Continued fractions and good approximations.

We will study how to find good approximations for important real life constants. A good approximation must be both accurate and easy to use. For instance, our current calendar was designed to approximate the solar year by means of simple rules for omitting some leap years. However, one can find other simple calendars that give a better approximation. To find good approximations we will introduce *continued fractions*—fascinating objects related to number theory.

1 Introduction

The numbers

$$\dots, -3, -2, -1, 0, 1, 2, \dots$$

are called integers. An integer a is *divisible* by another integer $b \neq 0$ if $a = bc$ for some integer c. Then b is said to be a *divisor* of a. E.g. 10 is divisible by 5 since $10 = 5 \cdot 2$.

Clearly, any integer is divisible by itself and by 1. A positive integer $p \neq 1$ is called *prime* if it has no positive divisors except for 1 and p. The first primes are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 37, 41, 43, 47, 53, \dots$$

The fundamental theorem of arithmetics asserts that every positive integer, except 1, can be expressed as a product of primes in one way only (up to rearrangement of factors). Such expression is called the *prime decomposition*.

Problem 1.1 Find the prime decomposition of 36, 60, 91, 198, 361.

The fractions $\frac{p}{q}$, where p and $q \neq 0$ are integers, are called rational numbers. Not every number is rational. E.g. the equation

$$x^2 = 2$$

does not have rational solutions, so $\sqrt{2}$ is not rational. Numbers that are not rational are called *irrational*.

Problem 1.2 **(a)** Show that the *golden ratio* $\frac{1+\sqrt{5}}{2}$ is irrational.
(b) Is $\sqrt{2} + \sqrt{3}$ rational? Why or why not?

Some of the useful constants such as $e = 2.718281\dots$ and $\pi = 3.141592\dots$ are irrational. For practical purposes such constants are usually approximated by rational numbers (preferably with small denominators). E.g. Archimedes found that π is approximately $\frac{22}{7}$. This is a good approximation because it is both simple and accurate. Indeed, the error $\pi - \frac{22}{7}$ is less than $2 \cdot 10^{-3}$. Moreover, for $q < 10$ all approximations $\frac{p}{q}$ give much bigger error! I.e. the approximation $\frac{22}{7}$ is the best among all approximations with small denominators.

Problem 1.3 Find the best rational approximation of $\sqrt{2}$ with the denominator less than 10.

In general, for any irrational number α and a positive integer n there exists a good rational approximation $\frac{p}{q}$ such that the denominator q is less than n and

$$\alpha - \frac{p}{q} < \frac{1}{nq}.$$

To find such an approximation we need *continued fractions.*

Euclidean algorithm and continued fractions. Two numbers α and β are called *commensurable* if they have a *common divisor*, that is, a number x such that $\alpha = mx$ and $\beta = nx$ for some integers m and n. E.g. any rational number p/q and 1 are commensurable; $1/q$ being a common divisor.

Problem 1.4 Are numbers $\sqrt{2}$ and $1/\sqrt{2}$ commensurable? And numbers $\sqrt{2}$ and $2+\sqrt{2}$?

There is an algorithm due to Euclid that allows to find the greatest common divisor $gcd(\alpha, \beta)$ of any two commensurable numbers α and β. This algorithm uses the following fact.

Problem 1.5 If n is any integer, then $gcd(\alpha, \beta) = gcd(\alpha - n\beta, \beta)$.

Euclidean algorithm. Assume that $\alpha > \beta$.

1. Find an integer n such that $n\beta \leq \alpha < (n+1)\beta$. Note that the *remainder* $r = \alpha - n\beta$ is smaller than β, and $gcd(\alpha, \beta) = gcd(\beta, r)$!

2. If $r = 0$, then we are finished, and $gcd(\alpha, \beta) = \beta$. If $r \neq 0$ repeat steps 1, 2 for β and r.

Example Take $\alpha = 1071$ and $\beta = 1029$.

$$1071 = 1 \cdot 1029 + 42$$
$$1029 = 24 \cdot 42 + 21$$
$$42 = 2 \cdot 21 + 0$$

The algorithm terminates in three steps, and we get $gcd(1071, 1029) = 21$.

Euclidean algorithm can also be used to write a rational number p/q as a *continued fraction.* E.g. the above calculations for 1071 and 1029 can be rewritten as

$$1071/1029 = 1 + 42/1029$$
$$1029/42 = 24 + 21/42$$
$$42/21 = 2$$

This can be presented in the following nice way:

$$\frac{1071}{1029} = 1 + \frac{1}{24 + \frac{1}{2}}.$$

In particular, the truncated continued fraction $1 + 1/24 = 25/24 = 1.041(6)$ approximates $1071/1029 = 1,0408\ldots$ very well.

Problem 1.6 Find $gcd(125, 35)$, $gcd(1001, 891)$ and $gcd(355, 113)$. Write the corresponding continued fractions for 125/35, 1001/891 and 355/113.

2 More on Continued fractions

A simple continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}.$$

We will assume that the a_i are nonnegative integers with $a_i > 0$ for $i > 0$. These expressions can be finite or infinite. Moreover every positive number has such an expression. We calculate it by a version of the Euclidean algorithm. Here is how. We will use $\lfloor x \rfloor$ to denote the largest integer $\leq x$. Thus $\lfloor 2.1 \rfloor = 2$, $\lfloor 1.9 \rfloor = 1$.

Given x write

$$x = \lfloor x \rfloor + x_1, \quad \text{and set} \quad a_0 = \lfloor x \rfloor.$$

Then by definition the "remainder" x_1 satisfies $0 \leq x_1 < 1$. If $x_1 = 0$, stop. Otherwise, note that $1/x_1 > 1$ so that

$$\frac{1}{x_1} = \lfloor \frac{1}{x_1} \rfloor + x_2$$

where $a_1 = \lfloor \frac{1}{x_1} \rfloor > 0$, and $0 \leq x_2 < 1$. So

$$\frac{1}{x_1} = a_1 + x_2, \quad x_1 = \frac{1}{a_1 + x_2}.$$

So

$$x = a_0 + x_1 = a_0 + \frac{1}{a_1 + x_2}.$$

If $x_2 = 0$, stop. Otherwise, follow the above procedure to write

$$\frac{1}{x_2} = a_2 + x_3, \quad x_2 = \frac{1}{a_2 + x_3},$$

so that

$$x = a_0 + \frac{1}{a_1 + x_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2+x_3}}.$$

For example,

$$\begin{array}{rclcr}
\dfrac{33}{14} & = & 2 + \dfrac{5}{14}, & & 33 = 2 \cdot 14 + 5 \\
\dfrac{14}{5} & = & 2 + \dfrac{4}{5} & & 14 = 2 \cdot 5 + 4 \\
\dfrac{5}{4} & = & 1 + \dfrac{1}{4} & & 5 = 1 \cdot 4 + 1 \\
\dfrac{4}{1} & = & 4. & & 4 = 4 \cdot 1 + 0.
\end{array}$$

(The Euclidean algorithm is on the right, so you can compare the two directly.) Hence

$$\frac{33}{14} = 2 + \frac{1}{2 + \frac{1}{1+\frac{1}{4}}}.$$

For short we write $x = [a_0; a_1, a_2]$ if

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}.$$

Thus

$$\frac{33}{14} = [2; 2, 1, 4].$$

Proposition 2.1 *The continued fraction expansion of a number x is finite if and only if x is rational.*

Proof: Observe that when you construct the continued fraction expansion of a fraction p/q the successive "remainders" $x_1, x_2, x_3, \ldots$ are rational numbers < 1. Moreover, if $x_k = b_k/c_k < 1$ then we get x_{k+1} from the equation

$$\frac{c_k}{b_k} = \text{integer } + x_{k+1}.$$

It follows that the denominator of x_{k+1} is b_k which is strictly less than c_k. Therefore, the denominators of the remainders are positive integers that decrease strictly. Hence the process must stop.

Thus rational numbers have finite continued fraction expansions. The converse is left to you. □

Example In the expansion we found above for $33/14$ the remainders are $x_1 = 5/14, x_2 = 4/5, x_3 = 1/4$ which have strictly decreasing denominators.

Problem 2.2 Experiment a bit with different rational numbers. Must the numerators of the remainders decrease?

Some numbers have very pretty continued fraction expansions.

Lemma 2.3 $\sqrt{2} = [1; 2, 2, 2, 2 \ldots]$ *or*

$$\sqrt{2} = 1 + \cfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}}$$

Proof: There are three ways of seeing this.
Method 1: Find the continued expansion directly, using your calculator.

Method 2: Calculate by hand. (This is possible only for a few numbers.)

Write $\sqrt{2} = 1 + x_1$. Since

$$(\frac{4}{3})^2 = \frac{16}{9} < 2, \qquad (\frac{3}{2})^2 = \frac{9}{4} > 2$$

we must have $\frac{1}{3} < x_1 < \frac{1}{2}$. So

$$2 < \frac{1}{x_1} < 3, \text{ that is } \frac{1}{x_1} = \lfloor \frac{1}{x_1} \rfloor + x_2 = 2 + x_2.$$

We claim that $x_2 = x_1$ so that the calculation repeats itself. (This is very special.) Since $x_1 = \sqrt{2} - 1$,

$$x_2 = \frac{1}{x_1} - 2 = \frac{1}{\sqrt{2}-1} - 2 = \frac{1 - 2(\sqrt{2}-1)}{\sqrt{2}-1} = \frac{3 - 2\sqrt{2}}{\sqrt{2}-1} = \frac{(3-2\sqrt{2})(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \sqrt{2} - 1 = x_1.$$

Hence $\frac{1}{x_2} = 2 + x_3$ where $x_3 = x_1$ again. Thus $\sqrt{2} = [1; 2, 2, 2, \ldots]$.

Method 3: (A good way to check the answer, if you guessed it beforehand.) Consider the continued fraction $x = [1; 2, 2, 2, 2 \ldots]$ ie

$$x = 1 + \cfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}} = 1 + \cfrac{1}{1 + \left(1 + \frac{1}{2 + \frac{1}{2 + \ldots}}\right)}.$$

Notice that

$$x = 1 + \frac{1}{1 + x}.$$

Thus $x - 1 = \frac{1}{1+x}$, and so $x^2 - 1 = 1$, that is $x^2 = 2$. Since $x = [1; 2, 2, 2, \ldots]$ is positive, we must have $x = \sqrt{2}$. $\square$

Problem 2.4 (i) Find the continued fraction expansions of $\sqrt{3}, \sqrt{7}$, using your calculator. (Note: the calculation should repeat after a few steps.)
(ii) Show that your calculation is correct by finding the equation satisfied by these numbers as in Method 3 above.

Problem 2.5 (i) Calculate $[1;1]$, $[1;1,1]$,$[1;1,1,1]$, and $[1;1,1,1,1]$. On the basis of this, what do you think the number $[1;1,\dots,1]$ is (where there are n 1s)?
(ii) What is the infinite fraction $[1;1,1,1,\dots]$?

Problem 2.6 Use Method 3 above to find the numbers represented by the infinite fractions $y = [2;3,3,3,\dots]$ and $y = [3;4,4,4,\dots]$. Can you make a general rule?

A continued fraction $[a_0; a_1, a_2, a_3, \dots]$ is said to be periodic if there is n such that the numbers $a_n, a_{n+1}, a_{n+2}, \dots$ consist of a infinitely repreated finite sequence. eg [2; 3, 4, 1, 5, 1, 5, 1, 5, ...].

Proposition 2.7 *The number x has a periodic continued fraction expansion if and only if it is a quadratic irrational, ie is the root of a quadratic equation with integer coefficients.*

I am not going to prove this now. Here is an example that illustrates why periodic continued fractions give rise to quadratic irrationals. Suppose that

$$x = [0;2,3,1,3,1,3,\dots] = \frac{1}{2 + \frac{1}{3+\frac{1}{1+\dots}}}.$$

Let $y = [0;0,3,1,3,\dots]$ be the repeating part of x. Then y satisfies the equation

$$y = \frac{1}{3+\frac{1}{1+y}} = \frac{1}{\frac{3y+4}{y+1}} = \frac{1+y}{3y+4}.$$

Thus y satisfies a quadratic equation with integer coeffs. and so has the form $r + s\sqrt{t}$ for r, s rational and t integral. But

$$x = \frac{1}{2+y}$$

and so also has this form. □

3 The convergents of a continued fraction

Given an infinite continued fraction $[a_0; a_1, a_2, \dots]$ the rational numbers

$$\frac{h_n}{k_n} = [a_0; a_1, a_2, \dots, a_n]$$

are called its **convergents**. (As always when writing a rational number we suppose that p_n, q_n are mutually prime so that they are defined without ambiguity.) For example, the first few convergents to $\pi = [3; 7, 15, 1, 292, \ldots] = 3.141592654$ are:

$$\begin{aligned}
\frac{h_0}{k_0} &= 3, \\
\frac{h_1}{k_1} &= 3 + \frac{1}{7} = \frac{22}{7} = 3.142857143, \\
\frac{h_2}{k_2} &= 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106} = 3.141509434, \\
\frac{h_3}{k_3} &= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = \frac{355}{113} = 3.14159292.
\end{aligned}$$

(This last approximation 355/133 to π was known to the Chinese mathematician Tsu Chung-Chi. It is the best approximation of any fraction below the next convergent which is 103993/33102!) The first few convergents to $\sqrt{2} = [1; 2, 2, 2, \ldots] = 1.41423562$ are

$$\begin{array}{ll}
\frac{h_0}{k_0} = 1, & \frac{h_1}{k_1} = 1 + \frac{1}{2} = \frac{3}{2} = 1.5, \\
\frac{h_2}{k_2} = 1 + \frac{1}{2+\frac{1}{12}} = \frac{7}{5} = 1.4, & \frac{h_3}{k_3} = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2}}} = \frac{17}{12} = 1.4166667\cdots \\
\frac{h_4}{k_4} = \frac{41}{29} = 1.413793103\cdots & \frac{h_5}{k_5} = \frac{99}{70} = 1.414285714\cdots.
\end{array}$$

If you look at the convergents above, you will notice that they do not simply increase or decrease. Rather we have

$$\frac{h_0}{k_0} < \frac{h_2}{k_2} < \frac{h_4}{k_4} < \cdots < \frac{h_5}{k_5} < \frac{h_3}{k_3} < \frac{h_1}{k_1}.$$

That is, the even ones increase and the odd ones decrease, and both get closer and closer to the number x that we are trying to approximate.

This is a general pattern. To explain why, we need to say more about the convergents. Their most important property is that that they can be calculated from the explansion $x = [a_0; a_1, a_2, \ldots]$ by the following rule. Put

$$h_{-1} = 1, \quad h_{-2} = 0, \qquad k_{-1} = 0, \quad k_{-2} = 1.$$

Then define:

$$\begin{aligned}
h_0 &:= a_0 h_{-1} + h_{-2} = a_0, & k_0 &:= a_0 k_{-1} + k_{-2} = 1 \\
h_1 &:= a_1 h_0 + h_{-1} = a_1 a_0 + 1, & k_1 &:= a_1 k_0 + k_{-1} = a_1
\end{aligned}$$

and for general $n = 2, 3, 4, \ldots$

$$h_n := a_n h_{n-1} + h_{n-2}, \quad k_n := a_n k_{n-1} + k_{n-2}.$$

NOTE This kind of formula for h_n in terms of the previous two elements in the sequence $h_{-2}, h_{-1}h_0, h_1, h_2, \ldots$ is called a *recursive* or *iterative* formula. It is reminiscent of the formula for the Fibonacci numbers F_k:

$$F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, \quad F_n = F_{n-1} + F_{n-2}.$$

Lemma 3.1 *(i)* $[a_0; a_1, \ldots, a_n] = \frac{h_n}{k_n}$.
(ii) $k_n h_{n-1} - k_{n-1}h_n = (-1)^n$ *for all* $n \geq 0$.

I won't prove this lemma. (Though if you are interested, this could be a project.) Instead let's look at an example.

Example For $x = \sqrt{2} = [1; 2, 2, 2, \ldots]$ we have $h_0 = a_0 = 1, k_0 = 1$; then

$$\begin{array}{ll} h_1 = a_1h_0 + h_{-1} = 2 + 1 = 3, & k_1 = a_1k_0 + k_{-1} = 2, \\ h_2 = a_2h_1 + h_0 = 2 \cdot 3 + 1 = 7, & k_2 = a_2k_1 + k_0 = 2 \cdot 2 + 1 = 5 \\ h_3 = a_3h_2 + h_1 = 2 \cdot 7 + 3 = 17, & k_3 = a_3k_2 + k_1 = 2 \cdot 5 + 2 = 12, \end{array}$$

So

$$\frac{h_1}{k_1} = \frac{3}{2}, \quad \frac{h_2}{k_2} = \frac{7}{5}, \quad \frac{h_3}{k_3} = \frac{17}{12},$$

This illustrates the formula in (i). As for (ii), notice that

$$\begin{array}{lcl} k_1h_0 - k_0h_1 = 2 \cdot 1 - 1 \cdot 3 & = & -1, \\ k_2h_1 - k_1h_2 = 5 \cdot 3 - 2 \cdot 7 & = & 1, \\ k_3h_2 - k_2h_3 = 12 \cdot 7 - 5 \cdot 17 & = & 84 - 85 = -1. \end{array}$$

Here are some conclusions we can draw from this lemma.

Conclusion 1 Each convergent is in its lowest terms, i.e. has greatest common divisor equal to 1. For if d divides both h_n and k_n it would also divide $k_nh_{n-1} - k_{n-1}h_n = (-1)^n$.

Conclusion 2 The difference between successive convergents is:

$$\frac{h_n}{k_n} - \frac{h_{n-1}}{k_{n-1}} = \frac{h_nk_{n-1} - k_nh_{n-1}}{k_nk_{n-1}} = \frac{(-1)^{n-1}}{k_nk_{n-1}}.$$

Conclusion 3

$$[a_0; a_1, a_2, \ldots, a_n] = a_0 - \frac{1}{k_1k_0} + \frac{1}{k_2k_1} - \frac{1}{k_3k_2} + \cdots \pm \frac{1}{k_nk_{n-1}}$$

(This is known as an alternating sum. The infinite continued fraction is given by an infinite alternating sum.) This holds because the convergent $\frac{h_n}{k_n}$ can be written as an alternating sum of the differences:

$$\frac{h_n}{k_n} = \left(\frac{h_n}{k_n} - \frac{h_{n-1}}{k_{n-1}}\right) + \left(\frac{h_{n-1}}{k_{n-1}} - \frac{h_{n-2}}{k_{n-2}}\right) + \cdots + \left(\frac{h_0}{k_0} - \frac{h_{-1}}{k_{-1}}\right)$$

Finally we can see why the convergents give such good approximations to x. For x is given by an alternating sum of the form

$$x = c_0 - c_1 + c_2 - c_3 + \cdots +, \quad s_n = c_0 - c_1 + c_2 - c_3 + \cdots \pm c_n$$

where the convergent is the sum s_n of the first $n+1$ terms. For any such series (assuming that c_n tends to 0) the difference $|x - s_n|$ is always bounded by the size $|c_{n+1}|$ of the next term. So:

Conclusion 4 For any x with convergents $\frac{h_n}{k_n}$,

$$\left|x - \frac{h_n}{k_n}\right| < \frac{1}{k_{n+1}k_n}.$$

Theorem 3.2 $\dfrac{h_n}{k_n}$ *is the best fractional approximation to x with denominator $\leq k_n$.*

Proof: Note that the convergents k_n increase strictly and so $k_{n+1} \geq 2$. So if $H := h_n, K := k_n$ we know that H/K approximates x to within $1/2K^2$, i.e.

$$\left|x - \frac{H}{K}\right| < \frac{1}{2K^2}.$$

Now suppose that p/q is a better approximation and that $q \leq K$. Then

$$\begin{aligned}
\left|\frac{p}{q} - \frac{H}{K}\right| &\leq \left|\frac{p}{q} - x\right| + \left|x - \frac{H}{K}\right| && \text{(triangle inequality)} \\
&\leq 2\left|x - \frac{H}{K}\right| && \text{(since } p/q \text{ is closer to } x\text{)} \\
&< \frac{1}{K^2}.
\end{aligned}$$

On the other hand

$$\left|\frac{p}{q} - \frac{H}{K}\right| = \left|\frac{pK - qH}{Kq}\right| > \frac{1}{Kq} \geq \frac{1}{K^2}.$$

A contradiction. So such p/q cannot exist.

Problems and Projects Here are some problems. Any of them could be the core of a project. You could also expand Problem 3.4, for example, into a project on the Fibonacci numbers (discuss completely different aspects of these if you want) or Problem 3.5 into a study of continued fractions in dynamical systems. Other possible projects involve applications (eg investigating different calendar systems, or investigating different musical scales, and in particular the well tempered scale.)

Problem 3.3 (i) Calculate the continued fraction expansion for $2\sqrt{2}$ and then find its first 6 convergents.
(ii) Can you figure out a way to multiply the continued fraction

$$1 + \cfrac{1}{2 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}}}$$

by 2, thus explaining the expansion you found in (i)?

Problem 3.4 (i) Find a formula for the convergents to the golden ratio τ in terms of the Fibonacci numbers. (cf Problem 2.5.)
(ii) Test the inequalities

$$|\tau - \frac{p_n}{q_n}| < \frac{1}{q_n^2}, \quad |\tau - \frac{p_n}{q_n}| < \frac{1}{2q_n^2}$$

for at least 5 different values of n. (You cannot take n too large, since your calculator will start making significant errors.)
(iii) Take a few random numbers q (ie ones that are not Fibonacci numbers) and show that the inequality

$$|\tau - \frac{p}{q}| < \frac{1}{2q^2}$$

has no integer solution p.

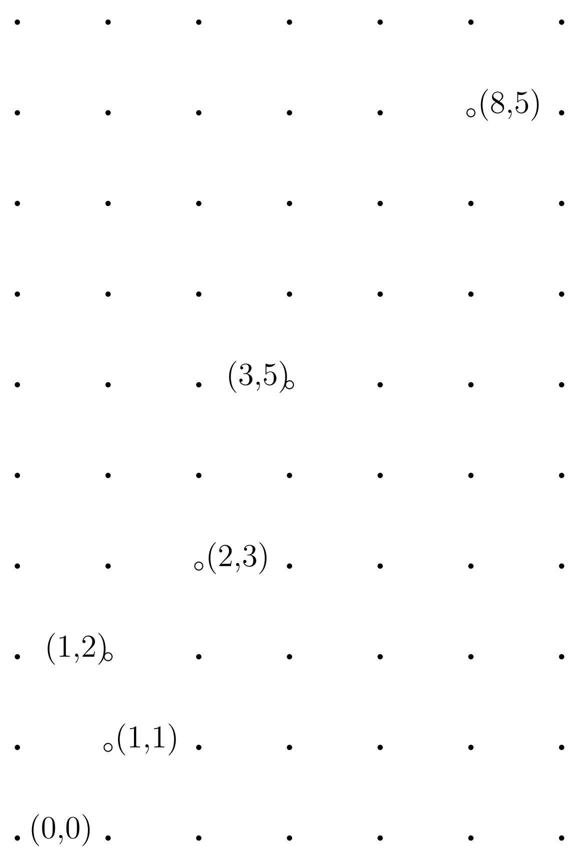
Problem 3.5 Let $x = \tau$ and mark the points $x - \lfloor x \rfloor, 2x - \lfloor 2x \rfloor, 3x - \lfloor 3x \rfloor, \dots, kx - \lfloor kx \rfloor, \dots$ on the interval $[0, 1]$. Try to do this fairly accurately: either do it on a computer or do it by hand on squared paper, scaling the interval to be as large as possible. (Note that $kx - \lfloor kx \rfloor]$ is the nonintegral part of the number kx.) For which k do you get values closest to the endpoints $0, 1$? Note which k give values close to 0 and which give values close to 1. Can you explain what you find? (In principle you can take x to be any irrational number, but you might have to take many more values of k before you see anything. However, if you have a computer that can do this kind of thing, you could see what happens for other x.

Problem 3.6 (cf Problem 2.4) Find the continued fraction expansions of $\sqrt{d}$ for d = 2, 3, 5, 6, 7, 8, 10. Make all possible general conjectures that are consistent with these data and that seem reasonable to you and test them against the cases $d = 11, 12, 13, 14$. Can you prove any of your conjectures?

4 Geometry of continued fractions.

Take the coordinate plane and mark all points, whose coordinates are integers. We will call such points *nodes*. Consider the line l given by the equation $y = \alpha x$. The line l passes through the origin. If $\alpha = p/q$ is rational, then the line l also passes through infinitely many other nodes, namely, through (p, q), $(2p, 2q)$, $(3p, 3q)$,... . However, if α is irrational, there are no other nodes on the line l. Nevertheless, we can always find nodes that are as close to the line l as we wish. Imagine that the line l is a string and nodes are pegs. Move the string down. The pegs it presses against are the closest nodes for the line l from below. If we move the string up, then we will get the closest nodes to l from above.

Example: Let us find the closest nodes with small coordinates in the case where $\alpha = (1 + \sqrt{5})/2$ is the golden ratio. The line l is given by the equation $y = (1 + \sqrt{5})/2x$



Theorem: *The closest nodes to the line $\{y = \alpha x\}$ are in one-to-one correspondence with the convergents of the continued fraction for α: the convergent p_n/q_n gives the node $A_n = (q_n, p_n)$. The node A_n is above the line, if n is odd, and below otherwise. In particular, there are no nodes between the line and the segment A_nA_{n+2}.*

In our example, the closest nodes correspond to the convergents 1, 2, 3/2, 5/3 and 8/5.

Problem: (a) Verify the statement of the theorem for 12/7 and 13/8 (since these numbers are rational we have only finite number of distinct convergents).

(b) Verify the statement of the theorem for the first few convergents of $\sqrt{2}$ and $e-2$.
(c) How to prove the theorem in general?

If we are given the nodes $A_0, A_1, A_2, \dots$ that are the closest to the line $\{y = \alpha x\}$, then we can find the coefficients $[a_0, a_1, a_2, a_3, \dots]$ of the continued fraction for α. Namely, a_n is the *integral* distance between the nodes A_n and A_{n+2}. To find the *integral* distance between two nodes A and B we count the number of nodes on the segment AB and subtract 1. E.g if $A = (1,1)$ and $B = (2,3)$, then the integral distance is 1. And if $A = (1,1)$, $B = (3,3)$, then the distance is 2. In our example, all integral distances between A_n and A_{n+1} are equal to 1.

Theorem: *Let $[a_0, a_1, a_2, \dots]$ be the continued fraction for α and let a_0, p_1/q_1, p_2/q_2,... be its convergents. Then a_n is the integral distance between the nodes (q_n, p_n) and (q_{n+2}, p_{n+2}).*

Problem: (a) Check this theorem for 12/7 and 13/8 and for the first few coefficients of the continued fraction for $\sqrt{2}$ and $e-2$.
(b) How to prove this theorem in the general case?

Very good approximations. We already know that the convergents of a continued fraction give the best approximations. Let $[a_0, a_1, a_2, \dots]$ be the continued fraction for a number α. Then taking $[a_0]$, $[a_o, a_1]$, $[a_0, a_1, a_2]$, ...we get better and better approximations of α. When is it reasonable to stop? Which of the convergents is accurate enough but still simple?

Example: The first few convergents of $\pi = [3; 7, 15, 1, 292, 1, 1, 1, \dots] = 3.141592654$ are:

$$\begin{aligned}
[3] &= 3, \\
[3,7] &= \frac{22}{7} = 3.142857143, \\
[3,7,15] &= \frac{333}{106} = 3.141509434, \\
[3,7,15,1] &= \frac{355}{113} = 3.141592920, \\
[3,7,15,1,292] &= \frac{103993}{33102} = 3.141592653.
\end{aligned}$$

Obviously, the approximation 355/133 is the best choice. Note that it is obtained by cutting the expansion $[3, 7, 15, 1, 292, 1, 1, 1, \dots]$ immediately before the largest (among the first eight) coefficient 292. The same is true in general. The greater the coefficient a_n, the better the approximation $[a_0, a_1, \dots, a_{n-1}]$. In particular, the golden ratio $[1, 1, 1, 1, \dots]$ has the worst possible approximations. Similarly, all quadratic irrationals like $\sqrt{2}$, $\sqrt{3}$ etc. can not be approximated very well. Indeed, their continued fractions are periodic so the coefficients $a_0, a_1, a_2, \dots$ are all bounded by the greatest element in the period.

Surprisingly, the number

$$\alpha = e^{\pi\sqrt{163}}$$

has an excellent approximation by an integer number. Moreover, even the cubic root of α is almost an integer (up to 9 decimal digits!):

$$e^{\pi\sqrt{163}/3} = 640320.0000000006\ldots$$

The continued fraction for $e^{\pi\sqrt{163}/3}$ is

$$[640320, 1653264929, 30, 1, 321, 2, 1, 1, 1, 4, 3, 4, 2, 1, 1, \ldots].$$

The second coefficient is so big that the simplest approximation 640320 is already very good.