2.4 Arc Length of a Curve and Surface Area

Learning Objectives

2.4.1 Determine the length of a curve, $y = f(x)$, between two points.
2.4.2 Determine the length of a curve, $x = g(y)$, between two points.
2.4.3 Find the surface area of a solid of revolution.

In this section, we use definite integrals to find the arc length of a curve. We can think of **arc length** as the distance you would travel if you were walking along the path of the curve. Many real-world applications involve arc length. If a rocket is launched along a parabolic path, we might want to know how far the rocket travels. Or, if a curve on a map represents a road, we might want to know how far we have to drive to reach our destination.

We begin by calculating the arc length of curves defined as functions of x, then we examine the same process for curves defined as functions of y. (The process is identical, with the roles of x and y reversed.) The techniques we use to find arc length can be extended to find the surface area of a surface of revolution, and we close the section with an examination of this concept.

Arc Length of the Curve *y* = *f*(*x*)

In previous applications of integration, we required the function $f(x)$ to be integrable, or at most continuous. However, for calculating arc length we have a more stringent requirement for $f(x)$. Here, we require $f(x)$ to be differentiable, and furthermore we require its derivative, $f'(x)$, to be continuous. Functions like this, which have continuous derivatives, are called *smooth*. (This property comes up again in later chapters.)

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0,\ 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Although it might seem logical to use either horizontal or vertical line segments, we want our line segments to approximate the curve as closely as possible. Figure 2.37 depicts this construct for $n = 5$.

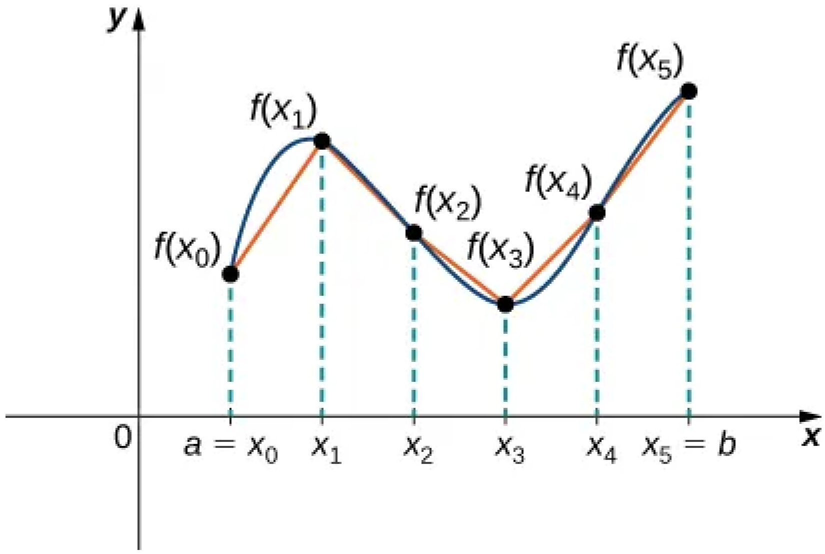


Figure 2.37 We can approximate the length of a curve by adding line segments.

To help us find the length of each line segment, we look at the change in vertical distance as well as the change in horizontal distance over each interval. Because we have used a regular partition, the change in horizontal distance over each interval is given by Δx. The change in vertical distance varies from interval to interval, though, so we use $\Delta y_i = f(x_i) - f(x_{i-1})$ to represent the change in vertical distance over the interval $[x_{i-1}, x_i]$, as shown in Figure 2.38. Note that some (or all) Δy_i may be negative.

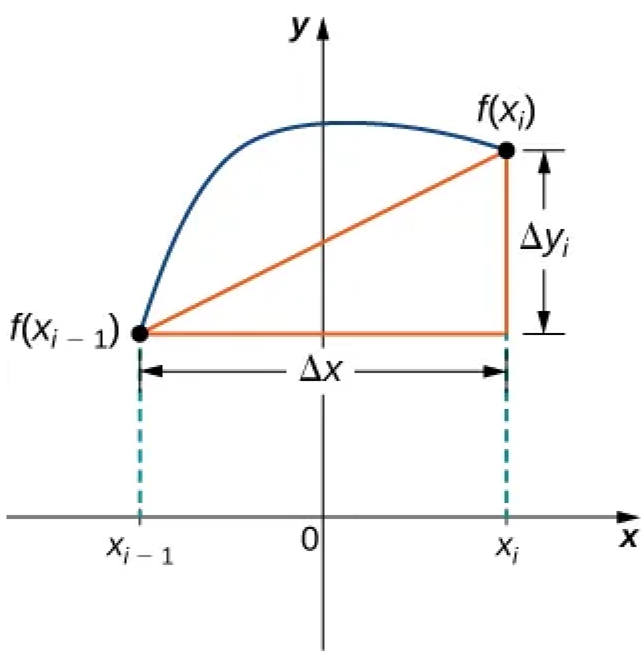


Figure 2.38 A representative line segment approximates the curve over the interval $[x_{i-1}, x_i]$.

By the Pythagorean theorem, the length of the line segment is $\sqrt{(\Delta x)^2 + (\Delta y_i)^2}$. We can also write this as $\Delta x \sqrt{1 + ((\Delta y_i) / (\Delta x))^2}$. Now, by the Mean Value Theorem, there is a point $x_i^* \in [x_{i-1}, x_i]$

such that $f'(x_i^*) = (\Delta y_i) / (\Delta x)$. Then the length of the line segment is given by $\Delta x \sqrt{1 + [f'(x_i^*)]^2}$. Adding up the lengths of all the line segments, we get

$$\text{Arc Length } \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we have

$$\text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We summarize these findings in the following theorem.

THEOREM 2.4

Arc Length for *y* = *f*(*x*)

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b,\ f(b))$ is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \tag{2.7}$$

Note that we are integrating an expression involving $f'(x)$, so we need to be sure $f'(x)$ is integrable. This is why we require $f(x)$ to be smooth. The following example shows how to apply the theorem.

EXAMPLE 2.18

Calculating the Arc Length of a Function of *x*

Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

[Show/Hide Solution]

CHECKPOINT 2.18

Let $f(x) = (4/3)\, x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Although it is nice to have a formula for calculating arc length, this particular theorem can generate expressions that are difficult to integrate. We study some techniques for integration in Introduction to Techniques of Integration. In some cases, we may have to use a computer or calculator to approximate the value of the integral.

EXAMPLE 2.19

Using a Computer or Calculator to Determine the Arc Length of a Function of *x*

Let $f(x) = x^2$. Calculate the arc length of the graph of $f(x)$ over the interval $[1, 3]$.

[Show/Hide Solution]

CHECKPOINT 2.19

Let $f(x) = \sin x$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, \pi]$. Use a computer or calculator to approximate the value of the integral.

Arc Length of the Curve *x* = *g*(*y*)

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of y, we can repeat the same process, except we partition the $y\text{-axis}$ instead of the $x\text{-axis}$. Figure 2.39 shows a representative line segment.

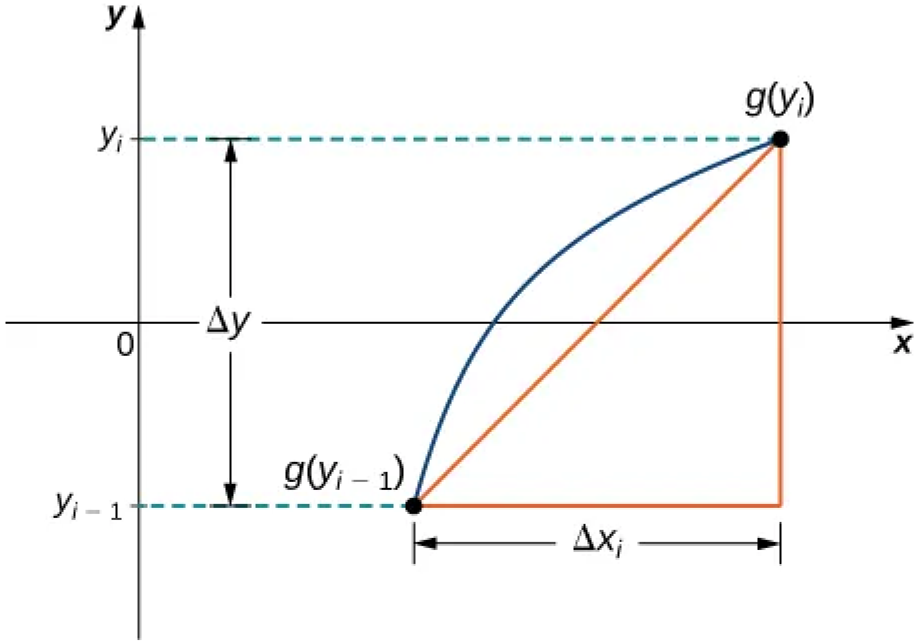


Figure 2.39 A representative line segment over the interval $[y_{i-1}, y_i]$.

Then the length of the line segment is $\sqrt{(\Delta y)^2 + (\Delta x_i)^2}$, which can also be written as $\Delta y\sqrt{1 + ((\Delta x_i) / (\Delta y))^2}$. If we now follow the same development we did earlier, we get a formula for arc length of a function $x = g(y)$.

THEOREM 2.5

Arc Length for *x* = *g*(*y*)

Let $g(y)$ be a smooth function over a y interval $[c, d]$. Then, the arc length of the graph of $g(y)$ from the point $(g(d), d)$ to the point $(g(c), c)$ is given by

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \tag{2.8}$$

EXAMPLE 2.20

Calculating the Arc Length of a Function of *y*

Let $g(y) = 3y^3$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 2]$.

[Show/Hide Solution]

CHECKPOINT 2.20

Let $g(y) = 1/y$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 4]$. Use a computer or calculator to approximate the value of the integral.

Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the $x\text{-axis}$ as shown in the following figure.

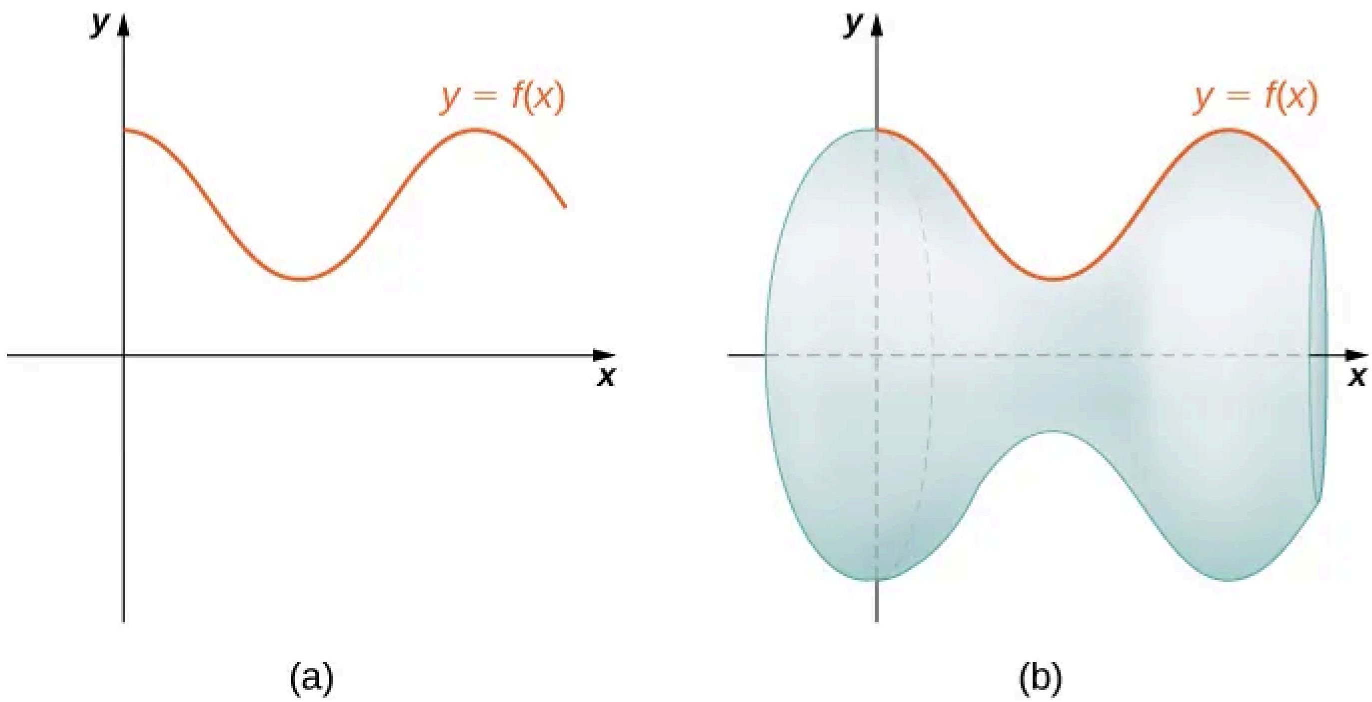


Figure 2.40 (a) A curve representing the function $f(x)$. (b) The surface of revolution formed by revolving the graph of $f(x)$ around the $x\text{-axis}$.

As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. We start by using line segments to approximate the curve, as we did earlier in this section. For $i = 0, 1, 2,..., n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2,..., n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the $x\text{-axis}$ to generate an approximation of the surface of revolution as shown in the following figure.

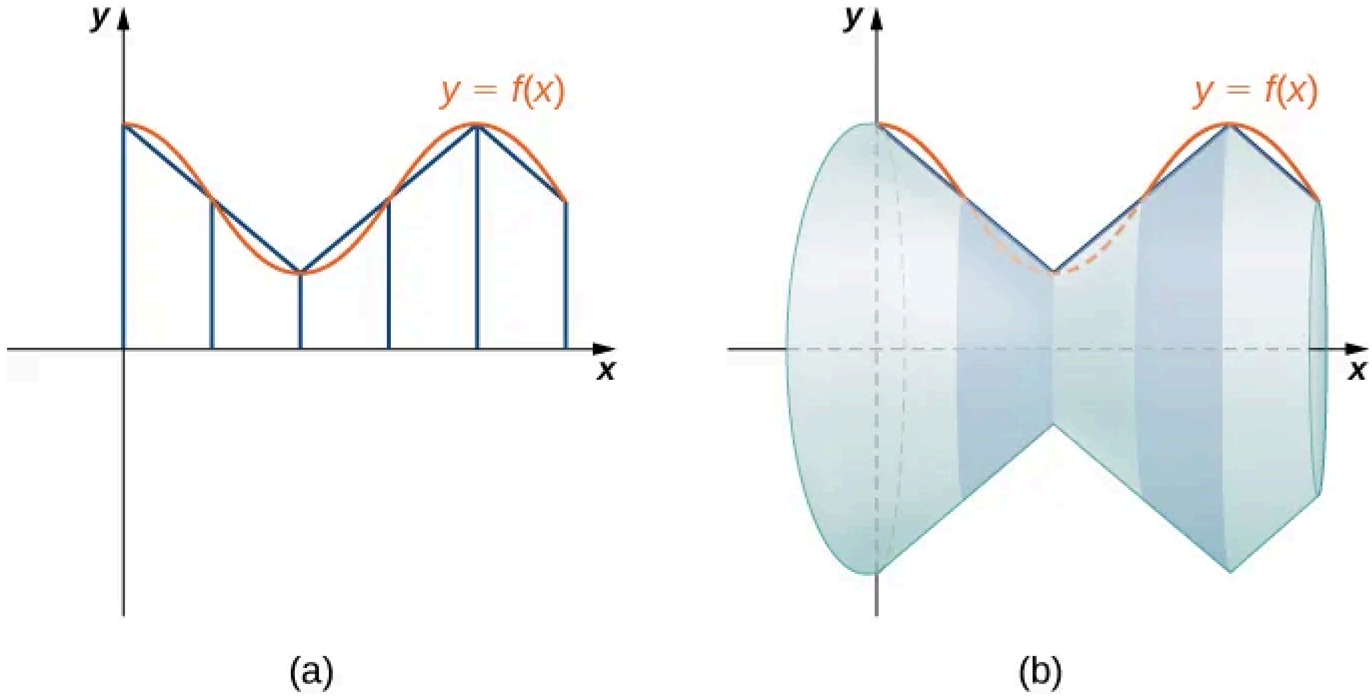


Figure 2.41 (a) Approximating $f(x)$ with line segments. (b) The surface of revolution formed by revolving the line segments around the x-axis.

Notice that when each line segment is revolved around the axis, it produces a band. These bands are actually pieces of cones (think of an ice cream cone with the pointy end cut off). A piece of a cone like this is called a **frustum** of a cone.

To find the surface area of the band, we need to find the lateral surface area, S, of the frustum (the area of just the slanted outside surface of the frustum, not including the areas of the top or bottom faces). Let r_1 and r_2 be the radii of the wide end and the narrow end of the frustum, respectively, and let l be the slant height of the frustum as shown in the following figure.

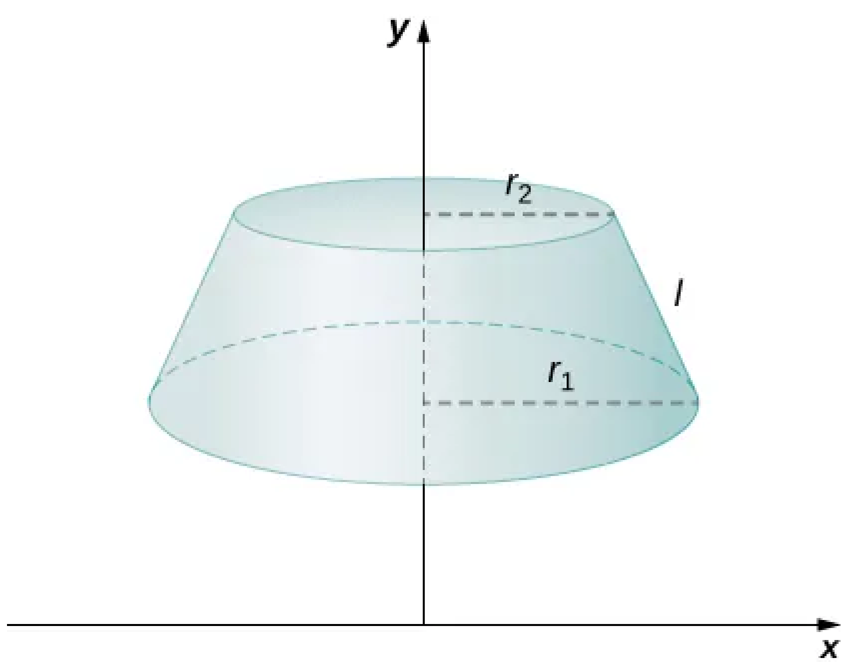


Figure 2.42 A frustum of a cone can approximate a small part of surface area.

We know the lateral surface area of a cone is given by

$$\text{Lateral Surface Area} = \pi r s,$$

where r is the radius of the base of the cone and s is the slant height (see the following figure).

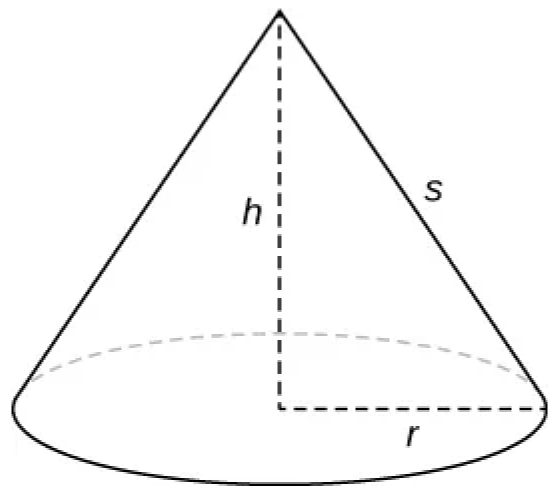


Figure 2.43 The lateral surface area of the cone is given by $\pi r s$.

Since a frustum can be thought of as a piece of a cone, the lateral surface area of the frustum is given by the lateral surface area of the whole cone less the lateral surface area of the smaller cone (the pointy tip) that was cut off (see the following figure).

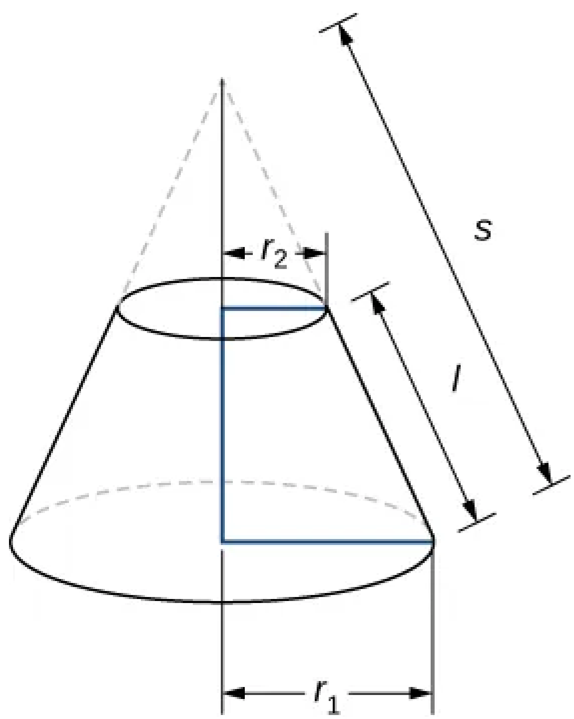


Figure 2.44 Calculating the lateral surface area of a frustum of a cone.

The cross-sections of the small cone and the large cone are similar triangles, so we see that

$$\frac{r_2}{r_1} = \frac{s-l}{s}.$$

Solving for s, we get

$$\begin{aligned} \frac{r_2}{r_1} &= \frac{s-l}{s} \\ r_2 s &= r_1 (s-l) \\ r_2 s &= r_1 s - r_1 l \\ r_1 l &= r_1 s - r_2 s \\ r_1 l &= (r_1 - r_2) s \\ \frac{r_1 l}{r_1 - r_2} &= s. \end{aligned}$$

Then the lateral surface area (SA) of the frustum is

$$\begin{aligned} S &= (\text{Lateral SA of large cone}) - (\text{Lateral SA of small cone}) \\ &= \pi r_1 s - \pi r_2 (s-l) \\ &= \pi r_1 \left(\frac{r_1 l}{r_1 - r_2}\right) - \pi r_2 \left(\frac{r_1 l}{r_1 - r_2} - l\right) \\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \pi r_2 l \\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_2 l (r_1 - r_2)}{r_1 - r_2} \\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_1 r_2 l}{r_1 - r_2} - \frac{\pi {r_2}^2 l}{r_1 - r_2} \\ &= \frac{\pi (r_1^2 - r_2^2) l}{r_1 - r_2} = \frac{\pi (r_1 - r_2)(r_1 + r_2) l}{r_1 - r_2} = \pi (r_1 + r_2) l. \end{aligned}$$

Let’s now use this formula to calculate the surface area of each of the bands formed by revolving the line segments around the $x\text{-axis}$. A representative band is shown in the following figure.

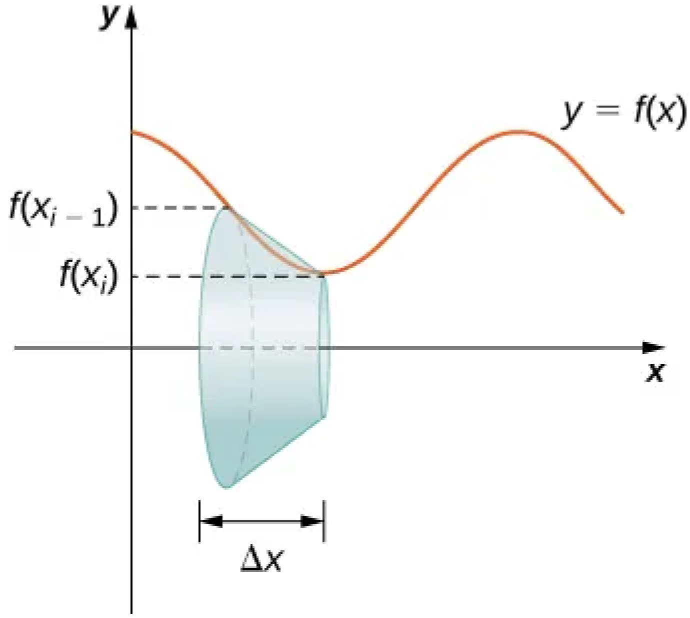


Figure 2.45 A representative band used for determining surface area.

Note that the slant height of this frustum is just the length of the line segment used to generate it. So, applying the surface area formula, we have

$$\begin{aligned} S &= \pi\left(r_1+r_2\right)l \\ &= \pi\left(f(x_{i-1})+f(x_i)\right)\sqrt{\Delta x^2+\left(\Delta y_i\right)^2} \\ &= \pi\left(f(x_{i-1})+f(x_i)\right)\Delta x\sqrt{1+\left(\frac{\Delta y_i}{\Delta x}\right)^2}. \end{aligned}$$

Now, as we did in the development of the arc length formula, we apply the Mean Value Theorem to select $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/\Delta x$. This gives us

$$S = \pi\left(f(x_{i-1})+f(x_i)\right)\Delta x\sqrt{1+\left(f'(x_i^*)\right)^2}.$$

Furthermore, since $f(x)$ is continuous, by the Intermediate Value Theorem, there is a point $x_i^{**} \in [x_{i-1}, x_i]$ such that $f(x_i^{**}) = (1/2)\left[f(x_{i-1})+f(x_i)\right]$, so we get

$$S = 2\pi f(x_i^{**})\Delta x\sqrt{1+\left(f'(x_i^*)\right)^2}.$$

Then the approximate surface area of the whole surface of revolution is given by

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1+\left(f'(x_i^*)\right)^2}.$$

This *almost* looks like a Riemann sum, except we have functions evaluated at two different points, x_i^* and x_i^{**}, over the interval $[x_{i-1}, x_i]$. Although we do not examine the details here, it turns out that because $f(x)$ is smooth, if we let $n \to \infty$, the limit works the same as a Riemann sum even with the two different evaluation points. This makes sense intuitively. Both x_i^* and x_i^{**} are in the interval $[x_{i-1}, x_i]$, so it makes sense that as $n \to \infty$, both x_i^* and x_i^{**} approach x. Those of you who are interested in the details should consult an advanced calculus text.

Taking the limit as $n \to \infty$, we get

$$\text{Surface Area} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1+(f'(x_i^*))^2} = \int_a^b \left(2\pi f(x)\sqrt{1+(f'(x))^2}\right) dx.$$

As with arc length, we can conduct a similar development for functions of y to get a formula for the surface area of surfaces of revolution about the $y\text{-axis}$. These findings are summarized in the following theorem.

THEOREM 2.6

Surface Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the *x*-axis is given by

$$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1+(f'(x))^2}\right) dx. \tag{2.9}$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c, d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the $y\text{-axis}$ is given by

$$\text{Surface Area} = \int_c^d \left(2\pi g(y)\sqrt{1+(g'(y))^2}\right) dy.$$

EXAMPLE 2.21

Calculating the Surface Area of a Surface of Revolution 1

Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the $x\text{-axis}$. Round the answer to three decimal places.

[Show/Hide Solution]

CHECKPOINT 2.21

Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the $x\text{-axis}$. Round the answer to three decimal places.

EXAMPLE 2.22

Calculating the Surface Area of a Surface of Revolution 2

Let $f(x) = y = \sqrt[3]{3x}$. Consider the portion of the curve where $0 \leq y \leq 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the $y\text{-axis}$.

[Show/Hide Solution]

CHECKPOINT 2.22

Let $g(y) = \sqrt{9-y^2}$ over the interval $y \in [0, 2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the $y\text{-axis}$.

Section 2.4 Exercises

For the following exercises, find the length of the functions over the given interval.

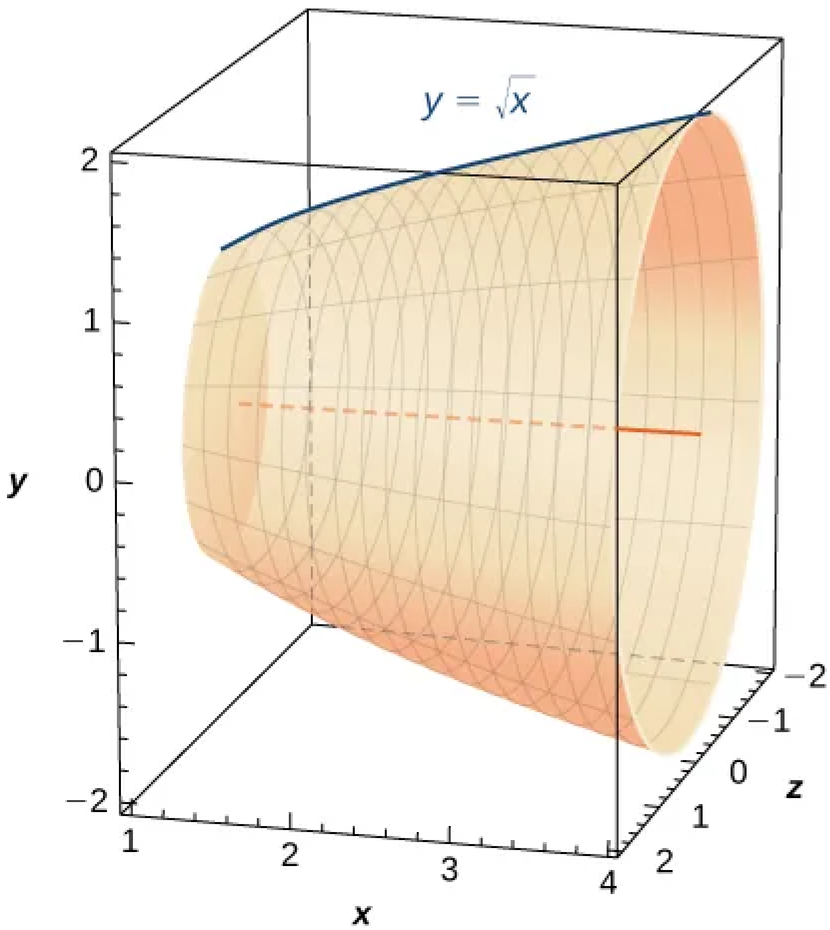
165. $y = 5x$ from $x = 0$ to $x = 2$

166. $y = -\frac{1}{2}x + 25$ from $x = 1$ to $x = 4$

167. $x = 4y$ from $y = -1$ to $y = 1$

168. Pick an arbitrary linear function $x = g(y)$ over any interval of your choice (y_1, y_2). Determine the length of the function and then prove the length is correct by using geometry.

169. Find the surface area of the volume generated when the curve $y = \sqrt{x}$ revolves around the $x\text{-axis}$ from $(1, 1)$ to $(4, 2)$, as seen here.



170. Find the surface area of the volume generated when the curve $y = x^2$ revolves around the

$y\text{-axis}$ from $(1,\ 1)$ to $(3, 9)$.

For the following exercises, find the lengths of the functions of x over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

171. $y = x^{3/2}$ from $(0, 0)$ to $(1, 1)$

172. $y = x^{2/3}$ from $(1, 1)$ to $(8, 4)$

173. $y = \frac{1}{3}\left(x^2 + 2\right)^{3/2}$ from $x = 0$ to $x = 1$

174. $y = \frac{1}{3}\left(x^2 - 2\right)^{3/2}$ from $x = 2$ to $x = 4$

175. **[T]** $y = e^x$ on $x = 0$ to $x = 1$

176. $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 3$

177. $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$

178. $y = \frac{2x^{3/2}}{3} - \frac{x^{1/2}}{2}$ from $x = 1$ to $x = 4$

179. $y = \frac{1}{27}\left(9x^2 + 6\right)^{3/2}$ from $x = 0$ to $x = 2$

180. **[T]** $y = \sin x$ on $x = 0$ to $x = \pi$

For the following exercises, find the lengths of the functions of y over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

181. $y = \frac{5-3x}{4}$ from $y = 0$ to $y = 4$

182. $x = \frac{1}{2}\left(e^y + e^{-y}\right)$ from $y = -1$ to $y = 1$

183. $x = 5y^{3/2}$ from $y = 0$ to $y = 1$

184. **[T]** $x = y^2$ from $y = 0$ to $y = 1$

185. $x = \sqrt{y}$ from $y = 0$ to $y = 1$

186. $x = \frac{2}{3}\left(y^2 + 1\right)^{3/2}$ from $y = 1$ to $y = 3$

187. **[T]** $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$

188. **[T]** $x = \cos^2 y$ from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$

189. **[T]** $x = 4^y$ from $y = 0$ to $y = 2$

190. **[T]** $x = \ln(y)$ on $y = \frac{1}{e}$ to $y = e$

For the following exercises, find the surface area of the volume generated when the following curves revolve around the $x\text{-axis}$. If you cannot evaluate the integral exactly, use your calculator to approximate it.

191. $y = \sqrt{x}$ from $x = 2$ to $x = 6$

192. $y = x^3$ from $x = 0$ to $x = 1$

193. $y = 7x$ from $x = -1$ to $x = 1$

194. [T] $y = \frac{1}{x^2}$ from $x = 1$ to $x = 3$

195. $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$

196. $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$

197. $y = 5x$ from $x = 1$ to $x = 5$

198. [T] $y = \tan x$ from $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$

For the following exercises, find the surface area of the volume generated when the following curves revolve around the $y\text{-axis}$. If you cannot evaluate the integral exactly, use your calculator to approximate it.

199. $y = x^2$ from $x = 0$ to $x = 2$

200. $y = \frac{1}{2}x^2 + \frac{1}{2}$ from $x = 0$ to $x = 1$

201. $y = x + 1$ from $x = 0$ to $x = 3$

202. [T] $y = \frac{1}{x}$ from $x = \frac{1}{2}$ to $x = 1$

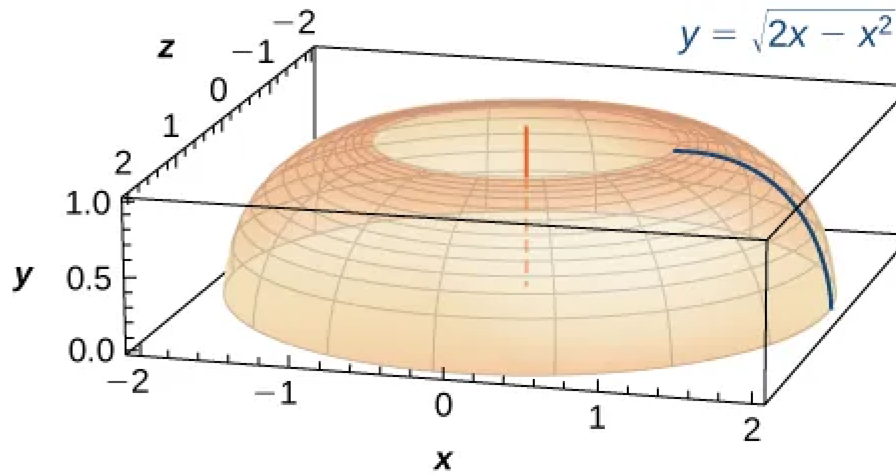
203. $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$

204. [T] $y = 3x^4$ from $x = 0$ to $x = 1$

205. [T] $y = \frac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$

206. [T] $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$

207. The base of a lamp is constructed by revolving a quarter circle $y = \sqrt{2x - x^2}$ around the $y\text{-axis}$ from $x = 1$ to $x = 2$, as seen here. Create an integral for the surface area of this



curve and compute it.

208. A light bulb is a sphere with radius $1/2$ in. with the bottom sliced off to fit exactly onto a cylinder of radius $1/4$ in. and length $1/3$ in., as seen here. The sphere is cut off at the bottom to fit exactly onto the cylinder, so the radius of the cut is $1/4$ in. Find the surface area (not

including the top or bottom of the cylinder).

209. **[T]** A lampshade is constructed by rotating $y = 1/x$ around the x-axis from $y = 1$ to $y = 2$, as seen here. Determine how much material you would need to construct this lampshade—that is, the surface area—accurate to four decimal places.

210. **[T]** An anchor drags behind a boat according to the function $y = 24e^{-x/2} - 24$, where y represents the depth beneath the boat and x is the horizontal distance of the anchor from the back of the boat. If the anchor is 23 ft below the boat, how much rope do you have to pull to reach the anchor? Round your answer to three decimal places.

211. **[T]** You are building a bridge that will span 10 ft. You intend to add decorative rope in the shape of $y = 5\left|\sin\left((x\pi)/5\right)\right|$, where x is the distance in feet from one end of the bridge. Find out how much rope you need to buy, rounded to the nearest foot.

For the following exercises, find the exact arc length for the following problems over the given interval.

212. $y = \ln(\sin x)$ from $x = \pi/4$ to $x = (3\pi)/4.$ (*Hint:* Recall trigonometric identities.)

213. Draw graphs of $y = x^2$, $y = x^6$, and $y = x^{10}$. For $y = x^n$, as n increases, formulate a prediction on the arc length from $(0, 0)$ to $(1, 1)$. Now, compute the lengths of these three functions and determine whether your prediction is correct.

214. Compare the lengths of the parabola $x = y^2$ and the line $x = by$ from $(0, 0)\text{ to }(b^2, b)$ as b increases. What do you notice?

215. Solve for the length of $x = y^2$ from $(0, 0)\text{ to }(1, 1)$. Show that $x = (1/2)\,y^2$ from $(0, 0)$ to $(2,\ 2)$ is twice as long. Graph both functions and explain why this is so.

216. **[T]** Which is longer between $(1,\ 1)$ and $(2, 1/2)$: the hyperbola $y = 1/x$ or the graph of $x + 2y = 3$?

217. Explain why the surface area is infinite when $y = 1/x$ is rotated around the $x\text{-axis}$ for $1 \le x < \infty$, but the volume is finite.

2.5 Physical Applications

Learning Objectives

2.5.1 Determine the mass of a one-dimensional object from its linear density function.
2.5.2 Determine the mass of a two-dimensional circular object from its radial density function.
2.5.3 Calculate the work done by a variable force acting along a line.
2.5.4 Calculate the work done in pumping a liquid from one height to another.
2.5.5 Find the hydrostatic force against a submerged vertical plate.

In this section, we examine some physical applications of integration. Let's begin with a look at calculating mass from a density function. We then turn our attention to work, and close the section with a study of hydrostatic force.

Mass and Density

We can use integration to develop a formula for calculating mass based on a density function. First we consider a thin rod or wire. Orient the rod so it aligns with the $x\text{-axis},$ with the left end of the rod at $x = a$ and the right end of the rod at $x = b$ (Figure 2.48). Note that although we depict the rod with some thickness in the figures, for mathematical purposes we assume the rod is thin enough to be treated as a one-dimensional object.

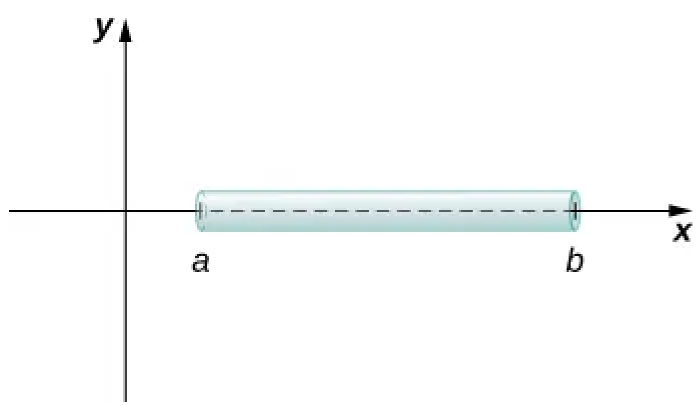


Figure 2.48 We can calculate the mass of a thin rod oriented along the $x\text{-axis}$ by integrating its density function.

If the rod has constant density ρ, given in terms of mass per unit length, then the mass of the rod is just the product of the density and the length of the rod: $(b - a)\,\rho$. If the density of the rod is not constant, however, the problem becomes a little more challenging. When the density of the rod varies from point to point, we use a linear **density function**, $\rho(x)$, to denote the density of the rod at any point, x. Let $\rho(x)$ be an integrable linear density function. Now, for $i = 0, 1, 2, \ldots, n$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure 2.49 shows a representative segment of the rod.

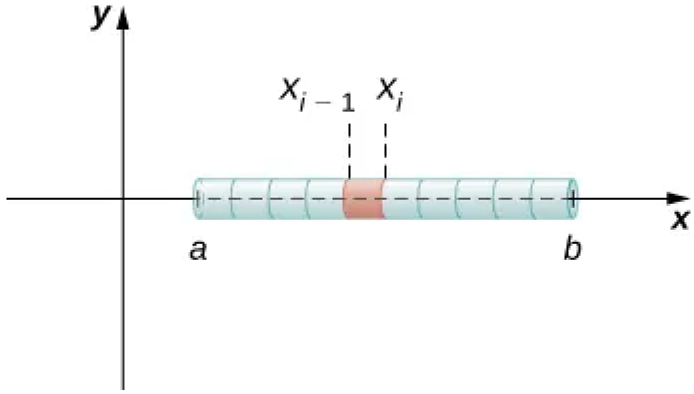


Figure 2.49 A representative segment of the rod.

The mass m_i of the segment of the rod from x_{i-1} to x_i is approximated by

$$m_i \approx \rho(x_i^*)\,(x_i - x_{i-1}) = \rho(x_i^*)\Delta x.$$

Adding the masses of all the segments gives us an approximation for the mass of the entire rod:

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we get an expression for the exact mass of the rod:

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x = \int_a^b \rho(x)dx.$$

We state this result in the following theorem.

THEOREM 2.7

Mass–Density Formula of a One-Dimensional Object

Given a thin rod oriented along the $x\text{-axis}$ over the interval $[a, b]$, let $\rho(x)$ denote a linear density function giving the density of the rod at a point *x* in the interval. Then the mass of the rod is given by

$$m = \int_a^b \rho(x)dx. \qquad (2.10)$$

We apply this theorem in the next example.

EXAMPLE 2.23

Calculating Mass from Linear Density

Consider a thin rod oriented on the *x*-axis over the interval $[\pi/2, \pi]$. If the density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

[Show/Hide Solution]

CHECKPOINT 2.23

Consider a thin rod oriented on the *x*-axis over the interval $[1, 3]$. If the density of the rod is given by $\rho(x) = 2x^2 + 3$, what is the mass of the rod?

We now extend this concept to find the mass of a two-dimensional disk of radius r. As with the rod we looked at in the one-dimensional case, here we assume the disk is thin enough that, for mathematical purposes, we can treat it as a two-dimensional object. We assume the density is given in terms of mass per unit area (called *area density*), and further assume the density varies only along the disk's radius (called *radial density*). We orient the disk in the $xy\text{-plane}$, with the center at the origin. Then, the density of the disk can be treated as a function of x, denoted $\rho(x)$. We assume $\rho(x)$ is integrable. Because density is a function of x, we partition the interval from $[0, r]$ along the $x\text{-axis}$. For $i = 0, 1, 2, ..., n$, let $P = \{x_i\}$ be a regular partition of the interval $[0, r]$, and for $i = 1, 2, ..., n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Now, use the partition to break up the disk into thin (two-dimensional) washers. A disk and a representative washer are depicted in the following figure.

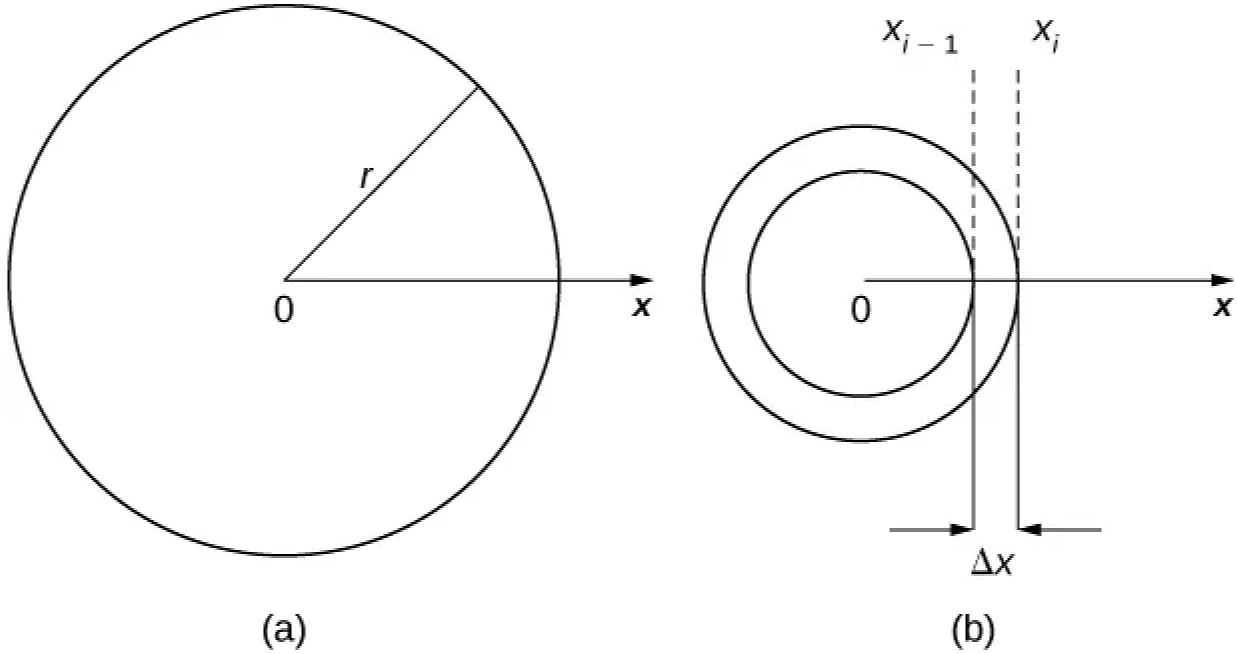


Figure 2.50 (a) A thin disk in the *xy*-plane. (b) A representative washer.

We now approximate the density and area of the washer to calculate an approximate mass, m_i. Note that the area of the washer is given by

$$\begin{aligned} A_i &= \pi(x_i)^2 - \pi(x_{i-1})^2 \\ &= \pi\left[x_i^2 - x_{i-1}^2\right] \\ &= \pi\left(x_i + x_{i-1}\right)\left(x_i - x_{i-1}\right) \\ &= \pi(x_i + x_{i-1})\Delta x. \end{aligned}$$

You may recall that we had an expression similar to this when we were computing volumes by shells. As we did there, we use $x_i^* \approx (x_i + x_{i-1})/2$ to approximate the average radius of the washer. We obtain

$$A_i = \pi(x_i + x_{i-1})\Delta x \approx 2\pi x_i^* \Delta x.$$

Using $\rho(x_i^*)$ to approximate the density of the washer, we approximate the mass of the washer by

$$m_i \approx 2\pi x_i^* \rho(x_i^*)\Delta x.$$

Adding up the masses of the washers, we see the mass m of the entire disk is approximated by

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x.$$

We again recognize this as a Riemann sum, and take the limit as $n \to \infty$. This gives us

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*) \Delta x = \int_0^r 2\pi x \rho(x) dx.$$

We summarize these findings in the following theorem.

THEOREM 2.8

Mass–Density Formula of a Circular Object

Let $\rho(x)$ be an integrable function representing the radial density of a disk of radius r. Then the mass of the disk is given by

$$m = \int_0^r 2\pi x \rho(x) dx. \tag{2.11}$$

EXAMPLE 2.24

Calculating Mass from Radial Density

Let $\rho(x) = \sqrt{x}$ represent the radial density of a disk. Calculate the mass of a disk of radius 4.

[Show/Hide Solution]

CHECKPOINT 2.24

Let $\rho(x) = 3x + 2$ represent the radial density of a disk. Calculate the mass of a disk of radius 2.

Work Done by a Force

We now consider work. In physics, work is related to force, which is often intuitively defined as a push or pull on an object. When a force moves an object, we say the force does work on the object. In other words, work can be thought of as the amount of energy it takes to move an object. According to physics, when we have a constant force, work can be expressed as the product of force and distance.

In the English system, the unit of force is the pound and the unit of distance is the foot, so work is given in foot-pounds. In the metric system, kilograms and meters are used. One newton is the force needed to accelerate 1 kilogram of mass at the rate of 1 m/sec^2. Thus, the most common unit of work is the newton-meter. This same unit is also called the *joule*. Both are defined as kilograms times meters squared over seconds squared $\left(\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2\right)$.

When we have a constant force, things are pretty easy. It is rare, however, for a force to be constant. The work done to compress (or elongate) a spring, for example, varies depending on how far the spring has already been compressed (or stretched). We look at springs in more detail later in this section.

Suppose we have a variable force $F(x)$ that moves an object in a positive direction along the *x*-axis from point a to point b. To calculate the work done, we partition the interval $[a, b]$ and estimate the work done over each subinterval. So, for $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. To calculate the work done to move an object from point x_{i-1} to point x_i, we assume the force is roughly constant over the interval, and use $F(x_i^*)$ to approximate the force. The work done over the interval $[x_{i-1}, x_i]$, then, is given by

$$W_i \approx F(x_i^*)\,(x_i - x_{i-1}) = F(x_i^*)\Delta x.$$

Therefore, the work done over the interval $[a, b]$ is approximately

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

Taking the limit of this expression as $n \to \infty$ gives us the exact value for work:

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i^*)\Delta x = \int_a^b F(x)dx.$$

Thus, we can define work as follows.

DEFINITION

If a variable force $F(x)$ moves an object in a positive direction along the *x*-axis from point *a* to point *b*, then the **work** done on the object is

$$W = \int_a^b F(x)dx. \tag{2.12}$$

Note that if *F* is constant, the integral evaluates to $F \cdot (b - a) = F \cdot d$, which is the formula we stated at the beginning of this section.

Now let's look at the specific example of the work done to compress or elongate a spring. Consider a block attached to a horizontal spring. The block moves back and forth as the spring stretches and compresses. Although in the real world we would have to account for the force of friction between the block and the surface on which it is resting, we ignore friction here and assume the block is resting on a frictionless surface. When the spring is at its natural length (at rest), the system is said to be at equilibrium. In this state, the spring is neither elongated nor compressed, and in this equilibrium position the block does not move until some force is introduced. We orient the system such that $x = 0$ corresponds to the equilibrium position (see the following figure).

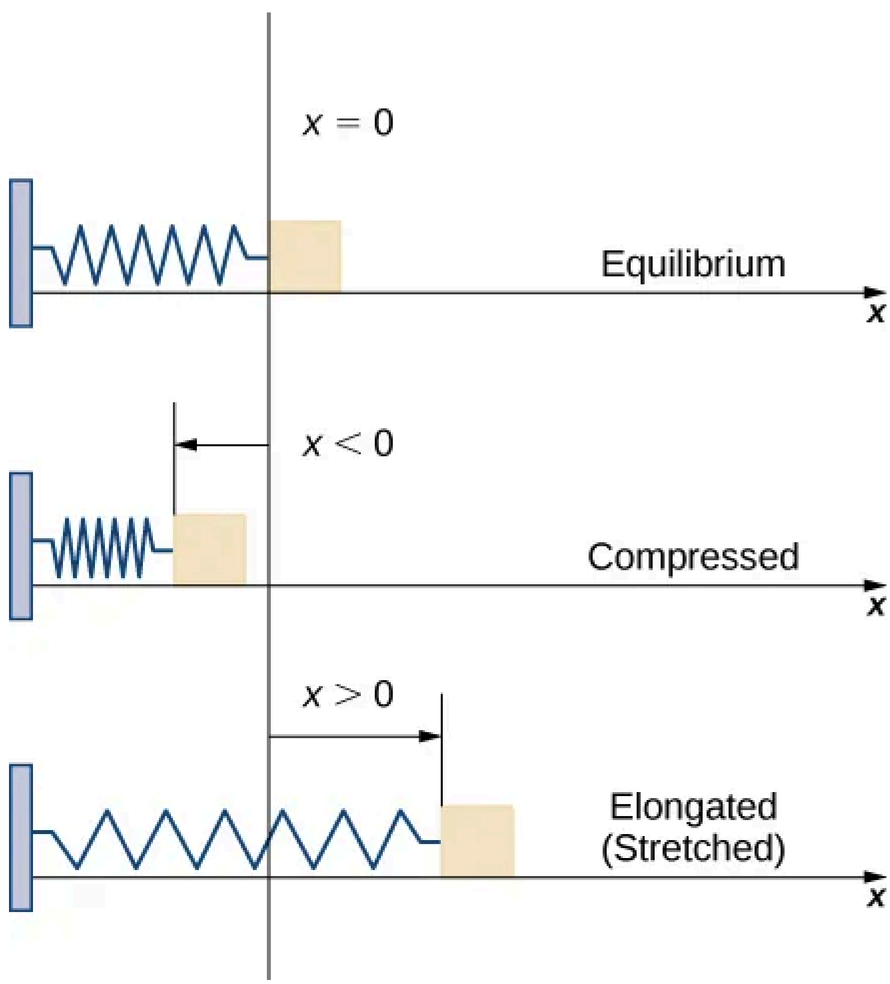


Figure 2.51 A block attached to a horizontal spring at equilibrium, compressed, and elongated.

According to **Hooke's law**, the force required to compress or stretch a spring from an equilibrium position is given by $F(x) = kx$, for some constant k. The value of k depends on the physical characteristics of the spring. The constant k is called the *spring constant* and is always positive. We can use this information to calculate the work done to compress or elongate a spring, as shown in the following example.

EXAMPLE 2.25

The Work Required to Stretch or Compress a Spring

Suppose it takes a force of 10 N (in the negative direction) to compress a spring 0.2 m from the equilibrium position. How much work is done to stretch the spring 0.5 m from the equilibrium position?

[Show/Hide Solution]

CHECKPOINT 2.25

Suppose it takes a force of 8 lb to stretch a spring 6 in. from the equilibrium position. How much work is done to stretch the spring 1 ft from the equilibrium position?

Work Done in Pumping

Consider the work done to pump water (or some other liquid) out of a tank. Pumping problems are a little more complicated than spring problems because many of the calculations depend on the shape and size of the tank. In addition, instead of being concerned about the work done to move a single mass, we are looking at the work done to move a volume of water, and it takes more work to move the water from the bottom of the tank than it does to move the water from the top of the tank.

We examine the process in the context of a cylindrical tank, then look at a couple of examples using tanks of different shapes. Assume a cylindrical tank of radius 4 m and height 10 m is filled to a depth of 8 m. How much work does it take to pump all the water over the top edge of the tank?

The first thing we need to do is define a frame of reference. We let x represent the vertical distance below the top of the tank. That is, we orient the $x\text{-axis}$ vertically, with the origin at the top of the tank and the downward direction being positive (see the following figure).

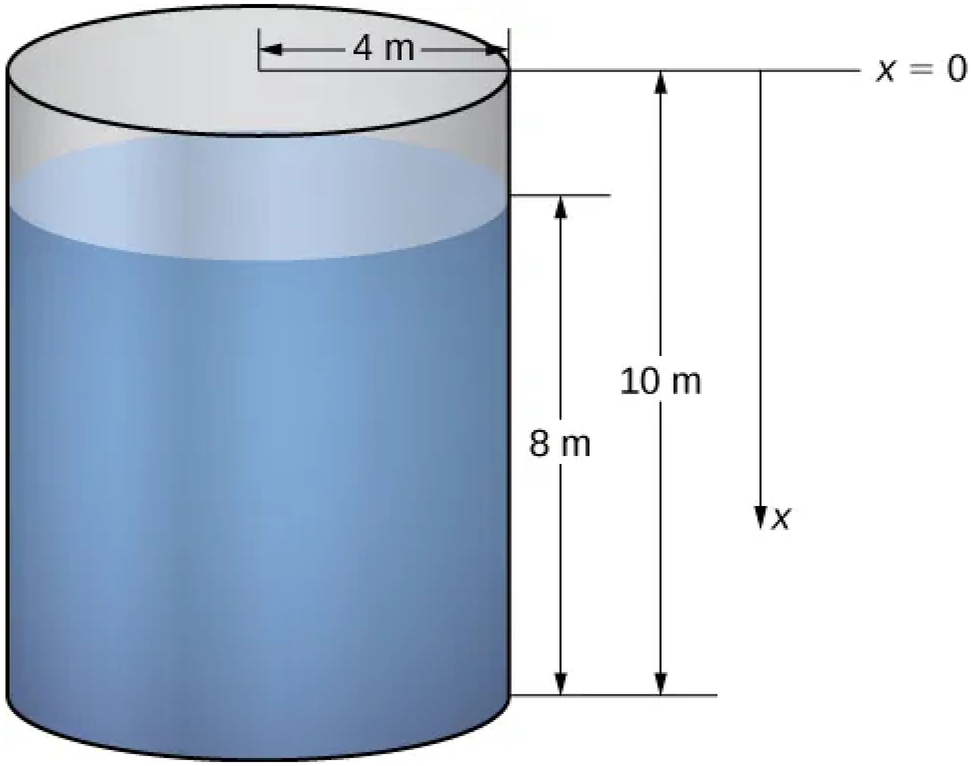


Figure 2.52 How much work is needed to empty a tank partially filled with water?

Using this coordinate system, the water extends from $x = 2$ to $x = 10$. Therefore, we partition the interval $[2,\ 10]$ and look at the work required to lift each individual "layer" of water. So, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[2,\ 10]$, and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure 2.53 shows a representative layer.

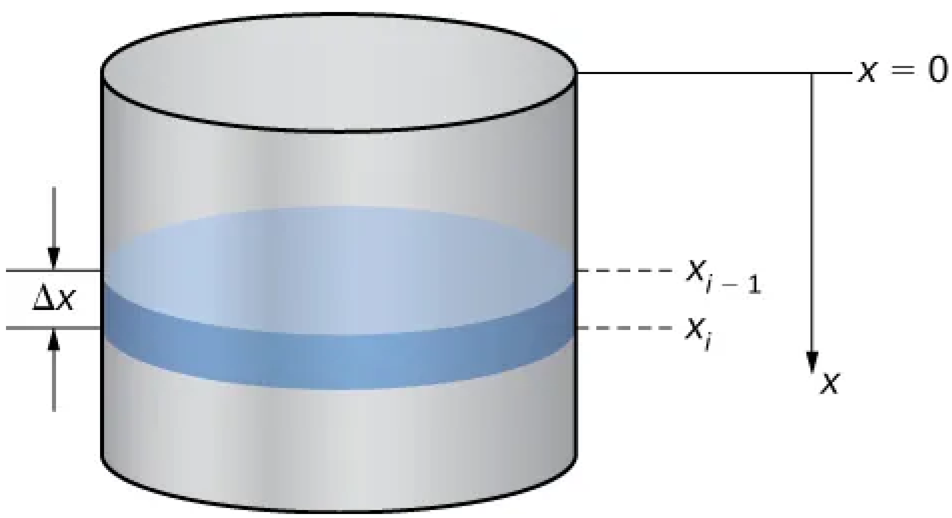


Figure 2.53 A representative layer of water.

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. Given that the weight-density of water is 9800 N/m^3, or 62.4 lb/ft^3, calculating the volume of each layer gives us the weight. In this case, we have

$$V = \pi(4)^2 \Delta x = 16\pi \Delta x.$$

Then, the force needed to lift each layer is

$$F = 9800 \cdot 16\pi \Delta x = 156{,}800\pi \Delta x.$$

Note that this step becomes a little more difficult if we have a noncylindrical tank. We look at a noncylindrical tank in the next example.

We also need to know the distance the water must be lifted. Based on our choice of coordinate systems, we can use x_i^* as an approximation of the distance the layer must be lifted. Then the work to lift the $i\text{th}$ layer of water W_i is approximately

$$W_i \approx 156{,}800\pi x_i^* \Delta x.$$

Adding the work for each layer, we see the approximate work to empty the tank is given by

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 156{,}800\pi x_i^* \Delta x.$$

This is a Riemann sum, so taking the limit as $n \to \infty$, we get

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 156{,}800\pi x_i^* \Delta x \\ &= 156{,}800\pi \int_2^{10} x\,dx \\ &= 156{,}800\pi \left[\frac{x^2}{2}\right]\Big|_2^{10} = 7{,}526{,}400\pi \approx 23{,}644{,}883. \end{aligned}$$

The work required to empty the tank is approximately 23,650,000 J.

For pumping problems, the calculations vary depending on the shape of the tank or container. The following problem-solving strategy lays out a step-by-step process for solving pumping problems.

PROBLEM-SOLVING STRATEGY

Solving Pumping Problems

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.

4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to get an estimate of the work needed to lift the layer of water.
6. Sum the work required to lift all the layers. This expression is an estimate of the work required to pump out the desired amount of water, and it is in the form of a Riemann sum.
7. Take the limit as $n \to \infty$ and evaluate the resulting integral to get the exact work required to pump out the desired amount of water.

We now apply this problem-solving strategy in an example with a noncylindrical tank.

EXAMPLE 2.26

A Pumping Problem with a Noncylindrical Tank

Assume a tank in the shape of an inverted cone, with height 12 ft and base radius 4 ft. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4 ft. How much work is required to pump out that amount of water?

[Show/Hide Solution]

CHECKPOINT 2.26

A tank is in the shape of an inverted cone, with height 10 ft and base radius 6 ft. The tank is filled to a depth of 8 ft to start with, and water is pumped over the upper edge of the tank until 3 ft of water remain in the tank. How much work is required to pump out that amount of water?

Hydrostatic Force and Pressure

In this last section, we look at the force and pressure exerted on an object submerged in a liquid. In the English system, force is measured in pounds. In the metric system, it is measured in newtons. Pressure is force per unit area, so in the English system we have pounds per square foot (or, perhaps more commonly, pounds per square inch, denoted psi). In the metric system we have newtons per square meter, also called *pascals*.

Let's begin with the simple case of a plate of area A submerged horizontally in water at a depth *s* (Figure 2.56). Then, the force exerted on the plate is simply the weight of the water above it, which is given by $F = \rho As$, where ρ is the weight density of water (weight per unit volume). To find the **hydrostatic pressure**—that is, the pressure exerted by water on a submerged object—we divide the force by the area. So the pressure is $p = F/A = \rho s$.

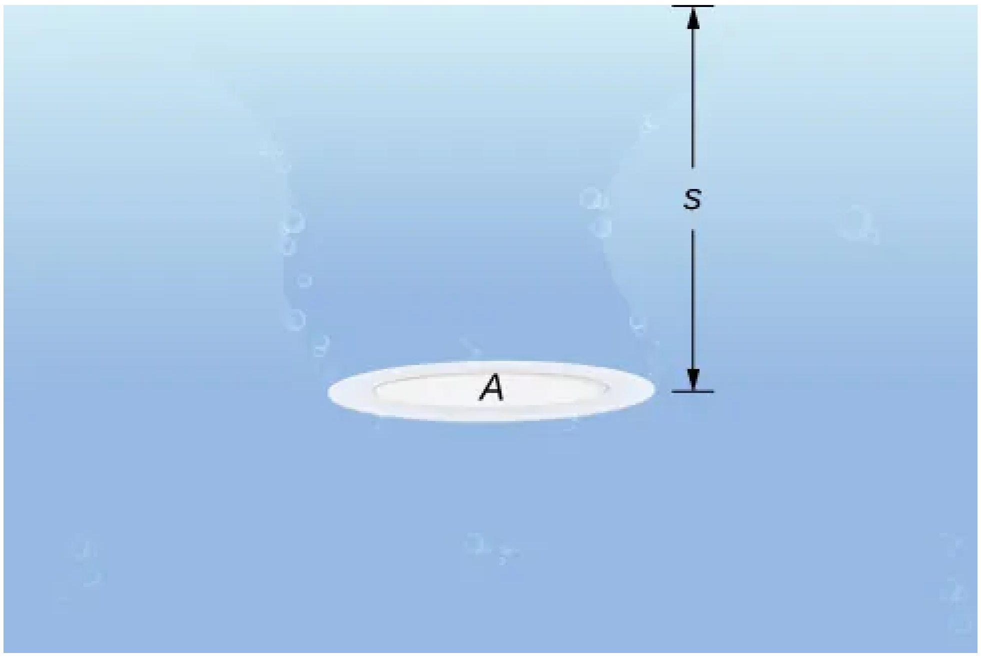


Figure 2.56 A plate submerged horizontally in water.

By Pascal's principle, the pressure at a given depth is the same in all directions, so it does not matter if the plate is submerged horizontally or vertically. So, as long as we know the depth, we know the pressure. We can apply Pascal's principle to find the force exerted on surfaces, such as dams, that are oriented vertically. We cannot apply the formula $F = \rho As$ directly, because the depth varies from point to point on a vertically oriented surface. So, as we have done many times before, we form a partition, a Riemann sum, and, ultimately, a definite integral to calculate the force.

Suppose a thin plate is submerged in water. We choose our frame of reference such that the *x*-axis is oriented vertically, with the downward direction being positive, and point $x = 0$ corresponding to a logical reference point. Let $s(x)$ denote the depth at point *x*. Note we often let $x = 0$ correspond to the surface of the water. In this case, depth at any point is simply given by $s(x) = x$. However, in some cases we may want to select a different reference point for $x = 0$, so we proceed with the development in the more general case. Last, let $w(x)$ denote the width of the plate at the point x.

Assume the top edge of the plate is at point $x = a$ and the bottom edge of the plate is at point $x = b$. Then, for $i = 0, 1, 2,..., n$, let $P = \{x_i\}$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2,..., n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. The partition divides the plate into several thin, rectangular strips (see the following figure).

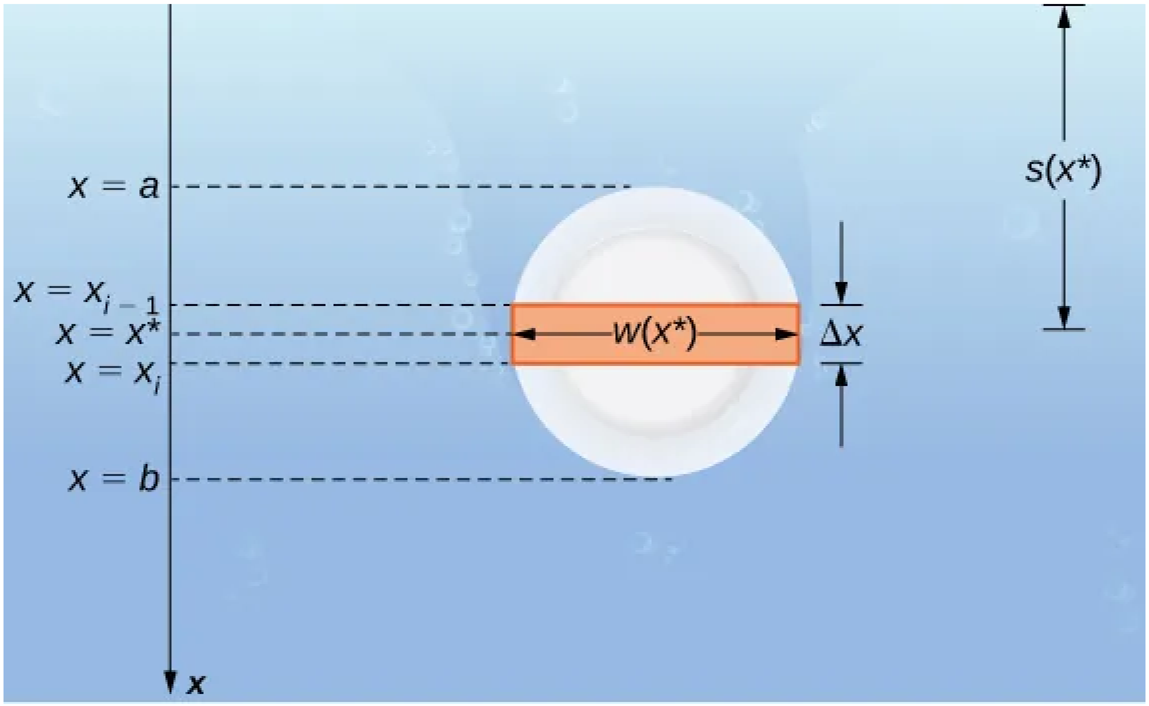


Figure 2.57 A thin plate submerged vertically in water.

Let's now estimate the force on a representative strip. If the strip is thin enough, we can treat it as if it is at a constant depth, $s(x_i^*)$. We then have

$$F_i = \rho As = \rho \left[w(x_i^*)\Delta x\right] s(x_i^*).$$

Adding the forces, we get an estimate for the force on the plate:

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \rho \left[w(x_i^*)\Delta x\right] s(x_i^*).$$

This is a Riemann sum, so taking the limit gives us the exact force. We obtain

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \left[w(x_i^*)\Delta x\right] s(x_i^*) = \int_a^b \rho w(x)s(x)dx. \tag{2.13}$$

Evaluating this integral gives us the force on the plate. We summarize this in the following problem-solving strategy.

PROBLEM-SOLVING STRATEGY

Finding Hydrostatic Force

1. Sketch a picture and select an appropriate frame of reference. (Note that if we select a frame of reference other than the one used earlier, we may have to adjust Equation 2.13 accordingly.)
2. Determine the depth and width functions, $s(x)$ and $w(x)$.
3. Determine the weight-density of whatever liquid with which you are working. The weight-density of water is 62.4 lb/ft^3, or 9800 N/m^3.
4. Use the equation to calculate the total force.

EXAMPLE 2.27

Finding Hydrostatic Force

A water trough 15 ft long has ends shaped like inverted isosceles triangles, with base 8 ft and height 3 ft. Find the force on one end of the trough if the trough is full of water.

[Show/Hide Solution]

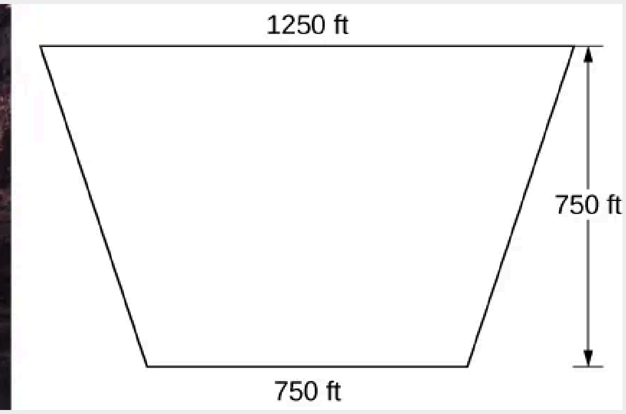
CHECKPOINT 2.27

A water trough 12 m long has ends shaped like inverted isosceles triangles, with base 6 m and height 4 m. Find the force on one end of the trough if the trough is full of water.

EXAMPLE 2.28

Chapter Opener: Finding Hydrostatic Force

We now return our attention to the Hoover Dam, mentioned at the beginning of this chapter. The actual dam is arched, rather than flat, but we are going to make some simplifying assumptions to help us with the calculations. Assume the face of the Hoover Dam is shaped like an isosceles trapezoid with lower base 750 ft, upper base 1250 ft, and height 750 ft (see the following figure).



When the reservoir is full, Lake Mead's maximum depth is about 530 ft, and the surface of the lake is about 10 ft below the top of the dam (see the following figure).

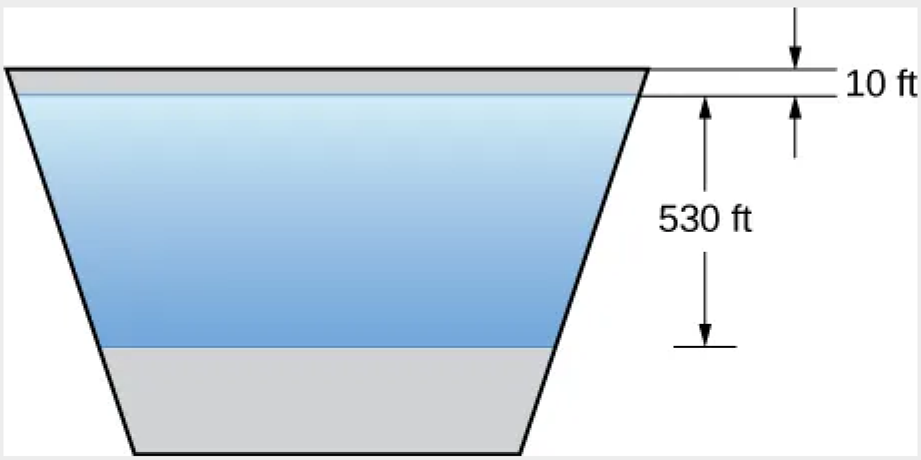


Figure 2.59 A simplified model of the Hoover Dam with assumed dimensions.

a. Find the force on the face of the dam when the reservoir is full.
b. The southwest United States has been experiencing a drought, and the surface of Lake Mead is about 125 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

[Show/Hide Solution]

CHECKPOINT 2.28

When the reservoir is at its average level, the surface of the water is about 50 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

MEDIA

To learn more about Hoover Dam, see this article published by the History Channel.

Section 2.5 Exercises

For the following exercises, find the work done.

218. Find the work done when a constant force $F = 12$ lb moves a chair from $x = 0.9$ to $x = 1.1$ ft.

219. How much work is done when a person lifts a 50 lb box of comics onto a truck that is 3 ft off the ground?

220. What is the work done lifting a 20 kg child from the floor to a height of 2 m? (Note that a mass of1 kg weighs 9.8 N near the surface of the Earth.)

221. Find the work done when you push a box along the floor 2 m, when you apply a constant force of $F = 100\ \text{N}$.

222. Compute the work done for a force $F = 12/x^2$ N from $x = 1$ to $x = 2$ m.

223. What is the work done moving a particle from $x = 0$ to $x = 1$ m if the force acting on it is $F = 3x^2$ N?

For the following exercises, find the mass of the one-dimensional object.

224. A wire that is 2 ft long (starting at $x = 0$) and has a density function of $\rho(x) = x^2 + 2x$ lb/ft

225. A car antenna that is 3 ft long (starting at $x = 0$) and has a density function of $\rho(x) = 3x + 2$ lb/ft

226. A metal rod that is 8 in. long (starting at $x = 0$) and has a density function of $\rho(x) = e^{1/2x}$ lb/in.

227. A pencil that is 4 in. long (starting at $x = 2$) and has a density function of $\rho(x) = 5/x$ oz/in.

228. A ruler that is 12 in. long (starting at $x = 5$) and has a density function of $\rho(x) = \ln(x) + (1/2)x^2$ oz/in.

For the following exercises, find the mass of the two-dimensional object that is centered at the origin.

229. An oversized hockey puck of radius 2 in. with density function $\rho\left(x\right) = x^3 - 2x + 5$

230. A frisbee of radius 6 in. with density function $\rho\left(x\right) = e^{-x}$

231. A plate of radius 10 in. with density function $\rho\left(x\right) = 1 + \cos\left(\pi x\right)$

232. A jar lid of radius 3 in. with density function $\rho(x) = \ln(x+1)$

233. A disk of radius 5 cm with density function $\rho\left(x\right) = \sqrt{3x}$

234. A 12-in. spring is stretched to 15 in. by a force of 75 lb. What is the spring constant?

235. A spring has a natural length of 10 cm. It takes 2 J to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm?

236. A 1-m spring requires 10 J to stretch the spring to 1.1 m. How much work would it take to stretch the spring from 1 m to 1.2 m?

237. A spring requires 5 J to stretch the spring from 8 cm to 12 cm, and an additional 4 J to stretch the spring from 12 cm to 14 cm. What is the natural length of the spring?

238. A shock absorber is compressed 1 in. by a weight of 1 t. What is the spring constant?

239. A force of $F = 20x - x^3$ N stretches a nonlinear spring by x meters. What work is required to stretch the spring from $x = 0$ to $x = 2$ m?

240. Find the work done by winding up a hanging cable of length 100 ft and weight-density 5 lb/ft.

241. For the cable in the preceding exercise, how much work is done to lift the cable 50 ft?

242. For the cable in the preceding exercise, how much additional work is done by hanging a 200 lb weight at the end of the cable?

243. **[T]** A pyramid of height 500 ft has a square base 800 ft by 800 ft. Find the area A at height h. If the rock used to build the pyramid weighs approximately $w = 100\,\text{lb/ft}^3$, how much work did it take to lift all the rock?

244. **[T]** For the pyramid in the preceding exercise, assume there were 1000 workers each working 10 hours a day, 5 days a week, 50 weeks a year. If the workers, on average, lifted 10 100 lb rocks 2 ft/hr, how long did it take to build the pyramid?

245. **[T]** The force of gravity on a mass m is $F = -\left(\left(GMm\right)/x^2\right)$ newtons. For a rocket of mass $m = 1000\,\text{kg}$, compute the work to lift the rocket from $x = 6400$ to $x = 6500$ km.

State your answers with three significant figures. (*Note*: $G = 6.67 \times 10^{-11}\,\text{N m}^2/\text{kg}^2$ and $M = 6 \times 10^{24}\,\text{kg.}$)

246. **[T]** For the rocket in the preceding exercise, find the work to lift the rocket from $x = 6400$ to $x = \infty$.

247. **[T]** A rectangular dam is 40 ft high and 60 ft wide. Assume the weight density of water is 62.5 lbs/ft^3. Compute the total force F on the dam when

a. the surface of the water is at the top of the dam and
b. the surface of the water is halfway down the dam.

248. **[T]** Find the work required to pump all the water out of a cylinder that has a circular base of radius 5 ft and height 200 ft. Use the fact that the density of water is 62 lb/ft^3.

249. **[T]** Find the work required to pump all the water out of the cylinder in the preceding exercise if the cylinder is only half full.

250. **[T]** How much work is required to pump out a swimming pool if the area of the base is 800 ft^2, the water is 4 ft deep, and the top is 1 ft above the water level? Assume that the density of water is 62 lb/ft^3.

251. A cylinder of depth H and cross-sectional area A stands full of water at density ρ. Compute the work to pump all the water to the top.

252. For the cylinder in the preceding exercise, compute the work to pump all the water to the top if the cylinder is only half full.

253. A cone-shaped tank has a cross-sectional area that increases with its depth: $A = \left(\pi r^2 h^2\right)/H^3$. Show that the work to empty it is half the work for a cylinder with the same height and base.

2.6 Moments and Centers of Mass

Learning Objectives

2.6.1 Find the center of mass of objects distributed along a line.
2.6.2 Locate the center of mass of a thin plate.
2.6.3 Use symmetry to help locate the centroid of a thin plate.
2.6.4 Apply the theorem of Pappus for volume.

In this section, we consider centers of mass (also called *centroids*, under certain conditions) and moments. The basic idea of the center of mass is the notion of a balancing point. Many of us have seen performers who spin plates on the ends of sticks. The performers try to keep several of them spinning without allowing any of them to drop. If we look at a single plate (without spinning it), there is a sweet spot on the plate where it balances perfectly on the stick. If we put the stick anywhere other than that sweet spot, the plate does not balance and it falls to the ground. (That is why performers spin the plates; the spin helps keep the plates from falling even if the stick is not exactly in the right place.) Mathematically, that sweet spot is called the *center of mass of the plate*.

In this section, we first examine these concepts in a one-dimensional context, then expand our development to consider centers of mass of two-dimensional regions and symmetry. Last, we use centroids to find the volume of certain solids by applying the theorem of Pappus.

Center of Mass and Moments

Let's begin by looking at the center of mass in a one-dimensional context. Consider a long, thin wire or rod of negligible mass resting on a fulcrum, as shown in Figure 2.62(a). Now suppose we place objects having masses m_1 and m_2 at distances d_1 and d_2 from the fulcrum, respectively, as shown in Figure 2.62(b).

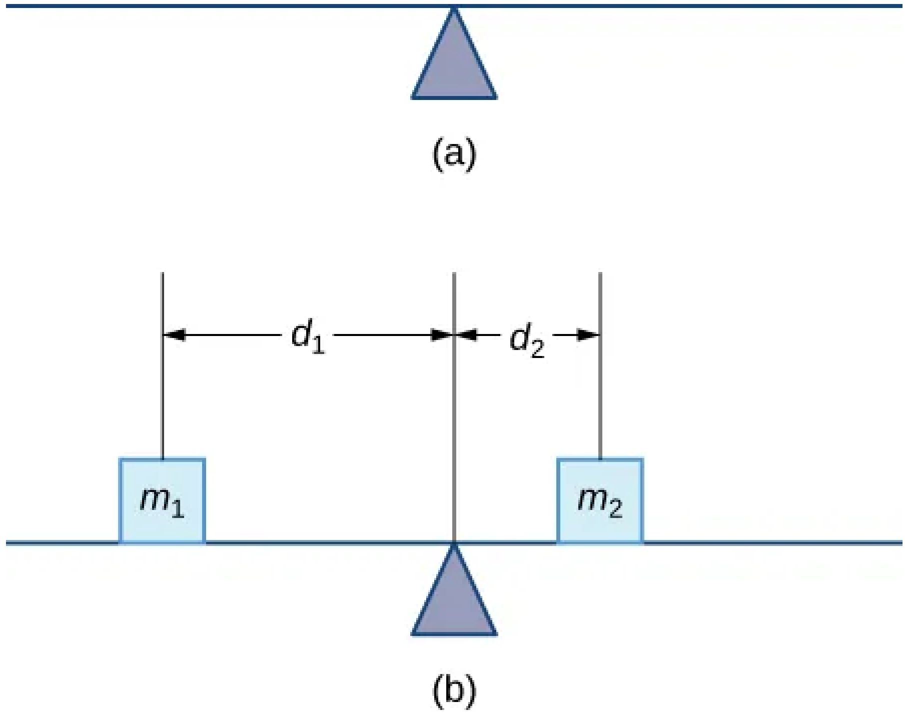


Figure 2.62 (a) A thin rod rests on a fulcrum. (b) Masses are placed on the rod.

The most common real-life example of a system like this is a playground seesaw, or teeter-totter, with children of different weights sitting at different distances from the center. On a seesaw, if one child sits at each end, the heavier child sinks down and the lighter child is lifted into the air. If the heavier child slides in toward the center, though, the seesaw balances. Applying this concept to the masses on the rod, we note that the masses balance each other if and only if $m_1 d_1 = m_2 d_2$.

In the seesaw example, we balanced the system by moving the masses (children) with respect to the fulcrum. However, we are really interested in systems in which the masses are not allowed to move, and instead we balance the system by moving the fulcrum. Suppose we have two point masses, m_1 and m_2, located on a number line at points x_1 and x_2, respectively (Figure 2.63). The center of mass, $\bar{x}$, is the point where the fulcrum should be placed to make the system balance.

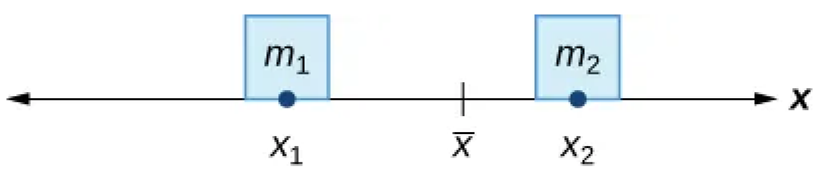


Figure 2.63 The center of mass $\bar{x}$ is the balance point of the system.

Thus, we have

$$\begin{aligned} m_1 \left| \bar{x} - x_1 \right| &= m_2 \left| x_2 - \bar{x} \right| \\ m_1 \left(\bar{x} - x_1 \right) &= m_2 \left(x_2 - \bar{x} \right) \\ m_1 \bar{x} - m_1 x_1 &= m_2 x_2 - m_2 \bar{x} \\ \bar{x} \left(m_1 + m_2 \right) &= m_1 x_1 + m_2 x_2 \\ \bar{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \end{aligned}$$

The expression in the numerator, $m_1x_1 + m_2x_2$, is called the *first moment of the system with respect to the origin.* If the context is clear, we often drop the word *first* and just refer to this expression as the **moment** of the system. The expression in the denominator, $m_1 + m_2$, is the total mass of the system. Thus, the **center of mass** of the system is the point at which the total mass of the system could be concentrated without changing the moment.

This idea is not limited just to two point masses. In general, if *n* masses, $m_1, m_2, \ldots, m_n$, are placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, then the center of mass of the system is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.$$

THEOREM 2.9

Center of Mass of Objects on a Line

Let $m_1, m_2, \ldots, m_n$ be point masses placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then, the moment of the system with respect to the origin is given by

$$M = \sum_{i=1}^{n} m_i x_i \tag{2.14}$$

and the center of mass of the system is given by

$$\bar{x} = \frac{M}{m}. \tag{2.15}$$

We apply this theorem in the following example.

EXAMPLE 2.29

Finding the Center of Mass of Objects along a Line

Suppose four point masses are placed on a number line as follows:

$$m_1 = 30\,\text{kg, placed at } x_1 = -2\,\text{m} \qquad m_2 = 5\,\text{kg, placed at } x_2 = 3\,\text{m}$$
$$m_3 = 10\,\text{kg, placed at } x_3 = 6\,\text{m} \qquad m_4 = 15\,\text{kg, placed at } x_4 = -3\,\text{m}.$$

Find the moment of the system with respect to the origin and find the center of mass of the system.

[Show/Hide Solution]

CHECKPOINT 2.29

Suppose four point masses are placed on a number line as follows:

$$m_1 = 12\,\text{kg, placed at } x_1 = -4\,\text{m} \qquad m_2 = 12\,\text{kg, placed at } x_2 = 4\,\text{m}$$
$$m_3 = 30\,\text{kg, placed at } x_3 = 2\,\text{m} \qquad m_4 = 6\,\text{kg, placed at } x_4 = -6\,\text{m}.$$

Find the moment of the system with respect to the origin and find the center of mass of the system.

We can generalize this concept to find the center of mass of a system of point masses in a plane. Let m_1 be a point mass located at point (x_1, y_1) in the plane. Then the moment M_x of the mass with respect to the *x*-axis is given by $M_x = m_1 y_1$. Similarly, the moment M_y with respect to the *y*-axis is given by $M_y = m_1 x_1$. Notice that the *x*-coordinate of the point is used to calculate the moment with respect to the *y*-axis, and vice versa. The reason is that the *x*-coordinate gives the distance from the point mass to the *y*-axis, and the *y*-coordinate gives the distance to the *x*-axis (see the following figure).

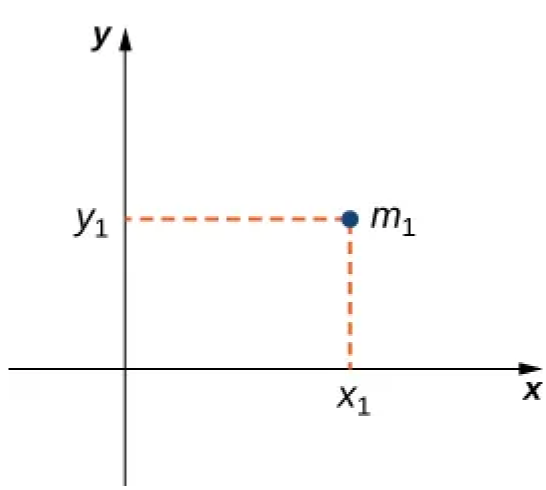


Figure 2.64 Point mass m_1 is located at point (x_1, y_1) in the plane.

If we have several point masses in the *xy*-plane, we can use the moments with respect to the *x*- and *y*-axes to calculate the *x*- and *y*-coordinates of the center of mass of the system.

THEOREM 2.10

Center of Mass of Objects in a Plane

Let $m_1, m_2, \ldots, m_n$ be point masses located in the *xy*-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then the moments M_x and M_y of the system with respect to the *x*- and *y*-axes, respectively, are given by

$$M_x = \sum_{i=1}^{n} m_i y_i \quad \text{and} \quad M_y = \sum_{i=1}^{n} m_i x_i. \tag{2.16}$$

Also, the coordinates of the center of mass $(\bar{x}, \bar{y})$ of the system are

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}. \tag{2.17}$$

The next example demonstrates how to apply this theorem.

EXAMPLE 2.30

Finding the Center of Mass of Objects in a Plane

Suppose three point masses are placed in the *xy*-plane as follows (assume coordinates are given in meters):

$$\begin{aligned} m_1 &= 2\,\text{kg, placed at } (-1,3)\,, \\ m_2 &= 6\,\text{kg, placed at } (1,1)\,, \\ m_3 &= 4\,\text{kg, placed at } (2,-2)\,. \end{aligned}$$

Find the center of mass of the system.

[Show/Hide Solution]

CHECKPOINT 2.30

Suppose three point masses are placed on a number line as follows (assume coordinates are given in meters):

$$\begin{aligned} m_1 &= 5\,\text{kg, placed at } (-2,-3)\,, \\ m_2 &= 3\,\text{kg, placed at } (2,3)\,, \\ m_3 &= 2\,\text{kg, placed at } (-3,-2)\,. \end{aligned}$$

Find the center of mass of the system.

Center of Mass of Thin Plates

So far we have looked at systems of point masses on a line and in a plane. Now, instead of having the mass of a system concentrated at discrete points, we want to look at systems in which the mass of the system is distributed continuously across a thin sheet of material. For our purposes, we assume the sheet is thin enough that it can be treated as if it is two-dimensional. Such a sheet is called a **lamina**. Next we develop techniques to find the center of mass of a lamina. In this section, we also assume the density of the lamina is constant.

Laminas are often represented by a two-dimensional region in a plane. The geometric center of such a region is called its **centroid**. Since we have assumed the density of the lamina is constant, the center of mass of the lamina depends only on the shape of the corresponding region in the plane; it does not depend on the density. In this case, the center of mass of the lamina corresponds to the centroid of the delineated region in the plane. As with systems of point masses, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the *x*- and *y*-axes.

We first consider a lamina in the shape of a rectangle. Recall that the center of mass of a lamina is the point where the lamina balances. For a rectangle, that point is both the horizontal and vertical center of the rectangle. Based on this understanding, it is clear that the center of mass of a rectangular lamina is the point where the diagonals intersect, which is a result of the **symmetry principle**, and it is stated here without proof.

THEOREM 2.11

The Symmetry Principle

If a region *R* is symmetric about a line *l*, then the centroid of *R* lies on *l*.

Let's turn to more general laminas. Suppose we have a lamina bounded above by the graph of a continuous function $f(x)$, below by the *x*-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in the following figure.

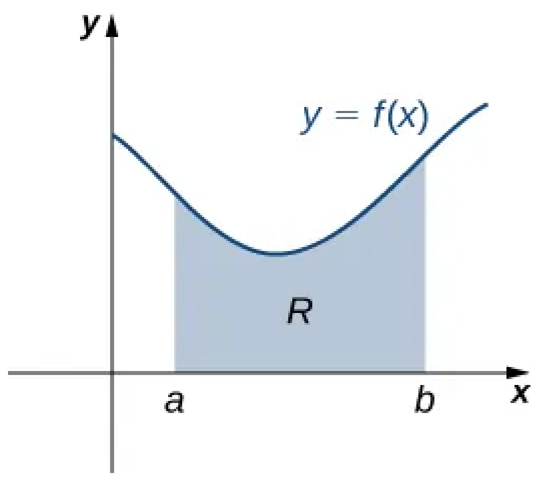


Figure 2.65 A region in the plane representing a lamina.

As with systems of point masses, to find the center of mass of the lamina, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the *x*- and *y*-axes. As we have done many times before, we approximate these quantities by partitioning the interval $[a, b]$ and constructing rectangles.

For $i = 0, 1, 2,..., n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Recall that we can choose any point within the interval $[x_{i-1}, x_i]$ as our x_i^*. In this case, we want x_i^* to be the *x*-coordinate of the centroid of our rectangles. Thus, for $i = 1, 2,..., n$, we select $x_i^* \in [x_{i-1}, x_i]$ such that x_i^* is the midpoint of the interval. That is, $x_i^* = (x_{i-1} + x_i)/2$. Now, for $i = 1, 2,..., n$, construct a rectangle of height $f(x_i^*)$ on $[x_{i-1}, x_i]$. The center of mass of this rectangle is $(x_i^*, (f(x_i^*))/2)$, as shown in the following figure.

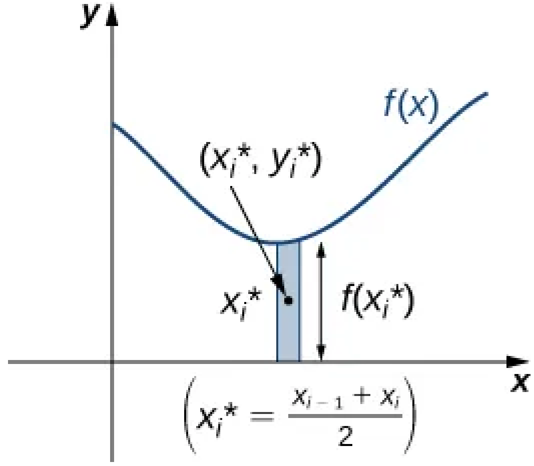


Figure 2.66 A representative rectangle of the lamina.

Next, we need to find the total mass of the rectangle. Let ρ represent the density of the lamina (note that ρ is a constant). In this case, ρ is expressed in terms of mass per unit area. Thus, to find the total mass of the rectangle, we multiply the area of the rectangle by ρ. Then, the mass of the rectangle is given by $\rho f(x_i^*)\Delta x$.

To get the approximate mass of the lamina, we add the masses of all the rectangles to get

$$m \approx \sum_{i=1}^{n} \rho f(x_i^*)\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$ gives the exact mass of the lamina:

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho f(x_i^*)\Delta x = \rho \int_a^b f(x)dx.$$

Next, we calculate the moment of the lamina with respect to the *x*-axis. Returning to the representative rectangle, recall its center of mass is $\left(x_i^*, \left(f(x_i^*)\right)/2\right)$. Recall also that treating the rectangle as if it is a point mass located at the center of mass does not change the moment. Thus, the moment of the rectangle with respect to the *x*-axis is given by the mass of the rectangle, $\rho f(x_i^*)\Delta x$, multiplied by the distance from the center of mass to the *x*-axis: $\left(f(x_i^*)\right)/2$. Therefore, the moment with respect to the *x*-axis of the rectangle is $\rho\left(\left[f(x_i^*)\right]^2/2\right)\Delta x$. Adding the moments of the rectangles and taking the limit of the resulting Riemann sum, we see that the moment of the lamina with respect to the *x*-axis is

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \frac{\left[f(x_i^*)\right]^2}{2}\Delta x = \rho \int_a^b \frac{\left[f(x)\right]^2}{2}dx.$$

We derive the moment with respect to the *y*-axis similarly, noting that the distance from the center of mass of the rectangle to the *y*-axis is x_i^*. Then the moment of the lamina with respect to the *y*-axis is given by

$$M_y = \lim_{n \to \infty} \sum_{i=1}^{n} \rho x_i^* f(x_i^*) \Delta x = \rho \int_a^b x f(x) dx.$$

We find the coordinates of the center of mass by dividing the moments by the total mass to give $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$. If we look closely at the expressions for $M_x, M_y,$ and m, we notice that the constant ρ cancels out when $\bar{x}$ and $\bar{y}$ are calculated.

We summarize these findings in the following theorem.

THEOREM 2.12

Center of Mass of a Thin Plate in the *xy*-Plane

Let *R* denote a region bounded above by the graph of a continuous function $f(x)$, below by the *x*-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b f(x) dx. \tag{2.18}$$

ii. The moments M_x and M_y of the lamina with respect to the *x*- and *y*-axes, respectively, are

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx \text{ and } M_y = \rho \int_a^b x f(x) dx. \tag{2.19}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}. \tag{2.20}$$

In the next example, we use this theorem to find the center of mass of a lamina.

EXAMPLE 2.31

Finding the Center of Mass of a Lamina

Let *R* be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the *x*-axis over the interval $[0, 4]$. Find the centroid of the region.

[Show/Hide Solution]

CHECKPOINT 2.31

Let *R* be the region bounded above by the graph of the function $f(x) = x^2$ and below by the *x*-axis over the interval $[0, 2]$. Find the centroid of the region.

We can adapt this approach to find centroids of more complex regions as well. Suppose our region is bounded above by the graph of a continuous function $f(x)$, as before, but now, instead of having the lower bound for the region be the *x*-axis, suppose the region is bounded below by the graph of a second continuous function, $g(x)$, as shown in the following figure.

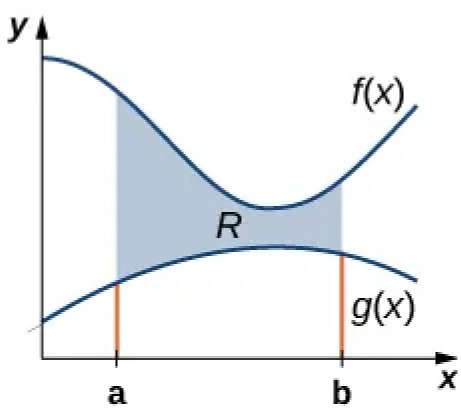


Figure 2.68 A region between two functions.

Again, we partition the interval $[a, b]$ and construct rectangles. A representative rectangle is shown in the following figure.

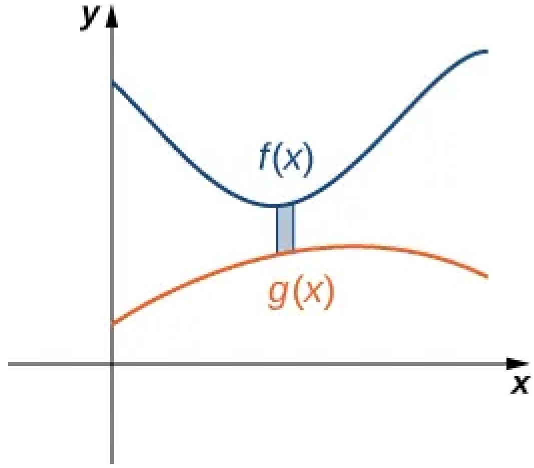


Figure 2.69 A representative rectangle of the region between two functions.

Note that the centroid of this rectangle is $\left(x_i^*, \left(f(x_i^*) + g(x_i^*)\right)/2\right)$. We won't go through all the details of the Riemann sum development, but let's look at some of the key steps. In the development of the formulas for the mass of the lamina and the moment with respect to the *y*-axis, the height of each rectangle is given by $f(x_i^*) - g(x_i^*)$, which leads to the expression $f(x) - g(x)$ in the integrands.

In the development of the formula for the moment with respect to the *x*-axis, the moment of each rectangle is found by multiplying the area of the rectangle, $\rho\left[f(x_i^*) - g(x_i^*)\right]\Delta x$, by the distance of the centroid from the *x*-axis, $\left(f(x_i^*) + g(x_i^*)\right)/2$, which gives $\rho\,(1/2)\left\{\left[f(x_i^*)\right]^2 - \left[g(x_i^*)\right]^2\right\}\Delta x$. Summarizing these findings, we arrive at the following theorem.

THEOREM 2.13

Center of Mass of a Lamina Bounded by Two Functions

Let *R* denote a region bounded above by the graph of a continuous function $f(x)$, below by the graph of the continuous function $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b \left[f(x) - g(x)\right] dx. \tag{2.21}$$

ii. The moments M_x and M_y of the lamina with respect to the *x*- and *y*-axes, respectively, are

$$M_x = \rho \int_a^b \frac{1}{2}\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right) dx \text{ and } M_y = \rho \int_a^b x\left[f(x) - g(x)\right] dx. \tag{2.22}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}. \tag{2.23}$$

We illustrate this theorem in the following example.

EXAMPLE 2.32

Finding the Centroid of a Region Bounded by Two Functions

Let *R* be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by the graph of the function $g(x) = x - 1$. Find the centroid of the region.

[Show/Hide Solution]

CHECKPOINT 2.32

Let *R* be the region bounded above by the graph of the function $f(x) = 6 - x^2$ and below by the graph of the function $g(x) = 3 - 2x$. Find the centroid of the region.

The Symmetry Principle

We stated the symmetry principle earlier, when we were looking at the centroid of a rectangle. The symmetry principle can be a great help when finding centroids of regions that are symmetric. Consider the following example.

EXAMPLE 2.33

Finding the Centroid of a Symmetric Region

Let *R* be the region bounded above by the graph of the function $f(x) = 4 - x^2$ and below by the *x*-axis. Find the centroid of the region.

[Show/Hide Solution]

CHECKPOINT 2.33

Let *R* be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by *x*-axis. Find the centroid of the region.

STUDENT PROJECT

The Grand Canyon Skywalk

The Grand Canyon Skywalk opened to the public on March 28, 2007. This engineering marvel is a horseshoe-shaped observation platform suspended 4000 ft above the Colorado River on the West Rim of the Grand Canyon. Its crystal-clear glass floor allows stunning views of the canyon below (see the following figure).

Figure 2.72 The Grand Canyon Skywalk offers magnificent views of the canyon. (credit: 10da_ralta, Wikimedia Commons)

The Skywalk is a cantilever design, meaning that the observation platform extends over the rim of the canyon, with no visible means of support below it. Despite the lack of visible support posts or struts, cantilever structures are engineered to be very stable and the Skywalk is no exception. The observation platform is attached firmly to support posts that extend 46 ft down into bedrock. The structure was built to withstand 100-mph winds and an 8.0-magnitude earthquake within 50 mi, and is capable of supporting more than 70,000,000 lb.

One factor affecting the stability of the Skywalk is the center of gravity of the structure. We are going to calculate the center of gravity of the Skywalk, and examine how the center of gravity changes when tourists walk out onto the observation platform.

The observation platform is U-shaped. The legs of the U are 10 ft wide and begin on land, under the visitors' center, 48 ft from the edge of the canyon. The platform extends 70 ft over the edge of the canyon.

To calculate the center of mass of the structure, we treat it as a lamina and use a two-dimensional region in the *xy*-plane to represent the platform. We begin by dividing the region into three subregions so we can consider each subregion separately. The first region, denoted

R_1, consists of the curved part of the U. We model R_1 as a semicircular annulus, with inner radius 25 ft and outer radius 35 ft, centered at the origin (see the following figure).

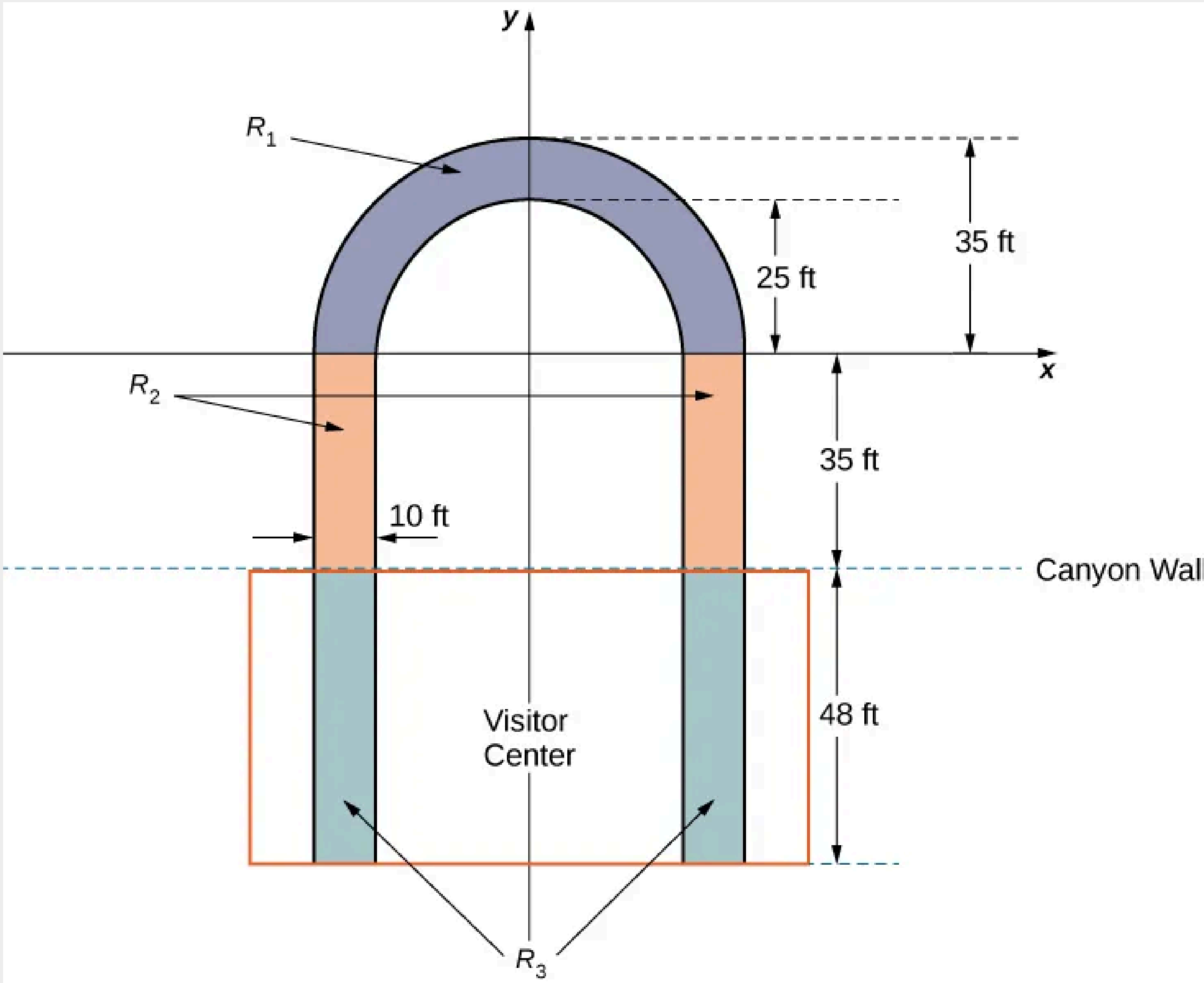


Figure 2.73 We model the Skywalk with three sub-regions.

The legs of the platform, extending 35 ft between R_1 and the canyon wall, comprise the second sub-region, R_2. Last, the ends of the legs, which extend 48 ft under the visitor center, comprise the third sub-region, R_3. Assume the density of the lamina is constant and assume the total weight of the platform is 1,200,000 lb (not including the weight of the visitor center; we will consider that later). Use $g = 32\,\text{ft/sec}^2$.

1. Compute the area of each of the three sub-regions. Note that the areas of regions R_2 and R_3 should include the areas of the legs only, not the open space between them. Round answers to the nearest square foot.
2. Determine the mass associated with each of the three sub-regions.
3. Calculate the center of mass of each of the three sub-regions.
4. Now, treat each of the three sub-regions as a point mass located at the center of mass of the corresponding sub-region. Using this representation, calculate the center of mass of

the entire platform.

5. Assume the visitor center weighs 2,200,000 lb, with a center of mass corresponding to the center of mass of R_3. Treating the visitor center as a point mass, recalculate the center of mass of the system. How does the center of mass change?
6. Although the Skywalk was built to limit the number of people on the observation platform to 120, the platform is capable of supporting up to 800 people weighing 200 lb each. If all 800 people were allowed on the platform, and all of them went to the farthest end of the platform, how would the center of gravity of the system be affected? (Include the visitor center in the calculations and represent the people by a point mass located at the farthest edge of the platform, 70 ft from the canyon wall.)

Theorem of Pappus

This section ends with a discussion of the **theorem of Pappus for volume**, which allows us to find the volume of particular kinds of solids by using the centroid. (There is also a theorem of Pappus for surface area, but it is much less useful than the theorem for volume.)

THEOREM 2.14

Theorem of Pappus for Volume

Let *R* be a region in the plane and let *l* be a line in the plane that does not intersect *R*. Then the volume of the solid of revolution formed by revolving *R* around *l* is equal to the area of *R* multiplied by the distance *d* traveled by the centroid of *R.*

Proof

We can prove the case when the region is bounded above by the graph of a function $f(x)$ and below by the graph of a function $g(x)$ over an interval $[a, b]$, and for which the axis of revolution is the *y*-axis. In this case, the area of the region is $A = \int_a^b [f(x) - g(x)]\,dx$. Since the axis of rotation is the *y*-axis, the distance traveled by the centroid of the region depends only on the *x*-coordinate of the centroid, $\bar{x}$, which is

$$\bar{x} = \frac{M_y}{m},$$

where

$$m = \rho \int_a^b [f(x) - g(x)]\, dx \text{ and } M_y = \rho \int_a^b x\,[f(x) - g(x)]\, dx.$$

Then,

$$d = 2\pi \frac{\rho \int_a^b x\,[f(x) - g(x)]\, dx}{\rho \int_a^b [f(x) - g(x)]\, dx}$$

and thus

$$d \cdot A = 2\pi \int_a^b x\,[f(x) - g(x)]\, dx.$$

However, using the method of cylindrical shells, we have

$$V = 2\pi \int_a^b x\,[f(x) - g(x)]\, dx.$$

So,

$$V = d \cdot A$$

and the proof is complete.

□

EXAMPLE 2.34

Using the Theorem of Pappus for Volume

Let *R* be a circle of radius 2 centered at $(4, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving *R* around the *y*-axis.

[Show/Hide Solution]

CHECKPOINT 2.34

Let *R* be a circle of radius 1 centered at $(3, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving *R* around the *y*-axis.

Section 2.6 Exercises

For the following exercises, calculate the center of mass for the collection of masses given.

254. $m_1 = 2$ at $x_1 = 1$ and $m_2 = 4$ at $x_2 = 2$

255. $m_1 = 1$ at $x_1 = -1$ and $m_2 = 3$ at $x_2 = 2$

256. $m = 3$ at $x = 0, 1, 2, 6$

257. Unit masses at $(x, y) = (1, 0), (0, 1), (1, 1)$

258. $m_1 = 1$ at $(1, 0)$ and $m_2 = 4$ at $(0, 1)$

259. $m_1 = 1$ at $(1, 0)$ and $m_2 = 3$ at $(2, 2)$

For the following exercises, compute the center of mass $\overline{x}$.

260. $\rho = 1$ for $x \in (-1, 3)$

261. $\rho = x^2$ for $x \in (0, L)$

262. $\rho = 1$ for $x \in (0, 1)$ and $\rho = 2$ for $x \in (1, 2)$

263. $\rho = \sin x$ for $x \in (0, \pi)$

264. $\rho = \cos x$ for $x \in \left(0, \frac{\pi}{2}\right)$

265. $\rho = e^x$ for $x \in (0, 2)$

266. $\rho = x^3 + xe^{-x}$ for $x \in (0, 1)$

267. $\rho = x \sin x$ for $x \in (0, \pi)$

268. $\rho = \sqrt{x}$ for $x \in (1, 4)$

269. $\rho = \ln x$ for $x \in (1, e)$

For the following exercises, compute the center of mass $(\overline{x}, \overline{y})$. Use symmetry to help locate the center of mass whenever possible.

270. $\rho = 7$ in the square $0 \leq x \leq 1, 0 \leq y \leq 1$

271. $\rho = 3$ in the triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$

272. $\rho = 2$ for the region bounded by $y = \cos(x)$, $y = -\cos(x)$, $x = -\frac{\pi}{2}$, and $x = \frac{\pi}{2}$

For the following exercises, use a calculator to draw the region, then compute the center of mass $(\overline{x}, \overline{y})$. Use symmetry to help locate the center of mass whenever possible.

273. **[T]** The region bounded by $y = \cos(2x)$, $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$

274. **[T]** The region between $y = 2x^2$, $y = 0$, $x = 0$, and $x = 1$

275. **[T]** The region between $y = \frac{5}{4}x^2$ and $y = 5$

276. **[T]** Region between $y = \sqrt{x}$, $y = \ln(x)$, $x = 1$, and $x = 4$

277. **[T]** The region bounded by $y = 0$, $\frac{x^2}{4} + \frac{y^2}{9} = 1$

278. **[T]** The region bounded by $y = 0$, $x = 0$, and $\frac{x^2}{4} + \frac{y^2}{9} = 1$

279. **[T]** The region bounded by $y = x^2$ and $y = x^4$ in the first quadrant

For the following exercises, use the theorem of Pappus to determine the volume of the shape.

280. Rotating $y = mx$ around the x-axis between $x = 0$ and $x = 1$

281. Rotating $y = mx$ around the y-axis between $x = 0$ and $x = 1$

282. A general cone created by rotating a triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$ around the y-axis. Does your answer agree with the volume of a cone?

283. A general cylinder created by rotating a rectangle with vertices $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b) around the y-axis. Does your answer agree with the volume of a cylinder?

284. A sphere created by rotating a semicircle with radius a around the y-axis. Does your answer agree with the volume of a sphere?

For the following exercises, use a calculator to draw the region enclosed by the curve. Find the area M and the centroid $(\overline{x}, \overline{y})$ for the given shapes. Use symmetry to help locate the center of mass whenever possible.

285. **[T]** Quarter-circle: $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$

286. **[T]** Triangle: $y = x$, $y = 2 - x$, and $y = 0$

287. **[T]** Lens: $y = x^2$ and $y = x$

288. **[T]** Ring: $y^2 + x^2 = 1$ and $y^2 + x^2 = 4$

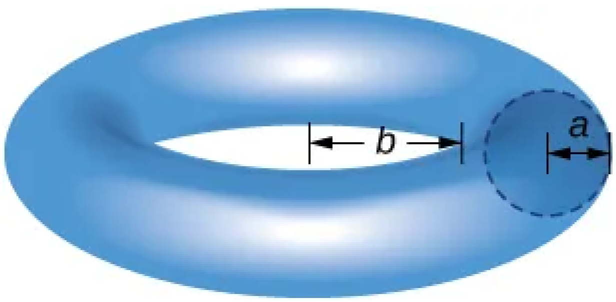
289. **[T]** Half-ring: $y^2 + x^2 = 1$, $y^2 + x^2 = 4$, and $y = 0$

290. Find the generalized center of mass in the sliver between $y = x^a$ and $y = x^b$ with $a > b$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

291. Find the generalized center of mass between $y = a^2 - x^2$, $x = 0$, and $y = 0$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

292. Find the generalized center of mass between $y = b \sin(ax)$, $x = 0$, and $x = \frac{\pi}{a}$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

293. Use the theorem of Pappus to find the volume of a torus (pictured here). Assume that a disk of radius a is positioned with the left end of the circle at $x = b$, $b > 0$, and is rotated around



the *y*-axis.

294. Find the center of mass $(\overline{x}, \overline{y})$ for a thin wire along the semicircle $y = \sqrt{1 - x^2}$ with unit mass. (*Hint:* Use the theorem of Pappus.)