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A Menon-type Identity with Multiplicative and Additive Characters

Yan Li, Xiaoyu Hu and Daeyeoul Kim*

Abstract. This paper studies Menon-type identities involving both multiplicative characters and additive characters. In the paper, we shall give the explicit formula of the following sum

$$\sum_{\substack{a \in \mathbb{Z}_n^* \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \gcd(a-1, b_1, \dots, b_k, n) \chi(a) \lambda_1(b_1) \cdots \lambda_k(b_k),$$

where for a positive integer n , \mathbb{Z}_n^* is the group of units of the ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, \gcd represents the greatest common divisor, χ is a Dirichlet character modulo n , and for a nonnegative integer k , $\lambda_1, \dots, \lambda_k$ are additive characters of \mathbb{Z}_n . Our formula further extends the previous results by Sury [13], Zhao-Cao [17] and Li-Hu-Kim [4].

1. Introduction

In 1965, P. K. Menon [7] proved the following beautiful identity:

$$(1.1) \quad \sum_{a \in \mathbb{Z}_n^*} \gcd(a-1, n) = \varphi(n) \tau(n),$$

where for a positive integer n , \mathbb{Z}_n^* is the group of units of the ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, \gcd represents the greatest common divisor, φ is the Euler's totient function and $\tau(n)$ is the number of positive divisors of n .

The Menon's identity (1.1) is very interesting and appealing. Many mathematicians made contributions on it. It has been proved by B. Sury [13] that

$$(1.2) \quad \sum_{\substack{a \in \mathbb{Z}_n^* \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \gcd(a-1, b_1, \dots, b_k, n) = \varphi(n) \sigma_k(n),$$

where $\sigma_k(n) = \sum_{d|n} d^k$ by using the Cauchy-Frobenius-Burnside lemma. It is also interesting to note that Miguel [8,9] extended identities (1.1) and (1.2) from \mathbb{Z} to any residually finite Dedekind domain.

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Recently, Zhao and Cao [17] derived the following elegant Menon-type identity with a Dirichlet character

$$(1.3) \quad \sum_{a \in \mathbb{Z}_n^*} \gcd(a-1, n) \chi(a) = \varphi(n) \tau\left(\frac{n}{d}\right),$$

where χ is a Dirichlet character modulo n and d is the conductor of χ .

From the point of view of Fourier analysis on finite Abelian groups, Zhao and Cao's results in fact give the explicit expression of Fourier transformation of the function $f(a) = \gcd(a-1, n)$ on the Abelian group $(\mathbb{Z}/n\mathbb{Z})^*$. Therefore, the identity (1.3) is not only graceful but also gives more information.

In [4], Li, Hu and Kim further extended identities (1.2) and (1.3). They obtained the following identity with Dirichlet character χ :

$$(1.4) \quad \sum_{\substack{a \in \mathbb{Z}_n^* \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \gcd(a-1, b_1, \dots, b_k, n) \chi(a) = \varphi(n) \sigma_k\left(\frac{n}{d}\right),$$

where d is the conductor of χ and k is a nonnegative integer.

For other related works on Menon's identity, see [1–3, 5, 6, 10–12, 14–16] and references therein.

Denote

$$(1.5) \quad S_{\chi, \underline{\lambda}}(n, k) = \sum_{\substack{a \in \mathbb{Z}_n^* \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \gcd(a-1, b_1, \dots, b_k, n) \chi(a) \lambda_1(b_1) \cdots \lambda_k(b_k),$$

where $\lambda_1, \dots, \lambda_k$ are additive characters of \mathbb{Z}_n and $\underline{\lambda}$ represents the vector $(\lambda_1, \dots, \lambda_k)$. For $1 \leq i \leq k$, each λ_i can be uniquely written as

$$(1.6) \quad \lambda_i(b) = \exp(2\pi\sqrt{-1}w_i b/n), \quad 0 \leq w_i \leq n-1, \quad w_i \in \mathbb{Z}$$

where $b \in \mathbb{Z}_n$ and $\sqrt{-1}$ is the square root of -1 whose imaginary part is positive. Denote the order of λ_i by d_i , that is,

$$(1.7) \quad d_i = \frac{n}{\gcd(w_i, n)}.$$

Theorem 1.1. *Let n be a positive integer and χ be a Dirichlet character modulo n whose conductor is d . Assume k is a nonnegative integer. Let $\lambda_1, \dots, \lambda_k$ be additive characters of \mathbb{Z}_n , explicitly given in (1.6). Let d_1, \dots, d_k as in (1.7) be the orders of $\lambda_1, \dots, \lambda_k$, respectively. Then, we have the following identity:*

$$\sum_{\substack{a \in \mathbb{Z}_n^* \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \gcd(a-1, b_1, \dots, b_k, n) \chi(a) \lambda_1(b_1) \cdots \lambda_k(b_k) = \varphi(n) \sigma_k\left(\frac{n}{\text{lcm}(d, d_1, \dots, d_k)}\right)$$

where lcm represents the least common multiple. Equivalently, it can also be written as

$$\sum_{\substack{a \in \mathbb{Z}_n^* \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \gcd(a-1, b_1, \dots, b_k, n) \chi(a) \lambda_1(b_1) \cdots \lambda_k(b_k) = \varphi(n) \sigma_k \left(\gcd \left(\frac{n}{d}, w_1, \dots, w_k \right) \right).$$

From the point of view of Fourier analysis, Theorem 1.1 gives the explicit expression of Fourier coefficients of the function $f(a, b_1, \dots, b_k) = \gcd(a-1, b_1, \dots, b_k, n)$ on the Abelian group $(\mathbb{Z}/n\mathbb{Z})^* \times (\mathbb{Z}/n\mathbb{Z})^k$.

Remark 1.2. If additive characters $\lambda_1, \dots, \lambda_k$ are trivial, then Theorem 1.1 reduces to identity (1.4). If both additive characters $\lambda_1, \dots, \lambda_k$ and multiplicative character χ are trivial, Theorem 1.1 reduces to Sury's identity (1.2). If $k = 0$, Theorem 1.1 reduces to Zhao and Cao's identity (1.3).

The rest of paper is organized as follows. In Section 2, we prove Theorem 1.1 in the special case of n being a prime power. The general case is treated in Section 3 by combining prime power cases with the Chinese remainder theorem.

2. Prime power case

In this section, we assume $n = p^m$, where p is a prime number and m is a positive integer. Let χ be a Dirichlet character modulo n with conductor d . Since $d \mid n$, we denote $d = p^t$, where $0 \leq t \leq m$. Let $\lambda_1, \dots, \lambda_k$ be additive characters of \mathbb{Z}_n with orders d_1, \dots, d_k , respectively. For $1 \leq i \leq k$, since $d_i \mid n$, we denote $d_i = p^{v_i}$, where $0 \leq v_i \leq m$.

Since $n = p^m$ is a prime power, the whole subgroups of \mathbb{Z}_n form a chain:

$$0 = p^m \mathbb{Z}_n \subset p^{m-1} \mathbb{Z}_n \subset \cdots \subset p \mathbb{Z}_n \subset \mathbb{Z}_n.$$

Clearly, for $0 \leq s \leq m$, $\#(p^s \mathbb{Z}_n) = p^{m-s}$, where $\#$ denote the cardinality of sets.

In the following, we adopt the similar method as in [4] to calculate $S_{\chi, \Delta}(p^m, k)$. From (1.5), we obtain

$$\begin{aligned} & \sum_{\substack{a \in \mathbb{Z}_n^* \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \gcd(a-1, b_1, \dots, b_k, n) \chi(a) \lambda_1(b_1) \cdots \lambda_k(b_k) \\ (2.1) \quad &= \sum_{s=0}^m \sum_{\substack{\gcd(b_1, \dots, b_k, n) = p^s \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \sum_{a \in \mathbb{Z}_n^*} \gcd(a-1, p^s) \chi(a) \lambda_1(b_1) \cdots \lambda_k(b_k) \\ &= \sum_{s=0}^m \left(\sum_{a \in \mathbb{Z}_n^*} \gcd(a-1, p^s) \chi(a) \right) \left(\sum_{\substack{\gcd(b_1, \dots, b_k, n) = p^s \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \lambda_1(b_1) \cdots \lambda_k(b_k) \right). \end{aligned}$$

Therefore, we need to compute

$$\sum_{a \in \mathbb{Z}_n^*} \gcd(a-1, p^s) \chi(a) \quad \text{and} \quad \sum_{\substack{\gcd(b_1, \dots, b_k, n) = p^s \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \lambda_1(b_1) \cdots \lambda_k(b_k)$$

explicitly. The first summation is already treated in [4]. We quote it here as Lemma 2.1. The second summation is computed in Lemma 2.3.

Lemma 2.1. [4, Lemma 2.2] *Let $n = p^m$ and χ be a Dirichlet character modulo n with conductor p^t , where $0 \leq t \leq m$. Let s be an integer such that $0 \leq s \leq m$. Then we obtain*

$$\sum_{a \in \mathbb{Z}_n^*} \gcd(a-1, p^s) \chi(a) = \begin{cases} (s-t+1)(p^m - p^{m-1}) & \text{if } s \geq t, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in Lemma 2.1, if $s = m$, this is just Lemma 3.1 of [17].

The following lemma is important to prove Lemma 2.3. It is a standard fact on characters of finite Abelian groups. For convenience of readers, we give a concrete proof here.

Lemma 2.2. *Let $n = p^m$ and λ be an additive character of \mathbb{Z}_n with order p^v . Then, for $0 \leq s \leq m$, we have*

$$\sum_{b \in p^s \mathbb{Z}_n} \lambda(b) = p^{m-s} [s \geq v],$$

where $[s \geq v]$ is the Iverson bracket, that is,

$$[s \geq v] = \begin{cases} 1 & \text{if } s \geq v, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $b \in p^s \mathbb{Z}_n$, write $b = p^s b'$ with some $b' \in \mathbb{Z}_n$. Then $\lambda(b) = \lambda^{p^s}(b')$. If $s \geq v$, then λ^{p^s} is a trivial character. In this case, $\lambda(b) = 1$ for every $b \in p^s \mathbb{Z}_n$. Therefore, we obtain

$$\sum_{b \in p^s \mathbb{Z}_n} \lambda(b) = \#(p^s \mathbb{Z}_n) = p^{m-s}.$$

Otherwise, λ^{p^s} is nontrivial on \mathbb{Z}_n . Hence, there exists some $b_0 = p^s b'_0 \in p^s \mathbb{Z}_n$ such that $\lambda(b_0) = \lambda^{p^s}(b'_0) \neq 1$. We have

$$\sum_{b \in p^s \mathbb{Z}_n} \lambda(b) = \sum_{b \in p^s \mathbb{Z}_n} \lambda(b + b_0) = \lambda(b_0) \sum_{b \in p^s \mathbb{Z}_n} \lambda(b).$$

As a result, we obtain

$$\sum_{b \in p^s \mathbb{Z}_n} \lambda(b) = 0.$$

□

Lemma 2.3. *Let $n = p^m$ be a prime power and $0 \leq s \leq m$ be an integer. Assume $k \geq 0$ is an integer. Let $\lambda_1, \dots, \lambda_k$ be additive characters of \mathbb{Z}_n with orders p^{v_1}, \dots, p^{v_k} , respectively. Denote $v = \max\{v_1, \dots, v_k\}$. Then for $0 \leq s \leq m-1$,*

$$\sum_{\substack{b_1, \dots, b_k \in \mathbb{Z}_n \\ \gcd(b_1, \dots, b_k, p^m) = p^s}} \lambda_1(b_1) \cdots \lambda_k(b_k) = p^{(m-s)k} [s \geq v] - p^{(m-s-1)k} [s+1 \geq v],$$

where $[s \geq v]$ is the Iverson bracket. Otherwise, for $s = m$, it is equal to 1.

Proof. The case $k = 0$ is obvious. Thus, we assume $k \geq 1$. Clearly,

$$(2.2) \quad \sum_{b_1, \dots, b_k \in p^s \mathbb{Z}_n} \lambda_1(b_1) \cdots \lambda_k(b_k) = \prod_{i=1}^k \sum_{b_i \in p^s \mathbb{Z}_n} \lambda_i(b_i).$$

Substituting Lemma 2.2 into (2.2), we get that

$$(2.3) \quad \sum_{b_1, \dots, b_k \in p^s \mathbb{Z}_n} \lambda_1(b_1) \cdots \lambda_k(b_k) = \prod_{i=1}^k p^{m-s} [s \geq v_i] = p^{(m-s)k} [s \geq v].$$

Note that $p^s \mid \gcd(b_1, \dots, b_k, p^m)$ if and only if $b_1, \dots, b_k \in p^s \mathbb{Z}_n$ holds. Therefore, for $0 \leq s \leq m-1$,

$$\gcd(b_1, \dots, b_k, p^m) = p^s \iff (b_1, \dots, b_k) \in (p^s \mathbb{Z}_n)^k - (p^{s+1} \mathbb{Z}_n)^k.$$

Hence, we obtain

$$(2.4) \quad \begin{aligned} & \sum_{\substack{b_1, \dots, b_k \in \mathbb{Z}_n \\ \gcd(b_1, \dots, b_k, p^m) = p^s}} \lambda_1(b_1) \cdots \lambda_k(b_k) \\ &= \sum_{b_1, \dots, b_k \in p^s \mathbb{Z}_n} \lambda_1(b_1) \cdots \lambda_k(b_k) - \sum_{b_1, \dots, b_k \in p^{s+1} \mathbb{Z}_n} \lambda_1(b_1) \cdots \lambda_k(b_k). \end{aligned}$$

It then follows from (2.3) and (2.4) that

$$\sum_{\substack{b_1, \dots, b_k \in \mathbb{Z}_n \\ \gcd(b_1, \dots, b_k, p^m) = p^s}} \lambda_1(b_1) \cdots \lambda_k(b_k) = p^{(m-s)k} [s \geq v] - p^{(m-s-1)k} [s+1 \geq v].$$

Thus, the case $0 \leq s \leq m-1$ is done.

Clearly, for $s = m$, one can readily check that

$$\gcd(b_1, \dots, b_k, p^m) = p^s \iff (b_1, \dots, b_k) \in (p^s \mathbb{Z}_n)^k.$$

In this case, there is only one summation term $\lambda_1(p^m) \cdots \lambda_k(p^m)$, which is equal to 1. This concludes the proof. \square

Finally, we prove the following result, which is a special case of Theorem 1.1.

Theorem 2.4. *Let $n = p^m$ be a prime power and χ be a Dirichlet character whose conductor is $d = p^t$. Assume k is a nonnegative integer. Let λ_i be an additive character of \mathbb{Z}_n with order $d_i = p^{v_i}$ such that $0 \leq v_i \leq m$, where $1 \leq i \leq k$. Then, the following identity holds*

$$\sum_{\substack{a \in \mathbb{Z}_n^* \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \gcd(a-1, b_1, \dots, b_k, n) \chi(a) \lambda_1(b_1) \cdots \lambda_k(b_k) = \varphi(n) \sigma_k \left(\frac{n}{\text{lcm}(d, d_1, \dots, d_k)} \right),$$

where $\text{lcm}(d, d_1, \dots, d_k)$ is the least common multiple of d, d_1, \dots, d_k .

Proof. By equation (2.1), $S_{\chi, \underline{\lambda}}(p^m, k)$ equals to

$$(2.5) \quad \sum_{s=0}^m \left(\sum_{a \in \mathbb{Z}_n^*} \gcd(a-1, p^s) \chi(a) \right) \left(\sum_{\substack{b_1, \dots, b_k \in \mathbb{Z}_n \\ \gcd(b_1, \dots, b_k, p^m) = p^s}} \lambda_1(b_1) \cdots \lambda_k(b_k) \right).$$

Substituting Lemma 2.1 into (2.5), we get

$$(2.6) \quad S_{\chi, \underline{\lambda}}(p^m, k) = \sum_{s=t}^m (s-t+1)(p^m - p^{m-1}) \left(\sum_{\substack{b_1, \dots, b_k \in \mathbb{Z}_n \\ \gcd(b_1, \dots, b_k, p^m) = p^s}} \lambda_1(b_1) \cdots \lambda_k(b_k) \right).$$

Denote $v = \max\{v_1, \dots, v_k\}$. Then substituting Lemma 2.3 into (2.6), we have that $S_{\chi, \underline{\lambda}}(p^m, k)$ equals to

$$\begin{aligned} & \varphi(p^m) \left(\sum_{s=t}^{m-1} (s-t+1) \left(p^{(m-s)k} [s \geq v] - p^{(m-s-1)k} [s+1 \geq v] \right) + (m-t+1) \right) \\ &= \varphi(p^m) \left(\sum_{s=t}^m (s-t+1) p^{(m-s)k} [s \geq v] - \sum_{s=t}^{m-1} (s-t+1) p^{(m-s-1)k} [s+1 \geq v] \right) \\ &= \varphi(p^m) \left(\sum_{s=t}^m (s-t+1) p^{(m-s)k} [s \geq v] - \sum_{s=t+1}^m (s-t) p^{(m-s)k} [s \geq v] \right). \end{aligned}$$

The last equality is obtained by substituting $s+1$ with s in the posterior summation. It is easy to see that

$$\begin{aligned} S_{\chi, \underline{\lambda}}(p^m, k) &= \varphi(p^m) \sum_{s=t}^m p^{(m-s)k} [s \geq v] \\ &= \varphi(p^m) \sum_{s=\max\{t, v\}}^m p^{(m-s)k} \\ &= \varphi(p^m) \sum_{s=0}^{m-\max\{t, v\}} p^{sk}. \end{aligned}$$

Further, the last equality is obtained by substituting $m - s$ with s . Therefore,

$$S_{\chi, \lambda}(p^m, k) = \varphi(p^m) \sigma_k \left(\frac{p^m}{p^{\max\{t, v\}}} \right) = \varphi(p^m) \sigma_k \left(\frac{p^m}{\text{lcm}(p^t, p^{v_1}, \dots, p^{v_k})} \right),$$

which concludes the proof. \square

3. The general case

In this section, we will prove the main theorem. First, we show that $S_{\chi, \lambda}(n, k)$ is multiplicative with respect to n by the Chinese remainder theorem. Then, using multiplicative property, we prove Theorem 1.1 by combining prime power cases, which are already treated in Section 2.

Let $n = n_1 n_2$ be the product of positive integers n_1 and n_2 such that $\gcd(n_1, n_2) = 1$. By the Chinese remainder theorem, we have the ring isomorphism: $\mathbb{Z}_n \simeq \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$, which induces the multiplicative group isomorphism: $\mathbb{Z}_n^* \simeq \mathbb{Z}_{n_1}^* \times \mathbb{Z}_{n_2}^*$. Therefore, each Dirichlet character modulo n can be uniquely written as $\chi = \chi' \cdot \chi''$, where χ , χ' and χ'' are Dirichlet characters modulo n , n_1 and n_2 , respectively. Similarly, any additive character λ of \mathbb{Z}_n can be uniquely written as $\lambda = \lambda' \cdot \lambda''$, where λ' and λ'' are additive characters of \mathbb{Z}_{n_1} and \mathbb{Z}_{n_2} , respectively. Explicitly, we obtain that

$$(3.1) \quad \chi(c \bmod n) = \chi'(c \bmod n_1) \cdot \chi''(c \bmod n_2)$$

and

$$(3.2) \quad \lambda(c \bmod n) = \lambda'(c \bmod n_1) \cdot \lambda''(c \bmod n_2)$$

for any integer c such that $\gcd(c, n) = 1$. For $1 \leq i \leq k$, we denote $\lambda_i = \lambda'_i \cdot \lambda''_i$ with the same meaning as above.

To simplify notations, for $a \in \mathbb{Z}_n$, we let $a' \in \mathbb{Z}_{n_1}$ and $a'' \in \mathbb{Z}_{n_2}$ denote the image of a in \mathbb{Z}_{n_1} and \mathbb{Z}_{n_2} , respectively, i.e.,

$$a' \equiv a \pmod{n_1} \quad \text{and} \quad a'' \equiv a \pmod{n_2}.$$

Let d , d' and d'' be the conductors of χ , χ' and χ'' , respectively. It is well known that $d = d' d''$. For $1 \leq i \leq k$, let d_i , d'_i and d''_i be the orders of λ_i , λ'_i and λ''_i , respectively. Since d'_i and d''_i are coprime to each other, we have $d_i = d'_i \cdot d''_i$, where $1 \leq i \leq k$. Denote the vectors $(\lambda'_1, \dots, \lambda'_k)$ and $(\lambda''_1, \dots, \lambda''_k)$ by $\underline{\lambda}'$ and $\underline{\lambda}''$, respectively.

The following lemma shows that $S_{\chi, \lambda}(n, k)$ is multiplicative with respect to n .

Lemma 3.1. *With the above notations we have*

$$S_{\chi, \lambda}(n, k) = S_{\chi', \underline{\lambda}'}(n_1, k) \cdot S_{\chi'', \underline{\lambda}''}(n_2, k).$$

Proof. From (1.5), (3.1) and (3.2), we have

$$\begin{aligned}
& \sum_{\substack{a \in \mathbb{Z}_n^* \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \gcd(a-1, b_1, \dots, b_k, n) \chi(a) \lambda_1(b_1) \cdots \lambda_k(b_k) \\
&= \sum_{\substack{a \in \mathbb{Z}_n^* \\ b_1, \dots, b_k \in \mathbb{Z}_n}} \gcd(a-1, b_1, \dots, b_k, n_1) \gcd(a-1, b_1, \dots, b_k, n_2) \\
&\quad \times \chi'(a) \chi''(a) \lambda'_1(b_1) \lambda''_1(b_1) \cdots \lambda'_k(b_k) \lambda''_k(b_k) \\
&= \sum_{\substack{a' \in \mathbb{Z}_{n_1}^* \\ b'_1, \dots, b'_k \in \mathbb{Z}_{n_1}}} \gcd(a'-1, b'_1, \dots, b'_k, n_1) \chi'(a') \lambda'_1(b'_1) \cdots \lambda'_k(b'_k) \\
&\quad \times \sum_{\substack{a'' \in \mathbb{Z}_{n_2}^* \\ b''_1, \dots, b''_k \in \mathbb{Z}_{n_2}}} \gcd(a''-1, b''_1, \dots, b''_k, n_2) \chi''(a'') \lambda''_1(b''_1) \cdots \lambda''_k(b''_k).
\end{aligned}$$

The last equality is obtained by the Chinese remainder theorem. Indeed, as (a, b_1, \dots, b_k) runs over $\mathbb{Z}_n^* \times (\mathbb{Z}_n)^k$, $(a', b'_1, \dots, b'_k, a'', b''_1, \dots, b''_k)$ runs over $\mathbb{Z}_{n_1}^* \times (\mathbb{Z}_{n_1})^k \times \mathbb{Z}_{n_2}^* \times (\mathbb{Z}_{n_2})^k$, too. Therefore, we have

$$S_{\chi, \underline{\lambda}}(n, k) = S_{\chi', \underline{\lambda}'}(n_1, k) \cdot S_{\chi'', \underline{\lambda}''}(n_2, k). \quad \square$$

Remark 3.2. The proof of Lemma 3.1 is similar to that of Lemma 3.1 in [4]. Also see the proof of Theorem 1.1 and Theorem 1.2 in [17].

Proof of Theorem 1.1. We prove the first identity by induction on $\omega(n)$, where $\omega(n)$ is the number of distinct prime factors of n .

If $\omega(n) = 1$, i.e., n is a prime power, this is proved in Theorem 2.4. Assume it is true for $\omega(n) = u - 1$, where $u \geq 2$ is an integer. Now we consider the case $\omega(n) = u$.

Let p^m be a prime power, exactly dividing n . Denote $n_1 = p^m$ and $n_2 = n/p^m$. Then $\gcd(n_1, n_2) = 1$.

Factor $\chi = \chi' \cdot \chi''$, where χ' and χ'' are Dirichlet characters modulo n_1 and n_2 with conductors d' and d'' , respectively. Similarly, for $1 \leq i \leq k$, decompose $\lambda_i = \lambda'_i \cdot \lambda''_i$ where λ'_i and λ''_i are additive characters of \mathbb{Z}_{n_1} and \mathbb{Z}_{n_2} with orders d'_i and d''_i .

We note that

$$(3.3) \quad d = d' d'', \quad \gcd(d', d'') = 1 \quad \text{and} \quad d_i = d'_i d''_i, \quad \gcd(d'_i, d''_i) = 1,$$

where $1 \leq i \leq k$. Denote the vectors $(\lambda'_1, \dots, \lambda'_k)$ and $(\lambda''_1, \dots, \lambda''_k)$ by $\underline{\lambda}'$ and $\underline{\lambda}''$, respectively. By Theorem 2.4 and the assumption, we have

$$(3.4) \quad S_{\chi', \underline{\lambda}'} = \varphi(n_1) \left(\frac{n_1}{\text{lcm}(d', d'_1, \dots, d'_k)} \right) \quad \text{and} \quad S_{\chi'', \underline{\lambda}''} = \varphi(n_2) \left(\frac{n_2}{\text{lcm}(d'', d''_1, \dots, d''_k)} \right).$$

Combining Lemma 3.1 and (3.4), we get

$$\begin{aligned} S_{\chi, \underline{\lambda}}(n, k) &= S_{\chi', \underline{\lambda}'}(n_1, k) S_{\chi'', \underline{\lambda}''}(n_2, k) \\ &= \varphi(n_1) \varphi(n_2) \sigma_k \left(\frac{n_1}{\text{lcm}(d', d'_1, \dots, d'_k)} \right) \sigma_k \left(\frac{n_2}{\text{lcm}(d'', d''_1, \dots, d''_k)} \right). \end{aligned}$$

Since arithmetic functions φ , σ_k and lcm are multiplicative, by (3.3), we get the desired result

$$S_{\chi, \underline{\lambda}}(n, k) = \varphi(n) \sigma_k \left(\frac{n}{\text{lcm}(d, d_1, \dots, d_k)} \right).$$

The second identity can be justified as follows:

$$\begin{aligned} \frac{n}{\text{lcm}(d, d_1, \dots, d_k)} &= \frac{n}{\text{lcm}(n/(n/d), n/\gcd(w_1, n), \dots, n/\gcd(w_k, n))} \\ &= \frac{n}{n/\gcd(n/d, \gcd(w_1, n), \dots, \gcd(w_k, n))} \\ &= \gcd(n/d, w_1, \dots, w_k). \end{aligned}$$

This completes the proof of Theorem 1.1. □

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