[redacted]

Q1:

A

Let there be an arbitrary Graph $G = (V, E)$ that is $undirected$

Thus let's mark the following $(*)$ $(x_0, x_1, \ldots, x_n, y)$ as the path between Nodes $x_0, y \in V$

Where it's the shortest path from x_0 to y.

Let $(**)$ = $(x_i, \ldots, x_j)$ for some $0 \leq i < j \leq n+1$ (lets say that $x_{n+1} = y$) be be a sub-path of $(*)$

We will prove that $(**)$ is the shortest path from x_i to x_j

Let's assume by contradiction that $(**)$ isn't the shortest path from x_i to x_j $\implies$ There $\exists$ a path

$subP$ = $(x_i, x_{i+1}, \ldots, x_j)$ such that $|subP| < |(**)|$

Let's mark the lengths of the sub path $(x_0, \ldots, x_{i-1})$ as l_1

length of sub path as $(x_{j+1}, \ldots, y)$ as l_2

length of sub path of $(**)$ as l_3

length of $subP$ as l_4

by our contradiction assumption we get that $l_3 > l_4$

We will look at the Chained Path from the following paths of $A = (x_0, \ldots, x_i)$, $B = (x_j, \ldots, y)$

Where A, B are sub paths of $(*)$ and where the chained path we will mark as C.

length of C = $l_1 + l_2 + l_3 = |(*)| = l_1 + l_2 + l_4 \implies l_3 = l_4$ which is a contradiction! Thus proving that a sub path of a shortest path is a shortest path in it's own right $\blacksquare$

B) We will prove that given a Graph $G = (V, E)$ that is connected and undirected, that G is a bipartite Graph $\iff$ There are no cycles with an odd length in G

Let there G an arbitrary undirected and connected Graph.

$\implies$: Let G be a bipartite graph. Let's assume by contradiction that there exists in G a cycle with an odd length $(*)$

By definition there exists $v_1, v_2 \subseteq V$ such that $V = v_1 \cup v_2$ and $v_1 \cap v_2 = \emptyset$ such that $E \subseteq v_1 X v_2$

$(*)$ We will mark this cycle as $(x_0, \ldots, x_{k-1}, x_0)$ w.l.o.g such that k is the length of the cycle which is odd according to our assumption $(**)$

w.l.o.g that $x_0 \in v_1 \implies x_1 \in v_2$ Since there doesn't exist arcs between items that are in the same group v_1 or v_2 (#) $\implies x_1 \in v_1$

Let's assume by induction that the m item such that m is even belongs to v_1 (w.l.o.g), ($m < k+1$)
$\implies$ that the m+1 item belongs to v_2 because of (#). In the case where the m item is odd and belongs to v_1 (w.l.o.g), ($m < k+1$) $\implies$ that the item in placement $m+1$ belongs to v_2 according to (#) $\implies$ the items in odd placements in the cycle belong to v_1 and in the odd placements belong to v_2 $\implies$ x_0 is in v_1 (0 is even), but x_0 is in the placement k in the cycle which according to $(**)$ k is odd Thus $x_0 \in v_2$ which is a contradiction. ■

$\implies$: (other direction)

Let's assume that G is a graph that doesn't have any cycles with an odd length.
Let there be $v_1, v_2 = \emptyset$
let there be $v \in V$ and will induce v in v_1.
We will put the neighbors of v in v_2
Then we will put the neighbors of those neighbors in v_1 (Those who aren't yet in v_1)
We will continue this process until we split all the Nodes in the graph.

(we need to prove that G is a bipartite graph)

$\implies$ We will show that v_1, v_2 fulfill the requirements of a bipartite graph.

If G is connected $\implies$ There exists a path from v to every Node in the graph $\implies$ $v_1 \cup v_2 = V$ (since every Node belongs at some stage of splitting nodes to one of these two groups)
and $v_1 \cap v_2 = \emptyset$. We can see that through the process we don't place any Node twice and not in the same group as it's neighbors.

$\implies$ If G is connected then there is path in V for every Node in the graph.

Let's assume by contradiction that the division that we created for v_1, v_2 has an arc that connects two Nodes in the same group. Let's mark these two Nodes as q, m w.l.o.g $q, m \in v_2$

If $q, m \in v_2$ $\implies$ There exists a path with an odd length from v to q and an odd length path from v to m.
This claim is due to the construction of the group - we got to m, q through splitting of neighbors and their neighbors and so forth of v and we know that this path has an odd length since when we activated this process an odd amount of times until we got to q (same for m). We will mark these paths as a, b

$\implies$ There exists a cycle in the graph from the chain of a with the arc between m to q and the chain with it's length $|a| + |b| + 1$ is odd since $|a|, |b|$ have an odd length each.
Analogously, if $q, m \in v_1$ Then we will get $|a|, |b|$ are even
and that $|a| + |b| + 1$ is odd

which is a contradiction. ■

Q2:

A) Let's assume that $G=(V,E)$ is $K-Dalil$, Therefore, we can divide the Graph into k groups where each group is disconnected from the other groups which we can see from definition 2 of k-$Dalil$.

Claim (I)**:** the following is true: $|E| \leq |V|-1$ if G is a forest $G=(V,E)$
Proof:
Since if we split G into all it's connected components, let's assume that there are p connected components where $p \in \mathbb{N}$ (Thus $p>0$) , since G is a forest $\implies$ each connected component is a tree by definition. $\implies$ Let's mark each tree as an $G_i=(V_i,E_i)$ which is induced from G, From a theorem we proved in the lecture, $|E_i|=|V_i|-1 \implies$ if we add up all these separate trees which forms the G forest then we get that $|E|=\sum_{i=1}^{p}(|V_i|-1)$ $=|V|-p \leq |V|-1$ Thus proving this Claim.

$\frac{m}{n-1}=\frac{|E|}{n-1}=(*)\sum_{i=1}^{k}\frac{|E_i|}{n-1} \leq (**)1+1+...+1=k$
$(*)$ The move is true since $\forall i \neq j : i,j \in \{1,\ldots,k\}\ E_i \cap E_j=\emptyset$
$(**)$ Let's assume in every Sub Graph induced G_i has at most n nodes. Since G_i is a forest then according to claim (I) we get that $|E_i| \leq |V_i|-1 \leq n-1 \implies \frac{|E_i|}{n-1} \leq 1$. ■

B) Let's assume that $G=(V,E)$ is K-$Dalil$, Therefore, we can divide the Graph into k groups where each group is disconnected from the other groups which we can see from definition 2 of k-$Dalil$ and we also get that there is a group of subset arc groups $E_1,E_2,\ldots,E_k \subseteq E$ as well as each group G_i is a forest.

Let's assume by contradiction that each $Vertice$ in G has a degree greater than $2k$
$\implies \forall v \in V : deg(v) \geq 2k \geq (part\ a)\ \frac{2m}{n-1} > \frac{m}{n-1}$

$\frac{m}{n-1}$ represents the amount of arcs per Node but it's given that G is k-$Dalil$ but each k group is a forest thus it's impossible that the $deg(v)$ is greater than the total amount of arcs per Node for every Node. Thus the assumption is wrong $\implies$ There exists a Node in the Graph G such that it's degree is smaller than $2k$ ■

C) Given that $G=(V,E)$ and that G is k-$Dalil$
We need to prove that if G is $2k$ colored $\iff$ There exists a function $C:V->[2k]$: $\forall(u,v) \in E : C(v) \neq C(u)$

Given that there is a split of all the arcs such that:
I $E_1,E_2,\ldots,E_k \subseteq E$
II $\forall 1 \leq i<j \leq k : E_j \cap E_i=\emptyset$
III $G_i=(v_i,E_i)\ where\ \forall i \in [k] : G_i$ is a forest meaning no cycles.

We also know from part A, B of the question that $2k \geq \frac{m}{n-1}$, (*) $\exists v \in V : deg(v)<2k$

Since G is is k-$Dalil$ $\iff$ There are k different groups where each group has at least one Node, meaning there are no arcs between any two Nodes if they are in different groups (by definition of k-$Dalil$), Therefore, we will color one Node in each group in a different color. We receive from section B (*) Therefore, for a group i in V, there exists a $u \in V$ such that

$(v, u) \in E_i$ if u is colored then we choose a different Node otherwise we will color it with color K+1, From the clue given, We can take a sub Graph $V - v$ and this graph will remain $k\text{-}Dalil$.

$\iff$ we will iteratively do this method mentioned above until we get to 2k different colors on Nodes, such for every time we remove a Node the sub Graph will remain $k\text{-}Dalil$ and Thus from part B There is an arc between nodes $v, u \in E$ that if one is colored then we can color the other Node. Furthermore, we are taking the original G graph that will end up having 2k colors.

Therefore, we have proven the required since it goes both ways ■

Q3:

A) Let's assume that $G = (V, E)$ is $K - Dalil$, Therefore, we can divide the Graph into k groups where each group is disconnected from the other groups which we can see from definition 2 of $k\text{-}Dalil$.

We also know that $2|E| = |V| = n$

$m = |E| \leq \frac{n}{2} \leq \frac{n-1}{2} + \frac{1}{2} \leq$ (because $k \in \mathbb{Z}, k > 0) \leq \frac{(n-1)\cdot k}{2} + \frac{1}{2} \leq (n-1) \cdot k$ (it's trivial when $n \geq 2$)
transition is true because it's an undirected map and each k group is a forest.
$\implies m \leq k \cdot (n-1) \implies \frac{m}{n-1} \leq k$ ■

B) Let's assume that $G = (V, E)$ is $K\text{-}Dalil$, Therefore, we can divide the Graph into k groups where each group is disconnected from the other groups which we can see from definition 2 of $k\text{-}Dalil$ and we also get that there is a group of subset arc groups $E_1, E_2, \ldots, E_k \subseteq E$ as well as each group G_i is a forest.

Let's assume by contradiction that each $Vertice$ in G has a degree greater than $2k$
$\implies \forall v \in V : deg(v) \geq 2k \geq (part\ a)\ \frac{2m}{n-1} > \frac{m}{n-1}$

$\frac{m}{n-1}$ represents the amount of arcs per Node but it's given that G is $k\text{-}Dalil$ but each k group is a forest thus it's impossible that the $deg(v)$ is greater than the total amount of arcs per Node for every Node. Thus the assumption is wrong $\implies$ There exists a Node in the Graph G such that it's degree is smaller than $2k$ ■

C) Given that $G = (V, E)$ and that G is $k\text{-}Dalil$
We need to prove that if G is $2k$ colored $\iff$ There exists a function $C : V -> [2k]$: $\forall (u, v) \in E : C(v) \neq C(u)$

Given that there is a split of all the arcs such that:
I $E_1, E_2, \ldots, E_k \subseteq E$
II $\forall 1 \leq i < j \leq k : E_j \cap E_i = \emptyset$
III $G_i = (v_i, E_i)\ where\ \forall i \in [k]$: G_i is a forest meaning no cycles.

We also know from part A, B of the question that $2k \geq \frac{m}{n-1}$, (*) $\exists v \in V : deg(v) < 2k$

Since G is is $k\text{-}Dalil \iff$ There are k different groups where each group has at least one Node, meaning there are no arcs between any two Nodes if they are in different groups (by definition of $k\text{-}Dalil$), Therefore, we will color one Node in each group in a different color. We receive from section B (*) Therefore, for a group i in V, there exists a $u \in V$ such that $(v,u) \in E_i$ if u is colored then we choose a different Node otherwise we will color it with color K+1, From the clue given, We can take a sub Graph $V - v$ and this graph will remain $k\text{-}Dalil$.

$\iff$ we will iteratively do this method mentioned above until we get to 2k different colors on Nodes, such for every time we remove a Node the sub Graph will remain $k\text{-}Dalil$ and Thus from part B There is an arc between nodes $v, u \in E$ that if one is colored then we can color the other Node. Furthermore, we are taking the original G graph that will end up having 2k colors.

Therefore, we have proven the required since it goes both ways ■

Q3:

1. Let there be an arbitrary Graph $G = (V, E)$ that is undirected. G has connected components.
   We will prove that $|E| \geq |V| - |C|$

$(*)$ Let's assume that $C = 1$, according to a theorem that we proved in a lecture, for a undirected Graph G that is connected (thus where $C = 1$ there is only one connected component and thus the graph is connected). Therefore, the graph is connected and the number of arcs is at least $|V| - 1$. $\implies$ that for $C = 1$, such that $|E| \geq |V| - 1$

Let's assume that $C = k$, let's mark the Nodes in each connected component as the group V_i pay attention that the following is true from this division of nodes:
$I)\ |V_1| + |V_2| + \ldots + |V_k| = |V|$
$II)\ V_1 \cup V_2 \cup \ldots \cup V_k = V$
$III)\ V_i \cap V_j = \emptyset\ \forall i \neq j$

Therefore, the connected component $i \in \{1, \ldots, k\}$ where by $(*)$ the following happens:
$|E_i| \geq |V_i| - 1$

We will note that all the groups Nodes and Arcs in the connected components are foreign to each other meaning that $E = \bigcup_{j=1}^{k} E_i$, $\forall i \neq j : E_i \cap E_j = \emptyset$ such that E_j is the set of arcs in the graph that is induced by V_i from two different connected components.
$\implies$
$|E| = \sum_{i=1}^{k} |E_i| \geq \sum_{i=1}^{k} |v_i| - 1 = |V| - K \implies |E| \geq |V| - k = |V| - |C|$ ■

B) Firstly, we will build a tree from the given graph $G = (V, E)$ like so:

$I)$ If it happens and G is a tree then we're done.
else-

II) If there exists an arc in G where if we remove it from the Graph G and the graph stays connected. Otherwise, G is a tree by definition (according to considered definitions of a tree that we proved in the lecture Where G is a tree $\iff$ not $\exists$ an arc that if we remove it from the graph and the graph remains connected)

III) We will root G at some arbitrary Node, we will remove Nodes in the following manner:

I') if in G there exists k Nodes then we finished.
II') Otherwise, we will remove a leave from G
III') we will return on I'

We will end up with a Graph G' that is induced from G by removing Nodes as per previous moves stated above and G' graph will have N nodes. ■

C)

1. The sum of degrees is equal twice the amount of Nodes in the graph.
   Therefore, we get $\frac{18}{2} = 9 = |E| = |V| - 1 = 8$
   Where if it were a tree then $|E| = |V| - 1$. This condition is not fulfilled. Therefore, there doesn't exist such a tree.
2. Graph for example:
   A-->B
   C-->D
   E-->F
   G-->H-->I
   J-->K-->L
3. From Part A of question 3 we proved that $|E| \geq |V| - |C|$ but from the given, we get that $24 = |E| \geq |V| - |C| = 30 - 5 = 25 \implies 24 \geq 25$ which is incorrect thus such a graph does not exist. ■
4. Let's assume by contradiction that there exists a graph that fulfills the requirements. Let us remove an arbitrary arc. Therefore, we now have a graph with 8 arcs, 9 Nodes and no Cycles $(*)$. $\implies |E| = |V| - 1$ (#)

   (#) this condition is equivalent to a tree. Which is equivalent to a graph with no cycles but it's enough that we add one arc and there is a cycle $\implies$ the graph with the arc we removed has a cycle. This is a contradiction due to the fact that we were given a graph with no cycles.

   $(*)$ Let's assume by contradiction that the graph induced after removing an arbitrary arc has a cycle. If we add the arc that removed, the same cycle will remain since adding an arc won't change whether the graph has a cycle or not. Thus in contradiction to that there are no cycles in the graph, Thus the graph is acyclic.

Q4:

Pseudo Code:
We will mark right_sibling as RS, left_child as LC, degree as d

code:

1. $updateD(root, True)$
2. $updateD(curr, firstR)\{$
3. $\quad if\ curr.RS \neq NIL:$
4. $\qquad updateD(curr.RS, false)$
5. $\quad if\ curr.LC \neq Nil:$
6. $\qquad curr.d = 1$
7. $\qquad if\ firstR:$
8. $\qquad\quad updateD(curr.LC, True)$
9. $\}$

Line one calls the function where we start at the root of the tree, this takes $\Theta(1)$
Line 2 is the name of the function, where $curr$ is of type Node, which is the node we are currently checking, t being the current distance from the root meaning the degree from the root Node. $firstR$ is a condition that we use to check if we're on the the first Node of that row or not.

In this manner we traverse further into the depth of the tree only through the leftmost Node of each level, so that we don't go through a Node more then once. Once we're on a certain Layer, let's say layer k, we then update the degree of the current Node, which is one if it has a left child, if it doesn't have any other children then the degree of that Node is 0.

We then call the function on it's right sibling and to it's next right sibling until we go through all the nodes on that level and update their degrees. In this manner we do for each level.

Each line of code will run at a סיבכיות of $\Theta(1)$ and we go through each Node at most once. Therefore, our Algorithm will only go through each Node exactly Once and Therefore will reach the desired סיבכיות.