Journal of Economic Dynamics and Control 16 (1992) 403–426. North-Holland

# Theory of constant proportion portfolio insurance*

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We study constant proportion portfolio insurance (CPPI), a dynamic strategy that maintains the portfolio's risk exposure a constant multiple of the excess of wealth over a floor, up to a borrowing limit. We use this simple rule to investigate how transaction costs and borrowing constraints affect portfolio insurance-type strategies. Absent transaction costs, CPPI is equivalent to investing in perpetual American call options, and is optimal for a piecewise-HARA utility function with a minimum consumption constraint. As the multiple increases, the payoffs under CPPI approach those of a stop-loss strategy. The expected holding-period return is not monotonic in the multiple, and a higher expected return can be obtained under CPPI than with a stop-loss strategy.

## 1. Introduction

Portfolio insurance strategies are appropriate for investors who need downside protection and desire upside potential. The class of such strategies is large. Any rule that takes less risk at lower wealth levels and more risk at higher wealth levels is a candidate.

In principle, the best portfolio insurance strategy can be found by solving for the intertemporal investment–consumption rules that maximize expected utility. Traditionally, this has been done under the fairly standard assumptions of frictionless markets and no borrowing restrictions [see particularly Merton (1971), and also Brennan and Solanki (1981) and Leland (1980)]. Without these assumptions, solving for the optimal strategy is generally extremely difficult. Transaction costs and borrowing constraints induce path dependencies which greatly increase the complexity of the problem. Portfolio insurance strategies *are* trading-intensive, however, and they operate intrinsi-

*We thank John Long, Jr. for helpful comments on an earlier draft.

cally on the notion of leverage. Finding ways to introduce transaction costs and borrowing constraints is important, therefore, if the goal is to bring the level of analysis closer to decisionmaking in actual markets. That is what we try to do in this paper.

Our approach is not to try to solve for utility-maximizing strategies; rather, we take a stylized decision rule as given, and develop its properties. Our rule – constant proportion portfolio insurance (CPPI) – is a particularly simple one. It invests a constant multiple of the cushion in risky assets up to the borrowing limit, where the cushion is the difference between wealth and a specified floor. CPPI was proposed in Black and Jones (1987) and Perold (1986) as an alternative to the more complex approaches to portfolio insurance based on option replication.

In the absence of borrowing constraints and transaction costs, CPPI is a special case of the HARA utility-maximizing rules that have appeared in the literature at least as far back as Merton (1971). One of our results is that the borrowing-constrained rule too is utility-maximizing. The utility function is piecewise HARA, and intertemporal consumption is constrained above a subsistence level.

We employ a two-asset framework. Wealth is to be allocated among the 'active' and 'reserve' assets. The reserve asset is a safe asset, where 'safe' may mean that it closely tracks a liability stream [for example, the reserve asset might represent a dedicated bond portfolio, as discussed in Black (1988)]. The reserve asset may fluctuate in value. The active asset is relatively risky (perhaps representing a portfolio of stocks). Its expected return exceeds that of the reserve asset.

To allow for transaction costs, we employ a discretization. Trading occurs only after security price moves of a pre-specified size. For this to be feasible, we require continuous sample paths and the ability to instantaneously execute market orders. When included, transaction costs are assumed proportional to the size of the trade.

Our first results concern the behavior of a CPPI-managed portfolio as a function of the price paths of the active and reserve assets. In the absence of transaction costs and a borrowing limit, the portfolio payoff at time $t$ is shown to depend only on the values of the active and reserve assets at $t$ and on the number of trades along the way. This is a weak form of path independence. That is, holding fixed the values of the active and reserve assets at time $t$, all that matters about the sample paths prior to $t$ is the number of trades.

The number of trades is proportional to the number of reversals. Reversals are costly for portfolio insurance strategies because, ex post, they involve buying high and selling low. This is usually referred to as 'volatility cost'.

A closely-related characterization obtains in the case of limited borrowing. No trading takes place while at the borrowing limit, so that reversals are not costly while in this state. While the borrowing limit lowers volatility cost, it also reduces the potential upside.

With or without a borrowing limit, transaction costs do not affect the basic form of the portfolio payoff. Transaction costs do make reversals more expensive.

These results are not difficult to derive. Their simplicity and elegance stem from the decision rule as well as the type of discretization employed. Other discretizations – for example, trading at equally spaced points in time – make the analysis more difficult and less intuitive. [See Leland (1985) and Perold (1986).]

We next develop further properties of the borrowing-constrained case under stronger assumptions. We assume geometric Brownian motion processes and frictionless and continuous trading. In this case, the payoffs depend on the sample path not through the number of trades (which is infinite), but through how much time was spent at the borrowing limit. In addition, we can analyze the effect of increasing the multiple. A larger multiple means greater exposure to the active asset. It also means trading more quickly in order to protect the downside. We show that, as the multiple goes to infinity, CPPI becomes a stop-loss strategy – investing the maximum (up to the borrowing limit) in the active asset while wealth is above the floor, then switching completely into the reserve asset if and when wealth reaches the floor. The expected holding-period return is not monotonic in the multiple, however. In particular, higher expected returns can be obtained with CPPI strategies than with the stop-loss strategy. This is somewhat surprising since the stop-loss strategy is the most aggressive in terms of risk exposure. The stop-loss strategy also has a higher expected return than path-independent borrowing-constrained strategies (which include European option replication strategies). We are unaware of any simple characterization of the borrowing-constrained strategy that maximizes the expected holding-period return.

There is an equivalence between CPPI and perpetual American call options. Specifically, the payoffs under CPPI can be reconstructed with a position in perpetual American calls on a dividend-paying security. Early exercise of the calls corresponds to reaching the borrowing limit under CPPI. Reinvestment in options corresponds to moving away from the borrowing limit.

The equivalence between CPPI and perpetual calls motivates the need to introduce intertemporal consumption into the problem in order for CPPI to be utility-maxizing. To be contended with is the path dependency induced by the borrowing constraint. With intertemporal consumption, all path dependency can be removed. Path independence is necessary for borrowing-unconstrained strategies to be utility-maximizing. [See Cox and Leland (1982).] We exhibit the class of utility functions which, with a minimum consumption constraint, have CPPI as the optimal solution. The floor on the portfolio guarantees the minimum consumption level, and the size of the multiple is related to the investor's impatience to consume. The borrowing limit is binding when consumption exceeds the minimum.

## 2. Preliminaries

### 2.1. Basic notation

The values of the active and reserve assets at $t$ are $A(t)$ and $R(t)$, respectively. Unless otherwise stated, both are payout-protected. [Alternatively, dividends and/or interest payments are reinvested in the assets that yielded them. In that case, $A(t)$ and $R(t)$ are total rate of return indexes.]

$W(t)$ represents wealth at time $t$. This is also the portfolio value at time $t$.

There is a floor on the portfolio, $F(t)$. It equals the value of a fixed number of shares of the reserve asset. [$F(t)$ is proportional to $R(t)$.]

The cushion, $C(t)$, is the excess of $W(t)$ over $F(t)$.

The exposure, $E(t)$, is the investment in the active asset.

The CPPI decision rule calls for keeping the exposure a constant multiple, $m$, of the cushion. That is, $E(t) = m \cdot C(t)$. $m$ is greater than one.

When in effect, the maximum leverage ratio is $b$, where $1 \leq b < m$. That is, $b \cdot W(t)$ is the maximum that can be invested in the active asset at time $t$. Equivalently, $(b-1) \cdot W$ is the maximum amount of the reserve asset that can be borrowed (sold short). The borrowing limit is binding when $m \cdot C(t) \geq b \cdot W(t)$. The general rule is

$$E(t) = \min\{m \cdot C(t), b \cdot W(t)\}. \tag{2.1}$$

Note that (2.1) is equivalent to a CPPI strategy involving allocations among the reserve asset and a levered portfolio that holds the active and reserve assets in constant proportions $b:1-b$. The multiple in this equivalent CPPI strategy is $m' = m/b$, and the maximum leverage ratio is $b' = 1$. This follows simply by dividing both sides of (2.1) by $b$. There is little lost in generality, therefore, by assuming $b = 1$, i.e., no borrowing permitted. Until otherwise stated, the decision rule will be

$$E(t) = \min\{m \cdot C(t), W(t)\}, \tag{2.2}$$

where the borrowing limit is reached when $m \cdot C(t) = W(t)$, i.e., $W(t) = F(t) \cdot m/(m-1)$.

## *2.2. Distributional assumptions*

As noted earlier, we assume throughout that the values of the active and reserve assets evolve continuously over time (there are no discontinuous jumps).

At times, we will want to make the much stronger assumption that the active and reserve assets follow geometric Brownian motion. $\mu_A, \mu_R$ and $\sigma_A, \sigma_R$ will denote their (stationary) means and standard deviations of instantaneous return. The covariance $\sigma_{A,R}$ between instantaneous returns will be assumed constant.[1]

The assumption of geometric Brownian motion is fairly standard for securities such as stocks. It is nonstandard for securities such as bonds where, if nothing else, the volatility of instantaneous return necessarily dampens as the maturity shortens.

## *2.3. Choice of numeraire*

For most of this paper, it will simplify the exposition considerably to change the numeraire from dollars to *shares of the reserve asset*. This involves dividing all dollar quantities by $R(t)/R_0$. The value of the reserve asset then becomes a constant, as does the floor. The CPPI rule remains unchanged.

When the numeraire is shares of the reserve asset, the value of the active asset reflects fluctuations in the dollar values of both the active and reserve assets. To emphasize this, we will denote the restated value of the active asset by $S(t)$ $[=A(t)/R(t)]$, and call it the *index ratio*.

The index ratio increases (decreases) when the active asset outperforms (underperforms) the reserve asset.

We will use the same notation $C$, $F$, $E$, and $W$ for the restated cushion, floor, exposure, and wealth. Unless otherwise stated, the numeraire will be assumed to be shares of the reserve asset.

When the dollar values of the active and reserve assets each follow geometric Brownian motion, so too does the index ratio. Using Itô's Lemma, the instantaneous mean and standard deviation of the index ratio, $\mu_S$ and $\sigma_S$, may be expressed as

$$\mu_S = \mu_A - \mu_R + \sigma_R^2(1-\beta), \qquad \sigma_S^2 = \sigma_A^2 + \sigma_R^2(1-2\beta),$$

where $\beta$ $(=\sigma_{A,R}/\sigma_R^2)$ is the 'bond beta' of the active index, the sensitivity of the dollar return on the active asset to the dollar return on the reserve asset.

[1]Under these assumptions, the assets are lognormally distributed. For example, $\ln R(t)$ is normally distributed with mean $(\mu_R - \frac{1}{2}\sigma_R^2)t$ and variance $\sigma_R^2 t$. When the reserve asset is risk-free, $\sigma_R = 0$ and $\mu_R$ is the riskless rate of interest, say $r$. Then $R(t)/R_0 = \exp(rt)$.

To illustrate, consider the case where the active asset represents U.S. stocks and the reserve asset represents long-term U.S. government bonds. During the last ten years, the sensitivity of stock returns to bond returns has been about $\beta = 0.6$. At this level of $\beta$, $\mu_S > \mu_A - \mu_R$ – the expected change in the index ratio is greater than the risk premium of stocks over bonds. And $\sigma_S^2 < \sigma_A^2$ – the index ratio is less volatile than stocks alone.

In the remainder of the paper, we will use $\mu$ and $\sigma^2$ in place of either $\mu_S$ and $\sigma_S^2$ or $\mu_A$ and $\sigma_A^2$. Which it is will be clear from the context.

## 3. CPPI with discrete and continuous rebalancing

For the time being, we assume (1) that the portfolio is far away from its borrowing limit (if any) and (2) that there are no transaction costs. In particular, security prices are unaffected by market orders which execute instantaneously.

By rebalancing, we mean resetting the exposure to $m$ times the cushion. The need to rebalance will occur only when $E$ and $m \cdot C$ differ by some minimum amount. There are several equivalent metrics for measuring this difference. From the basic definitions, there follows a simple correspondence among moves in the index ratio, the cushion, the exposure, and the exposure-to-cushion ratio $(E/C)$. Specifically, since the last rebalancing, the fractional change in the index ratio is $\delta$ if and only if

$$\text{the fractional change in the exposure is } \delta, \tag{3.1}$$

$$\text{the fractional change in the cushion is } m\delta, \tag{3.2}$$

$$\text{the fractional change in } E/C \text{ is } -(m-1)\delta/(1+m\delta). \tag{3.3}$$

Thus, a rebalancing strategy based on moves in any one of the above-mentioned items translates directly into a rebalancing strategy based on moves in any other.

These relations illustrate general features of portfolio insurance strategies. (3.2) shows that moves in the index ratio (*differential* moves in the active versus reserve assets) are magnified ($m$ times) in the cushion. (3.3) shows that any move in the index ratio puts the portfolio out of balance. Specifically, an increase in the index ratio ($\delta > 0$) causes $E/C$ to fall below $m$. To restore it to $m$, $E$ must be increased, triggering purchases of the active asset and sales of the reserve asset; and conversely if the index ratio falls. What the rule is doing is shifting money from the worse-performing to the better-performing asset.

We will base our analysis on moves in the index ratio. We will rebalance after a fractional up-move in the index ratio of size $u$ or a fractional down-move of size $d$, whichever occurs sooner. Moreover, we will choose the tolerances $u$ and $d$ so that $(1+u)(1-d)=1$. This way, down-moves exactly cancel up-moves in the index ratio. A *reversal* is a pair of up- and down-moves.

From (3.2), we see that an up-move followed by a down-move in the index ratio affects the cushion by a factor $\alpha$,

$$\alpha = (1+mu)(1-md). \tag{3.4}$$

Since $m > 1$, $\alpha$ is less than one.

$1-\alpha$ represents the cost of a reversal, or 'volatility cost'. It is the fraction by which the cushion falls for each reversal in the index ratio.

To illustrate, suppose the tolerance for up-moves in the index ratio is 3%, so that $u = 0.03$. Suppose the multiple is $m = 4$. Then $\alpha$ evaluates to 0.9895. Hence, each reversal causes the cushion to fall by 1.05%. If the cushion is 10% of total assets, then a reversal results in a volatility cost of 10.5 basis points of wealth.

As long as the exposure limit is not reached, trading occurs after every up- or down-move in the index ratio. Hence the number of moves equals the number of trades. If we let $n$ denote the number of trades, and let $i$ and $j$, respectively, denote the number of up- and down-moves ($i+j=n$), then we may write the index ratio $S$ as

$$S = S_0(1+u)^i(1-d)^j \tag{3.5}$$

and the cushion $C$ as

$$C = C_0(1+mu)^i(1-md)^j. \tag{3.6}$$

From this we can obtain:

*Proposition 1. Let the index ratio have continuous sample paths and suppose that the exposure limit is not reached. Then, after n trades, the cushion is given by*

$$C = C_0 \cdot \alpha^{\frac{1}{2}n} \cdot (S/S_0)^{\gamma}, \tag{3.7}$$

*where*

$$\gamma = \tfrac{1}{2}\ln[(1+mu)/(1-md)]/\ln(1+u). \tag{3.8}$$

*$\gamma$ is greater than m but approaches m as $u \to 0$.*

*Proof.* Follows directly by eliminating $i$ and $j$ using (3.5) and $i+j=n$. ■

Note that (3.7) holds only at 'lattice' points, i.e., at points where trading occurs. More generally, (3.7) should include an additional factor, $(1+m\delta)/(1+\delta)^{\gamma}$, where $\delta$ is the change in the index ratio since the last trade, $-d<\delta<u$. This factor approaches unity as $\delta \to 0$.

Proposition 1 says that under the assumption of only continuous sample paths, the cushion (at a lattice point) depends ex post on just two aspects of the sample path: the final level of the index ratio and the number of trades during the interval.

This is a form of path independence, since the cushion does not depend on the order of the up- and down-moves. It is weaker than the usual form of path independence in which the terminal payoff is a function only of the terminal value of the underlying instrument.

The number of trades is a measure of ex post volatility. When the index ratio follows geometric Brownian motion, the volatility is known and constant ex ante. As $u \to 0$, it turns out the number of trades approaches infinity in a deterministic manner (approximately as $\sigma^2/u^2$).[2] Hence, the usual form of path independence obtains:

*Proposition 2. Let the index ratio follow geometric Brownian motion. If, during* $0$ *to* $t$, *the exposure limit is not reached, then as* $u \to 0$

$$C(t) \to C_0 \cdot \exp - \tfrac{1}{2}(m^2-m)\sigma^2 t \cdot (S(t)/S_0)^m, \tag{3.9}$$

*with probability one.*

*Proof.* See the appendix. ■

To compare the expression for the cushion in (3.7) with that in (3.9), consider a time horizon of one year ($t=1$), a multiple of 4, a tolerance for up-moves of 3% ($u=0.03$), and a lognormally distributed index ratio with $\sigma = 0.15$ (15% per annum). With these values, the power $\gamma$ is 4.01, which is very close to $m=4$, while the expected number of trades is roughly 25 per year or about one every two weeks ($n \approx \sigma^2/u^2$). Over a year, the volatility cost under discrete rebalancing will be about 12.3% of the cushion $(1-\alpha^{\frac{1}{2}n})$, while it is 12.6% of the cushion $(1-\exp-\frac{1}{2}(m^2-m)\sigma^2 t)$ under continuous rebalancing. These volatility costs will typically be much smaller when expressed in relation to total wealth.

### *The borrowing limit*

So far we have assumed that the borrowing limit is not a factor. With a large enough multiple, however, $m \cdot C$ will exceed $W$, and the portfolio will

[2] See the proof of Proposition 2 in the appendix.

begin at the borrowing limit, i.e., fully invested in the active asset. Even if it begins away from the borrowing limit, it may not take much of an up-move before the limit is reached.

When the borrowing limit is in effect, the portfolio is in a buy-and-hold state. Since no trading occurs during this state, there can be no volatility cost. But the cushion grows only linearly in $S$ as opposed to geometrically in $S$ when the limit is not in effect.

In general, the portfolio will alternate between being at the borrowing limit and being away from the borrowing limit. How it does so will be determined by the sample path.

Potentially, the cushion may depend in a very complex way on the pattern of transitions to and from the borrowing limit. We now show that this dependence is relatively straightforward.

First, as mentioned above, if the borrowing limit is always in effect, the cushion is simply linear in $S$. Second, if the portfolio leaves the borrowing limit at some point $t^*$, then the cushion at $t > t^*$ depends only on the sample path beyond $t^*$. This is because at $t^*$, $m \cdot C = W \equiv F + C$. That is, $C(t^*) = F/(m-1)$, a constant. And third, if the portfolio is at the borrowing limit at $t$, and last reached the limit at $t^* < t$, then $C(t) = (C(t^*) + F)S(t)/S(t^*) - F$, where again $C(t^*) = F/(m-1)$. The last two statements are precise in the case of continuous rebalancing (geometric Brownian motion). Under discrete rebalancing, the cushion at $t^*$ will differ slightly from $F/(m-1)$ (becoming exact as $u \to 0$) when reaching or leaving the borrowing limit.

This leaves the case of an interval over which the portfolio begins and ends partially invested. We obtain an appealing result in the case of continuous rebalancing, and also with discrete rebalancing under a slightly modified rebalancing rule. The modified rule transacts only while $m \cdot C \leq W$.[3] This modification is needed because, while rebalancing is occurring at proportionately spaced points along the $S$ axis, the cushion does not evolve along a uniform lattice structure.

*Proposition 3. If the portfolio starts away from the borrowing limit, and at t is also away from the borrowing limit, then*

(i) *under 'modified' discrete rebalancing, the expression for the cushion at t is identical to that given in (3.7),*

(ii) *under continuous rebalancing (with geometric Brownian motion), the cushion at t differs from that given in (3.9) only in the volatility cost factor,*

[3]The modified rule differs from the 'usual' rule as follows: When an up-move causes a transition from $mC < W$ to $mC' > W'$, the usual rule transacts to increase the exposure to $W'$. The modified rule does not. At such a point, the exposure under the modified rule is $mC(1+u) < W'$, so that there remains some holding in the reserve asset (unless per chance $mC = W$). For small $u$ this is a negligible amount.

*which now is* $\exp - \frac{1}{2}(m^2 - m)\sigma^2\tau$, *where* $\tau$ *is the total amount of time during* 0 *to* $t$ *that the portfolio is away from the borrowing limit.*

*Proof.* Sketched in the discussion below. ■

Proposition 3 says that the only effect of being temporarily at the borrowing limit is to reduce volatility cost. This is seen most easily in the case of discrete rebalancing. When the portfolio is in a buy-and-hold state, pairs of up- and down-moves cancel each other not only as regards the index ratio, but also as regards the cushion. The following case illustrates the point. Suppose $E_0 = m \cdot C_0 < W_0$, that the next two moves are an up-move followed by a down-move, and that after the up-move, $m \cdot C_1 > W_1$. Under the modified rule, no trading takes place after the up-move, so that $E_1 = E_0 \cdot (1+u)$. Thus, following the down-move, $W_2 = W_0$ and $C_2 = C_0$. Moreover, $E_2$ will equal $E_0$. Hence, $E_2 = m \cdot C_2$, and no rebalancing trade is required. Therefore, the number of trades is less than the number of moves in the index ratio. The number of trades exactly equals the number of moves in the index ratio that occur while the portfolio is away from the borrowing limit (at both the start and end of a move).

We note that Proposition 3 is one place where some generality has been lost in assuming a maximum leverage ratio of $b = 1$. (Recall the discussion in section 2.1.) That is so because trading is required to keep the portfolio at a leverage ratio $b > 1$. Volatility cost is incurred both while away from the borrowing limit and while at the borrowing limit. In the case of $b > 1$, Proposition 3 generalizes as follows:

- With continuous rebalancing, the volatility cost factor becomes $\exp - \frac{1}{2}\sigma^2 m\{(m-1)\tau + (b-1)(t-\tau)\}$.
- With discrete rebalancing, the volatility cost factor becomes approximately $\alpha^{\frac{1}{2}n_1} \cdot \tilde{\alpha}^{\frac{1}{2}n_2}$, where $n_1$ is the number of trades that occur while away from the borrowing limit and $n_2$ is the number of trades that occur while the borrowing limit is binding. $\tilde{\alpha}$ is given by $\{(1+bu)(1-bd)\}^{\gamma/\gamma'}$, where $\gamma' = \frac{1}{2}\ln[(1+bu)/(1-bd)]/\ln(1+u)$, and $\alpha$ and $\gamma$ are as in (3.4) and (3.8), respectively. The approximation stems from the fact that during transitions from $m \cdot C < b \cdot W$ to $m \cdot C > b \cdot W$, trades do not occur at exactly $m \cdot C = b \cdot W$.

## 4. Transaction costs

We consider the case of proportional transaction costs. The costs will be incurred in exchanging shares of the active asset for shares of the reserve asset. We think of these costs as brokerage fees.

Rebalancing will occur *net of transaction costs*. That is, the exposure will be set equal to $m$ times the cushion, where the cushion has been adjusted to reflect the cost of the transaction. Per our discussion above, if the borrowing limit is in effect, transactions will occur only while $m \cdot C < W$.

*Proposition 4. Assume the portfolio begins and ends away from the borrowing limit. Let the one-way proportional transaction cost be $k$ and the rebalancing tolerance for up-moves in the index ratio be $u$. Assume only continuous sample paths. Define $k_{\max} = [1-(m-1)u]/m < 1/m$. Suppose that $k \le k_{\max}$. Then, after $n$ trades, if the index ratio level is $S$, the cushion may be expressed as*

$$C = C_0 \cdot \hat{\alpha}^{\frac{1}{2}n} \cdot (S/S_0)^{\hat{\gamma}},$$

*where*

$$\hat{\alpha} = (1+m\hat{u})(1-m\hat{d}),$$

$$\hat{\gamma} = \tfrac{1}{2}\ln\left[(1+m\hat{u})/(1-m\hat{d})\right]\Big/\ln(1+u),$$

*and*

$$\hat{u} = u \cdot (1+k)/(1+mk),$$

$$\hat{d} = d \cdot (1-k)/(1-mk).$$

*Moreover, $\hat{\gamma} > \gamma > m$ and $\hat{\alpha} < \alpha$.*

*As $u \to 0$, $\hat{\gamma}$ converges to $m \cdot (1-mk^2)/(1-m^2k^2) > m$.*

*If the index ratio follows geometric Brownian motion, then $\hat{\alpha}^{\frac{1}{2}n} \to 0$ as $u \to 0$. That is, transaction costs destroy the cushion (but no more than the cushion) as the trading frequency increases.*

*If $k > k_{\max}$, then trading on a down-move will cause the portfolio to fall beneath its floor.*

*Proof.* See the appendix. ■

Proposition 4 says that sufficiently small transaction costs do not alter the basic form of the cushion. What does change is the values of the parameters. The principal result is that the cushion behaves as though rebalancing were occurring after a smaller up-move $\hat{u} < u$ and a larger down-move $\hat{d} > d$ (although the decision of when to trade and the number of trades are still based on the originally specified tolerances $u$ and $d$). Transaction costs thus show up principally in the volatility cost. This is not surprising, since the volatility cost arises as a direct consequence of trading, whether or not there are transaction costs.

To illustrate, consider our previous example with $u = 0.03$, $m = 4$. In order that the floor not be endangered, $k$ can be at most $k_{\max} = 0.2275$ (22.75%). With a transaction cost of 1% ($k = 0.01$), $\hat{\alpha}$ evaluates to 0.9731 versus $\alpha = 0.9895$. Hence the effect of a 1% transaction cost is to raise the cost of a reversal from 1.05% of the cushion to 2.69% of the cushion. The effect is also to increase the power: $\hat{\gamma}$ evaluates to 4.19 versus $\gamma = 4.01$. As $u \to 0$, $\hat{\gamma} \to 4.05$ (versus $\gamma \to 4$).

## 5. Behavior of CPPI as the multiple becomes large

We have seen that along any sample path, the behavior of a CPPI-managed portfolio depends on the pattern of transitions to and from the borrowing limit. While at the borrowing limit, the portfolio is in a buy-and-hold state. While away from the limit, rebalancing occurs. The greater the multiple, the more likely it is that the portfolio will be in a buy-and-hold state. And when not in that state, the more active will be the rebalancing.

In a discrete trading context, it is clear that CPPI becomes a stop-loss order for a sufficiently large multiple. This can be seen from (3.2) where the effect of a down-move is to multiply the cushion by the factor $1 - md$. When $m = 1/d$, a down-move destroys the cushion. Once the cushion is zero, the portfolio must remain fully invested in the reserve asset or the floor will be endangered. It therefore will not participate in any subsequent good performance of the active asset.

We now show that the same result holds true in the case of geometric Brownian motion and continuous, frictionless trading. The result is not a trivial extension of the foregoing discussion, since we are first letting the trading interval go to zero before letting the multiple go to infinity. Indeed, it is easy to construct continuous sample paths along which CPPI with continuous trading dominates the stop-loss strategy in the limit as $m \to \infty$.[4]

We have noted before that for large enough $m$, CPPI and the stop-loss strategy both will begin with the borrowing limit binding. The point of departure for the two strategies will come if and when the cushion reaches $F/(m-1)$, i.e., $m \cdot C = W$ and the borrowing limit is just binding under CPPI. As $m \to \infty$, this boundary goes to zero. Hence, in the limit, CPPI and the stop-loss strategy will be identical over the interval $[0, T)$, where, for a given sample path, $T$ is the point at which the cushion under the stop-loss strategy becomes zero and the portfolio invests completely in the reserve asset. (Possibly, $T = \infty$.)

[4]For example, let $S(t)$ be deterministic, given by $S_0 = W_0 = F + T^2$, and $\mathrm{d}S = 2(t-T)\mathrm{d}t$. $S(t)$ is the parabola $S(t) = F + (t-T)^2$. Under the stop-loss strategy, $C(t) = S(t) - F$ for $t < T$, and $C(t) = 0$ for $t \geq T$. Under CPPI with multiple $m$, the borrowing limit ceases to be binding when $C(t) < C^*(m) = F/(m-1)$. $C(t)$ evaluates to $S(t) - F$ for $(t-T)^2 \geq C^*(m)$ and $C^*\{S(t)/(C^* + F)\}^m$ for $(t-T)^2 < C^*(m)$.

Consider now the size of the cushion at some time $T' > T$. Under the stop-loss strategy, the cushion at $T'$ is zero. Under CPPI, and for finite $m$, the cushion at $T'$ is strictly greater than zero. However, with probability one, the index ratio will be beneath $S(T)$ at some point during the interval $(T, T']$. Let $\tau > 0$ be the total time for which $S(t) < S(T)$ during $(T, T']$. Note that the borrowing limit will not be binding while $S(t) < S(T)$. Hence, the cushion at $T'$ will contain the volatility cost factor $\exp - \frac{1}{2}m(m-1)\sigma^2\tau$ which goes to zero as $m \to \infty$.

This establishes:

*Proposition 5. Under the assumptions of geometric Brownian motion and continuous and frictionless trading, the payoff under CPPI approaches that of the stop-loss strategy with probability one as the multiple approaches infinity.*

### *The expected return as a function of m*

In practice, it is fairly common to see investors give consideration to the expected holding-period return when choosing a portfolio insurance strategy. It could be that expected return is simply one of several attributes of utility. For example, the investor's preferences might be approximated in terms of a tradeoff between expected holding-period return and downside protection. In the extreme, the investor may genuinely have infinite risk tolerance at all levels of wealth above his floor.

Calculating the expected holding period return for borrowing-constrained strategies (CPPI or otherwise) is not easy, however. This is not surprising, given the path dependence of the payoff. Obviously, the expected return is less than that obtainable by using maximum leverage at all times (assuming no floor), and greater than that obtainable by investing 100% in the reserve asset at all times. But, for a given floor, it is not even known which strategy has the maximum expected return, nor what the maximum itself is.

Path-independent, borrowing-constrained strategies are better understood. From within this class, expected return is maximized by strategies that replicate European call options. See Rubinstein (1985). But these strategies are not expected-return-maximizing if path-dependent strategies are admitted. In particular, path-independent strategies have lower expected returns than obtainable under the stop-loss strategy.

While there is no simple formula for the expected return under CPPI, it is relatively straightforward to compute the expected return under the stop-loss strategy, however. (See Proposition 6 below.) It is thus of interest to investigate how the expected return of a CPPI strategy approaches that of the stop-loss strategy as $m \to \infty$. We do so by deriving a partial differential equation for the expected return under CPPI. From this we compute a closed-form solution for its Laplace transform. We use this to show that the expected return under CPPI approaches that under a stop-loss strategy in a nonmonotonic fashion. Specifically, a higher expected return can be obtained with a finite multiple than with an infinite multiple.

This result is of independent interest, since it demonstrates that the stop-loss strategy is not expected-return-maximizing. It is tempting to think that the stop-loss strategy might maximize expected return since it is 'bang bang': At all times, it invests the maximum possible in the active asset. A direct proof that the stop-loss strategy is not expected-return-maximizing is given in the appendix.

The foregoing discussion is reflected formally in the propositions below. They assume geometric Brownian motion and continuous and frictionless trading.

*Proposition 6. Under the stop-loss strategy, the expected wealth at t is*

$$F \cdot \left[1 + \left\{B(w,t) - w^{1-a}B(w^{-1},t)\right\} \cdot \exp \mu t\right], \tag{5.1}$$

*where $w = W_0/F$ is the initial wealth-to-floor ratio, $a = 2\mu/\sigma^2$, and $B(P,t)$ is the Black–Scholes formula for a European call option on a security with initial price $P$, strike price one, interest rate $\mu$, volatility $\sigma^2$, and time until expiration $t$.*

*Proof.* See the appendix. ■

*Proposition 7.*[5] *Under a CPPI strategy with continuous rebalancing and multiple $m$, let $G(W,t)$ be the expected wealth at $t$ when the initial wealth is $W$. Then $G$ satisfies the partial differential equation*

$$\tfrac{1}{2}\sigma^2 E^2 G_{WW} + \mu E G_W - G_t = 0, \tag{5.2}$$

*with boundary conditions $G_W(\infty,t) = \exp \mu t$, $G(F,t) = F$, and $G(W,0) = W$, where $E = E(W) = \min\{W, m(W-F)\}$.*

*Proof.* See the appendix. ■

*Proposition 8. Under a CPPI strategy with continuous rebalancing, the expected wealth at time t is not monotonic in the multiple. In particular, the expected return under CPPI can exceed that under the stop-loss strategy.*

*Proof.* See the appendix. ■

[5]This result was derived independently in Rouhani (1987).

## 6. CPPI and perpetual American call options

In this section, we investigate whether there are simple securities that produce the same payoffs as CPPI. The answer turns out to involve the purchase and possible exercise and reinvestment in perpetual American call options on dividend-paying securities. This connection is more than just a 'nice' characterization of CPPI. It illustrates what kinds of options will achieve the same types of payoffs as dynamic investment strategies under borrowing constraints. They are options where early exercise is rational. In addition, the analysis that leads to this conclusion makes it easy to see why CPPI is utility-maximizing when intertemporal consumption is introduced.

Consider a security with value $Y(t)$ paying dividends continuously and in proportion to $Y$ at a constant rate $d$. Assume that $Y$ follows geometric Brownian motion with instantaneous variance $\sigma^2$. Let $Q$ be the value of a perpetual American call option on this security with strike price $K$. Because the numeraire is shares of the reserve asset, the dollar value of the strike price will vary over time. For example, if the reserve asset pays a riskless rate of interest $r$, the dollar value of the strike price will grow at rate $r$.

Samuelson and McKean (1965) and Samuelson and Merton (1969) show that $Q$ is a function only of $Y$, $d$, $\sigma^2$, and $K$: For $Y \leq Y^*$,

$$Q = (Y^* - K)(Y/Y^*)\gamma, \tag{6.1}$$

where $\gamma = 1 + 2d/\sigma^2$ and $Y^* = K\gamma/(\gamma - 1)$. It is optimal to exercise the opinion when $Y > Y^*$.

Expression (6.1) bears a close resemblance to the formula for the cushion given in (3.9). The difference lies essentially in the volatility cost factor $\exp - \frac{1}{2}(m^2 - m)\sigma^2 t$. Propositions 9 and 10 tie (3.9) and (6.1) together.

*Proposition 9* (CPPI with a dividend-paying security). *Suppose the active asset, with value $S(t)$, continuously pays a constant proportional dividend $d$. Assume the dividend is continuously reinvested in the portfolio, which is managed according to a CPPI strategy with multiple $m$. Assume continuous trading and geometric Brownian motion for $S$ and that the borrowing limit is not reached. Then the cushion at $t$ may be expressed as*

$$C = C_0 \cdot \exp m\left[d - \tfrac{1}{2}(m-1)\sigma^2\right]t \cdot \left(S(t)/S_0\right)^m. \tag{6.2}$$

*When $d = \frac{1}{2}(m-1)\sigma^2$, the cushion evaluates to*

$$C = C_0 \cdot \left(S(t)/S_0\right)^\gamma, \tag{6.3}$$

*where $\gamma = m = 1 + 2d/\sigma^2$ as in (6.1).*

*Proof.* Since the dividend is reinvested, there is no economic difference between, on one hand, investing in the active asset and receiving the dividend and, on the other, investing in a non-dividend-paying security with the same total return as the active asset. Let the value of this second security be $\hat{S}$. $\hat{S}(t)$ can be expressed in terms of $S(t)$ according to

$$\hat{S}(t) = \mathrm{e}^{dt} \cdot S(t).$$

The result follows by substituting this into (3.9). ∎

Proposition 9 shows that prior to reaching the borrowing limit, the payoff under a CPPI strategy differs only by a scale factor from that of a perpetual American call option prior to exercise, where the call is on a security that pays a dividend $d = \frac{1}{2}(m-1)\sigma^2$. The scale factor depends on the initial wealth, the floor, the strike price of the call, and the initial level of the active asset. By suitable choice of strike price, the point of exercise of the call option can be made to coincide with the point where the borrowing limit is reached under CPPI.

*Proposition 10. Let the active asset follow geometric Brownian motion and pay continuously a constant proportional dividend $d$. Consider a CPPI strategy with multiple $m$, continuous rebalancing, and reinvestment of the dividend. Assume that the portfolio is initially away from the borrowing limit.*

*If the parameters $m$, $d$, and $\sigma^2$ are such that $d = \frac{1}{2}(m-1)\sigma^2$, then the behavior of the cushion under CPPI can be replicated by purchasing $F/K$ perpetual American call options on the active asset. The floor, $F$, and the strike price, $K$, together must satisfy*

$$K = \left[\frac{F}{(m-1)(W_0 - F)}\right]^{1/m} \cdot S_0 \cdot (m-1)/m. \tag{6.4}$$

*Exercise of the calls is coincident with reaching the borrowing limit under CPPI.*

*If and when the wealth level returns to the point where the borrowing limit is barely slack ($m \cdot C = W$), $F/K'$ new calls must be purchased with strike price $K' = K\,\mathrm{e}^{-d\tau}$, where $\tau$ is the time spent at the borrowing limit to date. The same must be done at all subsequent transitions away from the borrowing limit.*

*Proof.* The equivalence up to the first point of exercise/reaching the borrowing limit follows directly by choosing the number of calls and the strike price $K$ to make (6.1) and (6.3) identical. The strike price of options purchased subsequently declines by the factor $\mathrm{e}^{-d\tau}$ since the accumulation of dividends once the option has been exercised reduces the level to which $S$ must fall before the borrowing limit becomes slack again. ■

Note in Proposition 10 that the active asset employed by CPPI need not be dividend paying per se. It must just have the same total return as the security on which the perpetual call is written. The latter security, of course, must be dividend-paying.

The relationship $d = \frac{1}{2}(m-1)\sigma^2$ can be interpreted in at least two ways. For given $m$, it says that the optioned security must be one whose dividend is proportional to the volatility cost under CPPI. For a given optionable security with dividend $d$, it says that only those CPPI strategies with $m = 1 + 2d/\sigma^2$ can be made equivalent to a perpetual call on that security.

*Intertemporal consumption and path independence*

Interpreting CPPI in terms of perpetual calls on a dividend-paying security (Proposition 10) gives us another way to think about path dependency. Under CPPI, the path dependency arises due to the reduction in volatility cost while the borrowing limit is binding. With perpetual call options, the path dependency arises out of reinvestment of the dividend following exercise of the option and taking delivery of the underlying security, and before reinvestment in new call options. It follows that if the dividend is consumed rather than reinvested, wealth becomes path-independent, as does consumption.[6] Specifically, let $S^*$ be the level of $S$ at which $m \cdot C = W$, where $C$ is given in (6.3). $S^*$ can be expressed as $mK/(m-1)$, where $K$ is the strike price given in (6.4). Then:

- For $S(t) \leq S^*$, $C(t) = C_0 \cdot (S(t)/S_0)^m$ as per (6.3).
- For $S(t) > S^*$, $C(t) = F \cdot (S(t)/K - 1)$, where consumption $= d \cdot F \cdot S(t)/K$ with $d = \frac{1}{2}(m-1)\sigma^2$.

Wealth and consumption are functions only of $S(t)$.

This result is of interest since path independence is a necessary condition for an investment–consumption rule to be utility-maximizing in the borrowing-unconstrained case. [See Cox and Leland (1982).] The question then becomes whether there is a borrowing-unconstrained utility function for which a perpetual call option with consumption of the dividend following exercise is optimal. We provide the answer in Proposition 11 below.

[6] With consumption of the dividend, the number of calls repurchased after transitions away from the borrowing limit becomes the constant $F/K$; the strike price of these calls is the constant $K$.

## 7. Utility maximization

In this final section, it will simplify the presentation to change the numeraire from shares of the reserve asset to dollars. We will assume the reserve asset pays a risk-free rate of interest $r$ and that the active asset follows geometric Brownian motion. Consider the utility of intertemporal consumption:

$$\mathrm{E}\left\{\int_0^{\infty} \mathrm{e}^{-\rho t} U(c(t))\, \mathrm{d}t\right\}, \tag{7.1}$$

which is to be maximized subject to consumption $c(t) \geq c_{\min}$. $\rho > 0$ measures the impatience to consume. (7.1) is the standard, time-additive, state-independent utility of lifetime consumption function considered in the literature [e.g., Merton (1971)]. The constraint on consumption is less standard. $c_{\min}$ can be interpreted as a subsistence level of consumption.

Define $U(c)$ as follows:

$$\begin{aligned} U(c) &= c^{1-\lambda}/(1-\lambda) && \text{for} \quad c \geq c^*, \\ &= U(c^*) - (c^* - c) \cdot U'(c^*) && \text{for} \quad c < c^*, \end{aligned}$$

where $\lambda > 0$ and $c^* > 0$ are given. The function $U(\cdot)$ is linear below $c^*$ and a power function above $c^*$. It is concave and differentiable at $c^*$.

Determine $c_{\min}$, $F$, $m$, $b$, and $d$ according to:

$$c_{\min} = c^*/\left(1 + \tfrac{1}{2} mb\sigma^2/r\right),$$

$$F = c_{\min}/r,$$

$$b = (\mu - r)/\lambda\sigma^2,$$

$$m = (\mu - r)/\lambda'\sigma^2,$$

$$d = (m-b)\left\{r/m + \tfrac{1}{2} b\sigma^2\right\},^7$$

where

$$\lambda' = \left[(r - \rho - \xi) + \sqrt{(r-\rho-\xi)^2 + 4r\xi}\,\right]\Big/2r$$

with $\xi = \tfrac{1}{2}(\mu - r)^2/\sigma^2$.

*Proposition 11.* *With $U(\cdot)$ and $c_{\min}$ as given above, the optimal consumption program for (7.1) subject to $c(t) \geq c_{\min}$ is $c(t) = c_{\min}$ for $W < W^*$ and $c(t) =$*

[7] When $r = 0$, as is the case when the numeraire is shares of a capital asset, and $b = 1$, $d$ reduces to $\frac{1}{2}(m-1)\sigma^2$ as in section 6.

*$d \cdot W(t)$ for $W \geq W^*$, where $W^* = c^*/d$. The optimal investment strategy is CPPI with constant floor $F$, multiple $m$, and maximum leverage ratio $b$.*

*Proof.* See the appendix. ■

We make several observations. First, consumption is proportional to wealth when $W(t) > W^*$. By construction, $W^*$ can also be expressed as $W^* = F \cdot m/(m-b)$, the level of wealth at which the borrowing limit is reached. Consumption is therefore proportional to wealth while the borrowing limit is binding, and fixed at $c_{\min}$ otherwise. $c_{\min} < c^*$ so that consumption is not continuous at $W^*$. $c_{\min}$ depends on the utility function through $c^*$ and $\lambda$.

Second, for $c \geq c^*$, $U(c)$ has constant relative risk aversion $\lambda$. The maximum leverage ratio $b$ will exceed unity for sufficiently small $\lambda$, i.e., an investor with sufficiently large risk tolerance. This leverage ratio can only become binding (in high wealth states) if the multiple $m$ exceeds $b$. $m$ will exceed $b$ for sufficiently large $\rho$, i.e., sufficient impatience to consume. The multiple depends on the utility function only through $\rho$.

Third, maximizing (7.1) for general utility $U(c)$ and subject to $c(t) \geq c_{\min}$ will lead to the same optimal investment strategy over a range $W \leq W^*$ for which the optimal $c(t) = c_{\min}$. That is, for $W(t) \leq W^*$, the optimal strategy is CPPI with $m$ dependent on the utility function only through $\rho$ as given above.[8] For $W(t) > W^*$, the solution depends on the function $U(c)$ over the range $c \geq c_{\min}$. If the optimal consumption is to be continuous in wealth, then both consumption and the exposure must be *nonlinear* in wealth for $W \geq W^*$.

For $b = 1$, the CPPI strategy in Proposition 11 can be interpreted in terms of perpetual calls as follows: Place \$$F$ in a bond paying riskless interest $c_{\min}$ (at rate $r$); place the remaining wealth \$$(W_0 - F)$ in $F/K$ perpetual calls on the active asset which pays dividends at rate $d$, where the strike price is $K$ as given in (6.4). Prior to exercise of the options, consume the interest paid by the bond. Upon exercise of the options, exchange the bond for $F/K$ shares of the active asset, consuming the dividends on these shares from then on. Reverse these steps if and when wealth falls beneath $W^*$. In the case of $b > 1$, the calls must be written on the active asset leveraged $b:1$.

## Appendix

### *A.1. Proof of Proposition 2*

The proposition is straightforward to establish if the limit is assumed to exist and to correspond to the limit under continuous trading. In that case Itô's Lemma provides a short proof: The relations $\mathrm{d}\ln(C) = \mathrm{d}C/C -$

[8] This is shown in Merton (1990) who solves (7.1) for the case $c_{\min} = 0$, $r > 0$, and $U'(0)$ finite.

$\frac{1}{2}(\mathrm{d}C/C)^2$, $\mathrm{d}S/S = \mathrm{d}\ln(S) + \frac{1}{2}(\mathrm{d}S/S)^2$, and $\mathrm{d}C/C = m \cdot \mathrm{d}S/S$ combine to yield $\mathrm{d}\ln(C) = m\,\mathrm{d}\ln(S) - \frac{1}{2}(m^2 - m)\sigma^2\,\mathrm{d}t$. Integrating both sides establishes the result.

However, convergence as $u \to 0$ is not obvious since the number and timing of trades is random. We need to resort to unorthodox methods to obtain a proof.

From Proposition 1, we have that $\gamma \to m$ as $u \to 0$. It remains to be shown that $\alpha^{\frac{1}{2}n} \to \exp - \frac{1}{2}(m^2 - m)\sigma^2 t$ with probability one as $u \to 0$.

Let $v = \ln(1 + u)$. Note, $v \approx u$ for small $u$.

Write $\ln(\alpha^{\frac{1}{2}n})$ as the product of $\frac{1}{2}\ln(\alpha)/v^2$ and $v^2 n$. It is straightforward to show that the first term has a limit $\frac{1}{2}(m - m^2)$ as $v \to 0$. The more difficult part is to show that $v^2 n \to \sigma^2 t$. We will show that this is true when $S$ has zero drift ($\mu = \frac{1}{2}\sigma^2$), unconditional on $S(t)$. Then it must be true when $S$ has zero drift, conditional on $S(t)$, from which it follows that it must be true for any drift, conditional on $S(t)$. The latter part of this assertion derives from the fact that, conditional on $S(t)$, the distribution of $S(\tau)$, $\tau \le t$, is independent of $\mu$.

Let $\Delta t(v)$ be the time that passes between a pair of successive moves. Note that the number of moves between 0 and $t$ depends on $v$ [$n = n(v)$] and goes to infinity as $v \to 0$. Moreover, $\sum \Delta t(v) \to t$ as $v \to 0$, where the sum is over $n(v)$ terms.

Harrison (1985, p. 41) shows that the Laplace transform of $\Delta t$ with dummy parameter $\lambda$ is $2\xi/(1 + \xi^2)$ where $\xi = \exp(-\sqrt{2\lambda}\, v/\sigma)$. It follows that $\Delta t(v)/v^2$ and $\Delta t(1)$ have the same probability distribution.

It is also the case that $\mathrm{E}\{\Delta t(1)\} = 1/\sigma^2$ [Harrison (1985, p. 52)].

Consider now the two summations $\sum \Delta t(v)/v^2 n(v)$ and $\sum \Delta t(1)/n(v)$. From the above, both have the same distribution. Being the sum of $n(v)$ iid variates with mean $1/\sigma^2$, the latter converges to $1/\sigma^2$ by the law of large numbers. Hence, so does the former. This implies that $v^2 n(v) \to \sigma^2 t$, which completes the proof. ■

### A.2. *Proof of Proposition 4*

Initially, $E = m \cdot C$. After a fractional move in the index ratio of size $\delta$, the holding in the active asset changes to $E \cdot (1 + \delta)$ and the cushion to $C \cdot (1 + m\delta)$.

Let $C'$ denote the cushion after rebalancing. $C'$ will differ from $C \cdot (1 + m\delta)$ by the cost of rebalancing. The cost of rebalancing is $k \cdot |mC' - E(1 + \delta)|$ which equals $km \cdot |C' - C(1 + \delta)|$. Hence,

$$C' = C \cdot (1 + m\delta) - km \cdot |C' - C(1 + \delta)|.$$

Dividing by $C$ gives

$$C'/C = 1 + m\delta - km \cdot |C'/C - 1 - \delta| .$$

For $\delta = u > 0$, the solution to this equation is $C'/C = 1 + m\hat{u}$. For $\delta = -d < 0$, the solution is $C'/C = 1 - m\hat{d}$.

The remaining parts of the proposition are straightforward to establish. ■

### A.3. Proof of Proposition 6

We give a direct proof. The result can also be established by checking that (5.1) satisfies the partial differential eq. (5.2) (in the case $m = \infty$) and its boundary conditions.

Assume $W_0 = S_0 = 1$, $F < 1$. Let $Y(t) = \min\{\ln(S(\tau)/F)\colon \tau \leq t\}$. Under a stop-loss strategy,

$$W(t) = F \cdot \exp\max\{Y(t), 0\} . \tag{A.1}$$

The probability distribution of $Y(t)$ is well-known [Harrison (1985, p. 46)]. Specifically,

$$\text{Prob}\{Y(t) > y\} = \Phi(c_1) - \exp(-2\nu x/\sigma^2)\Phi(c_2), \tag{A.2}$$

where $\nu = \mu - \frac{1}{2}\sigma^2$, $x = -\ln F$, $c_1 = (-y + x + \nu t)/\sigma\sqrt{t}$, $c_2 = c_1 - 2x/\sigma\sqrt{t}$, and $\Phi$ is the cumulative of the unit normal density. From this the probability of being 'stopped out' $[Y(t) \leq 0]$ follows immediately. Also, differentiating (A.2) yields the density of $Y(t)$: $[\phi(c_1) - \exp(-2\nu x/\sigma^2)\phi(c_2)]/\sigma\sqrt{t}$, where $\phi$ is the unit normal density.

The expected wealth can now be found by integrating (A.1) in a standard fashion. ■

### A.4. Proof of Proposition 7

$G(W,t) = \{W(t)|W\}_0$, where $\{\ \}_s$ denotes the expectation taken as of time $s$. The expectation can be written as $\{\{W(t)|W(\Delta t)\}_{\Delta t}|W\}_0$ which equals $\{G(W(\Delta t), t - \Delta t)|W\}_0$. Expanding to second order yields

$$G(W,t) \approx \{G(W,t) - G_t\,\Delta t + G_W\,\Delta W + \tfrac{1}{2}G_{WW}\,\Delta W^2\}_0, \tag{A.3}$$

where $\Delta W = W(\Delta t) - W$. Eq. (5.2) follows upon substituting $\{\Delta W\}_0 \approx E\mu\,\Delta t$ and $\{\Delta W^2\}_0 \approx E^2\sigma^2\,\Delta t$.

The boundary conditions require no explanation. ■

*A.5. Proof of Proposition 8*

Let $L(W,s)$ be the Laplace transform of $G(W,t)$.

If $G$ satisfies (5.2) and its associated boundary conditions, then $L$ satisfies the ordinary differential equation

$$\tfrac{1}{2}\sigma^2 E^2 L_{WW} + \mu E L_W - sL + W = 0, \tag{A.4}$$

subject to the boundary conditions $L_W(\infty, s) = 1/(s-\mu)$ and $L(1,s) = 1/s$.

Let $W^* = Fm/(m-1)$. It is readily checked that the solution satisfying (A.4) and the boundary conditions is of the form

$$L = [W - A\mu W^{\gamma}/s]/(s-\mu),$$

when $W > W^*$, and

$$L = (W-F)/(s-\mu m) + F/s + B(W-F)^{\alpha},$$

when $W < W^*$, where $A$ and $B$ are constants of integration, $\gamma$ is the negative root of $\frac{1}{2}\sigma^2\gamma(\gamma-1) + \mu\gamma - s = 0$, and $\alpha$ is the positive root of $\frac{1}{2}\sigma^2 m^2\alpha(\alpha-1) + \mu\alpha m - s = 0$. Note that $L$ is defined for $s > \mu m$.

We determine $A$ and $B$ by imposing that $L$ be continuously differentiable at $W^*$. It turns out that

$$A = \frac{m[(m-1)/m)]^{\gamma}(s - \alpha m\mu)}{(s-\mu m)(m\alpha - \gamma)},$$

$$B = \frac{\mu m(m-1)^{\alpha}(\mu\gamma - s)}{s(s-\mu)(s-\mu m)(m\alpha-\gamma)}.$$

By similar methods, the Laplace transform of the expected return under a stop-loss strategy can be derived. Call it $L_\infty(W,s)$:

$$L_\infty = [W - \mu W^{\gamma}/s]/(s-\mu).$$

$L_\infty$ is defined for $s > \mu$, and for $W > W^*$ differs from $L$ only in the factor $A$.

For large $m$, we are interested in the case $W > W^*$. Here, the dependence of $L$ on $m$ is solely through $A$. We now show that $A < 1$ for all large $m$ and $s > \mu m$.

Fix $s$ as a function of $m$ according to $s = \xi\mu m$ for $\xi > 1$. This makes $A$ and $\gamma$ functions only of $m$. As $m \to \infty$, $A \to 1$. The derivative of $A$ with respect to $m$ can be written as $\xi(\xi-1)\mu/(\sigma m)^2 + \mathrm{O}(m^{-5/2})$. Hence $A$ is increasing in $m$ for large $m$.

This proves that $A < 1$ for large enough $m$ and $s > \mu m$. In turn, this proves that $L > L_\infty$ for large enough $m$ and $s > \mu m$. Hence, it cannot be the case that $G(W,t) \leq G_\infty(W,t)$ for all $W > W^*$ and $t$, where $G_\infty$ is the expected return under the stop-loss strategy.

This completes the proof. ■

### *A.6. Direct proof that stop-loss strategies are not expected-return-maximizing*

Let $J(W,t)$ be the expected return that can be attained under the expected-return-maximizing policy. If the stop-loss strategy is expected-return-maximizing, then $J(W,t)$ must be the expression (5.1) given in Proposition 6.

With borrowing-unconstrained utility maximization, the optimal fraction $x$ of wealth to be invested in the active asset is given by

$$x = \frac{-J_W \mu}{W \cdot J_{WW} \sigma^2}. \tag{A.5}$$

[See, e.g., Merton (1971); recall that the riskless rate is zero in this context.] Hence, if the stop-loss strategy is optimal, it must be that the right-hand side of (A.5) is at least unity. However, under the stop-loss strategy, the right-hand side of (A.5) can be shown to approach $\frac{1}{2}$ as $W \to F$ so contradicting its optimality. ■

### *A.7. Proof of Proposition 11*

Following Merton (1971), let $J(W,t)$ denote the derived utility of wealth function given a particular investment–consumption strategy. For the strategy to be optimal, the Bellman equation must be satisfied:

$$0 = \max_{\{c,E\}} \{ \mathrm{e}^{-\rho t} U(c) + J_t + J_W [E(\mu - r) + rW - c] \\ + \tfrac{1}{2} E^2 \sigma^2 J_{WW} \}, \tag{A.6}$$

where $E$ is the exposure to the active asset.

Optimal consumption must maximize

$$\mathrm{e}^{-\rho t} U(c) - J_W c, \tag{A.7}$$

subject to $c \geq c_{\min}$, and the optimal exposure must satisfy

$$E = -(\mu - r)/\sigma^2 \cdot J_W / J_{WW}. \tag{A.8}$$

Try $J(W,t) = e^{-\rho t} I(W)$, where $I(W)$ is given by

$$I(W) = P \cdot W^{1-\lambda}/(1-\lambda) \qquad \text{for} \quad W \geq W^*,$$

$$= P' \cdot (W-F)^{1-\lambda'}/(1-\lambda') + Q' \quad \text{for} \quad W \leq W^*,$$

where $P = d^{-\lambda}$ and $P'$ and $Q'$ are constants chosen so that $I(W)$ is continuously differentiable at $W^*$. When so constructed, $I(W)$ turns out to be twice continuously differentiable at $W^*$.

From here, it is an algebraic exercise to show that the posited investment–consumption rule satisfies (A.7) and (A.8) and, when substituting these into (A.6), that the right-hand side evaluates to zero.

This completes the proof. ■

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