# 2.1 Areas between Curves

## Learning Objectives

2.1.1 Determine the area of a region between two curves by integrating with respect to the independent variable.
2.1.2 Find the area of a compound region.
2.1.3 Determine the area of a region between two curves by integrating with respect to the dependent variable.

In Introduction to Integration, we developed the concept of the definite integral to calculate the area below a curve on a given interval. In this section, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of $x$, beginning with the simple case in which one function value is always greater than the other. We then look at cases when the graphs of the functions cross. Last, we consider how to calculate the area between two curves that are functions of $y$.

## Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in the following figure.

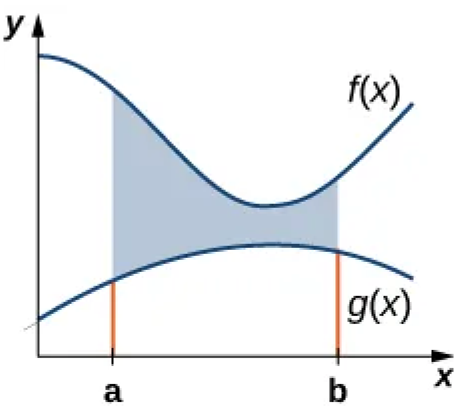



**Figure 2.2** The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$.

As we did before, we are going to partition the interval on the $x\text{-axis}$ and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2,..., n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2,..., n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. Figure 2.3(a) shows the rectangles when $x_i^*$ is selected to be the left endpoint of the interval and $n = 10$. Figure 2.3(b) shows a representative rectangle in detail.

**MEDIA**

Use this calculator to learn more about the areas between two curves.

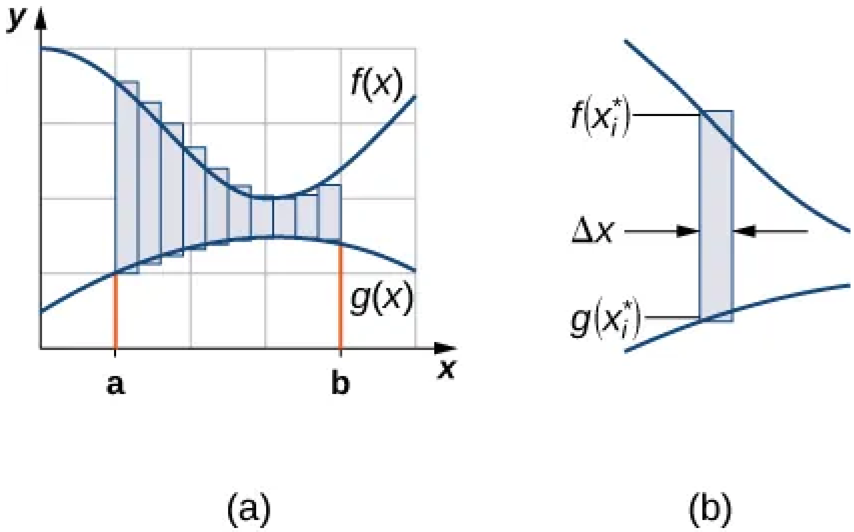



**Figure 2.3** (a)We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is $\Delta x$. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x.$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x = \int_a^b [f(x) - g(x)]\,dx.$$

These findings are summarized in the following theorem.

**THEOREM 2.1**

### Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let $R$ denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $R$ is given by

$$A = \int_a^b [f(x) - g(x)]\, dx. \tag{2.1}$$

We apply this theorem in the following example.

### EXAMPLE 2.1

#### Finding the Area of a Region between Two Curves 1

If $R$ is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region $R$.

[Show/Hide Solution]

### CHECKPOINT 2.1

If $R$ is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region $R$.

In Example 2.1, we defined the interval of interest as part of the problem statement. Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

### EXAMPLE 2.2

#### Finding the Area of a Region between Two Curves 2

If $R$ is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region $R$.

[Show/Hide Solution]

**CHECKPOINT 2.2**

If *R* is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4$, find the area of region $R$.

## Areas of Compound Regions

So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

**THEOREM 2.2**

### Finding the Area of a Region between Curves That Cross

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ . Let $R$ denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $R$ is given by

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

In practice, applying this theorem requires us to break up the interval $[a, b]$ and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example.

**EXAMPLE 2.3**

### Finding the Area of a Region Bounded by Functions That Cross

If *R* is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$ , find the area of region $R$.

**[Show/Hide Solution]**

## CHECKPOINT 2.3

If *R* is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[\pi/2, 2\pi]$ , find the area of region $R$.

## EXAMPLE 2.4

### Finding the Area of a Complex Region

Consider the region depicted in Figure 2.7. Find the area of $R$.

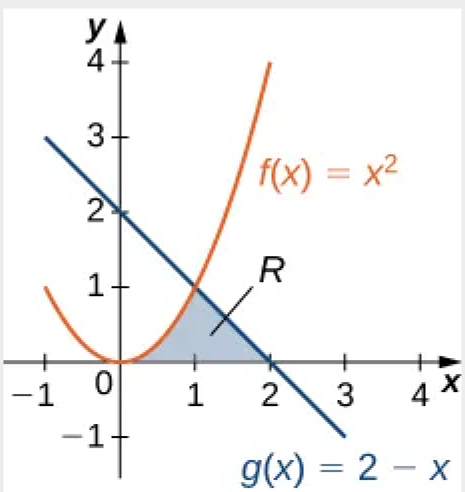
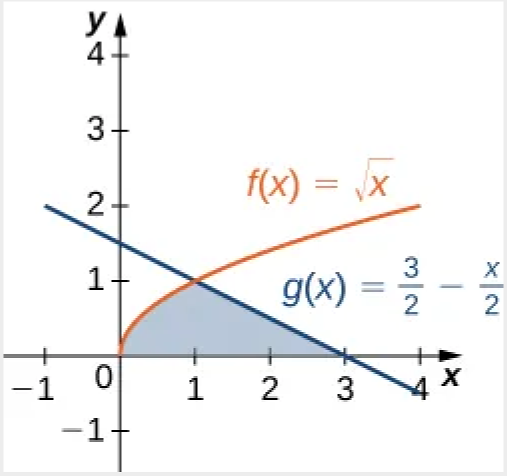



**Figure 2.7** Two integrals are required to calculate the area of this region.

[Show/Hide Solution]

## CHECKPOINT 2.4

Consider the region depicted in the following figure. Find the area of $R$.




## Regions Defined with Respect to *y*

In Example 2.4, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of $y$, instead of as functions of $x$? Review Figure 2.7. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for $x$ and represent the curve by the function $x = v(y) = \sqrt{y}$. (Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a function of $y$. However, based on the graph, it is clear we are interested in the positive square root.) Similarly, the right graph is represented by the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of $y$, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to $y$, we need to evaluate one integral only. Let's develop a formula for this type of integration.

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in the following figure.

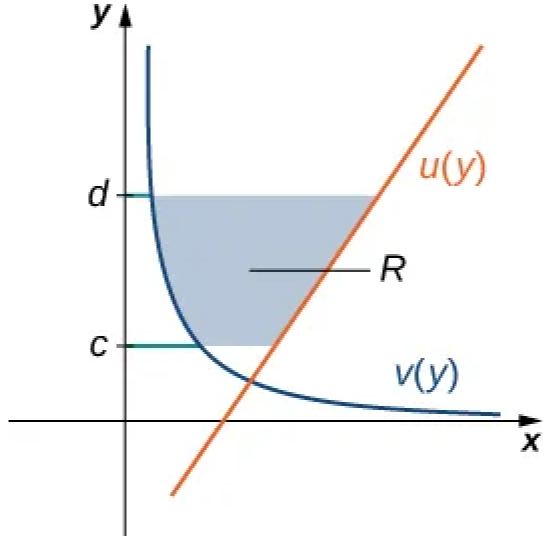



**Figure 2.8** We can find the area between the graphs of two functions, $u(y)$ and $v(y)$.

This time, we are going to partition the interval on the $y\text{-axis}$ and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \ldots, n$, let $Q = \{y_i\}$ be a regular partition of $[c, d]$ . Then, for $i = 1, 2, \ldots, n$, choose a point $y_i^* \in [y_{i-1}, y_i]$ , then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v\left(y_i^*\right)$ to $u\left(y_i^*\right)$ . Figure 2.9(a) shows the rectangles when $y_i^*$ is selected to be the lower endpoint of the interval and $n = 10$. Figure 2.9(b) shows a representative rectangle in detail.

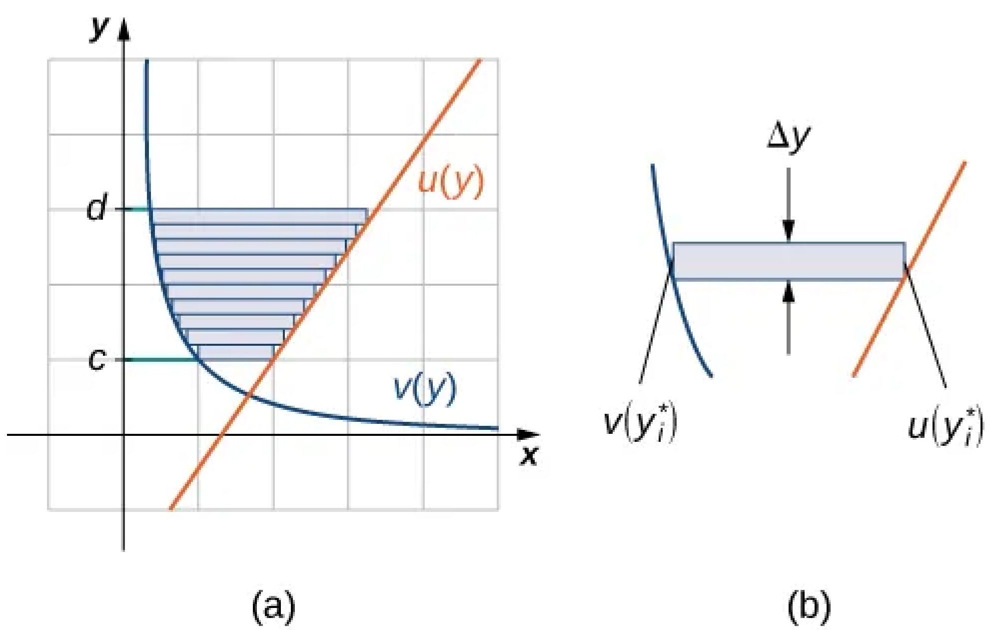



**Figure 2.9** (a) Approximating the area between the graphs of two functions, $u(y)$ and $v(y)$, with rectangles. (b) The area of a typical rectangle.

The height of each individual rectangle is $\Delta y$ and the width of each rectangle is $u\left(y_i^*\right) - v\left(y_i^*\right)$ . Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} \left[ u\left(y_i^*\right) - v\left(y_i^*\right) \right] \Delta y.$$

This is a Riemann sum, so we take the limit as $n \to \infty$, obtaining

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ u\left(y_i^*\right) - v\left(y_i^*\right) \right] \Delta y = \int_c^d \left[ u(y) - v(y) \right] dy.$$

These findings are summarized in the following theorem.

### THEOREM 2.3

#### Finding the Area between Two Curves, Integrating along the *y*-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$ . Let $R$ denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of $R$ is given by

$$A = \int_c^d \left[ u(y) - v(y) \right] dy. \tag{2.2}$$

### EXAMPLE 2.5

#### Integrating with Respect to *y*

Let's revisit Example 2.4, only this time let's integrate with respect to $y$. Let $R$ be the region depicted in Figure 2.10. Find the area of $R$ by integrating with respect to $y$.

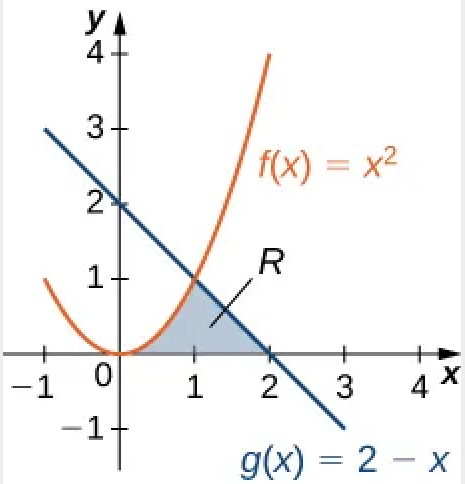
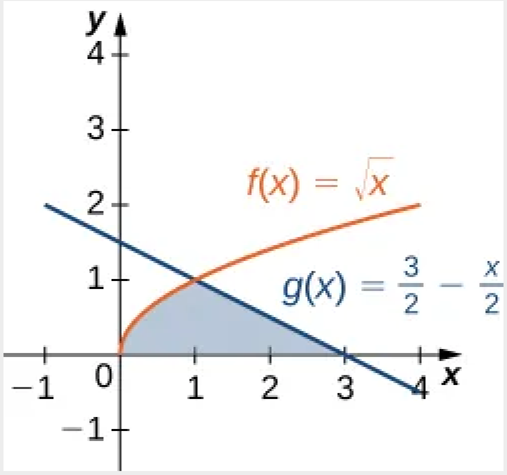



**Figure 2.10** The area of region $R$ can be calculated using one integral only when the curves are treated as functions of $y$.

[Show/Hide Solution]

### CHECKPOINT 2.5

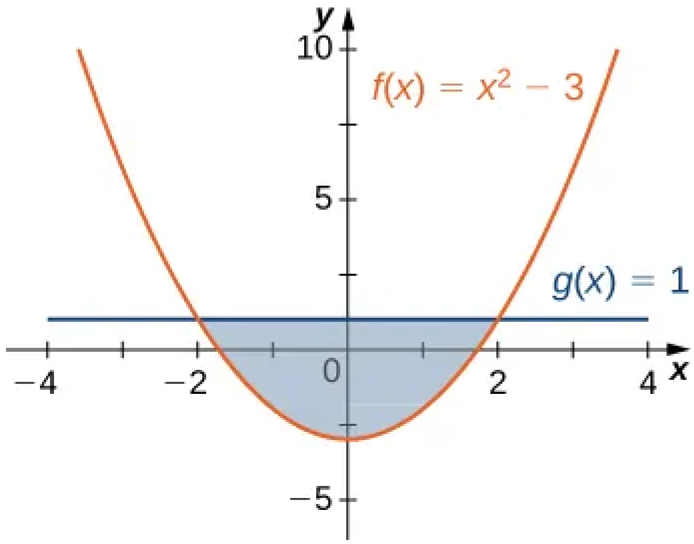
Let's revisit the checkpoint associated with Example 2.4, only this time, let's integrate with respect to $y$. Let $R$ be the region depicted in the following figure. Find the area of $R$ by integrating with respect to $y$.




## Section 2.1 Exercises

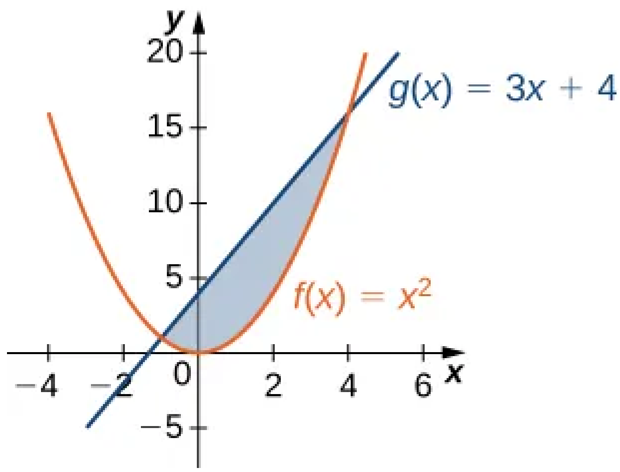
For the following exercises, determine the area of the region between the two curves in the given figure by integrating over the $x$-axis.

**1.**




$y = x^2 - 3 \text{ and } y = 1$

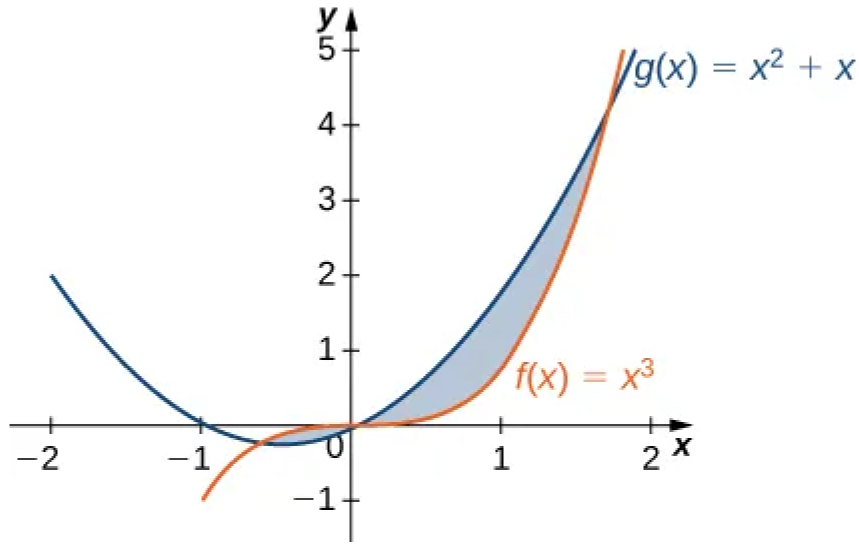
**2.**




$y = x^2 \text{ and } y = 3x + 4$

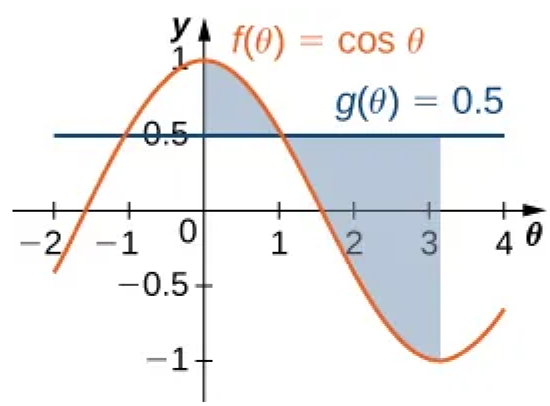
For the following exercises, split the region between the two curves into two smaller regions, then determine the area by integrating over the $x\text{-axis}$. Note that you will have two integrals to solve.

**3.**




$y = x^3$ and $y = x^2 + x$

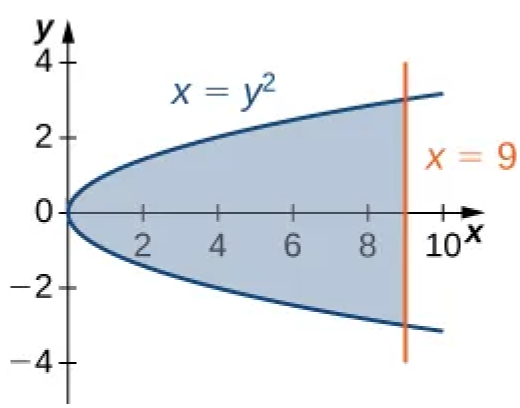
**4.**




$y = \cos\theta$ and $y = 0.5$, for $0 \leq \theta \leq \pi$

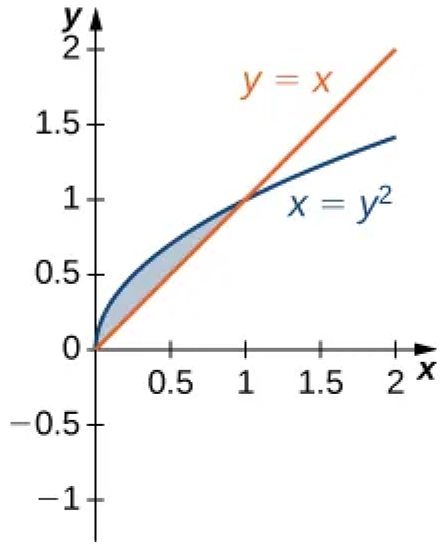
For the following exercises, determine the area of the region between the two curves by integrating over the $y$-axis.

**5.**




$x = y^2$ and $x = 9$

**6.**




$y = x$ and $x = y^2$

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the $x$-axis.

**7.** $y = x^2$ and $y = -x^2 + 18x$

**8.** $y = \frac{1}{x}, y = \frac{1}{x^2}$, and $x = 3$

**9.** $y = \cos x$ and $y = \cos^2 x$ on $x = [-\pi, \pi]$

**10.** $y = e^x, y = e^{2x-1}$, and $x = 0$

**11.** $y = e^x, y = e^{-x}, x = -1$ and $x = 1$

**12.** $y = e, y = e^x$, and $y = e^{-x}$

**13.** $y = |x|$ and $y = x^2$

For the following exercises, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.

**14.** $y = \sin(\pi x), y = 2x$, and $x > 0$

**15.** $y = 12 - x, y = \sqrt{x}$, and $y = 1$

**16.** $y = \sin x$ and $y = \cos x$ over $x = [-\pi, \pi]$

**17.** $y = x^3$ and $y = x^2 - 2x$ over $x = [-1, 1]$

**18.** $y = x^2 + 9$ and $y = 10 + 2x$ over $x = [-1, 3]$

**19.** $y = x^3 + 3x$ and $y = 4x$

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the $y$-axis.

**20.** $x = y^3$ and $x = 3y - 2$

**21.** $x = 2y$ and $x = y^3 - y$

**22.** $x = -3 + y^2$ and $x = y - y^2$

**23.** $y^2 = x$ and $x = y + 2$

**24.** $x = |y|$ and $2x = -y^2 + 2$

**25.** $x = \sin y, x = \cos(2y), y = \pi/2,$ and $y = -\pi/2$

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the *x*-axis or *y*-axis, whichever seems more convenient.

**26.** $x = y^4$ and $x = y^5$

**27.** $y = xe^x, y = e^x, x = 0,$ and $x = 1$

**28.** $y = x^6$ and $y = x^4$

**29.** $x = y^3 + 2y^2 + 1$ and $x = -y^2 + 1$

**30.** $y = |x|$ and $y = x^2 - 1$

**31.** $y = 4 - 3x$ and $y = \frac{1}{x}$

**32.** $y = \sin x, x = -\pi/6, x = \pi/6,$ and $y = \cos^3 x$

**33.** $y = x^2 - 3x + 2$ and $y = x^3 - 2x^2 - x + 2$

**34.** $y = 2\cos^3(3x), y = -1, x = \frac{\pi}{4},$ and $x = -\frac{\pi}{4}$

**35.** $y + y^3 = x$ and $2y = x$

**36.** $y = \sqrt{1 - x^2}$ and $y = x^2 - 1$

**37.** $y = \cos^{-1} x, y = \sin^{-1} x, x = -1,$ and $x = 1$

For the following exercises, find the exact area of the region bounded by the given equations if possible. If you are unable to determine the intersection points analytically, use a calculator to approximate the intersection points with three decimal places and determine the approximate area of the region.

**38.** **[T]** $x = e^y$ and $y = x - 2$

39. **[T]** $y = x^2$ and $y = \sqrt{1 - x^2}$

40. **[T]** $y = 3x^2 + 8x + 9$ and $3y = x + 24$

41. **[T]** $x = \sqrt{4 - y^2}$ and $y^2 = 1 + x^2$

42. **[T]** $x^2 = y^3$ and $x = 3y$

43. **[T]** $y = \sin^3 x + 2, y = \tan x, x = -1.5,$ and $x = 1.5$

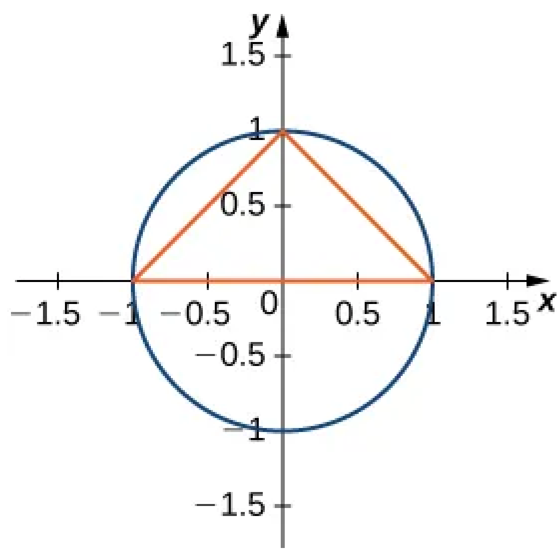
44. **[T]** $y = \sqrt{1 - x^2}$ and $y^2 = x^2$

45. **[T]** $y = \sqrt{1 - x^2}$ and $y = x^2 + 2x + 1$

46. **[T]** $x = 4 - y^2$ and $x = 1 + 3y + y^2$

47. **[T]** $y = \cos x, y = e^x, x = -\pi,$ and $x = 0$

48. The largest triangle with a base on the $x$-axis that fits inside the upper half of the unit circle $y^2 + x^2 = 1$ is given by $y = 1 + x$ and $y = 1 - x$. See the following figure. What is the area




inside the semicircle but outside the triangle?

49. A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where $x$ represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

50. An amusement park has a marginal cost function $C(x) = 1000e^{-x} + 5$, where $x$ represents the number of tickets sold, and a marginal revenue function given by $R(x) = 60 - 0.1x$. Find the total profit generated when selling $550$ tickets. Use a calculator to determine intersection points, if necessary, to two decimal places.

51. The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = 1 - \cos\left((\pi t)/2\right)$ whereas the speed of the tortoise is $T(t) = (1/2)\tan^{-1}(t/4)$, where $t$ is time measured in hours and the speed is measured in miles per hour. Find the area between the curves from time $t = 0$ to the first time after one hour when the tortoise and hare are traveling at the same speed. What does it represent? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

52. The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = (1/2) - (1/2)\cos(2\pi t)$ whereas the speed of the tortoise is $T(t) = \sqrt{t}$, where $t$ is time measured in hours and speed is measured in kilometers per hour. If the race is over in $1$ hour, who won the race and by how much? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

For the following exercises, find the area between the curves by integrating with respect to $x$ and then with respect to $y$. Is one method easier than the other? Do you obtain the same answer?

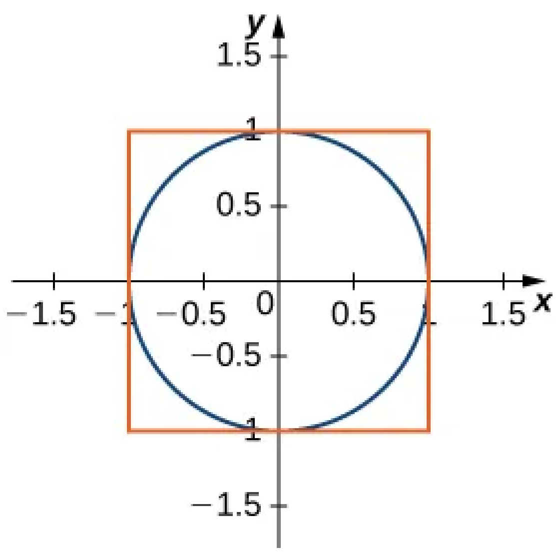
53. $y = x^2 + 2x + 1 \text{ and } y = -x^2 - 3x + 4$

54. $y = x^4 \text{and } x = y^5$

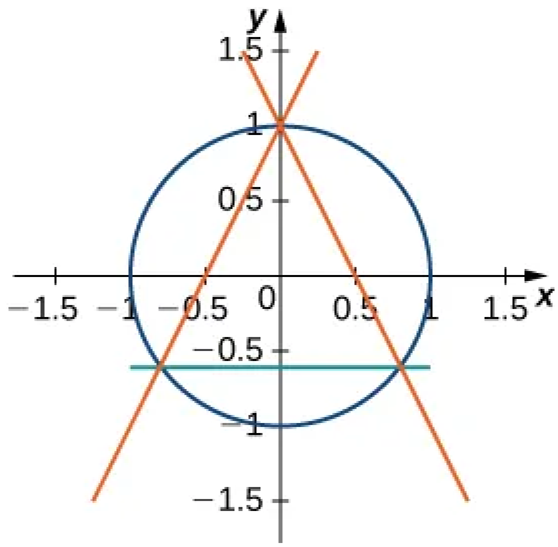
55. $x = y^2 - 2 \text{ and } x = 2y$

For the following exercises, solve using calculus, then check your answer with geometry.

56. Determine the equations for the sides of the square that touches the unit circle on all four sides, as seen in the following figure. Find the area between the perimeter of this square and the unit circle. Is there another way to solve this without using calculus?




57. Find the area between the perimeter of the unit circle and the triangle created from $y = 2x + 1, y = 1 - 2x$ and $y = -\frac{3}{5}$, as seen in the following figure. Is there a way to solve




this without using calculus?

# 2.2 Determining Volumes by Slicing

## Learning Objectives

2.2.1 Determine the volume of a solid by integrating a cross-section (the slicing method).
2.2.2 Find the volume of a solid of revolution using the disk method.
2.2.3 Find the volume of a solid of revolution with a cavity using the washer method.

In the preceding section, we used definite integrals to find the area between two curves. In this section, we use definite integrals to find volumes of three-dimensional solids. We consider three approaches—slicing, disks, and washers—for finding these volumes, depending on the characteristics of the solid.

## Volume and the Slicing Method

Just as area is the numerical measure of a two-dimensional region, volume is the numerical measure of a three-dimensional solid. Most of us have computed volumes of solids by using basic geometric formulas. The volume of a rectangular solid, for example, can be computed by multiplying length, width, and height: $V = lwh$. The formulas for the volume of a sphere $\left(V = \frac{4}{3}\pi r^3\right)$, a cone $\left(V = \frac{1}{3}\pi r^2 h\right)$, and a pyramid $\left(V = \frac{1}{3}Ah\right)$ have also been introduced. Although some of these formulas were derived using geometry alone, all these formulas can be obtained by using integration.

We can also calculate the volume of a cylinder. Although most of us think of a cylinder as having a circular base, such as a soup can or a metal rod, in mathematics the word *cylinder* has a more general meaning. To discuss cylinders in this more general context, we first need to define some vocabulary.

We define the **cross-section** of a solid to be the intersection of a plane with the solid. A *cylinder* is defined as any solid that can be generated by translating a plane region along a line perpendicular to the region, called the *axis* of the cylinder. Thus, all cross-sections perpendicular to the axis of a cylinder are identical. The solid shown in Figure 2.11 is an example of a cylinder with a noncircular base. To calculate the volume of a cylinder, then, we simply multiply the area of the cross-section by the height of the cylinder: $V = A \cdot h$. In the case of a right circular cylinder (soup can), this becomes $V = \pi r^2 h$.

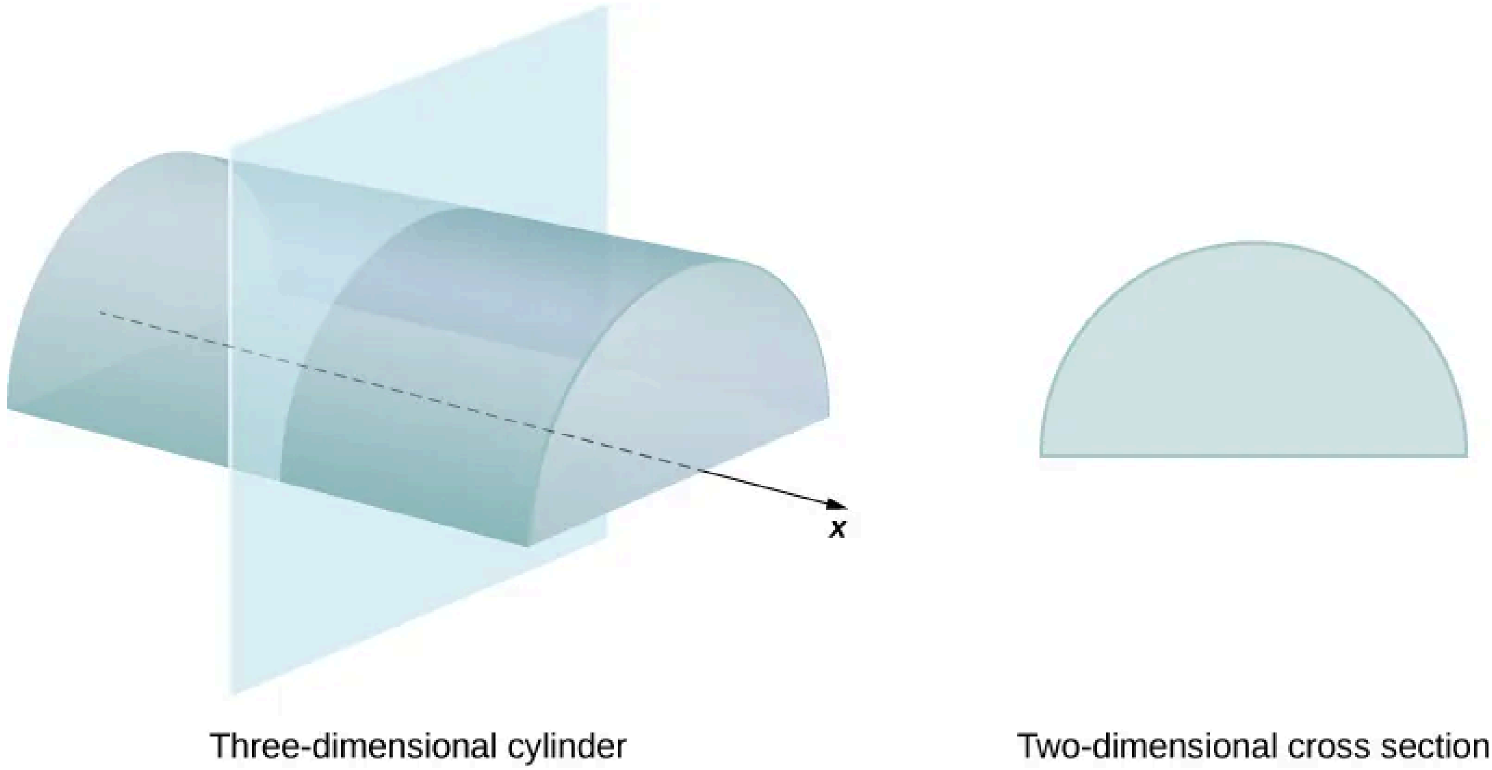



**Figure 2.11** Each cross-section of a particular cylinder is identical to the others.

If a solid does not have a constant cross-section (and it is not one of the other basic solids), we may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid. We do this by slicing the solid into pieces, estimating the volume of each slice, and then adding those estimated volumes together. The slices should all be parallel to one another, and when we put all the slices together, we should get the whole solid. Consider, for example, the solid *S* shown in Figure 2.12, extending along the $x$-axis.

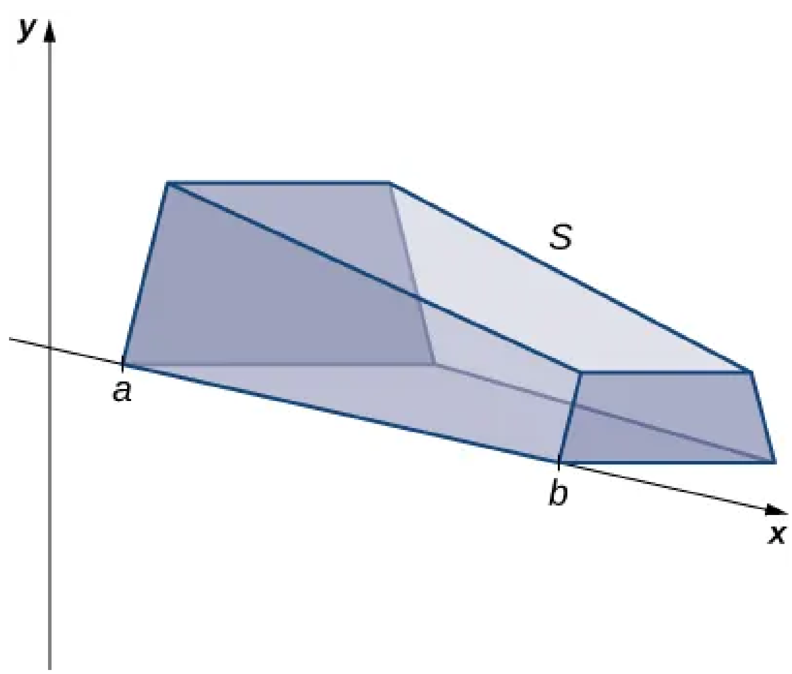



**Figure 2.12** A solid with a varying cross-section.

We want to divide $S$ into slices perpendicular to the $x\text{-axis}$. As we see later in the chapter, there may be times when we want to slice the solid in some other direction—say, with slices perpendicular to the *y*-axis. The decision of which way to slice the solid is very important. If we make the wrong choice, the computations can get quite messy. Later in the chapter, we examine some of these situations in detail and look at how to decide which way to slice the solid. For the purposes of this section, however, we use slices perpendicular to the $x\text{-axis}$.

Because the cross-sectional area is not constant, we let $A(x)$ represent the area of the cross-section at point $x$. Now let $P = \{x_0, x_1 ..., X_n\}$ be a regular partition of $[a, b]$, and for $i = 1, 2, ...n$, let $S_i$ represent the slice of $S$ stretching from $x_{i-1} \text{to } x_i$. The following figure shows the sliced solid with $n = 3$.

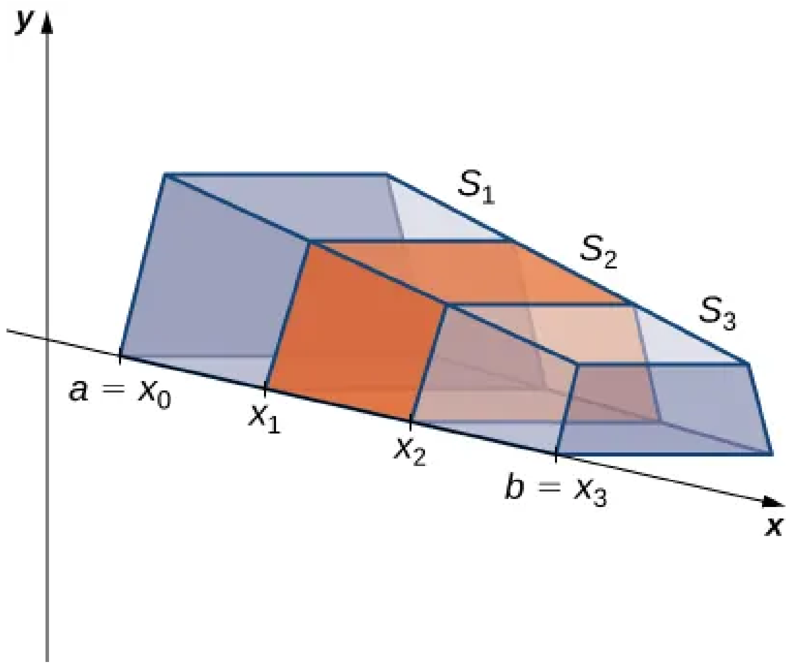



**Figure 2.13** The solid $S$ has been divided into three slices perpendicular to the $x\text{-axis}$.

Finally, for $i = 1, 2, ...n$, let $x_i^*$ be an arbitrary point in $[x_{i-1}, x_i]$. Then the volume of slice $S_i$ can be estimated by $V(S_i) \approx A(x_i^*)\,\Delta x$. Adding these approximations together, we see the volume of the entire solid $S$ can be approximated by

$$V(S) \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x.$$

By now, we can recognize this as a Riemann sum, and our next step is to take the limit as $n \to \infty$. Then we have

$$V(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx.$$

The technique we have just described is called the **slicing method**. To apply it, we use the following strategy.

**PROBLEM-SOLVING STRATEGY**

### Finding Volumes by the Slicing Method

1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
2. Determine a formula for the area of the cross-section.
3. Integrate the area formula over the appropriate interval to get the volume.

Recall that in this section, we assume the slices are perpendicular to the $x\text{-axis}$. Therefore, the area formula is in terms of *x* and the limits of integration lie on the $x\text{-axis}$. However, the problem-solving strategy shown here is valid regardless of how we choose to slice the solid.

**EXAMPLE 2.6**

### Deriving the Formula for the Volume of a Pyramid

We know from geometry that the formula for the volume of a pyramid is $V = \frac{1}{3}Ah$. If the pyramid has a square base, this becomes $V = \frac{1}{3}a^2h$, where $a$ denotes the length of one side of the base. We are going to use the slicing method to derive this formula.

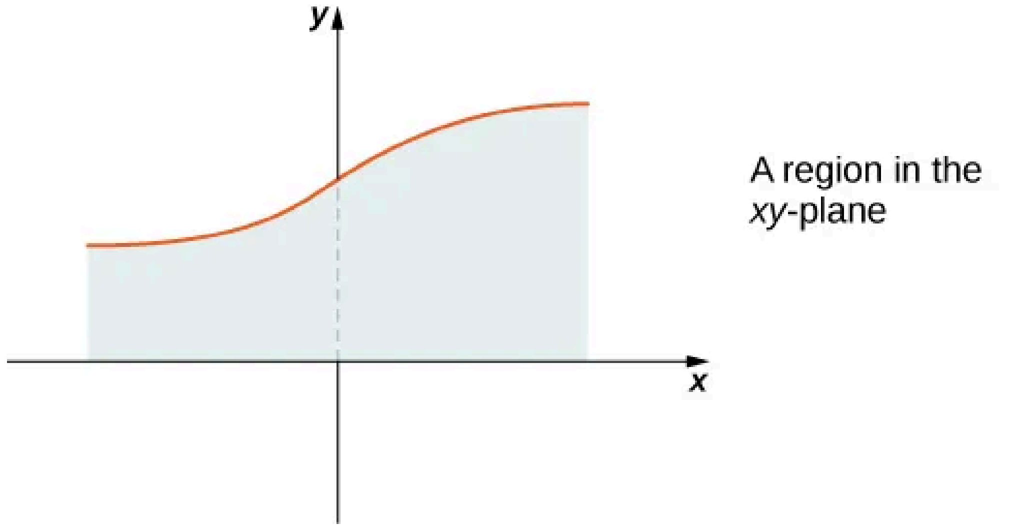
[Show/Hide Solution]

**CHECKPOINT 2.6**

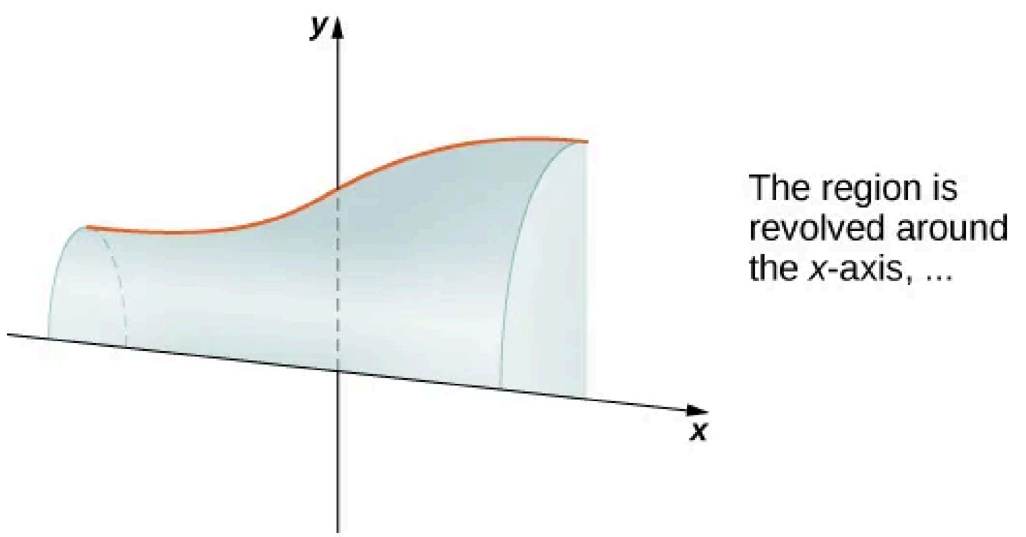
Use the slicing method to derive the formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a circular cone.

## Solids of Revolution

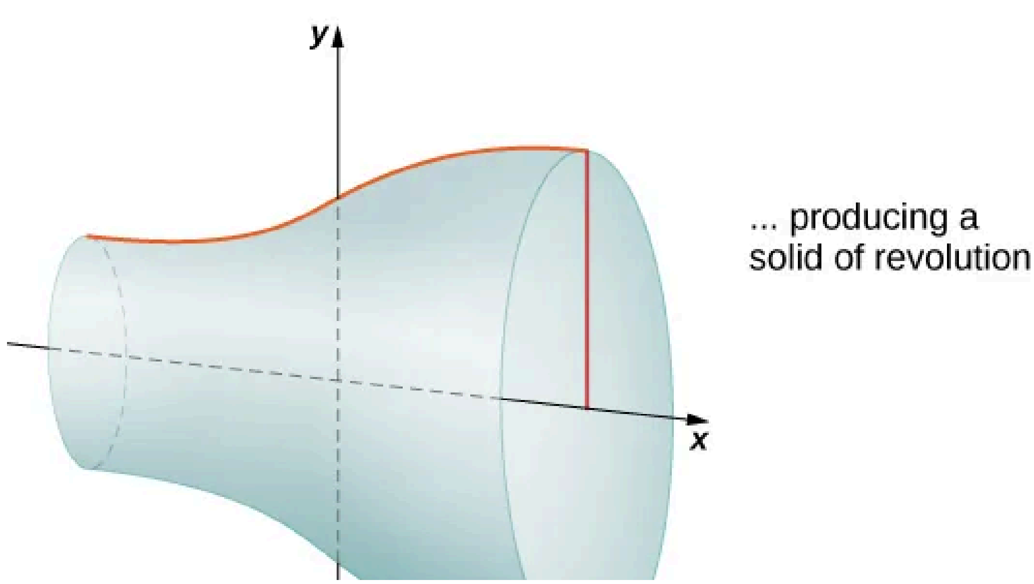
If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**, as shown in the following figure.


y
x
A region in the
xy-plane

(a)


y
x
The region is
revolved around
the x-axis, ...

(b)


y
x
... producing a
solid of revolution

(c)

**Figure 2.15** (a) This is the region that is revolved around the *x*-axis. (b) As the region begins to revolve around the axis, it sweeps out a solid of revolution. (c) This is the solid that results when the revolution is complete.

Solids of revolution are common in mechanical applications, such as machine parts produced by a lathe. We spend the rest of this section looking at solids of this type. The next example uses the slicing method to calculate the volume of a solid of revolution.

**MEDIA**

Use an online integral calculator to learn more.

**EXAMPLE 2.7**

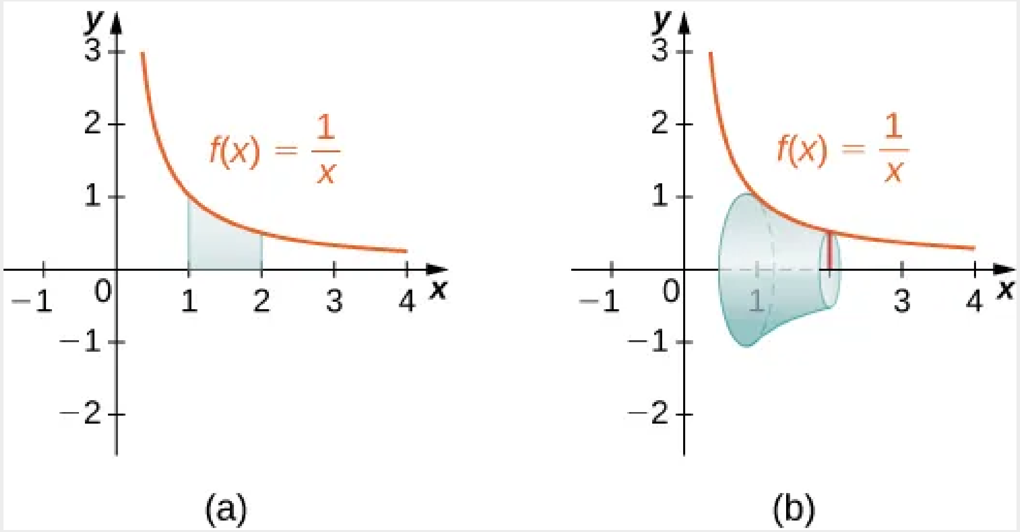
### Using the Slicing Method to find the Volume of a Solid of Revolution

Use the slicing method to find the volume of the solid of revolution bounded by the graphs of $f(x) = x^2 - 4x + 5, x = 1, \text{ and } x = 4$, and rotated about the $x\text{-axis}$.

[Show/Hide Solution]

**CHECKPOINT 2.7**

Use the method of slicing to find the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = 1/x$ and the $x\text{-axis}$ over the interval $[1, 2]$ around the $x\text{-axis}$. See the following figure.




## The Disk Method

When we use the slicing method with solids of revolution, it is often called the **disk method** because, for solids of revolution, the slices used to over approximate the volume of the solid are disks. To see this, consider the solid of revolution generated by revolving the region between the graph of the function $f(x) = (x-1)^2 + 1$ and the $x\text{-axis}$ over the interval $[-1, 3]$ around the $x\text{-axis}$. The graph of the function and a representative disk are shown in Figure 2.18(a) and (b). The region of revolution and the resulting solid are shown in Figure 2.18(c) and (d).

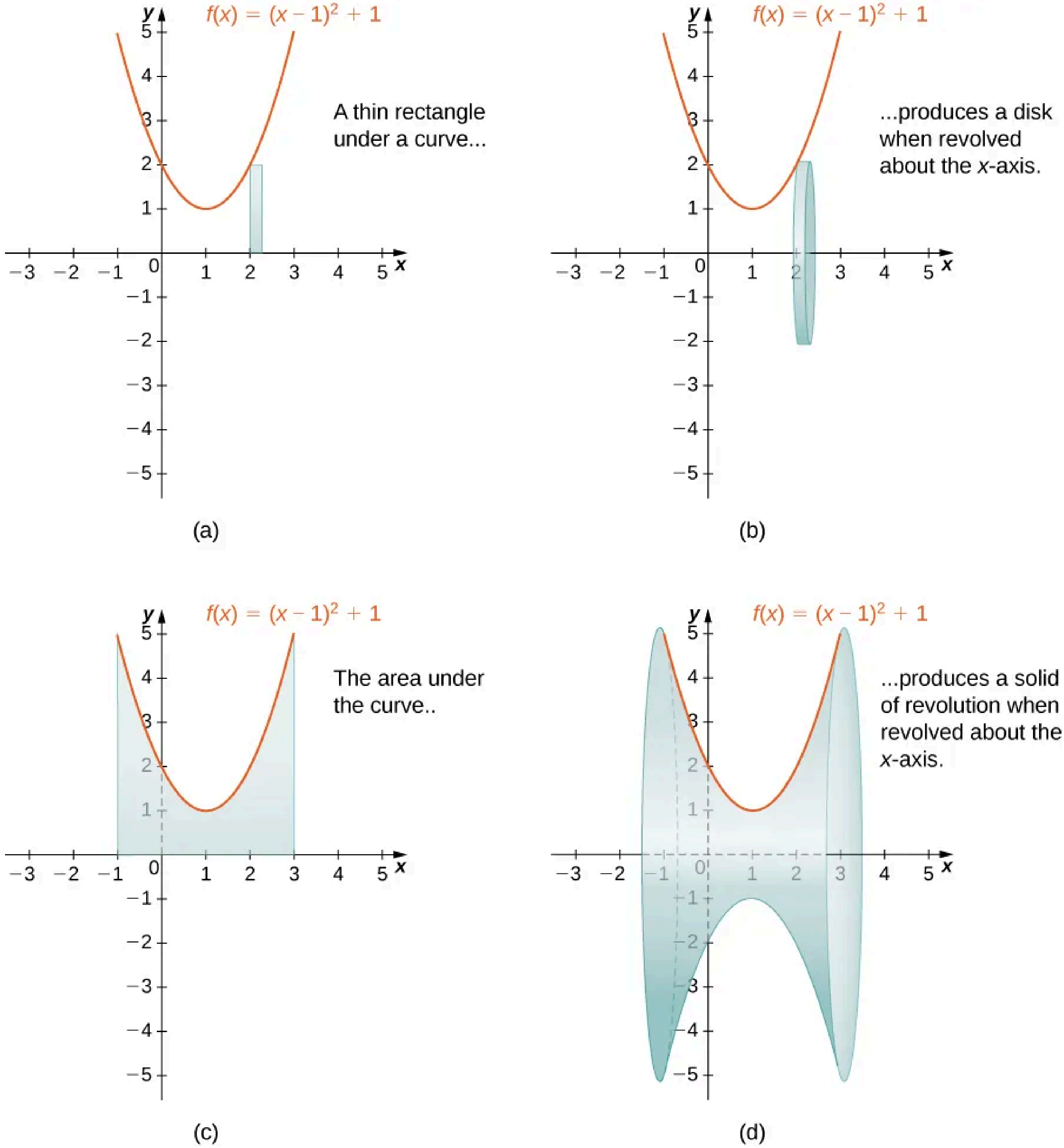



**Figure 2.18** (a) A thin rectangle for approximating the area under a curve. (b) A representative disk formed by revolving the rectangle about the $x\text{-axis}$. (c) The region under the curve is revolved about the $x\text{-axis}$, resulting in (d) the solid of revolution.

We already used the formal Riemann sum development of the volume formula when we developed the slicing method. We know that

$$V = \int_a^b A\left(x\right) dx.$$

The only difference with the disk method is that we know the formula for the cross-sectional area ahead of time; it is the area of a circle. This gives the following rule.

### RULE: THE DISK METHOD

Let $f(x)$ be continuous and nonnegative. Define $R$ as the region bounded above by the graph of $f(x)$, below by the $x\text{-axis}$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving $R$ around the $x\text{-axis}$ is given by

$$V = \int_a^b \pi [f(x)]^2 dx. \tag{2.3}$$

The volume of the solid we have been studying (Figure 2.18) is given by

$$\begin{aligned} V &= \int_a^b \pi [f(x)]^2 dx \\ &= \int_{-1}^3 \pi \left[(x-1)^2 + 1\right]^2 dx = \pi \int_{-1}^3 \left[(x-1)^4 + 2(x-1)^2 + 1\right] dx \\ &= \pi \left[\tfrac{1}{5}(x-1)^5 + \tfrac{2}{3}(x-1)^3 + x\right]\Big|_{-1}^3 = \pi \left[\left(\tfrac{32}{5} + \tfrac{16}{3} + 3\right) - \left(-\tfrac{32}{5} - \tfrac{16}{3} - 1\right)\right] = \tfrac{412\pi}{15} \text{ units}^3. \end{aligned}$$

Let's look at some examples.

### EXAMPLE 2.8

#### Using the Disk Method to Find the Volume of a Solid of Revolution 1

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{x}$ and the $x\text{-axis}$ over the interval $[1, 4]$ around the $x\text{-axis}$.

**[Show/Hide Solution]**

### CHECKPOINT 2.8

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{4-x}$ and the $x\text{-axis}$ over the interval $[0,\ 4]$ around the $x\text{-axis}.$

So far, our examples have all concerned regions revolved around the $x\text{-axis},$ but we can generate a solid of revolution by revolving a plane region around any horizontal or vertical line. In the next example, we look at a solid of revolution that has been generated by revolving a region around the $y\text{-axis}.$ The mechanics of the disk method are nearly the same as when the $x\text{-axis}$ is the axis of revolution, but we express the function in terms of $y$ and we integrate with respect to *y* as well. This is summarized in the following rule.

### RULE: THE DISK METHOD FOR SOLIDS OF REVOLUTION AROUND THE *Y*-AXIS

Let $g(y)$ be continuous and nonnegative. Define $Q$ as the region bounded on the right by the graph of $g(y)$ , on the left by the $y\text{-axis},$ below by the line $y = c,$ and above by the line $y = d.$ Then, the volume of the solid of revolution formed by revolving $Q$ around the $y\text{-axis}$ is given by

$$V = \int_c^d \pi[g(y)]^2 dy. \tag{2.4}$$

The next example shows how this rule works in practice.

### EXAMPLE 2.9

#### Using the Disk Method to Find the Volume of a Solid of Revolution 2

Let $R$ be the region bounded by the graph of $g(y) = \sqrt{4-y}$ and the $y\text{-axis}$ over the $y\text{-axis}$ interval $[0, 4]$ . Use the disk method to find the volume of the solid of revolution generated by rotating $R$ around the $y\text{-axis}.$

[Show/Hide Solution]

### CHECKPOINT 2.9

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $g(y) = y$ and the $y\text{-axis}$ over the interval $[1, 4]$ around the $y\text{-axis}.$

## The Washer Method

Some solids of revolution have cavities in the middle; they are not solid all the way to the axis of revolution. Sometimes, this is just a result of the way the region of revolution is shaped with respect to the axis of revolution. In other cases, cavities arise when the region of revolution is defined as the region between the graphs of two functions. A third way this can happen is when an axis of revolution other than the $x\text{-axis}$ or $y\text{-axis}$ is selected.

When the solid of revolution has a cavity in the middle, the slices used to approximate the volume are not disks, but washers (disks with holes in the center). For example, consider the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1$ over the interval $[1, 4]$. When this region is revolved around the $x\text{-axis},$ the result is a solid with a cavity in the middle, and the slices are washers. The graph of the function and a representative washer are shown in Figure 2.22(a) and (b). The region of revolution and the resulting solid are shown in Figure 2.22(c) and (d).

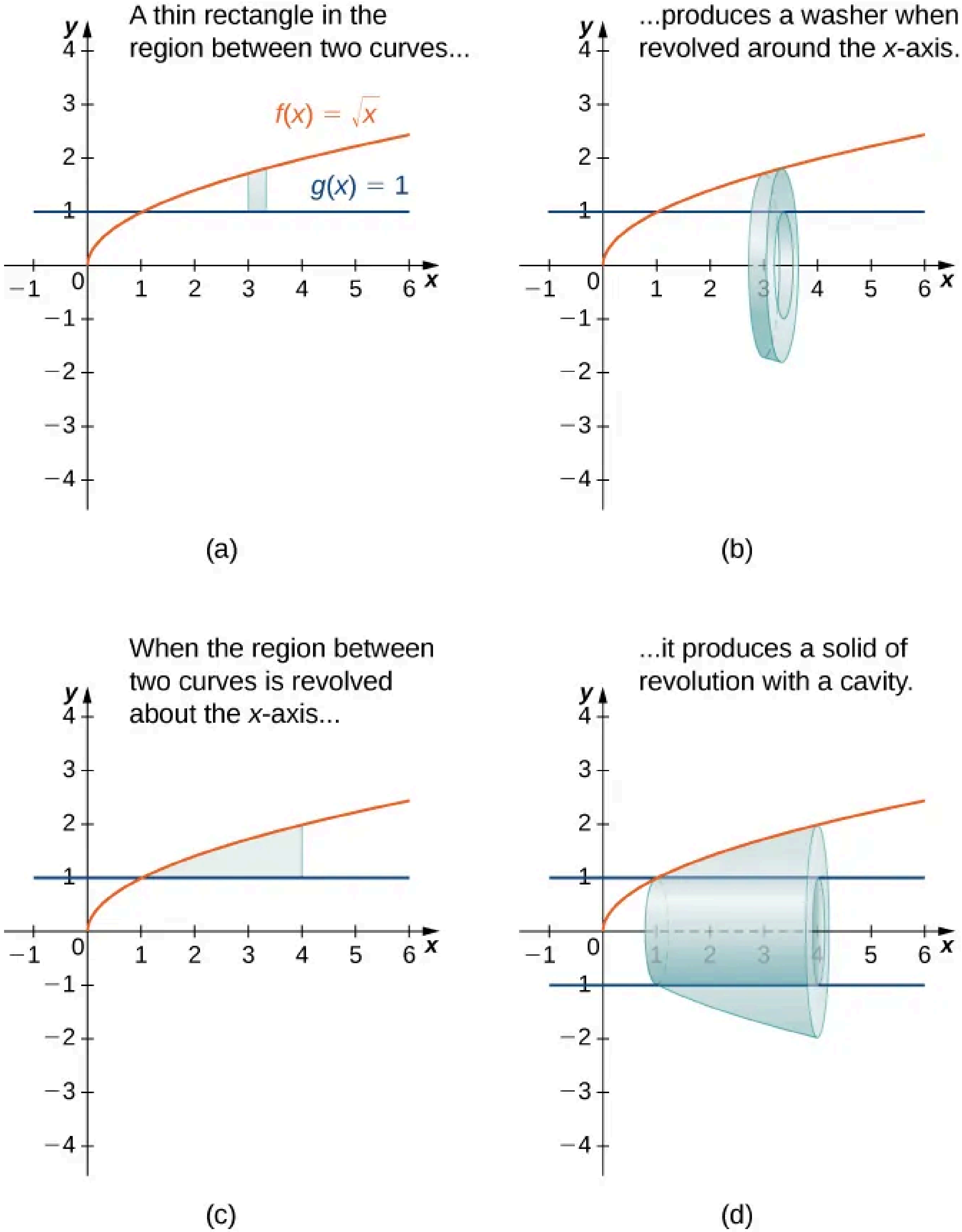



**Figure 2.22** (a) A thin rectangle in the region between two curves. (b) A representative washer formed by revolving the rectangle about the $x\text{-axis}$. (c) The region between the curves over the given interval. (d) The resulting solid of revolution.

The cross-sectional area, then, is the area of the outer circle less the area of the inner circle. In this case,

$$A(x) = \pi\left(\sqrt{x}\right)^2 - \pi(1)^2 = \pi(x-1).$$

Then the volume of the solid is

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \int_1^4 \pi(x-1)\,dx = \pi\left[\frac{x^2}{2} - x\right]\Big|_1^4 = \frac{9}{2}\pi\,\text{units}^3. \end{aligned}$$

Generalizing this process gives the **washer method**.

### RULE: THE WASHER METHOD

Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a, b]$ . Let $R$ denote the region bounded above by the graph of $f(x)$ , below by the graph of $g(x)$ , on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving $R$ around the $x$-axis is given by

$$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx. \tag{2.5}$$

### EXAMPLE 2.10

#### Using the Washer Method

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = 1/x$ over the interval $[1, 4]$ around the $x$-axis.

[Show/Hide Solution]

### CHECKPOINT 2.10

Find the volume of a solid of revolution formed by revolving the region bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = 1/x$ over the interval $[1, 3]$ around the $x$-axis.

As with the disk method, we can also apply the washer method to solids of revolution that result from revolving a region around the *y*-axis. In this case, the following rule applies.

**RULE: THE WASHER METHOD FOR SOLIDS OF REVOLUTION AROUND THE *Y*-AXIS**

Suppose $u(y)$ and $v(y)$ are continuous, nonnegative functions such that $v(y) \leq u(y)$ for $y \in [c, d]$ . Let $Q$ denote the region bounded on the right by the graph of $u(y)$ , on the left by the graph of $v(y)$ , below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving $Q$ around the $y\text{-axis}$ is given by

$$V = \int_c^d \pi \left[(u(y))^2 - (v(y))^2\right] dy.$$

Rather than looking at an example of the washer method with the $y\text{-axis}$ as the axis of revolution, we now consider an example in which the axis of revolution is a line other than one of the two coordinate axes. The same general method applies, but you may have to visualize just how to describe the cross-sectional area of the volume.

**EXAMPLE 2.11**

### The Washer Method with a Different Axis of Revolution

Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the $x\text{-axis}$ over the interval $[0, 4]$ around the line $y = -2.$

[Show/Hide Solution]

**CHECKPOINT 2.11**

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x + 2$ and below by the $x\text{-axis}$ over the interval $[0, 3]$ around the line $y = -1.$

## Section 2.2 Exercises

**58.** Derive the formula for the volume of a sphere using the slicing method.

**59.** Use the slicing method to derive the formula for the volume of a cone.

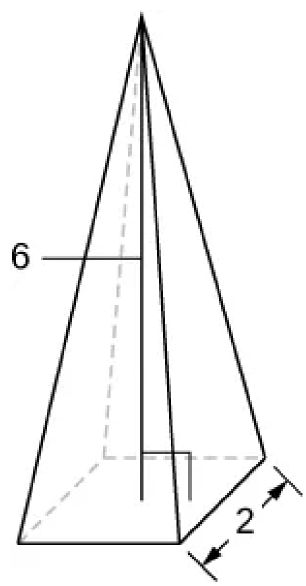
**60.** Use the slicing method to derive the formula for the volume of a tetrahedron with side length $a$.

**61.** Use the disk method to derive the formula for the volume of a trapezoidal cylinder.

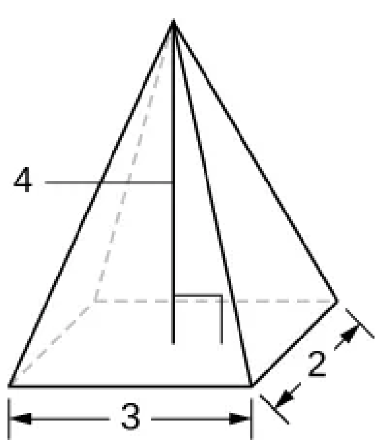
**62.** Explain when you would use the disk method versus the washer method. When are they interchangeable?

For the following exercises, draw a typical slice and find the volume using the slicing method for the given volume.

**63.** A pyramid with height 6 units and square base of side 2 units, as pictured here.

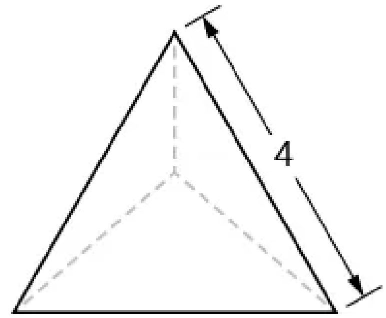



**64.** A pyramid with height 4 units and a rectangular base with length 2 units and width 3 units, as



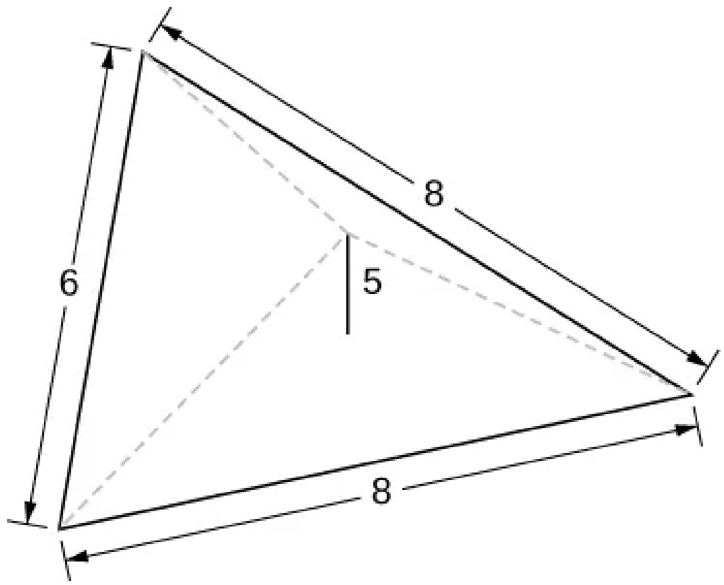

pictured here.

**65.**



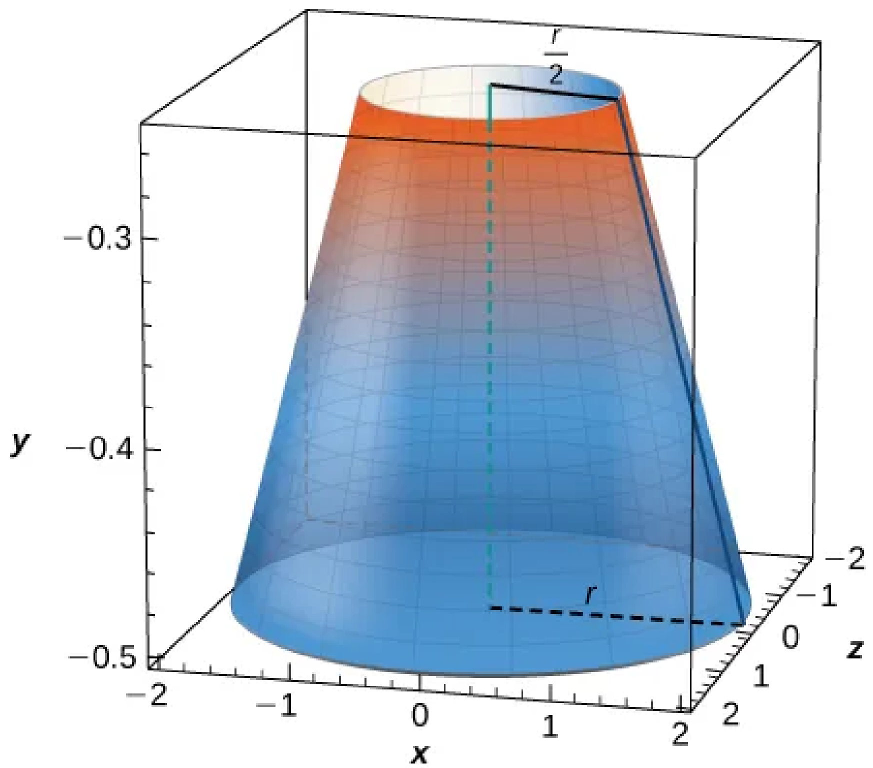

A tetrahedron with a base side of 4 units, as seen here.

**66.** A pyramid with height 5 units, and an isosceles triangular base with lengths of 6 units and 8 units,




as seen here.

**67.** A cone of radius $r$ and height $h$ has a smaller cone of radius $r/2$ and height $h/2$ removed from the top, as seen here. The resulting solid is called a *frustum*.




For the following exercises, draw an outline of the solid and find the volume using the slicing method.

**68.** The base is a circle of radius $a$. The slices perpendicular to the base are squares.

**69.** The base is a triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Slices perpendicular to the *x*-axis are semicircles.

**70.** The base is the region under the parabola $y = 1 - x^2$ in the first quadrant. Slices perpendicular to the *xy*-plane and parallel to the y-axis are squares.

**71.** The base is the region under the parabola $y = 1 - x^2$ and above the $x\text{-axis}$. Slices perpendicular to the $y\text{-axis}$ are squares.

**72.** The base is the region enclosed by $y = x^2$ and $y = 9$. Slices perpendicular to the *x*-axis are right isosceles triangles. The intersection of one of these slices and the base is the leg of the triangle.

**73.** The base is the area between $y = x$ and $y = x^2$. Slices perpendicular to the *x*-axis are semicircles.

For the following exercises, draw the region bounded by the curves. Then, use the disk method to find the volume when the region is rotated around the *x*-axis.

**74.** $x + y = 8, x = 0, \text{ and } y = 0$

**75.** $y = 2x^2, x = 0, x = 4, \text{ and } y = 0$

**76.** $y = e^x + 1, x = 0, x = 1, \text{ and } y = 0$

77. $y = x^4, x = 0, \text{ and } y = 1 \text{for } x \geq 0$

78. $y = \sqrt{x}, x = 0, x = 4, \text{ and } y = 0$

79. $y = \sin x, y = \cos x, \text{ and } x = 0$

80. $y = \frac{1}{x}, x = 2, \text{ and } y = 3$

81. $x^2 - y^2 = 9 \text{ and } x + y = 9, y = 0 \text{ and } x = 0$

For the following exercises, draw the region bounded by the curves. Then, find the volume when the region is rotated around the *y*-axis.

82. $y = 4 - \frac{1}{2}x, x = 0, \text{ and } y = 0$

83. $y = 2x^3, x = 0, x = 1, \text{ and } y = 0$

84. $y = 3x^2, x = 0, \text{ and } y = 3$

85. $y = \sqrt{4 - x^2}, y = 0, \text{ and } x = 0$

86. $y = \frac{1}{\sqrt{x+1}}, x = 0, \ x = 3, \text{ and } y = 0$

87. $x = \sec(y) \text{ and } y = \frac{\pi}{4}, \ y = 0 \text{ and } x = 0$

88. $y = \frac{1}{x+1}, x = 0, \ , \ x = 2, \text{ and } y = 0$

89. $y = 4 - x, y = x, \text{ and } x = 0$

For the following exercises, draw the region bounded by the curves. Then, find the volume when the region is rotated around the *x*-axis.

90. $y = x + 2, y = x + 6, x = 0, \text{ and } x = 5$

91. $y = x^2 \text{ and } y = x + 2$

92. $x^2 = y^3 \text{ and } x^3 = y^2$

93. $y = 4 - x^2 \text{ and } y = 2 - x$

94. **[T]** $y = \cos x, y = e^{-x}, x = 0, \text{ and } x = 1.2927$

95. $y = \sqrt{x} \text{ and } y = x^2$

96. $y = \sin x, y = 5 \sin x, x = 0 \text{ and } x = \pi$

97. $y = \sqrt{1 + x^2} \text{ and } y = \sqrt{4 - x^2}$

For the following exercises, draw the region bounded by the curves. Then, use the washer method to find the volume when the region is revolved around the *y*-axis.

**98.** $y = \sqrt{x}, x = 4, \text{ and } y = 0$

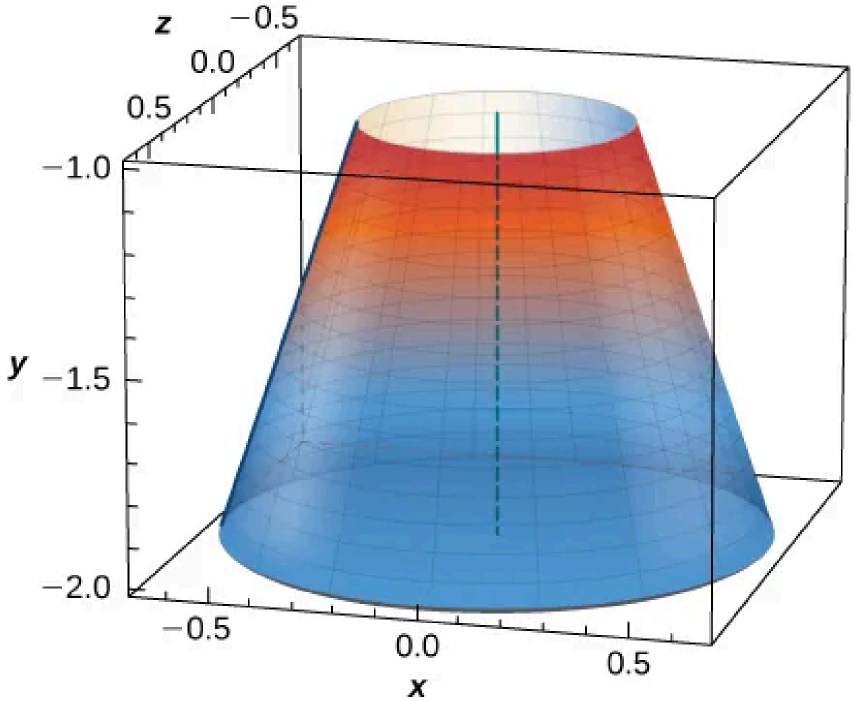
**99.** $y = x + 2, y = 2x - 1, \text{ and } x = 0$

**100.** $y = \sqrt[3]{x} \text{ and } y = x^3$

**101.** $x = e^{2y}, x = y^2, y = 0, \text{ and } y = \ln(2)$

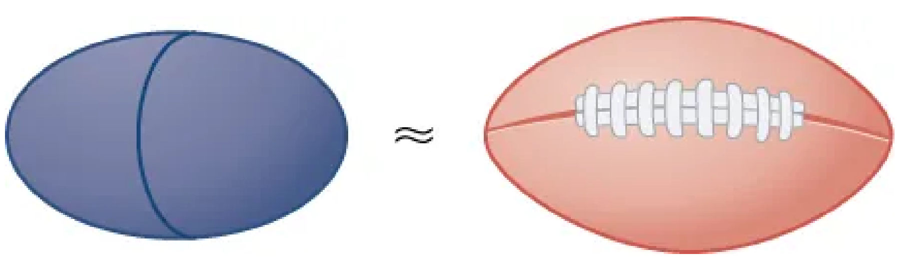
**102.** $x = \sqrt{9 - y^2}, x = e^{-y}, y = 0, \text{ and } y = 3$

**103.** Yogurt containers can be shaped like frustums. Rotate the line $y = \frac{1}{m}x$ around the *y*-axis to find the volume between $y = a$ and $y = b$.





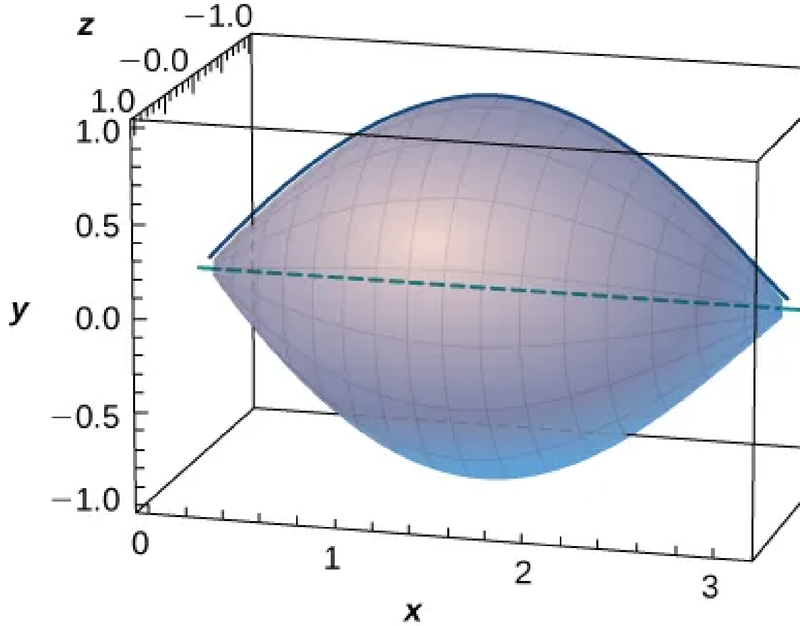
**104.** Rotate the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ around the *x*-axis to approximate the volume of a



football, as seen here.

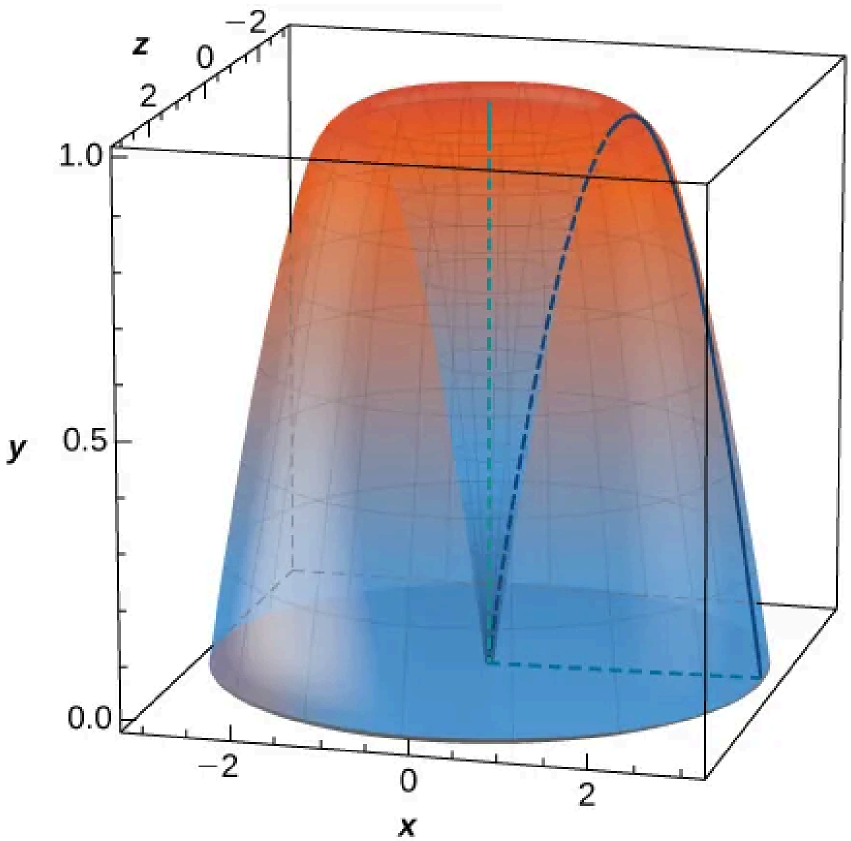
**105.** Rotate the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ around the *y*-axis to approximate the volume of a football.

**106.** A better approximation of the volume of a football is given by the solid that comes from rotating $y = \sin x$ around the *x*-axis from $x = 0$ to $x = \pi$. What is the volume of this football




approximation, as seen here?

**107**. What is the volume of the Bundt cake that comes from rotating $y = \sin x$ around the *y*-axis from



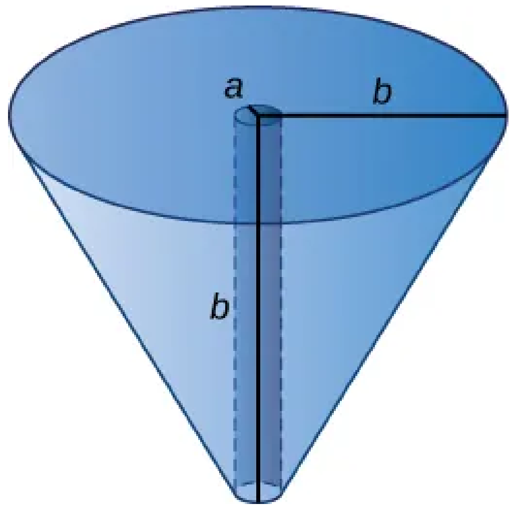

$x = 0$ to $x = \pi$?

For the following exercises, find the volume of the solid described.

**108**. The base is the region between $y = x$ and $y = x^2$. Slices perpendicular to the *x*-axis are semicircles.

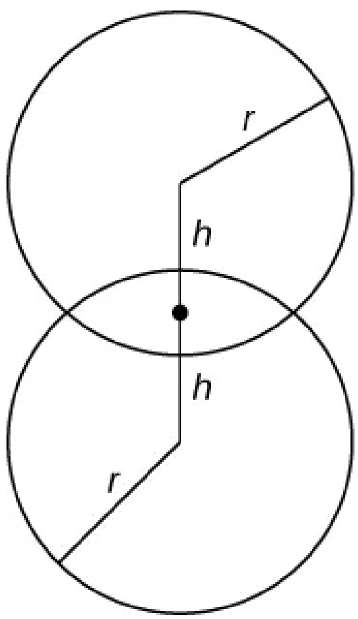
**109**. The base is the region enclosed by the generic ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Slices perpendicular to the *x*-axis are semicircles.

**110.** Bore a hole of radius $a$ down the axis of a right cone of height $b$ and radius $b$ through the base of



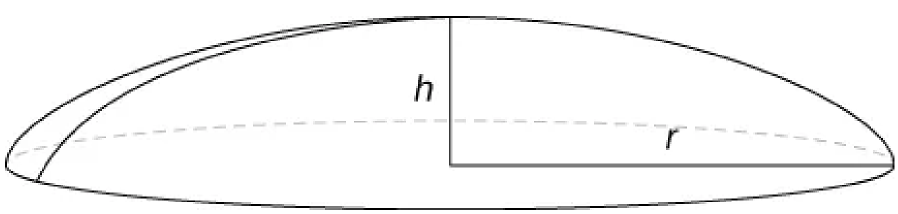

the cone as seen here.

**111.** Find the volume common to two spheres of radius $r$ with centers that are $2h$ apart, as shown

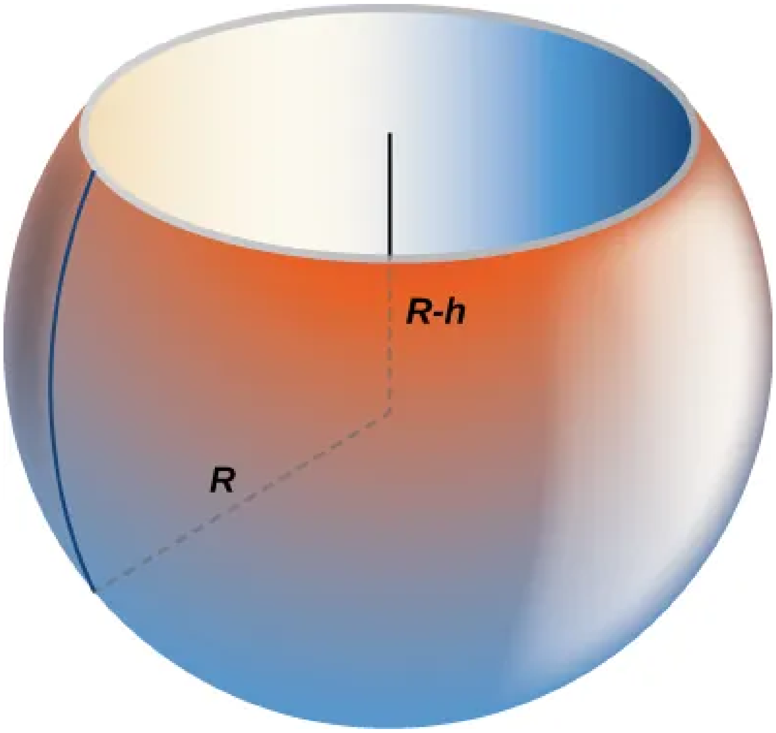



here.

**112.** Find the volume of a spherical cap of height $h$ and radius $r$ where $h < r$, as seen here.



**113.** Find the volume of a sphere of radius $R$ with a cap of height $h$ removed from the top, as seen

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# 2.3 Volumes of Revolution: Cylindrical Shells

## Learning Objectives

2.3.1 Calculate the volume of a solid of revolution by using the method of cylindrical shells.
2.3.2 Compare the different methods for calculating a volume of revolution.

In this section, we examine the method of cylindrical shells, the final method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the disk method or the washer method; however, with the disk and washer methods, we integrate along the coordinate axis parallel to the axis of revolution. With the method of cylindrical shells, we integrate along the coordinate axis *perpendicular* to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the washer method. In the last part of this section, we review all the methods for finding volume that we have studied and lay out some guidelines to help you determine which method to use in a given situation.

## The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region $R$, bounded above by the graph of a function $y = f(x)$, below by the $x$-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in Figure 2.25(a). We then revolve this region around the *y*-axis, as shown in Figure 2.25(b). Note that this is different from what we have done before. Previously, regions defined in terms of functions of $x$ were revolved around the $x\text{-axis}$ or a line parallel to it.

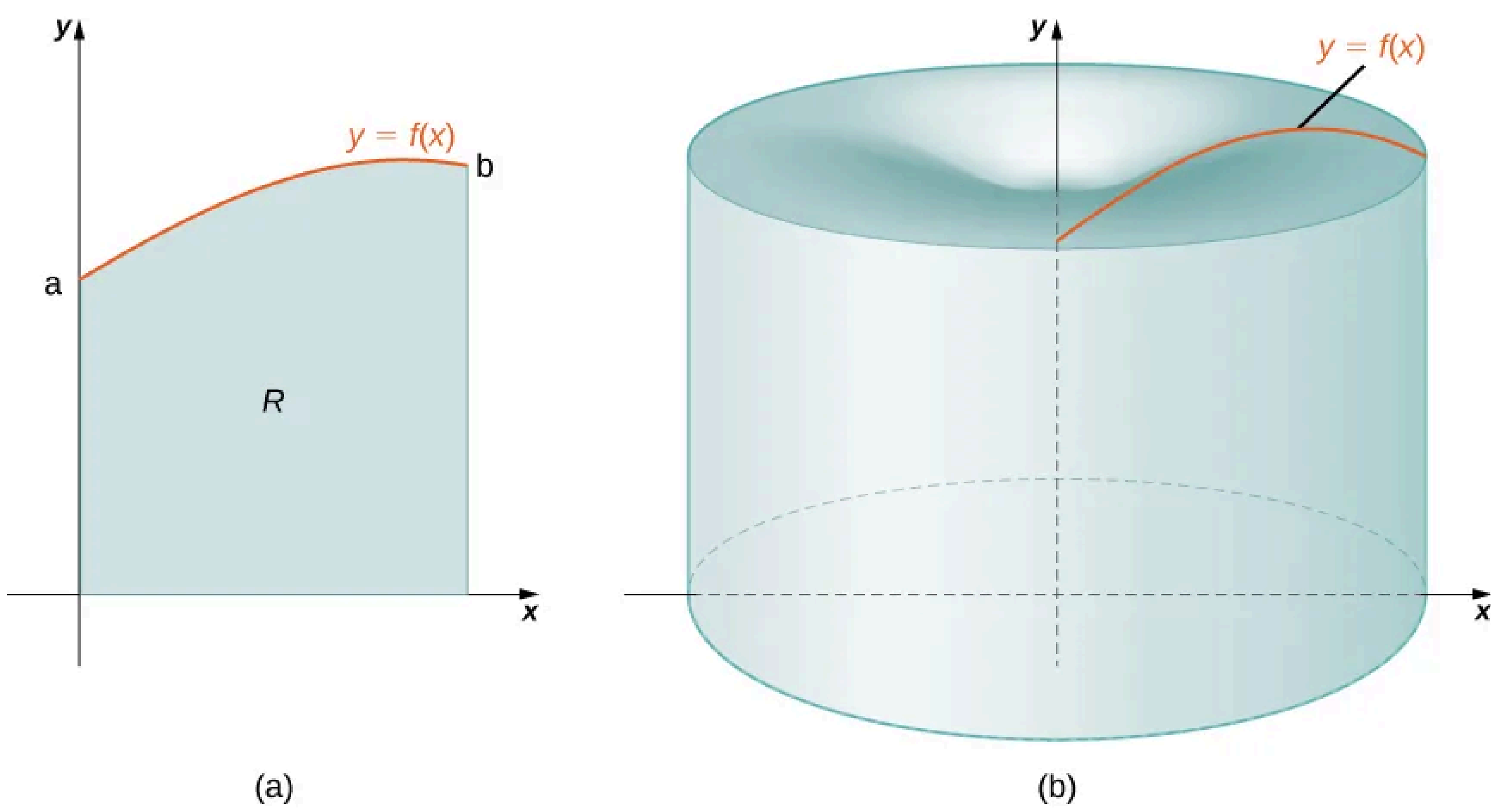


**Figure 2.25** (a) A region bounded by the graph of a function of $x$. (b) The solid of revolution formed when the region is revolved around the $y\text{-axis}.$

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = \{x_0, x_1, ..., x_n\}$ and, for $i = 1, 2, ..., n$, choose a point $x_i^* \in [x_{i-1}, x_i]$. Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width $\Delta x$. A representative rectangle is shown in Figure 2.26(a). When that rectangle is revolved around the *y*-axis, instead of a disk or a washer, we get a cylindrical shell, as shown in the following figure.

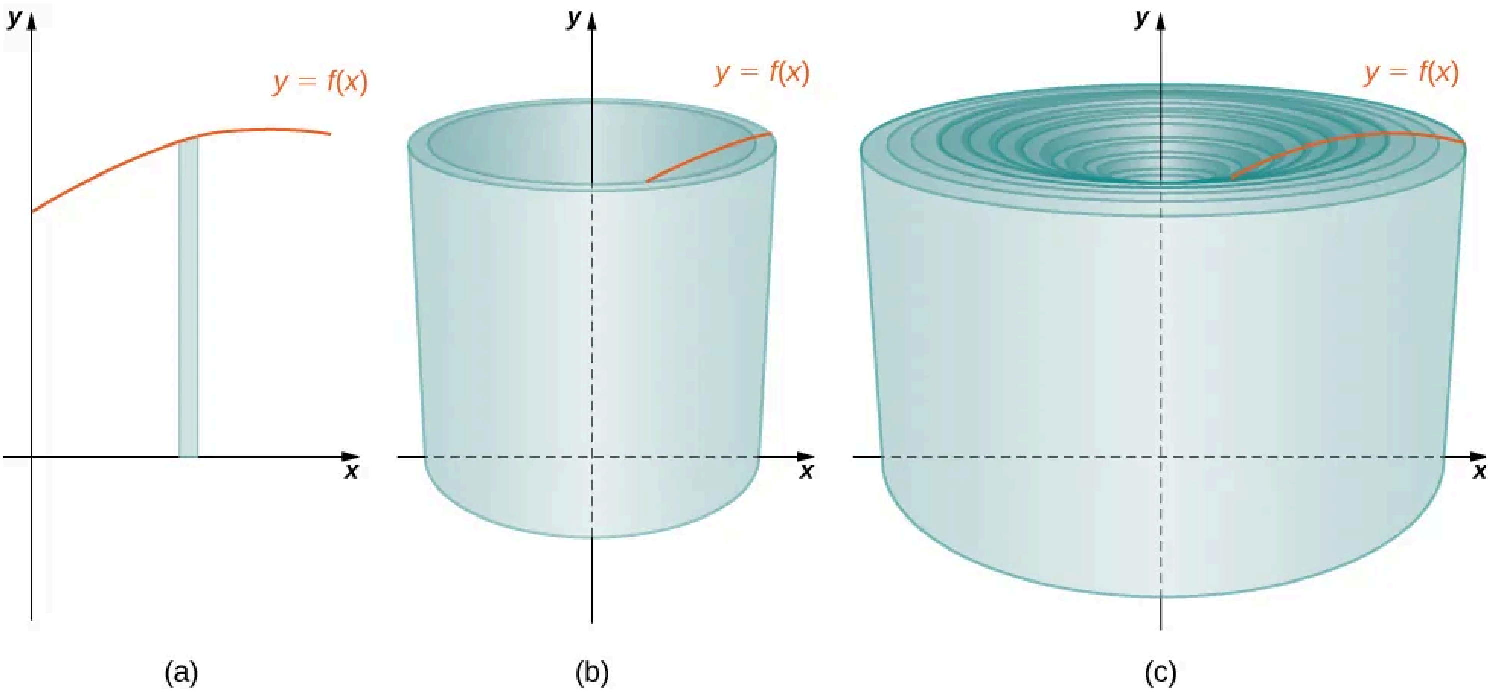



**Figure 2.26** (a) A representative rectangle. (b) When this rectangle is revolved around the $y\text{-axis},$ the result is a cylindrical shell. (c) When we put all the shells together, we get an approximation of the original solid.

To calculate the volume of this shell, consider Figure 2.27.

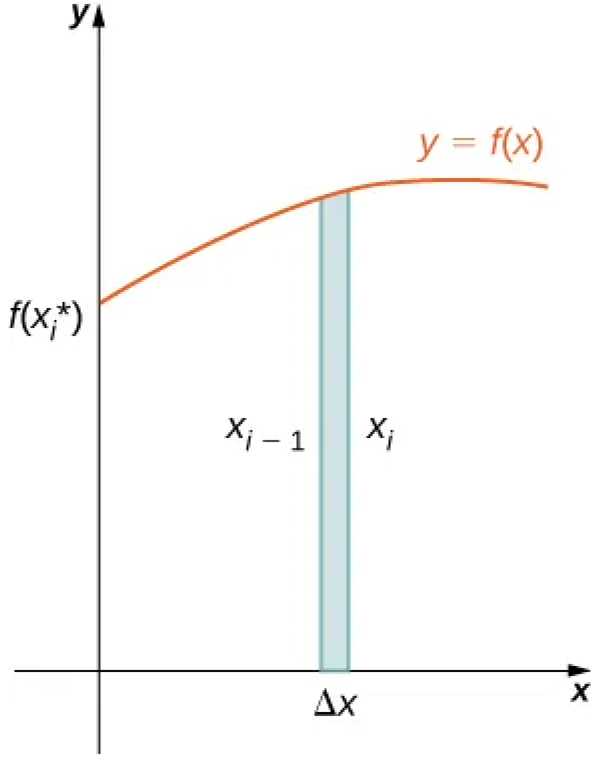



**Figure 2.27** Calculating the volume of the shell.

The shell is a cylinder, so its volume is the cross-sectional area multiplied by the height of the cylinder. The cross-sections are annuli (ring-shaped regions—essentially, circles with a hole in the center), with outer radius $x_i$ and inner radius $x_{i-1}$. Thus, the cross-sectional area is $\pi x_i^2 - \pi x_{i-1}^2$. The height of the cylinder is $f(x_i^*)$. Then the volume of the shell is

$$\begin{aligned} V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)\left(x_i^2 - x_{i-1}^2\right) \\ &= \pi f(x_i^*)\left(x_i + x_{i-1}\right)\left(x_i - x_{i-1}\right) \\ &= 2\pi f(x_i^*)\left(\tfrac{x_i + x_{i-1}}{2}\right)\left(x_i - x_{i-1}\right). \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x$, so we have

$$V_{\text{shell}} = 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x.$$

Furthermore, $\frac{x_i + x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by $x_i^*$. We then have

$$V_{\text{shell}} \approx 2\pi f(x_i^*)x_i^*\Delta x.$$

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate (Figure 2.28).

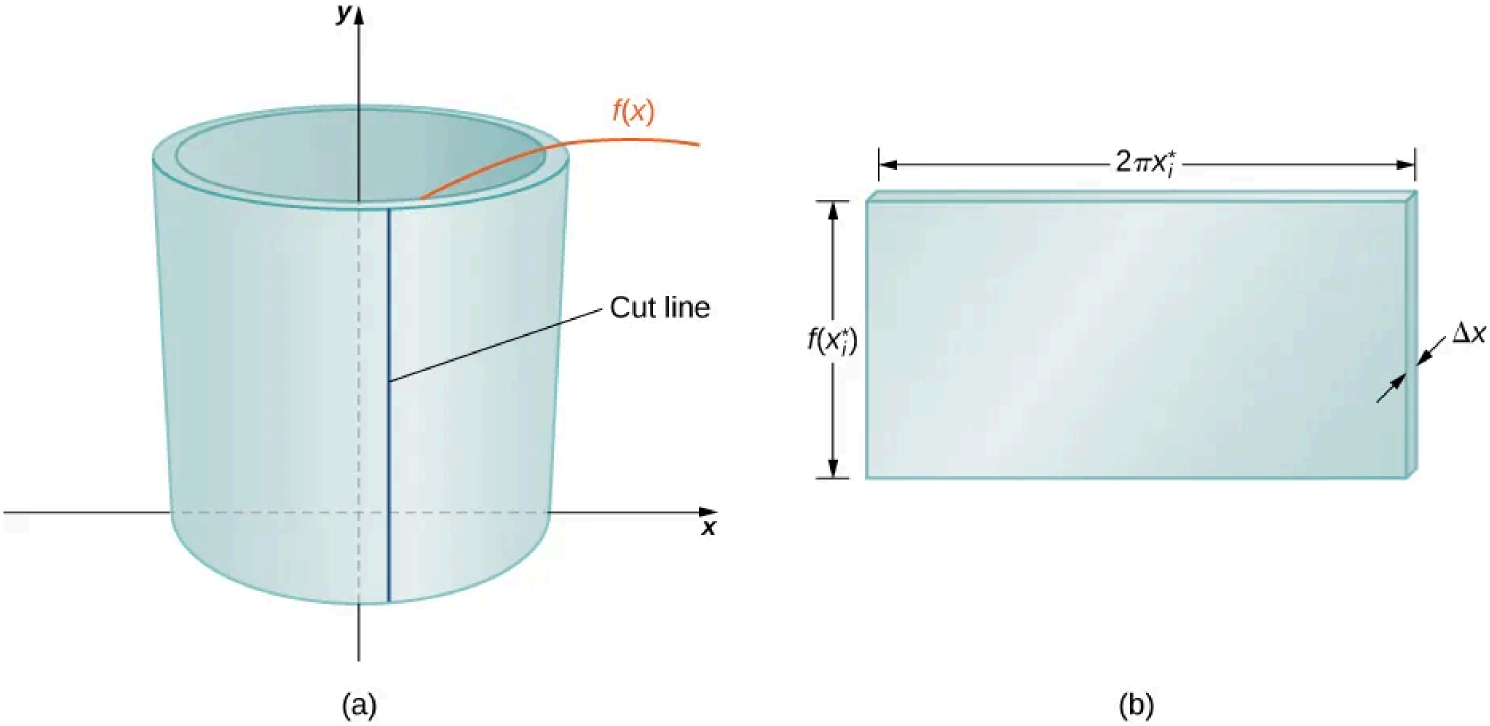



**Figure 2.28** (a) Make a vertical cut in a representative shell. (b) Open the shell up to form a flat plate.

In reality, the outer radius of the shell is greater than the inner radius, and hence the back edge of the plate would be slightly longer than the front edge of the plate. However, we can approximate the flattened shell by a flat plate of height $f(x_i^*)$, width $2\pi x_i^*$, and thickness $\Delta x$ (Figure 2.28). The volume of the shell, then, is approximately the volume of the flat plate. Multiplying the height, width, and depth of the plate, we get

$$V_{\text{shell}} \approx f(x_i^*)\,(2\pi x_i^*)\,\Delta x,$$

which is the same formula we had before.

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

$$V \approx \sum_{i=1}^{n} (2\pi x_i^* f(x_i^*)\Delta x)\,.$$

Here we have another Riemann sum, this time for the function $2\pi x f(x)$. Taking the limit as $n \to \infty$ gives us

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} (2\pi x_i^* f(x_i^*)\Delta x) = \int_a^b (2\pi x f(x))\,dx.$$

This leads to the following rule for the **method of cylindrical shells.**

### RULE: THE METHOD OF CYLINDRICAL SHELLS

Let $f(x)$ be continuous and nonnegative. Define $R$ as the region bounded above by the graph of $f(x)$, below by the $x\text{-axis}$, on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving $R$ around the *y*-axis is given by

$$V = \int_a^b (2\pi x f(x))\, dx. \tag{2.6}$$

Now let's consider an example.

### EXAMPLE 2.12

#### The Method of Cylindrical Shells 1

Define $R$ as the region bounded above by the graph of $f(x) = 1/x$ and below by the $x\text{-axis}$ over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y\text{-axis}$.

[Show/Hide Solution]

### CHECKPOINT 2.12

Define *R* as the region bounded above by the graph of $f(x) = x^2$ and below by the *x*-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y\text{-axis}$.

### EXAMPLE 2.13

#### The Method of Cylindrical Shells 2

Define *R* as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the $x\text{-axis}$ over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y\text{-axis}$.

[Show/Hide Solution]

### CHECKPOINT 2.13

Define $R$ as the region bounded above by the graph of $f(x) = 3x - x^2$ and below by the $x\text{-axis}$ over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y\text{-axis}$.

As with the disk method and the washer method, we can use the method of cylindrical shells with solids of revolution, revolved around the $x\text{-axis}$, when we want to integrate with respect to $y$. The analogous rule for this type of solid is given here.

### RULE: THE METHOD OF CYLINDRICAL SHELLS FOR SOLIDS OF REVOLUTION AROUND THE *X*-AXIS

Let $g(y)$ be continuous and nonnegative. Define $Q$ as the region bounded on the right by the graph of $g(y)$, on the left by the $y\text{-axis}$, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving $Q$ around the $x\text{-axis}$ is given by

$$V = \int_c^d (2\pi y g(y))\, dy.$$

### EXAMPLE 2.14

#### The Method of Cylindrical Shells for a Solid Revolved around the *x*-axis

Define $Q$ as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the $y\text{-axis}$ for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving $Q$ around the *x*-axis.

[Show/Hide Solution]

### CHECKPOINT 2.14

Define $Q$ as the region bounded on the right by the graph of $g(y) = 3/y$ and on the left by the $y\text{-axis}$ for $y \in [1, 3]$ . Find the volume of the solid of revolution formed by revolving $Q$ around the $x\text{-axis}$.

For the next example, we look at a solid of revolution for which the graph of a function is revolved around a line other than one of the two coordinate axes. To set this up, we need to revisit the development of the method of cylindrical shells. Recall that we found the volume of one of the shells to be given by

$$\begin{aligned} V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)\left(x_i^2 - x_{i-1}^2\right) \\ &= \pi f(x_i^*)\left(x_i + x_{i-1}\right)\left(x_i - x_{i-1}\right) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)\left(x_i - x_{i-1}\right). \end{aligned}$$

This was based on a shell with an outer radius of $x_i$ and an inner radius of $x_{i-1}$. If, however, we rotate the region around a line other than the $y\text{-axis}$, we have a different outer and inner radius. Suppose, for example, that we rotate the region around the line $x = -k$, where $k$ is some positive constant. Then, the outer radius of the shell is $x_i + k$ and the inner radius of the shell is $x_{i-1} + k$. Substituting these terms into the expression for volume, we see that when a plane region is rotated around the line $x = -k$, the volume of a shell is given by

$$\begin{aligned} V_{\text{shell}} &= 2\pi f(x_i^*)\left(\frac{(x_i+k)+(x_{i-1}+k)}{2}\right)\left((x_i + k) - (x_{i-1} + k)\right) \\ &= 2\pi f(x_i^*)\left(\left(\frac{x_i + x_{i-2}}{2}\right) + k\right)\Delta x. \end{aligned}$$

As before, we notice that $\frac{x_i + x_{i-1}}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$ and can be approximated by $x_i^*$. Then, the approximate volume of the shell is

$$V_{\text{shell}} \approx 2\pi\left(x_i^* + k\right) f(x_i^*)\Delta x.$$

The remainder of the development proceeds as before, and we see that

$$V = \int_a^b \left(2\pi\left(x + k\right) f(x)\right) dx.$$

We could also rotate the region around other horizontal or vertical lines, such as a vertical line in the right half plane. In each case, the volume formula must be adjusted accordingly. Specifically, the $x\text{-term}$ in the integral must be replaced with an expression representing the radius of a shell. To see how this works, consider the following example.

**EXAMPLE 2.15**

### A Region of Revolution Revolved around a Line

Define $R$ as the region bounded above by the graph of $f(x) = x$ and below by the $x\text{-axis}$ over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving $R$ around the line $x = -1$.

**[Show/Hide Solution]**

**CHECKPOINT 2.15**

Define $R$ as the region bounded above by the graph of $f(x) = x^2$ and below by the $x\text{-axis}$ over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving $R$ around the line $x = -2$.

For our final example in this section, let's look at the volume of a solid of revolution for which the region of revolution is bounded by the graphs of two functions.

**EXAMPLE 2.16**

### A Region of Revolution Bounded by the Graphs of Two Functions

Define $R$ as the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1/x$ over the interval $[1, 4]$. Find the volume of the solid of revolution generated by revolving $R$ around the $y\text{-axis}$.

**[Show/Hide Solution]**

### CHECKPOINT 2.16

Define $R$ as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving $R$ around the $y\text{-axis}$.

## Which Method Should We Use?

We have studied several methods for finding the volume of a solid of revolution, but how do we know which method to use? It often comes down to a choice of which integral is easiest to evaluate. Figure 2.34 describes the different approaches for solids of revolution around the $x\text{-axis}$. It's up to you to develop the analogous table for solids of revolution around the $y\text{-axis}$.

Comparing the Methods for Finding the Volume of a Solid Revolution around the *x*-axis

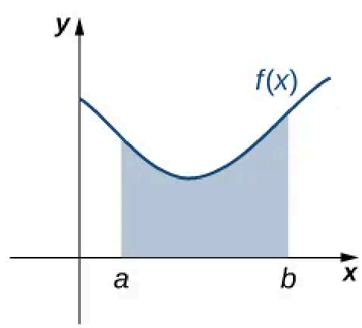
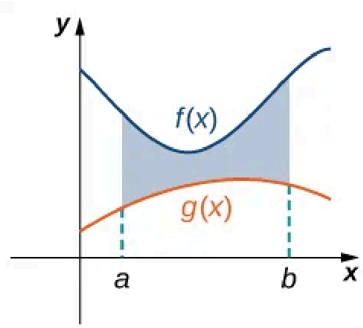
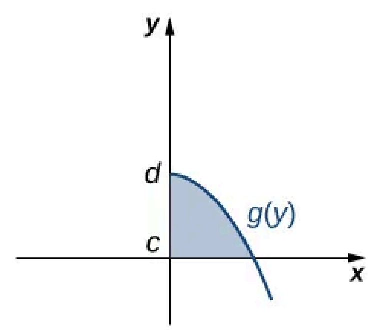
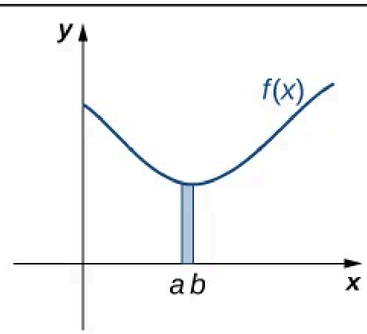
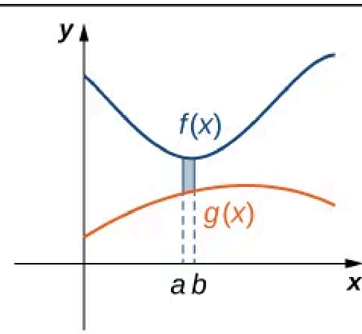
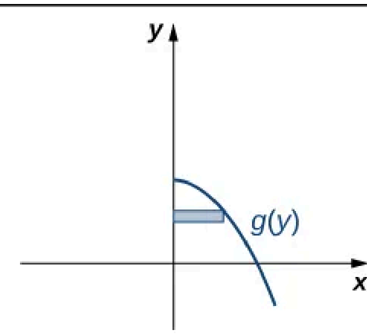
| Compare | Disk Method | Washer Method | Shell Method |
|---|---|---|---|
| Volume formula | $V = \int_a^b \pi[f(x)]^2\,dx$ | $V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$ | $V = \int_c^d 2\pi y\, g(y)\, dy$ |
| Solid | No cavity in the center | Cavity in the center | With or without a cavity in the center |
| Interval to partition | [*a*, *b*] on *x*-axis | [*a*, *b*] on *x*-axis | [*c*, *d*] on *y*-axis |
| Rectangle | Vertical | Vertical | Horizontal |
| Typical region |  |  |  |
| Typical element |  |  |  |

**Figure 2.34**

Let's take a look at a couple of additional problems and decide on the best approach to take for solving them.

**EXAMPLE 2.17**

### Selecting the Best Method

For each of the following problems, select the best method to find the volume of a solid of revolution generated by revolving the given region around the $x\text{-axis}$, and set up the integral to find the volume (do not evaluate the integral).

a. The region bounded by the graphs of $y = x$, $y = 2 - x$, and the $x\text{-axis}$.
b. The region bounded by the graphs of $y = 4x - x^2$ and the $x\text{-axis}$.

[Show/Hide Solution]

**CHECKPOINT 2.17**

Select the best method to find the volume of a solid of revolution generated by revolving the given region around the $x\text{-axis}$, and set up the integral to find the volume (do not evaluate the integral): the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$.

## Section 2.3 Exercises

For the following exercises, find the volume generated when the region between the two curves is rotated around the given axis. Use both the shell method and the washer method. Use technology to graph the functions and draw a typical slice by hand.

**114.** **[T]** Bounded by the curves $y = 3x, x = 0,$ and $y = 3$ rotated around the $y\text{-axis}$.

**115.** **[T]** Bounded by the curves $y = 3x, y = 0, \text{ and } x = 3$ rotated around the $y\text{-axis}$.

**116.** **[T]** Bounded by the curves $y = 3x, y = 0, \text{ and } y = 3$ rotated around the $x\text{-axis}$.

**117.** **[T]** Bounded by the curves $y = 3x, y = 0, \text{ and } x = 3$ rotated around the $x\text{-axis}$.

**118.** **[T]** Bounded by the curves $y = 2x^3, y = 0, \text{ and } x = 2$ rotated around the $y\text{-axis}$.

**119.** **[T]** Bounded by the curves $y = 2x^3, y = 0, \text{ and } x = 2$ rotated around the $x\text{-axis}$.

For the following exercises, use shells to find the volumes of the given solids. Note that the rotated regions lie between the curve and the $x$-axis and are rotated around the $y$-axis.

**120.** $y = 1 - x^2, x = 0, \text{ and } x = 1$

**121.** $y = 5x^3, x = 0, \text{ and } x = 1$

**122.** $y = \frac{1}{x}, x = 1, \text{ and } x = 100$

**123.** $y = \sqrt{1 - x^2}, x = 0, \text{ and } x = 1$

**124.** $y = \frac{1}{1+x^2}, x = 0, \text{ and } x = 3$

**125.** $y = \sin x^2, x = 0, \text{ and } x = \sqrt{\pi}$

**126.** $y = \frac{1}{\sqrt{1-x^2}}, x = 0, \text{ and } x = \frac{1}{2}$

**127.** $y = \sqrt{x}, x = 0, \text{ and } x = 1$

**128.** $y = \left(1 + x^2\right)^3, x = 0, \text{ and } x = 1$

**129.** $y = 5x^3 - 2x^4, x = 0, \text{ and } x = 2$

For the following exercises, use shells to find the volume generated by rotating the regions between the given curve and $y = 0$ around the $x$-axis.

**130.** $y = \sqrt{1 - x^2}, x = 0, \ x = 1$ and the *x*-axis

**131.** $y = x^2, x = 0, \ x = 2$ and the *x*-axis

**132.** $y = \frac{x^3}{2}, \ x = 0, \ x = 2$, and the *x*-axis

**133.** $y = \frac{2}{x^2}, \ x = 1, \ x = 2$, and the *x*-axis

**134.** $x = \frac{1}{1+y^2}, y = 1, \text{ and } y = 4$

**135.** $x = \frac{1+y^2}{y}, y = 1, \ y = 4$, and the *y*-axis

**136.** $x = \sqrt{4 - y^2}, x = 0, y = 0$

**137.** $x = y^3 - 2y^2, \ x = 0, \ x = 9$

**138.** $x = \sqrt{y} + 1, \ x = 1, \ x = 3$, and the *x*-axis

**139.** $x = \sqrt[3]{27y} \text{and } x = \frac{3y}{4}$

For the following exercises, find the volume generated when the region between the curves is rotated around the given axis.

**140.** $y = 3 - x, y = 0, x = 0, \text{ and } x = 2$ rotated around the $y$-axis.

**141.** $y = x^3, x = 0, \text{ and } y = 8$ rotated around the $y$-axis.

**142.** $y = x^2, y = x$, rotated around the $y$-axis.

**143.** $y = \sqrt{x}, y = 0, \text{ and } x = 1$ rotated around the line $x = 2$.

**144.** $y = \frac{1}{4-x}, x = 1, \ x = 2 \text{ and } y = 0$ rotated around the line $x = 4$.

**145.** $y = \sqrt{x} \text{ and } y = x^2$ rotated around the $y$-axis.

**146.** $y = \sqrt{x} \text{ and } y = x^2$ rotated around the line $x = 2$.

**147.** $x = y^3, x = \frac{1}{y}, x = 1, \text{ and } x = 2$ rotated around the $x$-axis.

**148.** $x = y^2 \text{ and } y = x$ rotated around the line $y = 2$.

**149.** **[T]** Left of $x = \sin(\pi y)$, right of $y = x$, around the $y$-axis.

For the following exercises, use technology to graph the region. Determine which method you think would be easiest to use to calculate the volume generated when the function is rotated around the specified axis. Then, use your chosen method to find the volume.

**150.** **[T]** $y = x^2$ and $y = 4x$ rotated around the $y$-axis.

**151.** **[T]** $y = \cos(\pi x), y = \sin(\pi x), x = \frac{1}{4}, \text{ and } x = \frac{5}{4}$ rotated around the $y$-axis. This exercise requires advanced technique. You may use technology to perform the integration.

**152.** **[T]** $y = x^2 - 2x, x = 2, \text{ and } x = 4$ rotated around the $y$-axis.

**153.** **[T]** $y = x^2 - 2x, x = 2, \text{ and } x = 4$ rotated around the $x$-axis.

**154.** **[T]** $y = 3x^3 - 2, y = x, \text{ and } x = 2$ rotated around the $x$-axis.

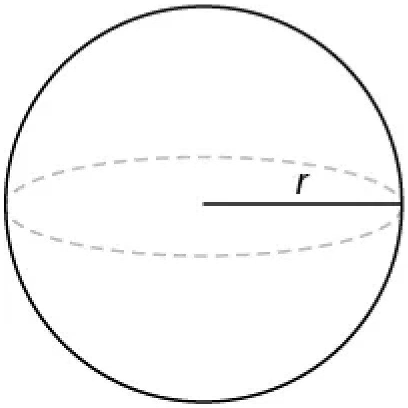
**155.** **[T]** $y = 3x^3 - 2, y = x, \text{ and } x = 2$ rotated around the $y$-axis.

**156.** **[T]** $x = \sin(\pi y^2)$ and $x = \sqrt{2}y$ rotated around the $x$-axis.

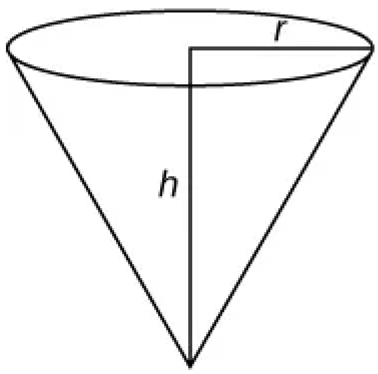
**157.** **[T]** $x = y^2, x = y^2 - 2y + 1, \text{ and } x = 2$ rotated around the $y$-axis.

For the following exercises, use the method of shells to approximate the volumes of some common objects, which are pictured in accompanying figures.

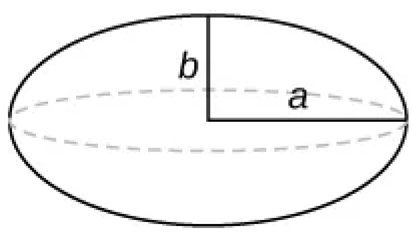
**158.** Use the method of shells to find the volume of a sphere of radius $r$.

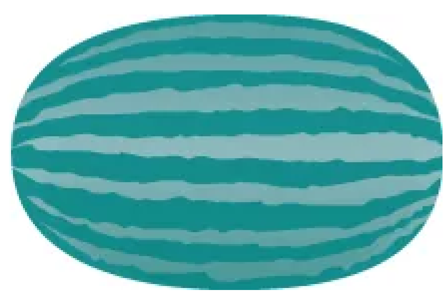


**159.** Use the method of shells to find the volume of a cone with radius $r$ and height $h$.



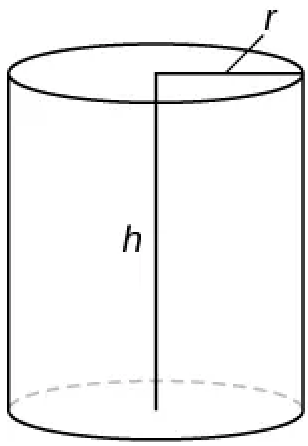
**160.** Use the method of shells to find the volume of an ellipsoid $(x^2/a^2) + (y^2/b^2) = 1$ rotated



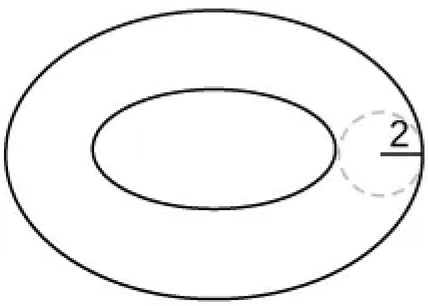


around the $x\text{-axis}$.

**161.** Use the method of shells to find the volume of a cylinder with radius $r$ and height $h$.




**162.** Use the method of shells to find the volume of the donut created when the circle $x^2 + y^2 = 4$ is rotated around the line $x = 4$.




**163.** Consider he region enclosed by the graphs of $y = f(x)$, $y = 1 + f(x)$, $x = 0$, $y = 0$, and $x = a > 0$. What is the volume of the solid generated when this region is rotated around the $y\text{-axis}$? Assume that the function is defined over the interval $[0, a]$.

**164.** Consider the function $y = f(x)$, which decreases from $f(0) = b$ to $f(1) = 0$. Set up the integrals for determining the volume, using both the shell method and the disk method, of the solid generated when this region, with $x = 0$ and $y = 0$, is rotated around the $y\text{-axis}$. Prove that both methods approximate the same volume. Which method is easier to apply? (*Hint:* Since $f(x)$ is one-to-one, there exists an inverse $f^{-1}(y)$.)