Accuracy of Convergents of Continued Fraction

From ProofWiki

Theorem

Let x be an irrational number.

Let $\langle C_n \rangle$ be the sequence of convergents of the simple infinite continued fraction of x.

Let $p_1, p_2, p_3, \ldots$ and $q_1, q_2, q_3, \ldots$ be its numerators and denominators.

Then:

$$\forall k \ge 1: \left| x - \frac{p_{k+1}}{q_{k+1}} \right| < \frac{1}{q_{k+1} q_{k+2}} \le \frac{1}{2 q_k q_{k+1}} < \left| x - \frac{p_k}{q_k} \right|$$

Thus:

> The left hand side of the inequality gives an indication of how close each convergent gets to its true value.
>
> The right hand side gives a bound that limits its accuracy.

Corollary

$$\forall k \ge 1: \frac{1}{q_k q_{k+1}} > \left| x - \frac{p_k}{q_k} \right| > \frac{1}{2 q_k q_{k+1}}$$

Proof

Let x have a simple infinite continued fraction of $\left[a_1, a_2, a_3, \ldots\right]$.

From Existence and Uniqueness of Simple Infinite Continued Fraction, $\left[a_1, a_2, a_3, \ldots\right]$ exists and is unique.

The Continued Fraction Algorithm gives the following system of equations:

$$\begin{aligned} x &= \left[a_1, x_2\right] \\ &= \left[a_1, a_2, x_3\right] \\ &= \ldots \\ &= \left[a_1, a_2, \ldots, a_n, x_{n+1}\right] \\ &= \ldots \end{aligned}$$

and

$$\begin{aligned} \left| x - \frac{p_n}{q_n} \right| &= \left| \left[a_1, a_2, \ldots, a_n, x_{n+1}\right] - \frac{p_n}{q_n} \right| \\ &= \left| \frac{x_{n+1} p_n + p_{n-1}}{x_{n+1} q_n + q_{n-1}} - \frac{p_n}{q_n} \right| && \text{Value of Simple Finite Continued Fraction} \\ &= \left| \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n \left(x_{n+1} q_n + q_{n-1}\right)} \right| \end{aligned}$$

\displaystyle	=	\displaystyle \frac 1 {q_n \left({x_{n + 1} q_n + q_{n - 1} }\right)}	\quad Difference between Adjacent Convergents of Simple Continued Fraction	\quad

Now from the Continued Fraction Algorithm:

> x_{n + 1} = \left[{a_{n + 1}, a_{n + 2}, a_{n + 3}, \ldots}\right]

So:

> a_{n + 1} < x_{n + 1} < a_{n + 1} + 1

Therefore:

$$\left\vert{x - \dfrac {p_n} {q_n} }\right\vert < \dfrac 1 {q_n \left({a_{n + 1} q_n + q_{n - 1} }\right)} = \dfrac 1 {q_n q_{n + 1} }$$

This gives the left hand side of the inequality when $n = k + 1$.

We also have:

$$\left\vert{x - \dfrac {p_n} {q_n} }\right\vert > \dfrac 1 {q_n \left({\left({a_{n + 1} + 1}\right) q_n + q_{n - 1} }\right)}$$

$$\begin{aligned}
\displaystyle \left\vert{x - \frac {p_n} {q_n} }\right\vert &> \displaystyle \frac 1 {q_n \left({\left({a_{n + 1} + 1}\right) q_n + q_{n - 1} }\right)} \quad\quad \\
&= \displaystyle \frac 1 {q_n \left({a_{n + 1} q_n + q_{n - 1} }\right) + q_n^2} \quad\quad \\
&= \displaystyle \frac 1 {q_n \left({q_{n + 1} + q_n}\right)} \quad\quad \\
&> \displaystyle \frac 1 {q_n \left({q_{n + 1} + q_{n + 1} }\right)} \quad\quad \\
&= \displaystyle \frac 1 {2 q_n q_{n + 1} } \quad\quad
\end{aligned}$$

This gives the right hand side of the inequality when $n = k$.

For the middle inequality, note that:

$$q_{k + 2} = a_{k + 2} q_{k + 1} + q_k > q_k + q_k = 2 q_k$$

So:

$$\dfrac 1 {q_{k + 1} q_{k + 2} } \le \dfrac 1 {2 q_k q_{k + 1} }$$

$\blacksquare$

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