PRICING AMERICAN OPTIONS USING MONTE CARLO SIMULATION

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Abstract. Option contracts are used by all major financial institutions and investors either to speculate on stock market trends or to control their level of risk from other investments. American options form the majority of those traded today. Yet pricing such options, even in the standard case of a lognormal process for the underlying asset, is still an area of active research.

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1. Introduction

An *options contract* on an *underlying asset*, such as a stock, allows the contract holder the right to purchase or sell the asset within a predetermined time period. The end of this period is known as the *maturity date* and, within this period, the holder of the option can purchase or sell the asset for an agreed amount - the *exercise price*. Options differ from other derivative securities, such as *forward contracts*, in that the holder has the right to exercise the option at some point in the future, but not the obligation. If exercising the option is unfavourable, it does not have to be exercised.

There are two basic types of options available. A *call option* allows the holder the right to purchase an asset at the exercise price by the maturity date. The owner of a *put option* can sell for the agreed amount within the agreed time period. *American options* can be exercised at any time up to the maturity date. *European options* can only be exercised on the maturity date itself.

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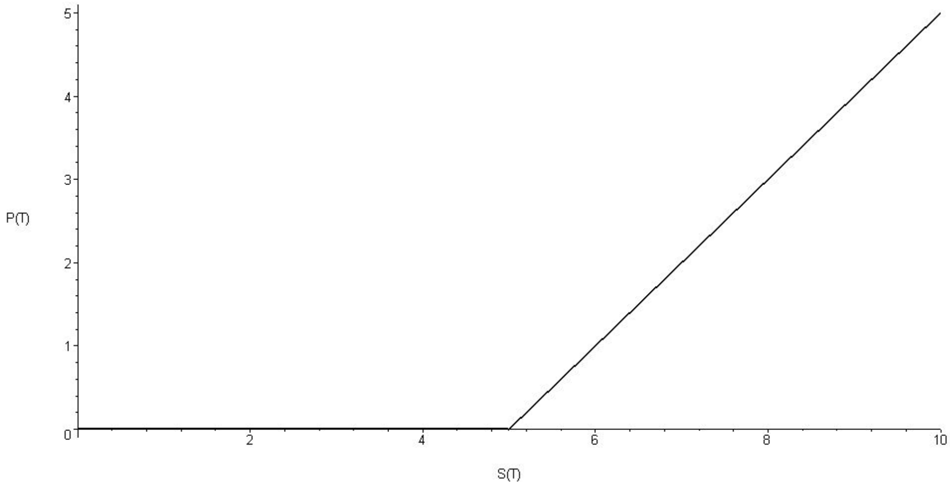

FIGURE 1. Payoff from the European call option.

Example 1. Suppose an investor buys a European call option on a non-dividend paying stock[1] and the exercise price is £5. The maturity date on the option is 1 year later.

From now on, let t represent time in years, with T representing maturity. Let K be the exercise price and S_t the value of the asset at time t.

If, at maturity, the asset price exceeds the exercise price, $S_T > K = £5$, the investor would exercise the option and buy the stock. They can then immediately sell the asset, realising a payoff of

$$P_T = S_T - K > 0$$

If the value at maturity is less than the exercise price, the stock could be bought for less elsewhere, so the option would not be exercised. The total payoff to the investor is

$$P_T = (S_T - K)^+ := \max(S_T - K, 0) := \begin{cases} S_T - K & S_T - K > 0 \\ 0 & \text{otherwise} \end{cases}$$

The *present value* of the payoff corresponds to the value of the payoff at the time the contract is sold. If r denotes the *risk-free interest rate*[2] compounded continuously, the present value of the payoff is

$$e^{-rT} P_T = e^{-rT}(S_T - K)^+$$

Figure 1 shows how the payoff varies with stock price at maturity. In the case of the option being a European put, the present value of the payoff is $e^{-rT}(K - S_T)^+$.

In the above scenario the investor can guarantee not to lose anything from holding the option. In the worst case the option is not exercised and the investor earns (and loses) nothing. If the stock price at maturity exceeds the exercise price, the investor makes a profit despite no initial investment.

[1]In this project, we shall assume all assets encountered do not pay any dividends.

[2]The risk-free rate is usually obtained from the treasury rate of short-dated government bonds in the required currency.

The notion of risk-less profit is known as *arbitrage*. To prevent arbitrage, the investor pays a *premium* to buy the option, regardless of whether it is exercised. The value of the premium is the present value of the expected payoff from the option.[3] The total payoff to the investor is $P_T - h$, where h denotes the option premium.

Example 2. Let $\widetilde{X}_t$ denote the present value of money worth X_t at time t. In the case of a European call option, the premium is

$$h = \mathbb{E}[\widetilde{P}_T] := \mathbb{E}[e^{-rT}(S_T - K)^+]$$

The present value of the expected payoff to the investor is then

$$\mathbb{E}[\widetilde{P}_T - h] = \mathbb{E}[\widetilde{P}_T - \mathbb{E}[\widetilde{P}_T]] = \mathbb{E}[\widetilde{P}_T] - \mathbb{E}[\mathbb{E}[\widetilde{P}_T]] = \mathbb{E}[\widetilde{P}_T] - \mathbb{E}[\widetilde{P}_T] = 0$$

Since there is no expected return (or loss) there is no arbitrage. With a European put option, the premium is $h = \mathbb{E}[e^{-rT}(K - S_T)^+]$, and the same result follows.

The aim of this project is to attempt to find the premium for other options - notably American options - by simulating the price of the asset over a specified time period. Monte Carlo methods can be used to obtain an estimate of the option price.

2. Simulating a stock price path

The project requires simulating asset prices over the time interval $[0, T]$. To do this, we require the following.

Definition 1. Let $\Omega = \{\omega_1, \omega_2, \ldots\}$ denote the *sample space*. The *σ-algebra* $\mathcal{F}$ is the set of all observable events for a single trial, with the following properties.

- $\emptyset, \Omega \in \mathcal{F}$, where $\emptyset$ is the empty set.
- If $A \in \mathcal{F}$, then $\Omega \setminus A \in \mathcal{F}$.
- If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

$\mathbb{P}$ is the *probability measure* on $\mathcal{F}$, where $\mathbb{P}(\omega) \in [0, 1], \forall \omega \in \Omega$ and $\mathbb{P}(\Omega) = 1$. The *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a *filtration* - a collection of σ-algebras $\{\mathcal{F}_t\}_{0 \leq t \leq T}$, where

$$\mathcal{F}_0 \subseteq \mathcal{F}_s \subset \mathcal{F}_t \subseteq \mathcal{F}_T$$

for $0 \leq s < t \leq T$. We assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$.[4]

Definition 2. A *stochastic process* $\{X_t\}_{0 \leq t \leq T}$ is a collection of random variables on $(\Omega, \mathcal{F})$. For each fixed $\omega \in \Omega$, $X_t(\omega)$ is the sample path of X_t associated with ω.

The following gives an example of a stochastic process.

Definition 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A one-dimensional *Brownian motion* is a continuous stochastic process $\{W_t\}_{0 \leq t \leq T}$ such that

[3] At the time the option is exercised. In the case of American options, this is not necessarily maturity time.

[4] The *power set* $\mathcal{P}(\Omega)$ is the set of all possible subsets of Ω. If Ω has n elements, then its power set contains 2^n distinct subsets.

- $W_0 = 0$ a.s.[5]
- For $0 \leq s < t \leq T$, $W_t - W_s$ is independent of $\mathcal{F}_s$ and is normally distributed with mean 0 and variance $t - s$.[6]
- For $0 = t_0 < t_1 < \ldots < t_m = T$, the increments $W_{t_j} - W_{t_{j-1}}$ are independent and their distribution depends only on the difference $t_j - t_{j-1}$. $\{W_t\}$ is said to have *independent and stationary increments.*

Definition 4. The *stochastic integral* is defined by the random variable

$$X_T := \int_0^T g(t)\, dW_t$$

that is

$$dX_T = g(t)\, dW_t$$

Here, dW_t can be thought of as an $N(0, dt)$ random variable.

We assume the stock price S_t follows a *geometric Brownian motion*, given by the stochastic differential equation (SDE)

$$dS_t = rS_t\, dt + \sigma S_t\, dW_t$$

where r denotes the risk-free rate and σ is the *volatility*, such that $\sigma\sqrt{\delta t}$ is the standard deviation of stock price return over a time period of length δt. By using Itô's Lemma (see Hull [2], p226-228), we obtain the equation below. This will be used to describe a simulated stock price path.

$$S_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t} = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z} \tag{2.1}$$

Here, Z is a standard normal random variable.[7] Since W_t is normally distributed, it follows that $\log \frac{S_t}{S_0} \sim N\left(\left(r - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$. Hence, S_t follows a *lognormal distribution.* We have

$$\mathbb{E}[S_t] = S_0 e^{rt} \quad \mathrm{Var}[S_t] = S_0^2\, e^{2rt}\left(e^{\sigma^2 t} - 1\right)$$

Example 3. Figure 2 shows a simulated path that follows equation 2.1. The initial stock price $S_0 = 50$, interest $r = 0.05$, volatility $\sigma = 0.35$. Maturity is $T = 1$ year from now.

3. Using Monte Carlo to price options

We will simulate n stock price paths $S_t(\omega_1), \ldots, S_t(\omega_n)$ over the time period $[0, T]$. We can obtain the payoff $h = h(S_T(\omega_i))$ of a particular option when the stock price is associated with the sample point $\omega_i \in \Omega$. By finding the sample mean of the discounted payoffs we obtain

$$\frac{1}{n}\sum_{i=1}^{n} \widetilde{h}(S_T(\omega_i)) \to \mathbb{E}[\widetilde{h}(S_T) \,|\, S_0] =: h_0$$

[5]A property holds *almost surely* if it holds everywhere except on a set whose probability is zero.

[6]We will write $X \sim N(\mu, \sigma^2)$ to denote the random variable X as normally distributed, with mean μ and variance σ^2.

[7]That is, $Z \sim N(0, 1)$.

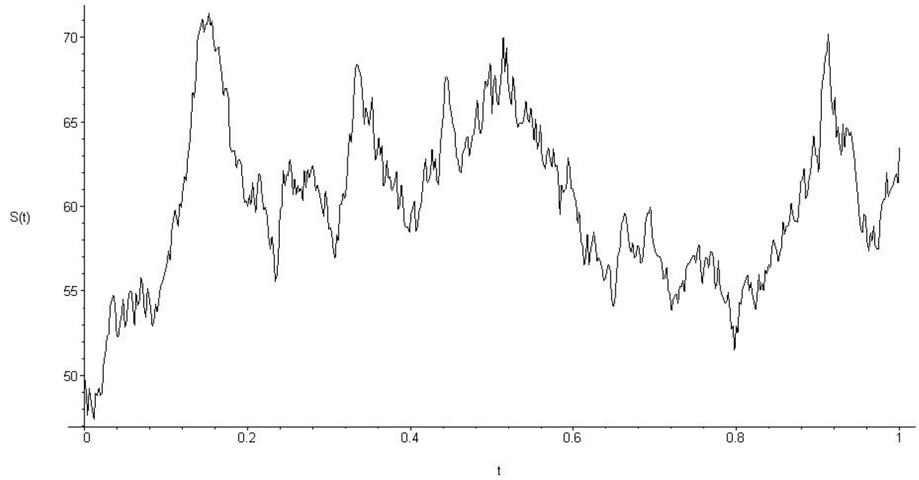

FIGURE 2. Simulated stock price path.

as $n \to \infty$. This allows us to obtain an estimate of the expected payoff at time 0 - an estimate of the option price.

Example 4 (Using Monte Carlo to price path-dependent options)**.** In order to simulate a path we shall consider the price of an asset on a finite set of $m+1$ evenly-spaced dates $0 = t_0, t_1, \dots, t_m = T$, where $t_j = j\delta t = j\frac{T}{m}$ is the time of the j^{th} observation. We shall use the following equation.

$$S_{t_j} = S_{t_{j-1}} e^{\left(r - \frac{\sigma^2}{2}\right)\delta t + \sigma\sqrt{\delta t} Z_j} \qquad Z_j \sim N(0,1) \qquad j = 1, \dots, m \tag{3.1}$$

From now on, r denotes the *risk-neutral* rate of interest.[8] σ represents volatility, as before. We shall assume the options are European in that they can only be exercised at maturity time. The payoffs for each option are given in table 1. $\overline{S}$, S_{max} and S_{min} denote the average, maximum and minimum values of a particular stock price path over the set of times $\{t_0, t_1, \dots, t_m\}$, respectively. By letting $m \to \infty$ (thus, letting $\delta t \to 0$), the path is observed more frequently, leading to the appropriate values over the continuous interval $[0, T]$.

Option type	Payoff
Asian call	$(\overline{S} - K)^+$
Asian put	$(K - \overline{S})^+$
Lookback call	$(S_{max} - K)^+$
Lookback put	$(K - S_{min})^+$
Floating lookback call	$(S_T - S_{min})^+$
Floating lookback put	$(S_{max} - S_T)^+$

TABLE 1. Payoffs from path-dependent options.

[8]In the risk-neutral world, investors are indifferent to risk. They will not pay to avoid risk and will not actively take any risks. The expected return on any investment is the risk-free rate

(1) For $i = 1, \ldots, n$
   (a) For $j = 1, \ldots, m$
      (i) Obtain the stock price at time t_j by using equation 3.1.
      (ii) Update the minimum and maximum values obtained so far if necessary.
   (b) We've reached maturity time for this particular path - obtain the value for $S_T = S_{t_m}(\omega_i)$ and the average value $\overline{S} = \frac{1}{m+1}\sum_{j=0}^{m} S_{t_j}(\omega_i)$.
   (c) Let $\widetilde{h}_i$ be the present value of the payoff from the selected option when path i is followed.
(2) Obtain the sample mean $\overline{h} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{h}_i$.
(3) Obtain the sample variance $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\widetilde{h}_i - \overline{h})^2$.
(4) Return $\overline{h}$ as the estimate of the option price, along with a 95% confidence interval $\overline{h} \pm 1.96\frac{s}{\sqrt{n}}$.

FIGURE 3. Algorithm for pricing path-dependent options.

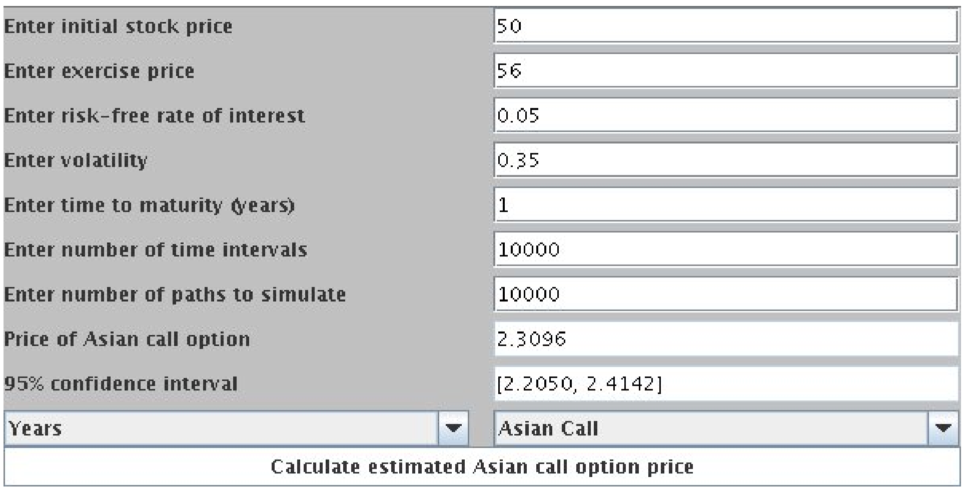

FIGURE 4. Estimating the price of an Asian call option.

We will use Java to simulate n stock price paths, each over $\{t_0, t_1, \ldots, t_m\}$, where m denotes the number of time intervals in $(0, T]$. Figure 3 provides a 'pseudo-code' representation of the algorithm used to price path-dependent options.

Figure 4 shows an estimate of the price of an Asian call option when $S_0 = 50$, $K = 56$, $r = 0.05$, $\sigma = 0.35$ and $T = 1$.[9] The time between observations is $\delta t = \frac{T}{m} = \frac{1}{10000} = 0.0001$ years, making $\{t_0, t_1, \ldots, t_m\}$ almost continuous. From figure 4 the option price is estimated as 2.310. Increasing the number of simulated paths should provide a smaller confidence interval.

[9]It is possible to specify time in days or months, as well as years. The program will convert time to the appropriate value for T in years. However, we assume that there are 252 trading days in a year and not 365.

4. Pricing American options

American options differ from European options in that they can be exercised at any time over $[0, T]$, not just at maturity. While the price of a European option depends only on the expected payoff at maturity, pricing an American option depends on expected payoff over the entire interval.

In this project, we consider $m+1 < \infty$ equally-spaced dates $t_0, t_1, \ldots, t_m$, $t_0 = 0$, $t_m = T$, where exercise is possible. Increasing m makes exercise possible at almost any point over $[0, T]$.[10]

We assume that the current asset price S_t is $\mathcal{F}_t$-measurable. That is, it is formed on the basis of prices over the period $[0, t]$. This follows since investors know present and previous values, but not future ones. S_t, and functions dependent on it, are said to be *adapted* to $\{\mathcal{F}_t\}_{0 \le t \le T}$.

Some useful properties, related to the *conditional expectation* of such adapted sequences - with their proofs - are discussed in Bingham and Kiesel [4], p51-53. A particularly important property is given below.

Definition 5. Let Y_t be adapted to $\{\mathcal{F}_t\}$.

- Y_t is said to be a *martingale* if $\forall s \in [t, T]$, $\mathbb{E}[Y_s \,|\, \mathcal{F}_t] = Y_t$.
- Y_t is a *supermartingale* if $\forall s \in [t, T]$, $\mathbb{E}[Y_s \,|\, \mathcal{F}_t] \le Y_t$.
- Y_t is a *submartingale* if $\forall s \in [t, T]$, $\mathbb{E}[Y_s \,|\, \mathcal{F}_t] \ge Y_t$.

Let $Z_t = Z(S_t)$ denote the payoff from exercising the American option at time t (if it hasn't been exercised already). In the case of a put option, $Z_t = (K - S_t)^+$. Let $h_t = h(S_t)$ be the price of the option at time t. We shall first consider the price at maturity $t_m = T$. Since this is the final opportunity to exercise, the payoff at this step is

$$h_T = Z_T$$

At the previous time step t_{m-1}, the option holder can either exercise at this point or wait until t_m, the value being the expected value, at time t_{m-1}, of the payoff at time t_m. Since the option holder wishes to maximise their payoff, the option price at time t_{m-1} is

$$h_{t_{m-1}} = \max(Z_{t_{m-1}}, e^{-r\delta t}\mathbb{E}[h_{t_m} \,|\, \mathcal{F}_{t_{m-1}}])$$

where $\delta t = \frac{T}{m} = t_j - t_{j-1}$ is the length of time from one step to the next.[11] Using backward induction (see Lamberton and Lapeyre [3], p11) and taking the value at time 0 gives the following dynamic programming (DP) recurrence.

$$\begin{aligned} \widetilde{h}_T &= \widetilde{Z}_T \\ \widetilde{h}_{t_{j-1}} &= \max(\widetilde{Z}_{t_{j-1}}, \mathbb{E}[\widetilde{h}_{t_j} \,|\, \mathcal{F}_{t_{j-1}}]) \quad j = 1, \ldots, m \end{aligned} \tag{4.1}$$

The problem is solved by finding $h_0 = \widetilde{h}_0$. From proposition 1.3.6 of Lamberton and Lapeyre [3], we find that $\{\widetilde{h}_{t_j}\}_{j=0}^m$ is a supermartingale.[12] As a result

[10] Such options, where exercise is only possible on discretely spaced dates are known as *Bermudan options*. By increasing m these options become American in style.

[11] Since all time steps are evenly-spaced.

[12] It is the smallest supermartingale that dominates the sequence $\{\widetilde{Z}_{t_j}\}$ of discounted early exercise prices. It is therefore said to be the *Snell envelope* of $\{\widetilde{Z}_{t_j}\}$.

$$\mathbb{E}[\widetilde{h}_t] = \mathbb{E}[\widetilde{h}_t \,|\, \mathcal{F}_0] \leq h_0 \qquad \forall t \in \{t_0, t_1, \ldots, t_m\}$$

The next objective to obtain a martingale from the sequence $\{\widetilde{h}_{t_j}\}$, in order to price American options.

5. Using stopping times to obtain a martingale

Definition 6. A *stopping time* ν is a random variable taking values in $\{t_0, t_1, \ldots, t_m\}$ such that

$$\{\nu \leq t_j\} := \{\omega \,|\, \nu(\omega) \leq t_j\} \in \mathcal{F}_{t_j} \qquad j = 0, 1, \ldots, m$$

In the case of American options, ν can be thought of as the time where exercise takes place. $\mathcal{T}_{0,T}$ is the set of all stopping times over the interval $[0, T]$. If $\nu = t_k$, we have

$$\widetilde{h}^{\nu}_{t_j} := \begin{cases} \widetilde{h}_{t_j} & 0 \leq j < k \\ \widetilde{h}_{t_k} & k \leq j \leq T \end{cases}$$

where $\widetilde{h}^{\nu}_{t_j}$ is the present value of the option price at time t_j when exercise takes place at time ν. Let $\tau = \inf\{\nu \in \mathcal{T}_{0,T} \,|\, \widetilde{h}_\nu = \widetilde{Z}_\nu\}$ denote the *smallest optimal stopping time* - the earliest time over $\{t_0, t_1, \ldots, t_m\}$ where exercise is preferable. From proposition 2.2.1 of Lamberton and Lapeyre [3], $\{\widetilde{h}^{\tau}_{t_j}\}_{j=0}^{m}$ is a martingale. Hence

$$h_0 = \mathbb{E}[\widetilde{h}^{\tau}_T] = \mathbb{E}[\widetilde{h}_\tau] = \mathbb{E}[\widetilde{Z}_\tau] = \sup_{\nu \in \mathcal{T}_{0,T}} \mathbb{E}[\widetilde{Z}_\nu]$$

The objective is then to find the smallest optimal stopping time, which we can use to find the present value of the option at time τ and, thus, the American option price.

6. Finding the optimal stopping time

The smallest optimal stopping time for a path $S_t(\omega_i)$ corresponds to the point where the option should be exercised. It is the earliest point where payoff from exercising the option exceeds the value of holding until a later date. In order to determine the best time to exercise when given $S_t(\omega_i)$, an *optimal stopping rule* needs to be found. Chapter 8 of Glasserman [1] suggests forming an *optimal exercise boundary* b^*_t. For each simulated path, τ is the time when S_t crosses b^*_t. That is

$$\tau = \begin{cases} \inf\{\nu \in \mathcal{T}_{0,T} \,|\, S_\nu \geq b^*_\nu\} & \text{for an American call option} \\ \inf\{\nu \in \mathcal{T}_{0,T} \,|\, S_\nu \leq b^*_\nu\} & \text{for an American put option} \end{cases}$$

Example 5. Figure 5 shows a possible exercise boundary for an American put option. The stock price is simulated up until the time t where $S_t \leq b_t$.

As before, we consider exercise on a finite set of evenly-spaced points $\{t_0, t_1, \ldots, t_m\}$. The stock price is treated as a *Markov process*, where

$$\mathbb{E}[S_q \,|\, \mathcal{F}_t] = \mathbb{E}[S_q \,|\, S_t] \qquad \forall q \in [t, T]$$

That is, the expected stock price in the future depends only on the current stock price, not on the history over $[0, t)$.

We consider three approaches to pricing American options.

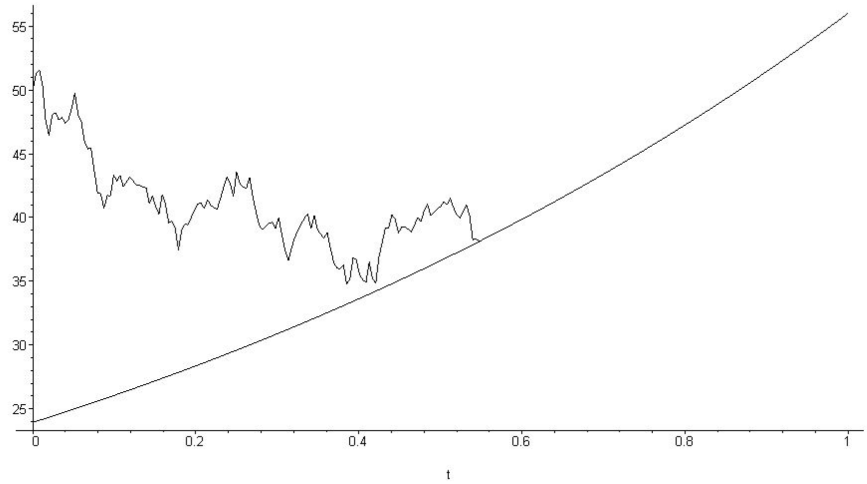

FIGURE 5. Exercise boundary for an American put option.

6.1. **Using the binomial tree model.** The Cox-Ross-Rubinstein method is a discrete-time approach to pricing options. In the case European options, it is a discrete time version of the Black-Scholes model. At each time step t_j, assume that the current asset price can go up by an amount $u = e^{\sigma\sqrt{\delta t}}$ or down by $d = e^{-\sigma\sqrt{\delta t}} = \frac{1}{u}$. The probability of an upward movement is

$$p := \mathbb{P}[S_{t_{j+1}} = S_{t_j} u \mid S_{t_j}] = \frac{e^{r\delta t} - d}{u - d} \tag{6.1}$$

with the probability of downward movement being $1-p$. At each time step, the stock price takes the value

$$S_{t_j} = S_0 u^i d^{j-i} \qquad i = 0, 1, \ldots, j$$

where i is the number of upward movements. Firstly, consider the value at time $t_m = T$. From equation 4.1

$$\widetilde{h}_T = \widetilde{Z}(S_0 u^i d^{m-i}) = \begin{cases} (S_0 u^i d^{m-i} - K)^+ & \text{for a call option} \\ (K - S_0 u^i d^{m-i})^+ & \text{for a put option} \end{cases}$$

We take T to be a stopping time, regardless of the stock price. This means the option ends by this time, if not before.[13] At time $t_j < T$, the expected value of the option at time t_{j+1} is

$$\mathbb{E}[h_{t_{j+1}} \mid S_{t_j} = S_0 u^i d^{j-i}] = p\, Z(S_0 u^{i+1} d^{j-i}) + (1-p)\, Z(S_0 u^i d^{j-i+1})$$

If $Z_{t_j} \geq e^{-r\delta t}\mathbb{E}[h_{t_{j+1}} \mid S_{t_j} = S_0 u^i d^{j-i}]$, then t_j is an optimal stopping time. Using equation 4.1, we have

$$\widetilde{h}(S_0 u^i d^{m-i}) = \widetilde{Z}(S_0 u^i d^{m-i})$$
$$\widetilde{h}_{t_j} = \max(\widetilde{Z}(S_0 u^i d^{j-i}), p\, \widetilde{Z}(S_0 u^{i+1} d^{j-i}) + (1-p)\, \widetilde{Z}(S_0 u^i d^{j-i+1}))$$
$$i = 0, 1, \ldots, j \quad j = 0, 1, \ldots, m$$

Using backwards induction finds the option price from this model. The points at which exercise is optimal can be used to form the exercise boundary

[13] We cannot have $\tau > T$, as the option expires after this time.

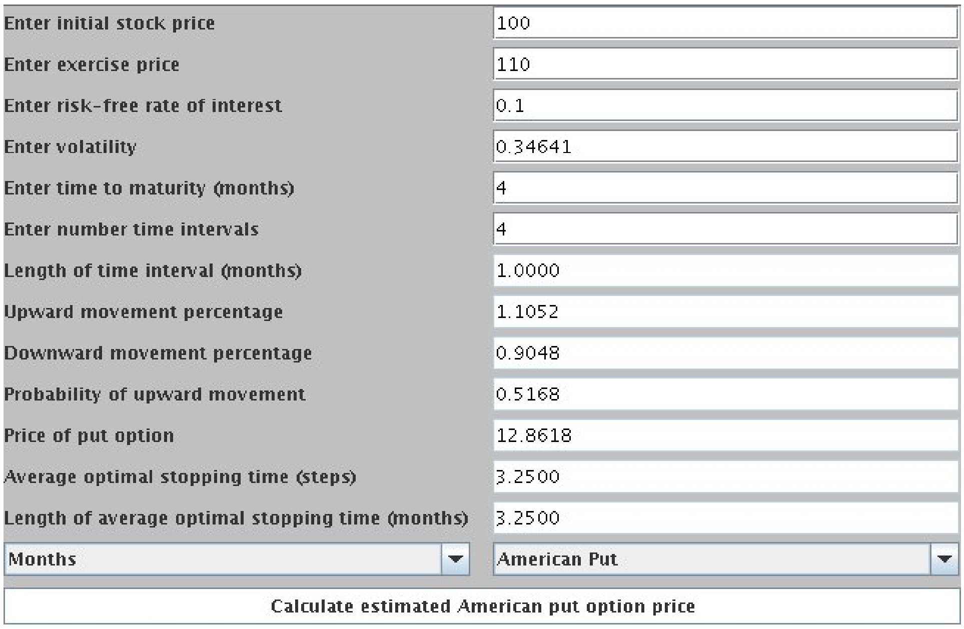

FIGURE 6. Using the binomial model to price an American put option.

b. Since the number of paths is finite, we can find both the price of the American option and the average optimal stopping time.

Example 6. Figure 6 gives an example of using this model to price an American put option. Here, $S_0 = 100$, $K = 110$, $r = 0.1$ and $\sigma = 0.34641$. The length of the option is $T = \frac{1}{3}$ years[14] and there are $m = 4$ intervals over $(0, T]$, each one month in length. The price of such an option is 12.862 and the average optimal stopping time is 3.25 months after the option is sold.

However, a problem with pricing options in this way follows from this being a discrete model. At each time step, it only considers a fixed set of possible stock prices. For example, at the first time step t_1, it will only consider the stock price as S_0u or S_0d, even though it could be between (or above or below) these two values. In theory, this can be solved by considering a larger value of m. However, the model requires all possible values at each time step $t_0, t_1, \ldots, t_m$.[15] Using $m + 1$ time steps requires storing $\frac{1}{2}(m + 1)(m + 2)$ values in memory and means that 2^m possible paths must be considered. Computing the average optimal stopping time in this way becomes a problem as m becomes very large.

[14] In this example, the time to maturity is specified as $l = 4$ months instead of $l = \frac{1}{3}$ years. The appropriate value for T is obtained by taking $T = \frac{l}{12}$. In the case of specifying the option length in days, we would take $T = \frac{l}{252}$.

[15] This is a result of trying to compute the option price using optimal stopping times. If we didn't need to know the average optimal stopping time, we would be able to compute the option price 'dynamically'. This means we would be able to take a much larger value of m.

(1) For $i = 1, \ldots, n$
  (a) Set $j = 0$, where $t = t_j$ is the current time step.
  (b) While $j \in \{0, 1, \ldots, m\}$ and an optimal stopping time for this path hasn't been found
    (i) Compute d_1 and d_2 using equation 6.3, given the parameters above, the current stock price $S_t = S_{t_j}$ and the remaining time until maturity $T - t = T - t_j$.
    (ii) Compute $\mathbb{E}[S_T \,|\, S_t]$ using equation 6.2.
    (iii) Compute the payoff, Z_t from exercise at time t.
    (iv) If $Z_t \geq \mathbb{E}[S_T \,|\, S_t]$, the boundary has been crossed. Let $\tau_i = t_j$ be the optimal stopping time for the path $S_t(\omega_i)$.
    (v) Otherwise, set $j = j + 1$, obtain the stock price at the next time step using equation 3.1 and continue.
  (c) If an optimal stopping time wasn't found for this path, $\tau_i = T$.
  (d) Set $\widetilde{h}_i = \widetilde{Z}_{\tau_i}$ to be the present value of payoff from exercise at time τ_i when path $S_t(\omega_i)$ is followed.
(2) Obtain the average optimal stopping time $\overline{\tau} = \frac{1}{n} \sum_{i=1}^{n} \tau_i$.
(3) Obtain the variance $v^2 = \frac{1}{n-1} \sum_{i=1}^{n} (\tau_i - \overline{\tau})^2$ for optimal stopping time.
(4) Obtain the sample mean $\overline{h} = \frac{1}{n} \sum_{i=1}^{n} \widetilde{h}_i$.
(5) Obtain the sample variance $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (\widetilde{h}_i - \overline{h})^2$.
(6) Return $\overline{\tau}$ as the estimate of stopping time, $\overline{h}$ as the estimate of the option price, along with a 95% confidence intervals $\overline{\tau} \pm 1.96 \frac{v}{\sqrt{n}}$, $\overline{h} \pm 1.96 \frac{s}{\sqrt{n}}$ for stopping time and option price, respectively.

FIGURE 7. Algorithm using the Black-Scholes model as an exercise boudnary.

6.2. **Making use of the Black-Scholes model.** The *Black-Scholes formula* for calculating the price of a European option at time t is

$$\mathbb{E}[S_T \,|\, S_t] = \begin{cases} S_t N(d_1) - K e^{-r(T-t)} N(d_2) & \text{for a call option} \\ K e^{-r(T-t)} N(-d_2) - S_t N(-d_1) & \text{for a put option} \end{cases} \tag{6.2}$$

where

$$d_1 = \frac{\log(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}} \qquad d_2 = d_1 - \sigma \sqrt{T - t} \tag{6.3}$$

and $N(x) = \mathbb{P}[Z < x]$ is the cumulative distribution function of an $N(0, 1)$ random variable. The Black-Scholes value of an option at time t will be used as a second example of an exercise boundary.

For each simulated path, the optimal stopping time is the first point $\tau \in \{t_0, t_1, \ldots, t_m\}$ where the value of the payoff at time t_j exceeds the Black-Scholes price, at that time, of an asset worth S_{t_j}, when K, r, σ and T are the exercise price, risk-neutral rate of interest, volatility and time of maturity, respectively.

Figure 7 shows the algorithm used to price American options in this way, when given S_0, r, σ, T, the number of intervals, m, over $(0, T]$ and the number of paths n.

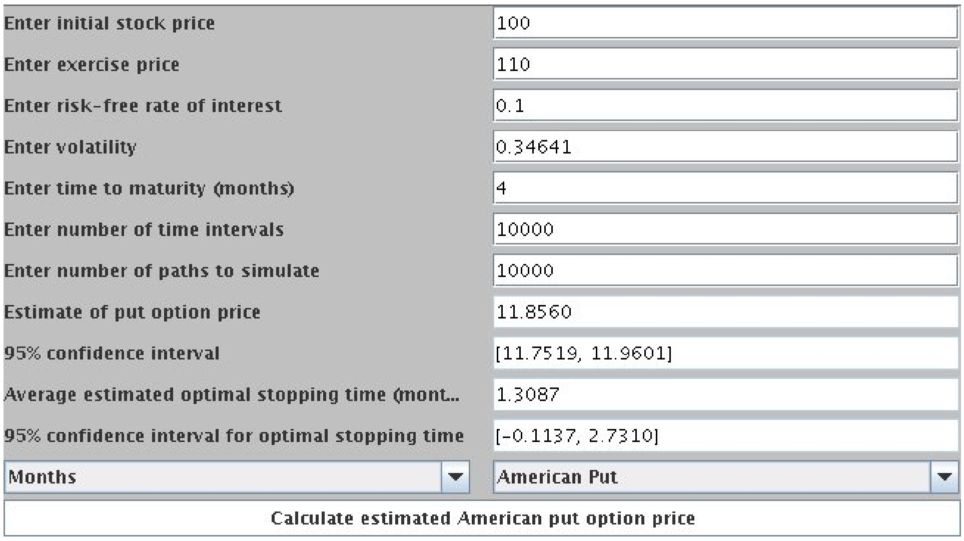

FIGURE 8. Using the Black-Scholes model to estimate the price of an American put option.

Example 7. Figure 8 gives an example of the model being used to price an American put option using the same parameters as in example 6. We use an interval length of $\delta t = 0.0004$ months, giving an almost continuous model, as opposed to the model we can use in section 6.1.

Notice that the option price from example 6 isn't included in the confidence interval in example 7. One problem from using this model is that it will usually underestimate the actual option price. The example below demonstrates why this is the case.

Example 8. Consider the subset of the binomial model in figure 9, where exercise is optimal at points 1, 2, 3 and 4. At points 1 and 2, we have

$$\widetilde{h}_{t_j} = \widetilde{Z}_{t_j} \geq \mathbb{E}[\widetilde{h}_{t_{j+1}} \mid S_{t_j}]$$

However. the Black-Scholes model assumes

$$\widetilde{h}_{t_j} = \mathbb{E}[\widetilde{h}_{t_{j+1}} \mid S_{t_j}]$$

Hence, the model underestimates the option price at time t_j when the stock price is at point 1 or 2. As a result, it underestimates the option price at time t_{j-1} when the stock price is at point 3.[16] It will continue backwards in this way. Since the price at time zero depends on paths that go through the numbered points (see equation 4.1), the estimate of the option price will be less than the actual value h_0.

6.3. **Using simulation to obtain an exercise boundary.** The final approach we will use to price American options is to obtain an estimate of the optimal exercise boundary using a sample of n_1 simulated paths. In this

[16]It will also underestimate at the unnumbered node above point 3.

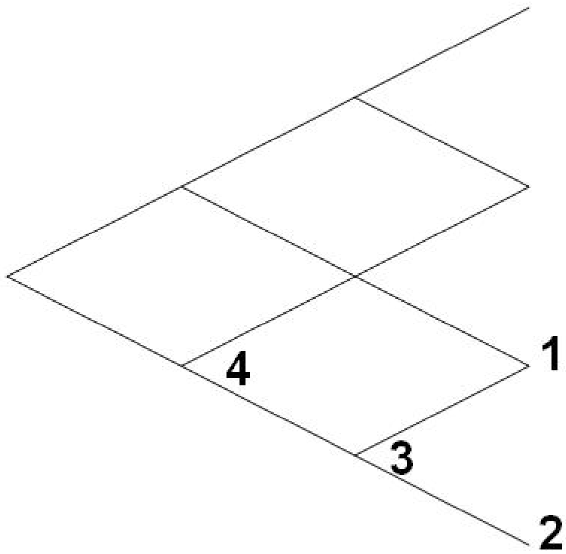

FIGURE 9. Binomial model showing a four early exercise points. Using the Black-Scholes model will underestimate the American option price.

section, we will only consider American put options.[17] Increasing n_1 should result in a more accurate estimate of b^*.

This approach uses the methods described in Glasserman [1], section 8.2. The first step is simulate n_1 stock price paths $S_t(\omega_1), \ldots, S_t(\omega_{n_1})$, $t \in \{t_0, t_1, \ldots, t_m\}$, using equation 3.1. Let θ_j denote the value of the exercise boundary at time t_j. Since we only exercise the option at maturity (if we haven't already done so) when the asset price is below the exercise price, we set $\theta_m = K$. The average present value of the payoff from exercising all options at this time is

$$\frac{1}{n_1} \sum_{i=1}^{n_1} \widetilde{Z}(S_T(\omega_i)) = \frac{e^{-rT}}{n_1} \sum_{i=1}^{n_1} (K - S_T(\omega_i))^+$$

We now consider the value of the boundary at time $t_j < T$. Since the present value of an option exercised at this point is greater than the present value of an option exercised at a later date when the stock price doesn't change[18], we have $\theta_j \leq \theta_{j+1}$, $\forall j = 0, 1, \ldots, m-1$. Therefore, we try $\theta_j = \theta_{j+1}$ as the first potential value for the exercise boundary at time t_j. For every simulated path where $S_{t_j} \leq \theta_j$, we exercise the option at this time. For all other paths, we exercise at the optimal stopping time, $\tau_j \in \mathcal{T}_{t_{j+1},T}$, over $\{t_{j+1}, \ldots, t_m\}$. The present value of the payoff from exercising the option

[17]Although it is possible to price them using sections 6.1 and 6.2 above, it is never optimal to exercise an American call option before maturity time. One can use the methods of section 6.1 to check this.

[18]Unless of course we have no interest $r = 0$. We always assume $r > 0$, however. Hence, if the stock price remains constant over $[t, t']$, we have $e^{q-t} S_q = e^{q-t} S_t < S_t, \forall q \in (t, t']$.

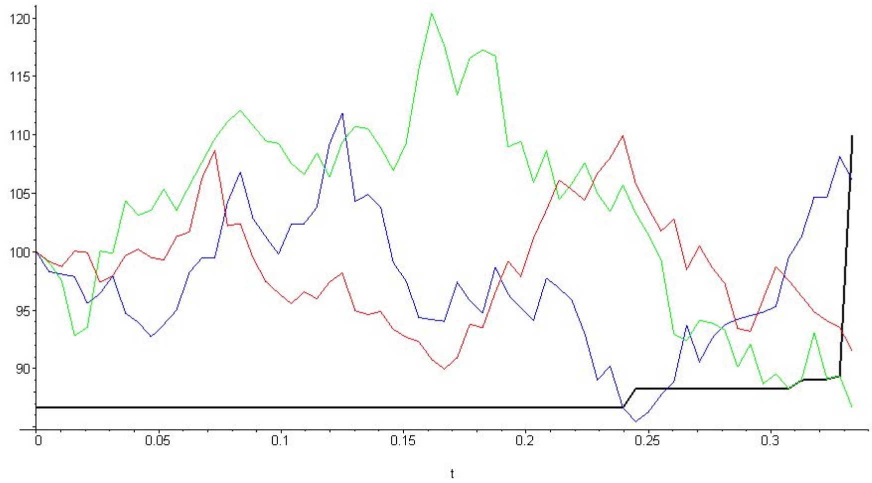

FIGURE 10. Using three paths to estimate b^*.

when given $S_t(\omega_i)$ is

$$\widetilde{Z}(S_{t_j}(\omega_i)) = \begin{cases} (K - S_{t_j})^+ & S_{t_j} \leq \theta_j \\ (K - S_{\tau_j})^+ & \text{otherwise} \end{cases} \tag{6.4}$$

We repeat this process using all stock prices $S_{t_j}(\omega_i) < \theta_{j+1}$ as potential values for θ_j, taking the θ_j that maximises

$$\frac{1}{n_1}\sum_{i=1}^{n_1} \widetilde{Z}(S_{t_j})$$

as the value of the exercise boundary at time t_j. This results in the following DP recurrence.

$$\begin{aligned} &\theta_m = K \\ &\theta_j = \max_{\theta_{j+1}, S_{t_j} < \theta_{j+1}} \{\tfrac{1}{n_1}\textstyle\sum_{i=1}^{n_1} \widetilde{Z}(S_{t_j})\} \quad j = 0, 1, \ldots, m-1 \end{aligned} \tag{6.5}$$

Finding $\theta_0, \theta_1, \ldots, \theta_m$ using equation 6.5 provides the estimate of the optimal exercise boundary b^*.

Example 9. Figure 10 shows the result of simulating three paths, using the same values for S_0, K, r, σ and T as in examples 6 and 7.[19] At time t_{m-1}, the potential values for θ_{m-1} are K and all three simulated values $S_{t_{m-1}}(\omega_1)$, $S_{t_{m-1}}(\omega_2)$ and $S_{t_{m-1}}(\omega_3)$ (represented by the red, green and blue lines, respectively). It turns out that $S_{t_{m-1}}(\omega_2)$ maximises the payoff over steps t_{m-1}, t_m so it becomes the value for θ_{m-1}. For the time steps immediately before this point, all three paths are greater than θ_{m-1}, so the boundary retains this value. It is only when $S_t(\omega_2)$ meets the boundary again that this becomes any smaller.

Once the boundary takes a value around 86.6557, it remains below all simulated paths. This is the value of the exercise boundary at time 0.

[19]In practice, we would always use more than three simulated paths and have done this for display purposes only.

(1) Simulate n_1 stock price paths $S_t(\omega_1), \dots, S_t(\omega_{n_1})$.
(2) Set $\theta_m = K$.
(3) For $j = m-1, m-2, \dots, 1, 0$
   (a) Set $\theta_j = \theta_{j+1}$ and $\hat{\theta}_j = \theta_j$ to be the maximising value of θ_j so far.
   (b) Set $\hat{P} = \frac{1}{n_1}\sum_{i=1}^{n_1} \widetilde{Z}(S_{t_j}(\omega_i))$, using equation 6.4 as the value for $\widetilde{Z}_{t_j}$.
   (c) For $i = 1, \dots, n_1$
      (i) If $S_{t_j}(\omega_i) < \theta_{j+1}$
         (A) Try $\theta_j = S_{t_j}(\omega_i)$.
         (B) Set $P = \frac{1}{n_1}\sum_{i=1}^{n_1} \widetilde{Z}(S_{t_j}(\omega_i))$ using equation 6.4 and this value of θ_j.
         (C) If $P > \hat{P}$ then $\hat{P} = P$ and $\hat{\theta}_j = \theta_j$
   (d) Set $\theta_j = \hat{\theta}_j$ as the exercise boundary value at time t_j.
(4) For $i = 1, \dots, m$
   (a) Set $j = 0$
   (b) While $j \in \{0, 1, \dots, m\}$ and an optimal stopping time for this path hasn't been found
      (i) If $S_{t_j}(\omega_i) \leq \theta_j$ then this path is below the boundary. Let $\tau_i = t_j$ be the optimal stopping time for this path.
      (ii) Otherwise, set $j = j+1$, use equation 3.1 to obtain the stock price for the next step and continue the while loop.
   (c) If an optimal stopping time for this path wasn't found, $\tau_i = T$.
   (d) Set $\widetilde{h}_i = (K - S_{\tau_i}(\omega_i))^+$ to be the present value from exercising this path at time τ_i.
(5) Obtain the average optimal stopping time $\overline{\tau} = \frac{1}{n}\sum_{i=1}^{n} \tau_i$.
(6) Obtain the variance $v^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\tau_i - \overline{\tau})^2$ for optimal stopping time.
(7) Obtain the sample mean $\overline{h} = \frac{1}{n}\sum_{i=1}^{n} \widetilde{h}_i$.
(8) Obtain the sample variance $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\widetilde{h}_i - \overline{h})^2$.
(9) Return $\overline{\tau}$ as the estimate of stopping time, $\overline{h}$ as the estimate of the option price, along with a 95% confidence intervals $\overline{\tau} \pm 1.96\frac{v}{\sqrt{n}}$, $\overline{h} \pm 1.96\frac{s}{\sqrt{n}}$ for stopping time and option price, respectively.

FIGURE 11. Algorithm that uses simulated paths to obtain the boundary, then estimates the price as before.

Having computed an estimate of the exercise boundary, we simulate n_2 new paths $S_t(\omega_1), \dots, S_t(\omega_{n_2})$ and estimate the American put option price as before. The optimal stopping time for a particular path is the time $\tau \in \{t_0, t_1, \dots, t_m\}$ when

$$S_\tau(\omega_i) \leq \theta_\tau$$

Figure 11 summarises this method of pricing American options.

Example 10. Figure 12 shows the result of using the above algorithm with the same parameters as in examples 6 and 7. As in 7, we can use a small

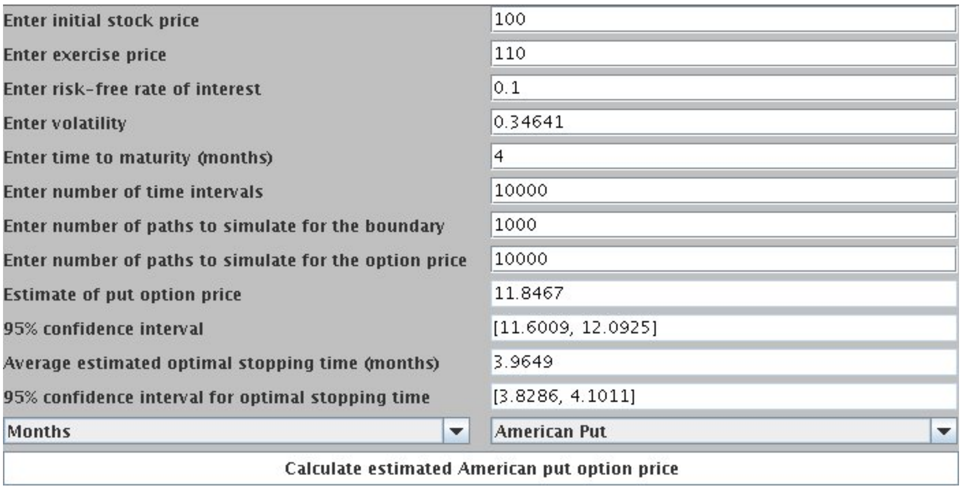

FIGURE 12. Using an estimated exercise boundary to price an American put option.

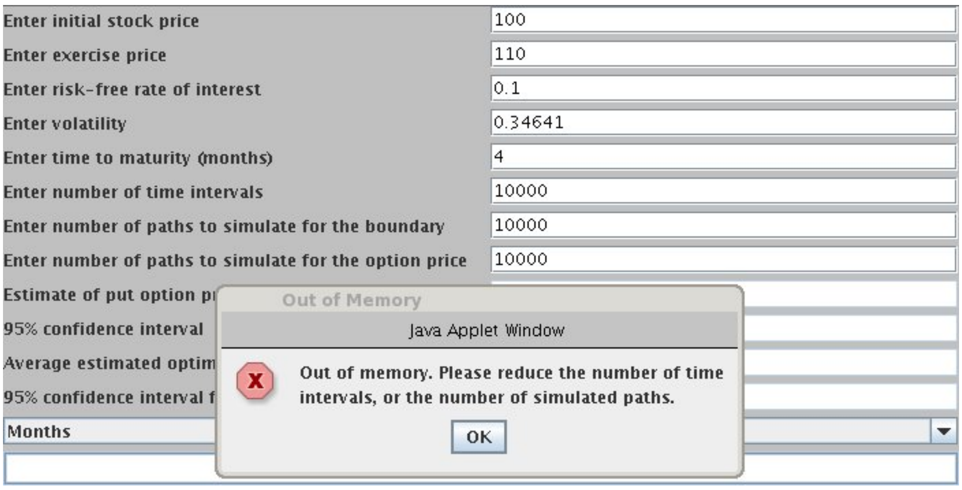

FIGURE 13. Specifying too many intervals or simulated paths results in failure to price the option

value for δt, giving an almost continuous model. Unlike 7, by letting $\delta t \to 0$ (from letting $n_1, n_2 \to \infty$), it ought to be the case that $\overline{h} \to h_0$.

However, in order to estimate the payoff boundary using this method, we need to store all simulated paths $S_t(\omega_1), \ldots, S_t(\omega_{n_1})$ in memory at one time. The case of using Java to estimate the boundary using n_1 paths with $m+1$ intervals $\{t_0, t_1, \ldots, t_m\}$ requires storing an $m+1 \times n_1$ array of double-precision decimal numbers.[20] Using the same value for m as in example 7 requires too much memory for even a high-end system, as figure 13 shows.

[20]In the above example, $m = 10000$ and $n_1 = 1000$. A double in Java is 8 bytes in length, resulting in 80,008,000 bytes $\approx$ 76.3 megabytes needed for the array alone.

7. Conclusion

The Black-Scholes model provides a way of pricing European call and put options. No such model exists for American options, so numerical procedures need to be used. In theory, we can price American options using optimal stopping times. In practice, finding the optimal stopping rule becomes a challenge, itself. Whilst the binomial tree method of Cox, Ross and Rubinstein provides a method for pricing American options in this way, it only considers a discrete set of stock price values at each time period. We can generalise to a continuous interval of possible stock prices by increasing the number of periods. However, doing this results in the number of possible paths we need to consider increasing exponentially.

Monte Carlo simulation estimates the option price by considering a sample of possible stock price paths. We can take one sample to create an estimate of the optimal exercise boundary, and use it with another sample to estimate the option price. However, the method of determining the boundary is, computationally, memory-intensive. Pricing options over a large continuous interval means we need to consider a larger number of time steps. As a consequence, a smaller sample must be used to determine the boundary, resulting in reduced accuracy. Whilst limited memory-usage is becoming less of a problem over time, it is still significant in the area of pricing American options.

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