Christian-Albrecht-University of Kiel

Faculty of Business, Economics and Social Sciences

Shrinkage Estimation of the Covariance Matrix for Portfolio Optimization

Master's thesis in MSc Quantitative Finance

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Abstract

This study investigates the efficiency of shrinkage techniques in covariance matrix estimation within portfolio optimization. Utilizing data from the Deutscher Aktienindex (DAX) 40 stocks, the study compares five covariance matrix estimators; the sample covariance matrix, single index, and common correlation models, alongside their shrinkage estimators. The focus is on their ability to construct the Global Minimum Variance Portfolio (GMVP) and their accuracy in forecasting portfolio variance. Empirical analysis indicates some advantages of shrinkage techniques in optimizing portfolios, such as reduced volatility and increased returns, although these benefits are not markedly significant. Nevertheless, the evaluative tests on forecasting portfolio variance confirm the effectiveness of shrinkage estimators as more accurate predictive models.

Contents

List of Tables

List of Figures

1 Introduction

Modern portfolio management has been significantly influenced by Markowitz (1952) revolutionary Expected Mean-Variance (EV) optimization approach. Although Markowitz's portfolio theory has received lots of criticism,[1] this honored framework is widely recognized for constructing portfolios that balances risk and return, with substantial dependence on input parameters such as stock returns and the covariance matrix estimation.

The sample covariance matrix, denoted $\hat{\Sigma}$, is not only a measure of the relationships between stock returns and their co-movements, but Markowitz (1952, p.77) also pointed out investors should consider "variance of return an undesirable thing". Inadvertently projecting that variance might be equated with risk, but while they aren't entirely the same, investors should be wary of it.

Moreover, numerous researchers have noted that these input parameters are susceptible to estimation errors due to problems such as sensitivity to noise, leading to suboptimal portfolio allocations (see for example, Merton (1980), Michaud (1989), Best and Grauer (1991), Britten-Jones (1999) [2] among others). Additionally, in high-dimensional settings, where the number of stocks N, exceeds the number of observations T, more problems arise (see for example, Kondor et al. (2007), Fan et al. (2008), Pantaleo et al. (2011)).

Reliance on the $\hat{\Sigma}$ for EV portfolio optimization, will amplify the inherrent errors, as Michaud (1989, p.33) highlighted, coining it estimation-error maximizing. Nevertheless, researchers have developed techniques, to address these challenges or at least lessen the impact of the estimation errors (see for example, Black and Litterman (1990, 1992), Ledoit and Wolf (2003, 2004,

[1] Cf. for instance, Vincent (2011, pp.5-7), Michaud (1989, pp.33-36).

[2] For instance Britten-Jones (1999, p.665) found significant sampling errors in EV efficient portfolio weight estimates. Largely due from suboptimal inputs.

2017) , DeMiguel et al. (2009) among others).

Ledoit and Wolf (2003, 2004) seminal work set a benchmark in the realm of covariance matrix shrinkage. This thesis, drawing inspiration from their approaches, aims to carry out a comparative analysis of different techniques used to estimate the covariance matrix, which is a crucial component in constructing Markowitz's EV efficient portfolios, where only expected returns and covariances are considered.

We centre our examination on the Deutscher Aktienindex (DAX) 40 stocks to form portfolios. We consider five estimators of the covariance matrix and construct their respective Global Minimum Variance Portfolio (GMVP):

- the sample estimator, a direct and unbiased estimator;
- the single index model (SI), which relates individual stock returns to an index;
- the common correlation model (CC), operates under the assumption that all asset pairs in the portfolio share the same correlation;
- a shrinkage estimator with SI as shrinkage target;
- a shrinkage estimator with CC as shrinkage target.

Subsequently, we conduct an evaluation of how well these covariance matrix estimators perform in forecasting the variance of a GMVP. The thesis extends to understand techniques suited for a continuously evolving financial market, shedding light on their practical relevance in real-world portfolio selection.

The remainder of this thesis is structured as follows: Chapter 2 provides a concise literature review of portfolio selection and covariance shrinkage techniques. Chapter 3 delves into the specifications of the competing covariance estimators employed in the portfolio selection problem addressed in this thesis, along with the method for evaluating their variance forecasts. Chapter 4 showcases the empirical results, while conclusions are drawn in Chapter 5.

2 Literature Review

In this chapter, we provide a concise review of Markowitz's portfolio selection and covariance matrix shrinkage estimation techniques, highlighting their relevance and limitations.

2.1 Review of Portfolio Selection

Portfolio selection, as defined by Goldfarb and Iyengar (2003, p.1), is the problem of allocating capital to assets such that return on the investment is maximized, while the risk minimized; this emphasizes the importance of returns and risk as central elements in the process of selecting a portfolio.

As eloquently stated by Markowitz (1952, p.77), the process of selecting a portfolio begins with forming beliefs from observations and concludes with the choice of portfolio. His concept highlights the linkage between proper information gathering, quantifiable metrics and the qualitative judgments required for appropriate portfolio selection.

Fabozzi et al. (2002, p.8) diagrammatically depicted the Markowitz portfolio selection process as shown in Figure 1. This structured representation captures the essence of Markowitz's framework, culminating in the creation of the EV efficient frontier that leads to the selection of an optimal portfolio best aligned with an investor's objectives.

Markowitz's portfolio selection has been profoundly influential. As acknowledged by Rubinstein (2002, p.1044); his concepts have been generalized and refined in countless ways and have become so deeply embedded in financial economics that they are now inseparable. He continued, not only has his pioneering effort paved the way for more sophisticated theories on the impact of risk on valuation, but it has also been practically applied in managing the portfolios of everyday investors.

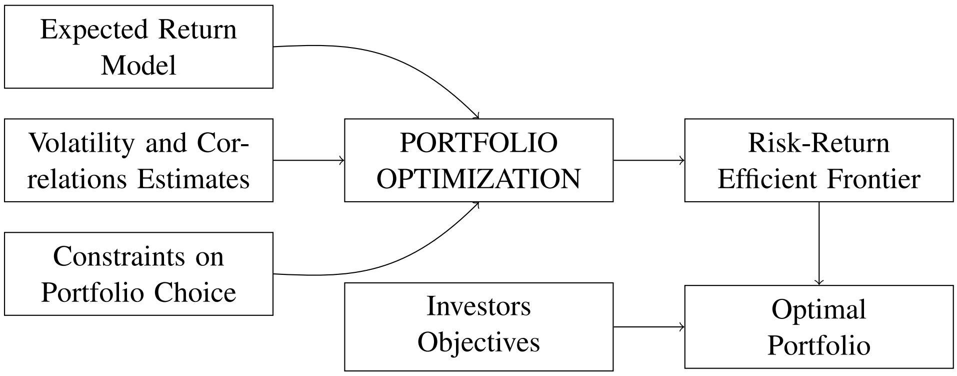

Figure 1: The Markowitz portfolio selection process depicting the pathway from primary financial metrics through optimization, leading to an optimal portfolio choice influenced by investor goals. Source: Fabozzi et al. (2002, p.8)

Moreover, Markowitz's framework, which preaches diversification, enables the reduction of risk. This principle's importance was emphasised by the financial crisis of 2008, highlighting the need for accurate risk assessment and diversification.

Markowitz's framework, while groundbreaking, is not without its limitations. Michaud (1989, p.34) noted that Markowitz's framework neglects fundamental investment considerations, such as the proportion of a company's market capital represented in portfolio holdings. The framework produces unique portfolios, but the uniqueness depends on the erroneous assumptions that the inputs are without statistical estimation errors (Michaud (1989, p.35)).

Furthermore Vincent (2011, p.5), pointed out while Markowitz's framework is grounded in robust theoretical and mathematical constructions, it demonstrates optimal performance primarily in environments that can be controlled. This observation however emphasizes the need for continuous validation and adjustment of these models in the face of ever-evolving financial market.

Additionally, Avramov and Zhou (2010, p.44) emphasized that "In making portfolio decisions, investors often encounter parameter estimation errors and potential model uncertainty." In general, these findings highlight how challenging it is to choose a portfolio and how important it is to employ cutting-edge techniques to address these fundamental problems.

From a technical standpoint, Markowitz already implied the input parameters, could be subject to estimation errors. He explained that initial statistical calculations should provide a provisional set of expected means and covariances. Subsequently, expert judgment should refine these values, considering factors that the computational methods might overlook (Markowitz (1952, p.92)).

Across time techniques such as shrinking have been introduced, to refine these values and mitigate issues hindering the selection of an optimal portfolio.

2.2 Review of Covariance Matrix Shrinkage

The covariance matrix estimation is central in extracting meaningful relationships from complex multivariate data. Lately, there's been a surge of innovative techniques and revolutionary findings in this area. These advancements span multiple disciplines such as Finance and Statistics (e.g., Tu and Zhou (2011), Ledoit and Wolf (2017, 2022), Machine Learning (e.g., Fan et al. (2018), Sihag et al. (2022)), Signal Processing (e.g., Olias et al. (2019)) among others.

Fan et al. (2018) for instance recognized the importance of accurate covariance matrices in machine learning and also highlighted the challenges posed by errors that impede accurate determination. To address this, they introduced an innovative technique, which they described as perturbation bound using the $l\infty$ norm specifically for the eigenvectors of low-rank and incoherent matrices. Building upon this, the authors subsequently proposed a new robust covariance estimator, especially designed for random variables with heavy tails.

The sample covariance matrix $\hat{\Sigma}$ is simple to compute and as aforementioned, it is however subject to errors. It's worth noting that $\hat{\Sigma}$ is the maximum likelihood estimate of the true and unknown covariance matrix, denoted Σ, derived under assumptions such as multivariate normality. Many empirical studies have shown that stock market returns are non-normal and are rather characterised by fat tails and asymmetry,[3] thus estimation errors in the covariance matrix will be further aggravated, when the normality assumption breakdown.

Although $\hat{\Sigma}$ is unbiased, it has high variance which makes it compelling to combine it with an estimator, known as the shrinkage target, that has nonzero bias but low variance (Coqueret and Milhau (2014)). This combination is realized through a technique called shrinkage.

Shrinkage estimation technique trace back to James and Stein (1961), with early applications in portfolio selection by Jobson (1979), Frost and Savarino (1986), Chopra et al. (1993) among others. For example Chopra et al. (1993) investigated three naive applications of Stein estimation using six individual country stock indexes. They concluded that portfolios with adjusted inputs yield higher mean returns, lower variance, and greater terminal wealth compared to those with unadjusted inputs in EV portfolios.

Shrinking $\hat{\Sigma}$ towards a shrinkage target has been a significant advancement in the field, with Ledoit and Wolf (2003, 2004), foundational contributions been well-known for their revolutionary work in this area. Their method basically addresses inaccuracies found in extreme coefficient estimates in the $\hat{\Sigma}$.

Specifically, they noted that extremely high coefficients in the $\hat{\Sigma}$ tend to be overestimated, necessitating an adjustment downwards. Likewise, when estimates are notably low, they typically harbor negative biases and thus necessitate an upward adjustment. This phenomenon is referred to as the "shrinkage of the extremes toward the centre" as described by Ledoit and Wolf (2004, p.111).

Following their exploration of sample covariance challenges, Ledoit and Wolf (2004, p.111) claimed that with the right implementation, shrinkage can effectively address the underlying problems in the $\hat{\Sigma}$. Their insights highlight the dual importance of execution and the selection of a structured shrinkage

[3]Cf. for instance Cont (2001, pp.224-232) for more on stylized facts.

target. Prominent examples of shrinkage targets include the previously mentioned single index model and the common correlation model. [4]

The single index model of Sharpe (1963), also called single-factor model, simplifies the covariance matrix estimation by relating each stock's return to a factor. In many contexts, especially those with high dimensional data, the $\hat{\Sigma}$ is not a favoured estimator due to its lack of structure as Ledoit and Wolf (2003 p.604) emphasized. They proposed imposing a specific structure on the estimator to rectify the situation. However, they acknowledged the accompanying challenges:

- How much structure should we impose?
- And what factors should we use?

In search of appropriate answers, Ledoit and Wolf explored alternatives to this conventional structure. One such approach they discussed involves integrating factor structure by leveraging a weighted average of the $\hat{\Sigma}$ and Sharpe (1963) single-factor model estimator, denoted $\hat{F}^{CC}$, which inherently carries more structure.

In their technique, the weight $\alpha \in [0, 1]$ holds significant importance. As noted by Ledoit and Wolf (2003, p.604), "the heavier the weight, the stronger the structure." This clever approach by Ledoit and Wolf has left an indelible mark on the field, blending precision with elegance.

Moreover Brandt (2010, p.304) highlighted that a major advantage of this technique is its ability to reduce the dimensionality of the portfolio problem to 3N+1 parameters. By estimating fewer parameters, there's a reduced risk of overfitting

According to Ledoit and Wolf (2003, p.614), the resulting shrinkage estimator, denoted $\hat{\Sigma}^{SF}$, which is also a weighted average of two positive semidefinite matrices; is invertible and also inherits the good-conditioning of the

[4] For more examples, one can refer to Coqueret and Milhau (2014)

single factor model estimator, thereby avoiding the ill-conditioning issues frequently associated with the $\hat{\Sigma}$. This innovation not only enhances the estimator's reliability but also positions it as a compelling alternative for portfolio optimization in varied financial market conditions.

The covariance matrix, implied by the single factor model, though innovative, may not capture the full complexity of financial market behavior. The limitation is that relying on a single factor might not encompass all covariation among assets, potentially resulting in not just a biased, but systematically biased estimate of the covariance matrix (Brandt (2010, p.304))

Equally, Fama and French (1993) contend that the return dynamics of stocks and bonds are influenced by multiple risk factors, not just a singular market factor. The argument challenges the single factor model's foundation, suggesting it oversimplifies financial market complexities. This limitation may hinder its efficiency as an ideal shrinkage target.

Shrinkage intensity α, introduced earlier as the weight, plays a pivotal role in the realization of the shrinkage estimator. As Ledoit and Wolf (2004, p.113) explained, any choice within its range [0,1] represents a compromise between the sample covariance matrix and the shrinkage target. Essentially, there exist an optimal intensity level, denoted $\hat{\alpha}^*$,[5] which minimizes the distance between the sample covariance matrix and its shrinkage target.

The second shrinkage target is the common correlation matrix, denoted $\hat{F}^{CC}$. Introduced by Elton and Gruber (1973), the common correlation is structured around the principle that all assets exhibit consistent pairwise correlations. By integrating this uniform correlation with individual variances, one can derive the covariance matrix. The idea of shrinking the sample covariance matrix towards the common correlation matrix, is attributed to Ledoit and Wolf (2004).

According to Ledoit and Wolf (2004, p.113), the common correlation model which is easy to estimate and implement, it also meets the two main criteria

[5] See Appendix 1 for the formula to derive optimal shrinkage intensity: $\hat{\alpha}^*$.

essential for an effective shrinkage target. Firstly, It operates with a minimal number of parameters $N+1$, and it accurately mirrors the traits of the unknown covariance matrix. A reduced parameter not only simplifies the model but, crucially, also lowers the risk of overfitting.

In the study by Ledoit and Wolf (2004, p.113), the common correlation model, when integrated with the sample covariance matrix, consistently outperformed other models in various scenarios. Specifically, the resulting shrinkage estimator, denoted $\hat{\Sigma}^{CC}$, produced the highest mean return and the most significant standard deviation of returns. Additionally, Ledoit and Wolf concluded that this model is particularly more effective than other shrinkage estimators when dealing with stocks equal to or fewer than 100.

While the common correlation model demonstrates considerable relevance in various scenarios, it's essential to consider its restrictions. Particularly, when dealing with stocks beyond threshold of 100, the integrated model might not maintain its level of precision and reliability. Coqueret and Milhau (2014) noted that the realism of the model is plausible within a single asset class, but becomes questionable for portfolios that span multiple asset classes. Given these considerations, the shrinkage estimator's effectiveness might be adversely impacted in a multi-asset portfolio context, potentially rendering it less suitable for such portfolios.

Despite the challenges and limitations in the targets introduced by Ledoit and Wolf, their innovative approach to shrinking the covariance matrix has significantly shaped portfolio optimization techniques. Coqueret and Milhau (2014) pointed out, while subsequent researchers have expanded upon Ledoit and Wolf's foundational concepts, there's yet to be a decisive study asserting the clear advantage of these improvements in portfolio optimization.

3 Methodology and Data

In this chapter, we detail the specifications of the covariance estimators under consideration. We then describe the dataset upon which our analysis is anchored and outline the method adopted to assess the efficiency of these estimators.

3.1 Competing Covariance Estimators

In this section, the assumptions and characteristics of the competing covariance matrices are detailed, largely followed from the work of Ledoit and Wolf (2003, pp.606-617).

Let $\mathbf{R}$ be a $T \times N$ matrix of stock returns, where R_{it} represents the return of asset i, for $i = 1, \ldots, N$, and $t = 1, \ldots, T$. Let $\hat{\mu}$ be a $1 \times N$ vector of returns, where $\hat{\mu}_i$ represents the mean return of asset i.

Assumption 1: Stock returns exhibit an independent and identically distributed (IID) pattern over time.

Assumption 2: Let N, be fixed and finite while T goes to infinity.
Assumption 3: Stocks returns have finite moments.

$$\forall i, j, k, l = 1, \ldots, N \quad \forall t = 1, \ldots, T : \quad \mathbb{E}[R_{it} R_{jt} R_{kt} R_{lt}] < \infty$$

3.1.1 Sample Covariance Matrix

We define the sample mean vector $\hat{\mu}$ and the sample covariance matrix $\hat{\Sigma}$ as follows:

$$\hat{\mu} = \hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} R_{it} \quad \text{for } i = 1, 2, \ldots, N \tag{3.1}$$

$$\hat{\Sigma} = \hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(R_{it} - \hat{\mu}_i)(R_{jt} - \hat{\mu}_j) \quad \text{for } i, j = 1, 2, \ldots, N \tag{3.2}$$

where $\hat{\sigma}_{ij}$ is the covariance between the i-th and j-th stock returns (in case $i = j$ it becomes the variance of the i-th stock returns).

3.1.2 Single Factor Model

The single factor model of Sharpe (1963) uses a market factor [6] in a linear regression model to derive the parameters that are used to calculate the expected returns and covariance inputs indirectly. In Sharpe's model the return of the i-th stock is assume to follow:

$$R_{it} = \alpha_i + \beta_i R_{0t} + \varepsilon_{it} \tag{3.3}$$

The residuals ε_{it} , are assumed to be uncorrelated with both the market returns R_{0t} and amongst themselves. Furthermore, the variance within stocks remains constant, represented as $\text{Var}(\varepsilon_{it}) = \delta_{ii}$. The estimator implied by this model is computed as:

$$\hat{F}^{SF} = \hat{\sigma}_{00}\beta\beta' + \Delta \tag{3.4}$$

Where $\hat{\sigma}_{00}$ is the sample variance of market returns R_{0t}, β represents the vector of slope estimates, and Δ is a diagonal matrix consisting of N residual variance estimates δ_{ii}. Let $\hat{f}_{ij}^{sf}$ denote the (i, j)-th entry of $\hat{F}^{SF}$.

The resulting shrinkage estimator, a weighted average of sample covariance and the single factor model, is computed as:

[6] The factor could also be any other key factor believed to majorly impact returns Sharpe (1963, p.281)

$$\hat{\Sigma}^{SF} = \hat{\alpha}^* \hat{F}^{SF} + (1 - \hat{\alpha}^*)\hat{\Sigma}. \tag{3.5}$$

3.1.3 Common Correlation Model

Let $\bar{r}_{ij}$ be the elements of a sample correlation matrix with $N(N-1)/2$ unique pairs of correlations r, thus the average sample correlation is computed as:

$$\bar{r} = \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} r_{ij} \tag{3.6}$$

The common correlation covariance matrix will be computed as:

$$\hat{F}^{CC} = \hat{f}_{ij}^{cc} \quad \text{with} \quad \hat{f}_{ij}^{CC} = \begin{cases} \hat{f}_{ii}^{cc} & = \hat{\sigma}_{ii} \quad \text{if } i = j, \\ \hat{f}_{ij}^{cc} & = \bar{r}\sqrt{\hat{\sigma}_{ii}\hat{\sigma}_{jj}} \quad \text{if } i \neq j. \end{cases} \tag{3.7}$$

Where σ_{ij} are the elements of 3.2.
The resulting shrinkage estimator, a weighted average of sample covariance and common correlation matrix, is computed as:

$$\hat{\Sigma}^{CC} = \hat{\alpha}^* \hat{F}^{CC} + (1 - \hat{\alpha}^*)\hat{\Sigma}. \tag{3.8}$$

3.2 Portfolio Selection Problem.

The framework presented here is grounded in Markowitz (1952) portfolio selection theory. The portfolio selection problem considered in this study is formulated in the following way:

$$\underset{\boldsymbol{x}}{\text{Minimize:}} \quad \boldsymbol{x}'\Sigma\boldsymbol{x} \tag{3.9}$$

$$\text{Subject to:} \quad \sum_{i=1}^{N} x_i = 1. \tag{3.10}$$

The optimization problem aims at minimizing the portfolios return variance 3.9 subject to the budget constraint 3.10; weights sum to unity. 3.10 further indicates the portfolio is fully invested.

The above construction leads us to the global minimum variance portfolio (GMVP) which has the lowest risk of all possible portfolios in the Expected mean-variance (EV) efficient portfolios. The GMVP has a well-known analytical solution given as:

$$\boldsymbol{x}_{GMVP} = \frac{\Sigma^{-1}\mathbf{1}_N}{\mathbf{1}_N'\Sigma^{-1}\mathbf{1}_N} \tag{3.11}$$

where $\mathbf{1}_N$ denotes an $N \times 1$ vector of ones.

3.11 illustrate that the weights of the GMVP depends on the inverse of the true covariance matrix Σ which is unobservable. In practice, to feasibly evaluate these quantities, the unknown matrix Σ is replaced by an estimator $\hat{\Sigma}$.

For each strategy, the GMVP is derived by substituting Σ^{-1} in 3.11, with the inverse of 3.2, 3.4, 3.5, 3.7 and 3.8.

3.3 Data

Data was accessed using the Refinitive Eikon Datastream via the Institute for Quantitative Business and Economics Research (QBER) remote connection. From September 2013 to June 2023, we implemented the following procedure: On the last Friday of September 2013, we retrieved the current list of DAX 40 constituents. For each constituent we downloaded the weekly (Friday to Friday) euro-denominated total return series spanning the previous 5 years.

After calculating the returns, the inverse of the competing estimators 3.2 3.4 3.5 3.7 3.8 were derived and plugged in 3.11 to obtain GMVP weights for each strategy. This portfolio was sustained until the last Friday of December 2013, when it was liquidated. Simultaneously, a new portfolio was assembled based on the refreshed DAX members [7] and sustained until the last Friday of March 2014 and this process continued in a similar fashion until the final set being established on Friday 30, June 2023.

Over the 10 year span from 2013 to 2023, we obtained 522 out-of-sample GMVP returns for each strategy. Our primary focus is on the out-of-sample returns, allowing us to evaluate the out-of-sample standard deviation and other performance attributes of each GMVP strategy over the decade.

Dorfleitner (2003, p.8) observes that using log returns as a substitution for simple returns when computing covariances may lead to notably suboptimal portfolio choices. This discrepancy arises, in part, because log returns can only effectively replace simple returns for values close to zero. Based on these insights, and considering simple returns satisfies portfolio additivity, we utilize simple returns in this process.

The in-sample and out-of-sample simple returns are computed using the following formula:

$$R_{it} = \frac{P_{it} - P_{it-1}}{P_{it-1}}. \tag{3.12}$$

where R_{it} represents the return of asset i at time t, and P_{it} and P_{it-1} are the total return series at times t and $t-1$, respectively.

After obtaining the in-sample returns, and the GMVP weights. For each strategy the out-of-sample GMVP returns are then calculated as:

$$r_t = \mathbf{R}\boldsymbol{x}. \tag{3.13}$$

In this context, $\mathbf{R}$ is a $T \times N$ matrix of out-of-sample simple returns, and $\boldsymbol{x}$ is an $N \times 1$ vector representing the portfolio weights.

[7]Thus the constituents list was continuously updated each quarter, as per https://www.stoxx.com/document/Indices/Common/Indexguide/Historical_Index_Compositions.pdf, last accessed August 25, 2023. Notably, it was previously composed of 30 members and expanded to 40 members in September 20, 2021.

3.4 Covariance Forecast Evaluation

Due to the inherent uncertainties in financial markets, portfolio managers must proactively evaluate the present vulnerabilities of their investment strategy, that might result in future reductions in portfolio value. Ensuring the accuracy of financial models through proper evaluation leading to correct volatility or covariance, can enhance performance without relying on strong assumptions (Patton and Sheppard (2009, p.830))

Patton and Sheppard (2009, p.831) demonstrated that if portfolio weights $\boldsymbol{x}$ are derived from the true covariance Σ, the variance of a GMVP from any other forecast must be greater. Thus, we can evaluate two competing covariance forecasts by comparing the volatility of the minimum variance portfolio from each.

In line with Patton and Sheppard, we evaluate the forecast accuracy of the variance forecast model using the Diebold Mariano West[8] (DMW) test as follows:

$$dt = r_t^a r_t^a - r_t^b r_t^b. \tag{3.14}$$

Here, r_t^a represents the out-sample GMVP returns obtained with the sample covariance matrix, while r_t^b represents the out-sample GMVP returns obtained with any of the other competing covariance estimator.

The null hypothesis that asserts equivalent predictive precision is represented as:

$$H_0 : \mathbb{E}[dt] = 0. \tag{3.15}$$

This is carried out using the Diebold Mariano West test statistic computed as:

[8]Cf. Diebold and Mariano (2002) and West (1996)

$$DMW_T = \frac{\sqrt{T}\bar{d}_T}{\sqrt{\widehat{AsyVar}[\bar{d}_T]}}. \tag{3.16}$$

Where $\bar{d}_T = \frac{1}{T}\sum_{t=1}^{T} d_t$, and $\widehat{AsyVar}[\bar{d}_T]$ is computed using a Newey-West variance estimator (see Newey and West (1987)) with the number of lags set to $[T^{1/3}]$. Under the null hypothesis, the test statistic DMW_T is asymptotically normally distributed.

The composite alternatives that indicate which forecast performs better are presented below:

$$H_{1a} : \mathbb{E}[dt] < 0, \tag{3.17}$$

$$H_{1b} : \mathbb{E}[dt] > 0. \tag{3.18}$$

If the DMW test statistic is significantly negative, we reject the null hypothesis of equal predictive precision in favor of the alternative hypothesis 3.17, indicating that the forecast r_t^a is better. Conversely, if the statistic is significantly positive, we reject the null hypothesis in favor of the alternative hypothesis 3.18, indicating that the forecast r_t^b performs better.

4 Empirical Results

Thus far, our efforts have been directed towards the construction of the GMVP using the five distinct estimators. Now, we aim to evaluate the quality of these estimators, particularly the effects of shrinkage, by examining their out-of-sample standard deviation. As noted by Elton and Gruber (1997, p.1752) modern theory highlights the importance of considering both risk and return when assessing performance. Consequently, we also examine additional metrics, including the Sharpe ratio and the wealth trajectory over the decade.

Dorfleitner (2003, p.5) emphasizes the advantages of utilizing log returns, when annualizing returns over varying time spans, and making comparisons. Based on his insights and considering the statistical properties and time additivity of log returns, the comparisons are based on out-of-sample log returns.

The out-of-sample performance of the five GMVPs based on their 522 out-of-sample weekly log returns are reported in Table 1. They include the sample covariance (hence, Sample), single factor model (SF Model), common correlation model (CC Model), shrinkage covariance with the target SF model (Shrink-SF), and shrinkage covariance with the target CC Model (Shrink-CC).

Table 1: Performance characteristics.

Metrics	Sample	SF Model	Shrink-SF	CC Model	Shrink-CC
Ann. Returns (%)	8.87	7.99	9.07	10.03	10.01
Ann. Std. Dev. (%)	17.29	18.88	16.98	19.21	17.16
Ann. Sharpe Ratio	0.51	0.42	0.53	0.52	0.58
Skewness	−2.02	−0.94	−1.92	−0.75	−1.78
Kurtosis	17.90	5.50	16.38	6.86	15.58

Notes: This table presents the performance characteristics based on the log returns. Annualized Return was calculated by taking the average of the log returns, then annualized by multiplying by 52. Annualized Standard Deviation is the standard deviation of the out-of-sample log returns, annualized by multiplying $\sqrt{52}$. Annualized Sharpe ratio is: Ann.Returns/Ann.Std.Dev. The Skewness and Kurtosis values differ from 0 and 3 respectively, showing evidence of negatively skewed and heavy tailed returns.

From Table 1, we observe that CC Model and the shrinkage estimators report annualized returns slightly higher than the Sample. The CC Model leads the pack with an annualized return of 10.03%, very closely followed by the Shrink-CC at 10.01%.

The annualized standard deviation, a measure of variability of returns, is the lowest for Shrink-SF at 16.98%. Interestingly, while both target matrices exhibit more volatility than the Sample, the act of shrinking the Sample results in tempered volatility, as evidenced by the reduced annualized standard deviations in the Shrink-SF and Shrink-CC models.

Figure 2 visualizes the annualized standard deviations, computed using a moving window of 52 weeks at a time, of the five different models across the decade. In the early years, leading up to 2017, all models display an increasing trend in variability. While the Sample, Shrink-CC and Shrink-SF initially demonstrate lower rolling volatilities, they were not immune to the general upward trajectory exhibited by all models.

The start of 2020, coinciding with the on-set of the COVID-19 pandemic, marks a pronounced surge in volatility across all models. Sample initially experienced the most dramatic spike, followed by Shrink-SF, Shrink-CC, SF Model and then CC Model.

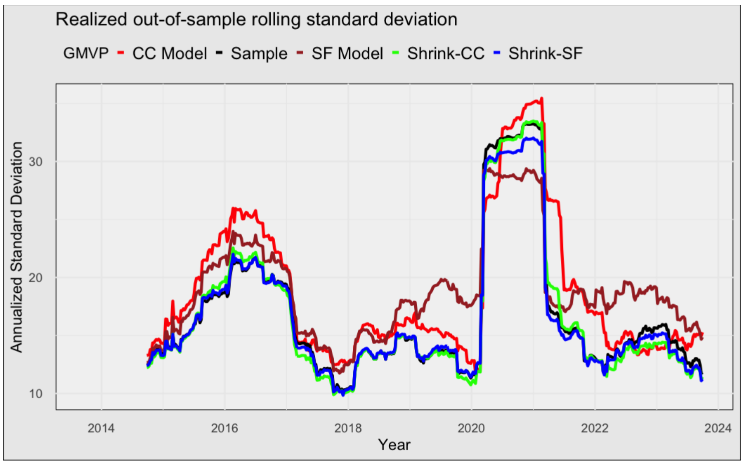

Figure 2: This graph depicts the annualized standard deviations, calculated from rolling 52 weekly log returns. Observing the y-axis, the models demonstrate fluctuations in volatility, ranging between 10% to 35%. All models reveal a synchronized spike in volatility around 2020, emphasizing the different impact of the COVID-19 crisis on various strategies.

As the world acclimated to the post-pandemic era in 2021, the volatility began its descent. Over the course of the decade, the Sample, Shrink-CC, and Shrink-SF models exhibit closely aligned trajectories. However, the shrinkage estimators, on average, tend to register slightly below the Sample.

The Sharpe ratio, a risk-adjusted performance metric, measures the excess return a strategy provides over a risk-free rate for each unit of standard deviation of returns, a higher positive value is better. We assumed a risk-free rate of zero; thus, the Sharpe ratio was computed as the ratio of the annualized return to the annualized standard deviation. Shrink CC and Shrink SF yields the highest Sharpe ratio 0.58 and 0.53 respectively, followed by the Sample 0.51 and the shrinkage target SF-Model the lowest 0.43 (see Table 1).

All models display negative skewness, indicating asymmetrical and negatively skewed returns. Their kurtosis values surpass 3, implying leptokurtic distributions with more outliers and extreme deviations than a normal distribution. Specifically, the Sample showcases the most pronounced skewness and kurtosis at -2.02 and 17.90, respectively, while the CC Model has lowest values of -0.75 and 6.86. The shrinkage estimators have reduced skewness and kurtosis when compared to the Sample.

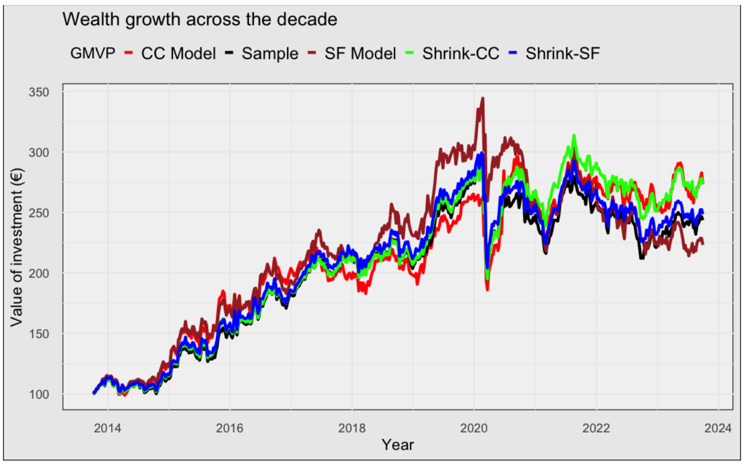

Figure 3: This graph depicts the growth of an initial €100 investment, using cumulative out-of-sample weekly simple returns over the decade. All strategies demonstrate upward trajectory, showing resilience despite intermittent market fluctuations. A pronounced decline is observable across all models at the onset of the COVID-19 pandemic in early 2020. The decade concluded with the Shrink-CC and its underlying target, the CC Model, outperforming the other strategies.

In terms of wealth growth over time, all strategies yielded positive values. The CC Model achieved the highest terminal wealth, accumulating €273.81 from a €100 initial investment over the decade, very closely followed by the Shrink-CC at €273.06. Subsequently, the Shrink-SF reached €248.45, the Sample attained €243.54, and the SF Model realized €223.07.

Let's now delve into the analysis of metrics that captured the negative tails of the out-of-sample GMVP return distributions (see Table 2). Maximum drawdown varied among the strategies, with SF Model experiencing the steepest decline at 42.7%. When comparing the Sample to the shrinkage estimators, the drawdown for the Sample was marginally greater than that of the shrinkage estimators. This suggests that shrinkage had a minimal impact on lessening the drawdowns.

Regarding the annualized downside deviation, which evaluates the variability of each returns below 0, the SF Model registered the highest at 13.64%, indicating the worst performance. In contrast, the shrinkage estimators demonstrated reduced downside volatility, showing better stability relative to the Sample and their respective underlying target matrix.

Turning to historical Value at Risk (VaR), a metric indicating the loss in value at 95% confidence level. The SF Model again assumes the highest risk position, while the Shrink-SF model emerges as the most conservative, implying a lower risk of extreme loss. The Sample and Shrink-CC aligned closely.

Table 2: Downside risk characteristics.

Metrics	Sample	SF Model	Shrink SF	CC Model	Shrink CC
Max. Drawdown (%)	35.57	42.70	34.47	32.06	34.92
Ann. Down. Dev. (%)	12.95	13.64	12.66	13.52	12.64
VaR (95%)	3.49	3.90	3.24	3.86	3.49
AVaR (95%)	5.47	5.83	5.35	6.00	5.38

Notes: This table outlines downside metrics from 522 weekly log returns. Higher values are unfavorable. Maximum Drawdown depicts the largest GMVP decline from peak to trough. Annualized Downside Deviation gauges annualized standard deviation of returns below 0. Value at Risk shows the worst loss at a 95% confidence level, while Average VaR assesses average losses beyond the VaR.

For Average Value at Risk (AVaR), also known as Expected Shortfall, which captured the average of the worst losses beyond the VaR threshold over the decade, the CC Model had the highest at 6%, indicating larger losses on average in adverse conditions. Conversely, the Shrink-SF Model showed the smallest AVaR at 5.35%, reflecting its effectiveness in mitigating losses.

Portfolio weight concentration involves allocating a significant portion of capital to a few assets, as opposed to diversification, which spreads capital across numerous assets to minimize the risk tied to individual assets. Each boxplot in Figure 4 provides a visual summary of model's weight distribution, with the Sample attributing extreme weights. The Sample portfolio significantly short-sold an asset at -26%, and in another period, heavily invested in an asset at 38.8%. Interestingly, the -26% was allocated to an asset with an in-sample volatility of 44%, higher than 75% of other assets in that estimation window. Conversely, a weight of 38% was given to an asset with much lower volatility, 16.6%, the lowest among assets in its window.

This allocation pattern aligns with Michaud (1989, p.34) research and illustrates a common limitation of Markowitz portfolios: overweighting low-volatility assets while underweighting high-volatility ones.

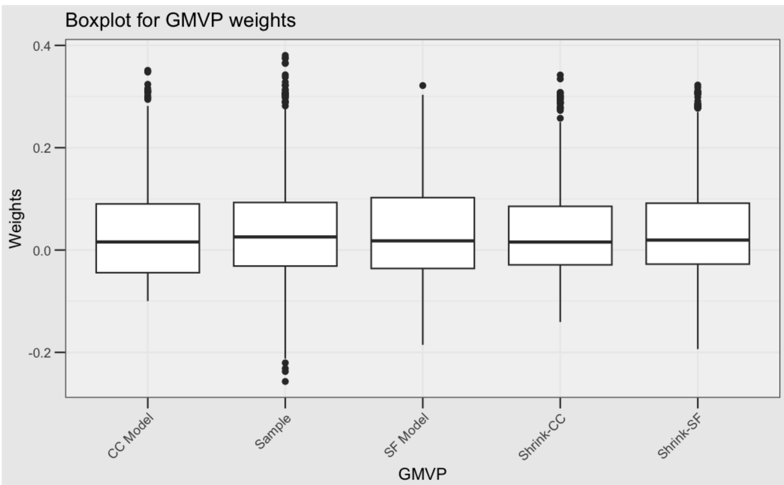

Figure 4: The box plot illustrates the distribution of GMVP weights from various models, obtained on the initial GMVP creation day and throughout subsequent rebalancing dates. The Sample model displays extreme weights, with notable short-selling at -26%, introducing potential risks. In comparison, CC and SF Models offer more moderated weights, indicating greater diversification and reduced extremities, thereby mitigating portfolio risks.

Table 3: Variance forecast evaluation.

Metrics	SF Model	Shrink-SF	CC Model	Shrink-CC
DMW_T	-47.10^{***}	39.55^{***}	-42.97^{***}	9.39^{***}
Newey SE	5.35×10^{-5}	1.17×10^{-5}	7.22×10^{-5}	1.80×10^{-5}
Decision	worse	better	worse	better

Notes: Asterisks *** indicate significance at the 1% level. The Decision is based on the significance of the Diebold-Mariano-West test statistics (DMW_T), computed using Newey-West standard errors (Newey SE). The decision of 'worse' indicates that the competing model has poorer forecasting performance than the Sample.

Table 3 presents the outcomes of a covariance forecast evaluation. The Diebold-Mariano-West test statistics (DMW_T) are statistically significant at the 1% level, suggesting strong evidence against the null hypothesis of equal predictive accuracy. The associated Newey-West standard errors for each model are notably small.

Moreover, DMW_T for both target matrices are significantly negative, indicating that these models perform worse than the Sample. Conversely, both shrinkage estimators have significantly positive DMW_T, implying they are superior forecasting models compared to the Sample.

5 Conclusion

In this study, we conduct a comparative analysis of five distinct covariance matrix estimators, specifically examining the effects of applying shrinkage to the sample covariance matrix. The results reveal that GMVPs using shrinkage estimators slightly improve in risk-adjusted performance, measured by the Sharpe ratio, over their targets and the Sample.

Our findings also show that all models demonstrate deviations from normality. These findings are consistent with the extensive financial literature that recognizes financial returns as typically exhibiting heavy tails. The Jarque Bera test results, detailed in Appendix 2, conclusively validate the non-normality of the returns across all models. Consequently, this implies that assuming a normal distribution for the innovations in volatility models, such as the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model, may be unreasonable.

In terms of downside risk, our analysis indicates that the shrinkage estimators exhibit lower susceptibility to downside risk than their targets counterpart and show an improvement over the Sample model. Furthermore, variance forecast evaluations show that, unlike the target matrices, shrinkage estimators provide superior predictive performance relative to the Sample model. This evidence suggests that shrinking the Sample is not only beneficial for achieving a more stable portfolio but also for enhancing the accuracy of variance forecasts.

Conclusively, the study advocates the adoption of shrinkage estimators in portfolio management, highlighting their ability to enhance risk-return, and improve both risk management and variance forecasting. Furthermore, considering the evident departure from a normal distribution in returns, and the tendency for stock returns to have fat tails, the robust covariance estimator proposed by Fan et al. (2018) is particularly recommended for future empirical exploration, its inclusion could refine the portfolio construction process.

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A Appendix

A.1 Formula for the Optimal Shrinkage Intensity

Let $\mathbf{R}$ be a $T \times N$ matrix of stock returns, where R_{it} represents the return of asset i, for $i = 1, \ldots, N$, and $t = 1, \ldots, T$. Let the average return of asset i be $\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} R_{it}$, while the elements of $\hat{\Sigma}$, $\hat{F}^{SF}$, $\hat{F}^{CC}$, be denoted $\hat{\sigma}_{ij}$, $\hat{f}_{ij}^{sf}$, $\hat{f}_{ij}^{cc}$ over the estimation window.

The formula for the optimal shrinkage intensity α^* is given as:

$$\alpha^* = \max\left\{0, \min\left\{\frac{\kappa}{T}, 1\right\}\right\}, \quad \text{where} \quad \kappa = \frac{\pi - \rho}{\gamma}. \tag{A.1}$$

Ledoit and Wolf (2003, 2004) explained, κ is unknown and therefore needs to be estimated. They showed that the consistent estimator for π, ρ, γ is given by $\hat{\pi}$, $\hat{\rho}$, $\hat{\gamma}$ as presented below.
$\hat{\pi}$ is the asymptotic variance of the entries of $\hat{\Sigma}$, $\hat{\rho}$ is the sum of asymptotic covariances of the entries of $\hat{F}^{SF}$; $\hat{F}^{CC}$, and $\hat{\gamma}$ measures the misspecifications of the target matrix; $\hat{F}^{SF}$, $\hat{F}^{CC}$.

The consistent estimator for π is given as:

$$\hat{\pi} = \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{\pi}_{ij}, \quad \text{where} \quad \hat{\pi}_{ij} = \frac{1}{T}\sum_{t=1}^{T}\left\{(R_{it} - \hat{\mu}_i)(R_{jt} - \hat{\mu}_j) - \hat{\sigma}_{ij}\right\}^2. \tag{A.2}$$

$\hat{\rho}$ and $\hat{\gamma}$ computed under SF Model and CC Model is as follows:

SF Model as target: Let R_{0t}, $\hat{\mu}_0$, and $\hat{\sigma}_{00}$ denote market returns, mean of market returns and variance of market returns respectively. Let $\hat{\sigma}_{i0}$ denote the sample covariance between i-th stock returns and the market returns.

$$\hat{\rho} = \sum_{i=1}^{N} \hat{\pi}_{ii} + \frac{1}{T} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} r_{ij,t} \tag{A.3}$$

Where:

$$r_{ij,t} = \left\{ \frac{(\hat{\sigma}_{j0}\hat{\sigma}_{00}(R_{it} - \hat{\mu}_i) + \hat{\sigma}_{i0}\hat{\sigma}_{00}(R_{jt} - \hat{\mu}_j) - \hat{\sigma}_{i0}\hat{\sigma}_{j0}(R_{0t} - \hat{\mu}_0))}{\hat{\sigma}_{00}^2} \right\}$$
$$\times (R_{0t} - \hat{\mu}_0)(R_{it} - \hat{\mu}_i)(R_{jt} - \hat{\mu}_j) - \hat{f}_{ij}^{sf} \hat{\sigma}_{ij}.$$

$$\hat{\gamma} = \sum_{i=1}^{N} \sum_{j=1}^{N} (\hat{\sigma}_{ij} - \hat{f}_{ij}^{sf})^2. \tag{A.4}$$

.

CC Model as target:

$$\hat{\rho} = \sum_{i=1}^{N} \hat{\pi}_{ii} + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\bar{r}}{2} \left(\sqrt{\frac{\hat{\sigma}_{jj}}{\hat{\sigma}_{ii}}} \hat{\theta}_{ii,ij} + \sqrt{\frac{\hat{\sigma}_{ii}}{\hat{\sigma}_{jj}}} \hat{\theta}_{jj,ij} \right) \tag{A.5}$$

Where:

$$\hat{\theta}_{ii,ij} = \frac{1}{T} \sum_{t=1}^{T} \left\{ (R_{it} - \hat{\mu}_i)^2 - \hat{\sigma}_{ii} \right\} \times \left\{ (R_{jt} - \hat{\mu}_j)(R_{it} - \hat{\mu}_i) - \hat{\sigma}_{ij} \right\}$$

$$\hat{\theta}_{jj,ij} = \frac{1}{T} \sum_{t=1}^{T} \left\{ (R_{jt} - \hat{\mu}_j)^2 - \hat{\sigma}_{jj} \right\} \times \left\{ (R_{jt} - \hat{\mu}_j)(R_{it} - \hat{\mu}_i) - \hat{\sigma}_{ij} \right\}$$

and $\bar{r}$ is the pairwise common correlation 3.6.

$$\hat{\gamma} = \sum_{i=1}^{N} \sum_{j=1}^{N} (\hat{\sigma}_{ij} - \hat{f}_{ij}^{cc})^2. \tag{A.6}$$

Ledoit and Wolf (2003, 2004) showed that an estimator for α^* is given by:

$$\hat{\alpha}^* = \max\left\{0, \min\left\{\frac{\hat{\kappa}}{T}, 1\right\}\right\}, \quad \text{where} \quad \hat{\kappa} = \frac{\hat{\pi} - \hat{\rho}}{\hat{\gamma}}. \tag{A.7}$$

A.2 Impact of size T-weekly observations on performance

Table 4: Evaluation Metrics for the out-sample log returns across differnt values of T-weekly observations.

Desc.	Weeks	Sample	SF Mod	Shrink-SF	CC Mod	Shrink-CC
	T=260	243.54	223.07	248.45	273.81	273.06
Terminal Wealth (€)	T=155	309.12	212.36	273.11	208.17	282.17
	T=80	332.40	207.11	274.57	209.51	278.87
	T=260	17.29	18.88	16.98	19.21	17.16
Ann. StdDev (%)	T=155	18.16	18.58	17.08	19.12	17.57
	T=80	19.40	18.25	17.26	19.03	18.08
	T=260	7320***	735***	6153***	1073***	5556***
Jarque Bera t-statistics	T=155	6564***	1400***	6885***	1612***	5613***
	T=80	1680***	2014***	4407***	2407***	3352***
	T=260	10.16	16.47*	11.89	13.29	10.68
Ljung-Box t-statistics	T=155	12.20	19.00*	13.50	11.34	9.39
	T=80	13.98	23.20***	20.43**	17.33*	15.07
	T=260	3.49	3.90	3.24	3.86	3.49
VaR (95%)	T=155	3.67	3.67	3.35	3.83	3.32
	T=80	3.68	3.57	3.22	3.76	3.37

Notes: This table presents the evaluation metrics for the 522 out-of-sample weekly log return for various T-week in-sample observation used. The Sample model excels in terminal wealth, but carries higher risk as T decreases, as indicated by the rising Annualized Standard Deviation. The shrinkage estimators show slightly enhanced performance in balancing risk and terminal wealth.

Asterisks *, **, *** indicate significance level at the 10%, 5% and 1% respectively. The Jarque Bera test-statistics indicates that normality is rejected for all models across all levels of T, emphasizing the non-normal nature of returns across the period. The Ljung-Box test-statistics conducted at 8 lag, which test for autocorrelation, show varied results across different models and T levels. Still, a general observation is that SF models' returns exhibit significant autocorrelation, especially when fewer observations are used in the estimation process.

As T reduces, the Historical Value Risk (VaR) values tend to fluctuate, but not uniformly. It's observed that while the other models present increased risk with shorter Ts, the Shrinkage estimators showcase slightly less risk, indicating the potential stabilizing effect of shrinkage techniques during turbulent times. To conclude, while volatility and downside risk amplifies as T decreases for Sample, the shrinkage estimators narrowly outpace the Sample model in this regards.

The out-of-sample weekly log returns, spanning from the quarter before COVID-19's onset in Germany to two years thereafter, were extracted and used to assess the pandemic's impact (see Table 5). According to Statista [9], COVID-19 began in Germany in March 2020.

Table 5: Descriptive Statistics for the out-of-sample weekly log returns during the COVI-19 period.

Desc.	Weeks	Sample	SF Mod	Shrink-SF	CC Mod	Shrink-CC
Max Draw. (%)	T=260	35.57	39.50	34.47	32.06	34.92
	T=155	34.87	35.63	33.94	32.69	34.31
	T=80	39.51	39.04	32.55	33.63	33.30
Ann. Down Dev (%)	T=260	21.49	19.82	20.70	20.96	20.91
	T=155	21.78	20.05	20.55	21.15	20.72
	T=80	22.08	20.64	20.16	22.10	20.99
VaR (99%)	T=260	13.31	13.09	13.24	12.79	13.26
	T=155	11.85	12.28	11.91	12.45	11.93
	T=80	9.60	12.30	11.34	12.27	11.24
AVaR (99%)	T=260	18.19	15.87	17.67	16.01	17.60
	T=155	18.03	16.29	17.39	16.64	17.30
	T=80	15.37	16.58	16.51	17.08	16.49

Notes: The table displays the descriptive statistics for the out-sample weekly log returns during the covid-19 period across different T values. As T decreases, models (except Shrink-SF and Shrink-CC) tend to experience increased drawdowns. For instance, at T=80, the Sample performs worse, and the Shrink-CC model is less adversely affected. The annualized downside deviation, which measures the variability of returns below zero during the pandemic, indicates that all models were quite volatile, and the impact of T is not noticeable here.
Both historical Value at Risk (VaR) and AVaR values decline across all models. Results further show that shrinkage estimators typically aid in moderating risk, with the Shrink-SF and Shrink-CC models being somewhat more stable and less extreme on average during the crisis period. Utilizing the most recent weekly returns data and applying shrinkage to the Sample can be beneficial in enhancing risk management during market turmoil

[9] https://www.statista.com/statistics/1100823/coronavirus-cases-development-germany/ last accessed 10 October 10, 2023.

Table 6: Descriptive Statistics for weights **x**.

Desc.	Weeks	Sample	SF Mod.	Shrink-SF	CC Mod.	Shrink-CC
Minimum	T=260	-0.2569	-0.1853	-0.1935	**-0.0997**	-0.1407
	T=155	-0.3848	-0.2324	-0.2636	**-0.1078**	-0.1921
	T=80	-0.6996	-0.2502	-0.3292	**-0.1265**	-0.2668
1Q	T=260	-0.0313	-0.0360	**-0.0275**	-0.0443	-0.0289
	T=155	-0.0378	-0.0345	**-0.0275**	-0.0421	-0.0303
	T=80	-0.0554	-0.0300	**-0.0269**	-0.0343	-0.0319
Median	T=260	0.0256	0.0180	0.0195	0.0158	**0.0156**
	T=155	0.0270	0.0165	0.0207	0.0157	**0.0134**
	T=80	0.0310	0.0206	0.0258	**0.0157**	0.0161
Mean	T=260	0.0337	0.0337	0.0337	0.0337	0.0337
	T=155	0.0337	0.0337	0.0337	0.0337	0.0337
	T=80	0.0337	0.0337	0.0337	0.0337	0.0337
3Q	T=260	0.0929	0.1024	0.0915	0.0901	**0.0854**
	T=155	0.0995	0.0952	0.0924	0.0866	**0.0864**
	T=80	0.1178	0.0871	0.0892	0.0797	**0.0788**
Maximum	T=260	0.3805	0.3215	0.3225	0.3512	**0.3421**
	T=155	0.5320	0.3164	0.3994	**0.3724**	0.4064
	T=80	0.6845	0.4650	0.4829	**0.3920**	0.4880

Notes: This table displays the weight characteristics of different estimators across various T values. Bold figures indicate the lowest weight at each T-level across the models. The weights assigned by all models become more extreme as fewer observations are used in the optimization process. The CC Model consistently shows the least negative minimum weights, suggesting a more conservative stance on short-selling assets.

Surprisingly, all estimators have identical mean weights across the different Ts. The third quartile weights for the Sample and Shrink-CC generally increase with shorter T, this could imply a higher allocation to assets that have performed better recently.

The reported values suggest that the CC Model assigns more stable weights to its constituents across different Ts compared to other models, while shrinkage estimators offer a more tempered approach to asset allocation across various timeframes compared to the Sample model. The reported data further suggest that using fewer observations in the optimization process will lead to a concentrated and less diversified portfolio.

Table 7: Descriptive Statistics for Shrinkage intensity $\hat{\alpha}^*$.

	Shrink-SF			Shrink-CC		
Description	T=260	T=155	T=80	T=260	T=155	T=80
Min	0.1945	0.2102	**0.2750**	0.1456	0.1911	0.1765
1st Quartile	0.2187	0.2666	0.3601	0.1970	0.2627	**0.3809**
Median	0.2393	0.3238	**0.4550**	0.2524	0.2970	0.4264
Mean	0.2509	0.3232	0.4416	**0.2520**	**0.3313**	**0.4806**
3rd Quartile	0.2668	**0.3688**	0.5226	**0.2717**	0.3611	**0.5153**
Max	0.4184	0.4812	0.6082	**0.4446**	**0.8143**	**1**

Note: This table provides the descriptive statistics for shrinkage intensity values. Values were obtained at each estimation window throughout the decade. Bold figures indicate highest values at each T-level across the models. As expected, the intensity value consistently fall between zero and one. Values closer to one suggest significant estimation errors in the sample covariance matrix compared to the bias in the underlying target.

The data shows an increasing trend of estimation error relative to bias as the weekly observations T decreases. The mean of shrinkage values increase as T decreases, this trend suggests heightened errors in the sample covariance matrix as the observations size T decreases on average. This is in line with findings that in high dimensional settings, that is as T approaches the number of stocks N, there exist more errors in the Sample.

For matrices computed with 80 weekly returns (T=80), reported median shrinkage values are 0.455 (0.4264) under Shrink-SF (Shrink-CC). Drawing from Ledoit and Wolf (2003, p.618) insights, this implies there are 2.275 (2.132) times more estimation errors in the Sample than there is bias in SF (CC) Model.

A.3 R code explanation

This section explains the details of the main functions implemented in R.

Data Acquisition: Both in-sample and out-sample data were sourced from the Refinitiv Eikon via the QBER institute remote connection. The data spanned 40 periods, from inception to the last rebalancing date, and were saved in the Excel workbook `D.Data.xlsx` comprising 40 sheets.

Data Cleaning: Initial data processing involved deleting stocks with more than 15 missing values. For stocks with fewer missing values, nearest-neighbor interpolation was employed to fill gaps. The cleaned dataset was then separeted into in-sample `Data_IS.xlsx` and out-sample `Data_OS.xlsx` datasets.

Data Importation in R: Transitioning to R, in-sample and out-sample data: (`Data_IS.xlsx`), (`Data_OS.xlsx`), were imported using the `read_excel()` function from the `readxl` library within a loop, facilitated by the `lapply()` function. The in and out-sample data was saved as: `Data_IS` and `Data_OS`

Covariance Functions: Two main functions were scripted for covariance computation:
`Covariance_SF` and `Covariance_CC`, adapted from Ledoit Olivier GitHub: https://github.com/oledoit/covShrinkage/blob/main/covMarket.m. Modifications were made, for instance instead of using average returns as a proxy for market returns (equally weighted market), as the initial code implied, we directly utilized the DAX returns. Thus, the code was updated to compute the elements of the shrinkage target F^{SF}, as described (see equation 3.4). The `Covariance_SF` function outputs sigma_hat, F_SF, and sigmahat_SF (equantion 3.2, 3.4 and 3.5), while `Covariance_CC` yields F_CC and sigmahat_CC (equation 3.7 and 3.8).

Optimal Portfolio Weights: The `optimalX` function computes the GMVP weights, corresponding to the specifications in equation 3.11.

Return Calculation: A `compute_returns` function was crafted to convert the data to xts (Extensible Time Series) objects and calculate simple returns using the `Return.calculate()` function from the `PerformanceAnalytics` library. The in-sample returns were then computed using `lapply()` to iterate over `Data_IS`, stored as `returns_list_260`.

Covariance and Weights Calculation: `returns_list_260` was utilized to compute covariance matrices using `Covariance_SF` and `Covariance_CC`, storing results in `CovMatx_260_SF` and `CovMatx_260_CC`. Subsequently, the optimal weights were computed using the `optimalX` function, with results stored in `Weights_260_SF` and `Weights_260_CC`. The weights were compiled leading to `df_Weights_260_SF` and `df_Weights_260_CC`.

Portfolio Return Calculations: `df_Weights_260_SF`, `df_Weights_260_CC`, were combined to create list of matrices, `all_weights_260` containing weights of the five estimators, across the 40 periods. Out-sample returns were computed from `Data_OS` using `compute_returns` function.
The `portfolio_returns` function was then scripted to compute out-sample portfolio returns using `returns_data` and `all_weights_260`. Afterwards the portfolio returns were calculated with the `portfolio_returns` function, looping through both `returns_data` and `all_weights_260`, stored in `all_returns_260`. `all_returns_260` was then concatenated and saved as, `port_returns_260`, containing 5 columns of out-sample GMVP returns.

Out-of-sample Characteristics : These out-sample GMVP returns were converted into log returns, stored in `log.port_returns_260`. Performance metrics, downside risk metrics, Jarque Bera test were then computed based on `log.port_returns_260`.

Variance Forecast Evaluation: The `DMW_test_newey` function was crafted to implement equations 3.14 - 3.18. The computation of Newey-West standard errors utilizes the `NeweyWest` function from the `sandwich` library, while hypothesis testing is facilitated by the `lmtest` library.

Declaration of Authorship

I hereby declare that I have composed my thesis titled "Shrinkage Estimation of the Covariance Matrix for Portfolio Optimization" independently using only those resources mentioned, and that I have as such identified all passages which I have taken from publications verbatim or in substance. I am informed that my thesis might be controlled by anti-plagiarism software. Neither this thesis, nor any extract of it, has been previously submitted to an examining authority, in this or a similar form.

I have ensured that the written version of this thesis is identical to the version saved on the enclosed storage medium.

November 2, 2023

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