Stochastic Simulation (MIE1613H) - Homework 4

Due: **April 4th**

- Submit your homework to Portal in PDF format by the above deadline. Late submissions are penalized.
- At the top of your homework include your name, student number, department, program and and e-mail address.
- You may discuss the assignment with other students, but each student must solve the problems, write the code and the solutions individually.
- Code must be in Python 2.7. unless you have permission from instructor to use another language. You must include both the source code (including comments to make it easy to follow) and the output.
- Full mark is given to answers that are correct and clearly explained. Write a brief and clear explanation of your solution approach for each problem.

Problem 1. An online retailer sells a product which it keeps in inventory. Customers make purchasing requests one at a time according to a Poisson process with rate $\lambda = 1$. Each request corresponds to the customer wishing to buy a random number of items, B, which follows a Geometric distribution with probability of success $p = 0.2$, i.e. $P(B = k) = (1-p)^{k-1}p$ for $k \geq 1$. If at the time of a demand arrival t_o the inventory level $I(t_o) \geq B$, then the customer is sold all B items at a price of $r = 1$ each. Otherwise if $I(t_o) < B$, the customer is only sold $I(t_o)$ items at a price of $r = 1$ each and the rest of the order is lost. That is, the customer is sold $m = \min\{I(t_o), B\}$ items and pays the company rm. The inventory level then drops to $I(t_o) - m$. All demand when $I = 0$ is lost. Whenever the inventory level $I(t)$ falls below a pre-specified value $s > 0$, the company places an order for $Y = S - I(t)$ new items that will be delivered L units of time later (L is called the lead time). L follows an Exponential distribution with rate $\mu = 1$. Placing orders for Y items costs the company oY where $o = 0.5$ is the unit ordering cost. There is a holding cost of $c = 0.2$ per item per unit time. We define the profit function over a time horizon of T as

$$R(T) = \sum_{i=1}^{N(T)} r\min\{B_i, I(t_i)\} - oM(T) - c\int_0^T I(t)dt,$$

where $N(T)$ is the number of purchase requests that arrive over $[0, T]$ and $M(T)$ is the total number of items ordered by the company over $[0, T]$. Note that the profit function is comprised of the total revenue minus the total holding and ordering costs up to time T.

(a) Estimate the expected profit $E[R(240)]$ assuming $s = 4$ and $S = 10$ and with initial inventory level $I(0) = S$. Use $n = 100$ replications and provide a 95% confidence interval for your estimate.

(b) Assume that the acceptable relative error for the estimate is $\kappa = 0.01$. Automatically control the number of replications in your simulation to achieve the desired relative error.

(c) Write down the problem of choosing parameters s and S to maximize the expected profit over the time-horizon $[0, 240]$ as a Simulation Optimization (SO) problem. (Make sure to clearly specify the objective function and constraints.)

(d) Apply the Selection of the Best procedure to find the best scenario among the following proposed scenarios. Start with $n_0 = 50$ replications for each scenario and use $1 - \alpha = 0.95$ and $\delta = 0.1$.

$\mathbf{x}_1$:$s = 6$ and $S = 8$

$\mathbf{x}_2$:$s = 4$ and $S = 6$

$\mathbf{x}_3$:$s = 3$ and $S = 6$

$\mathbf{x}_4$:$s = 2$ and $S = 5$

$\mathbf{x}_5$:$s = 2$ and $S = 6$

HINTS: The inventory process $I(t)$ (which keeps track of the number of items available in the inventory) is a continuous-time output similar to the number of customers in a queue. You may use the queue class from VBASim or define and keep track of your own state variable in your simulation model. If you use your own state variable, you may still want to use the VBASim's Calendar object to schedule delivery of orders. To generate a Geometric random variable with success probability p (for the size of purchase requests B) you can use the fact that a Geometric random variable is the number of attempts until the first success in a sequence of independent Bernoulli experiments with success probability p.

Problem 2. Consider a contact center modeled as an $M/G/s$ queue with arrival rate $\lambda = 9.2$ per minute, service times that are Erlang-3 distributed with mean $\tau = 1$ minute, and $s = 10$ servers.

(a) What is the long-run proportion of customers that wait longer than 3 minutes before their call is answered?

(b) What maximum delay guarantee T can we provide to the customers such that 95% of the calls are answered within T minutes?

Provide a reasonable (considering the application) 95% confidence interval for your estimates. Discuss your choice of the deletion point and number of replications.

HINTS: In part (a) you are estimating the probability that the steady-state waiting time in the queue in greater than 3, and in part (b) you are estimating the 95% quantile of the steady-state waiting time.

Problem 3. For the Asian Option pricing example from the class (and assuming the same parameters), we are interested in estimating the sensitivity of the value of the option to the initial price value $X(0) = 50$. That is, if we denote the option value as a function of the initial price, i.e.,

$$\nu(X(0)) = E[Y(X(0))],$$

with

$$Y(X(0)) = e^{-rt}(\bar{X}(T) - K)^+,$$

then we are interested in estimating the derivative $d\nu(X(0))/dX(0)$ at $X(0) = 50$.

(a) Use the finite difference method to estimate the above derivative using simulation. Change the value of δ in your finite-difference estimator, i.e.,

$$FD(X(0)) = \frac{Y(X(0)+\delta) - Y(X(0))}{\delta},$$

and discuss how it affects your estimate and its variability by reporting a confidence interval for each estimate. (The goal is to observe the bias-variance trade-off that we discussed in the class.)

HINTS: It helps to use common random numbers here to reduce the variance. That is, for each replication, you should use the same standard normal random numbers when evaluating the outcome of the option for initial price $X(0)$ and $X(0)+\delta$, but change the random numbers across replications. Note that the mean and variance of the estimator are

$$\begin{aligned} E[FD(X(0))] &= \frac{1}{\delta} E[Y(X(0)+\delta) - Y(X(0))], \\ Var(FD(X(0))) &= \frac{1}{\delta^2} Var[Y(X(0)+\delta) - Y(X(0))]. \end{aligned}$$

To obtain a confidence interval for each value of δ (and when using common random numbers) you can use the paired-t confidence interval discussed in the class (see page 205 of the textbook; In the formula for S_D^2, the term D_j^2 should be D_j)

Problem 4. (Bonus Question +10%) Exercise 27 from Chapter 8 of the textbook. Report a confidence interval for your estimate. (Use the same parameters as in the original Asian Option pricing example from the class. The formula for the control-variate estimator confidence-interval is given on pages 232-233 of the textbook. See also Section 8.5.3)

HINTS: Φ is the CDF of the standard normal distribution. Other parameters are as defined for the Asian Option example in Section 3.5.