# Approximate Solution of Logistic Models of Population

## Introduction

The recent past has led to the merging of mathematics and many areas of the sciences in the approximate solution of model in physics, chemistry, biology, geology, and other scientific questions. We will consider models of population density. These modules will focus on the development of the logistic model that is commonly used to study problems in ecology and biology.

We will start with a simpler model called the Malthusian model. This will produce an exponential population density model that grows, decays, or remains unchanged. The Malthusian model will be modified to include terms that limit the growth or decay of the population density.

As a part of the presentation, numerical approximations for the models will be developed to solve differential equations that result from these mathematical models developed.

## Malthusian Model of Population

One of the oldest models of population growth involves the assumption that the rate of change in the number of individuals in a population is proportional to the number individuals in the population. Let's translate this into a mathematical problem.

Suppose that we are modeling the number of voles in some region where the voles are not receiving any pressure due to predation or disease. Note that this situation is highly unlikely. If there are cats or dogs or owls and hawks or other predatory species that live in the neighborhood, the result will always be that the voles will be killed and will need to watch out for these predators. However, it will be easier to model the population in this simple setting.

**Note:** We will modify the simple Malthusian model below. The result will be the Logistic model.

**Notation:** Let's define some mathematical notation for the mathematical problem. Let

- $t$ - will represent time in days (365 days per year) or months (12 months per year)
- $V(t)$ - the total number of voles in the region of of interest. This is a positive integer.
- $A$ - the total area of the region in the model (e.g, 10 acres).
- $v(t)$ - the population density of the voles in.

We can relate the number of voles in the region to the population density as follows:

$$v(t) = \frac{V(t)}{A}$$

**Note:** If we use the total number voles, $V(t)$, we are limited to using integers to keep track of the population. So, if $V(0) = 21,227$, then what is the value predicted by the nodel at later times. That is, the number of voles after 1 day, 1 month, 1 year, 1 decade, and so on. In this case, we are asking for a a value that is a real number. It is easier to use the population density instead.

## Logistic Models of Population

Based on very simple calculations, the simple Malthusian model is clearly good on a small interval. To get a better population density model, we can add a term that acts to limit the population density.

One way to do this is to consider the linear term

$$L = \alpha \ P$$

where $\alpha > 0$. The initial population density will be a positive value, the population will continue grow, unchecked. However, if we add a term of the form

$$N = -\beta \ P^2$$

Due to the fact that the population density squared will be positive. We will choose $\beta \geq 0$, So the condition, $N \leq 0$ is true. This implies that the term will reduce the population density.

The Logistic model can be written as

$$\frac{dP}{dt} = \alpha \ P - \beta \ P^2$$

with an initial density value defined by $P(0) = P_0$. This again defines an initial value problem for population density.

**Note:** Even though this is an initial value problem, it is a nonlinear initial value problem due to the quadratic nonlinearity in the unknown solution, $P$. That is the $\beta P^2$ term in the differential equation.

## Analytic Solutions For Malthusian and Logistic Models

Both models described in this module admit analytic solutions for the initial value problem. This is one of the reasons that these types of models appear in calculus books.

**Analytic Solution using Separation of Variables**

Separation of variables is a technique used to solve a wide range of linear and nonlinear differential equations. We will use the simple equation

$$\frac{dP}{dt} = \alpha \ P$$

from the Malthus population model as a means to review the method of separating variables.

**Step 1: Multiply the Differential in $dt$** The first step involves multiplying the equation by $dt$ to obtain

$$\frac{dP}{dt} \ dt = \alpha \ P \ dt$$

Using the rules of differentials,

$$\frac{dP}{dt} \ dt = dP$$

and the equation can be written as

$$dP = \alpha \ P \ dt$$

**Step 2: Solve for $P$ and $dP$** The next thing we do in separation of variables is to move all of the dependence on the dependent variable, $P$, to one side of the equation. This means dividing both sides by $P$ in this case. The equation can be written as

$$\frac{1}{P} \ dP = \alpha \ dt$$

The left side of the equation only depends on $P$ while the right hand side of the equation is dependent on $P$.

**Step 3: Integrate** The next step involves integrating the left hand side with respect to $P$ and the right hand side with respect to $t$. That is,

$$\int \frac{1}{P} \ dP = \int \alpha \ dt$$

Now integrate both sides of the equation to obtain

$$ln|P| + C_1 = \alpha \ t + C_2$$

or, combining contants

$$ln|P| = \alpha \ t + (C_2 - C_1) = \alpha \ t + C_3$$

This is an implicit relationship between the dependent variable, $P$, and the independent variable, $t$.

**Step 4: Compute an Explicit Solution Form** We can determine an explicit expression for $P(t)$ by exponentiating both sides of the equation. This results in

$$P(t) = e^{\alpha\ t + C_3} = e^{\alpha\ t}\ e^{C_3} = A\ e^{\alpha\ t}$$

So, the exact solution is a simple exponential function.

**Step 5: Apply the Initial Condition** For an initial population, $P_0$, we can write

$$P(0) = P_0 = A\ e^0 = A$$

The final form of the solution as a function of the independent variable, $t$, and the initial population, $P_0$. That form is

$$P(t) = P_0\ e^{\alpha\ t}$$

We also know that $\alpha$ is also a part of the original problem and will give a rate of change in the population density.

**Note:** This method is suited to initial value problems where one integral is needed to produce the solution. The hardest part is going to be evaluating the integral. Most cases will involve an integral that may not be "doable". However, the work will result in some sort of implicit relationship.

### Extending to the Analytic Solution of Logistic Models

For the "better" model of population densities contained in the logistic model, we would like to apply the same separation of variables to produce an analytic solution. We can follow steps used on the Malthusian model. Recall that the logistic equation can be written as

$$\frac{dP}{dt} = \alpha\ P - \beta\ P^2$$

We will start from this form and come up with a unique solution.

**Step 1: Multiply by the differential** For an analytic solution, we start by mutiplying the equation by a differential in the independent variable. The result is the following.

$$\frac{dP}{dt}\ dt = \left(\alpha\ P - \beta\ P^2\right)\ dt$$

**Step 2: Solve for $P$ dependence** The result in this case is the following

$$\left(\frac{1}{\alpha\ P - \beta\ P^2}\right) \frac{dP}{dt}\ dt = dt$$

This can be modified to an equation of the form

$$\left(\frac{1}{P\ (\alpha - \beta\ P)}\right) \frac{dP}{dt}\ dt = dt$$

This form lends itself to an integration by partial fraction decomposition - which is the next step.

**Step 3: Rewrite the Integration using Partial Fractions** The next step is to apply partial fraction decompositions. This gives

$$\frac{1}{P\ (\alpha - \beta\ P)} = \frac{a}{P} + \frac{b}{\alpha - \beta\ P}$$

and by clearing fractions we obtain

$$1 = a\ (\alpha - \beta\ P) + b\ P$$

In partial fraction decomposition we can do one of two things. We can evaluate the expression above at special values of the unknown $P$ or we can compare coefficients in the (linear) polynomial in the unknown, $P$.

We will use the first method. Evaluating this expression for the value $P = 0$ results in

$$1 = a\ \alpha$$

This implies $a = \frac{1}{\alpha}$. For the value $P = \frac{\alpha}{\beta}$ the equation becomes

$$1 = b\ \frac{\alpha}{\beta}$$

So, $b = \frac{\beta}{\alpha}$ must be true.

We can substitute the expression into the separated equation to end with

$$\left(\frac{1}{\alpha\ P} + \frac{\beta}{\alpha}\ \frac{1}{\alpha - \beta P}\right) \frac{dP}{dt}\ dt = dt$$

The work to this point has been mostly a bit of algebra.

**Step 4: Solve for $P$ and $dP$** Using the calculus of differentials the work produces an equation with explicit $P$ dependence on the left hand side of the equation and explicit $t$ dependence on the right hand side of the equation. The equation is

$$\left(\frac{1}{\alpha\ P} + \frac{\beta}{\alpha}\ \frac{1}{\alpha - \beta P}\right)\ dP = dt$$

**Step 5: Integrate** Integrating the left hand side with respect to $P$ and the right hand side with respect to $t$ gives

$$\frac{1}{\alpha}\ ln(P) - \frac{1}{\alpha}\ ln(\alpha - \beta P) = t + C$$

where $C$ is a generic constant of integration. In the algebra below, we will reuse the samem $C$ to indicate the constant we need. The initial condition will be used below to determine the final value.

**Step 6: An Explicit Expression for the Solution** It is always a good idea to get an explicit solution form. Thhis can be done for our logistic model with a bit of work.

Multiplying through by $\alpha$ will result in

$$ln(P) - ln(\alpha - \beta P) = \alpha\ t + C$$

The terms on the left hand side of the equation can be combined using properties of logarithms.This means

$$ln(\left(\frac{P}{\alpha - \beta\ P}\right) = \alpha\ t + C$$

Next, exponentiating both sides will get us closer to an explicit expression for the solution.

The exponentiation results in

$$\frac{P}{\alpha - \beta\ P} = Ce^{\alpha t}$$

and clearing fractions results in

$$P = (\alpha - \beta\ P)\left(C\ e^{\alpha t}\right)$$

or

$$\left(1 + C\beta\ e^{\alpha t}\right)\ P = C\ \alpha\ e^{\alpha t}$$

We need only one more step to write

$$P(t) = \frac{C\ \alpha\ e^{\alpha t}}{1 + C\beta\ e^{\alpha t}}$$

So, after a few steps we have a solution.

**Note:** The steps are a bit tedious. However, the work results in an explicit solution form for the **nonlinear** ordinary differential equation. This type of approach will not work for most nonlinear differential equations.

## Approximate Solutions for Malthusian and Logistic Models

So, we have analytic solutions for both models. However, if we choose to develop more complicated models for the problem, there may be no way to compute an exact solution for the new model. In the more complicated cases it may be necessary to settle for an approximate solution.

There are a couple of reasons for pursuing an approximate solution for the two models that are presented in this module.

**First**, we will be developing numerical models for more complicated models. Doing the initial work on the Malthusian and Logistic models will be good practice on simple models where we have an analytic solution to compare results from the approximation.

**Second**, even in case of the simple models, evaluation of the functions involving exponential functions, the evaluations are at best done by approximation methods. In any computational environment the evaluation of most functions that we see in calculus involve approximations using some sort of series expansion or an approximation using integration formulas and the like.

### Finite Difference Approximations for the First Derivative

For both models, we can use an approximation of the first derivative to approximate the differential equation. Given the definition of the derivative,

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{P(t + \Delta t) - P(t)}{\Delta t}$$

we can define an approximation of the derivative of the form

$$\frac{dP}{dt} \approx \frac{P(t + \Delta t) - P(t)}{\Delta t}$$

**Note:** The approximation is valid for any $\Delta t$ that is greater than zero. How good the approximation is, will depend on the value of $\Delta t$

For each of the models obtained above, this specific derivative approximation will be used and an algorithm will be defined to compute the approximation. This will be done over the next couple of sections of the module.

### An Explicit Euler Method

The Euler method uses the first two terms in a Taylor series to approximate the exact solution of an ordinary differential equation. For the Malthusian model, we can use

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} \approx \alpha \ P(t)$$

Solving for $P(t + \Delta t)$ gives the formula

$$P(t + \Delta t) \approx P(t) + \alpha \ P(t) \ \Delta t = (1 + \alpha \ \Delta t) \ P(t)$$

Let's see what we can do with this.

If we apply the initial condition from the original differential condition with $t = 0$ and $P(0)$ we can write

$$P(\Delta t) \approx (1 + \alpha \ \Delta t) \ P_0$$

We are given both $\alpha$, the growth rate in the population density model and $P_0$, the initial population density. What we get from this is an approximate value for the solution at $t = \Delta t$.

We can assign this value to a parameter, $P_1$. That is,

$$P_1 = (1 + \alpha\ \Delta t)\ P_1$$

We can use this value to **bootstrap** our approximate solution, at $t = 2\Delta t$, using the formula

$$P_2 = (1 + \alpha\ \Delta t)\ P_1$$

and so on. We can continue this process for as many steps forward as possible. This process can be written as a recursive formula of the form

$$P_{k+1} = (1 + \alpha\ \Delta t) P_k$$

with $k = 0, 1, \ldots$ until done.

There is one last thing we can do with the approximation formula. That is, we can write down an analytic solution to the approximation. To do this, write

$$P_{k+1} = (1 + \alpha\ \Delta t) P_k = (1 + \alpha\ \Delta t)^2 P_{k-1} = \cdots = (1 + \alpha \Delta t)^k\ P_0$$

The fate of the population density value is dependent on the discrete growth/decay rate $r = 1 + \alpha\ \Delta t$.

**Note:** In computational mathematics, the factor is called an **amplification factor** and determines how the approximation behaves.

### An Implicit Euler Method

The implicit method for producing approximations for the models comes out of evaluating the right side of the first approximation above. That is,

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} \approx \alpha\ P(t + \Delta t)$$

Solving for $P(t + \Delta t)$ gives

$$(1 - \alpha\ \Delta t) P(t + \Delta t) \approx P(t)$$

So, the value at $t + \Delta t$ is given by

$$P(t + \Delta t) = \frac{1}{1 - \alpha \Delta t}\ P(t)$$

Given $P(t)$ we will use this approximation to come up with an approximation for $P(t + \Delta t)$.

As in the case of the explicit Euler method we will define a sequence of approximations at $t = \Delta t, 2\Delta t, \ldots$. Using this idea will produce $P_1, P_2, \ldots$ using

$$P_{k+1} = \frac{1}{1 - \alpha \Delta t}\ P_k$$

with $P_0 = P(0)$.

## Application of the Methods to Population Density Models

To complete this module, we will need to apply the approximation methods (explicit Euler and implicit Euler) to the population density models described at the beginning of the module. The formulas for the methods are listed without derivation.

### The Malthusian Model

**Explicit Euler:** Given $P_0 = P(0)$

$$P_{k+1} = (1 + \alpha\ \Delta t)P_k$$

**Implicit Euler:** Given $P_0 = P(0)$

$$P_{k+1} = \frac{1}{1 - \alpha\ \Delta t}\ P_k$$

### The Logistic Model

**Explicit Euler:** Given $P_0 = P(0)$

$$P_{k+1} = P_k + \Delta t\ (\alpha P_k - \beta P_k^2)$$

**Implicit Euler:** Given $P_0 = P(0)$. Solve for $P_{k+1}$ in the following.

$$(\beta\ \Delta t)\ P_{k+1}^2 + (1 - \alpha\ \Delta t)\ P_{k+1} - P_k = 0$$

**Note:** The equation in the Implicit Euler method for the logistic model is a simple quadratic of the form

$$a\ p^2 + b\ p + c = 0$$

where $a = \beta\ \Delta t$, $b = 1 - \alpha\ \Delta t$, and $c = -P_k$. Apply the quadratic formula to the resultant polynomial.

**Note:** One of the two roots will be meaningful and the other will not make sense.

$$P_{k+1} = \frac{(1 - \alpha\ \Delta t) \pm \sqrt{(1 - \alpha\ \Delta t)^2 + 4(\beta\ \Delta t)}}{2\ \beta\ \Delta t}$$

There are any number of ways to simplify the formula to minimize the number of calculations. However, for this module, the number of computations is not the major issue.

## Content Review

## Module Vocabulary

## The Main Question for Math 4610 at USU

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The question in this module is can we modify the terms to define a new relationship between the Malthus model for the evolution of population density and the corresponding logistic model. Suppose instead of the quadratic term that we subtract a term of the form $\gamma\ ln(P)\ P$. This will create an equation of the form

$$\frac{dP}{dt} = \alpha\ P - \gamma\ P\ ln(P)$$

with $P(0) = P_0$. Compare this model to the other two models. Use $\alpha = 0.2$, and $\gamma = 0.0003$. Determine the carrying capacity of the new model and compute approximations for the new model between $P = 0$ and the carrying capacity. Use both the explicit and implicit Euler methods to compute approximations. Print out numerical values for each of the cases you use to compare.

**Note:** There is no closed form analytic solution for the modified model in the exercise. We must be satisfied with the approximate solutions.