COURSE NAME : CH5019

Gambler’s Ruin Problem

Introduction

The Gambler's ruin problem is one of the classic problems in probability theory. In this project, we plan to stochastically model the gambler’s ruin problem.

Firstly we plan to model the gambler’s ruin problem as a one dimensional random walk process. Then we plan to calculate some important things related to the problem like probability that gambler goes broke before reaching the target of amount N,probability that gambler reaches the target of amount N before going broke, ruin probabilities of all gamblers, expected duration of the game and distribution of time in gambler’s ruin problem.

The model and results of gambler’s ruin problem are replicated and used in lot of different applications like :

1. Modeling the risk of bankruptcy for companies.It predicts the likelihood that a company will deplete its capital due to continuous losses before it can recover.
2. Insurance companies use it to model the risk of their reserves being depleted by a series of claims. This helps in setting premiums and reserves to ensure long-term solvency.
3. It can also be used to model the probability of the machine breaking down beyond repair before reaching a certain target number of operations.
4. This model can also be used to simulate sediment transport. Model is mathematically formulated to determine the probability of reaching the maximum state of the sediment carrying capacity, and the mean time spent in transient states before reaching the maximum sediment carrying capacity.

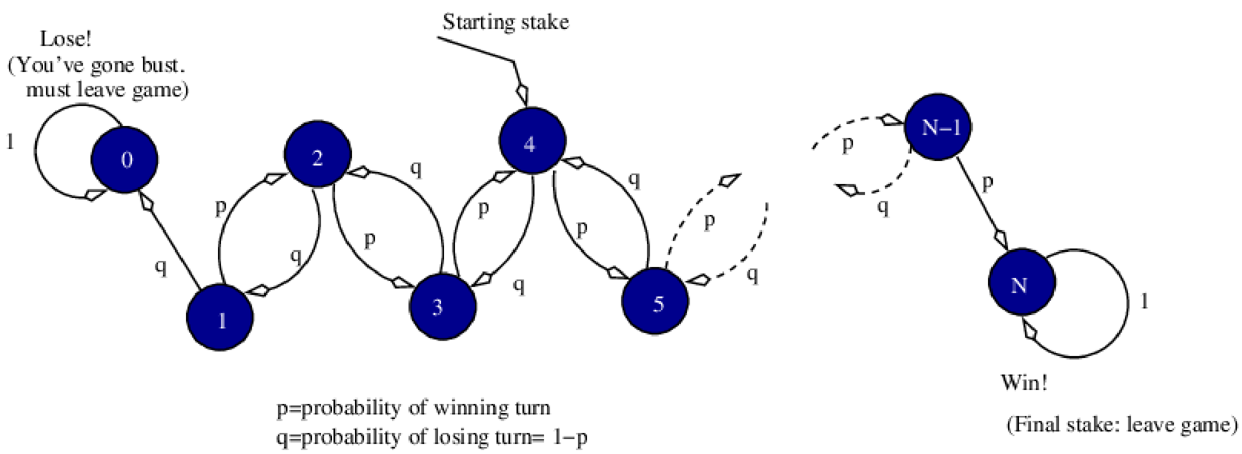
Problem Statement

The actual problem statement of the classical gambler's ruin problem is like this -

Consider a gambling game of a simple unfair coin toss. Alice has A dollars and Bob has B dollars. For convenience, the total amount of money can be written as M = A + B. At each step of the game, both players bet 1 dollar. Alice wins 1 dollar off Bob with probability , or Bob wins 1 dollar off Alice with probability .The game continues until one player is out of money (or is "ruined").

Methodology

Gambler's ruin problem can be formulated as an absorbing markov state process with the two absorbing states being when the player has 0 dollars or the target amount of dollars. A state is called absorbing if it is impossible to leave that state.



The total fortune of the gambler is X_n after the nth gambling step. While the game proceeds, $X_n > 0$ for all n>=0 forms a simple random walk.

At step n:

$$X_n = I_1 + I_2 + \dots\dots\dots + I_n$$

Where $\{ I_i \}$ forms a random variable sequence distributed as P(I = +1) = p and P(I = -1) = q = 1- p.

Game stops when X_n is either 0 or N. So the time in which game will end after its start can be given as :

$$\tau_i = min\{n \geq 0 : X_n \in \{0, N\} \mid X_0 = i\}$$

So simply put, if $X_{\tau_i} = N$, the gambler wins and if $X_{\tau_i} = 0$, the gambler is ruined.

Let $P_i = P(X_{\tau i} = N)$ denote the probability that the gambler wins when $X_{\tau i} = i$. Clearly $P_0 = 0$ and $P_N = 1$ by definition, and we next proceed to compute P_i , $1 \leq i \leq N - 1$.

By recurrence relation we can write :

$$P_i = pP_{i+1} + qP_{i-1}$$

Since $p + q = 1$, the above equation can be rewritten as :

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}) \, for \ 0 < i < N$$

Solving all these linear difference equations we get,

$$P_{i+1} = P_1 \sum_{i=0}^{i} (\frac{q}{p})^i$$

$$if\, p \neq q \quad P_{i+1} = P_1(\frac{1-(\frac{q}{p})^{i+1}}{1-(\frac{q}{p})})$$

$$if\, p = q \quad P_{i+1} = P_1(i + 1)$$

Choosing i = N − 1 and using the fact that $P_N = 1$ yields :

$$if\, p \neq q \quad P_1 = \frac{1-(\frac{q}{p})}{1-(\frac{q}{p})^N}$$

$$if\ p = q \quad P_1 = \frac{1}{N}$$

So the final result for P_i can be given as :

$$if\ p \neq q \quad P_i = \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N}$$

$$if\ p = q \quad P_i = \frac{i}{N}$$

Wald's Identity - Expected Duration of Game

The number of tosses before the end is obviously random. So we want to find the expected number of tosses before the game ends (ET) . To solve for the expected exit time ET,there is a well known approach called Wald's identity.

Let's say that we have S1 ,S2 ,... which is the sequence of partial sums of the increments I_i .The game ends at the random time T = min{n : S_n = 0 or S_n = A + B}. Clearly, for every n = 1, 2,...

$$ES_n = nEX_1 = n(p - q)$$

Wald's identity theorem basically says that for sequence of random variable S_i each with a stopping time , expected time T is related to the expected total sum at T as :

$$ES_n = (p - q)ET$$

Assuming the maximum amount of dollars that can be obtained at the termination state to be A + B and assuming initial amount was A, we calculate ES_n as :

$$ES_n = 0 * P(S_T = 0) + (A + B) * P(S_T = A + B)$$

So, $$ES_n = (A + B) * \frac{(1-(\frac{q}{p})^A)}{(1-(\frac{q}{p})^{A+B})}$$

Wald's identity gives the expected time :

$$ET = \frac{A+B}{p-q} * \frac{(1-(\frac{q}{p})^{A})}{(1-(\frac{q}{p})^{A+B})}$$

This gives the expected number of coin tosses before the gambler either wins the maximum amount or is ruined completely.

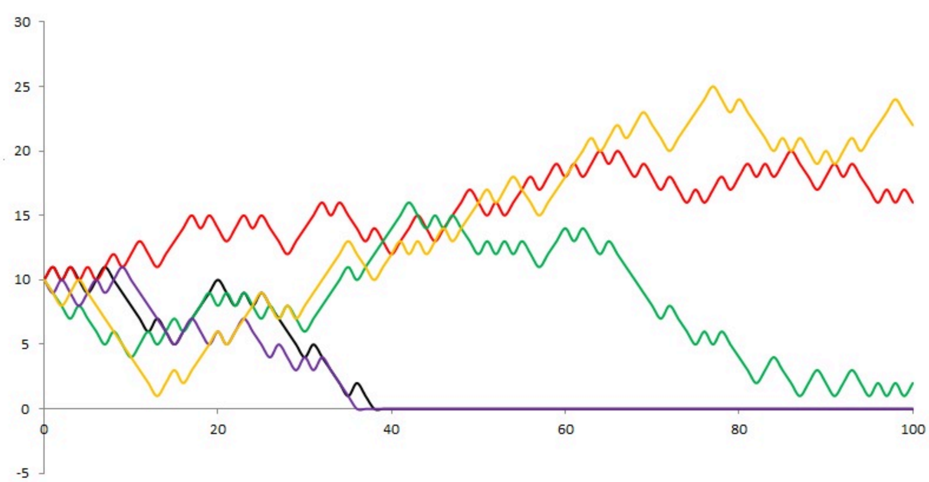
Results & Applications

These formulas give us some really cool and interesting results. For eg. Let's say John has 2 dollars initially and the casino's gambling probability is 0.6. The probability that he will get infinitely rich if he keeps playing is -

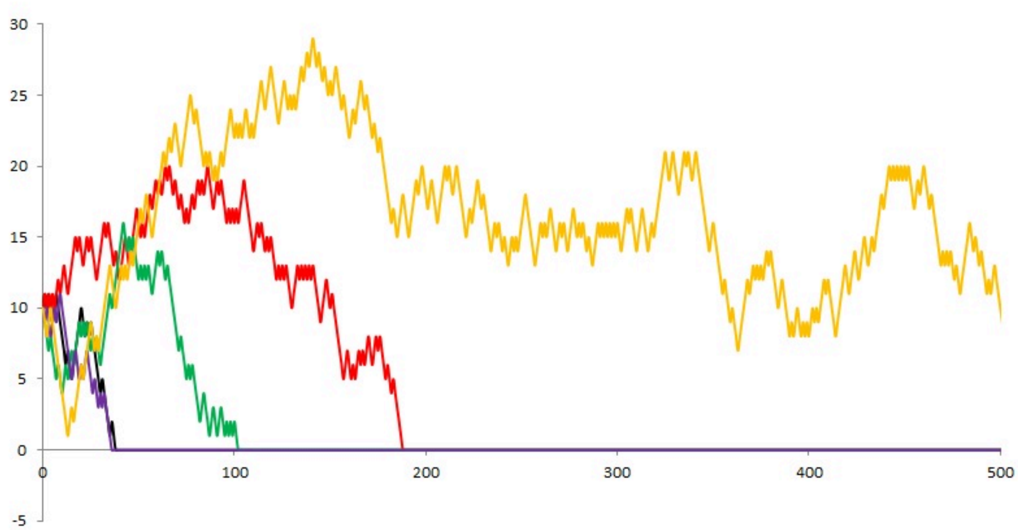
$$P_2(\infty) = 1 - (2/3)^2 = 5/9 = 0.56$$

So with these odds, he has a 56 % chance of becoming infinitely rich if he keeps on playing.

Here are some examples of generated random walks for the gambler's ruin problem with initial capital of 10 dollars :



This figure shows 5 different simulations of 100 plays each. Two of the simulations end in ruin (the purple at 36th play and the black at 38th play) fairly early in the process. At the end of 100 plays, the other three simulations are not yet in ruin, even though the random walk in green appears to be close to being in ruin. Let's extend the random walks up to 500 plays of the gambling games.



In all 500 plays of the game, no random walks have reached the maximum target of 100 units. In fact, the gambler is in ruin in four of the simulations, the purple and the black mentioned earlier along with the green and the red. The yellow random walk will likely be in ruin too if the game continues.

This is usually the case for most of the casinos. They design their probabilities in such a way that if you keep gambling for long periods of time, you will almost certainly lose money and hit 0.

This has many cool applications like risk insurance business. Consider an insurance company that earns \$1 per day (from interest), but on each day, independent of the past, might suffer a claim against it for the amount \$2 with probability $q = 1 - p$. Whenever such a claim is suffered, \$2 is removed from the reserve of money. Thus on the n th day, the net income for that day is exactly Δ_n as in the gamblers' ruin problem: 1 with probability p, −1 with probability q. Using the gambler's ruin problem formula, the company can calculate the probability of ruin.

Conclusions

1. A fundamental understanding of risk and reward in probabilistic circumstances can be gained from the Gambler's Ruin dilemma.
2. It emphasizes how losing money in fair games is mathematically inevitable over time and offers resources for estimating risk in financial and gaming endeavors.
3. The problem can be solved by modeling it as a simple absorbing state one dimensional random walk.
4. This problem is frequently visited in many different fields like finance, insurance,population genetics and computer algorithms.
5. There are more complex variations of gambler's ruin problem in higher dimensions which can not be solved by modeling them as brownian motions. There are more intelligent and complex algorithms based on matrix-analytic methods.

References

1. Generalized Gambler's Ruin Problem
   (https://sites.math.rutgers.edu/~zeilberg/EM20/GamblersRuin.pdf)
2. Probability of time to ruin distribution
   (https://stats.stackexchange.com/questions/514193/what-is-the-distribution-of-times-to-ruin-in-the-gamblers-ruin-problem-random)
3. MATH2750 notes
   (https://mpaldridge.github.io/math2750/S03-gamblers-ruin.html#ruin-chain)
4. RANDOM WALKS IN ONE DIMENSION
   (https://galton.uchicago.edu/~lalley/Courses/251/RW-251.pdf)
5. Columbia University - gambler's ruin problem.
   (https://www.columbia.edu/~ks20/stochastic-I/stochastic-I-GRP.pdf)