February 2021

Compressible Flow: Final Project

Shahar Meir, Gil Sasson
Mechanical Engineering at Ben-Gurion University

1 Solution

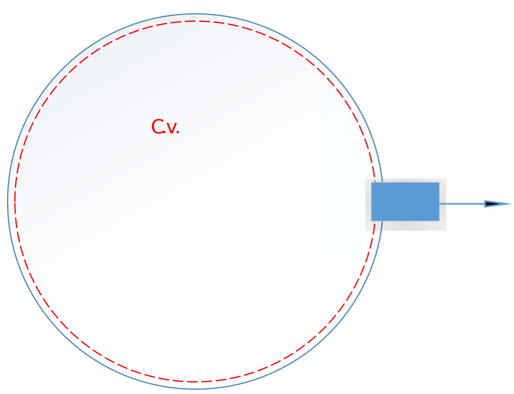
The nozzle remains choked as long as ($\gamma = 1.4$ for air):

$$\frac{P_t}{P_0} \geq \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

Let us show that the nozzle indeed remains choked in the process:

$$\frac{P_t}{P_0} \geq \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = 0.528 \;\; \forall \; P_t \;\because\; 375 \; [kPa] \; \leq P_t \leq \; 750 \; [kPa] \; \& \; P_0 = 100[kPa].$$

Because the volume of the container V is equal to 40 $[m^3]$ and the hole's diamater d is equal to 0.05 [m] and $A_{hole} << S_{container}$, we can then assume that the hole resembles a convergent nozzle of minimum cross-sectional area.



Energy equation (V is constant):

$$\underbrace{\frac{\partial}{\partial t}\int_{c.v.} \rho \left(e + \cancelto{negligible}{\frac{u_1^2}{2}} \right)}_{V \cdot C_v \frac{\partial}{\partial t}(\rho T)} V + \underbrace{\int_{c.s.} \rho \left(e + \frac{u_2^2}{2} + \frac{P}{\rho} \right) \vec{q} \cdot d\vec{A}}_{\dot{m}\underbrace{\left(h_2 + \frac{u_2^2}{2}\right)}_{h_t}} = 0$$

Assuming ideal gas:

$$\begin{aligned} \rho T &= \frac{P}{R} \\ R &= C_v(\gamma - 1) \\ h_t &= \frac{\gamma R T}{\gamma - 1} = C_p T \end{aligned}$$

Therefore:

$$\dot{m} = \frac{V}{\gamma R T_t} \frac{dP}{dt}. \tag{1}$$

On the other hand:

$$\dot{m} = \rho^* u^* A^*,$$

where $A*$ is the cross sectional area @ $M = 1$.
We've shown in class that:

$$\begin{aligned} \dot{m} &= \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{T_t}} \frac{M}{\left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}} A \\ &\overset{(a)}{=} \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{T_t}} \frac{1}{\left(1 + \frac{\gamma-1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}} A^* \\ &= \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{T_t}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} A^*. \end{aligned} \tag{2}$$

where (a) follows by substituting $M = 1$. From (2) and (1) we get:

$$\frac{1}{\gamma R T_t} \frac{dp_t}{dt} V = \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{T_t}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} A^*$$

Let us define T_{t_1} to be the temperature inside the container time t and p_{t_1} to be the pressure inside the container at time t and so on. We know how pressure relates to temperature in an isentropic, one dimensional process. It is given by:

$$\frac{T_t}{T_{t_1}} = \left(\frac{p_t}{p_{t_1}}\right)^{\frac{\gamma-1}{\gamma}}.$$

From the two expressions above:

$$\frac{1}{\gamma R\left(\frac{p_t}{p_{t_1}}\right)^{\frac{\gamma-1}{\gamma}} T_{t_1}} \frac{dp_t}{dt} V = \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{\left(\frac{p_t}{p_{t_1}}\right)^{\frac{\gamma-1}{\gamma}} T_{t_1}}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} A^*$$

$$\frac{V}{\gamma R\left(\frac{p_t}{p_{t_1}}\right)^{\frac{\gamma-1}{\gamma}} T_{t_1}} dp_t = \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{\left(\frac{p_t}{p_{t_1}}\right)^{\frac{\gamma-1}{\gamma}} T_{t_1}}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} A^* dt$$

$$dt = \frac{V}{\gamma^{\frac{3}{2}} A^* \sqrt{RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} p_t^{\frac{1-3\gamma}{2\gamma}} p_{t_1}^{\frac{\gamma-1}{2\gamma}} dp_t.$$

Integrating both sides:

$$\int_0^t dt = \frac{V p_{t_1}^{\frac{\gamma-1}{2\gamma}}}{\gamma^{\frac{3}{2}} A^* \sqrt{RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \int_{p_{t_1}}^{p_t} p_t^{\frac{1-3\gamma}{2\gamma}} dp_t$$

$$t = \frac{V p_{t_1}^{\frac{\gamma-1}{2\gamma}}}{\gamma^{\frac{3}{2}} A^* \sqrt{RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{2\gamma}{1-\gamma} p_t^{\frac{1-\gamma}{2g\gamma}} \Big|_{p_{t_1}}^{p_t}$$

$$= \frac{V p_{t_1}^{\frac{\gamma-1}{2\gamma}}}{\gamma^{\frac{3}{2}} A^* \sqrt{RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{2\gamma}{1-\gamma} \left[p_t^{\frac{1-\gamma}{2\gamma}} - p_{t_1}^{\frac{1-\gamma}{2\gamma}}\right]$$

$$= \frac{V p_{t_1}^{\frac{\gamma-1}{2\gamma}}}{\gamma^{\frac{3}{2}} A^* \sqrt{RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{2\gamma}{1-\gamma} \left[p_t^{\frac{1-\gamma}{2\gamma}} - p_{t_1}^{\frac{1-\gamma}{2g\gamma}}\right]$$

$$= \frac{V}{A^* \sqrt{\gamma RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{2}{1-\gamma} \left[\left(\frac{p_{t_1}}{p_t}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right]. \tag{3}$$

Substituting the given parameters into (3), that is:

$$\begin{aligned}
P_{t1} &= 375[kPa] \\
P_t &= 750[kPa] \\
A^* &= \frac{\pi}{4}d^2 = 1.9635 \cdot 10^{-3}[m^2] \\
R_{air} &= 286.69\left[\frac{J}{kgK}\right] \\
V &= 40[m^3] \\
T_{t1} &= T_t \cdot \left(\frac{P_{t1}}{P_t}\right)^{\frac{\gamma-1}{\gamma}} = 246.1[^\circ K] \\
\gamma &= 1.4
\end{aligned}$$

yields:

$$t = 52.79[sec].$$

2 Solution

In this sub-section we'll show the relevant derivation as indicated in our Matlab file:

$$\begin{aligned}
\frac{T_{t2}}{T_{t1}} &= \frac{\left(\frac{T_{t2}}{T^*}\right)_{M_2}}{\left(\frac{T_{t1}}{T^*}\right)_{M_1}} \\
&= \frac{\frac{2\gamma}{3\gamma-1}\left(1+\frac{\gamma-1}{2}M_2^2\right)}{\frac{2\gamma}{3\gamma-1}\left(1+\frac{\gamma-1}{2}M_1^2\right)} \\
&= \frac{\left(1+\frac{\gamma-1}{2}M_2^2\right)}{\left(1+\frac{\gamma-1}{2}M_1^2\right)}
\end{aligned}$$

Calculation of the entropy:

$$\frac{dS}{R/M_w} = \frac{\gamma}{\gamma-1}\frac{dT}{T} - \frac{dP}{P}$$

In an isothermal process there is no change in temperature therefore $\frac{dT}{T} = 0$. Integrating both sides:

$$\begin{aligned}
\int dS &= -\frac{R}{M_w}\int\frac{dP}{P} \\
\Delta S = S_2 - S_1 &= -\frac{R}{M_w}\ln\left(\frac{P_2}{P_1}\right)
\end{aligned}$$

For isothermal flow:

$$\Delta S = S_2 - S_1 = -\frac{R}{M_w} \ln\left(\frac{M_1}{M_2}\right)$$

3 Solution

Because any atom or molecule (i.e. noble gas in our case) has three degrees of freedom associated with translational motion. Therefore:

$$\begin{aligned} C_v &= \frac{1}{2} \underbrace{(Degrees\ of\ Freedom)}_{3} \cdot R \\ &= \frac{3}{2} \cdot R. \end{aligned}$$

And

$$\begin{aligned} C_p - C_v &= R \\ C_p &= \frac{5}{2} \cdot R \\ \therefore \gamma_{helium} &= \frac{C_p}{C_v} = \frac{5}{3} \end{aligned}$$

3.1 a)

Isothermal flow:

$$\begin{aligned} 4\bar{f}\frac{L_{max}}{D} &= \frac{1-\gamma M^2}{\gamma M^2} + \ln\left(\gamma M^2\right) \\ &= \frac{1-\frac{5}{3}\cdot 0.3^2}{\frac{5}{3}\cdot 0.3^2} + \ln\left(\frac{5}{3}\cdot 0.3^2\right) \\ &= 3.76954 \\ L_{max} &= \frac{3.76954 \cdot 0.15}{4 \cdot 0.005} \\ &= 28.2716[m] \end{aligned}$$

At the exit of the pipe:

$$\begin{aligned} M_2 &= \frac{1}{\sqrt{\gamma}} \\ \frac{P_2}{P_1} &= \frac{M_1}{M_2} \\ P_2 &= P_1 \cdot \frac{M_1}{M_2} \end{aligned}$$

$$= 0.7745 \cdot 10^5 \left[\frac{N}{m^2}\right]$$

3.2 b)

Adiabatic flow:

$$\begin{aligned}
4\bar{f}\frac{L_{max}}{D} &= \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma} \cdot \ln \frac{(\gamma+1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)} \\
&= 4.3468 \\
L_{max} &= \frac{4.3468 \cdot 0.15}{4 \cdot 0.005} \\
&= 32.601[m]
\end{aligned}$$

Using Matlab we find:

$$\left(\frac{P}{P^*}\right)_{M=0.3} = 3.7925$$

$$\therefore P_2 = P^* = 0.5273 \cdot 10^5 \left[\frac{N}{m^2}\right]$$

4 Solution

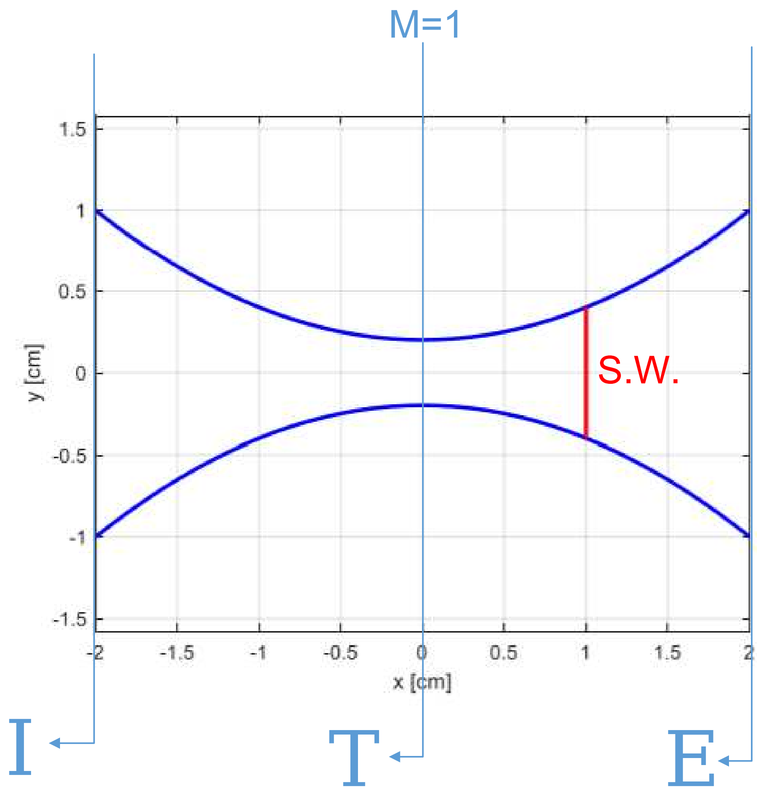
First, let us find the areas at the nozzle's inlet and exit, at the neck and at the shock-wave:

$$\begin{aligned}
r &= 0.2x^2 + 0.2 \ [cm] \\
r|_{x=-2} &= 0.2(-2)^2 + 0.2 = 1 \ [cm] = r|_{x=2}. \\
r|_{x=1} &= 0.4 \ [cm] \\
\therefore A_I &= A_E = \pi \ [cm^2] \\
A_{sw} &= \frac{4\pi}{25} \ [cm^2] \\
A_T &= \frac{\pi}{25} \ [cm^2]
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{A_E}{A_T} &= \frac{A_I}{A_T} = 25. \\
\frac{A_{sw}}{A_T} &= 4.
\end{aligned}$$

Because in this specific problem argon is the working gas, we can't use the isentropic flow table that is given in the course's website, where the specific heat ratio for air is $\gamma = 1.4$, the specific heat ratio of Argon is $\gamma = \frac{5}{3}$ because any atom or molecule (i.e. noble gas in our case) has three



degrees of freedom associated with translational motion. Therefore:

$$C_v = \frac{1}{2} \underbrace{(Degrees\ of\ Freedom)}_{3} \cdot R$$
$$= \frac{3}{2} \cdot R.$$

And

$$C_p - C_v = R$$
$$C_p = \frac{5}{2} \cdot R$$
$$\therefore \gamma_{argon} = \frac{C_p}{C_v} = \frac{5}{3}$$

We have shown in class that for an isentropic, one dimensional flow of ideal gas:

$$\frac{p_t}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$$
$$\frac{A}{A^*} = \frac{1}{M} \left[\frac{2}{\gamma + 1} \cdot \left(1 + \frac{\gamma - 1}{2} M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{T_t}{T_{t_1}} = \left(\frac{p_t}{p_{t_1}}\right)^{\frac{\gamma-1}{\gamma}}.$$

Using the isentropic relation of area-to-neck-area ratio we can find the corresponding mach number just before the shock-wave, using Matlab:

$$\underbrace{\frac{A_{sw}}{A_T} = 4\ ;\ \gamma = \frac{5}{3}}_{M = 3.441}$$

Now using the normal shock-wave relation ($\xi = 90°$, $\delta = 0$)

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)$$

We then know what happens to the total pressure:

$$\begin{aligned}\frac{P_{t2}}{p_{t1}} &= \frac{P_{t2}}{P_2}\cdot\frac{P_2}{P_1}\cdot\frac{P_1}{P_{t1}}\\ &= \underbrace{\left(1 + \frac{\gamma-1}{2}M_2^2\right)^{\frac{\gamma}{\gamma-1}}}_{Isentropic\ flow\ after\ the\ S.W.}\cdot\underbrace{\left[1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)\right]}_{normal\ S.W.}\cdot\underbrace{\frac{1}{\left(1 + \frac{\gamma-1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}}}}_{isentropic\ flow\ before\ the\ S.W.}\end{aligned}$$

In this manner we can find $\frac{P(x)}{P_I}$ using Matlab. We shall first define the area of the nozzle A(x). Then, we shall find the matrices M_sol and M_sol2 corresponding to $\frac{A(x)}{A^*}$, which both's dimensions are 1001x2 (to include zero) because we are taking 1001 samples from $x \in [-2, 2]$ and for each location there are two solutions (sub-sonic and super-sonic). Using these two matrices, we create two new matrices M_new and M_new2 in a for-loop with the right conditions (location and flow-type). With the new mach number matrices we can find the pressure profile. It is done in a for-loop that inserts into the matrix PPt(i) i.e. $\frac{P(x)}{P_I}$ the right mach number that is dependant on x. This results in the following pressure profile:

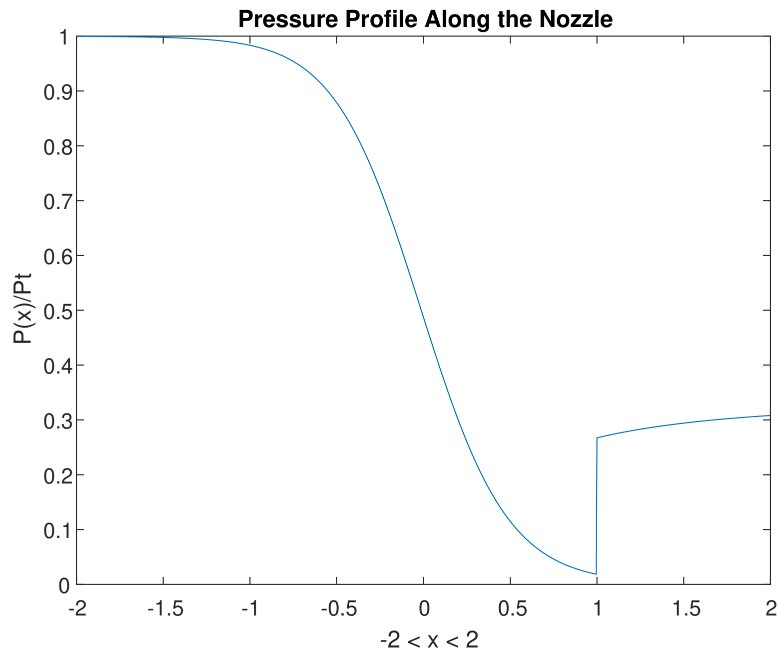
4.1 a)

To find the needed tank pressure in this case: for $M_1 = 3.441$ we then find that $\frac{P_{t2}}{P_{t1}} = 0.3278$. Then,

$$\left(\frac{A_E}{A^*}\right)_2 = \frac{A_E}{A_T}\cdot\frac{P_{t2}}{P_{t1}} = 8.195.$$

$$\therefore (isentropic\ flow) \longrightarrow \frac{P_E}{P_{t2}} = 0.9961$$

$$\frac{P_E}{P_{t1}} = \frac{P_E}{P_{t2}}\cdot\frac{P_{t2}}{P_{t1}} = 0.3265$$



Hence

$$P_{t1} = 3.06278 \cdot P_E.$$

4.2 c)

We can graph the temperature profile using the following isentropic equation:

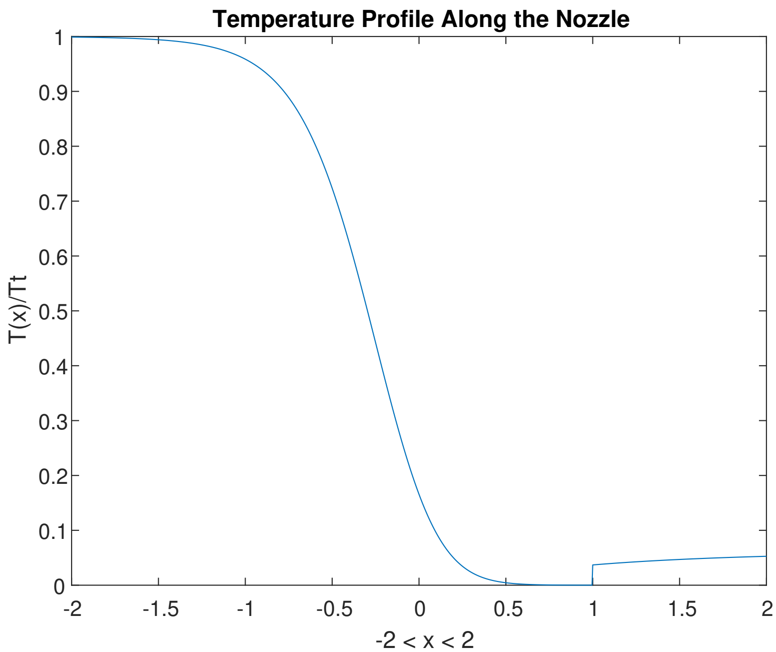
$$\frac{T}{T_t} = \left(\frac{P}{P_t}\right)^{\frac{\gamma-1}{\gamma}}$$

The following graph is then produced:

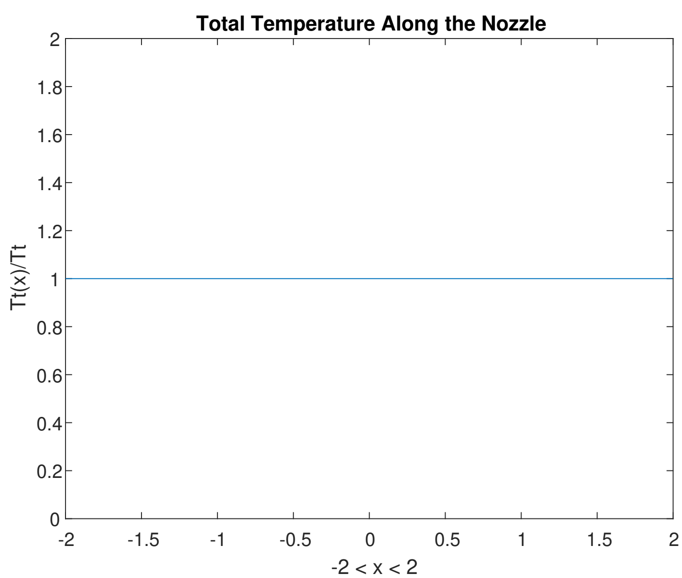
Because a shock wave does no work and because the nozzle does no thermodynamic work, and there is no heat addition, the total enthalpy and the total temperature are constant. We can also show that by:

$$\frac{T_{t2}}{T_{t1}} = \frac{T_{t2}}{T_2} \cdot \frac{T_2}{T_1} \cdot \frac{T_1}{T_{t1}} = 1$$

Hence the following graph is produced:
Note: the run time depends on N, i.e. the number of samples.



Temperature Profile Along the Nozzle
T(x)/Tt
-2 < x < 2



Total Temperature Along the Nozzle
Tt(x)/Tt
-2 < x < 2

5 Solution

We have shown in class that because the first law of thermodynamics requires that $\frac{dT_t}{dt} = 0$, the entropy change is given by:

$$\frac{ds_t}{R} = -\frac{dP_t}{P_t}$$

Therefore

$$\frac{ds_t}{C_p} = -\frac{\gamma - 1}{\gamma} \cdot \frac{dP_t}{P_t} \tag{4}$$

Integration of (4) yields:

$$\frac{\Delta s}{C_p} = \ln\left[M^2 \sqrt{\left(\frac{\gamma + 1}{2M^2\left(1 + \frac{\gamma-1}{2}M^2\right)}\right)^{\frac{\gamma+1}{\gamma}}}\right]$$

And the following equation can be used to plot the Fanno line:

$$H = \frac{h}{h_t} = \frac{C_p T}{C_p T_t} = \frac{T}{T_t}$$

$$\Delta S = \frac{\Delta s}{C_p} = \ln\left[\left(\frac{1}{H} - 1\right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{2}{\gamma - 1}\right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma+1}{2\gamma}} (H)^{\frac{\gamma+1}{2\gamma}}\right]$$