Financial Derivatives and High-Frequency Trading - Volatility modeling: Sergio Pulido

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Question 1.1

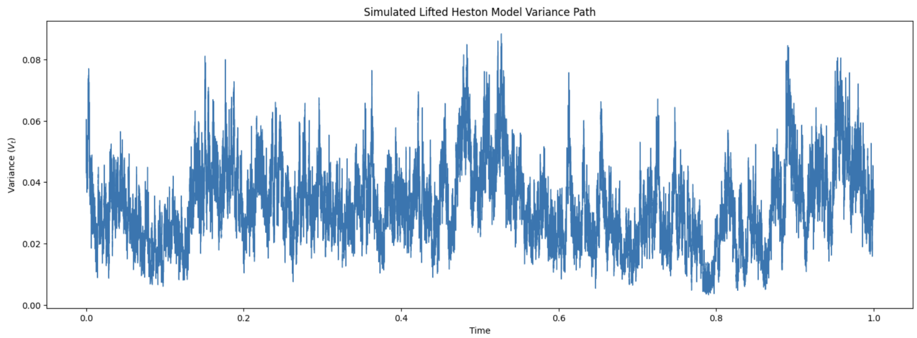

Figure 1: Simulated variance process V_t

Question 1.2

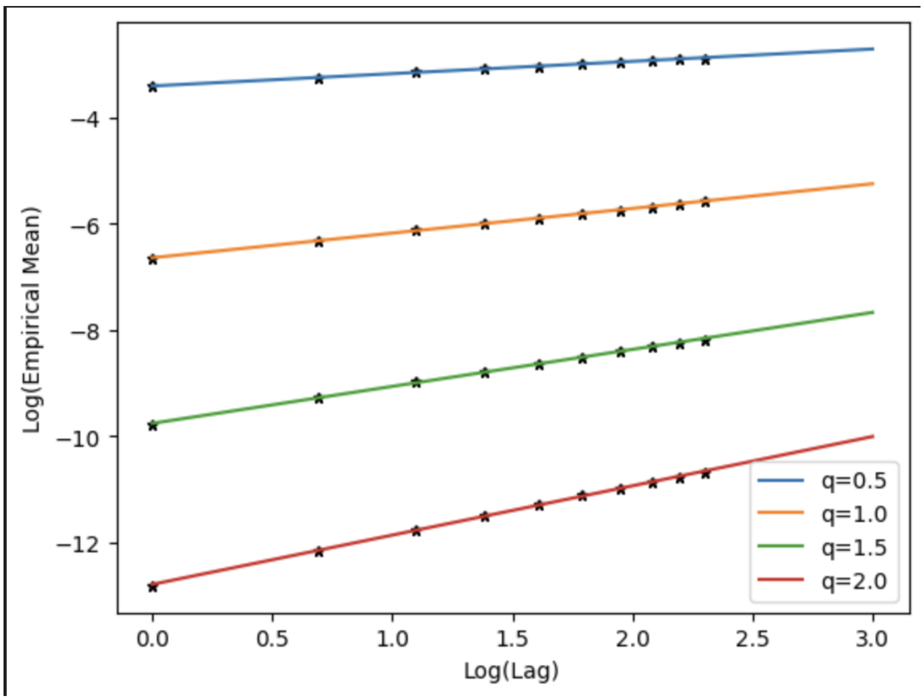

Figure 2: Log-log plot for moment estimation of H

Question 1.3

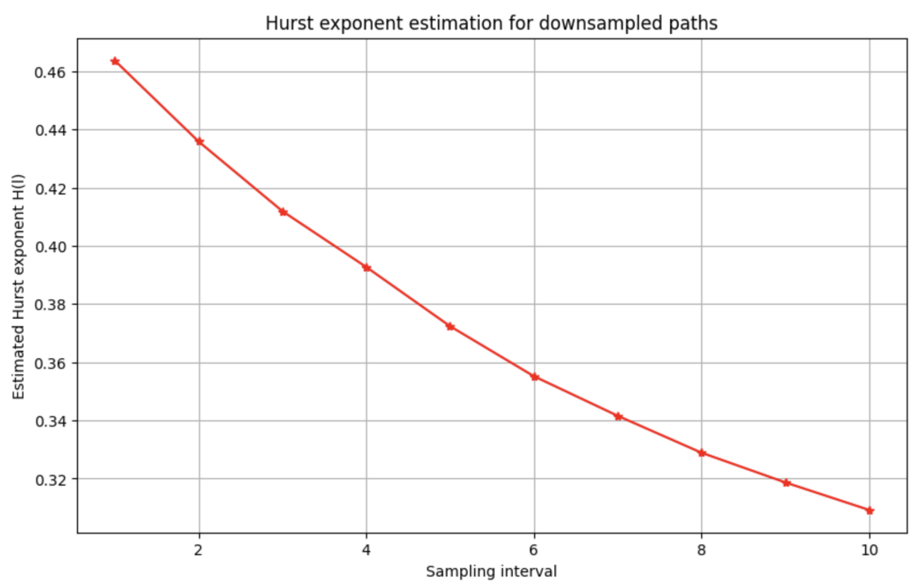

Figure 3: Hurst exponent estimation for downsampled paths

Question 1.4

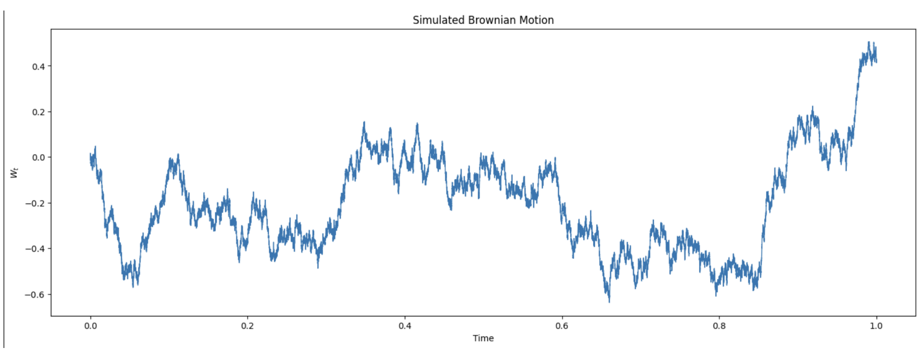

Figure 4: Simulated Brownian Motion

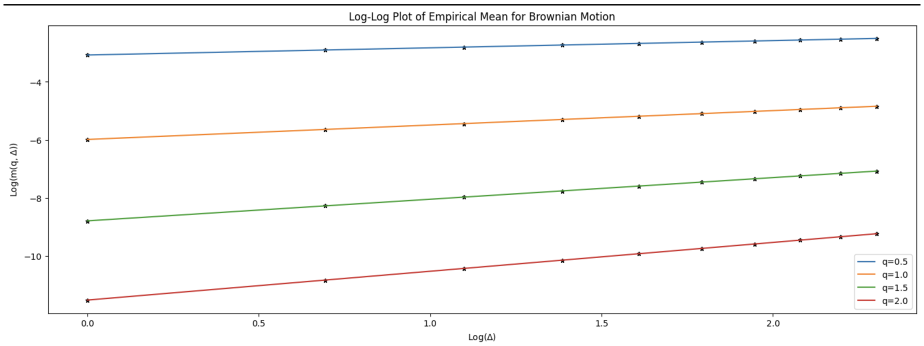

Figure 5: Log-Log Plot of Empirical Mean for Brownian Motion

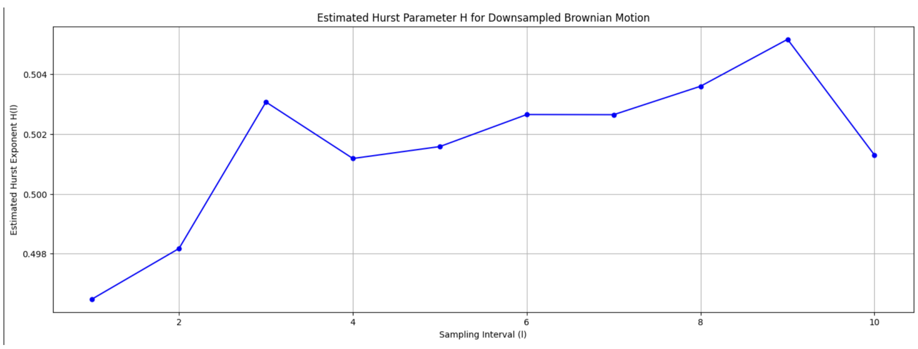

Figure 6: Estimated Hurst Parameter H for DOwnsampled Brownian Motion

Question 1.5

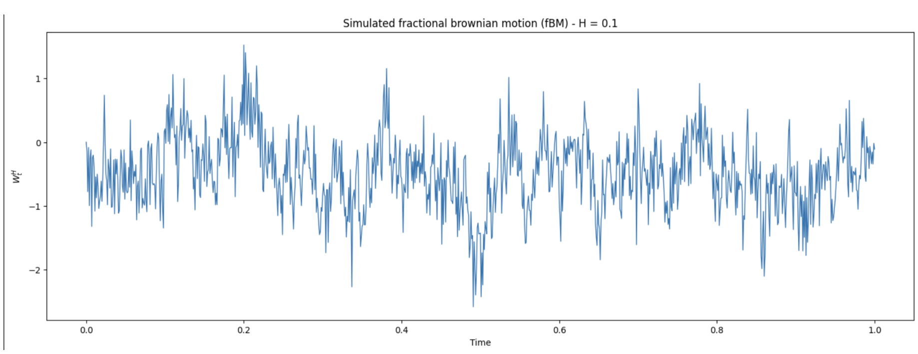

Figure 7: Simulated fractional brownian motion

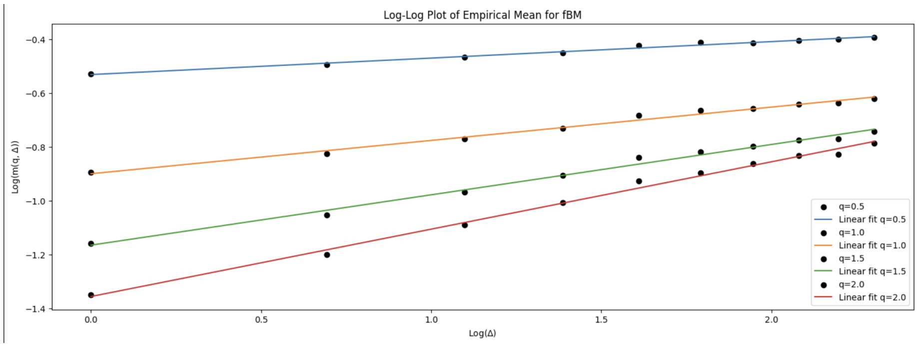

Figure 8: Log-Log plot of empirical mean for fBM

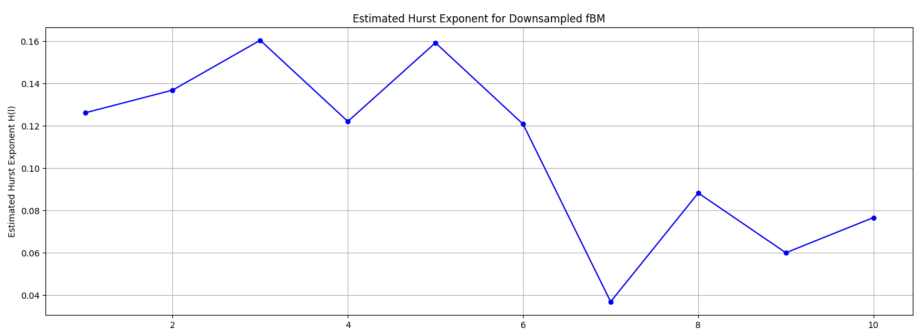

Figure 9: Estimated Hurst Exponent for Downsampled fBM

Question 2.1

(a) We consider the financial process M_t defined by the expression:

$$M_t = \exp\left(u\log(S_t) + \phi(t,T) + \sum_{i=1}^{n} c_i \psi^i (T-t) U_t^i\right),$$

where u is a complex number, ϕ is a function describing the time-value component, ψ^i are state-dependent functions, and U_t^i are stochastic processes.

To model this process, we employ a function f:

$$f : (t, x, u_1, \ldots, u_n) \mapsto f(t, x, u_1, \ldots, u_n),$$

allowing us to rewrite M_t as:

$$M_t = f(t, S_t, U_{1t}, \ldots, U_{nt}).$$

Given the twice-differentiable nature of f, Itô's formula is applied to determine the differential dM_t:

$$\begin{aligned} dM_t = &\frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dS_t + \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} dU_{it} \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2} d\langle S\rangle_t + \frac{1}{2}\frac{\partial^2 f}{\partial u_i^2} d\langle U\rangle_t \\ &+ \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x \partial u_i} d\langle S, U_i\rangle_t + \sum_{i\neq j} \frac{\partial^2 f}{\partial u_i \partial u_j} d\langle U_i, U_j\rangle_t. \end{aligned}$$

The condition for M_t being a local martingale hinges on the absence of a drift component. Concentrating on the terms that multiply dt, we consider the stochastic dynamics of the underlying processes:

$$\begin{aligned} dS_t &= S_t \sqrt{V_t} dB_t, \\ V_t &= g_0(t) + \sum_{i=1}^{n} c_i U_{it}, \\ dU_{it} &= (-x_i U_{it} - \lambda V_t) dt + \nu \sqrt{V_t} dW_t, \end{aligned}$$

where B_t and W_t denote Brownian motions, and $g_0(t)$ represents an initial volatility structure. Assuming ψ^i solves the Riccati differential equation, the condition for M_t to be a local martingale is that the drift part is zero. Therefore,

$$\frac{\partial \phi}{\partial t} - \sum_{i=1}^{n} c_i (\psi_i)' U_{it} - \sum_{i=1}^{n} c_i \psi_i x_i U_{it} + F\left(U, \sum_{i=1}^{n} c_i \psi_i\right) V = 0.$$

Thus, M_t is a local martingale if the drift is zero.

We have the following identities:

$$(\psi^i)' = -x_i\psi^i + F\left(U, \sum_{j=1}^{n} c_j\psi^j\right).$$

$$-\psi^i - x_i\psi^i = -F\left(U, \sum_{j=1}^{n} c_j\psi^j\right).$$

If we multiply by $c_i * U_t^i$ and sum we can use the previous identity to simplify the drift term to:

$$\left(\frac{\partial\phi}{\partial t} - F\left(U, \sum_{j=1}^{n} c_j\psi_j\right)\left(\sum_{i=1}^{n} c_i U_i^i\right) + F\left(U, \sum_{i=1}^{n} c_i\psi_i\right) V\right) dt.$$

The partial derivative of ϕ with respect to t is given by:

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= -F\left(U, \sum_{j=1}^{n} c_j\psi_j\right) g_0(t)\\ &= -F\left(U, \sum_{j=1}^{n} c_j\psi_j\right)(V - \sum_{i=1}^{n} c_i U_t^i).\end{aligned}$$

Therefore we find that the drift term is zero, so M_t is a local martingale.

(b) If M_t is a martingale, then we have:

$$E\left[M_T|\mathcal{F}_t\right] = M_t$$

Or, as $\psi(0) = \phi(T,T) = 0$, we can state that:

$$M_T = \exp\left(u\log(S_T)\right)$$

Then, we must have:

$$E\left[\exp\left(u\log(S_T)\right)|\mathcal{F}_t\right] = M_t$$

Question 2.2

$$C_0 = \frac{e^{-\tau T - \frac{\alpha_2^2}{2}}}{2\pi} \int_{-\infty}^{\infty} \phi_T(u - i(\alpha_2+1)) e^{-iulog(k)} \frac{(du)}{(\alpha_2 + iU)((\alpha_2+1)+iU)}$$

Such that

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{it(e^x - k)}\, dx \quad \text{because we deal with the density of } \log(S_T)$$

$$= \int_{log(k)}^{\infty} e^{it(e^x-k)}\, dx$$

$$= \int_{log(k)}^{\infty} [e^{itx+1} - ke^{itx}]\, dx.$$

$$\hat{f}(t) = -\frac{e^{(it+1)log(k)}}{1+it} + \frac{ke^{itlog(k)}}{it}$$

$$= \frac{ite^{(it+1)log(k)} - k(it+1)e^{itlog(k)}}{(1+it)it}$$

$$= \frac{e^{itlog(k)}[kit(e^{-1}) - e^{log(k)}(it)]}{(1+it)it}$$

$$= \frac{e^{(it+1)log(k)}}{(it+1)it}$$

Let $t = u + iw = u + i(\alpha_2 + 1)$
So $(it + 1 = iU - \alpha_2 - 1 + 1 = iU - \alpha_2)$
$it = iU - (\alpha_2 + 1)$
So

$$\hat{f}(t) = \frac{e^{(it-\alpha_2)log(k)}}{(\alpha_2 - iU)((\alpha_2+1)+iU)}$$

$$\hat{f}(t) = \frac{e^{-\alpha_2 log(k)} e^{-iUlog(k)}}{(\alpha_2 + iU)((\alpha_2+1)+iU)}$$

Then

$$C_0 = \frac{e^{-\tau T - \alpha_2 log(k)}}{2\pi} \int_{-\infty}^{\infty} \frac{\phi_T(u - i(\alpha_2+1))e^{-iUlog(k)}}{(\alpha_2+iU)((\alpha_2+1)+iU)}\, dU$$

$$\phi_T(U) = E[e^{iUlog(S_T)}] = \hat{q}(U)$$

So let

$$Z(U) = \frac{\phi_T(u - i(\alpha_2+1))e^{-iUlog(k)}}{(\alpha_2+iU)((\alpha_2+1)-iU)}$$

So for $u < 0$ we pose $U = -x$; $x > 0$

$$Z(-x) = \frac{\phi_T(-x - i(\alpha_2+1))e^{ixlog(k)}}{(\alpha_2 - ix)((\alpha_2+1)-ix)}$$

$$= \frac{\phi_T(x - i(\alpha_2 + 1))e^{-ixlog(k)}}{(\alpha_2 + ix)((\alpha_2 + 1) + ix)} = Z^*(x).$$

So in the integral we will have :

$$\int (Z(-x) + Z^*(x))\,dx = 2Re(Z)\,dx = 2Re(\overline{Z})\,dx$$

So

$$\int_{-\infty}^{\infty} Z(U)\,dU = 2\int_{-\infty}^{\infty} Re(Z^*(U))\,dU$$

Hence:

$$C_0 = \frac{e^{-\tau T - \alpha_2 log(k)}}{\pi} \int_0^{\infty} Re\left(\frac{\phi_T(u - i(\alpha_2 + 1))e^{-iUlog(k)}}{(\alpha_2 - iU)((\alpha_2 + 1) - iU)}\right) dU$$

Question 2.5

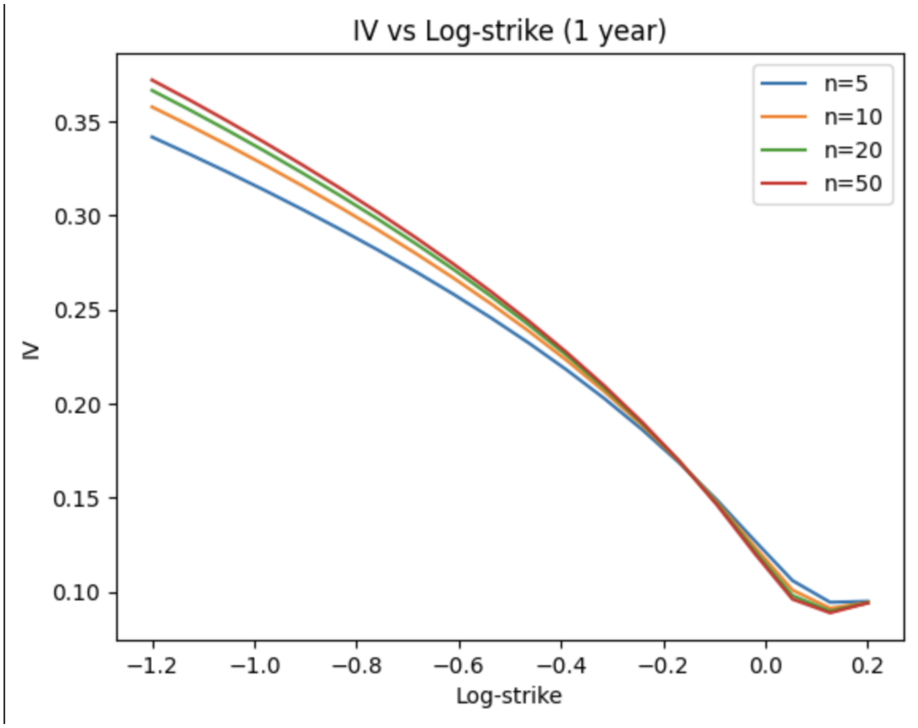

Figure 10: IV vs Log-Strike (1 Year)

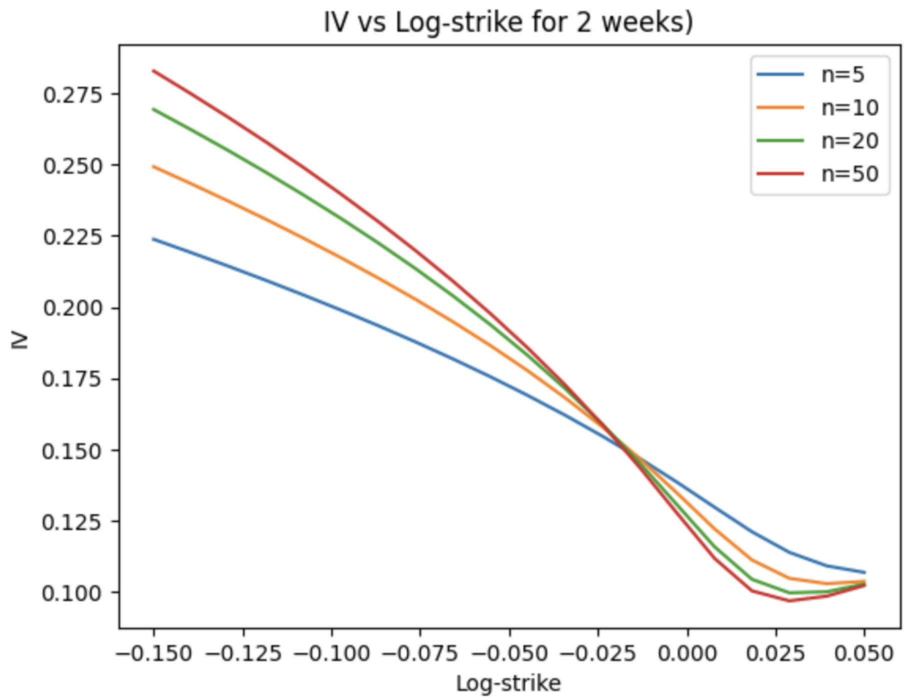

Figure 11: IV vs Log-strike for 2 weeks

Question 3.3

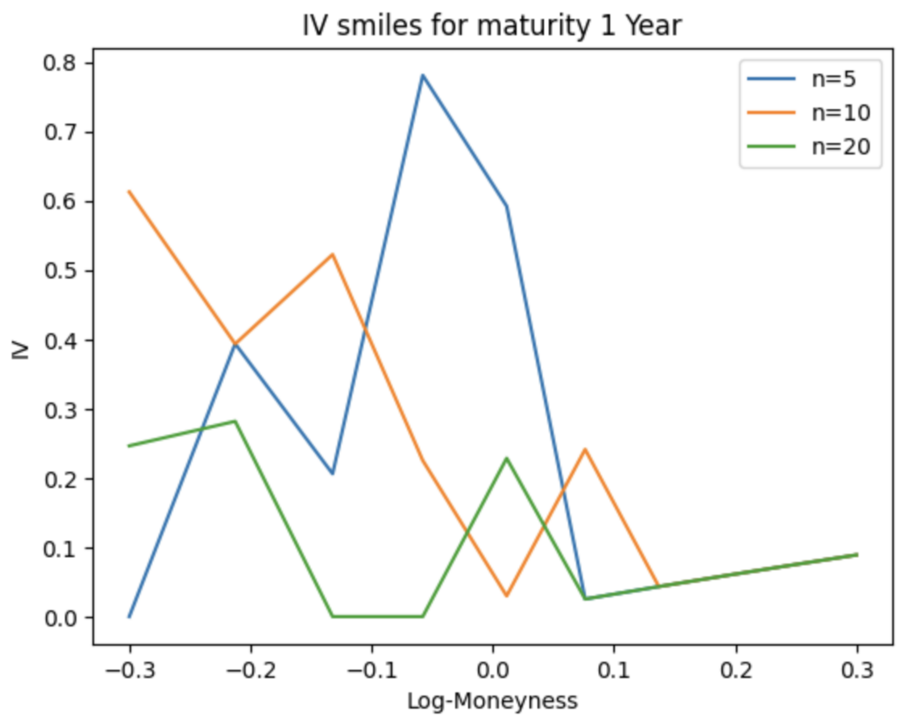

Figure 12: Enter Caption

Question 4.1

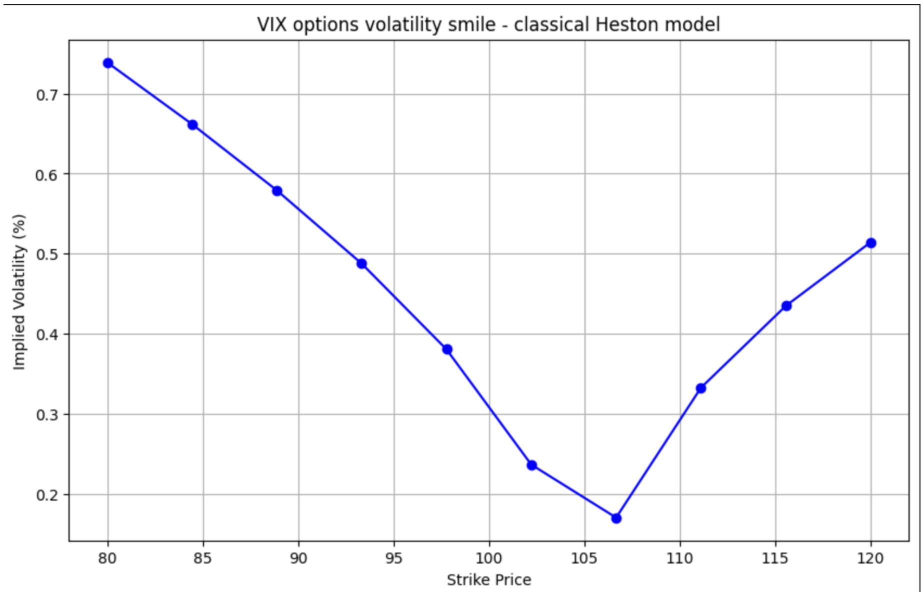

Figure 13: Enter Caption