COURSE NAME : CH5019

Stochastic Processes in Financial Markets

Stock prices prediction and risk analysis using stochastic process modeling

Introduction

Equity or stock markets are known for their randomness and uncertainty, making them very difficult to predict and model accurately . Many time series and machine learning algorithms tend to be deterministic and don't really give much attention to the uncertainty in the stock data. On the other hand,stochastic modeling is inherently random, and the uncertain factors are built into the model. The model produces many answers, estimations, and outcomes which helps us model the uncertainty in our predictions.

One of the most widely used models for stock price behavior is geometric brownian motion. In this project we will focus on modeling l the stock return of a chosen stock on the financial market using the geometric Brownian motion model, and compare it against the actual stock return to ascertain if the model is a good fit.

We will also discuss the limitations of geometric brownian motion and how these problems are rectified. In that context, we will briefly introduce the Merton jump diffusion model.

Objective & Data collection

We are interested in finding the return of the stock price **x(t)** , given any $\mathbf{x}_i$ for i = 0, 1, 2..., i−1, where $\mathbf{x}_i$ is defined as the stock closing price at the end of the i th trading period.

For data collection of this model, we will mainly select stock of large companies that have been in operation for a very long period of time or the companies that are dominant leaders in their respective industries. The reason being the stock prices of such companies are not as volatile as compared to smaller companies, and are less likely to be affected by financial news which might otherwise cause large deviations in stock prices of smaller companies. This will help us focus more on the systematic risk or uncertainty. Ultimately in problems like alpha trading and portfolio optimization, we are interested more in systematic uncertainties in stocks because of market volatility. Unsystematic risks that happen due to factors like company specific financial news are not really relevant in important problems like portfolio optimization.

So in the case of our problem, we will be using Microsoft for our primary analysis.

Simulation methodology

By construction, geometric brownian motion (**GBM**)is a simple continuous-time stochastic process in which the logarithm of the randomly varying quantity of interest follows a brownian motion with drift.

Before going into the methodology of actual GBM, we will first try to give an overview of what the famous “Ito’s lemma” is. The reason being it is the central concept for solving the stochastic differential equation.

Ito’s lemma

One of the most fundamental tools from ordinary calculus is the chain rule. It allows the calculation of the derivative of chained functional composition :

Chain rule works like this :

$$dW(t) = \mu(W(t), t)dt$$

$$d(f(W(t))) = f'(W(t))\mu(W(t), t)dt$$

In the similar sense, Ito's Lemma is a key component in the Ito Calculus, used to determine the derivative of a time-dependent function of a stochastic process. It performs the role of the chain rule in a stochastic setting, analogous to the chain rule in ordinary differential calculus.

Theorem states that :

Let $B(t)$ be a Brownian motion and $W(t)$ be an Ito drift-diffusion process which satisfies the stochastic differential equation:

$$dW(t) = \mu(W(t), t)dt + \sigma(W(t), t)dB(t)$$

If $f(w,t) \in C^2(R^2, R)$ then $f(W(t),t)$ is also an Ito drift-diffusion process, with its differential given by:

$$d(f(W(t), t)) = \frac{\partial f}{\partial t}(W(t), t)dt + f'(W(t), t)dW + \frac{1}{2}f''(W(t), t)dW(t)^2$$

Getting back on the main GBM model methodology , we can represent it by a Langevin equation :

$$d(x(t)) = x(t)\, [\mu dt + \sigma d(B(t))] \text{ \_\_\_\_\_\_ (Eq 1)}$$

As mentioned before ,

$x(t) = stock\ price\ at\ time\ t$

$\mu = drift\ term$

$\sigma > 0 => volatility$

$B(t) = standard\ brownian\ motion$

Now using the Ito's calculus formula we used before, we obtain the solution to the (Eq 1) :

$$x(t) = x_0 e^{(\mu - \sigma^2/2)t + \sigma B(t)} \quad where\ x_0 = x(0) \text{ \_\_\_\_\_\_ (Eq 2)}$$

By applying Ito's Lemma to this SDE and then equating it with the Fokker-Planck equation, you can derive the corresponding drift and diffusion coefficients. For GBM, the drift coefficient

$$A(x) = \mu x$$

$$B(x) = \frac{1}{2}\sigma^2 x^2$$

When the dynamics of the asset price follows a GBM, then a risk-neutral distribution (probability distribution which takes into account the risk of future price fluctuations) can be easily found by solving the Fokker-Planck equation with the above mentioned characteristic equations :

$$\frac{\partial P(x,t)}{\partial t} = -\mu\frac{\partial[xP(x,t)]}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2[x^2P(x,t)]}{\partial x^2} \quad \text{\_\_\_\_\_\_ (Eq 3)}$$

with initial condition P(x,t = 0) = δ(x − x_0). The solution of Eq. (3) is the famed log-normal distribution

$$P(x,t) = \frac{1}{x\sqrt{2\pi\sigma^2 t}}exp(-\frac{\left[log\,x - log x_0 - \bar{\mu}\right]^2}{2\sigma^2 t})$$

where $\bar{\mu} = \mu - \sigma^2/2$.

From the solution, it follows that the mean value and the mean square displacement (MSD) have exponential dependence on time :

$$< x(t) > = x_0 e^{\mu t} \quad \text{\_\_\_\_\_\_ (Eq 4)}$$

and

$$< x^2(t) > = x_0^2 e^{(\sigma^2+2\mu)t} \quad \text{\_\_\_\_\_\_ (Eq 5)}$$

and thus, the variance becomes

$$\langle x^2(t)\rangle - \langle x(t)\rangle^2 = x_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right) \quad \text{\_\_\_\_\_\_ (Eq 6)}$$

In GBM the diffusion coefficient scales proportionally with the square of the position of the particle, i.e , D(x) = $x^2\sigma^2/2$ and thus the MSD has an exponential dependence on time. A more convenient measure instead of the MSD for geometric processes is the behavior of the expectation of the logarithm of x(t). In the case of GBM the expectation of the logarithm of the particle position has a linear dependence on time.The mean value of the logarithm of x(t) becomes :

$$log(< x(t) >) = log(x_0) + \bar{\mu}\, t \quad \_\_\_\_\_\_ \text{(Eq 7)}$$

Also, the log-variance yields :

$$< log^2(x(t)) > - \, log(< x(t) >)^2 = \sigma^2 t \quad \_\_\_\_\_\_ \text{(Eq 8)}$$

Now using the A(x) and B(x) characteristic equations, we will do monte carlo simulation to generate different possible paths.

In our course we saw how to do this for continuous time processes. The similar logic is applied to generate the sample price trajectories.

Obviously , we will start our simulations from the last available historical stock price i.e.

$X(0) = X(T_{last\_historical})$

Results

Using the stochastic monte carlo simulation algorithm we simulate 2500 different price trajectories. You can see the algorithm for these simulations in this notebook.

We decided to do these simulations on the Microsoft stock price data from last year 13th May to this year's 13 th May.

The log returns and volatility graphs of the historical data can be seen here :

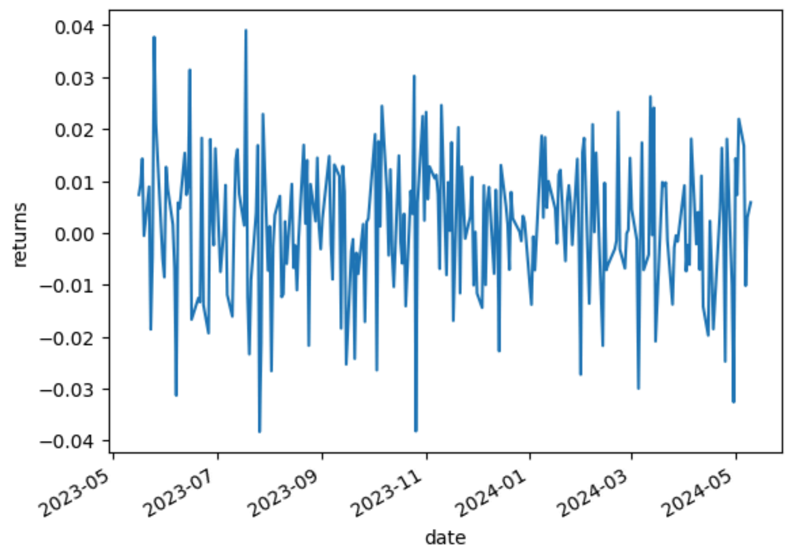

Fig 1 : Log returns vs date

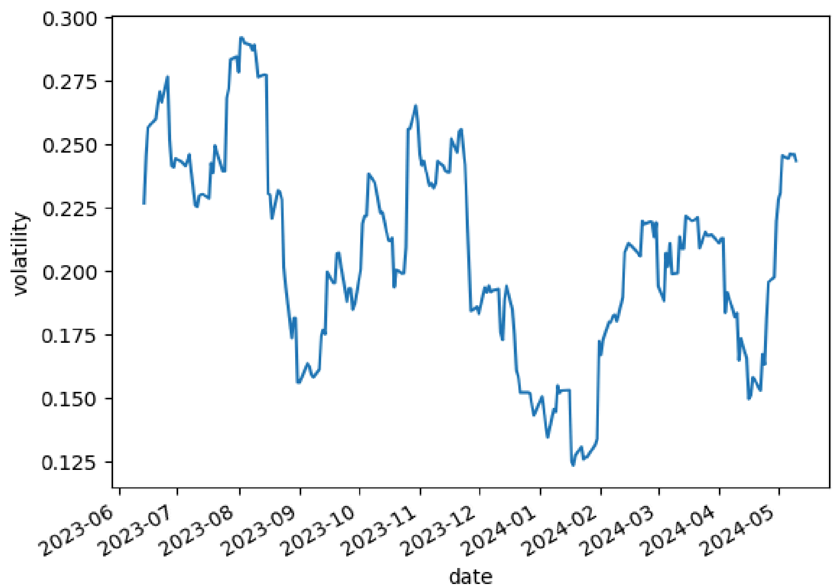

Fig 2 : Returns volatility vs date

As you can see the returns volatility of the Microsoft stock is fairly intermediate. That's why it is a fairly perfect choice for the simulation testing.

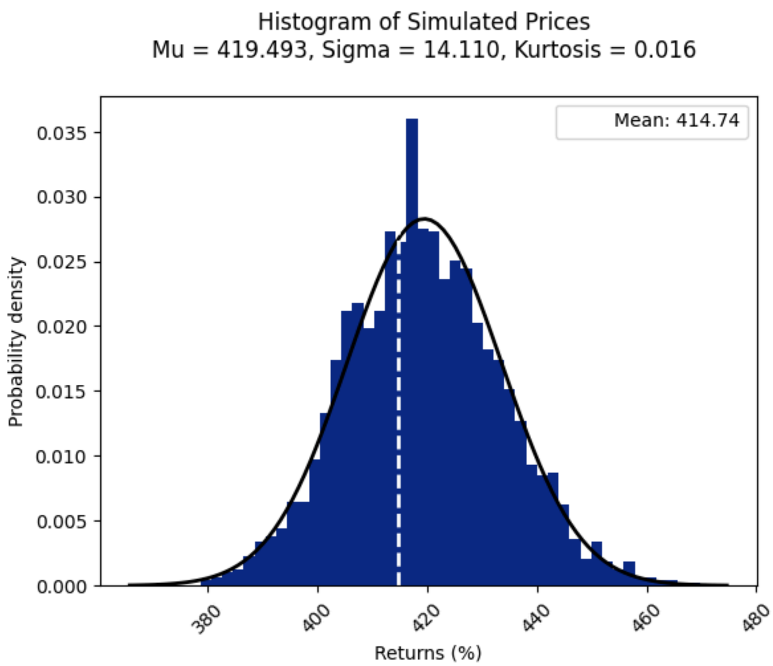

Fig 3 : Distribution of simulated prices

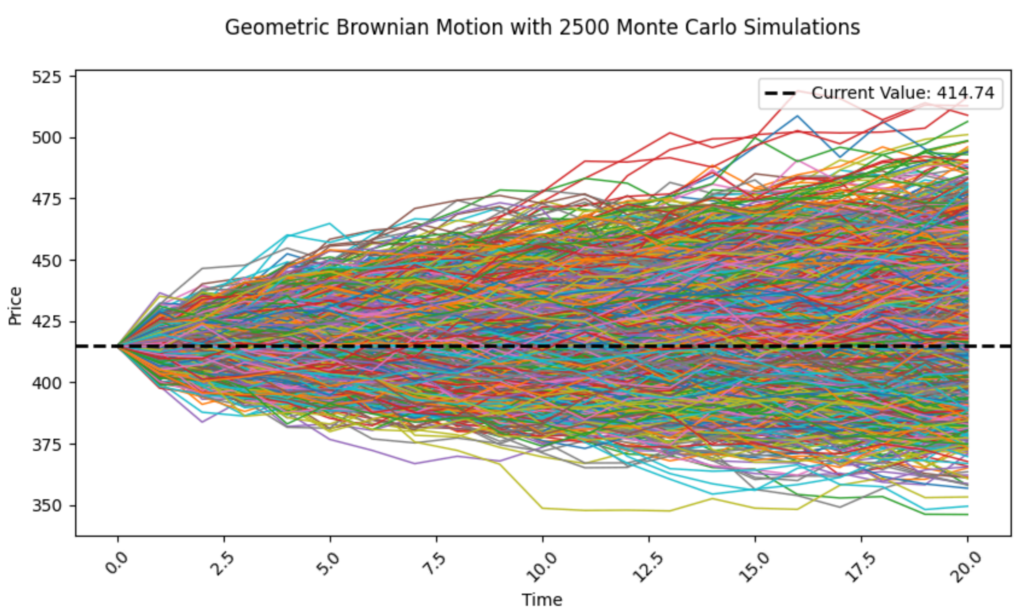

Fig 4 : Results of monte carlo simulations using GBM

Simulation graph gives us some really important information. For eg. One of the most important parameters needed in trading or investing is stop loss for that stock. Stop loss is basically a risk-management tool that automatically sells a security once it reaches a certain price. From these simulations, we can derive the function stop_loss(t). We can also get a general idea of stock price direction which can be used for decision making in portfolio optimization.We can also see the uncertainty for each prediction which can be quantified using the confidence interval limits.

Limitations and Improvements

The geometric brownian motion model has many advantages like :

- The model only assumes positive values (stock price at any given time is positive)
- Expected returns of the model are independent of stock prices at each time point

However, there are also some disadvantages to using this model.

- Stock price volatility tends to change over time, but volatility is assumed constant (constant σ) under the model. •
- Stock prices often show sharp jumps / dips due to unpredictable events or news, but the path is continuous under the model.

The second disadvantage is particularly hard to overcome. One of the ways it can be done is by using something called the **Merton Jump Diffusion Model.** It was proposed by Robert C. Merton who addressed this particular limitation of the geometric brownian motion presented by Black and Scholes. Geometric Brownian motion followed the normal distribution and didn't take into consideration the discontinuity occurring in the stochastic processes. Merton added a jump and diffusion component to the geometric Brownian motion formula, in which the model adds sudden jumps and discontinuities in the stochastic process to address highly-improbable events like stock crashes and therefore make the model more robust to tail risk by making the simulation results have heavier downside tails. An optional addition to this model is to add sudden upside jumps to explain the behavior of stocks with a more symmetric probability shape.

So we believe that using the Merton **Jump Diffusion Model** might be more intuitive and provide better simulation results than GBM.

Conclusions

In this project, we modeled the stock price at a future point in time by considering the movement of Microsoft stock prices as a stochastic process. We discussed the methodology of geometric brownian motion and use of monte carlo simulations to generate sample stock price trajectories. We also discussed the use of Ito's calculus for solving the stochastic differential equation of this process.
We investigated the probability distribution of simulated stock prices and also 2500 different stock price trajectories. We showed how it can be used to calculate some important parameters like stop loss,mean log returns and overall price drift direction.
Then we also discussed the main limitations of geometric brownian motion. One of the main avenues of improvement we found was inducing some kind of jump effect for modeling unpredictable market news and trader sentiments. So we discussed the Merton jump diffusion model that can help fix some major limitations of GBM.

References

1. Generalized geometric Brownian motion: Theory and applications to option pricing [Viktor Stojkoski1,2 , Trifce Sandev, Lasko Basnarkov,, Ljupco Kocarev2,5 and Ralf Metzler3,]
2. Stock price prediction using geometric Brownian motion [Farida Agustini W, Ika Restu Affianti, Endah RM Putri]
3. Introduction to Merton Jump Diffusion Model [Kazuhisa Matsuda]
4. Brownian Motion and Its Applications In The Stock Market [Angeliki Ermogenous]
5. Ito's calculas for GBM

Code

All the graphs were generated by me. So the code for these simulations and results can be found here :

https://drive.google.com/file/d/1dV6Th90j4gRU3mHK5PygqD6EVYCdC0LQ/view?usp=sharing