsf{b}))$ *is injective for any two objects* $\mathsf{a}, \mathsf{b} \in \mathcal{C}$*. It is called* full *if this induced function is surjective for any two objects* $\mathsf{a}, \mathsf{b} \in \mathcal{C}$*.*

Definition 26 *A functor* reflects *a property if, whenever the image (of objects or arrows) has a property, then the origin has the property too.*

Lemma 27 *Faithfull functors reflect epis and monos*

Lemma 28 *Full and Faithfull functors reflect property of being initial or terminal object.*

3 limits and colimits

Definition 29 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. A* cone *for F exists of an object $\mathsf{d} \in \mathcal{D}_0$ and a natural transformation $\mu : \Delta_{\mathsf{d}} \Rightarrow F$. So it is a family of morphisms $(\mu_{\mathsf{c}} : \mathsf{d} \to F(\mathsf{c}) | \mathsf{c} \in \mathcal{C}_0)$ such that for any morphism $f : \mathsf{c} \to \mathsf{c}'$ in $\mathcal{C}$ the following diagram commutes:*

$$\begin{tikzcd} & \mathsf{d} \arrow[dl, "\mu_{\mathsf{c}}"'] \arrow[dr, "\mu'_{\mathsf{c}}"] & \\ F(\mathsf{c}) \arrow[rr, "F(f)"] & & F(\mathsf{c}') \end{tikzcd}$$

This cone is denoted as (d, μ) and d is called the vertex *of the cone. In fact, cones of F form a category with morphisms $f : (\mathsf{d}, \mu) \to (\mathsf{d}', \mu')$ being a morphism $f : \mathsf{d} \to \mathsf{d}'$ in $\mathcal{C}_1$ such that $\mu'_{\mathsf{c}} g = \mu_{\mathsf{c}}$ for any $\mathsf{c} \in \mathcal{C}_0$. I.e. the following diagram commutes for any $\mathsf{c} \in \mathcal{C}_0$:*

$$\begin{tikzcd} \mathsf{d} \arrow[rr, "g"] \arrow[dr, "\mu_{\mathsf{c}}"] & & \mathsf{d}' \arrow[dl, "\mu'_{\mathsf{c}}"] \\ & F(\mathsf{c}) & \end{tikzcd}$$

So all cones over F form a category, which is denoted $Cone(F)$.

Note that the codomain of F in the previous definition is regularly called the *diagram of $\mathcal{C}$ in $\mathcal{D}$* and the category $\mathcal{C}$ is called the *index category* of this diagram.

Definition 30 *Let F be as in the previous definition. The terminal object in the category $Cone(F)$ is called the* limiting cone *for F or* limit *for the diagram F. It is also sometimes denoted $\boldsymbol{Lim}F$. Note that I say* the *terminal object because terminal objects are unique up to unique isomorphism.*

Now its natural to define the co-definition for cones and limits:

Definition 31 *Let $F : \mathcal{E} \to \mathcal{C}$ be a functor. A* cocone *for this functor is a pair (ν, d) where d is an object in $\mathcal{C}$ and $\nu : F \Rightarrow \Delta_D$ is a natural transformation. So a morphism $f : (\nu, \mathsf{d}) \to (\nu', \mathsf{d}')$ in the category of cocones of F are morphisms $f : \mathsf{d} \to \mathsf{d}'$ such that for any $\mathsf{e} \in \mathcal{E}_0$ the following diagram commutes:*

$$\begin{tikzcd} & F(\mathsf{c}) & \\ \mathsf{d} \arrow[ur, "\nu_{\mathsf{c}}"] \arrow[rr, "f"] & & \mathsf{d}' \arrow[ul, "\mu'_{\mathsf{c}}"'] \end{tikzcd}$$

The colimit *for F is the initial object in the category of cocones over F.*

Now I will give a few examples of limits and colimits:

Example 32 *Let* c, d *be two objects of a category* $\mathcal{C}$*. These two objects can be seen as the image of the functor* $F : \mathbf{2} \to \mathcal{C}$ *where* $\mathbf{2}$ *is the* discrete category with two objects *(the category with two objects and no arrows except for the identity arrows). Where the functor* F *sends the one element of* $\mathbf{2}$ *to* c *and the other to* d*. The* product *of* c *and* d *is the limit of the category of cones over* F*. I.e. it is an object* e *together with two morphisms* $p : \mathsf{e} \to \mathsf{c}$ *and* $q : \mathsf{e} \to \mathsf{d}$ *such that for any other such object* e' *and morphisms* p' *and* q' *there is a unique morphism* m *that factorizes* p' *and* q'*. In diagrams this looks as follows:*

$$\begin{tikzcd} & \mathsf{e}' \arrow[d, dashed, "\exists! m"] \arrow[ddl, "p'"'] \arrow[ddr, "q'"] & \\ & \mathsf{e} \arrow[dl, "p"] \arrow[dr, "q"'] & \\ \mathsf{c} & & \mathsf{d} \end{tikzcd}$$

The morphisms p *and* q *are called the* projections *of the product. Its* coproduct *of* c *and* d *is the colimit of this functor* F*. (Write out the diagram yourself.)*

Example 33 *Let* $\bar{\mathbf{2}}$ *be the category* $\begin{tikzcd} \mathsf{x} \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & \mathsf{y} \end{tikzcd}$*. A functor* $\bar{\mathbf{2}} \to \mathcal{C}$ *is basically a parallel pair of morphisms* $f, g : \mathsf{a} \to \mathsf{b}$ *in* $\mathcal{C}$*. And a cone for this functor is basically a diagram*

$$\begin{tikzcd} & \mathsf{d} \arrow[dl, "\mu_{\mathsf{a}}"'] \arrow[dr, "\mu_{\mathsf{b}}"] & \\ \mathsf{a} \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & & \mathsf{b} \end{tikzcd}$$

but $\mu_{\mathsf{b}} = f\mu_{\mathsf{a}} = g\mu_{\mathsf{a}}$ *so this arrow is usually omitted, leaving the following diagram:* $\begin{tikzcd} \mathsf{d} \arrow[r, "\mu_{\mathsf{a}}"] & \mathsf{a} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \mathsf{b} \end{tikzcd}$*. The limit in this category of cones (I.e. the object* d *and morphism* μ_{a}*) is called the* equalizer *of the pair* f *and* g*. The definition of the* co equalizer *follows automatically.*

Example 34 *A* pullback *is the limiting cone in the category of cones over a diagram like this:* $\begin{tikzcd} \mathsf{a} \arrow[r, "f"] & \mathsf{b} & \mathsf{c} \arrow[l, "g"] \end{tikzcd}$*. Just to clarify, a cone for this diagram looks like this:*

$$\begin{tikzcd} & \mathsf{d} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathsf{a} \arrow[dr, "f"'] & & \mathsf{c} \arrow[dl, "g"] \\ & \mathsf{b} & \end{tikzcd}$$

. (The arrow from d *to* b *is omitted since it can be given as the composition* $gq = fp$*.) The co definitions of the pullback is called a* pushout*.*

mention: definition of limits, initial/terminal objects, products and coproducts, equalizer and co equalizer, pullbacks and pushouts

4 yoneda

mention lemma and usual application of it

5 monads, adjunctions and T-algebras

mention relations between those, adjunctions for free and forgetfull