Simple continued fractions arithmetic

Algorithm 1 (The simple continued fraction expansion of a rational number). The simple continued fraction expansion of a rational number is actually the Euclidean algorithm, so *Definition 3* can be directly applied.

Algorithm 2 (The simple continued fraction expansion of homographic transformations). Let $f(x)=\frac{ax+b}{cx+d}$ with integers a, b, c, d and $x = [x_0, x_1, ...]$ satisfying $cx + d \neq 0$. If x is not an *integer*, it must also satisfy $ad \neq bc$ and additionally, depending on its value:

- if $x < -1$ then $c(c-d) \geq 0$
- if $-1 < x < 0$ then $d(d-c) \geq 0$
- if $0 < x < 1$ then $d(c + d) \geq 0$
- if $x > 1$ then $c(c+d) \geq 0$

Also, if x is irrational, $f(x)$ also has to be irrational. In order to compute the simple continued fraction expansion of $f(x)$ take the following steps:

0. *if* x is an *integer then* use *Algorithm 1* to compute the simple continued fraction expansion of the rational $f(x)$ and *stop*
1. *if* $x < 0$ *then* let $f(y)=\frac{-ay+b}{-cy+d}$, $y = -x$; replace x with y
2. *if* $0 < x < 1$ *then* let $f(y)=\frac{by+a}{dy+c}$, $y=\frac{1}{x}$; replace x with y
3. *assert* $x \geq 1$; *assert* $ad \neq bc$ *and* $c(c + d) \geq 0$; *assert* $cx + d \neq 0$; *assert* $x \in Q \vee f(x) \notin Q$
   if x is an *integer then use Algorithm 1* to compute the simple continued fraction expansion of the rational $f(x)$ and *stop*.
4. *assert* $x > 1$
   if $c \neq 0$ *and* $c + d \neq 0$ *and* $floor(\frac{a}{c})=floor(\frac{a+b}{c+d})$ *then assert* $c(c + d) > 0$ and then *output* $q=floor(\frac{a}{c})$, let
   $$f(x)=\frac{1}{\frac{ax+b}{cx+d}-q}=\frac{cx+d}{(a-cq)x+b-dq}=\frac{cx+d}{(a \bmod c)x+(a+b) \bmod (c+d)-a \bmod c}$$
   and *goto* step 3
5. (*otherwise*)
   assert $x > 1$
   input x_0 and let $f(y)=\frac{a(x_0+\frac{1}{y})+b}{c(x_0+\frac{1}{y})+d}=\frac{(ax_0+b)y+a}{(cx_0+d)y+c}$ where $y = [x_1, x_2, ...]$;
   replace x with y and *goto* step 3.

Note: If x is a simple continued fraction expansion rather than a simple continued fraction, the *assert* $x \geq 1$ statement in step 3 can be changed to *assert* $x > 1$.

Proposition 1. All the assertion statements in *Algorithm 2* always succeed and step 5 is executed at most a finite number of times until an output is generated. Also, all the output partial quotients are correct.

Proof. Let's prove that the assertion statements always succeed. There only three possible paths by which the algorithm can arrive at step 3: *1-2-3*, *4-3* and *5-3*.

In the *1-2-3* path the assertion in step *3* evidently succeeds.

In the *4-3* path, because x is unchanged by step *4*, it remains true that $x > 1$ and $x \in Q \vee f(x) \notin Q$; $ad \neq bc$ implies $c(b - d\,q) \neq d(a - c\,q)$; $(a \bmod c)(a \bmod c + (a + b) \bmod (c + d) - a \bmod c) \geq 0$. Because $x > 1$, $\frac{ax+b}{cx+d} \neq q$ the homographic function being strictly monotonic. This implies $(a - cq)\,x + b - dq \neq 0$. Because the value of $f(x)$ is changed simply by subtracting an integer and inverting, step *4* does not change the state of $f(x)$ being or not rational *so* $x \in Q \vee f(x) \notin Q$ remains true.

In the *5-3* path, $y = [x_1, \ldots] \geq 1$ and $ad \neq bc$ implies $(ax_0 + b)c \neq (cx_0 + d)a$. Moreover, $(cx_0+d)y+c=(cx_0+d)\frac{1}{x-x_0}+c=\frac{cx+d}{x-x_0}\neq 0$. If $c = 0$, $(cx_0 + d)(cx_0 + d + c) = d^2 \geq 0$. If $c \neq 0$, $(cx_0 + d)(cx_0 + d + c) = (cx_0 + d)^2 + c(c + d) + c^2(x_0 - 1) \geq 0$.

In step 4, $c(c + d) > 0$ implies $(cy + d) \neq 0$ for all $y \geq 1$. As $f'(y)=\frac{ad-bc}{(cy+d)^2}$ has constant sign + or – over $[1, \infty)$, f is strictly monotonic on the definition domain, $x > 1$ implies $f(x)$ between $f(1)=\frac{a+b}{c+d}$ and $f(\infty)=\frac{a}{c}$. If both have the same floor q then *floor(f(x))* is certainly q and the output partial quotient is correct.

We claim that at any time step 4 is executed, the current step or the next one satisfy the conditions $c \neq 0$ and $c + d \neq 0$. Moreover, if the current step satisfies $c(c + d) > 0$, all the consecutive step 4 executions also satisfy $c(c + d) > 0$. Of course c and $c + d$ cannot be simultaneously zero, because of the assertion $ad \neq bc$. If $c = 0$, then $cx_0 + d \neq 0$ and $(cx_0 + d) + c \neq 0$. If $c + d = 0$ then $cx_0 + d = c(x_0 - 1)$ and $(cx_0 + d) + c = cx_0 \neq 0$. If $x_0 = 1$ the next step will have $c = 0$ and it has just been proved this is favorable. Otherwise, $x_0 \neq 1$ and that implies $c(x_0 - 1) \neq 0$, which ends our proof. If $c(c + d) > 0$, $(cx_0 + d)(cx_0 + d + c) = (cx_0 + d)^2 + c(cx_0 + d) = (cx_0 + d)^2 + c(c + d) + c^2(x_0 - 1) > 0$.

Now let's prove that statement *5* is executed at most a finite number of times for a single output partial quotient. If x is a rational number, it has a finite simple continued fraction expansion and the algorithm certainly finishes in a finite number of steps. Suppose x is irrational and $c(c + d) > 0$. If this is not true, then it becomes true at the next step so we can consider it true right here. Let the sequence of functions recursively defined as $f_n : [1, \infty) \to R,\ n \geq 0,$

$f_0(y)=f(y)=\frac{ay+b}{cy+d}; f_{n+1}(y)=f_n(x_n+\frac{1}{y})$. Every time is called, step *4* replaces the current homographic function f_n with f_{n+1} and changes x accordingly (actually swallowing an input partial quotient). The recurrence yields

$$f_n(y)=f(x_0+\cfrac{1}{x_1+\cfrac{1}{x_2+\cfrac{1}{\ddots+\cfrac{1}{x_{n-1}+\cfrac{1}{y}}}}})$$

for all $n \geq 0$. As $f_0'(y) = f'(y)$ and has constant sign, the relation $f_{n+1}'(y) = \frac{-g_n'(x_n + \frac{1}{y})}{y^2}$ yields that f_n' has constant sign on its definition domain.

We claim that $\exists N \geq 0, \forall n \geq N, floor(f_n(y)) = g(n)$ constant on *[1, ∞)* and actually prove that $\exists N \geq 0, \forall n \geq N$ *, floor(fₙ(1))) = floor(fₙ(∞))*. This is sufficient due to f_n being strictly monotonic on *[1, ∞)*. The formula for f_n yields

$$f_n(1) = f(x_0 + \cfrac{1}{x_1 + \cfrac{1}{\ddots + \cfrac{1}{x_{n-1} + \cfrac{1}{1}}}}) = f(x_0 + \cfrac{1}{x_1 + \cfrac{1}{\ddots + \cfrac{1}{x_{n-1}+1}}})$$ and

$$f_n(\infty) = f(x_0 + \cfrac{1}{x_1 + \cfrac{1}{\ddots + \cfrac{1}{x_{n-1}}}}) = f(c_{n-1})$$. By the continuity of the homographic function f on its definition domain, $\lim f(c_{n-3}) = \lim f(c_{n-2}) = \lim f(c_{n-1}) = f(\lim c_n) = f(x)$. According to *theorem 3*, each convergent of a continued fraction for $n \geq 3$ lies between the two preceding ones. Considering the expansion $[x_0, x_1, ..., x_{n-1} + 1]$, this number is between $[x_0, x_1, ..., x_{n-3}]$ and $[x_0, x_1, ..., x_{n-2}]$, for all $n \geq 3$. Taking limits, $\lim [x_0, x_1, ..., x_{n-1} + 1] = x$ and $\lim f([x_0, x_1, ..., x_{n-1} + 1]) = f(x)$. As $f(x)$ is irrational, $\varepsilon = \min(\{f(x)\}, 1 - \{f(x)\}) > 0$,

$\exists N \geq 0, \forall n \geq N, |f(c_{n-1}) - f(x)| < \epsilon \wedge |f([x_0, x_1, ... x_{n-1} + 1]) - f(x)| < \epsilon$ yielding $floor(f(c_{n-1})) = floor(f([x_0, x_1, ... x_{n-1} + 1])), \forall n \geq N$.

Algorithm 3 (The simple continued fraction expansion of bilinear transformations). Let $f(x, y) = \frac{a x y + b x + c y + d}{e x y + f x + g y + h}$ with *a, b, c, d, e, f, g, h integer* and $x = [x_0, x_1, ...]$, $y = [y_0, y_1, ...]$ satisfying $exy + fx + gy + h \neq 0$.

If x is not an *integer*, but y is an *integer*, then y must also satisfy $(ay + b)(gy + h) \neq (cy + d)(ey + f)$ and $(ey + f)(ey + f + gy + h) \geq 0$.

If y is not an *integer*, but x is an *integer*, then x must also satisfy $(ax + c)(fx + h) \neq (bx + d)(e x + g)$ and $(e x + g)(e x + g + fx + h) \geq 0$.

If neither x nor y is *integer*, they must satisfy $(ax + c)(fx + h) \neq (bx + d)(e x + g)$, $(ay + b)(gy + h) \neq (cy + d)(ey + f)$ and additionally, depending on their values:

- if $x < 1$ or $y < 1$, after the steps *1* and *2* both become greater than one and the next condition must be satisfied, but only after step 2 is executed first time:
- if $x > 1$, $y > 1$ then those of e, $e + f$, $e + g$, $e + f + g + h$ that are nonzero must have the same sign.

Also, if x or y is irrational, $f(x, y)$ also has to be irrational.

In order to compute the simple continued fraction expansion of $f(x, y)$ take the following steps:

0. *if* x is an *integer then* use *Algorithm 2* to compute the simple continued fraction expansion of $f(y) = f(x_0, y)$ and *stop.*
   if y is an *integer then* use *Algorithm 2* to compute the simple continued fraction expansion of $f(x) = f(x, y_0)$ and *stop.*

1. *if* $x < 0$ *then* let $f(z,y)=\frac{(-a)zy+(-b)z+cy+d}{(-e)zy+(-f)z+gy+h}$, $z = -x$; replace x with z;

   if $y < 0$ *then* let $f(x,z)=\frac{(-a)xz+bx+(-c)z+d}{(-e)xz+fx+(-g)z+h}$, $z = -y$; replace y with z;
2. *if* $0 < x < 1$ *then* let $f(z,y)=\frac{czy+dz+ay+b}{gzy+hx+ey+f}$, $z=\frac{1}{x}$; replace x with z;

   if $0 < y < 1$ *then* let $f(x,z)=\frac{bxz+ax+dz+c}{fxz+ex+hz+g}$, $z=\frac{1}{y}$; replace y with z;
3. *assert* $x \geq 1$ *and* $y \geq 1$; *assert* those of e, $e + f$, $e + g$, $e + f + g + h$ that are nonzero have the same sign; *assert* $(ax+c)(fx+h) \neq (bx+d)(ex+g)$ *and* $(ay+b)(gy+h) \neq (cy+d)(ey+f)$ *and* $(ex + g)(ex + g + fx + h) \geq 0$ *and* $(ey + f)(ey + f + gy + h) \geq 0$; *assert* $exy + fx + gy + h \neq 0$; *assert* $(x\in Q \wedge y\in Q)\vee f(x,y)\notin Q$;

   if x is an *integer then use Algorithm 1* to compute the simple continued fraction expansion of the rational $f(y) = f(x_0, y)$ and *stop*.

   if y is an *integer then* use *Algorithm 2* to compute the simple continued fraction expansion of $f(x) = f(x, y_0)$ and *stop*.
4. *assert* $x > 1$ *and* $y > 1$

   if $e \neq 0$ *and* $e + f \neq 0$ *and* $e + g \neq 0$ *and* $e + f + g + h \neq 0$ *and* $floor(\frac{a}{e})=floor(\frac{a+b}{e+f})=floor(\frac{a+c}{e+g})=floor(\frac{a+b+c+d}{e+f+g+h})$ *then assert* $e(e + f) > 0$ *and* $e(e + g) > 0$ *and* $e(e + f + g + h) > 0$ and then *output* $q=floor(\frac{a}{e})$,

   let $f(x)=\frac{1}{\frac{axy+bx+cy+d}{exy+fx+gy+h}-q}$,

   $$f(x)=\frac{exy+fx+gy+h}{(a-qe)xy+(b-qf)x+(c-qg)y+d-qh}$$ and then *goto* step 3
5. (*otherwise*)

   assert $x > 1$ *and* $y > 1$

   input y_0 and switch x, y:

   $$f(z,x)=\frac{ax(y_0+\frac{1}{z})+bx+c(y_0+\frac{1}{z})+d}{ex(y_0+\frac{1}{z})+fx+g(y_0+\frac{1}{z})+h}=\frac{(ay_0+b)zx+(cy_0+d)z+ax+c}{(ey_0+f)zx+(gy_0+h)z+ex+g}$$

   where $z = [y_1, y_2, ...]$ and *goto* step 3.

Note: If x and y are simple continued fraction expansions rather than simple continued fractions, the statement *assert* $x \geq 1$ *and* $y \geq 1$ in step 3 can be changed to *assert* $x > 1$ *and* $y > 1$.

Proposition 1. All the assertion statements in *Algorithm 3* always succeed and step 5 is executed at most a finite number of times until an output is generated. Also, all the output partial quotients are correct.

Proof. Let's prove that the assertion statements always succeed. There only three possible paths by which the algorithm can arrive at step 3: *1-2-3*, *4-3* and *5-3*.

We claim that if those of e, $e + f$, $e + g$, $e + f + g + h$ that are nonzero have the same sign s, then $(e\alpha + g)(e\alpha + g + f\alpha + h) \geq 0$, $(ey + f)(ey + f + gy + h) \geq 0$ and $e\alpha\beta + f\alpha + g\beta + h$ is either zero or has sign s for any $\alpha, \beta \geq 1$. We note that $e(e\alpha + g) =$

$e^2\alpha + eg \geq e^2 + eg = e(e + g) \geq 0$, $e(e\alpha + g + f\alpha + h) = e(e + f + g + h) + e(e + f)(\alpha - 1) \geq 0$ yields $e^2(e\alpha + g)(e\alpha + g + f\alpha + h) \geq 0$. If $e \neq 0$ then $(e\alpha + g)(e\alpha + g + f\alpha + h) \geq 0$. If $e = 0$, $(e\alpha + g)(e\alpha + g + f\alpha + h) = (e + g)(e + f + g + h) + (e + f)(e + g)(\alpha - 1) \geq 0$. So the statement $(e\alpha + g)(e\alpha + g + f\alpha + h) \geq 0$ is true. Similarly it can be proved that $(ey + f)(ey + f + gy + h) \geq 0$. Also $e\alpha\beta + f\alpha + g\beta + h = (e + f + g + h) + ((e + g) + e(\alpha - 1))(\beta - 1) + (e + f)(\alpha - 1)$ implies $e\alpha\beta + f\alpha + g\beta + h = 0$ or $sgn(e\alpha\beta + f\alpha + g\beta + h) = s$ for all $\alpha, \beta \geq 1$. The equality happens iff $e+f+g+h=0 \wedge (\alpha=1 \vee e+f=0) \wedge (\beta=1 \vee (e+g=0 \wedge (e=0 \vee \alpha=1)))$. For $\alpha, \beta > 1$, the equality holds iff $e+f+g+h=0 \wedge e+f=0 \wedge e+g=0 \wedge e=0$ equivalent to $e = f = g = h = 0$, but this is prohibited by the input condition $exy + fx + gy + h \neq 0$. So $sgn(e\alpha\beta + f\alpha + g\beta + h) = s \neq 0$ for all $\alpha, \beta > 1$.

In the *1-2-3* path, all the assertions are evidently satisfied.

In the *4-3* path, because the value of $f(x, y)$ is changed simply by subtracting an integer and inverting, step *4* does not change the state of $f(x)$ being or not rational so $(x \in Q \wedge y \in Q) \vee f(x, y) \notin Q$ remains true.

Also $(ey + f)((c - qg)y + (d - qh)) = (ey + f)(cy + d) - q(ey + f)(gy + h) \neq (ay + b)(gy + h) - q(ey + f)(gy + h) = (gy + h)((a - qe)y + (b - qf))$ and $(ex + g)((b - qf)x + (d - qh)) = (ex + g)(bx + d) - q(ex + g)(fx + h) \neq (ax + c)(fx + h) - q(ex + g)(fx + h) = (fx + h)((a - qe)x + (c - qq))$; $a \bmod c$, $(a + b) \bmod (e + f)$, $(a + c) \bmod (e + g)$ and $(a + b + c + d) \bmod (e + f + g + h)$ are all nonnegative numbers. Also $sgn(e\alpha\beta + f\alpha + g\beta + h) = sgn(e) \neq 0$ for all $\alpha, \beta \geq 1$.

In the *5-3* path, because y is only changed by subtracting the floor and inverting, while $f(x, y)$ keeps the same value, it remains true that $x > 1, y > 1$ and $(x \in Q \wedge y \in Q) \vee f(x, y) \notin Q$; $(ax + c)(fx + h) \neq (bx + d)(ex + g)$ implies $((ay_0 + b)x + (cy_0 + d))(ex + g) \neq (ax + c)((ey_0 + f)x + (gy_0 + h))$ while $(ay + b)(gy + h) \neq (cy + d)(ey + f)$ implies $((ay_0 + b) + a(y - y_0))((gy_0 + h) + g(y - y_0)) \neq ((cy_0 + d) + c(y - y_0))((ey_0 + f) + e(y - y_0))$ and by division with the nonzero number $(y - y_0)^2$ it becomes $((ay_0 + b)z + a)((gy_0 + h)z + g) \neq ((cy_0 + d)z + c)((ey_0 + f)z + e)$. Moreover,

$$(e y_0+f) z x+(g y_0+h) z+e x+g=(e y_0+f) \frac{x}{y-y_0}+\frac{(g y_0+h)}{y-y_0}+e x+g=\frac{exy+fx+gy+h}{y-y_0} \neq 0$$

. Because the value of $f(x, y)$ is unchanged, step *5* does not change the quality of $f(x, y)$, x or y being or not rational so $(x \in Q \wedge y \in Q) \vee f(x, y) \notin Q$ remains true.

In step 4, $e \neq 0$ implies $e\alpha\beta + f\alpha + g\beta + h \neq 0$ for all $\alpha, \beta \geq 1$. We claim that $f(\alpha, \beta)$ cannot be greater than or equal to the largest of the four values $f(1, 1)$, $f(1, \infty)$, $f(\infty, 1)$ and $f(\infty, \infty)$, nor can it be smaller then or equal to the smallest, justifying the output choice $q = floor(\frac{a}{e}) = floor(\frac{a+b}{e+f}) = floor(\frac{a+c}{e+g}) = floor(\frac{a+b+c+d}{e+f+g+h})$.

Let's name some often appearing expressions: $r = be - af$, $s = ce - ag$, $t = be + ce + de - af - ag - ah$, $u = af + ce + cf - ag - be - bg$, $v = ce + cf + de + df - ag - ah - bg - bh$, $w = be + bg + de + dg - af - ah - cf - ch$ implying $\frac{a+b}{e+f} - \frac{a}{e} = \frac{r}{e(e+f)}$, $\frac{a+c}{e+g} - \frac{a}{e} = \frac{s}{e(e+g)}$, $\frac{a+b+c+d}{e+f+g+h} - \frac{a}{e} = \frac{t}{e(e+f+g+h)}$, $\frac{a+c}{e+g} - \frac{a+b}{e+f} = \frac{u}{(e+f)(e+g)}$, $\frac{a+b+c+d}{e+f+g+h} - \frac{a+b}{e+f} = \frac{v}{(e+f)(e+f+g+h)}$ and $\frac{a+b+c+d}{e+f+g+h} - \frac{a+c}{e+g} = \frac{w}{(e+g)(e+f+g+h)}$. Note that all these fractions have positive denominators.

We use the identities $f(\alpha,\beta)-\frac{a}{e}=\frac{r(\alpha-1)+s(\beta-1)+t}{e(e\alpha\beta+f\alpha+g\beta+h)}$, $f(\alpha,\beta)-\frac{a+b}{e+f}=\frac{-r(\alpha-1)(\beta-1)+u(\beta-1)+v}{(e+f)(e\alpha\beta+f\alpha+g\beta+h)}$, $f(\alpha,\beta)-\frac{a+c}{e+g}=\frac{-s(\alpha-1)(\beta-1)-u(\alpha-1)+w}{(e+g)(e\alpha\beta+f\alpha+g\beta+h)}$, $f(\alpha,\beta)-\frac{a+b+c+d}{e+f+g+h}=\frac{-t(\alpha-1)(\beta-1)-v(\alpha-1)-w(\beta-1)}{(e+f+g+h)(e\alpha\beta+f\alpha+g\beta+h)}$ and $sgn(e) = sgn(e + f) = sgn(e + g) = sgn(e + f + g + h) = sgn(e\alpha\beta + f\alpha + g\beta + h)$ to find out bounds for $f(\alpha, \beta)$; again, the fraction have only positive denominators.

If $\frac{a}{e}=min(\frac{a}{e},\frac{a+b}{e+f},\frac{a+c}{e+g},\frac{a+b+c+d}{e+f+g+h})$ then $r, s, t \geq 0$ and $f(\alpha,\beta)\geq\frac{a}{e}$; if $\frac{a}{e}=max(\frac{a}{e},\frac{a+b}{e+f},\frac{a+c}{e+g},\frac{a+b+c+d}{e+f+g+h})$ then $r, s, t \leq 0$ and $f(\alpha,\beta)\leq\frac{a}{e}$. If $\frac{a+b}{e+f}=min(\frac{a}{e},\frac{a+b}{e+f},\frac{a+c}{e+g},\frac{a+b+c+d}{e+f+g+h})$ then $r \leq 0$ while $u, v \geq 0$ and $f(\alpha,\beta)\geq\frac{a+b}{e+f}$; if $\frac{a+b}{e+f}=max(\frac{a}{e},\frac{a+b}{e+f},\frac{a+c}{e+g},\frac{a+b+c+d}{e+f+g+h})$ then $r \geq 0$ while $u, v \leq 0$ and $f(\alpha,\beta)\leq\frac{a+b}{e+f}$. If $\frac{a+c}{e+g}=min(\frac{a}{e},\frac{a+b}{e+f},\frac{a+c}{e+g},\frac{a+b+c+d}{e+f+g+h})$ then $s, u \leq 0$ while $w \geq 0$ and $f(\alpha,\beta)\geq\frac{a+c}{e+g}$; if $\frac{a+c}{e+g}=max(\frac{a}{e},\frac{a+b}{e+f},\frac{a+c}{e+g},\frac{a+b+c+d}{e+f+g+h})$ then $s, u \geq 0$ while $w \leq 0$ and $f(\alpha,\beta)\leq\frac{a+c}{e+g}$. If $\frac{a+b+c+d}{e+f+g+h}=min(\frac{a}{e},\frac{a+b}{e+f},\frac{a+c}{e+g},\frac{a+b+c+d}{e+f+g+h})$ then $t, v, w \leq 0$ and $f(\alpha,\beta)\geq\frac{a+b+c+d}{e+f+g+h}$; if $\frac{a+b+c+d}{e+f+g+h}=max(\frac{a}{e},\frac{a+b}{e+f},\frac{a+c}{e+g},\frac{a+b+c+d}{e+f+g+h})$ then $t, v, w \geq 0$ and $f(\alpha,\beta)\leq\frac{a+b+c+d}{e+f+g+h}$. So in all cases $min(\frac{a}{e},\frac{a+b}{e+f},\frac{a+c}{e+g},\frac{a+b+c+d}{e+f+g+h})\leq f(\alpha,\beta)\leq max(\frac{a}{e},\frac{a+b}{e+f},\frac{a+c}{e+g},\frac{a+b+c+d}{e+f+g+h})$ justifying our output choice.

We claim that at any time step 4 is executed, in at most *4* iterations of the same step, e, $e + f$, $e + g$, $e + f + g + h$ all become nonzero (and, of course, have the same sign) and this property holds true for any number of subsequent consecutive iterations of step 4. If at the current iteration we are concerned with e, $e + f$, $e + g$, $e + f + g + h$, the next iteration concerns us with $ey_0 + f$, $ey_0 + f + gy_0 + h$, $ey_0 + f + e$, $ey_0 + f + gy_0 + h + e + g$. We denote by *0* the zero element, and by *1* a nonzero element. Note that all the *1*'s in a single case have the same sign.

1000) $e \neq 0$, $e + f = e + g = e + f + g + h = 0$; $f = g = -e \neq 0$, $h = e \neq 0$; at the next step $e(y_0 - 1)$, 0, ey_0, 0; if $y_0 = 1$, *0010* else *1010*

0100) $e = 0$, $e + f \neq 0$, $e + g = e + f + g + h = 0$; $e = g = 0$, $h = -f \neq 0$; at the next step f, 0, f, 0, always *1010*

0010) $e = e + f = 0$, $e + g \neq 0$, $e + f + g + h = 0$; $e = f = 0$, $h = -g \neq 0$; at the next step 0, $g(y_0 - 1)$, 0, gy_0; if $y_0 = 1$, *0001* else *0101*

0001) $e = e + f = e + g = 0,\ e + f + g + h \neq 0;\ e = f = g = 0,\ h \neq 0;$ at the next step $0, h, 0, h,$ always *0101*

1100) $e \neq 0,\ e + f \neq 0,\ e + g = 0,\ e + f + g + h = 0;\ e + f \neq 0,\ g = -e \neq 0,\ h = -f$; at the next step $e(y_0 - 1) + (e + f),\ 0,\ ey_0 + (e + f),\ 0$ always *1010*

1010) $e \neq 0,\ e + f = 0,\ e + g \neq 0,\ e + f + g + h = 0;\ f = -e \neq 0,\ e + g \neq 0,\ h = -g$; at the next step $e(y_0 - 1),\ (e+g)(y_0 - 1),\ ey_0,\ (e+g)y_0$; if $y_0 = 1$, *0011* else *1111.*

1001) $e \neq 0,\ e + f = 0,\ e + g = 0,\ e + f + g + h \neq 0;\ f = g = -e \neq 0,\ h \neq e$; at the next step $e(y_0 - 1),\ h - e,\ ey_0,\ h - e$; if $y_0 = 1$, *0111* else *1111.*

0110) $e = 0,\ e + f \neq 0,\ e + g \neq 0,\ e + f + g + h = 0;\ e = 0,\ f \neq 0,\ g \neq 0,\ h = -f - g$; at the next step $f,\ g(y_0 - 1),\ f,\ gy_0$; if $y_0 = 1$, *1011* else *1111.*

0101) $e = 0,\ e + f \neq 0,\ e + g = 0,\ e + f + g + h \neq 0;\ e = 0,\ f \neq 0,\ g = 0,\ h \neq -f$; at the next step $f,\ f + h,\ f,\ f + h$, always *1111.*

0011) $e = 0,\ e + f = 0,\ e + g \neq 0,\ e + f + g + h \neq 0;\ e = f = 0,\ g \neq 0,\ h \neq -g$; at the next step $0,\ g(y_0 - 1) + (g + h),\ 0,\ gy_0 + (g + h)$, always *0101*

1110) $e \neq 0,\ e + f \neq 0,\ e + g \neq 0,\ e + f + g + h = 0;\ e \neq 0,\ e + f \neq 0,\ e + g \neq 0,\ h \neq -e - f - g$; at the next step $e(y_0 - 1) + e + f,\ (e + g)(y_0 - 1),\ ey_0 + (e + f),\ (e + g)y_0$; if $y_0 = 1$, *1011* else *1111.*

1101) $e \neq 0,\ e + f \neq 0,\ e + g = 0,\ e + f + g + h \neq 0;\ e + f \neq 0,\ g = -e \neq 0,\ f + h \neq 0$; at the next step $e(y_0 - 1) + e + f,\ e(y_0 - 1) + f + h,\ ey_0 + (e + f),\ e(y_0 - 1) + f + h$, always *1111.*

1011) $e \neq 0,\ e + f = 0,\ e + g \neq 0,\ e + f + g + h \neq 0;\ f = -e \neq 0,\ e + g \neq 0,\ g + h \neq 0$; at the next step $e(y_0 - 1),\ (e+g)(y_0 - 1) + g + h,\ ey_0,\ (e + g)y_0 + g + h$; if $y_0 = 1$, *0111* else *1111.*

0111) $e = 0,\ e + f \neq 0,\ e + g \neq 0,\ e + f + g + h \neq 0;\ e = 0,\ f \neq 0,\ g \neq 0,\ f + g + h \neq 0$; at the next step $f,\ g(y_0 - 1) + (f + g + h),\ f,\ gy_0 + (f + g + h)$, always *1111.*

1111) $e \neq 0,\ e + f \neq 0,\ e + g \neq 0,\ e + f + g + h \neq 0$; at the next step $e(y_0 - 1) + e + f,\ (e + g)(y_0 - 1) + e + f + g + h,\ ey_0 + (e + f),\ (e + g)y_0 + e + f + g + h$, always *1111.*

It can be easily seen that this finite state machine arrives at the end state *1111* in at most *4* transitions, for any initial state and input combination and does not leave that state. In state *1111* there exist $f(\infty,\infty)=\frac{a}{e}$, $f(\infty,1)=\frac{a+b}{e+f}$, $f(1,\infty)=\frac{a+c}{e+g}$ and $f(1,1)=\frac{a+b+c+d}{e+f+g+h}$.

Now let's prove that statement *5* is executed at most a finite number of times for a single output partial quotient. If x or y is a rational number, it has a finite simple continued fraction expansion and the algorithm certainly finishes in finite time. Suppose both are irrational and even if the conditions for step 4 are not satisfied, $e,\ e + f,\ e + g,\ e + f + g + h$ are all nonzero and have the same sign. This is possible due to the previous claim. Let the sequence of functions recursively defined as $f_n : [1, \infty) \times [1, \infty) \to R,\ n \geq 0.$

$f_0(\alpha,\beta)=f(\alpha,\beta);\ f_{n+1}(\beta,\alpha)=f_n(\alpha, y_n+\frac{1}{\beta})$. At every execution, step *4* replaces the current bilinear function f_n with f_{n+1}, changes y accordingly (actually swallowing an input partial quotient) and switches x and y. The recurrence yields

$$f_n(\alpha,\beta)=f(x_0+\cfrac{1}{x_1+\cfrac{1}{\ddots+\cfrac{1}{x_{floor(\frac{n-2}{2})}+\cfrac{1}{\alpha}}}},\ y_0+\cfrac{1}{y_1+\cfrac{1}{\ddots+\cfrac{1}{y_{ceil(\frac{n-2}{2})}+\cfrac{1}{\beta}}}})$$ for all $n \geq 1$.

We claim that $\exists N\geq 0, \forall n\geq N, floor(f_n(\alpha,\beta))=g(n)$ constant on *[1, ∞)*x *[1, ∞)*. Because $x_0+\cfrac{1}{x_1+\cfrac{1}{\ddots+\cfrac{1}{x_{floor(\frac{n-2}{2})}+\cfrac{1}{\alpha}}}}$ is between $c_{floor(\frac{n-2}{2})}$ and $c_{floor(\frac{n-2}{2})-1}$, where c_n are the convergents of x, it uniformly converges to x.

Analogously, $y_0+\cfrac{1}{y_1+\cfrac{1}{\ddots+\cfrac{1}{y_{ceil(\frac{n-2}{2})}+\cfrac{1}{\beta}}}}$ uniformly converges uniformly to y and let d_n be the convergents of y. By the continuity of the linear function f in the point (x, y), $\lim f(c_n, d_n) = f(\lim c_n, \lim d_n) = f(x, y)$ and the sequence $f_n(\alpha, \beta)$ *uniformly converges* to $f(x, y)$ for $n \to \infty$. As $f(x, y)$ is irrational, $\varepsilon = \min(\{f(x, y)\}, 1 - \{f(x, y)\}) > 0$,

$\exists N\geq 0, \forall n\geq N, |f_n(\alpha,\beta)-f(x)|<\frac{\epsilon}{2}, \forall \alpha,\beta\geq 1$ yielding

$\forall n\geq N, f(x)-\frac{\epsilon}{2}<f_n(\alpha,\beta)<f(x)+\frac{\epsilon}{2}, \forall \alpha,\beta\geq 1$;

$f(x)-\frac{\epsilon}{2}\leq f_N(\infty,\infty), f_N(\infty,1), f_N(1,\infty), f_N(1,1), \leq f(x)+\frac{\epsilon}{2}$ implies $floor(f_n(1, 1)) = floor(f_n(1, \infty)) = floor(f_n(\infty, 1)) = floor(f_n(\infty, \infty))$ ending our proof.

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