2.7 Integrals, Exponential Functions, and Logarithms

Learning Objectives

2.7.1 Write the definition of the natural logarithm as an integral.
2.7.2 Recognize the derivative of the natural logarithm.
2.7.3 Integrate functions involving the natural logarithmic function.
2.7.4 Define the number e through an integral.
2.7.5 Recognize the derivative and integral of the exponential function.
2.7.6 Prove properties of logarithms and exponential functions using integrals.
2.7.7 Express general logarithmic and exponential functions in terms of natural logarithms and exponentials.

We already examined exponential functions and logarithms in earlier chapters. However, we glossed over some key details in the previous discussions. For example, we did not study how to treat exponential functions with exponents that are irrational. The definition of the number *e* is another area where the previous development was somewhat incomplete. We now have the tools to deal with these concepts in a more mathematically rigorous way, and we do so in this section.

For purposes of this section, assume we have not yet defined the natural logarithm, the number *e*, or any of the integration and differentiation formulas associated with these functions. By the end of the section, we will have studied these concepts in a mathematically rigorous way (and we will see they are consistent with the concepts we learned earlier).

We begin the section by defining the natural logarithm in terms of an integral. This definition forms the foundation for the section. From this definition, we derive differentiation formulas, define the number e, and expand these concepts to logarithms and exponential functions of any base.

The Natural Logarithm as an Integral

Recall the power rule for integrals:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1.$$

Clearly, this does not work when $n = -1$, as it would force us to divide by zero. So, what do we do with $\int \frac{1}{x} dx$? Recall from the Fundamental Theorem of Calculus that $\int_1^x \frac{1}{t} dt$ is an antiderivative of $1/x$. Therefore, we can make the following definition.

DEFINITION

For $x > 0$, define the natural logarithm function by

$$\ln x = \int_1^x \frac{1}{t} dt. \tag{2.24}$$

For $x > 1$, this is just the area under the curve $y = 1/t$ from 1 to x. For $x < 1$, we have $\int_1^x \frac{1}{t} dt = -\int_x^1 \frac{1}{t} dt$, so in this case it is the negative of the area under the curve from $x \text{ to } 1$ (see the following figure).

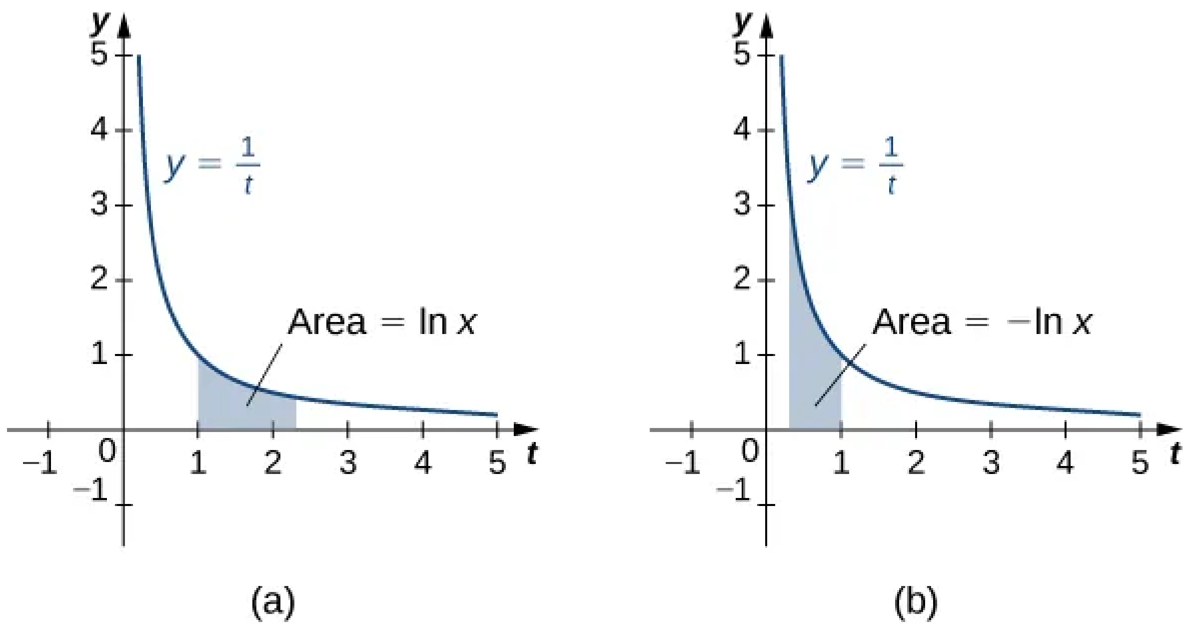


Figure 2.75 (a) When $x > 1$, the natural logarithm is the area under the curve $y = 1/t$ from $1 \text{ to } x$. (b) When $x < 1$, the natural logarithm is the negative of the area under the curve from x to 1.

Notice that $\ln 1 = 0$. Furthermore, the function $y = 1/t > 0$ for $x > 0$. Therefore, by the properties of integrals, it is clear that $\ln x$ is increasing for $x > 0$.

Properties of the Natural Logarithm

Because of the way we defined the natural logarithm, the following differentiation formula falls out immediately as a result of to the Fundamental Theorem of Calculus.

THEOREM 2.15

Derivative of the Natural Logarithm

For $x > 0$, the derivative of the natural logarithm is given by

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

THEOREM 2.16

Corollary to the Derivative of the Natural Logarithm

The function $\ln x$ is differentiable; therefore, it is continuous.

A graph of $\ln x$ is shown in Figure 2.76. Notice that it is continuous throughout its domain of $(0, \infty)$.

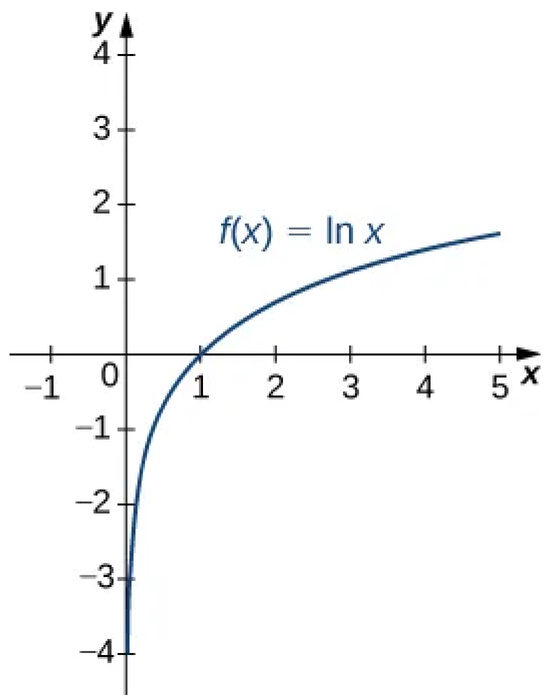


Figure 2.76 The graph of $f(x) = \ln x$ shows that it is a continuous function.

EXAMPLE 2.35

Calculating Derivatives of Natural Logarithms

Calculate the following derivatives:

a. $\frac{d}{dx}\ln\left(5x^3 - 2\right)$
b. $\frac{d}{dx}\left(\ln(3x)\right)^2$

[Show/Hide Solution]

CHECKPOINT 2.35

Calculate the following derivatives:

a. $\frac{d}{dx} \ln\left(2x^2 + x\right)$
b. $\frac{d}{dx} \left(\ln\left(x^3\right)\right)^2$

Note that if we use the absolute value function and create a new function $\ln\ |x|$, we can extend the domain of the natural logarithm to include $x < 0$. Then $(d/\left(dx\right)) \ln\ |x| = 1/x$. This gives rise to the familiar integration formula.

THEOREM 2.17

Integral of (1/*u*) *du*

The natural logarithm is the antiderivative of the function $f\left(u\right) = 1/u$:

$$\int \frac{1}{u} du = \ln\ |u| + C.$$

EXAMPLE 2.36

Calculating Integrals Involving Natural Logarithms

Calculate the integral $\int \frac{x}{x^2 + 4} dx.$

[Show/Hide Solution]

CHECKPOINT 2.36

Calculate the integral $\int \frac{x^2}{x^3+6}dx.$

Although we have called our function a "logarithm," we have not actually proved that any of the properties of logarithms hold for this function. We do so here.

THEOREM 2.18

Properties of the Natural Logarithm

If $a, b > 0$ and r is a rational number, then

i. $\ln 1 = 0$
ii. $\ln(ab) = \ln a + \ln b$
iii. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
iv. $\ln(a^r) = r \ln a$

Proof

i. By definition, $\ln 1 = \int_1^1 \frac{1}{t}dt = 0.$

ii. We have

$$\ln(ab) = \int_1^{ab} \frac{1}{t}dt = \int_1^a \frac{1}{t}dt + \int_a^{ab} \frac{1}{t}dt.$$

Use $u\text{-substitution}$ on the last integral in this expression. Let $u = t/a$. Then $du = (1/a)\,dt$. Furthermore, when $t = a, u = 1$, and when $t = ab, u = b$. So we get

$$\ln(ab) = \int_1^a \frac{1}{t}dt + \int_a^{ab} \frac{1}{t}dt = \int_1^a \frac{1}{t}dt + \int_a^{ab} \frac{a}{t} \cdot \frac{1}{a}dt = \int_1^a \frac{1}{t}dt + \int_1^b \frac{1}{u}du = \ln a + \ln b.$$

iv. Note that

$$\frac{d}{dx}\ln\left(x^r\right) = \frac{rx^{r-1}}{x^r} = \frac{r}{x}.$$

Furthermore,

$$\frac{d}{dx}(r\ln x) = \frac{r}{x}.$$

Since the derivatives of these two functions are the same, by the Fundamental Theorem of Calculus, they must differ by a constant. So we have

$$\ln\left(x^r\right) = r\ln x + C$$

for some constant C. Taking $x = 1$, we get

$$\begin{array}{rcl} \ln\left(1^r\right) & = & r\ln(1) + C \\ 0 & = & r(0) + C \\ C & = & 0. \end{array}$$

Thus $\ln\left(x^r\right) = r\ln x$ and the proof is complete. Note that we can extend this property to irrational values of r later in this section.
Part iii. follows from parts ii. and iv. and the proof is left to you.

□

EXAMPLE 2.37

Using Properties of Logarithms

Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 9 - 2\ln 3 + \ln\left(\frac{1}{3}\right).$$

[Show/Hide Solution]

CHECKPOINT 2.37

Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 8 - \ln 2 - \ln\left(\frac{1}{4}\right).$$

Defining the Number *e*

Now that we have the natural logarithm defined, we can use that function to define the number e.

DEFINITION

The number e is defined to be the real number such that

$$\ln e = 1.$$

To put it another way, the area under the curve $y = 1/t$ between $t = 1$ and $t = e$ is 1 (Figure 2.77). The proof that such a number exists and is unique is left to you. (*Hint*: Use the Intermediate Value Theorem to prove existence and the fact that $\ln x$ is increasing to prove uniqueness.)

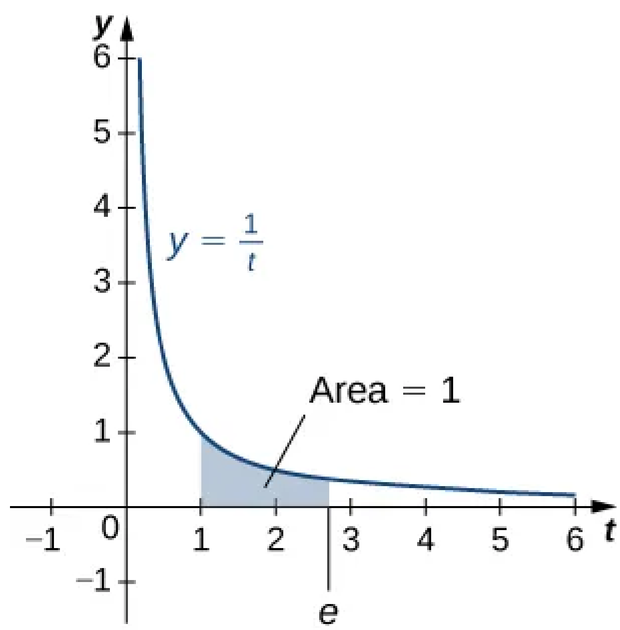


Figure 2.77 The area under the curve from 1 to e is equal to one.

The number e can be shown to be irrational, although we won't do so here (see the Student Project in Taylor and Maclaurin Series). Its approximate value is given by

$$e \approx 2.71828182846.$$

The Exponential Function

We now turn our attention to the function e^x. Note that the natural logarithm is one-to-one and therefore has an inverse function. For now, we denote this inverse function by $\exp x$. Then,

$$\exp(\ln x) = x \text{ for } x > 0 \text{ and } \ln(\exp x) = x \text{ for all } x.$$

The following figure shows the graphs of $\exp x$ and $\ln x$.

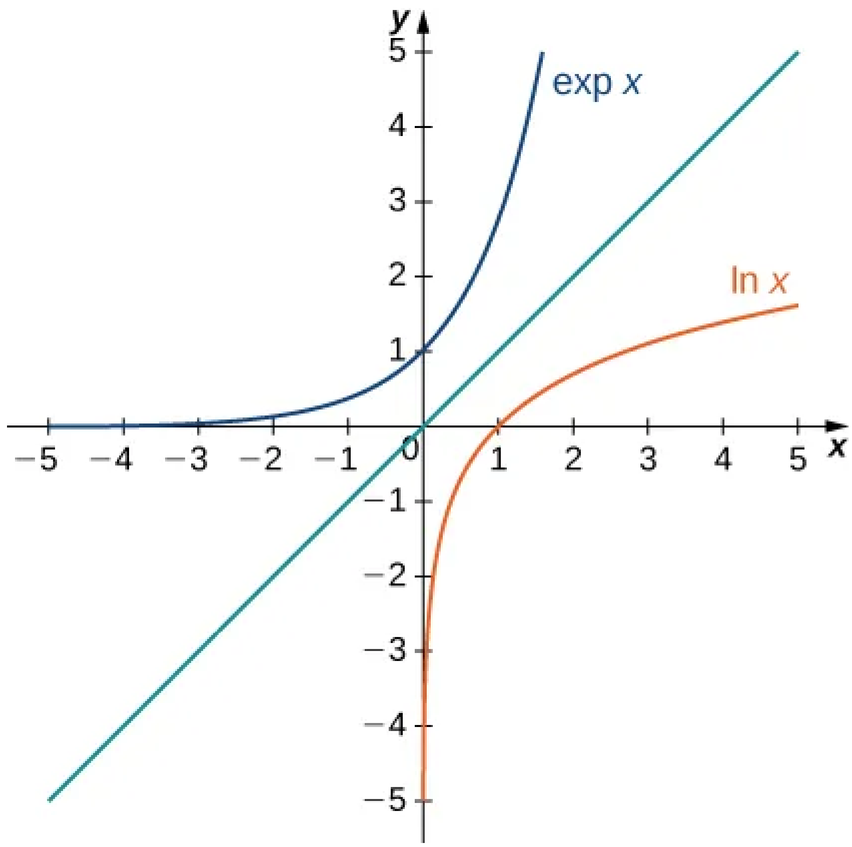


Figure 2.78 The graphs of $\ln x$ and $\exp x$.

We hypothesize that $\exp x = e^x$. For rational values of x, this is easy to show. If x is rational, then we have $\ln(e^x) = x \ln e = x$. Thus, when x is rational, $e^x = \exp x$. For irrational values of x, we simply define e^x as the inverse function of $\ln x$.

DEFINITION

For any real number x, define $y = e^x$ to be the number for which

$$\ln y = \ln\left(e^x\right) = x. \tag{2.25}$$

Then we have $e^x = \exp(x)$ for all x, and thus

$$e^{\ln x} = x \text{ for } x > 0 \text{ and } \ln\left(e^x\right) = x \tag{2.26}$$

for all x.

Properties of the Exponential Function

Since the exponential function was defined in terms of an inverse function, and not in terms of a power of e, we must verify that the usual laws of exponents hold for the function e^x.

THEOREM 2.19

Properties of the Exponential Function

If p and q are any real numbers and r is a rational number, then

i. $e^p e^q = e^{p+q}$
ii. $\frac{e^p}{e^q} = e^{p-q}$
iii. $\left(e^p\right)^r = e^{pr}$

Proof

Note that if p and q are rational, the properties hold. However, if p or q are irrational, we must apply the inverse function definition of e^x and verify the properties. Only the first property is verified here; the other two are left to you. We have

$$\ln\left(e^p e^q\right) = \ln\left(e^p\right) + \ln\left(e^q\right) = p + q = \ln\left(e^{p+q}\right).$$

Since $\ln x$ is one-to-one, then

$$e^p e^q = e^{p+q}.$$

□

As with part iv. of the logarithm properties, we can extend property iii. to irrational values of r, and we do so by the end of the section.

We also want to verify the differentiation formula for the function $y = e^x$. To do this, we need to use implicit differentiation. Let $y = e^x$. Then

$$\begin{aligned} \ln y &= x \\ \frac{d}{dx} \ln y &= \frac{d}{dx} x \\ \frac{1}{y} \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y. \end{aligned}$$

Thus, we see

$$\frac{d}{dx} e^x = e^x$$

as desired, which leads immediately to the integration formula

$$\int e^x dx = e^x + C.$$

We apply these formulas in the following examples.

EXAMPLE 2.38

Using Properties of Exponential Functions

Evaluate the following derivatives:

a. $\frac{d}{dt} e^{3t} e^{t^2}$
b. $\frac{d}{dx} e^{3x^2}$

[Show/Hide Solution]

CHECKPOINT 2.38

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right)$
b. $\frac{d}{dt}\left(e^{2t}\right)^3$

EXAMPLE 2.39

Using Properties of Exponential Functions

Evaluate the following integral: $\int 2xe^{-x^2}dx.$

[Show/Hide Solution]

CHECKPOINT 2.39

Evaluate the following integral: $\int \frac{4}{e^{3x}}dx.$

General Logarithmic and Exponential Functions

We close this section by looking at exponential functions and logarithms with bases other than e. Exponential functions are functions of the form $f(x) = a^x$. Note that unless $a = e$, we still do not have a mathematically rigorous definition of these functions for irrational exponents. Let's rectify that here by defining the function $f(x) = a^x$ in terms of the exponential function e^x. We then examine logarithms with bases other than e as inverse functions of exponential functions.

DEFINITION

For any $a > 0$, and for any real number x, define $y = a^x$ as follows:

$$y = a^x = e^{x \ln a}.$$

Now a^x is defined rigorously for all values of x. This definition also allows us to generalize property iv. of logarithms and property iii. of exponential functions to apply to both rational and irrational values of r. It is straightforward to show that properties of exponents hold for general exponential functions defined in this way.

Let's now apply this definition to calculate a differentiation formula for a^x. We have

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a}\ln a = a^x \ln a.$$

The corresponding integration formula follows immediately.

THEOREM 2.20

Derivatives and Integrals Involving General Exponential Functions

Let $a > 0$. Then,

$$\frac{d}{dx}a^x = a^x \ln a$$

and

$$\int a^x dx = \frac{1}{\ln a}a^x + C.$$

If $a \neq 1,$ then the function a^x is one-to-one and has a well-defined inverse. Its inverse is denoted by $\log_a x$. Then,

$$y = \log_a x \text{ if and only if } x = a^y.$$

Note that general logarithm functions can be written in terms of the natural logarithm. Let $y = \log_a x$. Then, $x = a^y$. Taking the natural logarithm of both sides of this second equation, we get

$$\begin{aligned} \ln x &= \ln\left(a^y\right) \\ \ln x &= y \ln a \\ y &= \tfrac{\ln x}{\ln a} \\ \log_a x &= \tfrac{\ln x}{\ln a}. \end{aligned}$$

Thus, we see that all logarithmic functions are constant multiples of one another. Next, we use this formula to find a differentiation formula for a logarithm with base a. Again, let $y = \log_a x$. Then,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\log_a x) \\ &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= \left(\frac{1}{\ln a}\right)\frac{d}{dx}(\ln x) \\ &= \frac{1}{\ln a}\cdot\frac{1}{x} \\ &= \frac{1}{x\ln a}. \end{aligned}$$

THEOREM 2.21

Derivatives of General Logarithm Functions

Let $a > 0$. Then,

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

EXAMPLE 2.40

Calculating Derivatives of General Exponential and Logarithm Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right)$
b. $\frac{d}{dx}\log_8\left(7x^2+4\right)$

[Show/Hide Solution]

CHECKPOINT 2.40

Evaluate the following derivatives:

a. $\frac{d}{dt}\,4^{t^4}$

b. $\frac{d}{dx}\log_3\left(\sqrt{x^2+1}\right)$

EXAMPLE 2.41

Integrating General Exponential Functions

Evaluate the following integral: $\int \frac{3}{2^{3x}}dx.$

[Show/Hide Solution]

CHECKPOINT 2.41

Evaluate the following integral: $\int x^2 2^{x^3}dx.$

Section 2.7 Exercises

For the following exercises, find the derivative $\frac{dy}{dx}$.

295. $y = \ln(2x)$

296. $y = \ln(2x+1)$

297. $y = \frac{1}{\ln x}$

For the following exercises, find the indefinite integral.

298. $\int \frac{dt}{3t}$

299. $\int \frac{dx}{1+x}$

For the following exercises, find the derivative dy/dx. (You can use a calculator to plot the function and the derivative to confirm that it is correct.)

300. **[T]** $y = \frac{\ln(x)}{x}$

301. **[T]** $y = x\ln(x)$

302. **[T]** $y = \log_{10} x$

303. **[T]** $y = \ln(\sin x)$

304. **[T]** $y = \ln(\ln x)$

305. **[T]** $y = 7\ln(4x)$

306. **[T]** $y = \ln\left((4x)^7\right)$

307. **[T]** $y = \ln(\tan x)$

308. **[T]** $y = \ln(\tan(3x))$

309. **[T]** $y = \ln(\cos^2 x)$

For the following exercises, find the definite or indefinite integral.

310. $\displaystyle\int_0^1 \frac{dx}{3+x}$

311. $\displaystyle\int_0^1 \frac{dt}{3+2t}$

312. $\displaystyle\int_0^2 \frac{x\,dx}{x^2+1}$

313. $\displaystyle\int_0^2 \frac{x^3 dx}{x^2+1}$

314. $\displaystyle\int_2^e \frac{dx}{x\ln x}$

315. $\displaystyle\int_2^e \frac{dx}{x(\ln x)^2}$

316. $\displaystyle\int \frac{\cos x\,dx}{\sin x}$

317. $\displaystyle\int_0^{\pi/4} \tan x\,dx$

318. $\displaystyle\int \cot(3x)\,dx$

319. $\int \frac{(\ln x)^2 dx}{x}$

For the following exercises, compute dy/dx by differentiating $\ln y$.

320. $y = \sqrt{x^2 + 1}$

321. $y = \sqrt{x^2 + 1}\sqrt{x^2 - 1}$

322. $y = e^{\sin x}$

323. $y = x^{-1/x}$

324. $y = e^{(ex)}$

325. $y = x^e$

326. $y = x^{(ex)}$

327. $y = \sqrt{x}\,\sqrt[3]{x}\,\sqrt[6]{x}$

328. $y = x^{-1/\ln x}$

329. $y = e^{-\ln x}$

For the following exercises, evaluate by any method.

330. $\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$

331. $\int_1^{e^\pi} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$

332. $\frac{d}{dx} \int_x^1 \frac{dt}{t}$

333. $\frac{d}{dx} \int_x^{x^2} \frac{dt}{t}$

334. $\frac{d}{dx} \ln(\sec x + \tan x)$

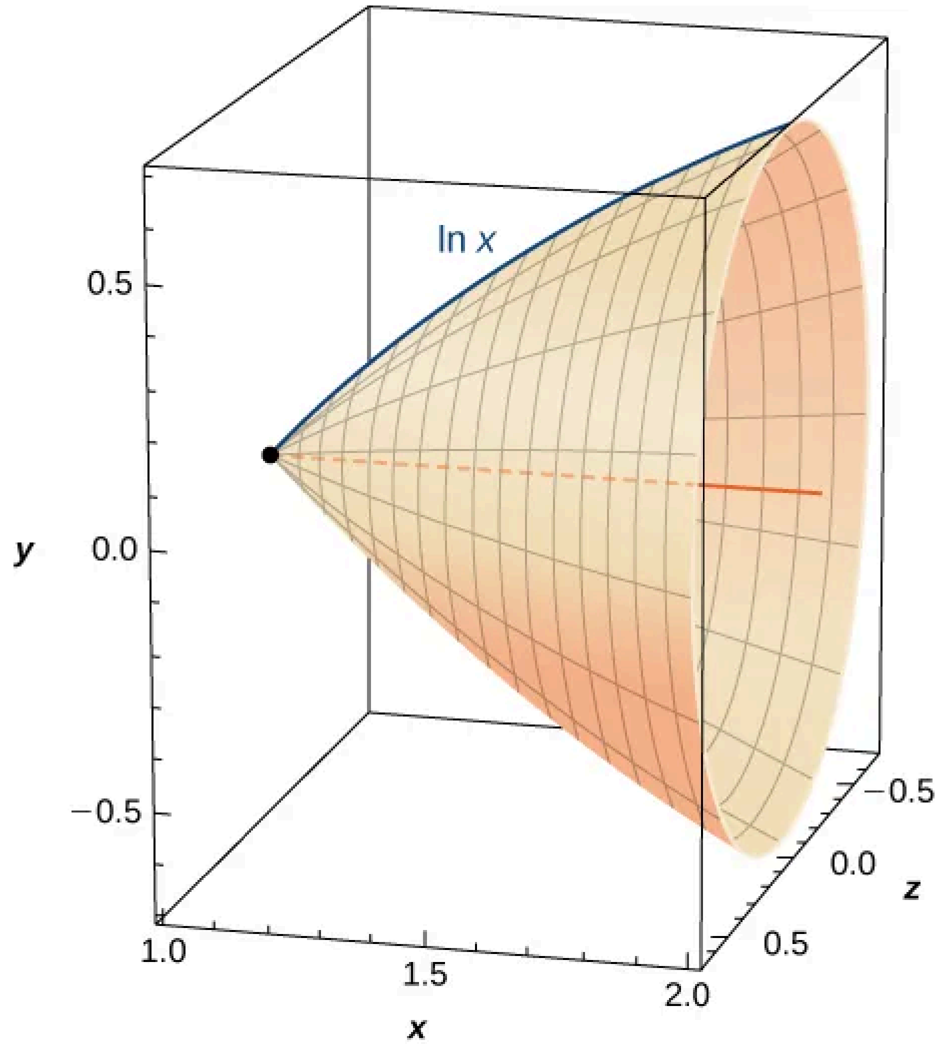
For the following exercises, use the function $\ln x$. If you are unable to find intersection points analytically, use a calculator.

335. Find the area of the region enclosed by $x = 1$ and $y = 5$ above $y = \ln x$.

336. **[T]** Find the arc length of $\ln x$ from $x = 1$ to $x = 2$.

337. Find the area between $\ln x$ and the *x*-axis from $x = 1 \text{ to } x = 2.$

338. Find the volume of the shape created when rotating this curve from $x = 1 \text{ to } x = 2$ around



the *x*-axis, as pictured here.

339. **[T]** Find the surface area of the shape created when rotating the curve in the previous exercise from $x = 1$ to $x = 2$ around the *x*-axis.

If you are unable to find intersection points analytically in the following exercises, use a calculator.

340. Find the area of the hyperbolic quarter-circle enclosed by $x = 2 \text{ and } y = 2$ above $y = 1/x.$

341. **[T]** Find the arc length of $y = 1/x$ from $x = 1 \text{ to } x = 4.$

342. Find the area under $y = 1/x$ and above the *x*-axis from $x = 1 \text{ to } x = 4.$

For the following exercises, verify the derivatives and antiderivatives.

343. $\frac{d}{dx}\ln\left(x + \sqrt{x^2+1}\right) = \frac{1}{\sqrt{1+x^2}}$

344. $\frac{d}{dx}\ln\left(\frac{x-a}{x+a}\right) = \frac{2a}{(x^2-a^2)}$

345. $\frac{d}{dx}\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) = -\frac{1}{x\sqrt{1-x^2}}$

346. $\frac{d}{dx}\ln\left(x+\sqrt{x^2-a^2}\right) = \frac{1}{\sqrt{x^2-a^2}}$

347. $\displaystyle\int \frac{dx}{x\ln(x)\ln(\ln x)} = \ln(\ln(\ln x)) + C$

2.8 Exponential Growth and Decay

Learning Objectives

2.8.1 Use the exponential growth model in applications, including population growth and compound interest.
2.8.2 Explain the concept of doubling time.
2.8.3 Use the exponential decay model in applications, including radioactive decay and Newton's law of cooling.
2.8.4 Explain the concept of half-life.

One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. From population growth and continuously compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature. In this section, we examine exponential growth and decay in the context of some of these applications.

Exponential Growth Model

Many systems exhibit exponential growth. These systems follow a model of the form $y = y_0 e^{kt}$, where y_0 represents the initial state of the system and k is a positive constant, called the *growth constant*. Notice that in an exponential growth model, we have

$$y' = k y_0 e^{kt} = k y. \tag{2.27}$$

That is, the rate of growth is proportional to the current function value. This is a key feature of exponential growth. Equation 2.27 involves derivatives and is called a *differential equation.* We learn more about differential equations in Introduction to Differential Equations.

RULE: EXPONENTIAL GROWTH MODEL

Systems that exhibit **exponential growth** increase according to the mathematical model

$$y = y_0 e^{kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *growth constant*.

Population growth is a common example of exponential growth. Consider a population of bacteria, for instance. It seems plausible that the rate of population growth would be proportional to the size of the population. After all, the more bacteria there are to reproduce, the faster the population grows. Figure

2.79 and Table 2.1 represent the growth of a population of bacteria with an initial population of 200 bacteria and a growth constant of 0.02. Notice that after only 2 hours $(120$ minutes), the population is 10 times its original size!

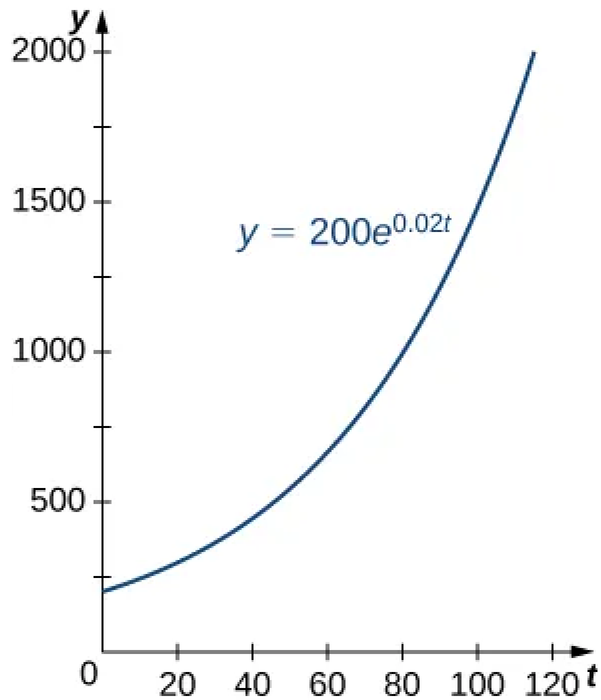


Figure 2.79 An example of exponential growth for bacteria.

Time (min)	Population Size (no. of bacteria)
10	244
20	298
30	364
40	445
50	544
60	664
70	811
80	991
90	1210
100	1478

Time (min)	Population Size (no. of bacteria)
110	1805
120	2205

Table 2.1 Exponential Growth of a Bacterial Population

Note that we are using a continuous function to model what is inherently discrete behavior. At any given time, the real-world population contains a whole number of bacteria, although the model takes on noninteger values. When using exponential growth models, we must always be careful to interpret the function values in the context of the phenomenon we are modeling.

EXAMPLE 2.42

Population Growth

Consider the population of bacteria described earlier. This population grows according to the function $f(t) = 200e^{0.02t}$, where *t* is measured in minutes. How many bacteria are present in the population after 5 hours $(300$ minutes)? When does the population reach $100{,}000$ bacteria?

[Show/Hide Solution]

CHECKPOINT 2.42

Consider a population of bacteria that grows according to the function $f(t) = 500e^{0.05t}$, where t is measured in minutes. How many bacteria are present in the population after 4 hours? When does the population reach 100 million bacteria?

Let's now turn our attention to a financial application: compound interest. Interest that is not compounded is called *simple interest*. Simple interest is paid once, at the end of the specified time period (usually 1 year). So, if we put $\$1000$ in a savings account earning 2% simple interest per year, then at the end of the year we have

$$1000\,(1 + 0.02) = \$1020.$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every 6 months, it credits half of the year's interest to the account after 6 months. During the second half of the year, the account earns interest not only on the initial $\$1000$, but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

$$1000\left(1+\frac{0.02}{2}\right)^2 = \$1020.10.$$

Similarly, if the interest is compounded every 4 months, we have

$$1000\left(1+\frac{0.02}{3}\right)^3 = \$1020.13,$$

and if the interest is compounded daily $(365$ times per year), we have $\$1020.20$. If we extend this concept, so that the interest is compounded continuously, after t years we have

$$1000\lim_{n\to\infty}\left(1+\frac{0.02}{n}\right)^{nt}.$$

Now let's manipulate this expression so that we have an exponential growth function. Recall that the number e can be expressed as a limit:

$$e = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m.$$

Based on this, we want the expression inside the parentheses to have the form $(1+1/m)$. Let $n = 0.02m$. Note that as $n \to \infty$, $m \to \infty$ as well. Then we get

$$1000\lim_{n\to\infty}\left(1+\frac{0.02}{n}\right)^{nt} = 1000\lim_{m\to\infty}\left(1+\frac{0.02}{0.02m}\right)^{0.02mt} = 1000\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^{0.02t}.$$

We recognize the limit inside the brackets as the number e. So, the balance in our bank account after t years is given by $1000e^{0.02t}$. Generalizing this concept, we see that if a bank account with an initial balance of $\$P$ earns interest at a rate of $r\%$, compounded continuously, then the balance of the account after t years is

$$\text{Balance} = Pe^{rt}.$$

EXAMPLE 2.43

Compound Interest

A 25-year-old student is offered an opportunity to invest some money in a retirement account that pays 5% annual interest compounded continuously. How much does the student need to invest today to have $\$1$ million when she retires at age 65? What if she could earn 6% annual interest compounded continuously instead?

[Show/Hide Solution]

CHECKPOINT 2.43

Suppose instead of investing at age 25, the student waits until age 35. How much would she have to invest at 5%? At 6%?

If a quantity grows exponentially, the time it takes for the quantity to double remains constant. In other words, it takes the same amount of time for a population of bacteria to grow from 100 to 200 bacteria as it does to grow from $10{,}000$ to $20{,}000$ bacteria. This time is called the doubling time. To calculate the doubling time, we want to know when the quantity reaches twice its original size. So we have

$$\begin{aligned} 2y_0 &= y_0 e^{kt} \\ 2 &= e^{kt} \\ \ln 2 &= kt \\ t &= \tfrac{\ln 2}{k}. \end{aligned}$$

DEFINITION

If a quantity grows exponentially, the **doubling time** is the amount of time it takes the quantity to double. It is given by

$$\text{Doubling time} = \frac{\ln 2}{k}.$$

EXAMPLE 2.44

Using the Doubling Time

Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches $10,000$. When will the owner's friends be allowed to fish?

[Show/Hide Solution]

CHECKPOINT 2.44

Suppose it takes 9 months for the fish population in Example 2.44 to reach 1000 fish. Under these circumstances, how long do the owner's friends have to wait?

Exponential Decay Model

Exponential functions can also be used to model populations that shrink (from disease, for example), or chemical compounds that break down over time. We say that such systems exhibit exponential decay, rather than exponential growth. The model is nearly the same, except there is a negative sign in the exponent. Thus, for some positive constant k, we have $y = y_0 e^{-kt}$.

As with exponential growth, there is a differential equation associated with exponential decay. We have

$$y' = -ky_0 e^{-kt} = -ky.$$

RULE: EXPONENTIAL DECAY MODEL

Systems that exhibit **exponential decay** behave according to the model

$$y = y_0 e^{-kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *decay constant*.

The following figure shows a graph of a representative exponential decay function.

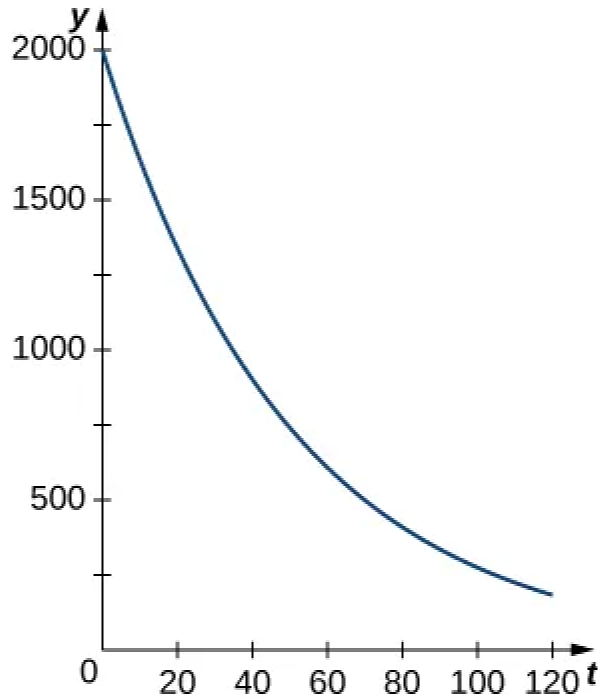


Figure 2.80 An example of exponential decay.

Let’s look at a physical application of exponential decay. Newton’s law of cooling says that an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. In other words, if T represents the temperature of the object and T_a represents the ambient temperature in a room, then

$$T' = -k\left(T - T_a\right).$$

Note that this is not quite the right model for exponential decay. We want the derivative to be proportional to the function, and this expression has the additional T_a term. Fortunately, we can make a change of variables that resolves this issue. Let $y(t) = T(t) - T_a$. Then $y'(t) = T'(t) - 0 = T'(t)$, and our equation becomes

$$y' = -ky.$$

From our previous work, we know this relationship between *y* and its derivative leads to exponential decay. Thus,

$$y = y_0 e^{-kt},$$

and we see that

$$\begin{aligned} T - T_a &= (T_0 - T_a)\, e^{-kt} \\ T &= (T_0 - T_a)\, e^{-kt} + T_a \end{aligned}$$

where T_0 represents the initial temperature. Let's apply this formula in the following example.

EXAMPLE 2.45

Newton's Law of Cooling

According to experienced baristas, the optimal temperature to serve coffee is between $155°\text{F}$ and $175°\text{F}$. Suppose coffee is poured at a temperature of $200°\text{F}$, and after 2 minutes in a $70°\text{F}$ room it has cooled to $180°\text{F}$. When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

[Show/Hide Solution]

CHECKPOINT 2.45

Suppose the room is warmer $(75°\text{F})$ and, after 2 minutes, the coffee has cooled only to $185°\text{F}$. When is the coffee first cool enough to serve? When is the coffee be too cold to serve? Round answers to the nearest half minute.

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{-kt} \\ \frac{1}{2} &= e^{-kt} \\ -\ln 2 &= -kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Note: This is the same expression we came up with for doubling time.

DEFINITION

If a quantity decays exponentially, the **half-life** is the amount of time it takes the quantity to be reduced by half. It is given by

$$\text{Half-life} = \frac{\ln 2}{k}.$$

EXAMPLE 2.46

Radiocarbon Dating

One of the most common applications of an exponential decay model is carbon dating. Carbon-14 decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon was originally present in an object and how much carbon remains, we can determine the age of the object. The half-life of carbon-14 is approximately 5730 years—meaning, after that many years, half the material has converted from the original carbon-14 to the new nonradioactive nitrogen-14. If we have 100 g carbon-14 today, how much is left in 50 years? If an artifact that originally contained 100 g of carbon now contains 10 g of carbon, how old is it? Round the answer to the nearest hundred years.

[Show/Hide Solution]

CHECKPOINT 2.46

If we have 100 g of carbon-14, how much is left after 500 years? If an artifact that originally contained 100 g of carbon now contains $20g$ of carbon, how old is it? Round the answer to the nearest hundred years.

Section 2.8 Exercises

True or False? If true, prove it. If false, find the true answer.

348. The doubling time for $y = e^{ct}$ is $(\ln\ (2))\,/\,(\ln\ (c))$.

349. If you invest $\$500$, an annual rate of interest of 3% yields more money in the first year than a 2.5% continuous rate of interest.

350. If you leave a $100°\text{C}$ pot of tea at room temperature $(25°\text{C})$ and an identical pot in the refrigerator $(5°\text{C})$, with $k = 0.02$, the tea in the refrigerator reaches a drinkable temperature $(70°\text{C})$ more than 5 minutes before the tea at room temperature.

351. If given a half-life of *t* years, the constant k for $y = e^{kt}$ is calculated by $k = \ln\,(1/2)\,/t$.

For the following exercises, use $y = y_0 e^{kt}$.

352. If a culture of bacteria doubles in 3 hours, how many hours does it take to multiply by 10?

353. If bacteria increase by a factor of 10 in 10 hours, how many hours does it take to increase by 100?

354. How old is a skull that contains one-fifth as much radiocarbon as a modern skull? Note that the half-life of radiocarbon is 5730 years.

355. If a relic contains 90% as much radiocarbon as new material, can it have come from the time of Christ (approximately 2000 years ago)? Note that the half-life of radiocarbon is 5730 years.

356. The population of Cairo grew from 5 million to 10 million in 20 years. Use an exponential model to find when the population was 8 million.

357. The populations of New York and Los Angeles are growing at 1% and 1.4% a year, respectively. Starting from 8 million (New York) and 6 million (Los Angeles), when are the populations equal?

358. Suppose the value of $\$1$ in Japanese yen decreases at 2% per year. Starting from $\$1 = ¥250$, when will $\$1 = ¥1$?

359. The effect of advertising decays exponentially. If 40% of the population remembers a new product after 3 days, how long will 20% remember it?

360. If $y = 1000$ at $t = 3$ and $y = 3000$ at $t = 4$, what was y_0 at $t = 0$?

361. If $y = 100$ at $t = 4$ and $y = 10$ at $t = 8$, when does $y = 1$?

362. If a bank offers annual interest of 7.5% or continuous interest of 7.25%, which has a better annual yield?

363. What continuous interest rate has the same yield as an annual rate of 9%?

364. If you deposit $\$5000$ at 8% annual interest, how many years can you withdraw $\$500$ (starting after the first year) without running out of money?

365. You are trying to save $\$50,000$ in 20 years for college tuition for your child. If interest is a continuous 10%, how much do you need to invest initially?

366. You are cooling a turkey that was taken out of the oven with an internal temperature of $165°\mathrm{F}$. After 10 minutes of resting the turkey in a $70°\mathrm{F}$ apartment, the temperature has reached $155°\mathrm{F}$. What is the temperature of the turkey 20 minutes after taking it out of the oven?

367. You are trying to thaw some vegetables that are at a temperature of $1°\mathrm{F}$. To thaw vegetables safely, you must put them in the refrigerator, which has an ambient temperature of $44°\mathrm{F}$. You check on your vegetables 2 hours after putting them in the refrigerator to find that they are now $12°\mathrm{F}$. Plot the resulting temperature curve and use it to determine when the vegetables reach $33°\mathrm{F}$.

368. You are an archaeologist and are given a bone that is claimed to be from a Tyrannosaurus Rex. You know these dinosaurs lived during the Cretaceous Era $($146 million years to 65 million years ago), and you find by radiocarbon dating that there is 0.000001% the amount of radiocarbon. Is this bone from the Cretaceous?

369. The spent fuel of a nuclear reactor contains plutonium-239, which has a half-life of $24,000$ years. If 1 barrel containing $10\ \mathrm{kg}$ of plutonium-239 is sealed, how many years must pass until only $10g$ of plutonium-239 is left?

For the next set of exercises, use the following table, which features the world population by decade.

Years since 1950	Population (millions)
0	$2,556$
10	$3,039$
20	$3,706$
30	$4,453$
40	$5,279$
50	$6,083$
60	$6,849$

Source: http://www.factmonster.com/ipka/A0762181.html.

370. **[T]** The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 2686e^{0.01604t}$. Use a graphing calculator to graph the data and the exponential curve together.

371. **[T]** Find and graph the derivative y' of your equation. Where is it increasing and what is the meaning of this increase?

372. **[T]** Find and graph the second derivative of your equation. Where is it increasing and what is the meaning of this increase?

373. **[T]** Find the predicted date when the population reaches 10 billion. Using your previous answers about the first and second derivatives, explain why exponential growth is unsuccessful in predicting the future.

For the next set of exercises, use the following table, which shows the population of San Francisco during the 19th century.

Years since 1850	Population (thousands)
0	21.00
10	56.80
20	149.5
30	234.0

Source: http://www.sfgenealogy.com/sf/history/hgpop.htm.

374. **[T]** The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 35.26e^{0.06407t}$. Use a graphing calculator to graph the data and the exponential curve together.

375. **[T]** Find and graph the derivative y' of your equation. Where is it increasing? What is the meaning of this increase? Is there a value where the increase is maximal?

376. **[T]** Find and graph the second derivative of your equation. Where is it increasing? What is the meaning of this increase?

2.9 Calculus of the Hyperbolic Functions

Learning Objectives

2.9.1 Apply the formulas for derivatives and integrals of the hyperbolic functions.
2.9.2 Apply the formulas for the derivatives of the inverse hyperbolic functions and their associated integrals.
2.9.3 Describe the common applied conditions of a catenary curve.

We were introduced to hyperbolic functions in Introduction to Functions and Graphs, along with some of their basic properties. In this section, we look at differentiation and integration formulas for the hyperbolic functions and their inverses.

Derivatives and Integrals of the Hyperbolic Functions

Recall that the hyperbolic sine and hyperbolic cosine are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions are then defined in terms of $\sinh x$ and $\cosh x$. The graphs of the hyperbolic functions are shown in the following figure.

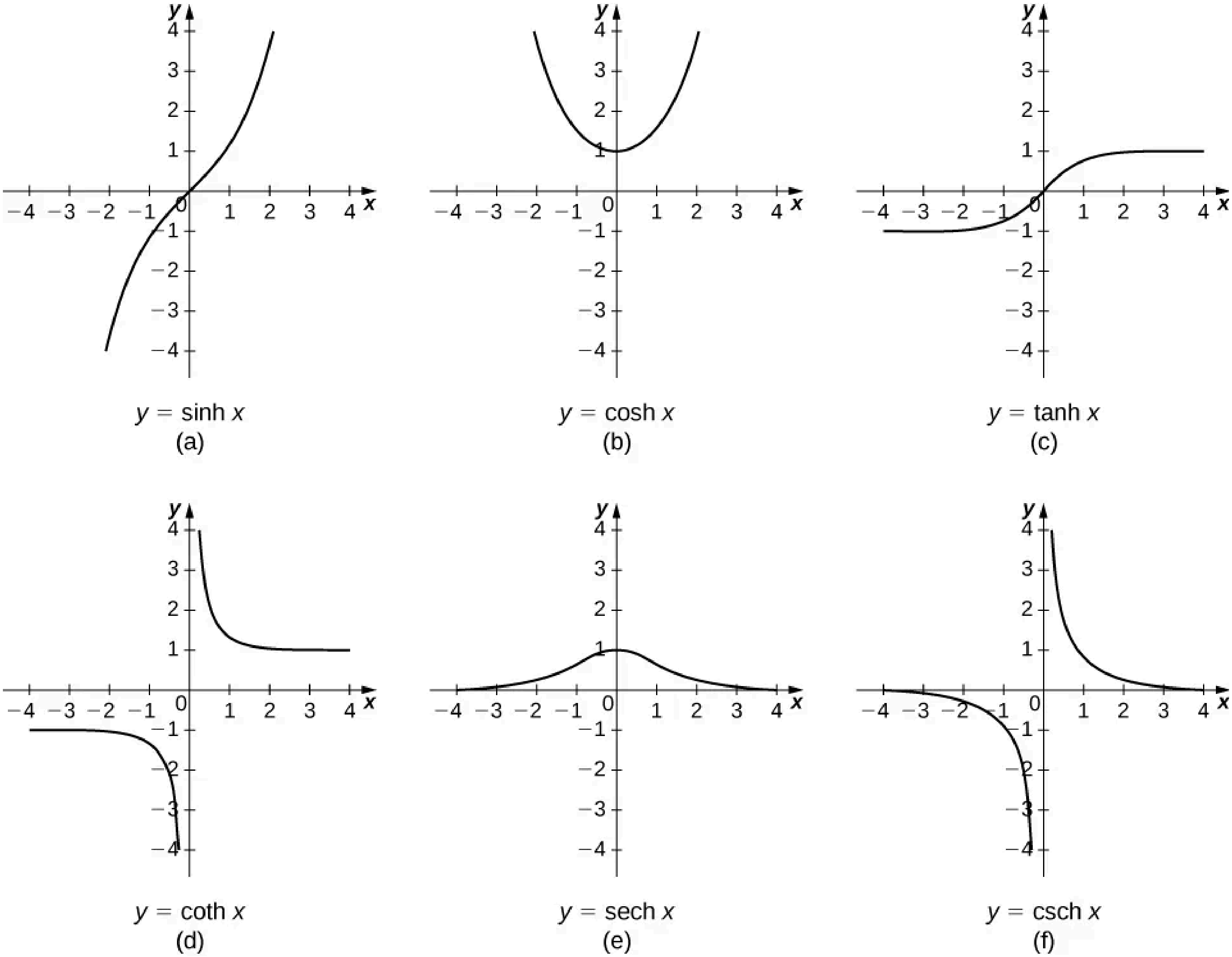


Figure 2.81 Graphs of the hyperbolic functions.

It is easy to develop differentiation formulas for the hyperbolic functions. For example, looking at $\sinh x$ we have

$$\begin{aligned} \frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\ &= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right] \\ &= \frac{1}{2}[e^x + e^{-x}] = \cosh x. \end{aligned}$$

Similarly, $(d/dx)\cosh x = \sinh x$. We summarize the differentiation formulas for the hyperbolic functions in the following table.

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh x$	$\cosh x$

$f(x)$	$\frac{d}{dx} f(x)$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{csch}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$

Table 2.2 Derivatives of the Hyperbolic Functions

Let's take a moment to compare the derivatives of the hyperbolic functions with the derivatives of the standard trigonometric functions. There are a lot of similarities, but differences as well. For example, the derivatives of the sine functions match: $(d/dx) \sin x = \cos x$ and $(d/dx) \sinh x = \cosh x$. The derivatives of the cosine functions, however, differ in sign: $(d/dx) \cos x = -\sin x$, but $(d/dx) \cosh x = \sinh x$. As we continue our examination of the hyperbolic functions, we must be mindful of their similarities and differences to the standard trigonometric functions.

These differentiation formulas for the hyperbolic functions lead directly to the following integral formulas.

$$\begin{aligned} \int \sinh u \, du &= \cosh u + C & \int \operatorname{csch}^2 u \, du &= -\coth u + C \\ \int \cosh u \, du &= \sinh u + C & \int \operatorname{sech} u \tanh u \, du &= -\operatorname{sech} u + C \\ \int \operatorname{sech}^2 u \, du &= \tanh u + C & \int \operatorname{csch} u \coth u \, du &= -\operatorname{csch} u + C \end{aligned}$$

EXAMPLE 2.47

Differentiating Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh\left(x^2\right)\right)$
b. $\frac{d}{dx}(\cosh x)^2$

[Show/Hide Solution]

CHECKPOINT 2.47

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\tanh\left(x^2+3x\right)\right)$
b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)$

EXAMPLE 2.48

Integrals Involving Hyperbolic Functions

Evaluate the following integrals:

a. $\int x\cosh\left(x^2\right)dx$
b. $\int \tanh x\,dx$

[Show/Hide Solution]

CHECKPOINT 2.48

Evaluate the following integrals:

a. $\int \sinh^3 x\cosh x\,dx$
b. $\int \operatorname{sech}^2(3x)\,dx$

Calculus of Inverse Hyperbolic Functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by close examination of the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in the following table.

Function	**Domain**	**Range**
$\sinh^{-1}x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\cosh^{-1}x$	$[1, \infty)$	$[0, \infty)$
$\tanh^{-1}x$	$(-1, 1)$	$(-\infty, \infty)$
$\coth^{-1}x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\operatorname{sech}^{-1}x$	$(0,\ 1]$	$[0, \infty)$
$\operatorname{csch}^{-1}x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Table 2.3 Domains and Ranges of the Inverse Hyperbolic Functions

The graphs of the inverse hyperbolic functions are shown in the following figure.

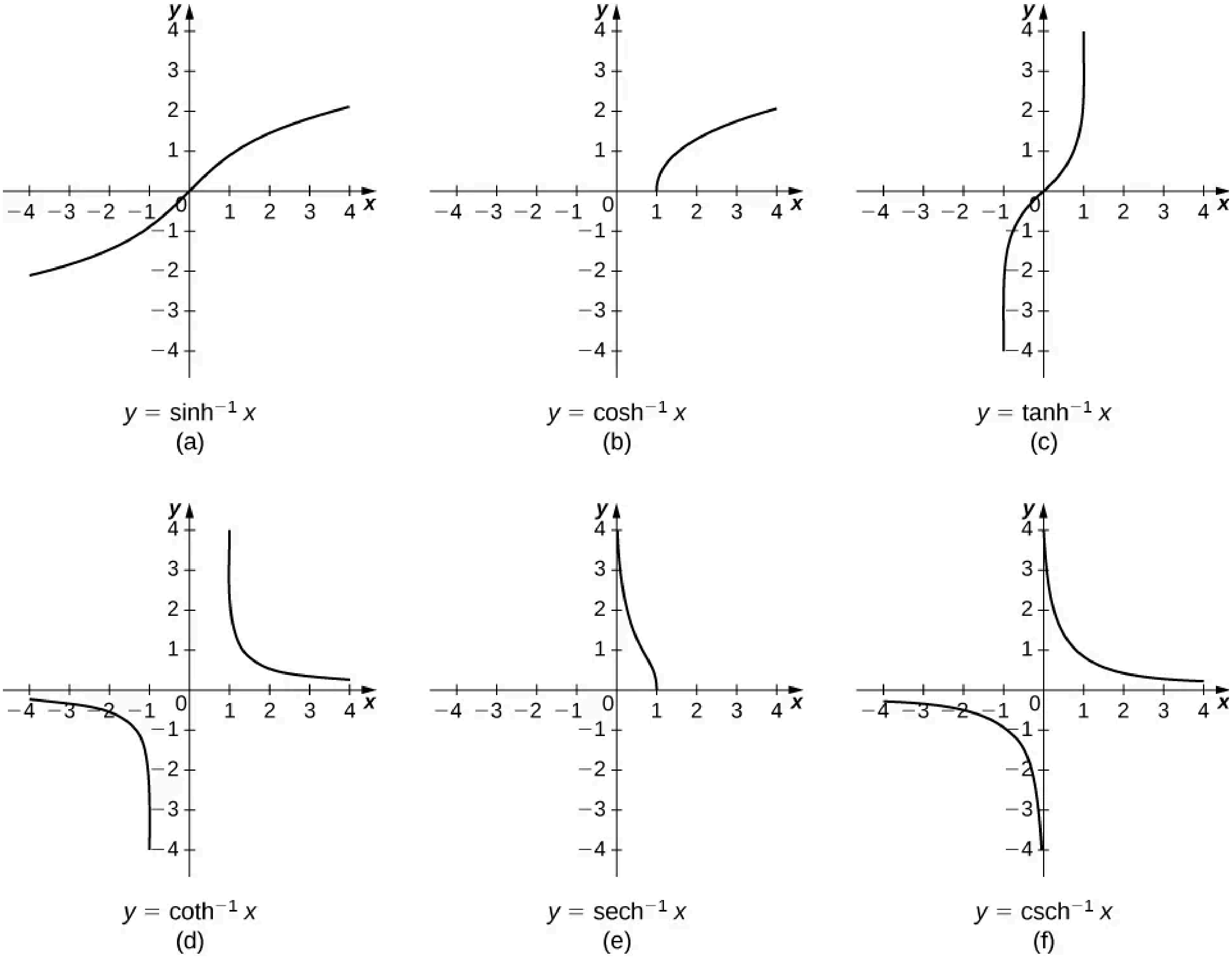


Figure 2.82 Graphs of the inverse hyperbolic functions.

To find the derivatives of the inverse functions, we use implicit differentiation. We have

$$\begin{aligned} y &= \sinh^{-1} x \\ \sinh y &= x \\ \frac{d}{dx}\sinh y &= \frac{d}{dx}x \\ \cosh y\frac{dy}{dx} &= 1. \end{aligned}$$

Recall that $\cosh^2 y - \sinh^2 y = 1$, so $\cosh y = \sqrt{1+\sinh^2 y}$. Then,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

We can derive differentiation formulas for the other inverse hyperbolic functions in a similar fashion. These differentiation formulas are summarized in the following table.

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh^{-1}x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1}x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1}x$	$\frac{1}{1-x^2}$
$\coth^{-1}x$	$\frac{1}{1-x^2}$
$\operatorname{sech}^{-1}x$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{csch}^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Table 2.4 Derivatives of the Inverse Hyperbolic Functions

Note that the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ are the same. Thus, when we integrate $1/\left(1-x^2\right)$, we need to select the proper antiderivative based on the domain of the functions and the values of x. Integration formulas involving the inverse hyperbolic functions are summarized as follows.

$$\int \frac{1}{\sqrt{1+u^2}}du = \sinh^{-1}u + C \qquad \int \frac{1}{u\sqrt{1-u^2}}du = -\operatorname{sech}^{-1}|u| + C$$

$$\int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1}u + C \qquad \int \frac{1}{u\sqrt{1+u^2}}du = -\operatorname{csch}^{-1}|u| + C$$

$$\int \frac{1}{1-u^2}du = \begin{cases} \tanh^{-1}u + C \text{ if } |u| < 1 \\ \coth^{-1}u + C \text{ if } |u| > 1 \end{cases}$$

EXAMPLE 2.49

Differentiating Inverse Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right)$
b. $\frac{d}{dx}\left(\tanh^{-1}x\right)^2$

[Show/Hide Solution]

CHECKPOINT 2.49

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\cosh^{-1}(3x)\right)$
b. $\frac{d}{dx}\left(\coth^{-1}x\right)^3$

EXAMPLE 2.50

Integrals Involving Inverse Hyperbolic Functions

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{4x^2-1}}dx$
b. $\int \frac{1}{2x\sqrt{1-9x^2}}dx$

[Show/Hide Solution]

CHECKPOINT 2.50

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{x^2-4}}dx, \;\; x > 2$
b. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$

Applications

One physical application of hyperbolic functions involves hanging cables. If a cable of uniform density is suspended between two supports without any load other than its own weight, the cable forms a curve called a **catenary**. High-voltage power lines, chains hanging between two posts, and strands of a spider's web all form catenaries. The following figure shows chains hanging from a row of posts.

Figure 2.83 Chains between these posts take the shape of a catenary. (credit: modification of work by OKFoundryCompany, Flickr)

Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a\cosh(x/a)$ are catenaries. Figure 2.84 shows the graph of $y = 2\cosh(x/2)$.

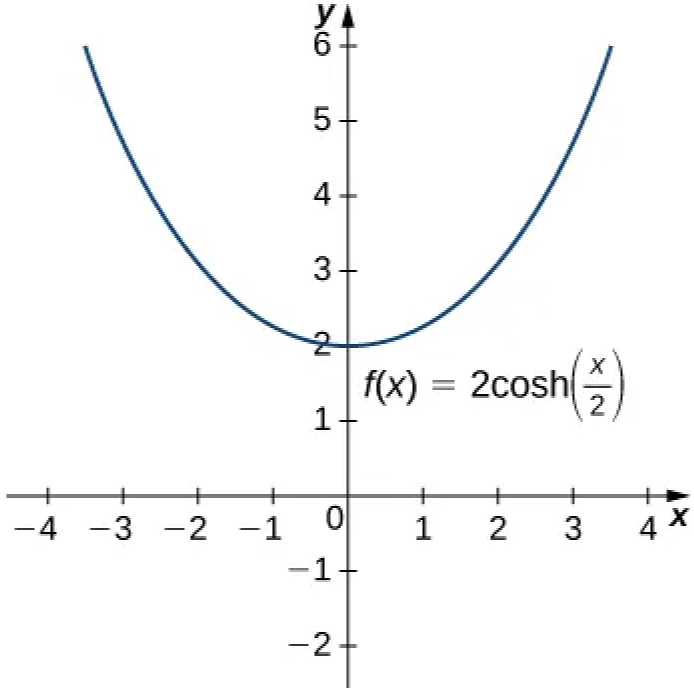


Figure 2.84 A hyperbolic cosine function forms the shape of a catenary.

EXAMPLE 2.51

Using a Catenary to Find the Length of a Cable

Assume a hanging cable has the shape $10 \cosh (x/10)$ for $-15 \leq x \leq 15$, where x is measured in feet. Determine the length of the cable (in feet).

[Show/Hide Solution]

CHECKPOINT 2.51

Assume a hanging cable has the shape $15 \cosh (x/15)$ for $-20 \leq x \leq 20$. Determine the length of the cable (in feet).

Section 2.9 Exercises

377. **[T]** Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$. Use a calculator to graph these functions and ensure your expression is correct.

378. From the definitions of $\cosh(x)$ and $\sinh(x)$, find their antiderivatives.

379. Show that $\cosh(x)$ and $\sinh(x)$ satisfy $y'' = y$.

380. Use the quotient rule to verify that $\tanh(x)' = \text{sech}^2(x)$.

381. Derive $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$ from the definition.

382. Take the derivative of the previous expression to find an expression for $\sinh(2x)$.

383. Prove $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ by changing the expression to exponentials.

384. Take the derivative of the previous expression to find an expression for $\cosh(x+y)$.

For the following exercises, find the derivatives of the given functions and graph along with the function to ensure your answer is correct.

385. **[T]** $\cosh(3x+1)$

386. **[T]** $\sinh(x^2)$

387. **[T]** $\frac{1}{\cosh(x)}$

388. **[T]** $\sinh(\ln(x))$

389. **[T]** $\cosh^2(x) + \sinh^2(x)$

390. **[T]** $\cosh^2(x) - \sinh^2(x)$

391. **[T]** $\tanh\left(\sqrt{x^2+1}\right)$

392. **[T]** $\frac{1+\tanh(x)}{1-\tanh(x)}$

393. **[T]** $\sinh^6(x)$

394. **[T]** $\ln(\text{sech}(x) + \tanh(x))$

For the following exercises, find the antiderivatives for the given functions.

395. $\cosh(2x+1)$

396. $\tanh(3x+2)$

397. $x\cosh(x^2)$

398. $3x^3\tanh(x^4)$

399. $\cosh^2(x)\sinh(x)$

400. $\tanh^2(x)\operatorname{sech}^2(x)$

401. $\frac{\sinh(x)}{1+\cosh(x)}$

402. $\coth(x)$

403. $\cosh(x)+\sinh(x)$

404. $(\cosh(x)+\sinh(x))^n$

For the following exercises, find the derivatives for the functions.

405. $\tanh^{-1}(4x)$

406. $\sinh^{-1}(x^2)$

407. $\sinh^{-1}(\cosh(x))$

408. $\cosh^{-1}(x^3)$

409. $\tanh^{-1}(\cos(x))$

410. $e^{\sinh^{-1}(x)}$

411. $\ln\left(\tanh^{-1}(x)\right)$

For the following exercises, find the antiderivatives for the functions.

412. $\int \frac{dx}{4-x^2}$

413. $\int \frac{dx}{a^2-x^2}$

414. $\int \frac{dx}{\sqrt{x^2+1}}$

415. $\int \frac{x\,dx}{\sqrt{x^2+1}}$

416. $\int -\frac{dx}{x\sqrt{1-x^2}}$

417. $\int \frac{e^x}{\sqrt{e^{2x}-1}}$

418. $\int -\frac{2x}{x^4-1}$

For the following exercises, use the fact that a falling body with friction equal to velocity squared obeys the equation $dv/dt = g - v^2$.

419. Show that $v(t) = \sqrt{g}\tanh\left(\left(\sqrt{g}\right)t\right)$ satisfies this equation.

420. Derive the previous expression for $v(t)$ by integrating $\frac{dv}{g - v^2} = dt$.

421. **[T]** Estimate how far a body has fallen in 12 seconds by finding the area underneath the curve of $v(t)$.

For the following exercises, use this scenario: A cable hanging under its own weight has a slope $S = dy/dx$ that satisfies $dS/dx = c\sqrt{1+S^2}$. The constant c is the ratio of cable density to tension.

422. Show that $S = \sinh(cx)$ satisfies this equation.

423. Integrate $dy/dx = \sinh(cx)$ to find the cable height $y(x)$ if $y(0) = 1/c$.

424. Sketch the cable and determine how far down it sags at $x = 0$.

For the following exercises, solve each problem.

425. **[T]** A chain hangs from two posts 2 m apart to form a catenary described by the equation $y = 2\cosh\left(x/2\right) - 1$. Find the slope of the catenary at the left fence post.

426. **[T]** A chain hangs from two posts four meters apart to form a catenary described by the equation $y = 4\cosh\left(x/4\right) - 3$. Find the total length of the catenary (arc length).

427. **[T]** A high-voltage power line is a catenary described by $y = 10\cosh\left(x/10\right)$. Find the ratio of the area under the catenary to its arc length. What do you notice?

428. A telephone line is a catenary described by $y = a\cosh\left(x/a\right)$. Find the ratio of the area under the catenary to its arc length. Does this confirm your answer for the previous question?

429. Prove the formula for the derivative of $y = \sinh^{-1}(x)$ by differentiating $x = \sinh(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

430. Prove the formula for the derivative of $y = \cosh^{-1}(x)$ by differentiating $x = \cosh(y)$.

(*Hint:* Use hyperbolic trigonometric identities.)

431. Prove the formula for the derivative of $y = \operatorname{sech}^{-1}(x)$ by differentiating $x = \operatorname{sech}(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

432. Prove that $(\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$.

433. Prove the expression for $\sinh^{-1}(x)$. Multiply $x = \sinh(y) = (1/2)\left(e^{y} + e^{-y}\right)$ by $2e^{y}$ and solve for y. Does your expression match the textbook?

434. Prove the expression for $\cosh^{-1}(x)$. Multiply $x = \cosh(y) = (1/2)\left(e^{y} - e^{-y}\right)$ by $2e^{y}$ and solve for y. Does your expression match the textbook?