November 2020

Compressible Flow:
Exercise 1 Student Solution

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1 Solution

1.1 a)

Using Knudsen number, which is defined as the ratio of the molecular mean free path length to a representative physical length scale, we get:

$$
\begin{aligned}
K_n &= \frac{\lambda}{l} \\
&= \lambda_0 \frac{\rho_0}{\rho l}\left(\frac{T}{T_0}\right)^{\frac{1}{2}} \\
&\overset{(a)}{=} \frac{\rho_0 \lambda_0 R T^{\frac{3}{2}}}{P l T_0^{\frac{1}{2}}} \\
&\overset{(b)}{=} \frac{\lambda_0 \overbrace{\rho_0 T_0 R}^{P_0} T^{\frac{3}{2}}}{P_0\left(1-\frac{\Gamma h}{T_0}\right)^{\frac{g}{\Gamma R}} l T_0^{\frac{3}{2}}} \\
&\overset{(c)}{=} \frac{\lambda_0\left(1-\frac{\Gamma h}{T_0}\right)^{\frac{3}{2}}}{l\left(1-\frac{\Gamma h}{T_0}\right)^{\frac{g}{\Gamma R}}} \\
&= \frac{\lambda_0}{l} \cdot \left(1-\frac{\Gamma h}{T_0}\right)^{\frac{3}{2}-\frac{g}{\Gamma R}}, \qquad (1)
\end{aligned}
$$

where step (a) follows by $\rho = \frac{P}{RT}$ (ideal gas), step (b) follows by substituting $P = P_0\left(1-\frac{\Gamma h}{T_0}\right)^{\frac{g}{\Gamma R}}$ and by multiplying by $1 = \frac{T_0}{T_0}$, and step (c) follows by substituting $T = T_0 - \Gamma h$.

Substituting the known parameters (i.e.: g, l, λ_0, Γ, R) in (1) yields:

$$
\begin{aligned}
K_n = 10^{-7} \cdot \left(1 - \frac{0.0065h}{T_0}\right)^{\frac{3}{2} - \frac{10}{0.0065 \cdot 290}} &\underbrace{< 0.01}_{Continuum flow} \\
\left(1 - \frac{0.0065h}{T_0}\right)^{-3.805} &< 10^5 \\
-3.805 \ln\left(1 - \frac{0.0065h}{T_0}\right) &> 11.5129 \\
\ln\left(1 - \frac{0.0065h}{T_0}\right) &> -3.0257 \\
1 - \frac{0.0065h}{T_0} &> 0.0485 \\
h &< 146.3846 \cdot T_0.
\end{aligned}
$$

1.2 b)

Given $T = T_0 - \Gamma h$ and $T = 0°K$ we obtain:

$$h = \frac{T_0}{\Gamma},$$

where $\Gamma = 0.0065[\frac{K}{m}]$.

1.3 c)

Continuum is what we refer to when there is a continuous presence of matter. This is important to us because assuming such a state helps us solve problems smoothly. The aforementioned mean free path is basically the mean distance traveled by a molecule between two successive collisions with other molecules. Free molecular flow refers to fluid dynamics of gases where the mean free path of the molecules is larger than the object under test. We say that the system is under free molecular flow if $K_n > 1$ and the continuum assumption does not hold. An intuition for why we can't use the classical ideal gas equation of state when free molecular flow occurs, let's say in outer space, would be to try and speak about the density of a single molecule or its volume.

1.4 d)

We can predict that the mean free path approaches zero as ρ approaches a critical density ρ_c.

2 Solution

In fluid mechanics we learned that the rate at which mass enters a system is equal to the rate at which mass leaves the system plus the accumulation of mass within the system. Assuming a hypothetical source of matter ρS:

$$\int\limits_{c.v.} \frac{\partial \rho}{\partial t} dV + \int\limits_{c.s.} \rho \left(\vec{q} \cdot d\vec{A} \right) = \int\limits_{c.v.} \rho S dV$$
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{q}) = \rho S$$

3 Solution

The thermodynamics properties of compressed gas in relatively low temperatures and high pressure is given by the following equation of state:

$$p = \frac{\rho R T}{1 - \beta \rho} - \alpha \rho^2. \tag{2}$$

We have shown in class that the speed of sound, in a compressible interior, of a small disturbance in an isentropic process is given by:

$$a^2 = \left. \frac{\partial P}{\partial \rho} \right|_s. \tag{3}$$

Therefore,

$$\begin{aligned} a^2 &\overset{(a)}{=} \frac{RT}{(1 - \beta\rho)^2} - 2\alpha\rho \\ &\overset{(b)}{=} \gamma \left[\frac{RT}{(1 - \beta\rho)^2} - 2\alpha\rho \right], \end{aligned} \tag{4}$$

where step (a) follows by taking the derivative of (2) with respect to ρ, and step (b) follows by the equality: $\left. \frac{\partial P}{\partial \rho} \right|_s = \gamma \left. \frac{\partial P}{\partial \rho} \right|_T$.
Substituting the known parameters (i.e.: p, R, T, α, β, γ) into (2) and solve for ρ we get:

$$\rho = 0.0726 \left[\frac{slug}{ft^3} \right].$$

Now we have all the parameters needed to solve for the speed of sound. Taking the square root of (4) we get:

$$a = 877.2 \left[\frac{ft}{sec} \right].$$

4 Solution

The stratosphere is considered to be a safe-zone for planes because of its stability. It lies just above the turbulent troposphere. Assuming the temperature in the stratosphere is around $-70°$F, the stagnation temperature T_t of each aircraft will be approximately:

$$\begin{aligned}
T_{t_{concord}} &= T_{stratosphere}\left(1+\frac{\gamma_{air}-1}{2}M_{concord}^2\right)\\
&= 216°K\left(1+\frac{1.4-1}{2}2.2^2\right)\\
&= 425°K,\\
T_{t_{SST}} &= T_{stratosphere}\left(1+\frac{\gamma_{air}-1}{2}M_{SST}^2\right)\\
&= 216°K\left(1+\frac{1.4-1}{2}2.7^2\right)\\
&= 531°K.
\end{aligned}$$

5 Solution

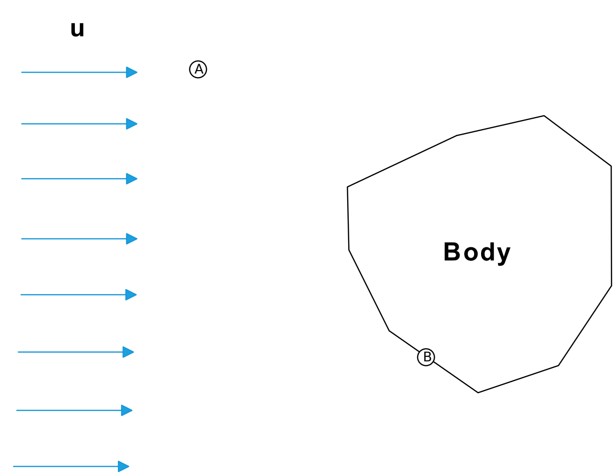

Figure 1: Body exposed to uniform isentropic flow of air

From the classical ideal gas equation of state:

$$\begin{aligned}
P_A &= \rho_A R_{air} T_A\\
T_A &= \frac{P_A}{\rho_A R_{air}}\\
&= \frac{14.7\cdot 144}{0.00237\cdot 1717}\\
&= 520.19°R.
\end{aligned}$$

In accord to (5):

$$a = \sqrt{1.4 \cdot 1717 \cdot 520.19}$$
$$= 1118.2[\frac{ft}{sec}].$$

On the other hand,

$$M \triangleq \frac{u}{a}$$
$$M_A = \frac{u_A}{a_A}$$
$$\overset{(a)}{=} \frac{1148.3}{1118.2}$$
$$= 1.027,$$

where (a) follows by the unit conversion $350[\frac{m}{sec}] = 1148.3[\frac{ft}{sec}]$.

According to Table A.1 (Isentropic flow properties of air) we find that:

$$\frac{P_t}{P_A}(M_A = 1.027) = 7.824,$$
$$P_t = 7.824 \cdot 14.7 \cdot 144$$
$$= 16,561.84[\frac{lbf}{ft^2}]$$

Therefore,

$$\frac{P_t}{P_B} = \frac{16,561.84}{1058.4}$$
$$= 15.65$$

We'll use Table A.1 once again to find:

$$M_B = 2.45$$

6 Solution

6.1 a)

According to Table A.1 (Isentropic flow properties of air) we find that:

$$\frac{P_t}{P}(M = 2) = 7.824,$$

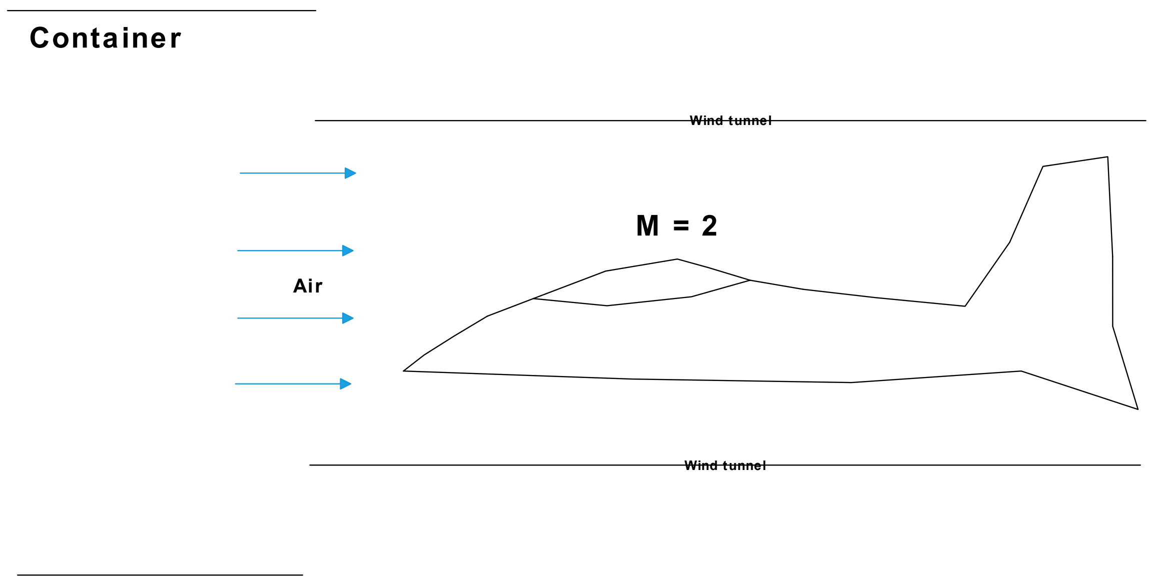

Figure 2: Wind tunnel with the scale model of an aircraft where $T_t = 70°F$, $P_{tunnel} = 14.7psi$

where $P = P_{tunnel} = 14.7 \cdot 144 = 2116.8 \left[\frac{lbf}{ft^2}\right]$. Therefore:

$$P_t = 16,561.84 \left[\frac{lbf}{ft^2}\right].$$

6.2 b)

We have shown in class that the speed of sound, in a compressible interior, of a small disturbance in an isentropic process is given:

$$a = \sqrt{\gamma RT}. \tag{5}$$

We have also shown that:

$$\frac{T}{T_t} = \left(\frac{P}{P_t}\right)^{\frac{\gamma-1}{\gamma}}.$$

Hence,

$$\begin{aligned} T_{tunnel} &= T_t \cdot \left(\frac{P_{tunnel}}{P_t}\right)^{\frac{\gamma-1}{\gamma}} \\ &= 529.2°R \cdot \left(\frac{2116.8}{16,561.84}\right)^{\frac{1.4-1}{1.4}} \\ &= 294°R. \end{aligned}$$

Substituting the parameters into (5):

$$a = \sqrt{1.4 \cdot 1717 \cdot 294} = 840.6 \left[\frac{ft}{sec}\right].$$

Finally,

$$\begin{aligned} M &\triangleq \frac{u}{a} \\ u &= Ma \\ &= 2 \cdot 840.6 \\ &= 1681.2 \left[\frac{ft}{sec}\right]. \end{aligned}$$

6.3 c)

$$\begin{aligned} \dot{m}_{tunnel} &= \rho_{tunnel} \cdot u \cdot A_{tunnel} \\ &= \frac{P_{tunnel}}{RT_{tunnel}} \cdot u \cdot A_{tunnel} \\ &= \frac{2116.8}{1717 \cdot 294} \cdot 1681.2 \cdot 2 \\ &= 14.1 \left[\frac{slug}{sec}\right] \end{aligned}$$

7 Solution

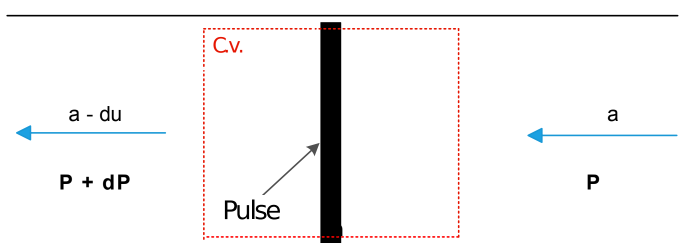

Figure 3: Galilean transformation to a system which moves with the disturbance (pulse)

7.1 a)

From conservation of mass:

$$\int_{c.s.} \rho \vec{q} \cdot d\vec{A} = 0$$

where A is the cross sectional area in 3.

$$\rho a \cancel{A} = (\rho + d\rho)(a - du)\cancel{A} \tag{6}$$

From conservation of momentum:

$$\begin{aligned}
&\int_{c.s.} \rho \vec{q} \cdot \left(\vec{q} \cdot d\vec{A}\right) = \Sigma \vec{F} = -\int_{c.s.} P dA \\
&\rho a (-aA) + \underbrace{(\rho + d\rho)(a - du)}_{\rho a}(a - du) A = PA - (P + dP) A \\
&-\rho a^2 + \rho a (a - du) = -dP \\
&dP = \rho a du \qquad (7)
\end{aligned}$$

From the classical ideal gas equation of state along with (5) and (7):

$$\begin{aligned}
du &= \frac{RT}{Pa} dP \\
&= \frac{RT}{P\sqrt{\gamma RT}} dP \\
&= \frac{\sqrt{\gamma RT}}{P\gamma} dP.
\end{aligned}$$

7.2 b)

$$\begin{aligned}
(P + dP) &= (\rho + d\rho) R (T + dT) \\
\left(\cancel{P} + \rho a du\right) &= \underbrace{\cancel{R\rho T}}_{P} + R\rho dT + Rd\rho T + R\cancel{d\rho dT}^{\,negligible}. \qquad (8)
\end{aligned}$$

Going back to (6):

$$\begin{aligned}
\cancel{\rho a} &= \cancel{\rho a} - \rho du + d\rho a - \cancel{d\rho du}^{\,negligible} \\
d\rho &= \frac{\rho du}{a}. \qquad (9)
\end{aligned}$$

Combining (8) and (9):

$$
\begin{aligned}
\rho a du &= R\rho dT + R\frac{\rho T du}{a} \\
dT &= \frac{1}{R}\left[a - R\frac{T}{a}\right] du \\
&= \frac{1}{R}\left[\sqrt{\gamma RT} - \frac{\sqrt{RT}}{\sqrt{\gamma}}\right] du \\
&= \frac{1}{R}\left[\sqrt{\gamma RT} - \frac{\sqrt{\gamma RT}}{\gamma}\right] du \\
&= \frac{1}{R\gamma}\left[\sqrt{\gamma RT}\,(\gamma - 1)\right] du \\
&= \frac{1}{\sqrt{\gamma RT}}\left[T\,(\gamma - 1)\right] du.
\end{aligned}
$$

8 Solution

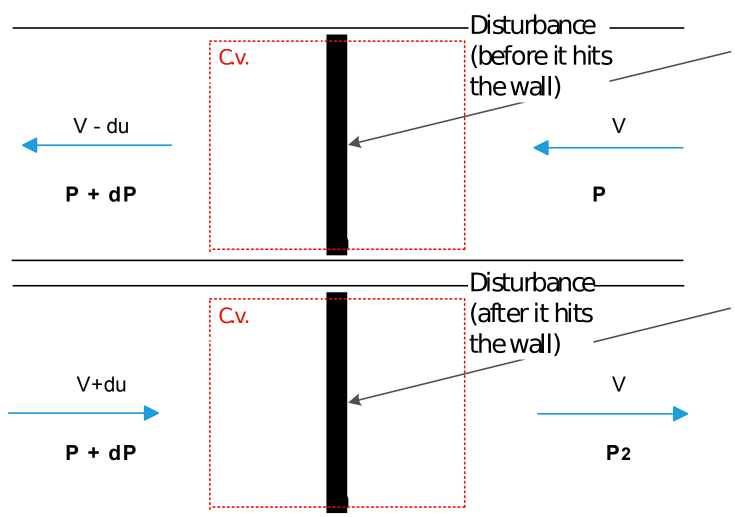

Figure 4: Galilean transformation to a system which moves with the disturbance before (and after) it hits the wall, where $P_2 = P + \Delta P$ and $\rho_2 = \rho + \Delta\rho$

A small pressure disturbance moves through a static fluid with velocity V until it hits a wall. The disturbance is then reflected back off the wall into the fluid with the same velocity V as before, as shown in 4. Before the disturbance hits the wall, we are in the same scenario as described in the previous question where we derived (7) from conservation of mass and momentum. Thus:

$$dP = \rho V du.$$

Conversation of mass after the disturbance hits the wall:

$$\int\limits_{c.s.} \rho\vec{q}\cdot d\vec{A} = 0$$

where A is the cross sectional area in 4.

$$(\rho + d\rho)(V + du)\cancel{A} = \rho_2\ V\cancel{A}$$

Conversation of momentum after the disturbance hits the wall:

$$\int\limits_{c.s.} \rho\vec{q}\cdot\left(\vec{q}\cdot d\vec{A}\right) = \Sigma\vec{F} = -\int\limits_{c.s.} PdA$$

In accord to 4:

$$\begin{aligned}
-(\rho + d\rho)(V + du)^2\cancel{A} + \rho_2 V^2\cancel{A} &= (P + dP)A - P_2\cancel{A}\\
\cancel{P} + \Delta P + (\rho + \Delta\rho)V^2 &= \cancel{P} + dP + \underbrace{(\rho + d\rho)(V + du)^2}_{(\rho+\Delta\rho)V(V+du)}\\
\Delta P &= \underbrace{dP}_{\rho V du} + (\rho + \Delta\rho)\left[V(V + du) - V^2\right]\\
\Delta P &= \rho V du + \rho V du + \cancel{\Delta\rho V du}\ \textit{negligible}\\
\Delta P &= 2\rho V du.\\
\Delta P &= 2dP.
\end{aligned}$$

9 Solution

In class we have shown the connection between the speed of sound and compressibility:

$$a^2 = \frac{\partial P}{\partial\rho}.$$

Assuming an isothermal process we can write:

$$a^2 = \left.\frac{\partial P}{\partial\rho}\right|_T, \tag{10}$$

where $T = T(P,\rho)$. Consequently, the total differential of (10) will be:

$$dT = \frac{dT}{dP}dP + \frac{dT}{d\rho}d\rho.$$

In an isothermal process $dT \equiv 0$. Thus,

$$\frac{dT}{dP}dP + \frac{dT}{d\rho}d\rho = 0$$
$$\frac{dP}{d\rho} = -\frac{\frac{dT}{d\rho}}{\frac{dT}{dP}}. \tag{11}$$

In class we've shown that for an isentropic process $\frac{P}{\rho^\gamma} = f(s) = constant$. Similarly, for an isothermal process $T = \frac{P}{\rho R} = f(T) = constant$.

$$\frac{\partial}{\partial \rho}\left(\frac{P}{\rho R}\right) = \frac{\partial f(T)}{\partial T}\frac{\partial T}{\partial \rho}$$
$$-\frac{P}{\rho^2 R} = f'(T)\frac{\partial T}{\partial \rho}$$
$$\frac{\partial T}{\partial \rho} = -\frac{P}{\rho^2 R}\frac{1}{f'(T)}. \tag{12}$$

$$\frac{\partial}{\partial P}\left(\frac{P}{\rho R}\right) = \frac{\partial f(T)}{\partial T}\frac{\partial T}{\partial P}$$
$$\frac{1}{\rho R} = f'(T)\frac{\partial T}{\partial P}$$
$$\frac{dT}{dP} = \frac{1}{\rho R}\frac{1}{f'(T)}. \tag{13}$$

Substituting (12) and (13) into (11) yields:

$$a^2 = \left.\frac{\partial P}{\partial \rho}\right|_T$$

$$= -\frac{-\frac{P}{\rho^{\cancel{2}}\cancel{R}}\cancel{\frac{1}{f'(T)}}}{\frac{1}{\cancel{\rho R}}\cancel{\frac{1}{f'(T)}}}$$

$$= \frac{P}{\rho}$$
$$= RT.$$

Therefore in an isothermal process:

$$a = \sqrt{RT}.$$

10 Solution

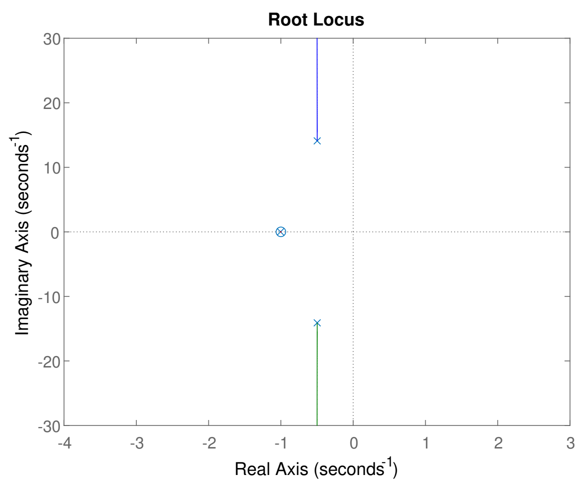

Figure 5: Isentropic flow of ideal gas in an adiabatic container with a convergent nozzle

$$p_t \geqslant \frac{p_o}{\frac{2}{\gamma+1}^{\frac{\gamma}{\gamma-1}}},$$

where p_0 is the external pressure and p_t is the total pressure within the container when the process begins.

We have shown in class that for an isentropic, one dimensional flow of ideal gas:

$$\frac{p_t}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$$

Or,

$$p_t = p\left(\frac{2 + (\gamma - 1)\, M^2}{2}\right)^{\frac{\gamma}{\gamma-1}}$$

For mach number $M = 1$:

$$p_t = \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$= \frac{p}{\frac{2}{\gamma+1}^{\frac{\gamma}{\gamma-1}}}$$

Thus, there is a choked flow at the throat of the convergent nozzle where we define:

$$\dot{m} = \rho^* a^* V A*,$$

where $A*$ is the cross sectional area @ $M = 1$.
On the other hand, we've shown in that:

$$\dot{m} = \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{T_t}} \frac{M}{\left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}} A$$

$$\begin{aligned}
&\overset{(a)}{=} \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{T_t}} \frac{1}{\left(1 + \frac{\gamma-1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}} A^* \\
&= \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{T_t}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} A^*.
\end{aligned} \tag{14}$$

where (a) follows by substituting $M = 1$.
Assuming the volume of the container remains a constant:

$$\begin{aligned}
\dot{m} &= \frac{dm}{dt} \\
&= \frac{d\rho_t}{dt} V \\
&= \frac{d\rho_t}{dp} \frac{dp}{dt} V
\end{aligned}$$

In an isentropic process, the speed of sound is given by (3) and (5) so that:

$$\dot{m} = \frac{1}{\gamma R T_t} \frac{dp}{dt} V. \tag{15}$$

From (14) and (15) we get:

$$\frac{1}{\gamma R T_t} \frac{dp_t}{dt} V = \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{T_t}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} A^*$$

Let us define T_{t_1} to be the temperature inside the container time t and p_{t_1} to be the pressure inside the container at time t and so on. We know how pressure relates to temperature in an isentropic, one dimensional process. It is given by:

$$\frac{T_t}{T_{t_1}} = \left(\frac{p_t}{p_{t_1}}\right)^{\frac{\gamma-1}{\gamma}}.$$

From the two expressions above:

$$\frac{1}{\gamma R\left(\frac{p_t}{p_{t_1}}\right)^{\frac{\gamma-1}{\gamma}} T_{t_1}} \frac{dp_t}{dt} V = \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{\left(\frac{p_t}{p_{t_1}}\right)^{\frac{\gamma-1}{\gamma}} T_{t_1}}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} A^*$$

$$\frac{V}{\gamma R\left(\frac{p_t}{p_{t_1}}\right)^{\frac{\gamma-1}{\gamma}} T_{t_1}} dp_t = \sqrt{\frac{\gamma}{R}} \frac{p_t}{\sqrt{\left(\frac{p_t}{p_{t_1}}\right)^{\frac{\gamma-1}{\gamma}} T_{t_1}}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} A^* dt$$

$$dt = \frac{V}{\gamma^{\frac{3}{2}} A^* \sqrt{RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} p_t^{\frac{1-3\gamma}{2\gamma}} p_{t_1}^{\frac{\gamma-1}{2\gamma}} dp_t.$$

Integrating both sides:

$$\int_0^t dt = \frac{V p_{t_1}^{\frac{\gamma-1}{2\gamma}}}{\gamma^{\frac{3}{2}} A^* \sqrt{RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \int_{p_{t_1}}^{p_t} p_t^{\frac{1-3\gamma}{2\gamma}} dp_t$$

$$t = \frac{V p_{t_1}^{\frac{\gamma-1}{2\gamma}}}{\gamma^{\frac{3}{2}} A^* \sqrt{RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{2\gamma}{1-\gamma} p_t^{\frac{1-\gamma}{2g\gamma}} \Big|_{p_{t_1}}^{p_t}$$

$$= \frac{V p_{t_1}^{\frac{\gamma-1}{2\gamma}}}{\gamma^{\frac{3}{2}} A^* \sqrt{RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{2\gamma}{1-\gamma} \left[p_t^{\frac{1-\gamma}{2\gamma}} - p_{t_1}^{\frac{1-\gamma}{2\gamma}}\right]$$

$$= \frac{V p_{t_1}^{\frac{\gamma-1}{2\gamma}}}{\gamma^{\frac{3}{2}} A^* \sqrt{RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{2\gamma}{1-\gamma} \left[p_t^{\frac{1-\gamma}{2\gamma}} - p_{t_1}^{\frac{1-\gamma}{2g\gamma}}\right]$$

$$= \frac{V}{A^* \sqrt{\gamma RT_{t_1}}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{2}{1-\gamma} \left[\left(\frac{p_{t_1}}{p_t}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right].$$