Mecánica Cuántica

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1 Problema 4

Problem 4

Use Equations 1, 2, and 3, to construct Y_0^0 and Y_2^1. Check that they are normalized and orthogonal.

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^m P_l(x) \tag{1}$$

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \tag{2}$$

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(cos\theta) \tag{3}$$

Solución: Usando la formula $Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(cos\theta)$

$$Y_0^0(\theta,\phi) = \sqrt{\frac{(2\cdot0+1)}{4\pi}\frac{(0-0)!}{(0+0)!}}e^{i\cdot0\phi}P_0^0(cos\theta)$$
$$P_0^0(x) = (-1)^0(1-x^2)^{0/2}(\tfrac{d}{dx})^0P_0(x)$$
$$P_0(x) = \tfrac{1}{2^0\cdot0!}(\tfrac{d}{dx})^0(x^2-1)^0 = 1 \Rightarrow P_0^0 = 1\cdot1\cdot1 = 1$$
$$Y_0^0(\theta,\phi) = \sqrt{\tfrac{1}{4\pi}\cdot1\cdot1\cdot1} = \sqrt{\tfrac{1}{4\pi}}$$

Verificamos que esté normalizada

$$\int_0^\pi\int_0^{2\pi}(\sqrt{\tfrac{1}{4\pi}}\sqrt{\tfrac{1}{4\pi}}sen\theta)d\theta d\phi = \tfrac{1}{4\pi}\int_0^\pi\int_0^{2\pi}sen\theta d\theta d\phi = \tfrac{4\pi}{4\pi} = 1$$

Para el siguiente usamos la misma formula

$$Y_2^1(\theta,\phi) = \sqrt{\frac{(2\cdot2+1)}{4\pi}\frac{(2-1)!}{(2+1)!}}e^{i\phi}P_2^1(cos\theta)$$
$$P_2^1(x) = (-1)^1(1-x^2)^{1/2}(\tfrac{d}{dx})P_2(x)$$
$$P_2(x) = \tfrac{1}{2^2\cdot2!}(\tfrac{d}{dx})^2(x^2-1)^2 = \tfrac{1}{2}(3x^2-1) \Rightarrow P_2^1(x) = -(1-x^2)^{1/2}\tfrac{d}{dx}\tfrac{1}{2}(3x^2-1) = -3x\sqrt{1-x^2}$$
$$Y_2^1(\theta,\phi) = \sqrt{\tfrac{5}{4\pi}\tfrac{1}{6}}e^{i\phi}(-3cos\theta\sqrt{1-cos^2\theta}) = -\sqrt{\tfrac{15}{8\pi}}e^{i\phi}cos\theta sen\theta$$

Verificamos que este normalizada

$$\int_0^{\pi}\int_0^{2\pi}(-\sqrt{\frac{15}{8\pi}}e^{-i\phi}cos\theta sen\theta)(-\sqrt{\frac{15}{8\pi}}e^{i\phi}cos\theta sen\theta)sen\theta d\theta d\phi$$
$$\frac{15}{8\pi}\int_0^{\pi}\int_0^{2\pi} sen^3\theta cos^2\theta d\phi d\theta = \frac{15}{8\pi}2\pi\int_0^{\pi} sen^3\theta cos^2\theta d\theta$$
$$\frac{15}{4}\int_0^{\pi} sen^3\theta(1-sen^2\theta)d\theta = \frac{15}{4}(\int_0^{\pi} sen^3\theta d\theta - \int_0^{\pi} sen^5\theta d\theta)$$
$$\frac{15}{4}(\frac{cos^3\theta}{3} - cos\theta + \frac{5cos^5\theta}{5} - \frac{2cos^3\theta}{3} + cos\theta)|_0^{\pi} = \frac{15}{5}\frac{4}{15} = 1$$

Ahora verificamos que sean ortogonales entre si

$$-\int_0^{\pi}\int_0^{2\pi}(\sqrt{\frac{1}{4\pi}}\sqrt{\frac{15}{8\pi}}e^{i\phi}cos\theta sen\theta sen\theta)d\theta d\phi = -\sqrt{\frac{15}{32\pi^2}}\int_0^{2\pi} e^{i\phi}d\phi\int_0^{\pi} sen^2\theta cos\theta d\theta$$
$$-\sqrt{\frac{15}{32\pi^2}}(-ie^{i\phi})|_0^{2\pi}(\frac{sen^3\theta}{3})|_0^{\pi} = -\sqrt{\frac{15}{32\pi^2}}\cdot 0\cdot 0 = 0$$

Lo que demuestra que son ortogonales entre si

2 Problema 5

Problem 5

Show that

$$\Theta(\theta) = Aln[tan(\theta/2)]$$

satisfies the θ equation (Equation 4), for . This is the unacceptable "second solution"—what's wrong with it?

$$sin\theta\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right) + [l(l+1)sin^2\theta - m^2]\Theta = 0 \tag{4}$$

Solución: La ecuación 4 para $l = m = 0$ es

$$sen\theta\frac{d}{d\theta}(sen\theta\frac{d\Theta}{d\theta}) + (0\cdot(0+1)sen^2\theta - 0^2)\Theta = sen\theta\frac{d}{d\theta}(sen\theta\frac{d\Theta}{d\theta})$$
$$\frac{d\Theta}{d\theta} = A\frac{sec^2(\theta/2)}{2tan(\theta/2)} = \frac{A}{2}\frac{1}{cos^2(\theta/2)}\frac{cos(\theta/2)}{sen(\theta/2)}$$
$$= \frac{A}{2}\frac{1}{cos(\theta/2)sen(\theta/2)} = \frac{A}{2\frac{sen\theta}{2}} = \frac{A}{sen\theta}$$
$$sen\theta\frac{d}{d\theta}(sen\theta\frac{d\Theta}{d\theta}) = sen\theta\frac{d}{d\theta}(sen\theta\frac{A}{sen\theta}) = sen\theta\frac{dA}{d\theta} = 0$$

Lo que cumple la ecuación 4, sin embargo podemos ver que Θ diverge tanto en 0 como en π, por lo que no es una solución aceptable

3 Problema 6

Problem 6

Using Equation 5 and footnote 5, show that

$$Y_l^{-m}(\theta,\phi) = (-1)^m\,(Y_l^m)^*$$

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(cos\theta) \tag{5}$$

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^m P_l(x) \tag{6}$$

Footnote 5 : Some books (including earlier editions of this one) do not include the factor $(-1)^m$ in the definition of P_l^m. Equation 6 assumes that $m \geq 0$; for negative values we define

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$$

Solución:

La ecuación (5) nos dice que :

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(cos\theta),$$

y el pie de nota 5 nos dice que para valores negativos $(-m)$ tenemos:

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x). \tag{7}$$

Veamos la ecuación (5) para valores $(-m)$:

$$Y_l^{-m}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-(-m)!}{(l+(-m)!}}e^{i(-m)\phi}P_l^{-m}(cos\theta),$$

pero de acuerdo a la ecuación (7) tenemos que:

$$P_l^{-m}(cos\theta) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(cos\theta).$$

Por lo que:

$$Y_l^{-m}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l+m)!}{(l-m)!}}e^{i(-m)\phi}(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(cos\theta),$$

$$\Longrightarrow Y_l^{-m}(\theta,\phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l+m)!}{(l-m)!}} e^{-im\phi} \sqrt{\frac{((l-m)!)^2}{((l+m)!)^2}} P_l^m(cos\theta)$$

$$\Longrightarrow Y_l^{-m}(\theta,\phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l+m)!}{(l-m)!}\frac{((l-m)!)^2}{((l+m)!)^2}} e^{-im\phi} P_l^m(cos\theta)$$

$$\Longrightarrow Y_l^{-m}(\theta,\phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{\cancel{(l+m)!}}{\cancel{(l-m)!}}\frac{\cancel{(l-m)!}(l-m)!}{\cancel{(l+m)!}(l+m)!}} e^{-im\phi} P_l^m(cos\theta)$$

$$\Longrightarrow Y_l^{-m}(\theta,\phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} e^{-im\phi} P_l^m(cos\theta), \tag{8}$$

Ahora notemos lo siguiente:

$$(e^{im\phi})^* = e^{-im\phi}$$

por otra parte, $P_l^m(cos\theta)$ es una función real por lo que:

$$(P_l^m(cos\theta))^* = P_l^m(cos\theta),$$

y como sabemos que l es un entero no negativo entonces tenemos que $\frac{(2l+1)}{4\pi}$ es un número real y $\frac{(l-m)!}{(l+m)!}$ es el cociente de dos factoriales, los cuales son siempre números reales. Por lo que si:

$$Y_l^m = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} e^{im\phi} P_l^m(cos\theta) \Longrightarrow (Y_l^m)^* = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} e^{-im\phi} P_l^m(cos\theta)$$

Sustituimos esto en la ecuación (8):

$$Y_l^{-m}(\theta,\phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} e^{-im\phi} P_l^m(cos\theta) = (-1)^m (Y_l^m)^*$$

por lo tanto concluimos que:

$$\boxed{Y_l^{-m}(\theta,\phi) = (-1)^m \left(Y_l^m\right)^*}$$

4 Problema 7

Problem 7

Using Equation 9, find $Y_l^l(\theta,\phi)$ and $Y_3^2(\theta,\phi)$. (You can take P_3^2 from Table 4.2, but you'll have to work P_l^l out from Equations 10 and 11.) Check that they satisfy the angular equation (Equation 12), for the appropriate values of l and m.

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(cos\theta) \tag{9}$$

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^m P_l(x) \tag{10}$$

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \tag{11}$$

$$sin\theta\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)sin^2\theta Y \tag{12}$$

Solución:

Usando la ecuación 9

$$Y_l^l(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-l)!}{(l+l)!}}e^{il\phi}P_l^l(cos\theta) = \sqrt{\frac{(2l+1)}{4\pi(2l)!}}e^{il\phi}P_l^l(cos\theta)$$

En donde al usar la Ecuación 10

$$P_l^l(x) = (-1)^l(1-x^2)^{l/2}\left(\frac{d}{dx}\right)^l P_l(x) = (-1)^l(1-x^2)^{l/2}\left(\frac{d}{dx}\right)^l\left[\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l(x^2-1)^l\right]$$

$$P_l^l(x) = (-1)^l(1-x^2)^{l/2}\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^{2l}\left[(x^2-1)^l\right]$$

Veamos como se comporta esta derivada

$$l=0\ ;\ \left(\frac{d}{dx}\right)^0(x^2-1)^0=(2(0))!$$
$$l=1\ ;\ \frac{d^2}{dx^2}(x^2-1)=2=(2(1))!$$
$$l=2\ ;\ \frac{d^4}{dx^4}(x^2-1)^2=24=(2(2))!$$
$$l=3\ ;\ \frac{d^6}{dx^6}(x^2-1)^3=720=(2(3))!$$

Así

$$\frac{d^{2l}}{dx^{2l}}(x^2-1)^l=(2l)!$$

Por lo tanto

$$P_l^l(x)=(-1)^l(1-x^2)^{l/2}\frac{1}{2^l l!}(2l)!$$

Hemos calculado que

$$Y_l^l(\theta,\phi)=\sqrt{\frac{(2l+1)}{4\pi(2l)!}}e^{il\phi}(-1)^l(1-cos^2\theta)^{l/2}\frac{(2l)!}{2^l l!}=\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)(2l)!}{4\pi}}e^{il\phi}(sen^2\theta)^{l/2}$$

$$Y_l^l(\theta,\phi)=\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}e^{il\phi}sen^l\theta=Be^{il\phi}sen^l\theta$$

En donde

$$B=\frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}$$

Veamos que cumple la ecuación angular, calcularemos las derivadas:

$$\frac{\partial}{\partial\phi}Y_l^l=Bsen^l\theta(il)e^{il\phi}\implies\frac{\partial^2}{\partial\phi^2}Y_l^l=(il)^2Y_l^l$$

$$sin\theta\frac{\partial Y_l^l}{\partial\theta}=lcos\theta sin\theta Be^{il\phi}sen^{l-1}\theta=cos\theta lBe^{il\phi}sen^l\theta=lcos\theta Y_l^l$$

$$\implies sin\theta\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial Y_l^l}{\partial\theta}\right)=sin\theta\left[lcos\theta\frac{\partial Y_l^l}{\partial\theta}-lY_1^1sin\theta\right]=\left[lcos\theta sin\theta\frac{\partial Y_l^l}{\partial\theta}-lY_1^1sin^2\theta\right]$$

$$\implies sin\theta\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial Y_1^1}{\partial\theta}\right)=\left[l^2cos^2\theta-lsin^2\theta\right]Y_1^1$$

Sustituimos los valores en la ecuación:

$$\left[l^2cos^2\theta-lsin^2\theta\right]Y_l^l-l^2Y_l^l=Y_l^l\left[l^2(cos^2\theta-1)-lsin^2\theta\right]$$

$$Y_1^1\Bigg[-l^2sin^2\theta - lsin^2\theta\Bigg] = -Y_1^1 sin^2\theta(l^2+l) = -l(l+1)sin^2\theta Y_1^1$$

Por lo tanto, vemos que sí es solución de la ecuación angular

$$sin\theta\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial Y_1^1}{\partial\theta}\right) + \frac{\partial^2 Y_1^1}{\partial\phi^2} = -l(l+1)sin^2\theta Y_1^1$$

Ahora, vamos a calcular $Y_3^2(\theta,\phi)$ y veremos que se satisfaga la ecuación angular.

$$Y_3^2(\theta,\phi) = \sqrt{\frac{(2(3)+1)}{4\pi}\frac{(3-2)!}{(3+2)!}}e^{i2\phi}P_3^2(cos\theta) = \sqrt{\frac{7}{4\pi}\frac{1}{5!}}e^{i2\phi}P_3^2(cos\theta)$$

$$Y_3^2(\theta,\phi) = \sqrt{\frac{7}{480\pi}}e^{i2\phi}P_3^2(cos\theta)$$

Calculamos $P_3^2(cos\theta)$

$$P_3^2(cos\theta) = (-1)^2(1-cos^2\theta)^{2/2}\frac{d^2}{d\theta^2}P_3(cos\theta) = (1-cos^2\theta)\frac{d^2}{d\theta^2}[P_3(cos\theta)]$$

En donde $P_3(cos\theta)$ se calcula como sigue:

$$P_3(cos\theta) = \frac{1}{2^3 3!}\frac{d^3}{d\theta^3}(cos^2\theta-1)^3 = \frac{1}{2^3 3!}\frac{d^3}{d\theta^3}((1-sen^2\theta)-1)^3 = -\frac{1}{2^3 3!}\frac{d^3}{d\theta^3}sen^6\theta$$

$$\Longrightarrow P_3(cos\theta) = -\frac{1}{2^3 3!}\frac{d^2}{d\theta^2}(6sen^5\theta cos\theta) = -\frac{6}{2^3 3!}\frac{d}{d\theta}(-sen^6\theta + 5sen^4\theta cos^2\theta)$$

$$\Longrightarrow P_3(cos\theta) = -\frac{6}{2^3 3!}[-6sen^5\theta cos\theta + 5(4sen^3\theta cos^3\theta - 2cos\theta sen^5\theta)]$$

$$\Longrightarrow P_3(cos\theta) = -\frac{6}{2^3 3!}[-16sen^5\theta cos\theta + 20sen^3\theta cos^3\theta]$$

$$\Longrightarrow P_3(cos\theta) = -\frac{6}{8(6)}[-16sen^5\theta cos\theta + 20sen^3\theta cos^3\theta] = 2sen^5\theta cos\theta - \frac{10}{4}sen^3\theta cos^3\theta$$

Así, sustituimos en la siguiente ecuación:

$$P_3^2(cos\theta) = (1-cos^2\theta)\frac{d^2}{d\theta^2}[P_3(cos\theta)] = sen^2\theta\frac{d^2}{d\theta^2}[2sen^5\theta cos\theta - \frac{10}{4}sen^3\theta cos^3\theta]$$

$$\Longrightarrow P_3^2(cos\theta) = sen^2\theta\frac{d}{d\theta}[2(5sen^4\theta cos^2\theta - sen^6\theta) - \frac{10}{4}(3sen^2\theta cos^4\theta - 3sen^4\theta cos^2\theta)]$$

$$\Longrightarrow P_3^2(cos\theta) = sen^2\theta\frac{d}{d\theta}[10sen^4\theta cos^2\theta - 2sen^6\theta - \frac{30}{4}sen^2\theta cos^4\theta + \frac{30}{4}sen^4\theta cos^2\theta]$$

$$\Longrightarrow P_3^2(cos\theta) = sen^2\theta\frac{d}{d\theta}\left[\frac{70}{4}sen^4\theta cos^2\theta - 2sen^6\theta - \frac{30}{4}sen^2\theta cos^4\theta\right]$$

$$\implies P_3^2(cos\theta) = sen^2\theta\Bigg[\frac{70}{4}(4sen^3cos^3\theta - 2cos\theta sen^5\theta) + 12sen^5\theta cos\theta$$

$$-\frac{30}{4}(4cos^3\theta sen^3\theta - 2sen\theta cos^5\theta)\Bigg]$$

$$P_3^2(cos\theta) = sen^2\theta\Bigg[40sen^3\theta cos^3\theta - 23cos\theta sen^5\theta + 15sen\theta cos^5\theta\Bigg]$$

$$P_3^2(cos\theta) = 40sen^5\theta cos^3\theta - 23sen^7\theta cos\theta + 15sen^3\theta cos^5\theta$$

Así finalmente, el armónico esferico resulta ser:

$$Y_3^2(\theta,\phi) = \sqrt{\frac{7}{480\pi}}e^{i2\phi}P_3^2(cos\theta)$$

$$\implies Y_3^2(\theta,\phi) = \sqrt{\frac{7}{480\pi}}e^{i2\phi}[40sen^5\theta cos^3\theta - 23sen^7\theta cos\theta + 15sen^3\theta cos^5\theta]$$

$$Y_3^2(\theta,\phi) = B[40sen^5\theta cos^3\theta - 23sen^7\theta cos\theta + 15sen^3\theta cos^5\theta]$$

En donde

$$B = \sqrt{\frac{7}{480\pi}}e^{i2\phi}$$

Ahora, vamos a corroborar que satisface la ecuación de ángulo.

$$\frac{\partial}{\partial\phi}Y_3^2 = 2iY_3^2$$

$$\implies \frac{\partial^2}{\partial\phi^2}Y_3^2 = 2i\frac{\partial}{\partial\phi}Y_3^2 = 2i(2iY_3^2) = -4Y_3^2$$

Calculamos la derivada con respecto a θ

$$\frac{\partial}{\partial\theta}Y_3^2 = B[40(5sen^4\theta cos^4\theta - 3cos^2\theta sen^6\theta) - 23(7sen^6\theta cos^2\theta - sen^8\theta)$$

$$+15(3sen^2\theta cos^6\theta - 5cos^4\theta sen^4\theta)]$$

$$\implies \frac{\partial}{\partial\theta}Y_3^2 = B[125sen^4\theta cos^4\theta - 281cos^2\theta sen^6\theta + 23sen^8\theta + 45sen^2\theta cos^6\theta]$$

$$\implies \frac{\partial}{\partial\theta}Y_3^2 = B[125sen^4\theta cos^4\theta - 281cos^2\theta sen^6\theta + 23sen^8\theta + 45sen^2\theta cos^6\theta]$$

Usando que $sen^2\theta + cos^2\theta = 1 \implies cos^2\theta = 1 - sen^2\theta$

$$\implies \frac{\partial}{\partial\theta}Y_3^2 = B[125sen^4\theta(cos^2\theta)^2 - 281cos^2\theta sen^6\theta + 23sen^8\theta + 45sen^2\theta(cos^2\theta)^3]$$

$$\implies \frac{\partial}{\partial\theta}Y_3^2 = B[125sen^4\theta(1 - sen^2\theta)^2 - 281(1 - sen^2\theta)sen^6\theta + 23sen^8\theta$$

$$+45sen^2\theta(1-sen^2\theta)^3]$$

$$\implies \frac{\partial}{\partial\theta}Y_3^2 = B[125sen^4\theta(1-2sen^2\theta+sen^4\theta)-281(1-sen^2\theta)sen^6\theta+23sen^8\theta$$

$$+45sen^2\theta(1-3sen^2\theta+3sen^4\theta-sen^6\theta)]$$

$$\implies \frac{\partial}{\partial\theta}Y_3^2 = B\big(45\sin^2\theta-10\sin^4\theta-396\sin^6\theta+384\sin^8\theta\big)$$

$$\implies sen\theta\frac{\partial}{\partial\theta}Y_3^2 = B\big(45\sin^3\theta-10\sin^5\theta-396\sin^7\theta+384\sin^9\theta\big)$$

$$\implies \frac{\partial}{\partial\theta}\left(sen\theta\frac{\partial}{\partial\theta}Y_3^2\right) = B\big(135\sin^2\theta-50\sin^4\theta-2583\sin^6\theta+3456\sin^8\theta\big)$$

$$\implies sen\theta\frac{\partial}{\partial\theta}\left(sen\theta\frac{\partial}{\partial\theta}Y_3^2\right) = B\big(135\sin^3\theta-50\sin^5\theta-2583\sin^7\theta+3456\sin^9\theta\big)$$

Con todo esto, podemos sustituir

$$sen\theta\frac{\partial}{\partial\theta}\left(sen\theta\frac{\partial}{\partial\theta}Y_3^2\right)+\frac{\partial^2}{\partial\phi^2}Y_3^2 = sen\theta\frac{\partial}{\partial\theta}\left(sen\theta\frac{\partial}{\partial\theta}Y_3^2\right)-4Y_3^2$$

Ahora, como:

$$Y_3^2 = B[40sen^5\theta cos^3\theta-23sen^7\theta cos\theta+15sen^3\theta cos^5\theta]$$

$$\implies Y_3^3 = [40sen^5\theta cos^3\theta-23sen^7\theta cos\theta+15sen^3\theta cos^5\theta]$$

5 Problema 8

Problem 8

Starting from the Rodrigues formula, derive the orthonormality condition for Legendre polynomials:

$$\int_{-1}^{1}P_l(x)P_{l'}(x)dx = \left(\frac{2}{2l+1}\right)\delta_{ll'}$$

Hint: Use integration by parts

Solución:

6 Problema 9

Problem 9

a) From the definition (Equation 13), construct $n_1(x)$ and $n_2(x)$.

b) Expand the sines and cosines to obtain approximate formulas for $n_1(x)$ and $n_2(x)$, valid when $x << 1$. Confirm that they blow up at the origin.

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right) \frac{sinx}{x} \ ; \ n_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{cosx}{x} \tag{13}$$

Solución:

Inciso a)

Usando la ecuación 13 calculamos $n_1(x)$

$$n_1(x) = -(-x)^1 \left(\frac{1}{x}\frac{d}{dx}\right)^1 \frac{cosx}{x}$$
$$n_1(x) = x\frac{1}{x}\frac{d}{dx}\left(\frac{cosx}{x}\right) = \frac{d}{dx}\left(\frac{cosx}{x}\right)$$
$$n_1(x) = \frac{-xsenx - cosx}{x^2}$$

Ahora, calculamos $n_2(x)$

$$n_2(x) = -(-x)^2 \left(\frac{1}{x}\frac{d}{dx}\right)^2 \frac{cosx}{x}$$
$$n_2(x) = -x^2\frac{1}{x^2}\frac{d^2}{dx^2}\left(\frac{cosx}{x}\right) = -\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{cosx}{x}\right)\right]$$
$$n_2(x) = -\frac{d}{dx}\left[\frac{-xsenx - cosx}{x^2}\right]$$
$$n_2(x) = -\frac{(-xcosx)x^2 - 2x(-xsenx - cosx)}{x^4}$$
$$n_2(x) = \frac{x^3cosx - 2x^2senx - 2xcosx}{x^4}$$
$$n_2(x) = \frac{x^2cosx - 2xsenx - 2cosx}{x^3}$$

Inciso b)

Sabemos que la expansión del coseno, resulta ser

$$cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Así

$$\frac{cos(x)}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k)!}$$

Con esto, podemos calcular $n_1(x)$ y $n_2(x)$

$$n_1(x) = -(-x)^1\left(\frac{1}{x}\frac{d}{dx}\right)^1 \frac{cosx}{x} = \frac{d}{dx}\left(\frac{cosx}{x}\right) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k)!}\right)$$

$$n_1(x) = \sum_{k=0}^{\infty}(2k-1)\frac{(-1)^k x^{2k-2}}{(2k)!}$$

$$n_2(x) = -(-x)^2\left(\frac{1}{x}\frac{d}{dx}\right)^2 \frac{cosx}{x} = -\frac{d^2}{dx^2}\left(\frac{cosx}{x}\right) = -\frac{d^2}{dx^2}\left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k)!}\right)$$

$$n_2(x) = \sum_{k=0}^{\infty}(1-2k)(2k-2)\frac{(-1)^k x^{2k-3}}{(2k)!}$$

Expandiendo la serie, tenemos lo siguiente:

$$n_1(x) \approx (-1)\frac{x^{-2}}{0!} + (1)\frac{(-1)^1 x^0}{2!} + (3)\frac{(-1)^2 x^2}{4!} + (5)\frac{(-1)^3 x^4}{6!} + (7)\frac{(-1)^4 x^6}{8!}$$

$$n_1(x) \approx -\frac{1}{x^2} - \frac{1}{2} + \frac{3x^2}{24} - \frac{5x^4}{720} + \frac{7x^6}{40320}$$

$$n_2(x) \approx (1)(-2)\frac{(-1)^0 x^{-3}}{0!} + (-3)(2)\frac{(-1)^2 x^1}{4!} + (-5)(4)\frac{(-1)^3 x^3}{6!} + (-7)(6)\frac{(-1)^4 x^5}{8!}$$

$$n_2(x) \approx -\frac{2}{x^3} - \frac{6x}{24} + \frac{20x^3}{720} - \frac{42x^5}{40320}$$

Puesto que $x << 1$, los términos que van a dominar en la aproximación resultan ser:

$$n_1(x) \approx -\frac{1}{x^2} - \frac{1}{2} \implies lim_{x\to 0} n_1(x) = -\infty$$

$$n_2(x) \approx -\frac{2}{x^3} \implies lim_{x\to 0} n_2(x) = -\infty$$

7 Problema 10

Problem 10

a) Check that $Arj_1(kr)$ satisfies the radial equation with $V(r) = 0$ and $l = 1$.

b) Determine graphically the allowed energies for the infinite spherical well, when $l = 1$. Show that for large N, $E_{N1} \approx (\hbar^2\pi^2/2ma^2)(N+1/2)^2$. Hint: First show that $j_1(x) = 0 \implies x = tanx$. Plot x and $tanx$ on the same graph, and locate the points of intersection.

Solución:

Inciso a)

Tenemos que la función para $l = 1$

$$u_1 = Arj_1(x) = Ar(-x)^1\left(\frac{1}{x}\frac{d}{dx}\right)^1\frac{sen(x)}{x} = (-Ar)\frac{cos(x)x - sen(x)}{(x)^2} = Ar\frac{sen(x) - xcos(x)}{x^2}$$

Tomamos $x = kr$

$$u_1 = Ar\frac{sen(kr) - krcos(kr)}{(kr)^2} = A\left[\frac{sen(kr)}{k^2r} - \frac{cos(kr)}{k}\right] = \frac{A}{kr}\left[\frac{sen(kr)}{k} - cos(kr)r\right]$$

La ecuación radial para $V(r) = 0$ es

$$\frac{d^2u_1}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2}\right]u_1 = 0$$

Calculamos las derivadas

$$\frac{du_1}{dr} = -\frac{A}{kr^2}\left[\frac{sen(kr)}{k} - cos(kr)r\right] + A\left[sen(kr)\right]$$

$$\frac{d^2u_1}{dr^2} = -\frac{A}{kr^2}\left[sen(kr)kr\right] + \frac{2A}{kr^3}\left[\frac{sen(kr)}{k} - cos(kr)r\right] + Acos(kr)k$$

$$\frac{d^2u_1}{dr^2} = \frac{A}{kr}\left[-sen(kr)k + k^2rcos(kr)\right] + \frac{2A}{kr^3}\left[\frac{sen(kr)}{k} - cos(kr)r\right]$$

$$\left[k^2 - \frac{l(l+1)}{r^2}\right]u_1 = \frac{Ak}{r}\left[\frac{sen(kr)}{k} - cos(kr)r\right] - \frac{l(l+1)}{r^2}\frac{A}{kr}\left[\frac{sen(kr)}{k} - cos(kr)r\right]$$

Como estamos en el caso en que $l = 1$

$$\left[k^2 - \frac{l(l+1)}{r^2}\right]u_1 = \frac{A}{kr}\left[sen(kr)k - cos(kr)k^2 r\right] - \frac{2A}{kr^3}\left[\frac{sen(kr)}{k} - cos(kr)r\right]$$

Así, sustituyendo las derivadas en la ecuación

$$\frac{d^2u_1}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2}\right]u_1 = \frac{A}{kr}\left[-sen(kr)k + k^2 r cos(kr)\right] + \frac{2A}{kr^3}\left[\frac{sen(kr)}{k} - cos(kr)r\right]$$

$$+\frac{A}{kr}\left[sen(kr)k - cos(kr)k^2 r\right] - \frac{2A}{kr^3}\left[\frac{sen(kr)}{k} - cos(kr)r\right] = 0$$

Por lo tanto se comprueba que la función es solución a la ecuación radial con $V(r) = 0$ y $l = 1$

Inciso b)

Sabemos que: $Aaj_1(ka) = 0 \implies j_1(ka) = 0$

$$\implies \frac{sen(ka) - (ka)cos(ka)}{(ka)^2} = 0$$

$$\implies tan(ka) - ka = 0$$

$$\implies tan(ka) = ka$$

o bien, como $x = ka$. Llegamos a la ecuación trascendental

$$tan(x) = x$$

La cual graficamos en la siguiente Figura **??**:

Notamos que para valores grandes de N, los valores en donde la función $tan(x)$ y la función x son iguales, estos se aproximan a $x = \frac{\pi}{2} + N\pi$ que es donde la función $tan(x)$ diverge.

Ahora como $x = ka$, para que la ecuación trascendental sea válida, se tiene que

$$ka = \frac{\pi}{2} + N\pi$$

$$a\sqrt{\frac{2mE_N}{\hbar^2}} = \frac{\pi}{2} + N\pi$$

$$\frac{2mE_N}{\hbar^2} = a^{-2}\pi^2(N + 1/2)^2$$

$$E_N = \frac{\hbar^2\pi^2}{2ma^2}(N + 1/2)^2$$

8 Problema 11

Problem 11

A particle of mass m is placed in a finite spherical well:

$V(r) = \begin{cases} -V_0. & r \leq a; \\ 0. & r > a \end{cases}$ Find the ground state, by solving the radial equation with $l = 0$. Show that there is no bound state if $V_0 a^2 < \pi^2 \hbar^2 / 8m$.

Solución: