Better X_0 estimation with likelihood approach

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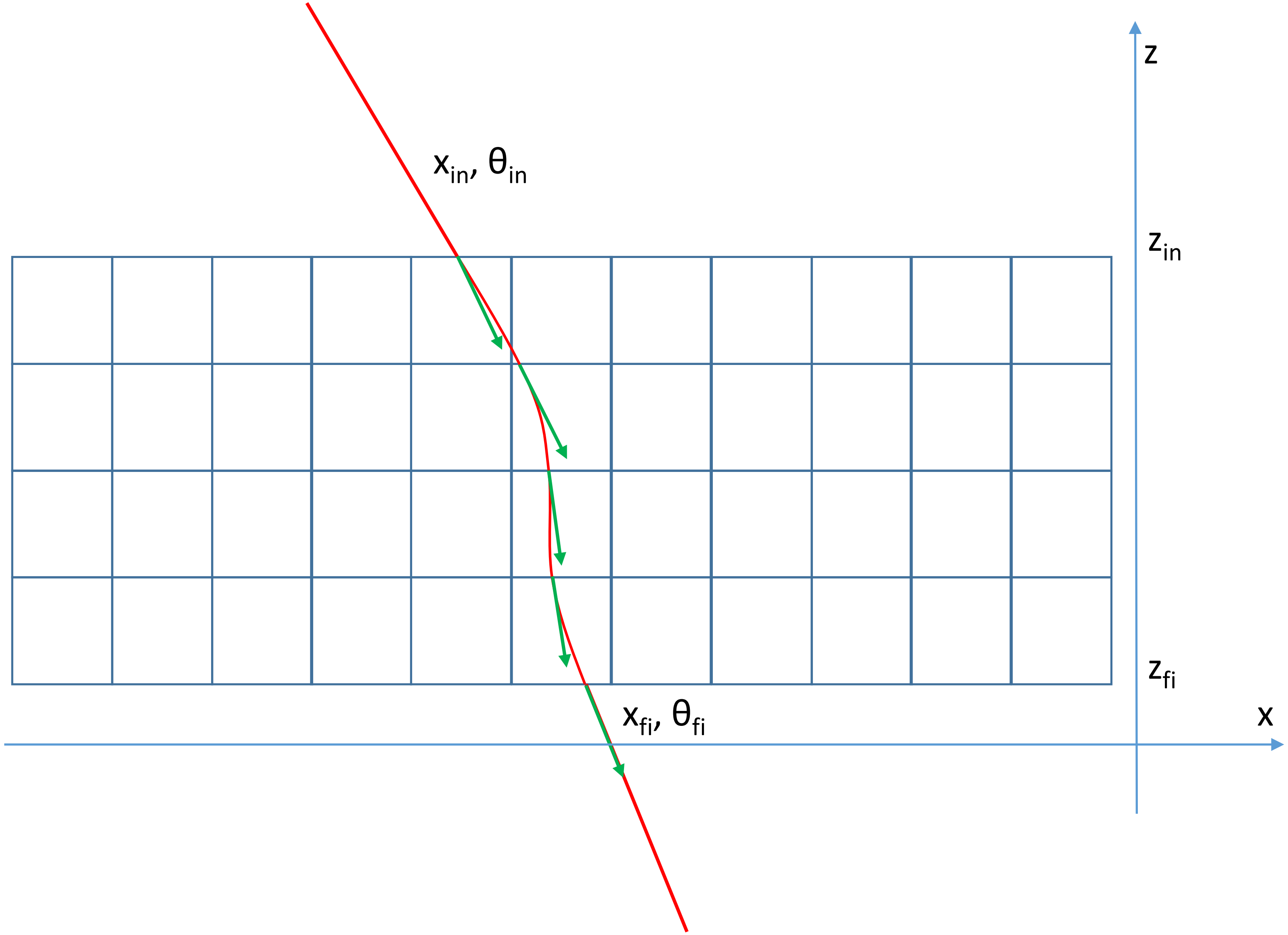
Statement of the problem of X_0 estimation

- Coordinate system: z vertical coordinate, xy describe surfaces
- Let us consider a voxelized volume impinged by muons from above ($z=z_{max}$), and their scattering and exit from below ($z=z_{min}$). Muons are detected at z_{max} and z_{min}, where x,y, θ_x, θ_y are measured with uncertainties $\sigma_x,\sigma_y,\sigma_{\theta x},\sigma_{\theta y}$.
- We may model the scattering as individual variations $\theta'_x=\theta_x+\Delta\theta_x$, $\theta'_y=\theta_y+\Delta\theta_y$ due to interactions within each voxel (we omit transverse displacements here). These have a probability $P(\Delta\theta|X_0)$ which here we consider a Gaussian with width scaling with the square root of X_0 (we omit for now non-Gaussian tails and momentum dependence for simplicity in this simplified analytical treatment).
- It is then straightforward to write down the likelihood of the most probable trajectory (see next slide), as a function of the measurements, the angular variations $\Delta\theta$, and the X_0 of the traversed voxels (see next slide):



x_{in}, θ_{in}
z
z_{in}
z_{fi}
x_{fi}, θ_{fi}
x

Likelihood for N layers

$$x_i = x_0 + h[tan(\theta_0) + tan(\theta_0 + \Delta\theta_0) + tan(\theta_0 + \Delta\theta_0 + \Delta\theta_1) + \cdots]$$

$$\theta_i = \theta_0 + \Delta\theta_0 + \Delta\theta_1 + \cdots$$

$$L = \left\{ \prod_1^N exp\left[-\frac{(\Delta\theta_i)^2}{2\sigma\left(X_{0,i}\right)^2} \right] \right\}$$

$$exp\left[-\frac{\left[x_{fi} - x_{in} - h(tan(\theta_0) + tan(\theta_0 + \Delta\theta_0) + tan(\theta_0 + \Delta\theta_0 + \Delta\theta_1) + \cdots)\right]}{2\left[\sigma\left(x_{fi}\right)^2 + \sigma(x_{in})^2\right]} \right]$$

$$exp\left[-\frac{\left[\theta_{fi} - \theta_{in} - \Delta\theta_0 - \Delta\theta_1 - \cdots \right]^2}{2\left[\sigma\left(\theta_{fi}\right)^2 \sigma(\theta_{in})^2\right]} \right]$$

Above, the first line in the L expression is the probability of a given angular scattering given the X_0 of the traversed voxel; the second line forces the muon starting at x_{in} to end up at x_{fi} after propagation, according to the first expression at the top. The third line similarly forces angles to match.

Note: Here we are considering only scattering in the xz plane. For the yz plane a corresponding set of terms exists. The only coupling of the two is in the X_0, which is the same for each traversed voxel in the two planes.

Solution for a muon and for a batch

The likelihood above is analytical if:

- we omit the p dependence
- we parametrize, as we did in the expression, scatterings as Gaussian; however, a more complex analytical parametrization still allows the analyticity of L

Note that we have taken also other approximations:

- all the scattering of a voxel is accounted for at the end of each voxel propagation (this is not an issue as long as we make voxels small)
- no displacement of trajectories in x and y occurs within a voxel

The solution for a single muon can be obtained for the displacements $\Delta\theta$ if we assume we know the X_0 of the layers traversed by the muon, and minimize -2logL.

If we obtain those displacements, and calculate the likelihood at those values, L now becomes a function of the X_0s. It is still analytical. We can thus accumulate the values of L that correspond to that maximum for a batch of muons. At the end, we differentiate -2logL as a function of each of the X_0, and obtain estimates for them.

The above method is approximated also because it only considers the most probable trajectory for each muon, given the X0 values it traverses. We can improve on this by randomly sampling variations of some of the $\Delta\theta$ for each muon, obtaining additional possible trajectories and their likelihood. Those values of L will be smaller than the maximum, and we can accumulate them in the global batch-wise likelihood by weighting each contribution by their respective probability. I expect this second method to be more accurate.