

1.1 Approximating Areas

Learning Objectives

- 1.1.1 Use sigma (summation) notation to calculate sums and powers of integers.
- 1.1.2 Use the sum of rectangular areas to approximate the area under a curve.
- 1.1.3 Use Riemann sums to approximate area.

Archimedes was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the *method of exhaustion*, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve, defined by a function $f(x)$, and the x -axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Let's start by introducing some notation to make the calculations easier. We then consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ , sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20.$$

We could probably skip writing a couple of terms and write

$$1 + 2 + 3 + 4 + \dots + 19 + 20,$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i,$$

which is much more compact.

Typically, sigma notation is presented in the form

$$\sum_{i=1}^n a_i$$

where a_i describes the terms to be added, and the i is called the *index*. Each term is evaluated, then we sum all the values, beginning with the value when $i = 1$ and ending with the value when $i = n$. For example, an

expression like $\sum_{i=2}^7 s_i$ is interpreted as $s_2 + s_3 + s_4 + s_5 + s_6 + s_7$. Note that the index is used only to keep

track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a *dummy variable*. We can use any letter we like for the index. Typically, mathematicians use i, j, k, m , and n for indices.

Let's try a couple of examples of using sigma notation.

EXAMPLE 1.1

Using Sigma Notation

- Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5$.
- Write the sum in sigma notation:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}.$$

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.1

Write in sigma notation and evaluate the sum of terms 2^i for $i = 3, 4, 5, 6$.

The properties associated with the summation process are given in the following rule.

RULE: PROPERTIES OF SIGMA NOTATION

Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m , with $1 \leq m \leq n$.

1.

$$\sum_{i=1}^n c = nc$$

(1.1)

2.

$$\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$$

(1.2)

3.

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

(1.3)

4.

$$\sum_{i=1}^n (a_i - b_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$$

(1.4)

5.

$$\sum_{i=1}^n a_i = \sum_{i=1}^m a_i + \sum_{i=m+1}^n a_i$$

(1.5)

Proof

We prove properties 2. and 3. here, and leave proof of the other properties to the Exercises.

2. We have

$$\begin{aligned}\sum_{i=1}^n ca_i &= ca_1 + ca_2 + ca_3 + \cdots + ca_n \\ &= c(a_1 + a_2 + a_3 + \cdots + a_n) \\ &= c \sum_{i=1}^n a_i.\end{aligned}$$

3. We have

$$\begin{aligned}\sum_{i=1}^n (a_i + b_i) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n) \\ &= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n) \\ &= \sum_{i=1}^n a_i + \sum_{i=1}^n b_i.\end{aligned}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we use them in the next set of examples.

RULE: SUMS AND POWERS OF INTEGERS

1. The sum of n integers is given by

$$\sum_{i=1}^n i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^n i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

EXAMPLE 1.2

Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

- a. The sum of the terms $(i-3)^2$ for $i = 1, 2, \dots, 200$.
- b. The sum of the terms $(i^3 - i^2)$ for $i = 1, 2, 3, 4, 5, 6$.

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.2

Find the sum of the values of $4 + 3i$ for $i = 1, 2, \dots, 100$.

EXAMPLE 1.3**Finding the Sum of the Function Values**

Find the sum of the values of $f(x) = x^3$ over the integers 1, 2, 3, ..., 10.

[Show/Hide Solution]

CHECKPOINT 1.3

Evaluate the sum indicated by the notation $\sum_{k=1}^{20} (2k + 1)$.

Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by $f(x)$ above, the x -axis below, the line $x = a$ on the left, and the line $x = b$ on the right ([Figure 1.2](#)).

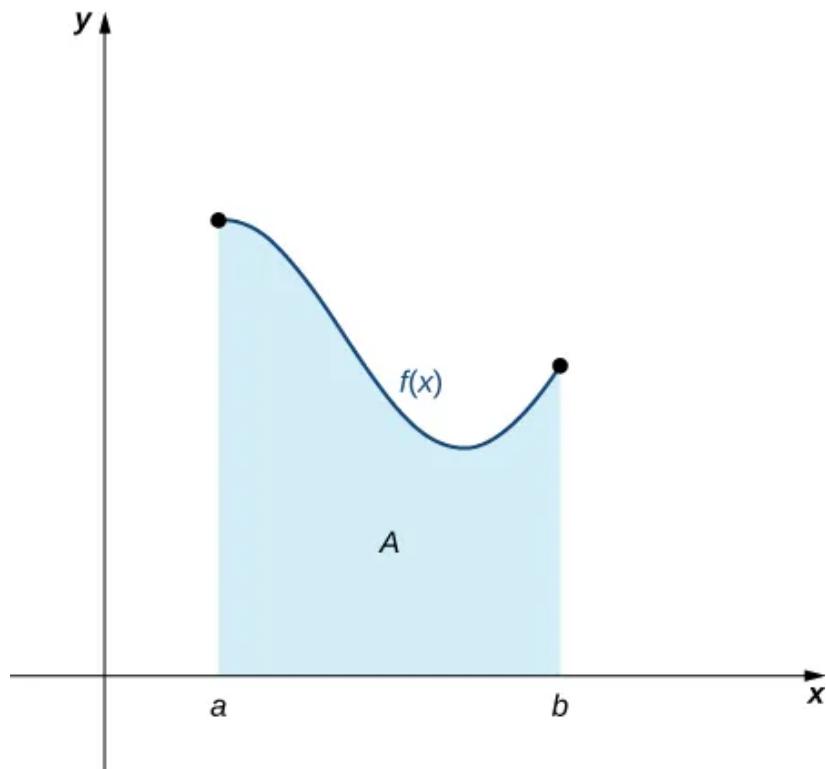


Figure 1.2 An area (shaded region) bounded by the curve $f(x)$ at top, the x -axis at bottom, the line $x = a$ to the left, and the line $x = b$ at right.

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\frac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \dots, x_n$ with $x_0 = a$, $x_n = b$, and

$$x_i - x_{i-1} = \frac{b-a}{n}$$

for $i = 1, 2, 3, \dots, n$.

We denote the width of each subinterval with the notation Δx , so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x$$

for $i = 1, 2, 3, \dots, n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

DEFINITION

A set of points $P = \{x_i\}$ for $i = 0, 1, 2, \dots, n$ with $a = x_0 < x_1 < x_2 < \dots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

RULE: LEFT-ENDPOINT APPROXIMATION

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \dots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1}) \Delta x$. Adding the areas of all these rectangles, we get an approximate value for A ([Figure 1.3](#)). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

$$\begin{aligned} A \approx L_n &= f(x_0) \Delta x + f(x_1) \Delta x + \dots + f(x_{n-1}) \Delta x \\ &= \sum_{i=1}^n f(x_{i-1}) \Delta x \end{aligned}$$

(1.6)

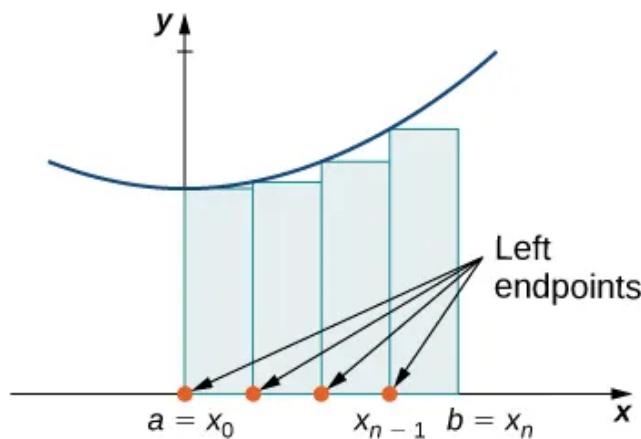


Figure 1.3 In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

RULE: RIGHT-ENDPOINT APPROXIMATION

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i) \Delta x$ and the approximation for A is given by

$$\begin{aligned} A \approx R_n &= f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_n) \Delta x \\ &= \sum_{i=1}^n f(x_i) \Delta x. \end{aligned} \quad (1.7)$$

The notation R_n indicates this is a **right-endpoint approximation** for A ([Figure 1.4](#)).

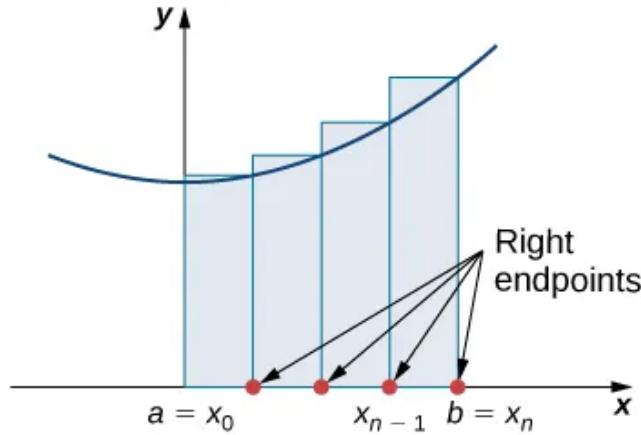


Figure 1.4 In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in [Figure 1.3](#).

The graphs in [Figure 1.5](#) represent the curve $f(x) = \frac{x^2}{2}$. In graph (a) we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width 0.5. Thus, $\Delta x = 0.5$. We then form six rectangles by drawing vertical lines perpendicular to x_{i-1} , the left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1, 2, 3, 4, 5, 6$. The intervals are $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$. We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between $f(x)$ and the x -axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation. Thus,

$$\begin{aligned} A \approx L_6 &= \sum_{i=1}^6 f(x_{i-1}) \Delta x = f(x_0) \Delta x + f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + f(x_4) \Delta x + f(x_5) \Delta x \\ &= f(0) 0.5 + f(0.5) 0.5 + f(1) 0.5 + f(1.5) 0.5 + f(2) 0.5 + f(2.5) 0.5 \\ &= (0) 0.5 + (0.125) 0.5 + (0.5) 0.5 + (1.125) 0.5 + (2) 0.5 + (3.125) 0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375. \end{aligned}$$

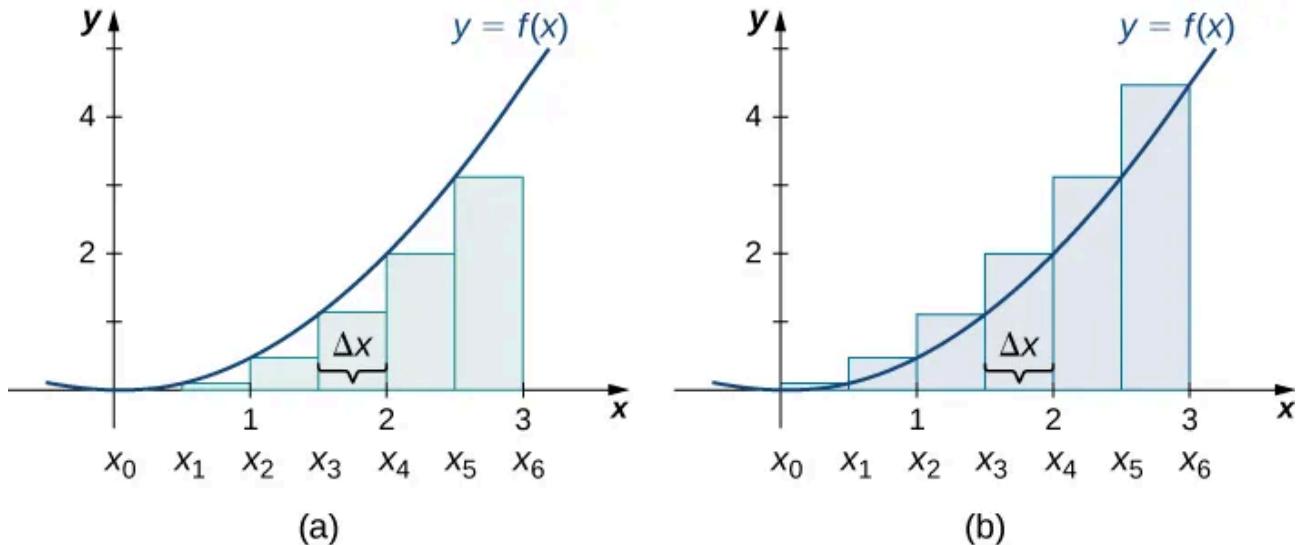


Figure 1.5 Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

In [Figure 1.5\(b\)](#), we draw vertical lines perpendicular to x_i such that x_i is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1, 2, 3, 4, 5, 6$. We multiply each $f(x_i)$ by Δx to find the rectangular areas, and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

$$\begin{aligned} A \approx R_6 &= \sum_{i=1}^6 f(x_i) \Delta x = f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + f(x_4) \Delta x + f(x_5) \Delta x + f(x_6) \Delta x \\ &= f(0.5) 0.5 + f(1) 0.5 + f(1.5) 0.5 + f(2) 0.5 + f(2.5) 0.5 + f(3) 0.5 \\ &= (0.125) 0.5 + (0.5) 0.5 + (1.125) 0.5 + (2) 0.5 + (3.125) 0.5 + (4.5) 0.5 \\ &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875. \end{aligned}$$

EXAMPLE 1.4**Approximating the Area Under a Curve**

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

[Show/Hide Solution]**CHECKPOINT 1.4**

Sketch left-endpoint and right-endpoint approximations for $f(x) = \frac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

Looking at [Figure 1.5](#) and the graphs in [Example 1.4](#), we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, it seems logical that if we increase the number of points in our partition, our estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing n , first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals, of the same curved region. [Figure 1.8](#) shows the area of the region under the curve $f(x) = (x - 1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

$$\Delta x = \frac{2 - 0}{4} = \frac{1}{2}.$$

The area is approximated by the summed areas of the rectangles, or

$$\begin{aligned} L_4 &= f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) \\ &= 7.5. \end{aligned}$$

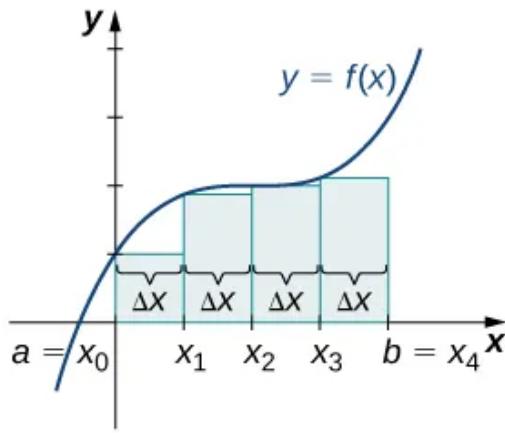


Figure 1.8 With a left-endpoint approximation and dividing the region from a to b into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.

[Figure 1.9](#) shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in [Figure 1.8](#) with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is area under the curve we are unable to include using our approximation. The area of the rectangles is

$$\begin{aligned} L_8 &= f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) \\ &\quad + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) \\ &= 7.75. \end{aligned}$$

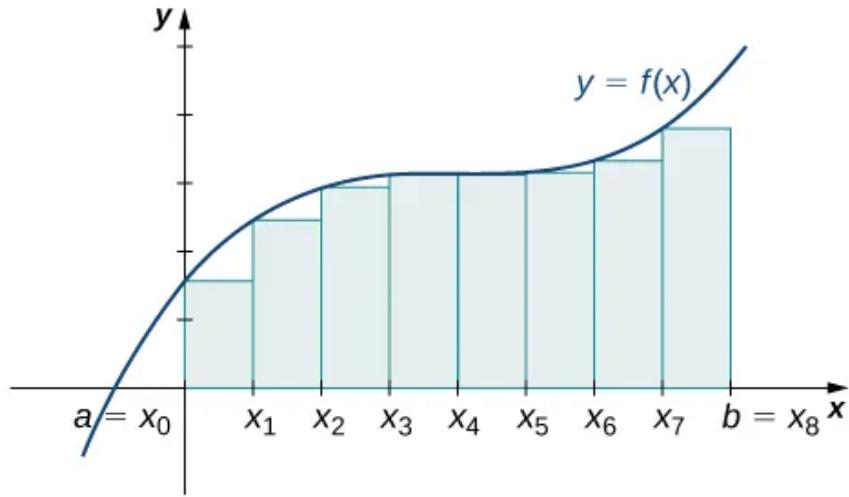


Figure 1.9 The region under the curve is divided into $n = 8$ rectangular areas of equal width for a left-endpoint approximation.

The graph in [Figure 1.10](#) shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

$$\begin{aligned} L_{32} &= f(0)(0.0625) + f(0.0625)(0.0625) + f(0.125)(0.0625) + \cdots + f(1.9375)(0.0625) \\ &= 7.9375. \end{aligned}$$

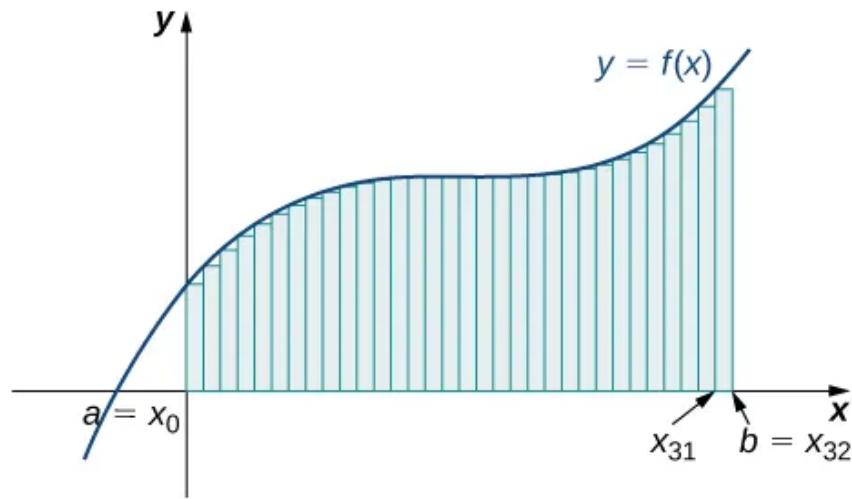


Figure 1.10 Here, 32 rectangles are inscribed under the curve for a left-endpoint approximation.

We can carry out a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles ([Figure 1.11](#)), yields an area

$$\begin{aligned} R_4 &= f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) \\ &= 8.5. \end{aligned}$$

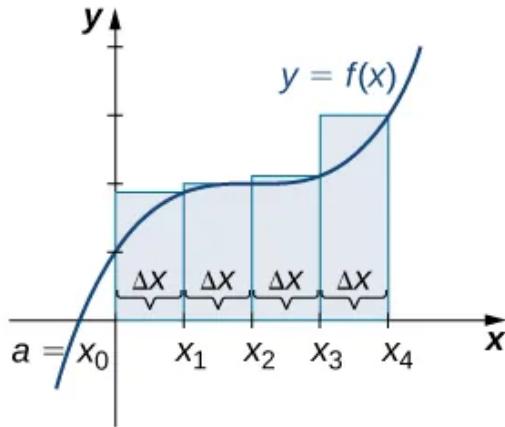


Figure 1.11 Now we divide the area under the curve into four equal subintervals for a right-endpoint approximation.

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \frac{2-0}{8} = 0.25$. The graph is shown in [Figure 1.12](#). The area is

$$\begin{aligned} R_8 &= f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) \\ &\quad + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) \\ &= 8.25. \end{aligned}$$

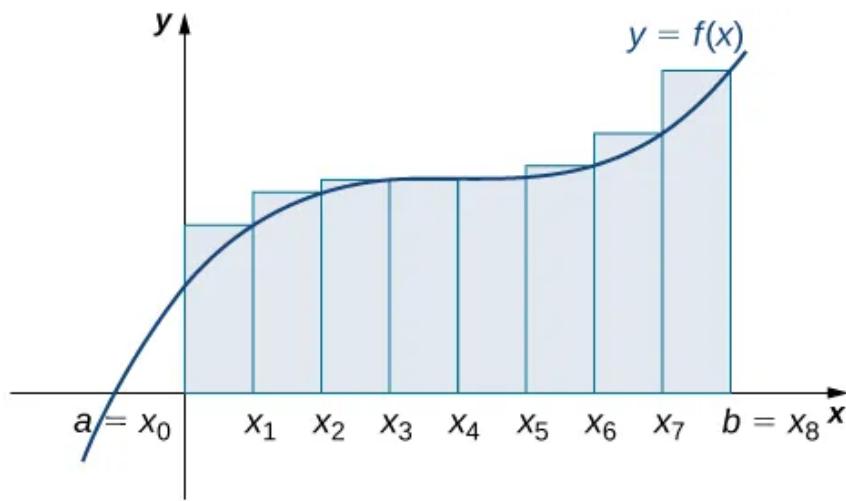


Figure 1.12 Here we use right-endpoint approximation for a region divided into eight equal subintervals.

Last, the right-endpoint approximation with $n = 32$ is close to the actual area ([Figure 1.13](#)). The area is approximately

$$\begin{aligned} R_{32} &= f(0.0625)(0.0625) + f(0.125)(0.0625) + f(0.1875)(0.0625) + \cdots + f(2)(0.0625) \\ &= 8.0625. \end{aligned}$$

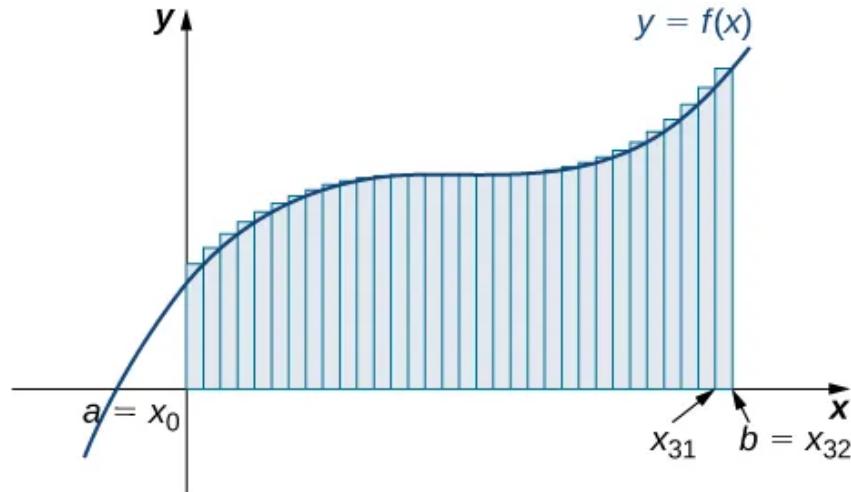


Figure 1.13 The region is divided into 32 equal subintervals for a right-endpoint approximation.

Based on these figures and calculations, it appears we are on the right track; the rectangles appear to approximate the area under the curve better as n gets larger. Furthermore, as n increases, both the left-endpoint and right-endpoint approximations appear to approach an area of 8 square units. [Table 1.1](#) shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as n gets larger and larger is very important, and we now explore this idea in more detail.

Values of n	Approximate Area L_n	Approximate Area R_n
$n = 4$	7.5	8.5

Values of n	Approximate Area L_n	Approximate Area R_n
$n = 8$	7.75	8.25
$n = 32$	7.94	8.06

Table 1.1 Converging Values of Left- and Right-Endpoint Approximations as n Increases

Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point x_i^* in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^n f(x_i^*) \Delta x.$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

DEFINITION

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be a regular partition of $[a, b]$. Let Δx be the width of each subinterval $[x_{i-1}, x_i]$ and for each i , let x_i^* be any point in $[x_{i-1}, x_i]$. A **Riemann sum** is defined for $f(x)$ as

$$\sum_{i=1}^n f(x_i^*) \Delta x.$$

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n . We are now ready to define the area under a curve in terms of Riemann sums.

DEFINITION

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^n f(x_i^*) \Delta x$ be a Riemann sum for $f(x)$. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

MEDIA

See a [graphical demonstration](#) of the construction of a Riemann sum.

Some subtleties here are worth discussing. First, note that taking the limit of a sum is a little different from taking the limit of a function $f(x)$ as x goes to infinity. Limits of sums are discussed in detail in the chapter on [Sequences and Series](#); however, for now we can assume that the computational techniques we used to compute limits of functions can also be used to calculate limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of $\{x_i^*\}$. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if

$f(x)$ is continuous on the closed interval $[a, b]$, then $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$ exists and is unique (in other words, it does not depend on the choice of $\{x_i^*\}$).

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for $\{x_i^*\}$. Although any choice for $\{x_i^*\}$ gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high (overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for $\{x_i^*\}$ to guarantee one result or the other.

If we want an overestimate, for example, we can choose $\{x_i^*\}$ such that for $i = 1, 2, 3, \dots, n$, $f(x_i^*) \geq f(x)$ for all $x \in [x_{i-1}, x_i]$. In other words, we choose $\{x_i^*\}$ so that for $i = 1, 2, 3, \dots, n$, $f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select $\{x_i^*\}$ in this way, then the Riemann sum $\sum_{i=1}^n f(x_i^*) \Delta x$ is called an

upper sum. Similarly, if we want an underestimate, we can choose $\{x_i^*\}$ so that for $i = 1, 2, 3, \dots, n$, $f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

EXAMPLE 1.5

Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.5

- a. Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$.
- b. Sketch the approximation.

EXAMPLE 1.6**Finding Lower and Upper Sums for $f(x) = \sin x$**

Find a lower sum for $f(x) = \sin x$ over the interval $[a, b] = [0, \frac{\pi}{2}]$; let $n = 6$.

[Show/Hide Solution]**CHECKPOINT 1.6**

Using the function $f(x) = \sin x$ over the interval $[0, \frac{\pi}{2}]$, find an upper sum; let $n = 6$.

Section 1.1 Exercises

1. State whether the given sums are equal or unequal.

- a. $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$
- b. $\sum_{i=1}^{10} i$ and $\sum_{i=6}^{15} (i - 5)$
- c. $\sum_{i=1}^{10} i(i - 1)$ and $\sum_{j=0}^9 (j + 1) j$
- d. $\sum_{i=1}^{10} i(i - 1)$ and $\sum_{k=1}^{10} (k^2 - k)$

In the following exercises, use the rules for sums of powers of integers to compute the sums.

2. $\sum_{i=5}^{10} i$

3. $\sum_{i=5}^{10} i^2$

Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$. In the following exercises, compute the sums.

4. $\sum_{i=1}^{100} (a_i + b_i)$

5. $\sum_{i=1}^{100} (a_i - b_i)$

6. $\sum_{i=1}^{100} (3a_i - 4b_i)$

7. $\sum_{i=1}^{100} (5a_i + 4b_i)$

In the following exercises, use summation properties and formulas to rewrite and evaluate the sums.

8. $\sum_{k=1}^{20} 100(k^2 - 5k + 1)$

9. $\sum_{j=1}^{50} (j^2 - 2j)$

10. $\sum_{j=11}^{20} (j^2 - 10j)$

11. $\sum_{k=1}^{25} [(2k)^2 - 100k]$

Let L_n denote the left-endpoint sum using n subintervals and let R_n denote the corresponding right-endpoint sum. In the following exercises, compute the indicated left and right sums for the given functions on the indicated interval.

12. L_4 for $f(x) = \frac{1}{x-1}$ on $[2, 3]$

13. R_4 for $g(x) = \cos(\pi x)$ on $[0, 1]$

14. L_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2, 5]$

15. R_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2, 5]$

16. R_4 for $\frac{1}{x^2+1}$ on $[-2, 2]$

17. L_4 for $\frac{1}{x^2+1}$ on $[-2, 2]$

18. R_8 for $x^2 - 2x + 1$ on $[0, 2]$

- 19.** L_8 for $x^2 - 2x + 1$ on $[0, 2]$
- 20.** Compute the left and right Riemann sums— L_4 and R_4 , respectively—for $f(x) = (2 - |x|)$ on $[-2, 2]$. Compute their average value and compare it with the area under the graph of f .
- 21.** Compute the left and right Riemann sums— L_6 and R_6 , respectively—for $f(x) = (3 - |3 - x|)$ on $[0, 6]$. Compute their average value and compare it with the area under the graph of f .
- 22.** Compute the left and right Riemann sums— L_4 and R_4 , respectively—for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ and compare their values.
- 23.** Compute the left and right Riemann sums— L_6 and R_6 , respectively—for $f(x) = \sqrt{9 - (x - 3)^2}$ on $[0, 6]$ and compare their values.

Express the following endpoint sums in sigma notation but do not evaluate them.

- 24.** L_{30} for $f(x) = x^2$ on $[1, 2]$
- 25.** L_{10} for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$
- 26.** R_{20} for $f(x) = \sin x$ on $[0, \pi]$
- 27.** R_{100} for $\ln x$ on $[1, e]$

In the following exercises, graph the function then use a calculator or a computer program to evaluate the following left and right endpoint sums. If the two agree, say "neither."

- 28.** [T] L_{100} and R_{100} for $y = x^2 - x + 3$ on the interval $[-1, 1]$
- 29.** [T] L_{100} and R_{100} for $y = x^2$ on the interval $[0, 1]$
- 30.** [T] L_{50} and R_{50} for $y = \frac{x+1}{x^2-1}$ on the interval $[2, 4]$
- 31.** [T] L_{100} and R_{100} for $y = x^3$ on the interval $[-1, 1]$
- 32.** [T] L_{50} and R_{50} for $y = \tan(x)$ on the interval $[0, \frac{\pi}{4}]$
- 33.** [T] L_{100} and R_{100} for $y = e^{2x}$ on the interval $[-1, 1]$
- 34.** Let t_j denote the time that it took Tejay van Garteren to ride the j th stage of the Tour de France in 2014. If there were a total of 21 stages, interpret $\sum_{j=1}^{21} t_j$.
- 35.** Let r_j denote the total rainfall in Portland on the j th day of the year in 2009. Interpret $\sum_{j=1}^{31} r_j$.
- 36.** Let d_j denote the hours of daylight and δ_j denote the increase in the hours of daylight from day $j - 1$ to day j in Fargo, North Dakota, on the j th day of the year. Interpret $d_1 + \sum_{j=2}^{365} \delta_j$.

- 37.** To help get in shape, Joe gets a new pair of running shoes. If Joe runs 1 mi each day in week 1 and adds $\frac{1}{10}$ mi to his daily routine each week, what is the total mileage on Joe's shoes after 25 weeks?
- 38.** The following table gives approximate values of the average annual atmospheric rate of increase in carbon dioxide (CO_2) each decade since 1960, in parts per million (ppm). Estimate the total increase in atmospheric CO_2 between 1964 and 2013.

Decade	Ppm/y
1964–1973	1.07
1974–1983	1.34
1984–1993	1.40
1994–2003	1.87
2004–2013	2.07

Table 1.2 Average Annual Atmospheric CO_2 Increase, 1964–2013 Source:
<http://www.esrl.noaa.gov/gmd/ccgg/trends/>.

- 39.** The following table gives the approximate increase in sea level in inches over 20 years starting in the given year. Estimate the net change in mean sea level from 1870 to 2010.

Starting Year	20-Year Change
1870	0.3
1890	1.5
1910	0.2
1930	2.8
1950	0.7
1970	1.1
1990	1.5

Table 1.3 Approximate 20-Year Sea Level Increases, 1870–1990 Source:
<http://link.springer.com/article/10.1007%2Fs10712-011-9119-1>

- 40.** The following table gives the approximate increase in dollars in the average price of a gallon of gas per decade since 1950. If the average price of a gallon of gas in 2010 was \$2.60, what was the average price of a gallon of gas in 1950?

Starting Year	10-Year Change
1950	0.03
1960	0.05
1970	0.86
1980	-0.03
1990	0.29
2000	1.12

Table 1.4 Approximate 10-Year Gas Price Increases, 1950–2000 *Source:* http://epb.lbl.gov/homepages/Rick_Diamond/docs/lbnl55011-trends.pdf.

41. The following table gives the percent growth of the U.S. population beginning in July of the year indicated. If the U.S. population was 281,421,906 in July 2000, estimate the U.S. population in July 2010.

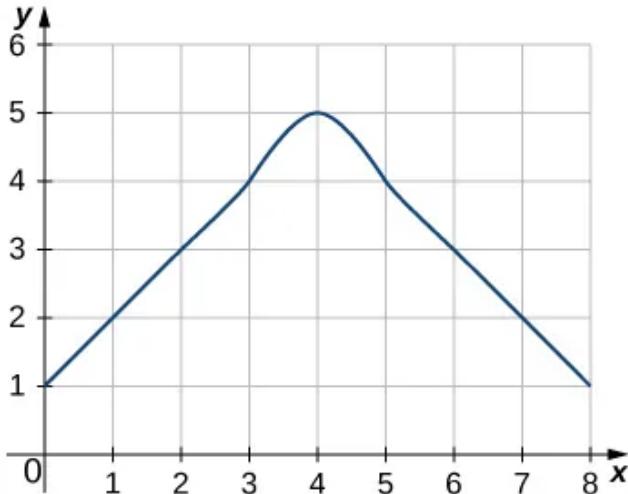
Year	% Change/Year
2000	1.12
2001	0.99
2002	0.93
2003	0.86
2004	0.93
2005	0.93
2006	0.97
2007	0.96
2008	0.95
2009	0.88

Table 1.5 Annual Percentage Growth of U.S. Population, 2000–2009 *Source:* <http://www.census.gov/popest/data>.

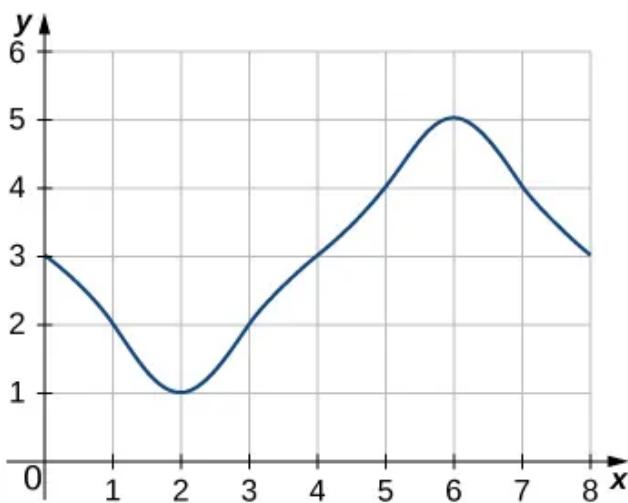
(*Hint:* To obtain the population in July 2001, multiply the population in July 2000 by 1.0112 to get 284,573,831.)

In the following exercises, estimate the areas under the curves by computing the left Riemann sums, L_8 .

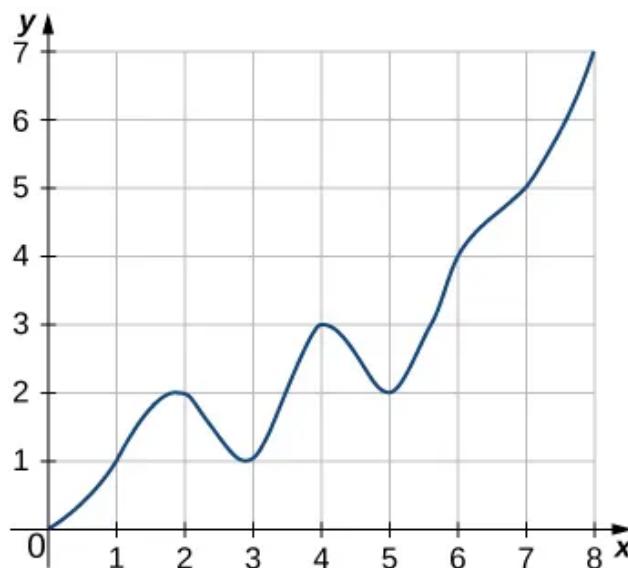
42.



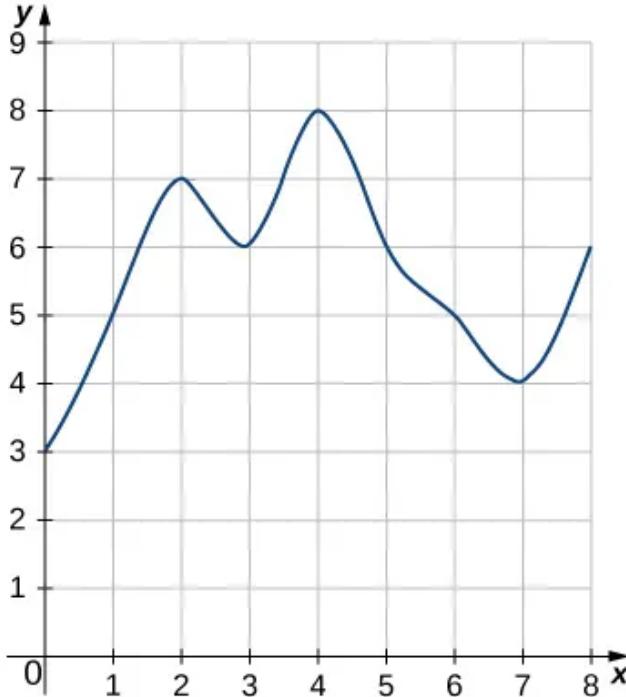
43.



44.



45.



46. [T] Use a computer algebra system to compute the Riemann sum, L_N , for $N = 10, 30, 50$ for $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$.
47. [T] Use a computer algebra system to compute the Riemann sum, L_N , for $N = 10, 30, 50$ for $f(x) = \frac{1}{\sqrt{1+x^2}}$ on $[-1, 1]$.
48. [T] Use a computer algebra system to compute the Riemann sum, L_N , for $N = 10, 30, 50$ for $f(x) = \sin^2 x$ on $[0, 2\pi]$. Compare these estimates with π .

In the following exercises, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$. How do these estimates compare with the exact answers, which you can find via geometry?

49. [T] $y = \cos(\pi x)$ on the interval $[0, 1]$

50. [T] $y = 3x + 2$ on the interval $[3, 5]$

In the following exercises, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$.

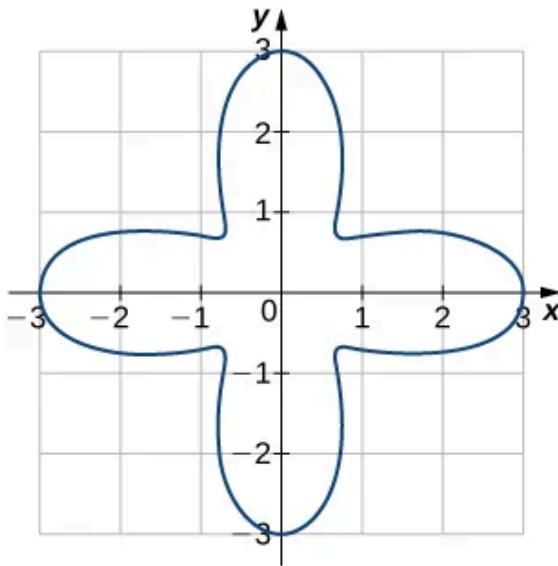
51. [T] $y = x^4 - 5x^2 + 4$ on the interval $[-2, 2]$, which has an exact area of $\frac{32}{15}$

52. [T] $y = \ln x$ on the interval $[1, 2]$, which has an exact area of $2\ln(2) - 1$

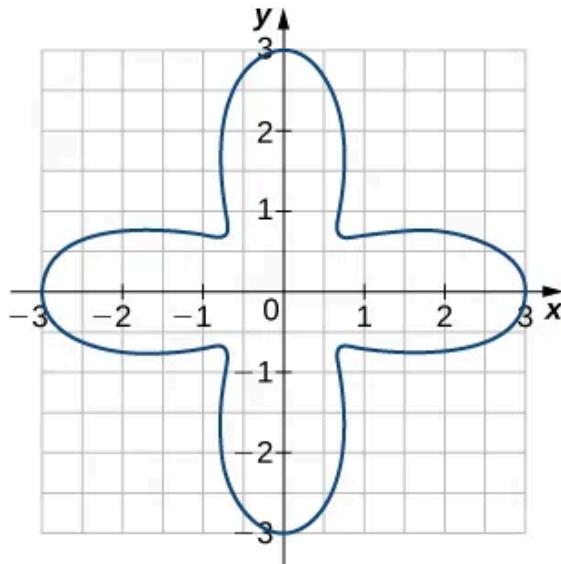
53. Explain why, if $f(a) \geq 0$ and f is increasing on $[a, b]$, that the left endpoint estimate is a lower bound for the area below the graph of f on $[a, b]$.

54. Explain why, if $f(b) \geq 0$ and f is decreasing on $[a, b]$, that the left endpoint estimate is an upper bound for the area below the graph of f on $[a, b]$.

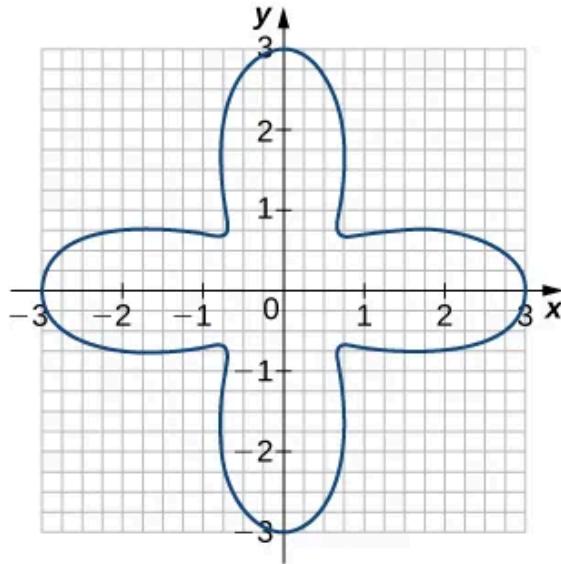
55. Show that, in general, $R_N - L_N = (b - a) \times \frac{f(b) - f(a)}{N}$.
56. Explain why, if f is increasing on $[a, b]$, the error between either L_N or R_N and the area A below the graph of f is at most $(b - a) \frac{f(b) - f(a)}{N}$.
57. For each of the three graphs:
- Obtain a lower bound $L(A)$ for the area enclosed by the curve by adding the areas of the squares *enclosed completely* by the curve.
 - Obtain an upper bound $U(A)$ for the area by adding to $L(A)$ the areas $B(A)$ of the squares *enclosed partially* by the curve.



Graph 1

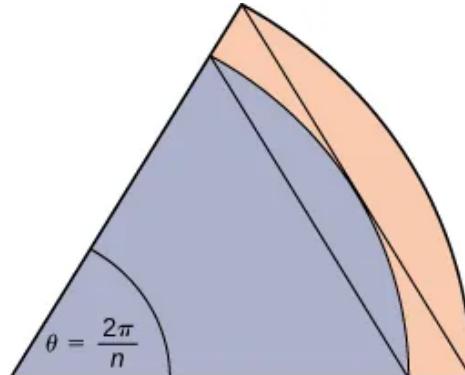


Graph 2



Graph 3

58. In the previous exercise, explain why $L(A)$ gets no smaller while $U(A)$ gets no larger as the squares are subdivided into four boxes of equal area.
59. A unit circle is made up of n wedges equivalent to the inner wedge in the figure. The base of the inner triangle is 1 unit and its height is $\sin\left(\frac{2\pi}{n}\right)$. The base of the outer triangle is $B = \cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)$ and the height is $H = B\sin\left(\frac{2\pi}{n}\right)$. Use this information to argue that



the area of a unit circle is equal to π .

1.2 The Definite Integral

Learning Objectives

- 1.2.1 State the definition of the definite integral.
- 1.2.2 Explain the terms integrand, limits of integration, and variable of integration.
- 1.2.3 Explain when a function is integrable.
- 1.2.4 Describe the relationship between the definite integral and net area.
- 1.2.5 Use geometry and the properties of definite integrals to evaluate them.
- 1.2.6 Calculate the average value of a function.

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

DEFINITION

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of f from a to b is given by

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x, \quad (1.8)$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an **integrable function**.

The integral symbol in the previous definition should look familiar. We have seen similar notation in the chapter on [Applications of Derivatives](#), where we used the indefinite integral symbol (without the a and b above and below) to represent an antiderivative. Although the notation for indefinite integrals may look similar to the notation for a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions. Later in this chapter we examine how these concepts are

related. However, close attention should always be paid to notation so we know whether we're working with a definite integral or an indefinite integral.

Integral notation goes back to the late seventeenth century and is one of the contributions of Gottfried Wilhelm Leibniz, who is often considered to be the codiscoverer of calculus, along with Isaac Newton. The integration symbol \int is an elongated S, suggesting sigma or summation. On a definite integral, above and below the summation symbol are the boundaries of the interval, $[a, b]$. The numbers a and b are x -values and are called the **limits of integration**; specifically, a is the lower limit and b is the upper limit. To clarify, we are using the word *limit* in two different ways in the context of the definite integral. First, we talk about the limit of a sum as $n \rightarrow \infty$. Second, the boundaries of the region are called the *limits of integration*.

We call the function $f(x)$ the **integrand**, and the dx indicates that $f(x)$ is a function with respect to x , called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a dummy variable, and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

$$\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(u) du$$

Previously, we discussed the fact that if $f(x)$ is continuous on $[a, b]$, then the limit $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$ exists and is unique. This leads to the following theorem, which we state without proof.

THEOREM 1.1

Continuous Functions Are Integrable

If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Functions that are not continuous on $[a, b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions continuous on a closed interval, apart from a finite number of jump discontinuities, are integrable.

It is also worth noting here that we have retained the use of a regular partition in the Riemann sums. This restriction is not strictly necessary. Any partition can be used to form a Riemann sum. However, if a nonregular partition is used to define the definite integral, it is not sufficient to take the limit as the number of subintervals goes to infinity. Instead, we must take the limit as the width of the largest subinterval goes to zero. This introduces a little more complex notation in our limits and makes the calculations more difficult without really gaining much additional insight, so we stick with regular partitions for the Riemann sums.

EXAMPLE 1.7

Evaluating an Integral Using the Definition

Use the definition of the definite integral to evaluate $\int_0^2 x^2 dx$. Use a right-endpoint approximation to generate the Riemann sum.

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.7

Use the definition of the definite integral to evaluate $\int_0^3 (2x - 1) dx$. Use a right-endpoint approximation to generate the Riemann sum.

Evaluating Definite Integrals

Evaluating definite integrals this way can be quite tedious because of the complexity of the calculations. Later in this chapter we develop techniques for evaluating definite integrals *without* taking limits of Riemann sums. However, for now, we can rely on the fact that definite integrals represent the area under the curve, and we can evaluate definite integrals by using geometric formulas to calculate that area. We do this to confirm that definite integrals do, indeed, represent areas, so we can then discuss what to do in the case of a curve of a function dropping below the x -axis.

EXAMPLE 1.8

Using Geometric Formulas to Calculate Definite Integrals

Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9 - (x - 3)^2} dx$.

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.8

Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x + 3) dx$.

Area and the Definite Integral

When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret “the area under the curve” when $f(x)$ is negative?

Net Signed Area

Let us return to the Riemann sum. Consider, for example, the function $f(x) = 2 - 2x^2$ (shown in [Figure 1.17](#)) on the interval $[0, 2]$. Use $n = 8$ and choose $\{x_i^*\}$ as the left endpoint of each interval. Construct a rectangle on each subinterval of height $f(x_i^*)$ and width Δx . When $f(x_i^*)$ is positive, the product $f(x_i^*) \Delta x$ represents the area of the rectangle, as before. When $f(x_i^*)$ is negative, however, the product $f(x_i^*) \Delta x$ represents the *negative* of the area of the rectangle. The Riemann sum then becomes

$$\sum_{i=1}^8 f(x_i^*) \Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$

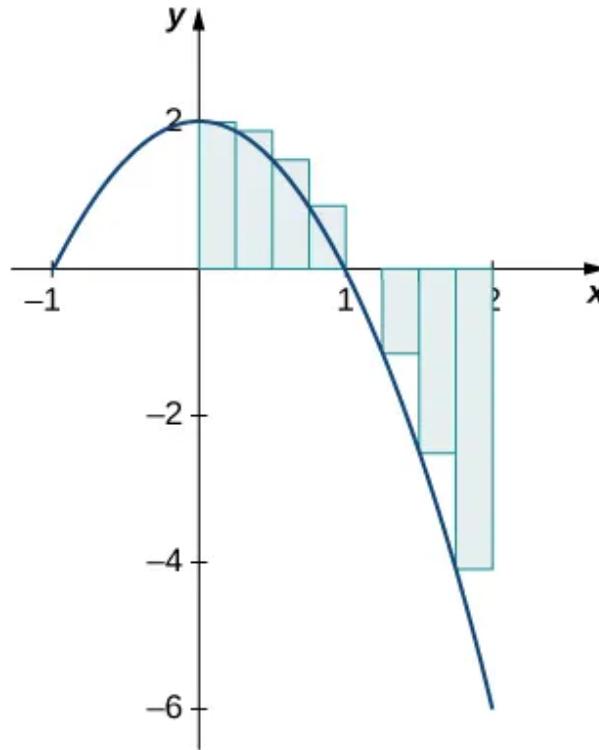


Figure 1.17 For a function that is partly negative, the Riemann sum is the area of the rectangles above the x -axis less the area of the rectangles below the x -axis.

Taking the limit as $n \rightarrow \infty$, the Riemann sum approaches the area between the curve above the x -axis and the x -axis, less the area between the curve below the x -axis and the x -axis, as shown in [Figure 1.18](#).

Then,

$$\int_0^2 f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ = A_1 - A_2.$$

The quantity $A_1 - A_2$ is called the **net signed area**.

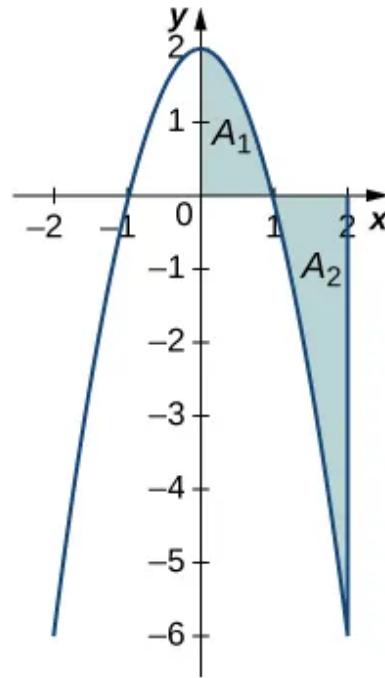


Figure 1.18 In the limit, the definite integral equals area A_1 less area A_2 , or the net signed area.

Notice that net signed area can be positive, negative, or zero. If the area above the x -axis is larger, the net signed area is positive. If the area below the x -axis is larger, the net signed area is negative. If the areas above and below the x -axis are equal, the net signed area is zero.

EXAMPLE 1.9

Finding the Net Signed Area

Find the net signed area between the curve of the function $f(x) = 2x$ and the x -axis over the interval $[-3, 3]$.

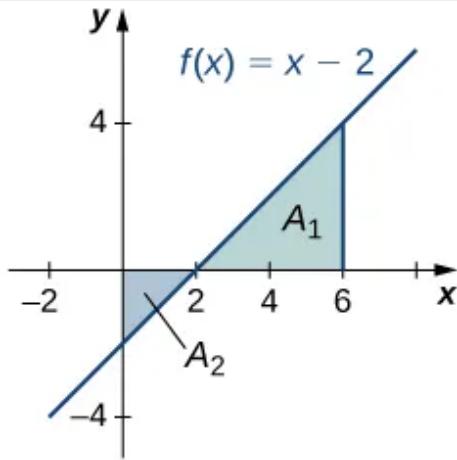
[\[Show/Hide Solution\]](#)

Analysis

If A_1 is the area above the x -axis and A_2 is the area below the x -axis, then the net area is $A_1 - A_2$. Since the areas of the two triangles are equal, the net area is zero.

CHECKPOINT 1.9

Find the net signed area of $f(x) = x - 2$ over the interval $[0, 6]$, illustrated in the following image.



Total Area

One application of the definite integral is finding displacement when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position. This is a very important application of the definite integral, and we examine it in more detail later in the chapter. For now, we're just going to look at some basics to get a feel for how this works by studying constant velocities.

When velocity is a constant, the area under the curve is just velocity times time. This idea is already very familiar. If a car travels away from its starting position in a straight line at a speed of 70 mph for 2 hours, then it is 140 mi away from its original position ([Figure 1.20](#)). Using integral notation, we have

$$\int_0^2 70 dt = 140.$$

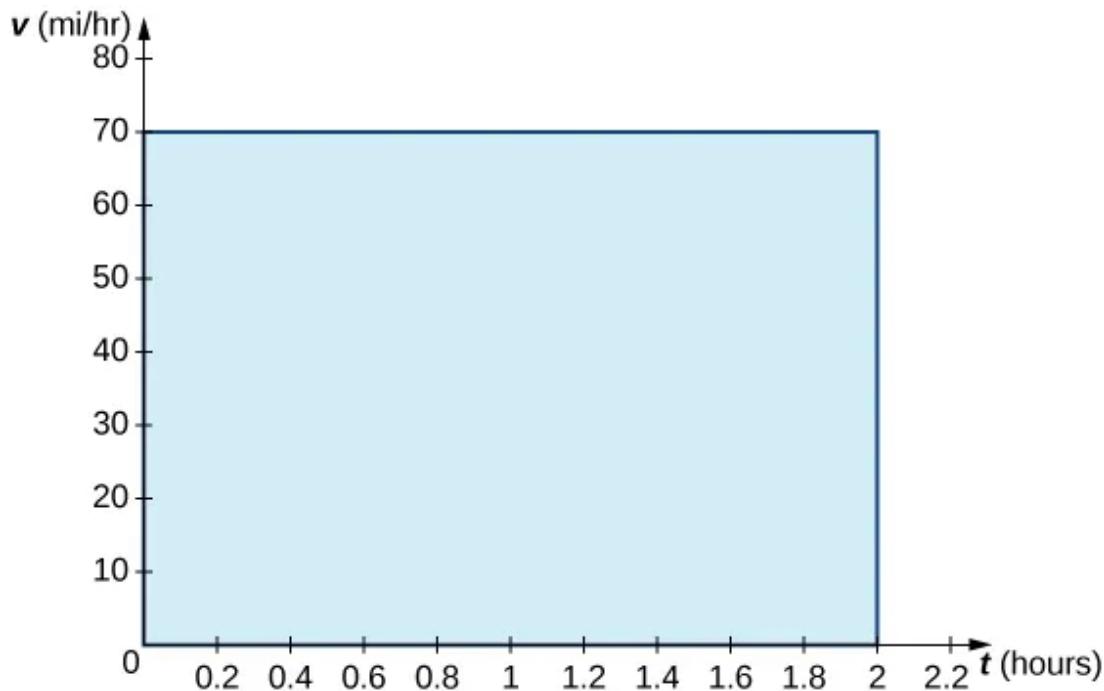


Figure 1.20 The area under the curve $v(t) = 70$ tells us how far the car is from its starting point at a given time.

In the context of displacement, net signed area allows us to take direction into account. If a car travels straight north at a speed of 60 mph for 2 hours, it is 120 mi north of its starting position. If the car then turns around and travels south at a speed of 40 mph for 3 hours, it will be back at its starting position ([Figure 1.21](#)). Again, using integral notation, we have

$$\int_0^2 60dt + \int_2^5 -40dt = 120 - 120 = 0.$$

In this case the displacement is zero.

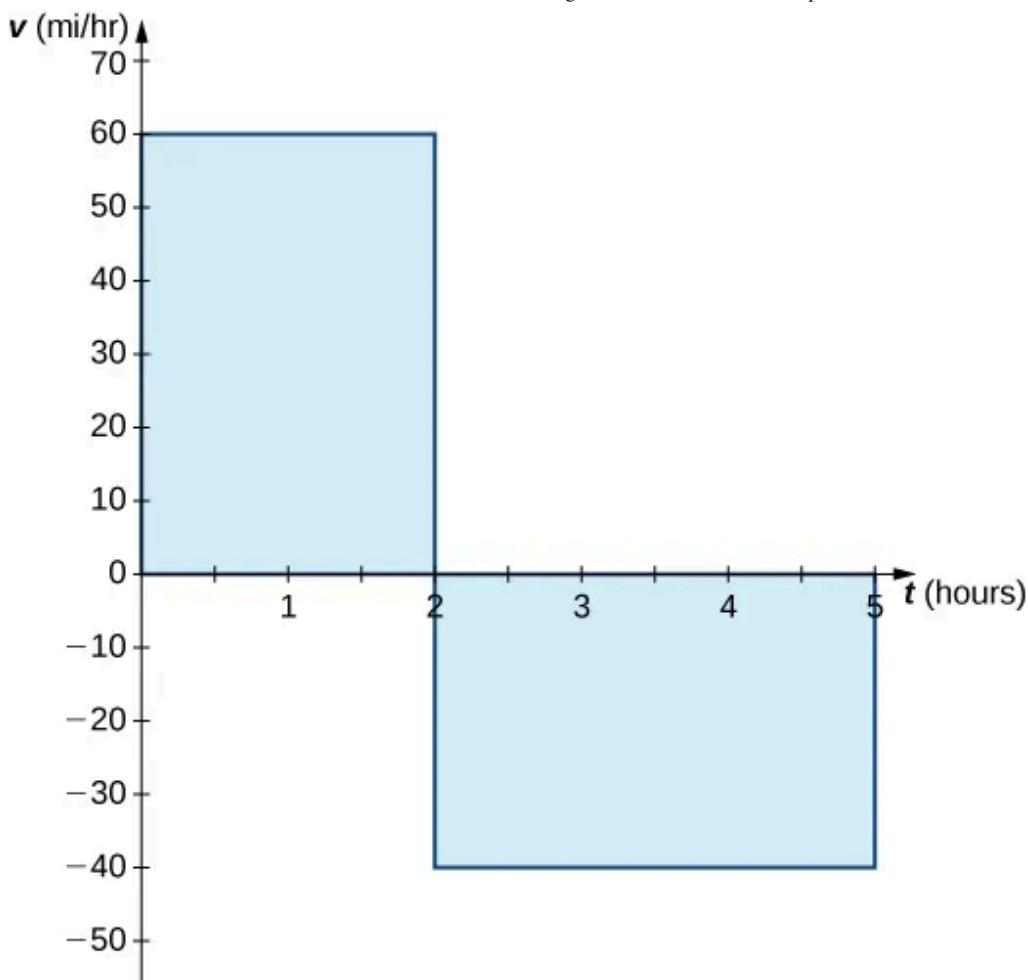


Figure 1.21 The area above the axis and the area below the axis are equal, so the net signed area is zero.

Suppose we want to know how far the car travels overall, regardless of direction. In this case, we want to know the area between the curve and the x -axis, regardless of whether that area is above or below the axis. This is called the **total area**.

Graphically, it is easiest to think of calculating total area by adding the areas above the axis and the areas below the axis (rather than subtracting the areas below the axis, as we did with net signed area). To accomplish this mathematically, we use the absolute value function. Thus, the total distance traveled by the car is

$$\begin{aligned} \int_0^2 |60| dt + \int_2^5 |-40| dt &= \int_0^2 60dt + \int_2^5 40dt \\ &= 120 + 120 \\ &= 240. \end{aligned}$$

Bringing these ideas together formally, we state the following definitions.

DEFINITION

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x -axis that lies *above* the axis and let A_2 represent the area between $f(x)$ and the x -axis that lies *below* the axis. Then, the **net signed area** between $f(x)$ and the x -axis is given by

$$\int_a^b f(x) dx = A_1 - A_2.$$

The **total area** between $f(x)$ and the x -axis is given by

$$\int_a^b |f(x)| dx = A_1 + A_2.$$

EXAMPLE 1.10

Finding the Total Area

Find the total area between $f(x) = x - 2$ and the x -axis over the interval $[0, 6]$.

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.10

Find the total area between the function $f(x) = 2x$ and the x -axis over the interval $[-3, 3]$.

Properties of the Definite Integral

The properties of indefinite integrals apply to definite integrals as well. Definite integrals also have properties that relate to the limits of integration. These properties, along with the rules of integration that we examine later in this chapter, help us manipulate expressions to evaluate definite integrals.

RULE: PROPERTIES OF THE DEFINITE INTEGRAL

1.

$$\int_a^a f(x) dx = 0 \quad (1.9)$$

If the limits of integration are the same, the integral is just a line and contains no area.

2.

$$\int_b^a f(x) dx = - \int_a^b f(x) dx \quad (1.10)$$

If the limits are reversed, then place a negative sign in front of the integral.

3.

$$\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx \quad (1.11)$$

The integral of a sum is the sum of the integrals.

4.

$$\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx \quad (1.12)$$

The integral of a difference is the difference of the integrals.

5.

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx \quad (1.13)$$

for constant c . The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

6.

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \quad (1.14)$$

Although this formula normally applies when c is between a and b , the formula holds for all values of a , b , and c , provided $f(x)$ is integrable on the largest interval.

EXAMPLE 1.11**Using the Properties of the Definite Integral**

Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2, 1]$ as the sum of three definite integrals.

[Show/Hide Solution]**CHECKPOINT 1.11**

Use the properties of the definite integral to express the definite integral of $f(x) = 6x^3 - 4x^2 + 2x - 3$ over the interval $[1, 3]$ as the sum of four definite integrals.

EXAMPLE 1.12**Using the Properties of the Definite Integral**

If it is known that $\int_0^8 f(x) dx = 10$ and $\int_0^5 f(x) dx = 5$, find the value of $\int_5^8 f(x) dx$.

[Show/Hide Solution]**CHECKPOINT 1.12**

If it is known that $\int_1^5 f(x) dx = -3$ and $\int_2^5 f(x) dx = 4$, find the value of $\int_1^2 f(x) dx$.

Comparison Properties of Integrals

A picture can sometimes tell us more about a function than the results of computations. Comparing functions by their graphs as well as by their algebraic expressions can often give new insight into the process of integration. Intuitively, we might say that if a function $f(x)$ is above another function $g(x)$,

then the area between $f(x)$ and the x -axis is greater than the area between $g(x)$ and the x -axis. This is true depending on the interval over which the comparison is made. The properties of definite integrals are valid whether $a < b$, $a = b$, or $a > b$. The following properties, however, concern only the case $a \leq b$, and are used when we want to compare the sizes of integrals.

THEOREM 1.2

Comparison Theorem

- i. If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x) dx \geq 0.$$

- ii. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x) dx \geq \int_a^b g(x) dx.$$

- iii. If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$\begin{aligned} m(b-a) &\leq \int_a^b f(x) dx \\ &\leq M(b-a). \end{aligned}$$

EXAMPLE 1.13

Comparing Two Functions over a Given Interval

Compare $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0, 1]$.

[\[Show/Hide Solution\]](#)

Average Value of a Function

We often need to find the average of a set of numbers, such as an average test grade. Suppose you received the following test scores in your algebra class: 89, 90, 56, 78, 100, and 69. Your semester grade is your average of test scores and you want to know what grade to expect. We can find the average by adding all the scores and dividing by the number of scores. In this case, there are six test scores. Thus,

$$\frac{89 + 90 + 56 + 78 + 100 + 69}{6} = \frac{482}{6} \approx 80.33.$$

Therefore, your average test grade is approximately 80.33, which translates to a B– at most schools.

Suppose, however, that we have a function $v(t)$ that gives us the speed of an object at any time t , and we want to find the object's average speed. The function $v(t)$ takes on an infinite number of values, so we can't use the process just described. Fortunately, we can use a definite integral to find the average value of a function such as this.

Let $f(x)$ be continuous over the interval $[a, b]$ and let $[a, b]$ be divided into n subintervals of width $\Delta x = (b - a)/n$. Choose a representative x_i^* in each subinterval and calculate $f(x_i^*)$ for $i = 1, 2, \dots, n$. In other words, consider each $f(x_i^*)$ as a sampling of the function over each subinterval. The average value of the function may then be approximated as

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n},$$

which is basically the same expression used to calculate the average of discrete values.

But we know $\Delta x = \frac{b-a}{n}$, so $n = \frac{b-a}{\Delta x}$, and we get

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{\frac{(b-a)}{\Delta x}}.$$

Following through with the algebra, the numerator is a sum that is represented as $\sum_{i=1}^n f(x_i^*)$, and we are dividing by a fraction. To divide by a fraction, invert the denominator and multiply. Thus, an approximate value for the average value of the function is given by

$$\begin{aligned} \frac{\sum_{i=1}^n f(x_i^*)}{\frac{(b-a)}{\Delta x}} &= \left(\frac{\Delta x}{b-a} \right) \sum_{i=1}^n f(x_i^*) \\ &= \left(\frac{1}{b-a} \right) \sum_{i=1}^n f(x_i^*) \Delta x. \end{aligned}$$

This is a Riemann sum. Then, to get the exact average value, take the limit as n goes to infinity. Thus, the average value of a function is given by

$$\frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \frac{1}{b-a} \int_a^b f(x) dx.$$

DEFINITION

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the **average value of the function** $f(x)$ (or f_{ave}) on $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

EXAMPLE 1.14

Finding the Average Value of a Linear Function

Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$.

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.13

Find the average value of $f(x) = 6 - 2x$ over the interval $[0, 3]$.

Section 1.2 Exercises

In the following exercises, express the limits as integrals.

60. $\lim_{n \rightarrow \infty} \sum_{i=1}^n (x_i^*) \Delta x$ over $[1, 3]$

61. $\lim_{n \rightarrow \infty} \sum_{i=1}^n (5(x_i^*)^2 - 3(x_i^*)^3) \Delta x$ over $[0, 2]$

62. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \sin^2(2\pi x_i^*) \Delta x$ over $[0, 1]$

63. $\lim_{n \rightarrow \infty} \sum_{i=1}^n \cos^2(2\pi x_i^*) \Delta x$ over $[0, 1]$

In the following exercises, given L_n or R_n as indicated, express their limits as $n \rightarrow \infty$ as definite integrals, identifying the correct intervals.

64. $L_n = \frac{1}{n} \sum_{i=1}^n \frac{i-1}{n}$

65. $R_n = \frac{1}{n} \sum_{i=1}^n \frac{i}{n}$

66. $L_n = \frac{2}{n} \sum_{i=1}^n \left(1 + 2\frac{i-1}{n}\right)$

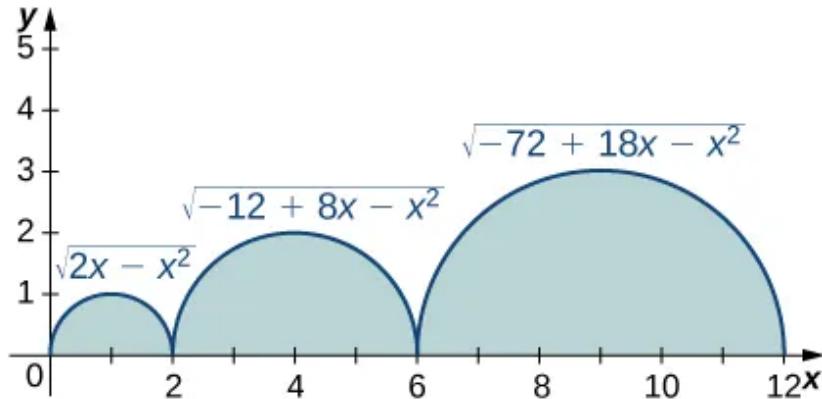
67. $R_n = \frac{3}{n} \sum_{i=1}^n \left(3 + 3\frac{i}{n}\right)$

68. $L_n = \frac{2\pi}{n} \sum_{i=1}^n 2\pi \frac{i-1}{n} \cos\left(2\pi \frac{i-1}{n}\right)$

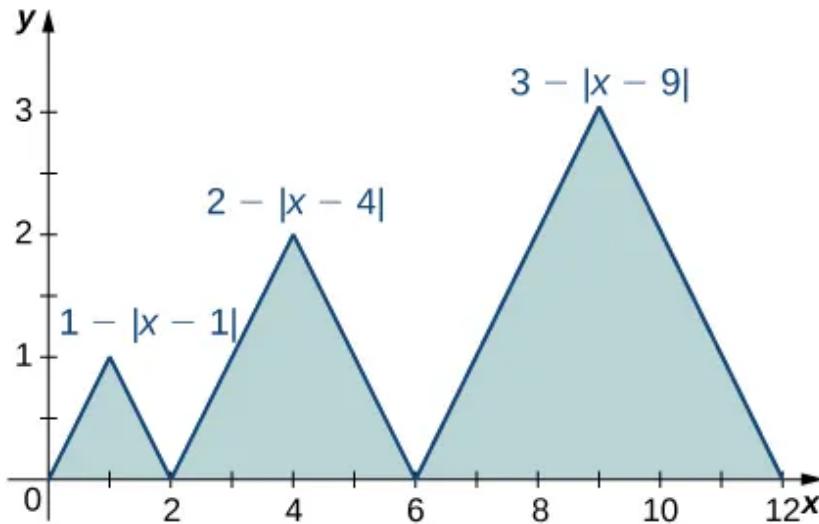
69. $R_n = \frac{1}{n} \sum_{i=1}^n \left(1 + \frac{i}{n}\right) \log\left(\left(1 + \frac{i}{n}\right)^2\right)$

In the following exercises, evaluate the integrals of the functions graphed using the formulas for areas of triangles and circles, and subtracting the areas below the x -axis.

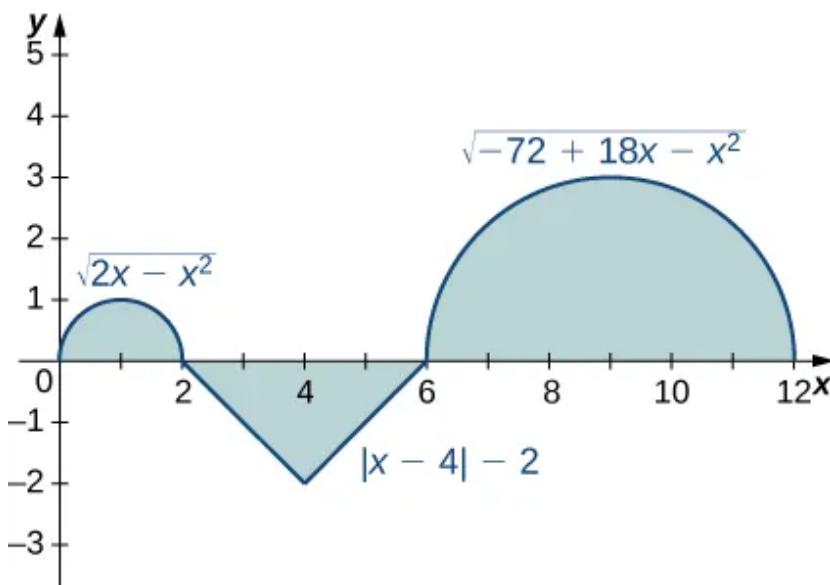
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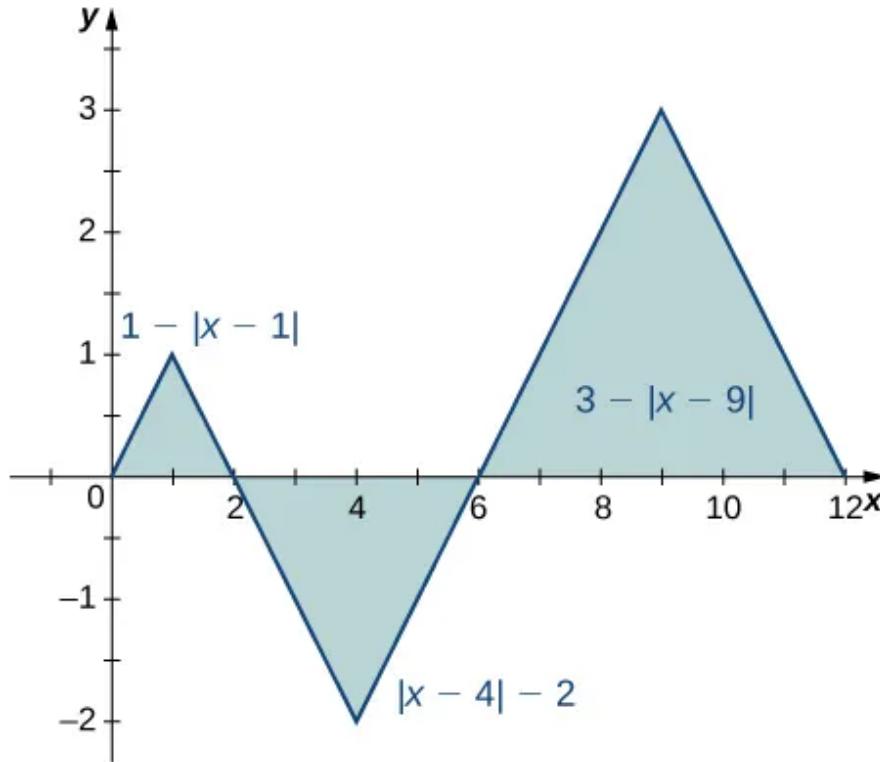
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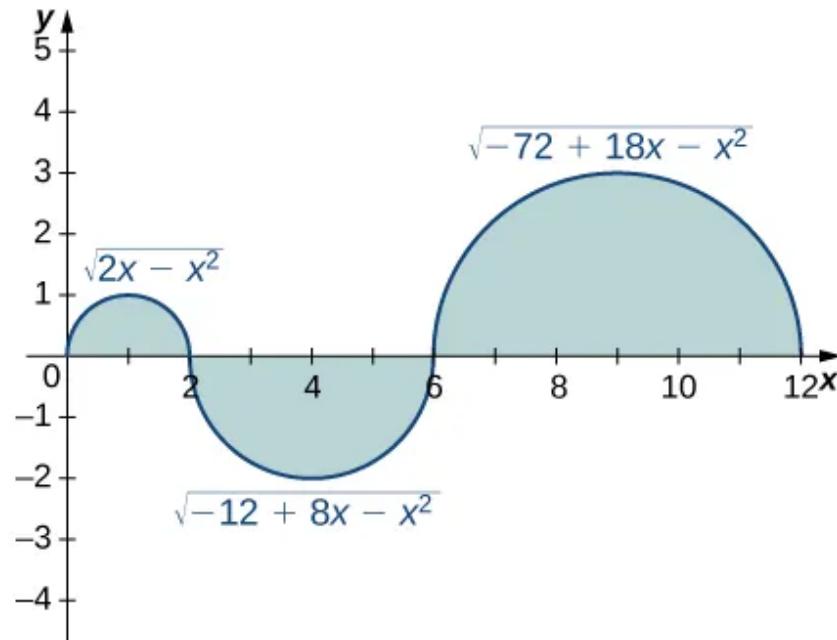
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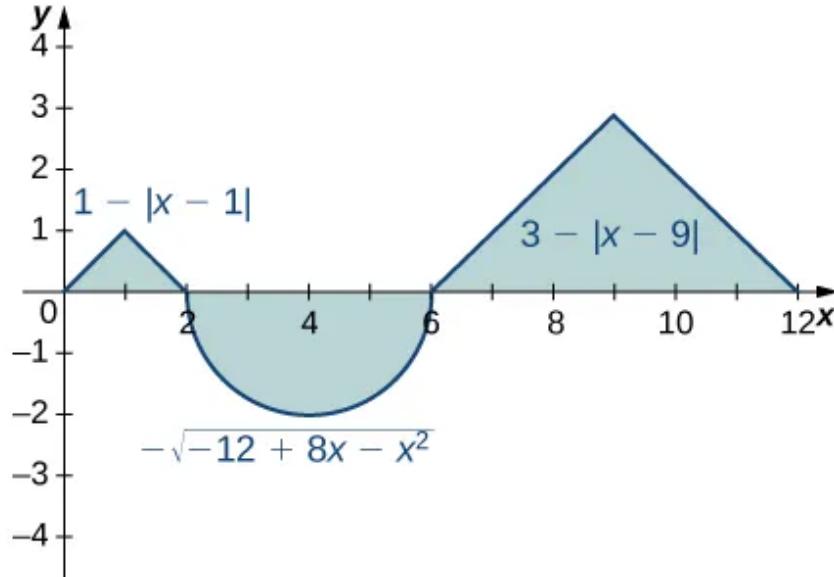


73.



74.



75.

In the following exercises, evaluate the integral using area formulas.

76. $\int_0^3 (3 - x) dx$

77. $\int_2^3 (3 - x) dx$

78. $\int_{-3}^3 (3 - |x|) dx$

79. $\int_0^6 (3 - |x - 3|) dx$

80. $\int_{-2}^2 \sqrt{4 - x^2} dx$

81. $\int_1^5 \sqrt{4 - (x - 3)^2} dx$

82. $\int_0^{12} \sqrt{36 - (x - 6)^2} dx$

83. $\int_{-2}^3 (3 - |x|) dx$

In the following exercises, use averages of values at the left (L) and right (R) endpoints to compute the integrals of the piecewise linear functions with graphs that pass through the given list of points over the indicated intervals.

84. $\{(0, 0), (2, 1), (4, 3), (5, 0), (6, 0), (8, 3)\}$ over $[0, 8]$

85. $\{(0, 2), (1, 0), (3, 5), (5, 5), (6, 2), (8, 0)\}$ over $[0, 8]$

86. $\{(-4, -4), (-2, 0), (0, -2), (3, 3), (4, 3)\}$ over $[-4, 4]$

87. $\{(-4, 0), (-2, 2), (0, 0), (1, 2), (3, 2), (4, 0)\}$ over $[-4, 4]$

Suppose that $\int_0^4 f(x) dx = 5$ and $\int_0^2 f(x) dx = -3$, and $\int_0^4 g(x) dx = -1$ and $\int_0^2 g(x) dx = 2$.

In the following exercises, compute the integrals.

88. $\int_0^4 (f(x) + g(x)) dx$

89. $\int_2^4 (f(x) + g(x)) dx$

90. $\int_0^2 (f(x) - g(x)) dx$

91. $\int_2^4 (f(x) - g(x)) dx$

92. $\int_0^2 (3f(x) - 4g(x)) dx$

93. $\int_2^4 (4f(x) - 3g(x)) dx$

In the following exercises, use the identity $\int_{-A}^A f(x) dx = \int_{-A}^0 f(x) dx + \int_0^A f(x) dx$ to compute the integrals.

94. $\int_{-\pi}^{\pi} \frac{\sin t}{1+t^2} dt$ (*Hint:* $\sin(-t) = -\sin(t)$)

95. $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{t}{1+\cos t} dt$

In the following exercises, find the net signed area between $f(x)$ and the x-axis.

96. $\int_1^3 (2-x) dx$ (*Hint:* Look at the graph of f .)

97. $\int_2^4 (x-3)^3 dx$ (*Hint:* Look at the graph of f .)

In the following exercises, given that $\int_0^1 x dx = \frac{1}{2}$, $\int_0^1 x^2 dx = \frac{1}{3}$, and $\int_0^1 x^3 dx = \frac{1}{4}$, compute the integrals.

98. $\int_0^1 (1 + x + x^2 + x^3) dx$

99. $\int_0^1 (1 - x + x^2 - x^3) dx$

100. $\int_0^1 (1 - x)^2 dx$

101. $\int_0^1 (1 - 2x)^3 dx$

102. $\int_0^1 \left(6x - \frac{4}{3}x^2\right) dx$

103. $\int_0^1 (7 - 5x^3) dx$

In the following exercises, use the [comparison theorem](#).

104. Show that $\int_0^3 (x^2 - 6x + 9) dx \geq 0$.

105. Show that $\int_{-2}^3 (x - 3)(x + 2) dx \leq 0$.

106. Show that $\int_0^1 \sqrt{1 + x^3} dx \leq \int_0^1 \sqrt{1 + x^2} dx$.

107. Show that $\int_1^2 \sqrt{1 + x} dx \leq \int_1^2 \sqrt{1 + x^2} dx$.

108. Show that $\int_0^{\pi/2} \sin t dt \geq \frac{\pi}{4}$. (*Hint:* $\sin t \geq \frac{2t}{\pi}$ over $[0, \frac{\pi}{2}]$)

109. Show that $\int_{-\pi/4}^{\pi/4} \cos t dt \geq \pi\sqrt{2}/4$.

In the following exercises, find the average value f_{ave} of f between a and b , and find a point c , where $f(c) = f_{\text{ave}}$.

110. $f(x) = x^2, a = -1, b = 1$

111. $f(x) = x^5, a = -1, b = 1$

112. $f(x) = \sqrt{4 - x^2}, a = 0, b = 2$

113. $f(x) = (3 - |x|), a = -3, b = 3$

114. $f(x) = \sin x, a = 0, b = 2\pi$

115. $f(x) = \cos x, a = 0, b = 2\pi$

In the following exercises, approximate the average value using Riemann sums L_{100} and R_{100} . How does your answer compare with the exact given answer?

116. [T] $y = \ln(x)$ over the interval $[1, 4]$; the exact solution is $\frac{\ln(256)}{3} - 1$.

117. [T] $y = e^{x/2}$ over the interval $[0, 1]$; the exact solution is $2(\sqrt{e} - 1)$.

118. [T] $y = \tan x$ over the interval $[0, \frac{\pi}{4}]$; the exact solution is $\frac{2\ln(2)}{\pi}$.

119. [T] $y = \frac{x+1}{\sqrt{4-x^2}}$ over the interval $[-1, 1]$; the exact solution is $\frac{\pi}{6}$.

In the following exercises, compute the average value using the left Riemann sums L_N for $N = 1, 10, 100$. How does the accuracy compare with the given exact value?

120. [T] $y = x^2 - 4$ over the interval $[0, 2]$; the exact solution is $-\frac{8}{3}$.

121. [T] $y = xe^{x^2}$ over the interval $[0, 2]$; the exact solution is $\frac{1}{4}(e^4 - 1)$.

122. [T] $y = (\frac{1}{2})^x$ over the interval $[0, 4]$; the exact solution is $\frac{15}{64\ln(2)}$.

123. [T] $y = x \sin(x^2)$ over the interval $[-\pi, 0]$; the exact solution is $\frac{\cos(\pi^2) - 1}{2\pi}$.

124. Suppose that $A = \int_0^{2\pi} \sin^2 t dt$ and $B = \int_0^{2\pi} \cos^2 t dt$. Show that $A + B = 2\pi$ and $A = B$.

125. Suppose that $A = \int_{-\pi/4}^{\pi/4} \sec^2 t dt = \pi$ and $B = \int_{-\pi/4}^{\pi/4} \tan^2 t dt$. Show that $A - B = \frac{\pi}{2}$.

126. Show that the average value of $\sin^2 t$ over $[0, 2\pi]$ is equal to $1/2$. Without further calculation, determine whether the average value of $\sin^2 t$ over $[0, \pi]$ is also equal to $1/2$.

127. Show that the average value of $\cos^2 t$ over $[0, 2\pi]$ is equal to $1/2$. Without further calculation, determine whether the average value of $\cos^2(t)$ over $[0, \pi]$ is also equal to $1/2$.

128. Explain why the graphs of a quadratic function (parabola) $p(x)$ and a linear function $\ell(x)$ can intersect in at most two points. Suppose that $p(a) = \ell(a)$ and $p(b) = \ell(b)$, and that

$\int_a^b p(t) dt > \int_a^b \ell(t) dt$. Explain why $\int_c^d p(t) > \int_c^d \ell(t) dt$ whenever $a \leq c < d \leq b$.

- 129.** Suppose that parabola $p(x) = ax^2 + bx + c$ opens downward ($a < 0$) and has a vertex of $y = \frac{-b}{2a} > 0$. For which interval $[A, B]$ is $\int_A^B (ax^2 + bx + c) dx$ as large as possible?

- 130.** Suppose $[a, b]$ can be subdivided into subintervals $a = a_0 < a_1 < a_2 < \dots < a_N = b$ such that either $f \geq 0$ over $[a_{i-1}, a_i]$ or $f \leq 0$ over $[a_{i-1}, a_i]$. Set $A_i = \int_{a_{i-1}}^{a_i} f(t) dt$.

a. Explain why $\int_a^b f(t) dt = A_1 + A_2 + \dots + A_N$.

b. Then, explain why $\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt$.

- 131.** Suppose f and g are continuous functions such that $\int_c^d f(t) dt \leq \int_c^d g(t) dt$ for every subinterval $[c, d]$ of $[a, b]$. Explain why $f(x) \leq g(x)$ for all values of x .

- 132.** Suppose the average value of f over $[a, b]$ is 1 and the average value of f over $[b, c]$ is 1 where $a < c < b$. Show that the average value of f over $[a, c]$ is also 1.

- 133.** Suppose that $[a, b]$ can be partitioned taking $a = a_0 < a_1 < \dots < a_N = b$ such that the average value of f over each subinterval $[a_{i-1}, a_i] = 1$ is equal to 1 for each $i = 1, \dots, N$. Explain why the average value of f over $[a, b]$ is also equal to 1.

- 134.** Suppose that for each i such that $1 \leq i \leq N$ one has $\int_{i-1}^i f(t) dt = i$. Show that $\int_0^N f(t) dt = \frac{N(N+1)}{2}$.

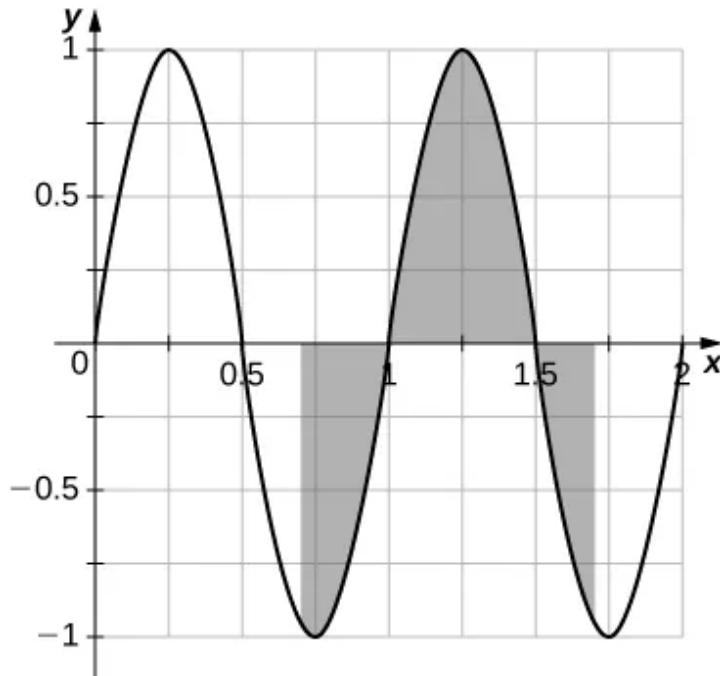
- 135.** Suppose that for each i such that $1 \leq i \leq N$ one has $\int_{i-1}^i f(t) dt = i^2$. Show that $\int_0^N f(t) dt = \frac{N(N+1)(2N+1)}{6}$.

- 136.** [T] Compute the left and right Riemann sums L_{10} and R_{10} and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t) = t^2$ over $[0, 1]$. Given that $\int_0^1 t^2 dt = 0.\bar{33}$, to how many decimal places is $\frac{L_{10}+R_{10}}{2}$ accurate?

- 137.** [T] Compute the left and right Riemann sums, L_{10} and R_{10} , and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t) = (4 - t^2)$ over $[1, 2]$. Given that $\int_1^2 (4 - t^2) dt = 1.\bar{66}$, to how many decimal places

is $\frac{L_{10}+R_{10}}{2}$ accurate?

- 138.** If $\int_1^5 \sqrt{1+t^4} dt = 41.7133\dots$, what is $\int_1^5 \sqrt{1+u^4} du$?
- 139.** Estimate $\int_0^1 t dt$ using the left and right endpoint sums, each with a single rectangle. How does the average of these left and right endpoint sums compare with the actual value $\int_0^1 t dt$?
- 140.** Estimate $\int_0^1 t dt$ by comparison with the area of a single rectangle with height equal to the value of t at the midpoint $t = \frac{1}{2}$. How does this midpoint estimate compare with the actual value $\int_0^1 t dt$?
- 141.** From the graph of $\sin(2\pi x)$ shown:
- Explain why $\int_0^1 \sin(2\pi t) dt = 0$.
 - Explain why, in general, $\int_a^{a+1} \sin(2\pi t) dt = 0$ for any value of a .



- 142.** If f is 1-periodic ($f(t+1) = f(t)$), odd, and integrable over $[0, 1]$, is it always true that $\int_0^1 f(t) dt = 0$?

- 143.** If f is 1-periodic and $\int_0^1 f(t) dt = A$, is it necessarily true that $\int_a^{1+a} f(t) dt = A$ for all A ?

1.3 The Fundamental Theorem of Calculus

Learning Objectives

- 1.3.1 Describe the meaning of the Mean Value Theorem for Integrals.
- 1.3.2 State the meaning of the Fundamental Theorem of Calculus, Part 1.
- 1.3.3 Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- 1.3.4 State the meaning of the Fundamental Theorem of Calculus, Part 2.
- 1.3.5 Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
- 1.3.6 Explain the relationship between differentiation and integration.

In the previous two sections, we looked at the definite integral and its relationship to the area under the curve of a function. Unfortunately, so far, the only tools we have available to calculate the value of a definite integral are geometric area formulas and limits of Riemann sums, and both approaches are extremely cumbersome. In this section we look at some more powerful and useful techniques for evaluating definite integrals.

These new techniques rely on the relationship between differentiation and integration. This relationship was discovered and explored by both Sir Isaac Newton and Gottfried Wilhelm Leibniz (among others) during the late 1600s and early 1700s, and it is codified in what we now call the **Fundamental Theorem of Calculus**, which has two parts that we examine in this section. Its very name indicates how central this theorem is to the entire development of calculus.

MEDIA

Isaac Newton's contributions to mathematics and physics changed the way we look at the world. The relationships he discovered, codified as Newton's laws and the law of universal gravitation, are still taught as foundational material in physics today, and his calculus has spawned entire fields of mathematics. To learn more, read a [brief biography](#) of Newton with multimedia clips.

Before we get to this crucial theorem, however, let's examine another important theorem, the Mean Value Theorem for Integrals, which is needed to prove the Fundamental Theorem of Calculus.

The Mean Value Theorem for Integrals

The **Mean Value Theorem for Integrals** states that a continuous function on a closed interval takes on its average value at some point in that interval. The theorem guarantees that if $f(x)$ is continuous, a point c exists in an interval $[a, b]$ such that the value of the function at c is equal to the average value of $f(x)$ over $[a, b]$. We state this theorem mathematically with the help of the formula for the average value of a function that we presented at the end of the preceding section.

THEOREM 1.3

The Mean Value Theorem for Integrals

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx. \quad (1.15)$$

This formula can also be stated as

$$\int_a^b f(x) dx = f(c) (b-a).$$

Proof

Since $f(x)$ is continuous on $[a, b]$, by the extreme value theorem (see [Maxima and Minima](#)), it assumes minimum and maximum values— m and M , respectively—on $[a, b]$. Then, for all x in $[a, b]$, we have $m \leq f(x) \leq M$. Therefore, by the comparison theorem (see [The Definite Integral](#)), we have

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a).$$

Dividing by $b-a$ gives us

$$m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M.$$

Since $\frac{1}{b-a} \int_a^b f(x) dx$ is a number between m and M , and since $f(x)$ is continuous and assumes the values m and M over $[a, b]$, by the Intermediate Value Theorem (see [Continuity](#)), there is a number c over $[a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx,$$

and the proof is complete.

□

EXAMPLE 1.15**Finding the Average Value of a Function**

Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0, 4]$ and find c such that $f(c)$ equals the average value of the function over $[0, 4]$.

[Show/Hide Solution]**CHECKPOINT 1.14**

Find the average value of the function $f(x) = \frac{x}{2}$ over the interval $[0, 6]$ and find c such that $f(c)$ equals the average value of the function over $[0, 6]$.

EXAMPLE 1.16**Finding the Point Where a Function Takes on Its Average Value**

Given $\int_0^3 x^2 dx = 9$, find c such that $f(c)$ equals the average value of $f(x) = x^2$ over $[0, 3]$.

[Show/Hide Solution]**CHECKPOINT 1.15**

Given $\int_0^3 (2x^2 - 1) dx = 15$, find c such that $f(c)$ equals the average value of $f(x) = 2x^2 - 1$ over $[0, 3]$.

Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

As mentioned earlier, the Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration, and gives us a way to evaluate definite integrals without using Riemann sums or calculating areas. The theorem is comprised of two parts, the first of which, the **Fundamental Theorem of Calculus, Part 1**, is stated here. Part 1 establishes the relationship between differentiation and integration.

THEOREM 1.4

Fundamental Theorem of Calculus, Part 1

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t) dt, \quad (1.16)$$

then $F'(x) = f(x)$ over (a, b) .

Before we delve into the proof, a couple of subtleties are worth mentioning here. First, a comment on the notation. Note that we have defined a function, $F(x)$, as the definite integral of another function, $f(t)$, from the point a to the point x . At first glance, this is confusing, because we have said several times that a definite integral is a number, and here it looks like it's a function. The key here is to notice that for any particular value of x , the definite integral is a number. So the function $F(x)$ returns a number (the value of the definite integral) for each value of x .

Second, it is worth commenting on some of the key implications of this theorem. There is a reason it is called the *Fundamental* Theorem of Calculus. Not only does it establish a relationship between integration and differentiation, but also it guarantees that any integrable function has an antiderivative.

Proof

Applying the definition of the derivative, we have

$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t) dt + \int_x^a f(t) dt \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt. \end{aligned}$$

Looking carefully at this last expression, we see $\frac{1}{h} \int_x^{x+h} f(t) dt$ is just the average value of the function $f(x)$ over the interval $[x, x + h]$. Therefore, by [The Mean Value Theorem for Integrals](#), there is some number c in $[x, x + h]$ such that

$$\frac{1}{h} \int_x^{x+h} f(x) dx = f(c).$$

In addition, since c is between x and $x + h$, c approaches x as h approaches zero. Also, since $f(x)$ is continuous, we have $\lim_{h \rightarrow 0} f(c) = \lim_{c \rightarrow x} f(c) = f(x)$. Putting all these pieces together, we have

$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(x) dx \\ &= \lim_{h \rightarrow 0} f(c) \\ &= f(x), \end{aligned}$$

and the proof is complete.

□

EXAMPLE 1.17

Finding a Derivative with the Fundamental Theorem of Calculus

Use the [Fundamental Theorem of Calculus, Part 1](#) to find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3 + 1} dt.$$

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.16

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of

$$g(r) = \int_0^r \sqrt{x^2 + 4} dx.$$

EXAMPLE 1.18

Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives

Let $F(x) = \int_1^{\sqrt{x}} \sin t dt$. Find $F'(x)$.

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.17

Let $F(x) = \int_1^{x^3} \cos t dt$. Find $F'(x)$.

EXAMPLE 1.19

Using the Fundamental Theorem of Calculus with Two Variable Limits of Integration

Let $F(x) = \int_x^{2x} t^3 dt$. Find $F'(x)$.

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.18

Let $F(x) = \int_x^{x^2} \cos t dt$. Find $F'(x)$.

Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The Fundamental Theorem of Calculus, Part 2, is perhaps the most important theorem in calculus. After tireless efforts by mathematicians for approximately 500 years, new techniques emerged that provided scientists with the necessary tools to explain many phenomena. Using calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate the bending strength of materials or the three-dimensional motion of objects. Our view of the world was forever changed with calculus.

After finding approximate areas by adding the areas of n rectangles, the application of this theorem is straightforward by comparison. It almost seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

THEOREM 1.5

The Fundamental Theorem of Calculus, Part 2

If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x) dx = F(b) - F(a). \quad (1.17)$$

We often see the notation $F(x)|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated limits a and b to indicate that we should evaluate the function $F(x)$ at the upper limit (in this case, b), and subtract the value of the function $F(x)$ evaluated at the lower limit (in this case, a).

The **Fundamental Theorem of Calculus, Part 2** (also known as the **evaluation theorem**) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Proof

Let $P = \{x_i\}, i = 0, 1, \dots, n$ be a regular partition of $[a, b]$. Then, we can write

$$\begin{aligned}
 F(b) - F(a) &= F(x_n) - F(x_0) \\
 &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \dots + [F(x_1) - F(x_0)] \\
 &= \sum_{i=1}^n [F(x_i) - F(x_{i-1})].
 \end{aligned}$$

Now, we know F is an antiderivative of f over $[a, b]$, so by the Mean Value Theorem (see [The Mean Value Theorem](#)) for $i = 0, 1, \dots, n$ we can find c_i in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x.$$

Then, substituting into the previous equation, we have

$$F(b) - F(a) = \sum_{i=1}^n f(c_i)\Delta x.$$

Taking the limit of both sides as $n \rightarrow \infty$, we obtain

$$\begin{aligned}
 F(b) - F(a) &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i)\Delta x \\
 &= \int_a^b f(x) dx.
 \end{aligned}$$

□

EXAMPLE 1.20

Evaluating an Integral with the Fundamental Theorem of Calculus

Use [The Fundamental Theorem of Calculus, Part 2](#) to evaluate

$$\int_{-2}^2 (t^2 - 4) dt.$$

[\[Show/Hide Solution\]](#)

Analysis

Notice that we did not include the “+ C” term when we wrote the antiderivative. The reason is that, according to the Fundamental Theorem of Calculus, Part 2, *any* antiderivative works. So,

for convenience, we chose the antiderivative with $C = 0$. If we had chosen another antiderivative, the constant term would have canceled out. This always happens when evaluating a definite integral.

The region of the area we just calculated is depicted in [Figure 1.28](#). Note that the region between the curve and the x -axis is all below the x -axis. Area is always positive, but a definite integral can still produce a negative number (a net signed area). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.

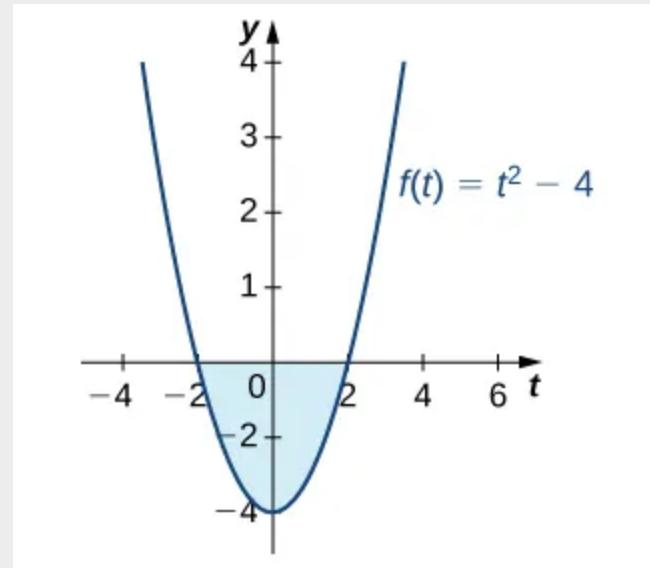


Figure 1.28 The evaluation of a definite integral can produce a negative value, even though area is always positive.

EXAMPLE 1.21

Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following integral using the Fundamental Theorem of Calculus, Part 2:

$$\int_1^9 \frac{x-1}{\sqrt{x}} dx.$$

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.19

Use [The Fundamental Theorem of Calculus, Part 2](#) to evaluate $\int_1^2 x^{-4} dx$.

EXAMPLE 1.22

A Roller-Skating Race

James and Kathy are racing on roller skates. They race along a long, straight track, and whoever has gone the farthest after 5 sec wins a prize. If James can skate at a velocity of $f(t) = 5 + 2t$ ft/sec and Kathy can skate at a velocity of $g(t) = 10 + \cos(\frac{\pi}{2}t)$ ft/sec, who is going to win the race?

[\[Show/Hide Solution\]](#)

CHECKPOINT 1.20

Suppose James and Kathy have a rematch, but this time the official stops the contest after only 3 sec. Does this change the outcome?

STUDENT PROJECT

A Parachutist in Free Fall



Figure 1.30 Skydivers can adjust the velocity of their dive by changing the position of their body during the free fall. (credit: Jeremy T. Lock)

Julie is an avid skydiver. She has more than 300 jumps under her belt and has mastered the art of making adjustments to her body position in the air to control how fast she falls. If she arches her back and points her belly toward the ground, she reaches a terminal velocity of approximately 120 mph (176 ft/sec). If, instead, she orients her body with her head straight down, she falls faster, reaching a terminal velocity of 150 mph (220 ft/sec).

Since Julie will be moving (falling) in a downward direction, we assume the downward direction is positive to simplify our calculations. Julie executes her jumps from an altitude of 12,500 ft. After she exits the aircraft, she immediately starts falling at a velocity given by $v(t) = 32t$. She continues to accelerate according to this velocity function until she reaches terminal velocity. After she reaches terminal velocity, her speed remains constant until she pulls her ripcord and slows down to land.

On her first jump of the day, Julie orients herself in the slower “belly down” position (terminal velocity is 176 ft/sec). Using this information, answer the following questions.

1. How long after she exits the aircraft does Julie reach terminal velocity?
2. Based on your answer to question 1, set up an expression involving one or more integrals that represents the distance Julie falls after 30 sec.
3. If Julie pulls her ripcord at an altitude of 3000 ft, how long does she spend in a free fall?

4. Julie pulls her ripcord at 3000 ft. It takes 5 sec for her parachute to open completely and for her to slow down, during which time she falls another 400 ft. After her canopy is fully open, her speed is reduced to 16 ft/sec. Find the total time Julie spends in the air, from the time she leaves the airplane until the time her feet touch the ground.

On Julie's second jump of the day, she decides she wants to fall a little faster and orients herself in the "head down" position. Her terminal velocity in this position is 220 ft/sec.

Answer these questions based on this velocity:

5. How long does it take Julie to reach terminal velocity in this case?

6. Before pulling her ripcord, Julie reorients her body in the "belly down" position so she is not moving quite as fast when her parachute opens. If she begins this maneuver at an altitude of 4000 ft, how long does she spend in a free fall before beginning the reorientation?

Some jumpers wear "wingsuits" (see [Figure 1.31](#)). These suits have fabric panels between the arms and legs and allow the wearer to glide around in a free fall, much like a flying squirrel. (Indeed, the suits are sometimes called "flying squirrel suits.") When wearing these suits, terminal velocity can be reduced to about 30 mph (44 ft/sec), allowing the wearers a much longer time in the air. Wingsuit flyers still use parachutes to land; although the vertical velocities are within the margin of safety, horizontal velocities can exceed 70 mph, much too fast to land safely.



Figure 1.31 The fabric panels on the arms and legs of a wingsuit work to reduce the vertical velocity of a skydiver's fall. (credit: Richard Schneider)

Answer the following question based on the velocity in a wingsuit.

7. If Julie dons a wingsuit before her third jump of the day, and she pulls her ripcord at an altitude of 3000 ft, how long does she get to spend gliding around in the air?

Section 1.3 Exercises

- 144.** Consider two athletes running at variable speeds $v_1(t)$ and $v_2(t)$. The runners start and finish a race at exactly the same time. Explain why the two runners must be going the same speed at some point.
- 145.** Two mountain climbers start their climb at base camp, taking two different routes, one steeper than the other, and arrive at the peak at exactly the same time. Is it necessarily true that, at some point, both climbers increased in altitude at the same rate?
- 146.** To get on a certain toll road a driver has to take a card that lists the mile entrance point. The card also has a timestamp. When going to pay the toll at the exit, the driver is surprised to receive a speeding ticket along with the toll. Explain how this can happen.
- 147.** Set $F(x) = \int_1^x (1-t) dt$. Find $F'(2)$ and the average value of F' over $[1, 2]$.

In the following exercises, use the Fundamental Theorem of Calculus, Part 1, to find each derivative.

148. $\frac{d}{dx} \int_1^x e^{-t^2} dt$

149. $\frac{d}{dx} \int_1^x e^{\cos t} dt$

150. $\frac{d}{dx} \int_3^x \sqrt{9-y^2} dy$

151. $\frac{d}{dx} \int_3^x \frac{ds}{\sqrt{16-s^2}}$

152. $\frac{d}{dx} \int_x^{2x} t dt$

153. $\frac{d}{dx} \int_0^{\sqrt{x}} t dt$

154. $\frac{d}{dx} \int_0^{\sin x} \sqrt{1-t^2} dt$

155. $\frac{d}{dx} \int_{\cos x}^1 \sqrt{1-t^2} dt$

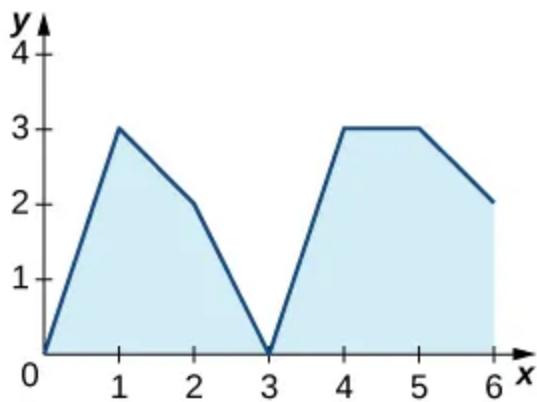
156. $\frac{d}{dx} \int_1^{\sqrt{x}} \frac{t^2}{1+t^4} dt$

157. $\frac{d}{dx} \int_1^{x^2} \frac{\sqrt{t}}{1+t} dt$

158. $\frac{d}{dx} \int_0^{\ln x} e^t dt$

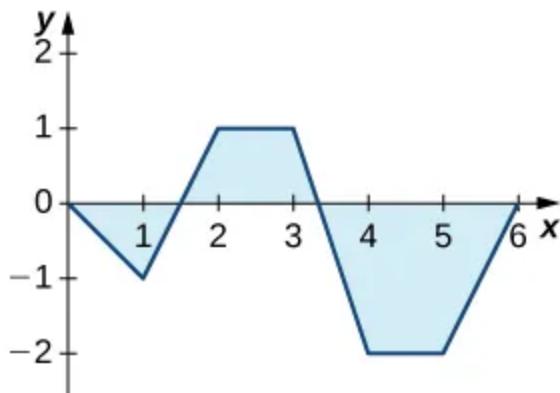
159. $\frac{d}{dx} \int_1^{e^x} \ln u^2 du$

- 160.** The graph of $y = \int_0^x f(t)dt$, where f is a piecewise constant function, is shown here.



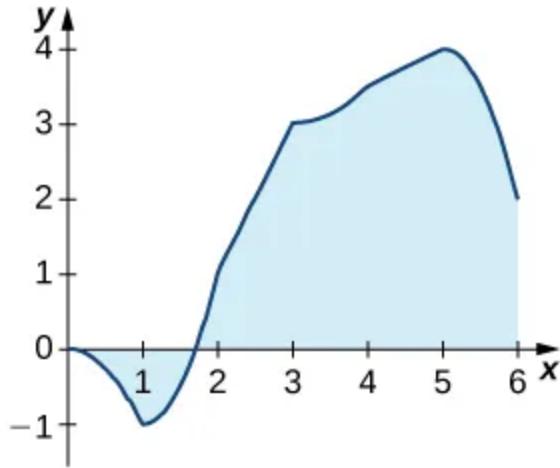
- Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?
- What are the maximum and minimum values of f ?
- What is the average value of f ?

- 161.** The graph of $y = \int_0^x f(t)dt$, where f is a piecewise constant function, is shown here.



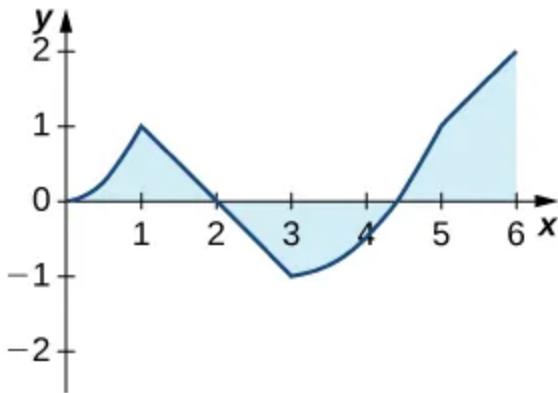
- a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?
- b. What are the maximum and minimum values of f ?
- c. What is the average value of f ?

- 162.** The graph of $y = \int_0^x \ell(t)dt$, where ℓ is a piecewise linear function, is shown here.



- a. Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?
- b. Over which intervals is ℓ increasing? Over which is it decreasing? Over which, if any, is it constant?
- c. What is the average value of ℓ ?

- 163.** The graph of $y = \int_0^x \ell(t)dt$, where ℓ is a piecewise linear function, is shown here.



- Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?
- Over which intervals is ℓ increasing? Over which is it decreasing? Over which intervals, if any, is it constant?
- What is the average value of ℓ ?

In the following exercises, use a calculator to estimate the area under the curve by computing T_{10} , the average of the left- and right-endpoint Riemann sums using $N = 10$ rectangles. Then, using the Fundamental Theorem of Calculus, Part 2, determine the exact area.

164. $[T] y = x^2$ over $[0, 4]$

165. $[T] y = x^3 + 6x^2 + x - 5$ over $[-4, 2]$

166. $[T] y = \sqrt{x^3}$ over $[0, 6]$

167. $[T] y = \sqrt{x} + x^2$ over $[1, 9]$

168. $[T] \int (\cos x - \sin x)dx$ over $[0, \pi]$

169. $[T] \int \frac{4}{x^2} dx$ over $[1, 4]$

In the following exercises, evaluate each definite integral using the Fundamental Theorem of Calculus, Part 2.

170. $\int_{-1}^2 (x^2 - 3x) dx$

171. $\int_{-2}^3 (x^2 + 3x - 5) dx$

172. $\int_{-2}^3 (t+2)(t-3) dt$

173. $\int_2^3 (t^2 - 9)(4 - t^2) dt$

174. $\int_1^2 x^9 dx$

175. $\int_0^1 x^{99} dx$

176. $\int_4^8 (4t^{5/2} - 3t^{3/2}) dt$

177. $\int_{1/4}^4 \left(x^2 - \frac{1}{x^2}\right) dx$

178. $\int_1^2 \frac{2}{x^3} dx$

179. $\int_1^4 \frac{1}{2\sqrt{x}} dx$

180. $\int_1^4 \frac{2 - \sqrt{t}}{t^2} dt$

181. $\int_1^{16} \frac{dt}{t^{1/4}}$

182. $\int_0^{2\pi} \cos \theta d\theta$

183. $\int_0^{\pi/2} \sin \theta d\theta$

184. $\int_0^{\pi/4} \sec^2 \theta d\theta$

185. $\int_0^{\pi/4} \sec \theta \tan \theta d\theta$

186. $\int_{\pi/3}^{\pi/4} \csc \theta \cot \theta d\theta$

187. $\int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta$

188. $\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3} \right) dt$

189. $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3} \right) dt$

In the following exercises, use the evaluation theorem to express the integral as a function $F(x)$.

190. $\int_a^x t^2 dt$

191. $\int_1^x e^t dt$

192. $\int_0^x \cos t dt$

193. $\int_{-x}^x \sin t dt$

In the following exercises, identify the roots of the integrand to remove absolute values, then evaluate using the Fundamental Theorem of Calculus, Part 2.

194. $\int_{-2}^3 |x| dx$

195. $\int_{-2}^4 |t^2 - 2t - 3| dt$

196. $\int_0^\pi |\cos t| dt$

197. $\int_{-\pi/2}^{\pi/2} |\sin t| dt$

- 198.** Suppose that the number of hours of daylight on a given day in Seattle is modeled by the function $-3.75 \cos\left(\frac{\pi t}{6}\right) + 12.25$, with t given in months and $t = 0$ corresponding to the winter solstice.

a. What is the average number of daylight hours in a year?

b. At which times t_1 and t_2 , where $0 \leq t_1 < t_2 < 12$, do the number of daylight hours equal the average number?

- c. Write an integral that expresses the total number of daylight hours in Seattle between t_1 and t_2 .
- d. Compute the mean hours of daylight in Seattle between t_1 and t_2 , where $0 \leq t_1 < t_2 < 12$, and then between t_2 and t_1 , and show that the average of the two is equal to the average day length.

199. Suppose the rate of gasoline consumption over the course of a year in the United States can be modeled by a sinusoidal function of the form $(11.21 - \cos(\frac{\pi t}{6})) \times 10^9$ gal/mo.

- a. What is the average monthly consumption, and for which values of t is the rate at time t equal to the average rate?
- b. What is the number of gallons of gasoline consumed in the United States in a year?
- c. Write an integral that expresses the average monthly U.S. gas consumption during the part of the year between the beginning of April ($t = 3$) and the end of September ($t = 9$).

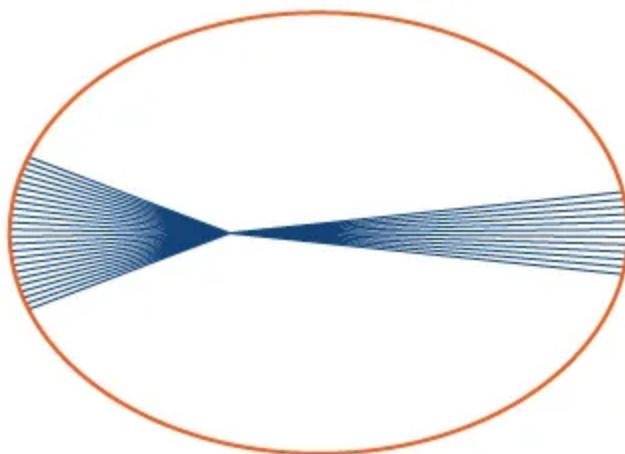
200. Explain why, if f is continuous over $[a, b]$, there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t) dt.$$

201. Explain why, if f is continuous over $[a, b]$ and is not equal to a constant, there is at least one point $M \in [a, b]$ such that $f(M) > \frac{1}{b-a} \int_a^b f(t) dt$ and at least one point $m \in [a, b]$ such that $f(m) < \frac{1}{b-a} \int_a^b f(t) dt$.

202. Kepler's first law states that the planets move in elliptical orbits with the Sun at one focus. The closest point of a planetary orbit to the Sun is called the *perihelion* (for Earth, it currently occurs around January 3) and the farthest point is called the *aphelion* (for Earth, it currently occurs around July 4). Kepler's second law states that planets sweep out equal areas of their elliptical orbits in equal times. Thus, the two arcs indicated in the following figure are swept

out in equal times. At what time of year is Earth moving fastest in its orbit? When is it moving



slowest?

- 203.** A point on an ellipse with major axis length $2a$ and minor axis length $2b$ has the coordinates $(a \cos \theta, b \sin \theta)$, $0 \leq \theta \leq 2\pi$.

- Show that the distance from this point to the focus at $(-c, 0)$ is $d(\theta) = a + c \cos \theta$, where $c = \sqrt{a^2 - b^2}$.
- Use these coordinates to show that the average distance \bar{d} from a point on the ellipse to the focus at $(-c, 0)$, with respect to angle θ , is a .

- 204.** As implied earlier, according to Kepler's laws, Earth's orbit is an ellipse with the Sun at one focus. The perihelion for Earth's orbit around the Sun is 147,098,290 km and the aphelion is 152,098,232 km.

- By placing the major axis along the x -axis, find the average distance from Earth to the Sun.
- The classic definition of an astronomical unit (AU) is the distance from Earth to the Sun, and its value was computed as the average of the perihelion and aphelion distances. Is this definition justified?

- 205.** The force of gravitational attraction between the Sun and a planet is $F(\theta) = \frac{GmM}{r^2(\theta)}$, where m is the mass of the planet, M is the mass of the Sun, G is a universal constant, and $r(\theta)$ is the distance between the Sun and the planet when the planet is at an angle θ with the major axis of its orbit. Assuming that M , m , and the ellipse parameters a and b (half-lengths of the major and minor axes) are given, set up—but do not evaluate—an integral that expresses in terms of G , m , M , a , b the average gravitational force between the Sun and the planet.

- 206.** The displacement from rest of a mass attached to a spring satisfies the simple harmonic motion equation $x(t) = A \cos(\omega t - \phi)$, where ϕ is a phase constant, ω is the angular frequency, and A is the amplitude. Find the average velocity, the average speed (magnitude of velocity), the average displacement, and the average distance from rest (magnitude of displacement) of the mass.