Simple continued fractions introduction

Definition 1. A *simple continued fraction* is an expression of the form $a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots}}$ which we abbreviate as $[a_0, a_1, a_2, \ldots]$ where $a_i \in Z$ for all $i \geq 0$, and $a_i \geq 1$, for all $i \geq 1$. The terms $a_0, a_1, a_2, \ldots$ are referred to as *partial quotients.* A simple continued fraction has at least one partial quotient.

Definition 2. Let n be a non-negative integer and a simple continued fraction $[a_0, a_1, a_2, \ldots]$. The simple continued fractions

$$c_n=\frac{p_n}{q_n}=[a_{0,}a_{1,}\ldots,a_n]=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\cdots+\cfrac{1}{a_n}}}}$$

are called the *convergents,* and the tails

$$\alpha_{n+1}=[a_{n+1},a_{n+2},\ldots]=a_{n+1}+\cfrac{1}{a_{n+2}+\cfrac{1}{a_{n+3}+\ddots}}$$

are known as its *complete quotients*.

Theorem 1 (Fundamental Correspondence). Given a sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ with $a_i \in Z$ for all $i \geq 0$, and $a_i \geq 1$, for all $i \geq 1$, and a sequence of reduced rationals $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}, \ldots$ with $q_i \geq 1$ for all $i \geq 1$, then

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \quad \text{for } n = 0..k$$

iff

$$\frac{p_n}{q_n}=[a_{0,}a_{1,}\ldots,a_n] \quad \text{for } n = 0..k$$

where k is an arbitrary nonnegative integer, less than the length of the a_n sequence. Defining $\begin{pmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the matrix based identity is actually equivalent to the recurrences

$$\begin{aligned} p_n&=a_n p_{n-1}+p_{n-2} \\ q_n&=q_n q_{n-1}+q_{n-2} \end{aligned}, \text{ for } n = 0 .. k.$$

Proof. We verify the claim by induction on n. The claim is easily seen true for $n = 0$ since, indeed $p_0 = a_0$ and $q_0 = 1$. Accordingly, we suppose that

$$\begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} b_n & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{pmatrix} \quad \text{iff} \quad \frac{x_n}{y_n}=[b_{1,}b_{2,}\ldots,b_n]$$

noting that this is just a case of n matrices. As all the individual matrices in the left are invertible, for $n \geq 1$,

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

is equivalent to

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$ or

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} q_n & q_{n-1} \\ p_n - a_0 q_n & p_{n-1} - a_0 q_{n-1} \end{pmatrix}$$. By the induction hypothesis, this is equivalent to $[a_1, a_2, \ldots, a_n] = \dfrac{q_n}{p_n - a_0 q_n}$ or

$$[a_0, a_1, a_2, \ldots, a_n] = a_0 + \frac{1}{[a_1, a_2, \ldots, a_n]} = a_0 + \frac{1}{\dfrac{q_n}{p_n - a_0 q_n}} = \frac{p_n}{q_n}$$ verifying the claim by induction.

Note that the formally described fundamental correspondence is independent of the actual nature of the partial quotients or their constraints. Taking the determinants in the (formal) fundamental correspondence immediately yields the *fundamental formula* $p_n q_{n-1} = p_{n-1} q_n = (-1)^{n+1}$ or $\dfrac{p_n}{q_n} = \dfrac{p_{n-1}}{q_{n-1}} + \dfrac{(-1)^{n-1}}{q_{n-1} q_n}$. It is then immediate that $\dfrac{p_n}{q_n} = a_0 + \dfrac{1}{q_0 q_1} - \dfrac{1}{q_1 q_2} + \cdots + \dfrac{(-1)^{n-1}}{q_{n-1} q_n}$. Furthermore, $\dfrac{p_n}{q_n} = \dfrac{p_{n+1} - p_{n-1}}{q_{n+1} - q_{n-1}}$ and the convergents $c_n = \dfrac{p_n}{q_n}$ also satisfy $c_n - c_{n-1} = \dfrac{(-1)^{n+1}}{q_n q_{n-1}}$ and $c_n - c_{n-2} = \dfrac{a_n (-1)^n}{q_n q_{n-2}}$.

Theorem 2. Given a simple continued fraction expansion $[a_0, a_1, a_2, \ldots]$, the even convergents c_{2n} form an increasing sequence, and the odd convergents c_{2n+1} form a decreasing sequence. Any even convergent is less than any odd convergent: $c_0 < c_2 < c_4 < \ldots < c_{2n-2} < c_{2n} < \ldots < c_{2n+1} < c_{2n-1} < \ldots < c_5 < c_3 < c_1$. Furthermore, each convergent for $n \geq 3$ lies between the two preceding ones: $c_{2n-2} < c_{2n} < c_{2n-1}$ and $c_{2n} < c_{2n+1} < c_{2n-1}$.

Proof. For n even, $n \geq 2$, $c_n - c_{n-2}$ is positive; for n odd, $n \geq 3$, $c_n - c_{n-2}$ is negative. The inequality $c_{2n} - c_{2n-1} = \dfrac{(-1)^{2n+1}}{q_{2n} q_{2n-1}} < 0$ yields $c_{2n} < c_{2n-1}$. Also $c_{2n+1} - c_{2n} = \dfrac{(-1)^{2n+2}}{q_{2n+1} q_{2n}} > 0$ yields $c_{2n} < c_{2n+1} < c_{2n-1}$. By induction on the statements $c_0 < c_2 < c_4 < \ldots < c_{2n-2} < c_{2n} < c_{2n+1} < c_{2n-1} < \ldots < c_5 < c_3 < c_1$ and $c_0 < c_2 < c_4 < \ldots < c_{2n-2} < c_{2n} < c_{2n-1} < \ldots < c_5 < c_3 < c_1$ it can be easily verified that any even convergent is less than any odd convergent.

Definition 3. We define the *simple continued fraction expansion of a real number* x as $[a_0, a_1, a_2, \ldots]$ where the first partial quotient is $a_0 = floor(x)$, the integral part of x. The remaining partial quotients are recursively defined as $r_0 = x$, $a_n = floor(r_n)$, $r_n = \dfrac{1}{r_{n-1} - a_{n-1}}$ stopping at the first integer r_n with $n \geq 1$.

Theorem 3. Given a real number x and its simple continued fraction expansion $[a_0, a_1, a_2, \ldots]$, than x is between c_n and c_{n+1} for all $n \geq 0$, if there exists such an $n + 1$'th convergent. If the expansion has a finite number $n + 1$ of partial quotients, then $x = c_n$.

Proof. We verify the claim by induction on n. For an expansion of length one and $n = 0$, $c_0 = floor(x) = x$; if the expansion has at least two elements, than x is certainly not an integer so

$$c_0 = floor(x) < x < floor(x) + \frac{1}{floor(\frac{1}{\{x\}})} = c_1$$. Now suppose the statement to be true for n and let's prove it for $n + 1$. If the expansion has $n + 2$ partial quotients, it means r_1 has the expansion

$[a_1, a_2, ..., a_{n+1}]$ with the convergents d_k satisfying the identity $c_{k+1}=a_0+\frac{1}{d_k}$. By the induction hypothesis, $d_n = r_1$ yielding $c_{n+1}=a_0+\frac{1}{d_n}=a_0+\frac{1}{r_1}=a_0+(x-a_0)=x$. If the expansion has more than $n + 2$ elements, then the expansion of r_1 has more than $n + 1$ elements, and so r_1 is between d_{n-1} and d_n. As r_1 is positive it implies $a_0+\frac{1}{r_1}=x$ is between $a_0+\frac{1}{d_{n-1}}=c_n$ and $a_0+\frac{1}{d_n}=c_{n+1}$ proving the induction step.

Theorem 4. An infinite simple continued fraction is always convergent. Additionally, let $x = [a_0, a_1, a_2, ...]$ and let c_n be the convergents. Then $c_0 < c_2 < c_4 < ... < c_{2n-2} < c_{2n} < ... < x < ... < c_{2n+1} < c_{2n-1} < ... < c_5 < c_3 < c_1$.

Proof. As $a_n \geq 1$ for all $n \geq 1$, and $q_{-1} = 0$, $q_0 = 1$, $q_n = a_n q_{n-1} + q_{n-2} \geq q_{n-1} + q_{n-2}$, one can easily prove by induction $q_n \geq F_{n+1}$, for all $n \geq 0$, where F_n is the Fibonacci sequence recursively defined as $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-2} + F_{n-1}$, for all $n \geq 2$. The infinite simple continued fraction $[a_0, a_1, a_2, ...]$ is $\lim \frac{p_n}{q_n}=a_0+\sum_{n\geq 0}\frac{(-1)^{n-1}}{q_n q_{n+1}}$ We prove the series is convergent. Let $x_n=\frac{(-1)^{n-1}}{q_n q_{n+1}}$ be the general term of the series to study. Then $|x_n|<\frac{1}{q_n q_{n+1}}\leq\frac{1}{F_{n+1}F_{n+2}}<\frac{1}{n(n+1)}$ for all $n \geq 4$. As $\sum_{n\geq 1}\frac{1}{n(n+1)}=1$ is a well known identity, it remains that $\sum_{n\geq 0}\frac{(-1)^{n-1}}{q_n q_{n+1}}$ is absolutely convergent. If a series converges absolutely, then it converges. The even convergents being increasing and converging to x, they are all less than x. The odd convergents being decreasing and converging to x, they are all greater than x.

Another proof can be easily deduced only from theorem 2.

Theorem 5. A real number is *rational* iff it has a *finite* simple continued fraction expansion. Moreover, the value of its continued fraction expansion is equal to the rational number.

Proof. By definition, the simple continued fraction expansion of a rational number p/q is the sequence of quotients from the Euclidean Algorithm applied to find the greatest common divisor of p and q, and the euclidean algorithm has a finite number of steps. Actually, Lamé showed that the number of steps needed to arrive at the greatest common divisor for two numbers less than m is

$steps\leq\frac{\ln m}{\ln \phi}+\frac{\ln\sqrt{5}}{\ln \phi}$ where Φ is the golden ratio. By theorem 3, the value of the finite simple continued fraction expansion of a real number x is actually equal to x. Applying Theorem 3 for $x = p / q$ ends the proof. Anyway, for completeness, we extracted from the proof of Theorem 3 only the induction on the length n of the continued fraction expansion. For $n = 1$, then p/q is an integer, and it's expansion is $[p/q]$. Now suppose the statement true for rational with an n length expansion and suppose p/q has a $n + 1$ length expansion $[a_0, a_1, a_2, ..., a_n]$. The first two steps of the euclidean algorithm are:

$r_0 = p/q$, $a_0 = floor(r_0)$; $r_1=\frac{1}{r_0-a_0}=\frac{q}{p-a_0 q}$, $a_1 = floor(r_1)$. It becomes clear that the expansion of the rational r_1 is $[a_1, a_2, ..., a_n]$. By the induction hypothesis, $r_1 = [a_1, a_2, ..., a_n]$. But

$r_0=\frac{p}{q}=a_0+\frac{1}{r_1}=a_0+\frac{1}{[a_1, a_2, \ldots, a_n]}=[a_0, a_1, \ldots, a_n]$ and the claim is true.

Now suppose a real number has a finite simple continued fraction expansion. According theorem 3, the number is equal to the last convergent, which is a rational number.

Theorem 6. A real number x is *irrational* iff it has an *infinite* simple continued fraction expansion. Moreover, its continued fraction expansion actually converges to x.

Proof. Theorem 2 implies that a real number x is irrational iff it has an infinite expansion. Given such an irrational number x and its expansion $[a_0, a_1, a_2, ...]$, which by theorem 1 is a convergent sequence, we have left to prove that $x = [a_0, a_1, a_2, ...]$. According to *theorem 3* x is between between any two consecutive elements of c_n, implying that $x = lim\ c_n = [a_0, a_1, a_2, ...]$.

From now on we say that the simple continued fraction expansion of a real number x is *equal* to x, in the infinite case actually meaning that it's limit is equal to x.

Theorem 7. Let $x = [a_0, a_1, a_2, ..., a_n]$ be a finite simple continued fraction and let $y = [b_0, b_1, b_2, ...]$ be any simple continued fraction. Then $x = y$ iff y is also a finite simple continued fraction and either y has the same length $n + 1$ as x and $a_i = b_i$ for all $0 \le i \le n$, or if $a_n > 1$ than y has length $n + 2$, $a_i = b_i$ for all $0 \le i \le n - 1$, $b_n = a_n - 1$, $b_{n+1} = 1$, or otherwise $a_n = 1$ and y has length n, $a_i = b_i$ for all $0 \le i \le n - 2$, $b_{n-1} = a_{n-1} + 1$.

Proof. Suppose y has an infinite length. If $a_i = b_i$ for all $0 \le i \le n$ than x is actually the $n + 1$'th convergent of y which by Theorem 4 is different from y. Otherwise there is an index $0 \le i \le n$ with $a_i \ne b_i$. If x were equal to y, then $\frac{1}{x-a_0}=\frac{1}{y-b_0}$ and repeating this procedure up to a_{i-1}, we get $a_i = [b_i, b_{i+1}, ...]$. This is absurd because by Theorem 4, $[b_i, b_{i+1}, ...]$ is between b_i and $b_i+\frac{1}{b_{i+1}}$, but this open interval contains no integers. Until now we have proved that y has to have finite length in order to be equal to x.

Suppose x and y have the same length. If $a_i = b_i$ for all $0 \le i \le n$, than it is clear that $x = y$. Otherwise, if only the last element differs, x cannot be equal to y. Otherwise, there is an index $0 \le i < n$ with $a_i \ne b_i$. Reversing the euclidean algorithm as above, if x were equal to y, it yields $[a_i, a_{i+1}, ..., a_n] = [b_i, b_{i+1}, ..., b_n]$. This implies the open intervals with limits a_i and $a_i+\frac{1}{a_{i+1}}$, respectively b_i and $b_i+\frac{1}{b_{i+1}}$ must not be disjoint. If $a_i < b_i$, it results that $a_i<b_i<a_i+\frac{1}{a_{i+1}}$ absurd. If $a_i > b_i$, it results that $b_i<a_i<b_i+\frac{1}{b_{i+1}}$ also absurd. So our claim that, in case x and y have the same length, they are equal iff $a_i = b_i$ for all $0 \le i \le n$ is true.

Suppose x and y are equal but have different lengths; more precisely, let x be shorter than y. By theorem 4 y cannot start with x, so there's an index $0 \le i \le n$ with $a_i \ne b_i$. Let i be the smallest index with this property. If $i < n$, by reversing the euclidean algorithm, it reveals to be absurd. So $i = n$ and $a_n = [b_n, b_{n+1}, ...]$. Note that $a_n > 1$. If y has more than $n + 2$ elements, than $[b_n, b_{n+1}, ...]$ again lies in the open inteval with limits b_n and $b_n+\frac{1}{b_{n+1}}$, and it cannot be equal to a_n. The only possibility that $a_n = [b_n, b_{n+1}, ...]$ is for y to have exactly $n + 2$ elements, and $a_n=[b_n, b_{n+1}]=b_n+\frac{1}{b_{n+1}}$, implying $b_{n+1} = 1$. The case when x is longer than y is similar, and this ends our proof.

Corollary 1. For any rational number x, there are exactly two simple continued fractions equal to it, both finite. One is its simple continued fraction expansion $[a_0, a_1, a_2, ..., a_n]$ and the other is $[a_0, a_1, a_2, ..., a_n - 1, 1]$ for $a_n > 1$, or $[a_0, a_1, a_2, ..., a_{n-1} + 1]$ for $a_n = 1$.

Corollary 2. Any finite simple continued fraction with the last partial quotient at least 2 in case the expansion has at least *2* elements is the (unique) expansion of an (unique) rational number.

Theorem 8. Let $x = [a_0, a_1, a_2, \ldots]$ be an infinite simple continued fraction and let $y = [b_0, b_1, b_2, \ldots]$ be any simple continued fraction. Then $x = y$ iff y is also an infinite simple continued fraction and $a_i = b_i$ for all $i \geq 0$.

Proof. By theorem 7, y cannot be finite. Now suppose $x = y$ and i is the smallest index such that $a_i \neq b_i$. Reversing the euclidean algorithm (working with convergent sequences) we get $[a_i, a_{i+1}, \ldots] = [b_i, b_{i+1}, \ldots]$ and that implies that the open intervals with limits a_i and $a_i + \frac{1}{a_{i+1}}$, respectively b_i and $b_i + \frac{1}{b_{i+1}}$ must not be disjoint, which is absurd having that $a_i \neq b_i$.

Corollary 1. For any irrational number x there is exactly one simple continued fraction equal to it. Evidently, it is the simple continued fraction expansion of x and it is infinite.

Corollary 2. Any infinite simple continued fraction is the expansion of an (unique) irrational number.

Theorem 9. The simple continued fraction expansion is a bijection between the real numbers and the set of simple continued fractions with the restriction that the finite ones with at least *2* partial quotients have the last partial quotient at least *2*.

Proof. This results from theorems 7 and 8 and their corollaries.

References

1. Enrico Bombieri, Alfred J. van der Poorten, *Continued Fractions of Agebraic Numbers*, http://www-centre.mpce.mq.edu.au/alfpapers/a113.pdf
2. *Continued Fractions*, http://www.calvin.edu/academic/math.old/confrac/theory.html
3. Alfred J. van der Poorten, *Notes on Continued Fractions and Recurrence Sequences*, http://www-centre.mpce.mq.edu.au/alfpapers/a094.pdf
4. Wolfram Research, Inc, *Eric Weisstein's World of Mathematics*, http://mathworld.wolfram.com

Project: ***PreciseFloating*** *- Author:* ***Daniel Aioanei*** *(aioaneid@go.ro)*