Category theory notes

December 10, 2021

1 Definition of a category

Definition 1 A category C has objects a and arrows $f: a \to b$ between objects. C_0 denotes the objects of category C and C_1 denotes its arrows. For an arrow $f: a \to b$ the object a is called its domain and b its codomain: also denoted a = dom(f) and b = cod(f). A category C must adhere to the following axioms:

• if f and g are two functions such that cod(f) = dom(g), then the composition qf (or q, f) is a function from dom(f) to cod(q):

$$(\mathsf{a} \overset{f}{\longrightarrow} \mathsf{b} \overset{g}{\longrightarrow} \mathsf{c}) \ \mapsto \ (\mathsf{a} \overset{gf}{\longrightarrow} \mathsf{c})$$

- composition of arrows is associative: i.e. given $f: a \to b$, $g: b \to c$ and $h: c \to d$, then $(hg)f = h(gf): a \to d$
- every element $a \in C_0$ has an identity arrow $id_a : a \to a$ satisfying $id_a . f = f \ \forall f \in C_1$ with cod(f) = a and $g.id_a = g \ \forall g \in C_1$ with dom(g) = a.

Example 2 A preorder is a set X together with a binary relation \leq which is

- reflexive: i.e. $x \le x \ \forall x \in X$
- transitive: i.e. $x \le y, y \le z \Rightarrow x \le z \ \forall x, y, z \in X$

any such a preorder can be seen as a category C with elements being the objects of X and a unique arrow $x \to y$ iff $x \le y$.

Example 3 A monoid is a set X with a binary operation, written like multiplication xy for $x, y \in X$ which is associative and has a unit element $e \in X$ such that $ex = xe = x \ \forall x \in X$. A monoid can be interpreted as a category with one object and one arrow x for every $x \in X$.

Example 4 Top, Grp, Mon, Rng, Grph, Pos are the categories with topological spaces, groups, monoids, rings, graphs, posets as objects and their respective homomorphisms (structure preserving functions) as arrows. Set is the category of sets with functions between sets as arrows.

Definition 5 A category C is called small iff both C_0 and C_1 are sets. The category is called locally small iff for any two objects $a, b \in C$ the set of morphisms between a and b (denoted Hom(a,b)) is a set.

Definition 6 Let C be a category, the oposite category is denoted C^{op} . This category has the same objects, but arrows pointing the other way. I.e. if $f: a \to b$ in C_1 , then $\bar{f}: b \to a$ in $(C^{op})_0$. Composition of arrows is defined as $\bar{f}\bar{g} = g\bar{f}$.

Definition 7 An arrow $f: a \to b$ is called a monomorphism if for any other object c and morphisms $g, h: c \to a$ fg = fh implies g = h. The arrow is called epimorphism if for any object c and morphisms $g, h: b \to c$ gf = hf implies g = h.

Lemma 8 Monomorphisms in Set correspond to injective functions, where epimorphisms in Set correspond to surjective functions.

Lemma 9 If gf is mono, then f is mono, so by duality: if fg is epi, then f is epi.

Definition 10 An epi $f: a \to b$ is called split epi if $\exists g: b \to a$ such that $fg = id_b$. Dually a mono $f: a \to b$ is called split mono if $\exists g: b \to a$ such that $gf = id_a$.

Definition 11 A morphism $f: a \to b$ is called an isomorphism if $\exists g: b \to a$ such that $fg = id_b$ and and $gf = id_a$. In this case the objects a and b are called isomorphic.

Lemma 12 • if two of f, g and fg are iso, then so is the third

- if f is epi and split mono, it is iso
- if f is split epi and mono, it is iso

Definition 13 An object $a \in C_0$ is called a terminal object if for any other object $b \in C_0$ there is a unique arrow $f : b \to a$. (E.g. singletons are terminal objects in Set) Similarly an object $a \in C_0$ is called a initial object if for ano other object $b \in C_0$ there is a unique arrow $f : a \to b$. (E.g. the empty set is the only initial object in Set.)

Lemma 14 Any two terminal objects are isomorphic. Same for any two initial objects.

2 Functors and natural morphisms

Definition 15 A functor F from a category C to a category D consists of operations $F_0: C_0 \to D_0$ and $F_1: C_1 \to D_1$ such that the following holds:

• for each $f : a \to b$ in $C \colon F_1(f) : F_0(a) \to F_0(b)$

- for each a \xrightarrow{f} b \xrightarrow{g} c : $F_1(gf) = F_1(g)F_1(f)$.
- $F_0(id_a) = id_{F_0(a)}$ for each $a \in \mathcal{C}$

Definition 16 An endofunctor is a functor F from a category C to itself.

Definition 17 A functor $F: \mathcal{C} \to \mathcal{D}$ is called covariant, whereas a functor $F: \mathcal{C}^{op} \to \mathcal{D}$ is called contravariant.

Since functors are composable, the following definition can be made:

Definition 18 Cat is the category whose objects are small categories and whose morphisms are functors between those categories.

Now that functors are defined, its time to define natural morphisms:

Definition 19 Let $F,G:\mathcal{C}\to\mathcal{D}$ be two functors of categories \mathcal{C} and \mathcal{D} : a natural morphism $\alpha:F\Rightarrow G$ is a set of morphisms $\alpha_{\mathsf{a}}:F(\mathsf{a})\to G(\mathsf{a})$ for any object $\mathsf{a}\in\mathcal{C}_0$ such that for any arrow $f:\mathsf{a}\to\mathsf{b}$ in \mathcal{C}_1 the following diagram in \mathcal{D} commutes:

$$F(\mathsf{a}) \xrightarrow{F(f)} F(\mathsf{b})$$

$$\downarrow^{\alpha_{\mathsf{a}}} \qquad \downarrow^{\alpha_{\mathsf{b}}}$$

$$G(\mathsf{a}) \xrightarrow{G(f)} G(\mathsf{b})$$

Definition 20 A natural transformation $\alpha: F \Rightarrow G$ is called a natural isomorphism iff its components $\alpha_a \in \mathcal{C}$ are isomorphisms.

Definition 21 Let C and D be categories. We can define the Functor category which is denoted as either D^{C} or [C, D] as follows:

- objects of this category are functors $F: \mathcal{C} \to \mathcal{D}$
- morphisms between two such functors F and G are natural transformations $\alpha: F \Rightarrow G$

Definition 22 Actually a category like Cat which has morphisms and natural transformations between morphisms is called a 2-category.

Definition 23 Let C and D be two categories, the product category or cartesian product of categories $C \times D$ has objects (a, b) where $a \in C_0$ and $b \in D_0$ and morphisms $(f, b) : (a, b) \to (a', b')$ where $f : a \to a' \in C_1$ and $g : b \to b' \in D_1$.

Definition 24 A functor from any product category to another category is called a bifunctor.

Definition 25 A functor $F: \mathcal{C} \to \mathcal{D}$ is called faithfull if its induced function $F_1: Hom(a,b) \to Hom(F(a),F(b))$ is injective for any two objects $a,b \in \mathcal{C}$. It is called full if this induced function is surjective for any two objects $a,b \in \mathcal{C}$.

Definition 26 A functor reflects a property if, whenever the image (of objects or arrows) has a property, then the origin has the property too.

Lemma 27 Faithfull functors reflect epis and monos

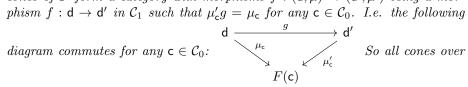
Lemma 28 Full and Faithfull functors reflect property of being initial or terminal object.

3 limits and colimits

Definition 29 Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between categories. A cone for F exists of an object $d \in \mathcal{D}_0$ and a natural transformation $\mu: \Delta_d \Rightarrow F$. So it is a family of morphisms $(\mu_c: d \to F(c)|c \in \mathcal{C}_0)$ such that for any morphism



This cone is denoted as (d, μ) and d is called the vertex of the cone. In fact, cones of F form a category with morphisms $f: (d, \mu) \to (d', \mu')$ being a morphism $f: d \to d'$ in C_1 such that $\mu'_c g = \mu_c$ for any $c \in C_0$. I.e. the following



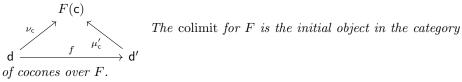
F form a category, which is denoted Cone(F).

Note that the codomain of F in the previous definition is regularly called the diagram of $\mathcal C$ in $\mathcal D$ and the category $\mathcal C$ is called the index category of this diagram.

Definition 30 Let F be as in the previous definition. The terminal object in the category Cone(F) is called the limiting cone for F or limit for the diagram F. It is also sometimes denoted LimF. Note that I say the terminal object because terminal objects are unique up to unique isomorphism.

Now its natural to define the co-definition for cones and limits:

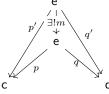
Definition 31 Let $F: \mathcal{E} \to \mathcal{C}$ be a functor. A cocone for this functor is a pair (ν, d) where d is an object in \mathcal{C} and $\nu: F \Rightarrow \Delta_D$ is a natural transformation. So a morphism $f: (\nu, \mathsf{d}) \to (\nu', \mathsf{d}')$ in the category of cocones of F are morphisms $f: \mathsf{d} \to \mathsf{d}'$ such that for any $\mathsf{e} \in \mathcal{E}_0$ the following diagram commutes:



Now I will give a few examples of limits and colimits:

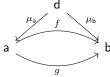
Example 32 Let c, d be two objects of a category C. These two objects can be seen as the image of the functor $F: \mathbf{2} \to \mathcal{C}$ where **2** is the discrete category with two objects (the category with two objects and no arrows except for the identity arrows). Where the functor F sends the one element of 2 to c and the other to d. The product of c and d is the limit of the category of cones over F. I.e. it is an object e together with two morphisms $p : e \rightarrow c$ and $q : e \rightarrow d$ such that for any other such object e' and morphisms p' and q' there is a unique morphism m

that factorizes p' and q'. In diagrams this looks as follows: $p' \stackrel{\exists !m}{\downarrow} q'$ e



The morphisms p and q are called the projections of the product. Its coproduct of c and d is the colimit of this functor F. (Write out the diagram yourself.)

Example 33 Let $\bar{\mathbf{2}}$ be the category \mathbf{x} $\overset{a}{\underset{b}{\smile}}$ \mathbf{y} . A functor $\bar{\mathbf{2}} \rightarrow \mathcal{C}$ is basically a parallel pair of morphisms $f,g: \mathbf{a} \rightarrow \mathbf{b}$ in \mathcal{C} . And a cone for this functor \mathbf{d} is basically a diagram \mathbf{d} \mathbf{d}



is usually omitted, leaving the following diagram: $d \xrightarrow{\mu_a} a \xrightarrow{f} b$. The

limit in this category of cones (I.e. the object d and morphism μ_a) is called the equalizer of the pair f and g. The definition of the co equalizer follows automatically.

Example 34 A pullback is the limiting cone in the category of cones over a diagram like this: $a \xrightarrow{f} b \leftarrow_{q} c$. Just to clarify, a cone for this diagram

looks like this: a \int_{g}^{q} c . (The arrow from d to b is omitted since it

can be given as the composition gq = fp.) The co definitions of the pullback is called a pushout.

mention: definition of limits, initial/terminal objects, products and coproducts, equalizer and co equalizer, pullbacks and pushouts

4 yoneda

mention lemma and usual application of it

5 monads, adjunctions and T-algebras

mention relations between those, adjunctions for free and forgetfull