1.4 Integration Formulas and the Net Change Theorem

Learning Objectives

1.4.1 Apply the basic integration formulas.
1.4.2 Explain the significance of the net change theorem.
1.4.3 Use the net change theorem to solve applied problems.
1.4.4 Apply the integrals of odd and even functions.

In this section, we use some basic integration formulas studied previously to solve some key applied problems. It is important to note that these formulas are presented in terms of *indefinite* integrals. Although definite and indefinite integrals are closely related, there are some key differences to keep in mind. A definite integral is either a number (when the limits of integration are constants) or a single function (when one or both of the limits of integration are variables). An indefinite integral represents a family of functions, all of which differ by a constant. As you become more familiar with integration, you will get a feel for when to use definite integrals and when to use indefinite integrals. You will naturally select the correct approach for a given problem without thinking too much about it. However, until these concepts are cemented in your mind, think carefully about whether you need a definite integral or an indefinite integral and make sure you are using the proper notation based on your choice.

Basic Integration Formulas

Recall the integration formulas given in the table in Antiderivatives and the rule on properties of definite integrals. Let's look at a few examples of how to apply these rules.

EXAMPLE 1.23

Integrating a Function Using the Power Rule

Use the power rule to integrate the function $\int_1^4 \sqrt{t}\,(1+t)\,dt.$

[Show/Hide Solution]

CHECKPOINT 1.21

Find the definite integral of $f(x) = x^2 - 3x$ over the interval $[1, 3]$.

The Net Change Theorem

The **net change theorem** considers the integral of a *rate of change*. It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways. The second is more familiar; it is simply the definite integral.

THEOREM 1.6

Net Change Theorem

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

or

$$\int_a^b F'(x)\,dx = F(b) - F(a). \qquad (1.18)$$

Subtracting $F(a)$ from both sides of the first equation yields the second equation. Since they are equivalent formulas, which one we use depends on the application.

The significance of the net change theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral. To illustrate, let’s apply the net change theorem to a velocity function in which the result is displacement.

We looked at a simple example of this in The Definite Integral. Suppose a car is moving due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m. We can graph this motion as shown in Figure 1.32.

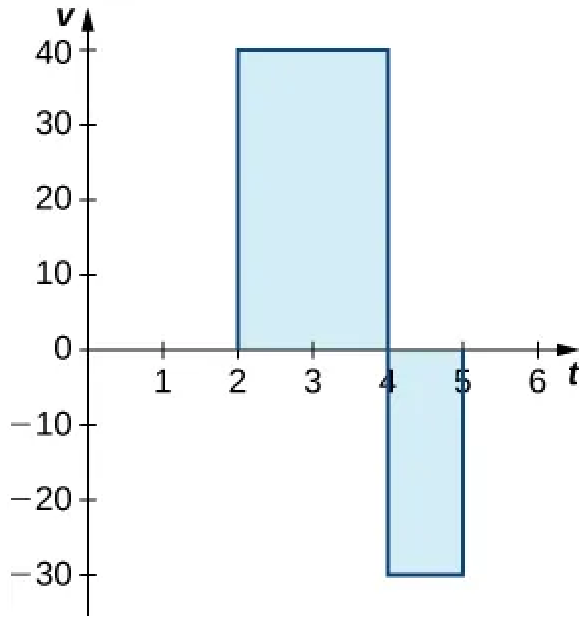


Figure 1.32 The graph shows speed versus time for the given motion of a car.

Just as we did before, we can use definite integrals to calculate the net displacement as well as the total distance traveled. The net displacement is given by

$$\begin{aligned}\int_2^5 v(t)dt &= \int_2^4 40dt + \int_4^5 -30dt \\ &= 80 - 30 \\ &= 50.\end{aligned}$$

Thus, at 5 p.m. the car is 50 mi north of its starting position. The total distance traveled is given by

$$\begin{aligned}\int_2^5 |v(t)|\,dt &= \int_2^4 40dt + \int_4^5 30dt \\ &= 80 + 30 \\ &= 110.\end{aligned}$$

Therefore, between 2 p.m. and 5 p.m., the car traveled a total of 110 mi.

To summarize, net displacement may include both positive and negative values. In other words, the velocity function accounts for both forward distance and backward distance. To find net displacement, integrate the velocity function over the interval. Total distance traveled, on the other hand, is always positive. To find the total distance traveled by an object, regardless of direction, we need to integrate the absolute value of the velocity function.

EXAMPLE 1.24

Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

[Show/Hide Solution]

EXAMPLE 1.25

Finding the Total Distance Traveled

Use Example 1.24 to find the total distance traveled by a particle according to the velocity function $v(t) = 3t - 5$ m/sec over a time interval $[0, 3]$.

[Show/Hide Solution]

CHECKPOINT 1.22

Find the net displacement and total distance traveled in meters given the velocity function $f(t) = \frac{1}{2}e^t - 2$ over the interval $[0, 2]$.

Applying the Net Change Theorem

The net change theorem can be applied to the flow and consumption of fluids, as shown in Example 1.26.

EXAMPLE 1.26

How Many Gallons of Gasoline Are Consumed?

If the motor on a motorboat is started at $t = 0$ and the boat consumes gasoline at the rate of $5 - 0.1t^3$ gal/hr, how much gasoline is used in the first 2 hours?

[Show/Hide Solution]

EXAMPLE 1.27

Chapter Opener: Iceboats

Figure 1.34 (credit: modification of work by Carter Brown, Flickr)

As we saw at the beginning of the chapter, top iceboat racers (Figure 1.1) can attain speeds of up to five times the wind speed. Andrew is an intermediate iceboater, though, so he attains speeds equal to only twice the wind speed. Suppose Andrew takes his iceboat out one morning when a light 5-mph breeze has been blowing all morning. As Andrew gets his iceboat set up, though, the wind begins to pick up. During his first half hour of iceboating, the wind speed increases according to the function $v(t) = 20t + 5$. For the second half hour of Andrew's outing, the wind remains steady at 15 mph. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5 & \text{for} \quad 0 \le t \le \frac{1}{2} \\ 15 & \text{for} \quad \frac{1}{2} \le t \le 1. \end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

[Show/Hide Solution]

CHECKPOINT 1.23

Suppose that, instead of remaining steady during the second half hour of Andrew's outing, the wind starts to die down according to the function $v(t) = -10t + 15$. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5 & \text{for} \quad 0 \le t \le \frac{1}{2} \\ -10t + 15 & \text{for} \quad \frac{1}{2} \le t \le 1. \end{cases}$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

Integrating Even and Odd Functions

We saw in Functions and Graphs that an even function is a function in which $f(-x) = f(x)$ for all *x* in the domain—that is, the graph of the curve is unchanged when *x* is replaced with –*x*. The graphs of even functions are symmetric about the *y*-axis. An odd function is one in which $f(-x) = -f(x)$ for all *x* in the domain, and the graph of the function is symmetric about the origin.

Integrals of even functions, when the limits of integration are from –*a* to *a*, involve two equal areas, because they are symmetric about the *y*-axis. Integrals of odd functions, when the limits of integration are similarly $[-a, a]$, evaluate to zero because the areas above and below the *x*-axis are equal.

RULE: INTEGRALS OF EVEN AND ODD FUNCTIONS

For continuous even functions such that $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

For continuous odd functions such that $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

EXAMPLE 1.28

Integrating an Even Function

Integrate the even function $\int_{-2}^{2}(3x^8-2)\,dx$ and verify that the integration formula for even functions holds.

[Show/Hide Solution]

EXAMPLE 1.29

Integrating an Odd Function

Evaluate the definite integral of the odd function $-5\sin x$ over the interval $[-\pi, \pi]$.

[Show/Hide Solution]

CHECKPOINT 1.24

Integrate the function $\int_{-2}^{2} x^4 dx.$

Section 1.4 Exercises

Use basic integration formulas to compute the following antiderivatives or definite integrals.

207. $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx$

208. $\int\left(e^{2x}-\frac{1}{2}e^{x/2}\right)dx$

209. $\int\frac{dx}{2x}$

210. $\int \frac{x-1}{x^2} dx$

211. $\int_0^{\pi} (\sin x - \cos x)\, dx$

212. $\int_0^{\pi/2} (x - \sin x)\, dx$

213. Write an integral that expresses the increase in the perimeter $P(s)$ of a square when its side length *s* increases from 2 units to 4 units and evaluate the integral.

214. Write an integral that quantifies the change in the area $A(s) = s^2$ of a square when the side length doubles from *S* units to 2*S* units and evaluate the integral.

215. A regular *N*-gon (an *N*-sided polygon with sides that have equal length *s*, such as a pentagon or hexagon) has perimeter *Ns*. Write an integral that expresses the increase in perimeter of a regular *N*-gon when the length of each side increases from 1 unit to 2 units and evaluate the integral.

216. The area of a regular pentagon with side length $a > 0$ is pa^2 with $p = \frac{1}{4}\sqrt{5 + \sqrt{5 + 2\sqrt{5}}}$. The Pentagon in Washington, DC, has inner sides of length 360 ft and outer sides of length 920 ft. Write an integral to express the area of the roof of the Pentagon according to these dimensions and evaluate this area.

217. A dodecahedron is a Platonic solid with a surface that consists of 12 pentagons, each of equal area. By how much does the surface area of a dodecahedron increase as the side length of each pentagon doubles from 1 unit to 2 units?

218. An icosahedron is a Platonic solid with a surface that consists of 20 equilateral triangles. By how much does the surface area of an icosahedron increase as the side length of each triangle doubles from *a* unit to 2*a* units?

219. Write an integral that quantifies the change in the area of the surface of a cube when its side length doubles from *s* unit to 2*s* units and evaluate the integral.

220. Write an integral that quantifies the increase in the volume of a cube when the side length doubles from *s* unit to 2*s* units and evaluate the integral.

221. Write an integral that quantifies the increase in the surface area of a sphere as its radius doubles from *R* unit to 2*R* units and evaluate the integral.

222. Write an integral that quantifies the increase in the volume of a sphere as its radius doubles from *R* unit to 2*R* units and evaluate the integral.

223. Suppose that a particle moves along a straight line with velocity $v(t) = 4 - 2t$, where $0 \leq t \leq 2$ (in meters per second). Find the displacement at time *t* and the total distance traveled up to $t = 2$.

224. Suppose that a particle moves along a straight line with velocity defined by $v(t) = t^2 - 3t - 18$, where $0 \leq t \leq 6$ (in meters per second). Find the displacement at time *t* and the total distance traveled up to $t = 6$.

225. Suppose that a particle moves along a straight line with velocity defined by $v(t) = |2t - 6|$, where $0 \leq t \leq 6$ (in meters per second). Find the displacement at time *t* and the total distance traveled up to $t = 6$.

226. Suppose that a particle moves along a straight line with acceleration defined by $a(t) = t - 3$, where $0 \leq t \leq 6$ (in meters per second). Find the velocity and displacement at time *t* and the total distance traveled up to $t = 6$ if $v(0) = 3$ and $d(0) = 0$.

227. A ball is thrown upward from a height of 1.5 m at an initial speed of 40 m/sec. Acceleration resulting from gravity is –9.8 m/sec^2. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball *t* seconds after it is thrown and before it returns to the ground.

228. A ball is thrown upward from a height of 3 m at an initial speed of 60 m/sec. Acceleration resulting from gravity is –9.8 m/sec^2. Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball *t* seconds after it is thrown and before it returns to the ground.

229. The area $A(t)$ of a circular shape is growing at a constant rate. If the area increases from 4π units to 9π units between times $t = 2$ and $t = 3$, find the net change in the radius during that time.

230. A spherical balloon is being inflated at a constant rate. If the volume of the balloon changes from 36π in.^{3} to 288π in.^{3} between time $t = 30$ and $t = 60$ seconds, find the net change in the radius of the balloon during that time.

231. Water flows into a conical tank with cross-sectional area πx^2 at height *x* and volume $\frac{\pi x^3}{3}$ up to height *x*. If water flows into the tank at a rate of 1 m^3/min, find the height of water in the tank after 5 min. Find the change in height between 5 min and 10 min.

232. A horizontal cylindrical tank has cross-sectional area $A(x) = 4(6x - x^2)\, m^2$ at height *x* meters above the bottom when $x \leq 3$.

a. The volume *V* between heights *a* and *b* is $\displaystyle\int_a^b A(x)\, dx$. Find the volume at heights between 2 m and 3 m.
b. Suppose that oil is being pumped into the tank at a rate of 50 L/min. Using the chain rule, $\frac{dx}{dt} = \frac{dx}{dV} \frac{dV}{dt}$, at how many meters per minute is the height of oil in the tank

changing, expressed in terms of x, when the height is at x meters?

c. How long does it take to fill the tank to 3 m starting from a fill level of 2 m?

233. The following table lists the electrical power in gigawatts—the rate at which energy is consumed—used in a certain city for different hours of the day, in a typical 24-hour period, with hour 1 corresponding to midnight to 1 a.m.

Hour	**Power**	**Hour**	**Power**
1	28	13	48
2	25	14	49
3	24	15	49
4	23	16	50
5	24	17	50
6	27	18	50
7	29	19	46
8	32	20	43
9	34	21	42
10	39	22	40
11	42	23	37
12	46	24	34

Find the total amount of energy in gigawatt-hours (gW-h) consumed by the city in a typical 24-hour period.

234. The average residential electrical power use (in hundreds of watts) per hour is given in the following table.

Hour	**Power**	**Hour**	**Power**
1	8	13	12
2	6	14	13

Hour	Power	Hour	Power
3	5	15	14
4	4	16	15
5	5	17	17
6	6	18	19
7	7	19	18
8	8	20	17
9	9	21	16
10	10	22	16
11	10	23	13
12	11	24	11

a. Compute the average total energy used in a day in kilowatt-hours (kWh).
b. If a ton of coal generates 1842 kWh, how long does it take for an average residence to burn a ton of coal?
c. Explain why the data might fit a plot of the form $p(t) = 11.5 - 7.5\sin\left(\frac{\pi t}{12}\right)$.

235. The data in the following table are used to estimate the average power output produced by Peter Sagan for each of the last 18 sec of Stage 1 of the 2012 Tour de France.

Second	Watts	Second	Watts
1	600	10	1200
2	500	11	1170
3	575	12	1125
4	1050	13	1100
5	925	14	1075
6	950	15	1000
7	1050	16	950

Second	Watts	Second	Watts
8	950	17	900
9	1100	18	780

Table 1.6 Average Power Output *Source*: sportsexercisengineering.com

Estimate the net energy used in kilojoules (kJ), noting that 1W = 1 J/s, and the average power output by Sagan during this time interval.

236. The data in the following table are used to estimate the average power output produced by Peter Sagan for each 15-min interval of Stage 1 of the 2012 Tour de France.

Minutes	Watts	Minutes	Watts
15	200	165	170
30	180	180	220
45	190	195	140
60	230	210	225
75	240	225	170
90	210	240	210
105	210	255	200
120	220	270	220
135	210	285	250
150	150	300	400

Table 1.7 Average Power Output *Source*: sportsexercisengineering.com

Estimate the net energy used in kilojoules, noting that 1W = 1 J/s.

237. The distribution of incomes as of 2012 in the United States in $5000 increments is given in the following table. The *k*th row denotes the percentage of households with incomes between $\$5000xk$ and $5000xk + 4999$. The row $k = 40$ contains all households with income between $200,000 and $250,000.

0	3.5	11	3.5	21	1.5	31	0.6
1	4.1	12	3.7	22	1.4	32	0.5
2	5.9	13	3.2	23	1.3	33	0.5
3	5.7	14	3.0	24	1.3	34	0.4
4	5.9	15	2.8	25	1.1	35	0.3
5	5.4	16	2.5	26	1.0	36	0.3
6	5.5	17	2.2	27	0.75	37	0.3
7	5.1	18	2.2	28	0.8	38	0.2
8	4.8	19	1.8	29	1.0	39	1.8
9	4.1	20	2.1	30	0.6	40	2.3
10	4.3						

Table 1.8 Income Distributions *Source*: http://www.census.gov/prod/2013pubs/p60-245.pdf

a. Estimate the percentage of U.S. households in 2012 with incomes less than $55,000.
b. What percentage of households had incomes exceeding $85,000?
c. Plot the data and try to fit its shape to that of a graph of the form $a\left(x+c\right)e^{-b(x+e)}$ for suitable a, b, c.

238. Newton's law of gravity states that the gravitational force exerted by an object of mass *M* and one of mass *m* with centers that are separated by a distance *r* is $F = G\frac{mM}{r^2}$, with *G* an empirical constant $G = 6.67x10^{-11}\,m^3/\left(kg\cdot s^2\right)$. The work done by a variable force over an interval $[a, b]$ is defined as $W = \displaystyle\int_a^b F\left(x\right)dx$. If Earth has mass $5.97219\ \times\ 10^{24}$ and radius 6371 km, compute the amount of work to elevate a polar weather satellite of mass 1400 kg to its orbiting altitude of 850 km above Earth.

239. For a given motor vehicle, the maximum achievable deceleration from braking is approximately 7 m/sec^2 on dry concrete. On wet asphalt, it is approximately 2.5 m/sec^2. Given that 1 mph corresponds to 0.447 m/sec, find the total distance that a car travels in meters on dry concrete after the brakes are applied until it comes to a complete stop if the initial velocity is 67 mph (30 m/sec) or if the initial braking velocity is 56 mph (25 m/sec). Find the corresponding distances if the surface is slippery wet asphalt.

240. John is a 25-year old man who weighs 160 lb. He burns $500 - 50t$ calories/hr while riding his bike for *t* hours. If an oatmeal cookie has 55 cal and John eats 4*t* cookies during the *t*th hour, how many net calories has he lost after 3 hours riding his bike?

241. Sandra is a 25-year old woman who weighs 120 lb. She burns $300 - 50t$ cal/hr while walking on her treadmill. Her caloric intake from drinking Gatorade is 100*t* calories during the *t*th hour. What is her net decrease in calories after walking for 3 hours?

242. A motor vehicle has a maximum efficiency of 33 mpg at a cruising speed of 40 mph. The efficiency drops at a rate of 0.1 mpg/mph between 40 mph and 50 mph, and at a rate of 0.4 mpg/mph between 50 mph and 80 mph. What is the efficiency in miles per gallon if the car is cruising at 50 mph? What is the efficiency in miles per gallon if the car is cruising at 80 mph? If gasoline costs $3.50/gal, what is the cost of fuel to drive 50 mi at 40 mph, at 50 mph, and at 80 mph?

243. Although some engines are more efficient at given a horsepower than others, on average, fuel efficiency decreases with horsepower at a rate of $1/25$ mpg/horsepower. If a typical 50-horsepower engine has an average fuel efficiency of 32 mpg, what is the average fuel efficiency of an engine with the following horsepower: 150, 300, 450?

244. **[T]** The following table lists the 2013 schedule of federal income tax versus taxable income.

Taxable Income Range	The Tax Is ...	... Of the Amount Over
$0–$8925	10%	$0
$8925–$36,250	$892.50 + 15%	$8925
$36,250–$87,850	$4,991.25 + 25%	$36,250
$87,850–$183,250	$17,891.25 + 28%	$87,850
$183,250–$398,350	$44,603.25 + 33%	$183,250
$398,350–$400,000	$115,586.25 + 35%	$398,350
> $400,000	$116,163.75 + 39.6%	$400,000

Table 1.9 Federal Income Tax Versus Taxable Income *Source*: http://www.irs.gov/pub/irs-prior/i1040tt--2013.pdf.

Suppose that Steve just received a $10,000 raise. How much of this raise is left after federal taxes if Steve's salary before receiving the raise was $40,000? If it was $90,000? If it was $385,000?

245. **[T]** The following table provides hypothetical data regarding the level of service for a certain highway.

Highway Speed Range (mph)	Vehicles per Hour per Lane	Density Range (vehicles/mi)
> 60	< 600	< 10
60–57	600–1000	10–20
57–54	1000–1500	20–30
54–46	1500–1900	30–45
46–30	1900–2100	45–70
<30	Unstable	70–200

Table 1.10

a. Plot vehicles per hour per lane on the *x*-axis and highway speed on the *y*-axis.
b. Compute the average decrease in speed (in miles per hour) per unit increase in congestion (vehicles per hour per lane) as the latter increases from 600 to 1000, from 1000 to 1500, and from 1500 to 2100. Does the decrease in miles per hour depend linearly on the increase in vehicles per hour per lane?
c. Plot minutes per mile (60 times the reciprocal of miles per hour) as a function of vehicles per hour per lane. Is this function linear?

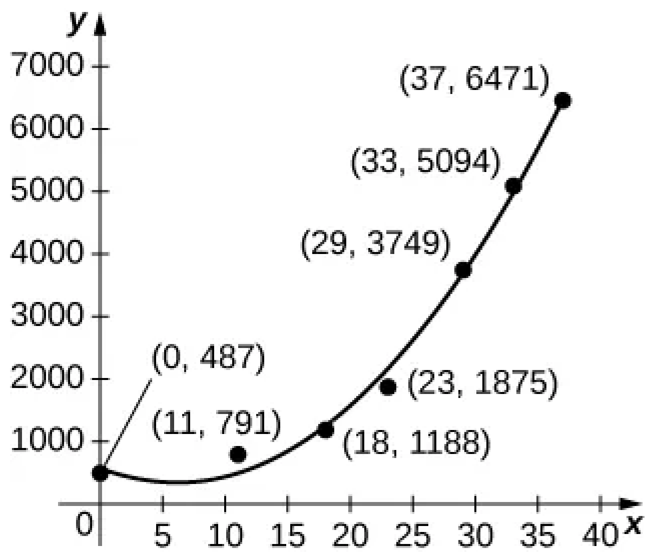
For the next two exercises use the data in the following table, which displays bald eagle populations from 1963 to 2000 in the continental United States.

Year	Population of Breeding Pairs of Bald Eagles
1963	487
1974	791
1981	1188
1986	1875
1992	3749
1996	5094

Year	Population of Breeding Pairs of Bald Eagles
2000	6471

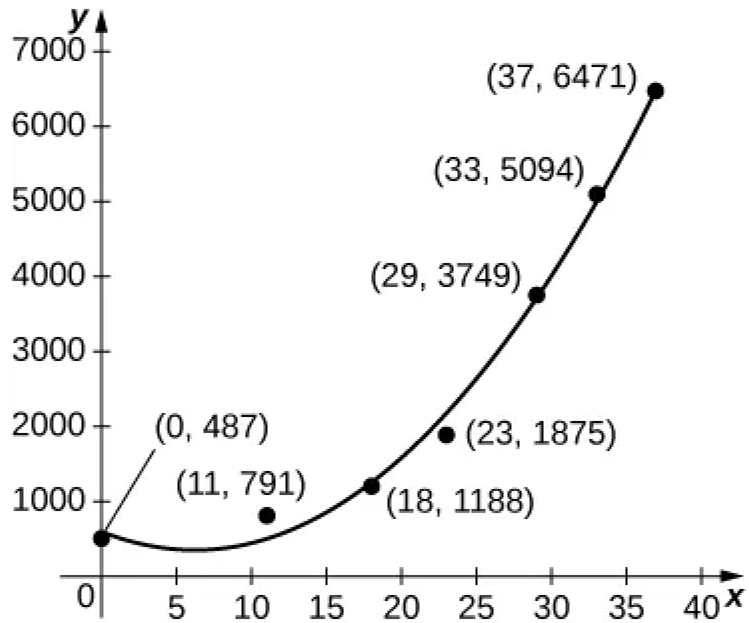
Table 1.11 Population of Breeding Bald Eagle Pairs *Source*: http://www.fws.gov/Midwest/eagle/population/chtofprs.html.

246. **[T]** The graph below plots the quadratic $p(t) = 6.48t^2 - 80.3\,1t + 585.69$ against the data in preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of *p*



over $[0, 37]$.

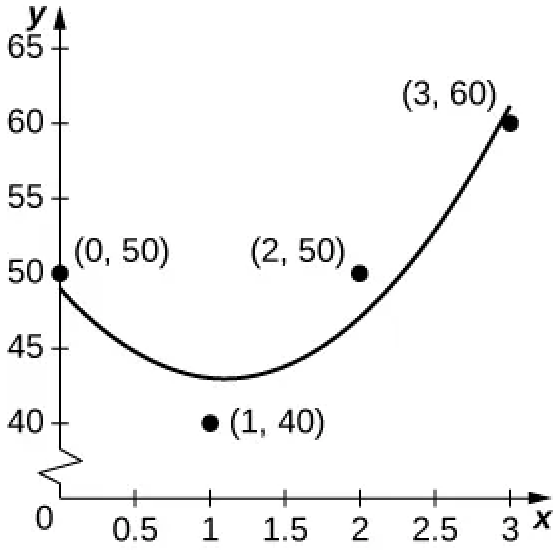
247. **[T]** The graph below plots the cubic $p(t) = 0.07t^3 + 2.42t^2 - 25.63t + 521.23$ against the data in the preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average



value of *p* over $[0, 37]$.

248. **[T]** Suppose you go on a road trip and record your speed at every half hour, as compiled in the following table. The best quadratic fit to the data is $q(t) = 5x^2 - 11x + 49$, shown in the accompanying graph. Integrate *q* to estimate the total distance driven over the 3 hours.

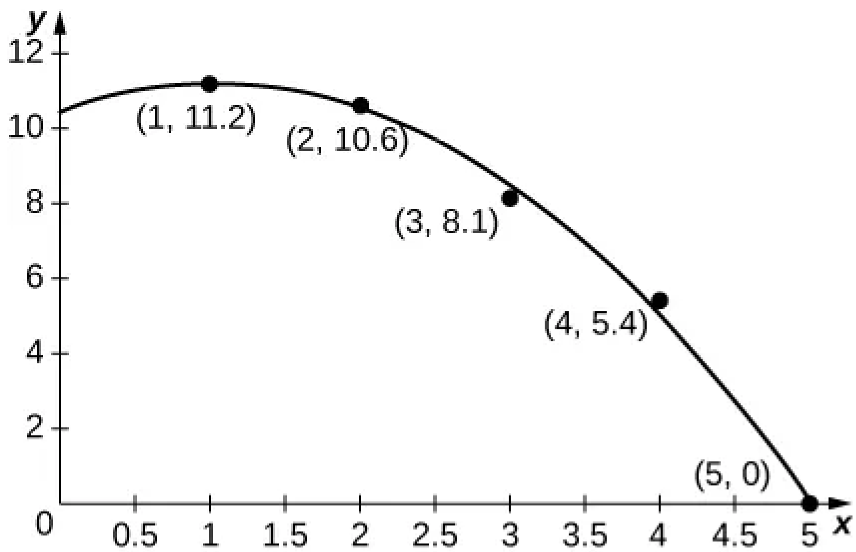
Time (hr)	**Speed (mph)**
0 (start)	50
1	40
2	50
3	60



As a car accelerates, it does not accelerate at a constant rate; rather, the acceleration is variable. For the following exercises, use the following table, which contains the acceleration measured at every second as a driver merges onto a freeway.

Time (sec)	Acceleration (mph/sec)
1	11.2
2	10.6
3	8.1
4	5.4
5	0

249. **[T]** The accompanying graph plots the best quadratic fit, $a(t) = -0.70t^2 + 1.44t + 10.44,$ to the data from the preceding table. Compute the average value of $a(t)$ to estimate the average acceleration between $t = 0$ and $t = 5$.

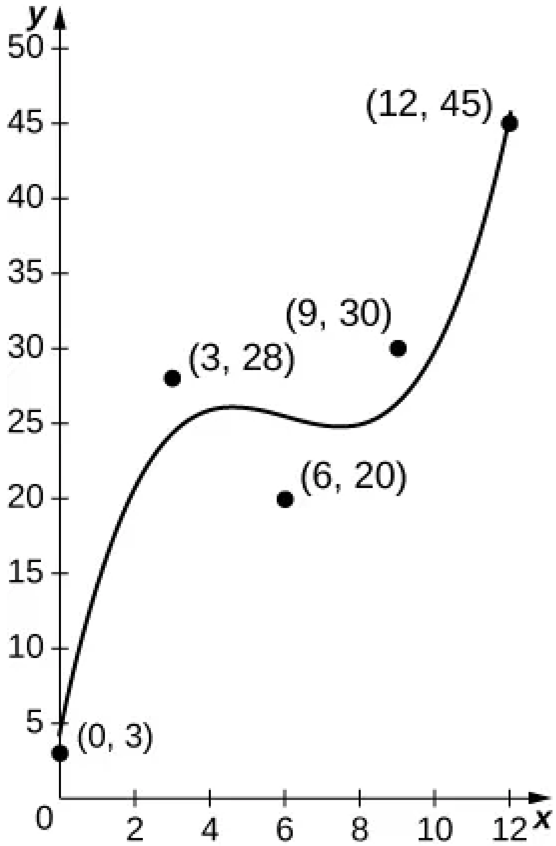


250. **[T]** Using your acceleration equation from the previous exercise, find the corresponding velocity equation. Assuming the initial velocity is 0 mph, find the velocity at time $t = 0$.

251. **[T]** Using your velocity equation from the previous exercise, find the corresponding distance equation, assuming your initial distance is 0 mi. How far did you travel while you accelerated your car? (*Hint:* You will need to convert time units.)

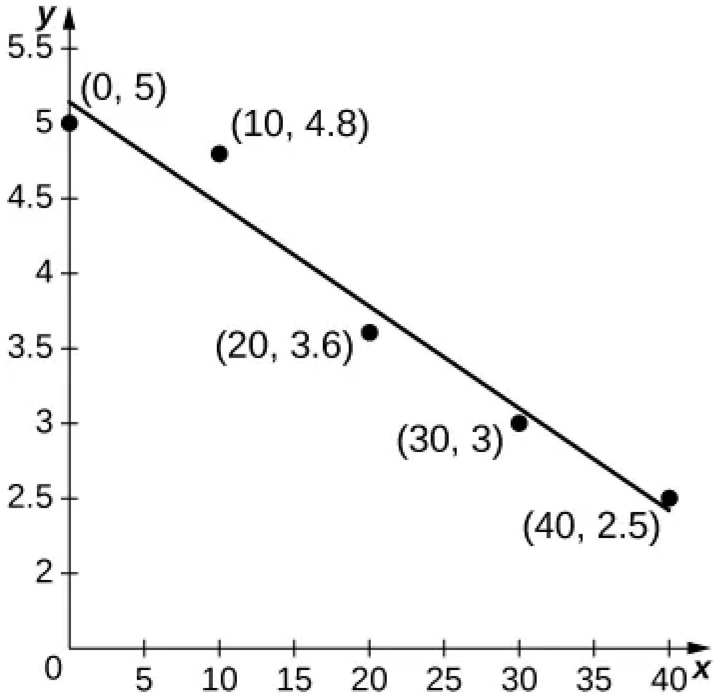
252. **[T]** The number of hamburgers sold at a restaurant throughout the day is given in the following table, with the accompanying graph plotting the best cubic fit to the data, $b\left(t\right) = 0.12t^3 - 2.13t^2 + 12.13t + 3.91$, with $t = 0$ corresponding to 9 a.m. and $t = 12$ corresponding to 9 p.m. Compute the average value of $b\left(t\right)$ to estimate the average number of hamburgers sold per hour.

Hours Past Midnight	No. of Burgers Sold
9	3
12	28
15	20
18	30
21	45



253. **[T]** An athlete runs by a motion detector, which records her speed, as displayed in the following table. The best linear fit to this data, $\ell(t) = -0.068t + 5.14$, is shown in the accompanying graph. Use the average value of $\ell(t)$ between $t = 0$ and $t = 40$ to estimate the runner's average speed.

Minutes	Speed (m/sec)
0	5
10	4.8
20	3.6
30	3.0
40	2.5



y
x
5.5
5
4.5
4
3.5
3
2.5
2
0
5
10
15
20
25
30
35
40
(0, 5)
(10, 4.8)
(20, 3.6)
(30, 3)
(40, 2.5)

1.5 Substitution

Learning Objectives

1.5.1 Use substitution to evaluate indefinite integrals.
1.5.2 Use substitution to evaluate definite integrals.

The Fundamental Theorem of Calculus gave us a method to evaluate integrals without using Riemann sums. The drawback of this method, though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called **integration by substitution**, to help us find antiderivatives. Specifically, this method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f\left[g\left(x\right)\right]g'\left(x\right)dx$. For example, in the integral $\int \left(x^2-3\right)^3 2x\,dx$, we have $f(x)=x^3$, $g(x)=x^2-3$, and $g'(x)=2x$. Then,

$$f\left[g\left(x\right)\right]g'\left(x\right)=\left(x^2-3\right)^3\left(2x\right),$$

and we see that our integrand is in the correct form.

The method is called *substitution* because we substitute part of the integrand with the variable *u* and part of the integrand with *du*. It is also referred to as **change of variables** because we are changing variables to obtain an expression that is easier to work with for applying the integration rules.

THEOREM 1.7

Substitution with Indefinite Integrals

Let $u=g\left(x\right)$, where $g'\left(x\right)$ is continuous over an interval, let $f\left(x\right)$ be continuous over the corresponding range of *g*, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned}\int f\left[g\left(x\right)\right]g'\left(x\right)dx &= \int f\left(u\right)du \\ &= F\left(u\right)+C \\ &= F\left(g\left(x\right)\right)+C.\end{aligned} \tag{1.19}$$

Proof

Let *f*, *g*, *u*, and *F* be as specified in the theorem. Then

$$\begin{aligned}\tfrac{d}{dx}F(g(x)) &= F'(g(x))g'(x)\\ &= f[g(x)]\,g'(x).\end{aligned}$$

Integrating both sides with respect to *x*, we see that

$$\int f[g(x)]\,g'(x)dx = F(g(x)) + C.$$

If we now substitute $u = g(x)$, and $du = g'(x)\,dx$, we get

$$\begin{aligned}\int f[g(x)]\,g'(x)dx &= \int f(u)\,du\\ &= F(u) + C\\ &= F(g(x)) + C.\end{aligned}$$

□

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2xdx$. Rewrite the integral in terms of *u*:

$$\int \underbrace{(x^2-3)}_{u}^{3}\underbrace{(2xdx)}_{du} = \int u^3 du.$$

Using the power rule for integrals, we have

$$\int u^3 du = \frac{u^4}{4} + C.$$

Substitute the original expression for *x* back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2-3)^4}{4} + C.$$

We can generalize the procedure in the following Problem-Solving Strategy.

PROBLEM-SOLVING STRATEGY

Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to *u*. Let's select $g(x)$ such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$ into the integral.
3. We should now be able to evaluate the integral with respect to *u*. If the integral can't be evaluated we need to go back and select a different expression to use as *u*.
4. Evaluate the integral in terms of *u*.
5. Write the result in terms of *x* and the expression $g(x)$.

EXAMPLE 1.30

Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative $\int 6x\left(3x^2+4\right)^4 dx.$

[Show/Hide Solution]

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for *C* of 1, we let $y = \frac{1}{5}\left(3x^2+4\right)^5 + 1$. We have

$$y = \frac{1}{5}\left(3x^2+4\right)^5 + 1,$$

so

$$\begin{aligned} y' &= \left(\tfrac{1}{5}\right)5\left(3x^2+4\right)^4 6x \\ &= 6x\left(3x^2+4\right)^4. \end{aligned}$$

This is exactly the expression we started with inside the integrand.

CHECKPOINT 1.25

Use substitution to find the antiderivative $\int 3x^2\left(x^3-3\right)^2 dx.$

Sometimes we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

EXAMPLE 1.31

Using Substitution with Alteration

Use substitution to find $\int z\sqrt{z^2 - 5}dz.$

[Show/Hide Solution]

CHECKPOINT 1.26

Use substitution to find $\int x^2\left(x^3 + 5\right)^9 dx.$

EXAMPLE 1.32

Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t}dt.$

[Show/Hide Solution]

CHECKPOINT 1.27

Use substitution to evaluate the integral $\int \frac{\cos t}{\sin^2 t}dt.$

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When we are done, *u* should be the only variable in the integrand. In some cases, this means solving for the original variable in terms of *u*. This technique should become clear in the next example.

EXAMPLE 1.33

Finding an Antiderivative Using *u*-Substitution

Use substitution to find the antiderivative $\int \frac{x}{\sqrt{x-1}}dx.$

[Show/Hide Solution]

CHECKPOINT 1.28

Use substitution to evaluate the indefinite integral $\int \cos^3 t \sin t \, dt.$

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

THEOREM 1.8

Substitution with Definite Integrals

Let $u = g(x)$ and let g' be continuous over an interval $[a, b]$, and let *f* be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f(g(x))\, g'(x)dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Although we will not formally prove this theorem, we justify it with some calculations here. From the substitution rule for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f(g(x))\,g'(x)dx = F(g(x)) + C.$$

Then

$$\begin{aligned}\int_a^b f[g(x)]\,g'(x)dx &= F(g(x))\big|_{x=a}^{x=b}\\ &= F(g(b)) - F(g(a))\\ &= F(u)\big|_{u=g(a)}^{u=g(b)}\\ &= \int_{g(a)}^{g(b)} f(u)du,\end{aligned} \tag{1.20}$$

and we have the desired result.

EXAMPLE 1.34

Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate $\int_0^1 x^2(1+2x^3)^5 dx.$

[Show/Hide Solution]

CHECKPOINT 1.29

Use substitution to evaluate the definite integral $\int_{-1}^{0} y(2y^2-3)^5 dy.$

EXAMPLE 1.35

Using Substitution with an Exponential Function

Use substitution to evaluate $\int_0^1 xe^{4x^2+3}dx.$

[Show/Hide Solution]

CHECKPOINT 1.30

Use substitution to evaluate $\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)dx.$

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution. Also, we have the option of replacing the original expression for *u* after we find the antiderivative, which means that we do not have to change the limits of integration. These two approaches are shown in Example 1.36.

EXAMPLE 1.36

Using Substitution to Evaluate a Trigonometric Integral

Use substitution to evaluate $\int_0^{\pi/2} \cos^2\theta\, d\theta.$

[Show/Hide Solution]

Section 1.5 Exercises

254. Why is *u*-substitution referred to as *change of variable*?

255. 2. If $f = g \circ h$, when reversing the chain rule, $\frac{d}{dx}(g \circ h)(x) = g'(h(x))h'(x)$, should you take $u = g(x)$ or $u = h(x)$?

In the following exercises, verify each identity using differentiation. Then, using the indicated *u*-substitution, identify *f* such that the integral takes the form $\int f(u)\,du.$

256. $\int x\sqrt{x+1}dx = \frac{2}{15}(x+1)^{3/2}(3x-2) + C; u = x+1$

257. For $x > 1:$ $\int \frac{x^2}{\sqrt{x-1}}dx = \frac{2}{15}\sqrt{x-1}\left(3x^2+4x+8\right) + C; u = x-1$

258. $\int x\sqrt{4x^2+9}dx = \frac{1}{12}\left(4x^2+9\right)^{3/2} + C; u = 4x^2+9$

259. $\int \frac{x}{\sqrt{4x^2+9}}dx = \frac{1}{4}\sqrt{4x^2+9} + C; u = 4x^2+9$

260. $\int \frac{x}{\left(4x^2+9\right)^2}dx = -\frac{1}{8(4x^2+9)}; u = 4x^2+9$

In the following exercises, find the antiderivative using the indicated substitution.

261. $\int (x+1)^4 dx; u = x+1$

262. $\int (x-1)^5 dx; u = x-1$

263. $\int (2x-3)^{-7} dx; u = 2x-3$

264. $\int (3x-2)^{-11} dx; u = 3x-2$

265. $\int \frac{x}{\sqrt{x^2+1}}dx; u = x^2+1$

266. $\int \frac{x}{\sqrt{1-x^2}}dx; u = 1-x^2$

267. $\int (x-1)\left(x^2-2x\right)^3 dx; u = x^2-2x$

268. $\int \left(x^2-2x\right)\left(x^3-3x^2\right)^2 dx; u = x^3–3x^2$

269. $\int \cos^3\theta d\theta; u = \sin\theta \; (Hint\colon \cos^2\theta = 1-\sin^2\theta)$

270. $\int \sin^3\theta d\theta; u = \cos\theta \; (Hint\colon \sin^2\theta = 1-\cos^2\theta)$

In the following exercises, use a suitable change of variables to determine the indefinite integral.

271. $\int x(1-x)^{99}dx$

272. $\int t\left(1-t^2\right)^{10}dt$

273. $\int (11x-7)^{-3}dx$

274. $\int (7x-11)^4dx$

275. $\int \cos^3\theta\sin\theta d\theta$

276. $\int \sin^7\theta\cos\theta d\theta$

277. $\int \cos^2(\pi t)\sin(\pi t)\,dt$

278. $\int \sin^2x\cos^3x dx\ (Hint\colon \sin^2x+\cos^2x=1)$

279. $\int t\sin\left(t^2\right)\cos\left(t^2\right)dt$

280. $\int t^2\cos^2\left(t^3\right)\sin\left(t^3\right)dt$

281. $\int \frac{x^2}{\left(x^3-3\right)^2}dx$

282. $\int \frac{x^3}{\sqrt{1-x^2}}dx$

283. $\int \frac{y^5}{\left(1-y^3\right)^{3/2}}dy$

284. $\int \cos\theta(1-\cos\theta)^{99}\sin\theta d\theta$

285. $\int \left(1-\cos^3\theta\right)^{10}\cos^2\theta\sin\theta d\theta$

286. $\int (\cos\theta - 1)\left(\cos^2\theta - 2\cos\theta\right)^3 \sin\theta d\theta$

287. $\int \left(\sin^2\theta - 2\sin\theta\right)\left(\sin^3\theta - 3\sin^2\theta\right)^3 \cos\theta d\theta$

In the following exercises, use a calculator to estimate the area under the curve using left Riemann sums with 50 terms, then use substitution to solve for the exact answer.

288. **[T]** $y = 3(1-x)^2$ over $[0, 2]$

289. **[T]** $y = x(1-x^2)^3$ over $[-1, 2]$

290. **[T]** $y = \sin x(1-\cos x)^2$ over $[0, \pi]$

291. **[T]** $y = \frac{x}{(x^2+1)^2}$ over $[-1, 1]$

In the following exercises, use a change of variables to evaluate the definite integral.

292. $\int_0^1 x\sqrt{1-x^2}dx$

293. $\int_0^1 \frac{x}{\sqrt{1+x^2}}dx$

294. $\int_0^2 \frac{t}{\sqrt{5+t^2}}dt$

295. $\int_0^1 \frac{t^2}{\sqrt{1+t^3}}dt$

296. $\int_0^{\pi/4} \sec^2\theta \tan\theta d\theta$

297. $\int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}d\theta$

In the following exercises, evaluate the indefinite integral $\int f(x)\,dx$ with constant $C = 0$ using *u*-substitution. Then, graph the function and the antiderivative over the indicated interval. If possible, estimate a value of *C* that would need to be added to the antiderivative to make it equal to the definite integral $F(x) = \int_a^x f(t)\,dt$, with *a* the left endpoint of the given interval.

298. **[T]** $\int (2x+1)\, e^{x^2+x-6} dx$ over $[-3, 2]$

299. **[T]** $\int \frac{\cos(\ln(2x))}{x} dx$ on $[0, 2]$

300. **[T]** $\int \frac{3x^2+2x+1}{\sqrt{x^3+x^2+x+4}} dx$ over $[-1, 2]$

301. **[T]** $\int \frac{\sin x}{\cos^3 x} dx$ over $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

302. **[T]** $\int (x+2)\, e^{-x^2-4x+3} dx$ over $[-5, 1]$

303. **[T]** $\int 3x^2\sqrt{2x^3+1} dx$ over $[0, 1]$

304. If $h(a) = h(b)$ in $\int_a^b g'(h(x))\, h'(x)\, dx$, what can you say about the value of the integral?

305. Is the substitution $u = 1 - x^2$ in the definite integral $\int_0^2 \frac{x}{1-x^2} dx$ okay? If not, why not?

In the following exercises, use a change of variables to show that each definite integral is equal to zero.

306. $\int_0^{\pi} \cos^2(2\theta)\sin(2\theta)\, d\theta$

307. $\int_0^{\sqrt{\pi}} t\cos\left(t^2\right)\sin\left(t^2\right) dt$

308. $\int_0^1 (1-2t)\, dt$

309. $\int_0^1 \frac{1-2t}{\left(1+\left(t-\frac{1}{2}\right)^2\right)} dt$

310. $\int_0^{\pi} \sin\left(\left(t-\frac{\pi}{2}\right)^3\right)\cos\left(t-\frac{\pi}{2}\right) dt$

311. $\int_0^2 (1-t)\cos(\pi t)\, dt$

312. $\displaystyle\int_{\pi/4}^{3\pi/4} \sin^2 t \cos t\,dt$

313. Show that the average value of $f(x)$ over an interval $[a, b]$ is the same as the average value of $f(cx)$ over the interval $\left[\frac{a}{c}, \frac{b}{c}\right]$ for $c > 0$.

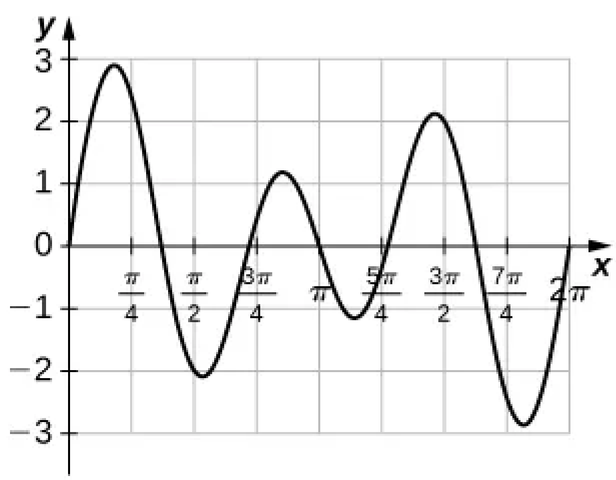
314. Find the area under the graph of $f(t) = \frac{t}{(1+t^2)^a}$ between $t = 0$ and $t = x$ where $a > 0$ and $a \neq 1$ is fixed, and evaluate the limit as $x \to \infty$.

315. Find the area under the graph of $g(t) = \frac{t}{(1-t^2)^a}$ between $t = 0$ and $t = x$, where $0 < x < 1$ and $a > 0$ is fixed. Evaluate the limit as $x \to 1$.

316. The area of a semicircle of radius 1 can be expressed as $\displaystyle\int_{-1}^{1} \sqrt{1 - x^2}\,dx$. Use the substitution $x = \cos t$ to express the area of a semicircle as the integral of a trigonometric function. You do not need to compute the integral.

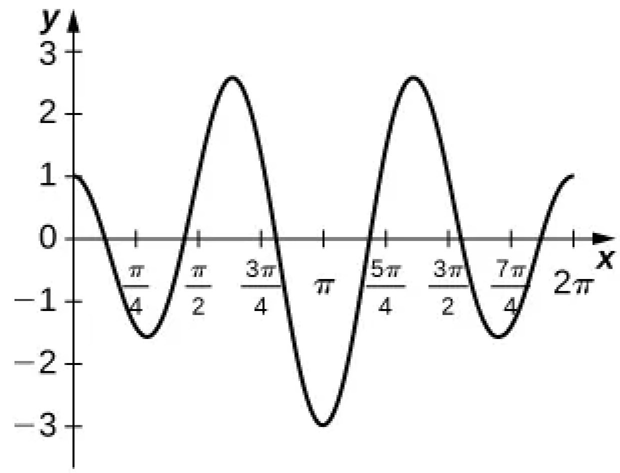
317. The area of the top half of an ellipse with a major axis that is the *x*-axis from $x = -a$ to $x = a$ and with a minor axis that is the *y*-axis from $y = -b$ to $y = b$ can be written as $\displaystyle\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx$. Use the substitution $x = a\cos t$ to express this area in terms of an integral of a trigonometric function. You do not need to compute the integral.

318. **[T]** The following graph is of a function of the form $f(t) = a\sin(nt) + b\sin(mt)$. Estimate the coefficients *a* and *b*, and the frequency parameters *n* and *m*. Use these estimates to



approximate $\displaystyle\int_0^{\pi} f(t)\,dt$.

319. **[T]** The following graph is of a function of the form $f(x) = a\cos(nt) + b\cos(mt)$. Estimate the coefficients *a* and *b* and the frequency parameters *n* and *m*. Use these estimates



to approximate $\int_0^{\pi} f(t)\, dt$.

1.6 Integrals Involving Exponential and Logarithmic Functions

Learning Objectives

1.6.1 Integrate functions involving exponential functions.
1.6.2 Integrate functions involving logarithmic functions.

Exponential and logarithmic functions are used to model population growth, cell growth, and financial growth, as well as depreciation, radioactive decay, and resource consumption, to name only a few applications. In this section, we explore integration involving exponential and logarithmic functions.

Integrals of Exponential Functions

The exponential function is perhaps the most efficient function in terms of the operations of calculus. The exponential function, $y = e^x$, is its own derivative and its own integral.

RULE: INTEGRALS OF EXPONENTIAL FUNCTIONS

Exponential functions can be integrated using the following formulas.

$$\begin{aligned} \int e^x dx &= e^x + C \\ \int a^x dx &= \frac{a^x}{\ln a} + C \end{aligned} \tag{1.21}$$

EXAMPLE 1.37

Finding an Antiderivative of an Exponential Function

Find the antiderivative of the exponential function e^{-x}.

[Show/Hide Solution]

CHECKPOINT 1.31

Find the antiderivative of the function using substitution: $x^2 e^{-2x^3}$.

A common mistake when dealing with exponential expressions is treating the exponent on *e* the same way we treat exponents in polynomial expressions. We cannot use the power rule for the exponent on *e*. This can be especially confusing when we have both exponentials and polynomials in the same expression, as in the previous checkpoint. In these cases, we should always double-check to make sure we're using the right rules for the functions we're integrating.

EXAMPLE 1.38

Square Root of an Exponential Function

Find the antiderivative of the exponential function $e^x\sqrt{1+e^x}$.

[Show/Hide Solution]

CHECKPOINT 1.32

Find the antiderivative of $e^x(3e^x-2)^2$.

EXAMPLE 1.39

Using Substitution with an Exponential Function

Use substitution to evaluate the indefinite integral $\int 3x^2 e^{2x^3}\,dx$.

[Show/Hide Solution]

CHECKPOINT 1.33

Evaluate the indefinite integral $\int 2x^3 e^{x^4}\,dx$.

As mentioned at the beginning of this section, exponential functions are used in many real-life applications. The number *e* is often associated with compounded or accelerating growth, as we have seen in earlier sections about the derivative. Although the derivative represents a rate of change or a growth rate, the integral represents the total change or the total growth. Let's look at an example in which integration of an exponential function solves a common business application.

A price–demand function tells us the relationship between the quantity of a product demanded and the price of the product. In general, price decreases as quantity demanded increases. The marginal price–demand function is the derivative of the price–demand function and it tells us how fast the price changes at a given level of production. These functions are used in business to determine the price–elasticity of demand, and to help companies determine whether changing production levels would be profitable.

EXAMPLE 1.40

Finding a Price–Demand Equation

Find the price–demand equation for a particular brand of toothpaste at a supermarket chain when the demand is 50 tubes per week at $2.35 per tube, given that the marginal price—demand function, $p'(x)$, for *x* number of tubes per week, is given as

$$p'(x) = -0.015e^{-0.01x}.$$

If the supermarket chain sells 100 tubes per week, what price should it set?

[Show/Hide Solution]

EXAMPLE 1.41

Evaluating a Definite Integral Involving an Exponential Function

Evaluate the definite integral $\int_1^2 e^{1-x} dx$.

[Show/Hide Solution]

CHECKPOINT 1.34

Evaluate $\int_0^2 e^{2x}\,dx.$

EXAMPLE 1.42

Growth of Bacteria in a Culture

Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t$, where *t* is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with 10,000 bacteria, find a function $Q(t)$ that gives the number of bacteria in the Petri dish at any time *t*. How many bacteria are in the dish after 2 hours?

[Show/Hide Solution]

CHECKPOINT 1.35

From Example 1.42, suppose the bacteria grow at a rate of $q(t) = 2^t$. Assume the culture still starts with 10,000 bacteria. Find $Q(t)$. How many bacteria are in the dish after 3 hours?

EXAMPLE 1.43

Fruit Fly Population Growth

Suppose a population of fruit flies increases at a rate of $g(t) = 2e^{0.02t}$, in flies per day. If the initial population of fruit flies is 100 flies, how many flies are in the population after 10 days?

[Show/Hide Solution]

CHECKPOINT 1.36

Suppose the rate of growth of the fly population is given by $g(t) = e^{0.01t}$, and the initial fly population is 100 flies. How many flies are in the population after 15 days?

EXAMPLE 1.44

Evaluating a Definite Integral Using Substitution

Evaluate the definite integral using substitution: $\int_1^2 \frac{e^{1/x}}{x^2} dx.$

[Show/Hide Solution]

CHECKPOINT 1.37

Evaluate the definite integral using substitution: $\int_1^2 \frac{1}{x^3} e^{4x^{-2}} dx.$

Integrals Involving Logarithmic Functions

Integrating functions of the form $f(x) = x^{-1}$ result in the absolute value of the natural log function, as shown in the following rule. Integral formulas for other logarithmic functions, such as $f(x) = \ln x$ and $f(x) = \log_a x$, are also included in the rule.

RULE: INTEGRATION FORMULAS INVOLVING LOGARITHMIC FUNCTIONS

The following formulas can be used to evaluate integrals involving logarithmic functions.

$$\begin{aligned} \int x^{-1} dx &= \ln |x| + C \\ \int \ln x \, dx &= x \ln x - x + C = x (\ln x - 1) + C \\ \int \log_a x \, dx &= \frac{x}{\ln a} (\ln x - 1) + C \end{aligned} \tag{1.22}$$

EXAMPLE 1.45

Finding an Antiderivative Involving $\ln x$

Find the antiderivative of the function $\frac{3}{x-10}$.

[Show/Hide Solution]

CHECKPOINT 1.38

Find the antiderivative of $\frac{1}{x+2}$.

EXAMPLE 1.46

Finding an Antiderivative of a Rational Function

Find the antiderivative of $\frac{2x^3+3x}{x^4+3x^2}$.

[Show/Hide Solution]

EXAMPLE 1.47

Finding an Antiderivative of a Logarithmic Function

Find the antiderivative of the log function $\log_2 x$.

[Show/Hide Solution]

CHECKPOINT 1.39

Find the antiderivative of $\log_3 x$.

Example 1.48 is a definite integral of a trigonometric function. With trigonometric functions, we often have to apply a trigonometric property or an identity before we can move forward. Finding the right form of the integrand is usually the key to a smooth integration.

EXAMPLE 1.48

Evaluating a Definite Integral

Find the definite integral of $\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx$.

[Show/Hide Solution]

Section 1.6 Exercises

In the following exercises, compute each indefinite integral.

320. $\int e^{2x}\,dx$

321. $\int e^{-3x}\,dx$

322. $\int 2^x\,dx$

323. $\int 3^{-x}\,dx$

324. $\int \frac{1}{2x} dx$

325. $\int \frac{2}{x} dx$

326. $\int \frac{1}{x^2} dx$

327. $\int \frac{1}{\sqrt{x}} dx$

In the following exercises, find each indefinite integral by using appropriate substitutions.

328. $\int \frac{\ln x}{x} dx$

329. $\int \frac{dx}{x(\ln x)^2}$

330. $\int \frac{dx}{x \ln x} \; (x > 1)$

331. $\int \frac{dx}{x \ln x \ln(\ln x)}$

332. $\int \tan \theta \, d\theta$

333. $\int \frac{\cos x - x \sin x}{x \cos x} dx$

334. $\int \frac{\ln (\sin x)}{\tan x} dx$

335. $\int \ln(\cos x) \tan x dx$

336. $\int x e^{-x^2} dx$

337. $\int x^2 e^{-x^3} dx$

338. $\int e^{\sin x} \cos x dx$

339. $\int e^{\tan x} \sec^2 x dx$

340. $\int e^{\ln x} \frac{dx}{x}$

341. $\int \frac{e^{\ln(1-t)}}{1-t} dt$

In the following exercises, verify by differentiation that $\int \ln x \, dx = x(\ln x - 1) + C$, then use appropriate changes of variables to compute the integral.

342. $\int x \ln x dx$ $(Hint\text{: } \int x \ln x dx = \frac{1}{2} \int x \ln(x^2) \, dx)$

343. $\int x^2 \ln(x^2) \, dx$

344. $\int \frac{\ln x}{x^2} dx$ $(Hint\text{: Set } u = \frac{1}{x}.)$

345. $\int \frac{\ln x}{\sqrt{x}} dx$ $(Hint\text{: Set } u = \sqrt{x}.)$

346. Write an integral to express the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to e^x and evaluate the integral.

347. Write an integral to express the area under the graph of $y = e^t$ between $t = 0$ and $t = \ln x$, and evaluate the integral.

In the following exercises, use appropriate substitutions to express the trigonometric integrals in terms of compositions with logarithms.

348. $\int \tan(2x) \, dx$

349. $\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)} dx$

350. $\int \frac{x \sin(x^2)}{\cos(x^2)} dx$

351. $\int x \csc(x^2) \, dx$

352. $\int \ln(\cos x) \tan x \, dx$

353. $\int \ln(\csc x) \cot x dx$

354. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

In the following exercises, evaluate the definite integral.

355. $\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3} dx$

356. $\int_0^{\pi/4} \tan x \, dx$

357. $\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x} dx$

358. $\int_{\pi/6}^{\pi/2} \csc x dx$

359. $\int_{\pi/4}^{\pi/3} \cot x dx$

In the following exercises, integrate using the indicated substitution.

360. $\int \frac{x}{x - 100} dx; u = x - 100$

361. $\int \frac{y - 1}{y + 1} dy; u = y + 1$

362. $\int \frac{1 - x^2}{3x - x^3} dx; u = 3x - x^3$

363. $\int \frac{\sin x + \cos x}{\sin x - \cos x} dx; u = \sin x - \cos x$

364. $\int e^{2x} \sqrt{1 - e^{2x}} dx; u = e^{2x}$

365. $\int \ln(x) \frac{\sqrt{1 - (\ln x)^2}}{x} dx; u = \ln x$

In the following exercises, does the right-endpoint approximation overestimate or underestimate the exact area? Calculate the right endpoint estimate R_{50} and solve for the exact area.

366. **[T]** $y = e^x$ over $[0, 1]$

367. **[T]** $y = e^{-x}$ over $[0, 1]$

368. **[T]** $y = \ln(x)$ over $[1, 2]$

369. **[T]** $y = \frac{x+1}{x^2+2x+6}$ over $[0, 1]$

370. **[T]** $y = 2^x$ over $[-1, 0]$

371. **[T]** $y = -2^{-x}$ over $[0, 1]$

In the following exercises, $f(x) \geq 0$ for $a \leq x \leq b$. Find the area under the graph of $f(x)$ between the given values *a* and *b* by integrating.

372. $f(x) = \frac{\log_{10}(x)}{x}; a = 10, b = 100$

373. $f(x) = \frac{\log_2(x)}{x}; a = 32, b = 64$

374. $f(x) = 2^{-x}; a = 1, b = 2$

375. $f(x) = 2^{-x}; a = 3, b = 4$

376. Find the area under the graph of the function $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

377. Compute the integral of $f(x) = xe^{-x^2}$ and find the smallest value of *N* such that the area under the graph $f(x) = xe^{-x^2}$ between $x = N$ and $x = N + 1$ is, at most, 0.01.

378. Find the limit, as *N* tends to infinity, of the area under the graph of $f(x) = xe^{-x^2}$ between $x = 0$ and $x = N$.

379. Show that $\displaystyle\int_a^b \frac{dt}{t} = \int_{1/b}^{1/a} \frac{dt}{t}$ when $0 < a \leq b$.

380. Suppose that $f(x) > 0$ for all *x* and that *f* and *g* are differentiable. Use the identity $f^g = e^{g \ln f}$ and the chain rule to find the derivative of f^g.

381. Use the previous exercise to find the antiderivative of $h(x) = x^x(1 + \ln x)$ and evaluate $\displaystyle\int_2^3 x^x(1 + \ln x)\,dx$.

382. Show that if $c > 0$, then the integral of $1/x$ from *ac* to *bc* $(0 < a < b)$ is the same as the integral of $1/x$ from *a* to *b*.

The following exercises are intended to derive the fundamental properties of the natural log starting from the *definition* $\ln(x) = \displaystyle\int_1^x \frac{dt}{t}$, using properties of the definite integral and making no further assumptions.

383. Use the identity $\ln(x) = \int_1^x \frac{dt}{t}$ to derive the identity $\ln\left(\frac{1}{x}\right) = -\ln x$.

384. Use a change of variable in the integral $\int_1^{xy} \frac{1}{t}dt$ to show that $\ln xy = \ln x + \ln y \text{ for } x, y > 0.$

385. Use the identity $\ln x = \int_1^x \frac{dt}{t}$ to show that $\ln(x)$ is an increasing function of *x* on $[0, \infty)$, and use the previous exercises to show that the range of $\ln(x)$ is $(-\infty, \infty)$. Without any further assumptions, conclude that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$.

386. Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln(x)$, but keep in mind that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$. Call it *E*. Use the identity $\ln xy = \ln x + \ln y$ to deduce that $E(a+b) = E(a)E(b)$ for any real numbers *a*, *b*.

387. Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln x$, but keep in mind that $\ln x$ has an inverse function defined on $(-\infty, \infty)$. Call it *E*. Show that $E'(t) = E(t)$.

388. The sine integral, defined as $S(x) = \int_0^x \frac{\sin t}{t}dt$ is an important quantity in engineering. Although it does not have a simple closed formula, it is possible to estimate its behavior for large *x*. Show that for $k \geq 1, |S(2\pi k) - S(2\pi(k+1))| \leq \frac{1}{k(2k+1)\pi}$.
$(Hint\colon \sin(t+\pi) = -\sin t)$

389. **[T]** The normal distribution in probability is given by $p(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$, where *σ* is the standard deviation and *μ* is the average. The *standard normal distribution* in probability, p_s, corresponds to $\mu = 0 \text{ and } \sigma = 1$. Compute the right endpoint estimates $R_{10} \text{ and } R_{100}$ of $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx.$

390. **[T]** Compute the right endpoint estimates $R_{50} \text{ and } R_{100}$ of $\int_{-3}^{5} \frac{1}{2\sqrt{2\pi}}e^{-(x-1)^2/8}.$