SIR

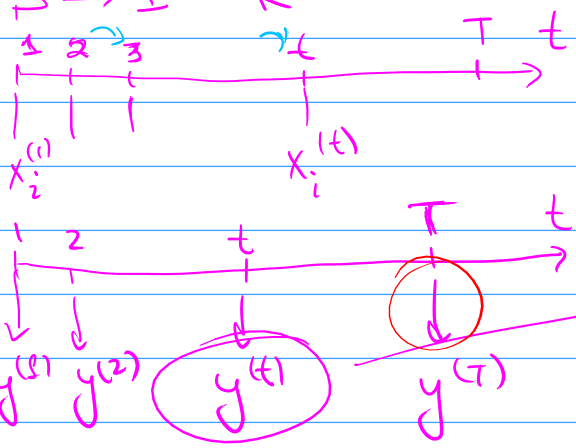
susceptible - infected - recovered

$X = \{x_1, \dots x_N\}$

$x_i^{(t)} \in \{S, I, R\}$

(1) $S \to I \to R$

(2)



SSS I I R . . R

SSS . . . SS

$x_i \sim x_j$

$I \to$

(3)

число зарегистр. больных в момент t

$X = \{x_1 \dots x_N\}$

$t: \ x_i^{(t)} \in \{S, I, R\}$

$y^{(t)}$ — число больных

$S^{(t)} + I^{(t)} + R^{(t)} = N$

1) $p(x_i^{(1)} = I) = \pi_i, \quad p(x_i^{(1)} = S) = 1 - \pi_i$

2) $p(x_i^{(t)} \in y^{(t)} \mid x_i^{(t)} = I) = p$

$p(y^{(t)} \mid I^{(t)}, p) = \text{Binomial}(y^{(t)} \mid I^{(t)}, p)$

3) $p(x_i^{(t+1)} = R \mid x_i^{(t)} = I) = \mu$

$\boxed{\theta = \{\pi, p, \mu, \beta\}}$

4) $\beta = p$(зараж. от одного контакта)

$p(x_i^{(t+1)} = S \mid x_i^{(t)} = S) = (1-\beta)^{I^{(t)}}$

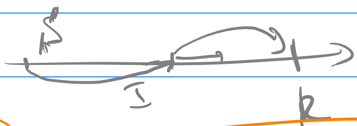
$p(x_i^{(t+1)} = I \mid x_i^{(t)} = S) = 1 - (1-\beta)^{I^{(t)}}$

$$p(x_i^{(t+1)} \mid x_i^{(t)}) = \begin{array}{cc} \begin{array}{ccc} S & I & R \end{array} & t+1 \\ \begin{pmatrix} (1-\beta)^{I^{(t)}} & 1-(1-\beta)^{I^{(t)}} & 0 \\ 0 & 1-\mu & \mu \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} S \\ I \\ R \end{array} \\ & t \end{array}$$

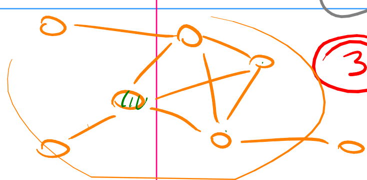
Расширения

(1) $SIR \to SEIR$, $SEIRS$

(2) Непрерывное время



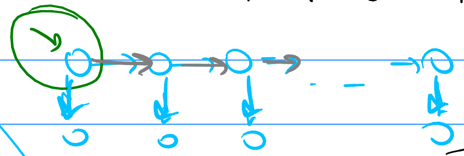
(3) Модель заражения



$|I^{(t)} \cap Nei(x_i)|$

$R_0 = \beta \cdot \boxed{E[\#\text{ контактов}]}$

$$p(X, y \mid \theta) = p(x^{(1)} \mid \pi)\, p(x^{(2)} \mid x^{(1)}, \beta, \mu)\, p(y^{(1)} \mid x^{(1)}, \rho)\, p(y^{(2)} \mid x^{(2)}, \rho) \dots p(x^{(T)} \mid x^{(T-1)}, \beta, \mu)\, p(y^{(T)} \mid x^{(T)}, \rho)$$



$$= \left(\prod_{i=1}^{N} \pi^{[x_i^{(1)} = I]} (1-\pi)^{[x_i^{(1)} = S]} \right) \times \prod_{t=1}^{T-1} \left[\prod_{i=1}^{N} p(x_i^{(t+1)} \mid \beta, \mu, I^{(t)}) \right] \prod_{t=1}^{T} \left[\binom{I^{(t)}}{y^{(t)}} \rho^{y^{(t)}} (1-\rho)^{I^{(t)} - y^{(t)}} \right]$$

θ "=" ρ, β, μ, π; y "=" $\{y^{(1)} \dots y^{(T)}\}$

$$p(\theta \mid y) \xrightarrow[\theta]{} \max$$

$$\propto p(\theta)\, p(y \mid \theta) = p(\theta) \int p(X, y \mid \theta)\, dX =$$

$$= \int p(\theta)\, p(X \mid \theta)\, p(y \mid X, \theta)\, dX$$

EM: $Q(\theta, \theta_k) = E_{p(X \mid \theta_k, y)}\left[p(X, y \mid \theta) \right] \xrightarrow[\theta]{} \max$

θ_{k+1}

$\boxed{\text{Monte Carlo EM}}$

max over θ

$$2.\ p(\theta \mid y) \approx \frac{1}{R} \sum_{r=1}^{R} p(X_r, y \mid \theta)$$

где $X_r \sim p(X \mid \theta_k, y)$

EM

$$\theta_0 \xrightarrow{\text{Sample}} X_{0,r}$$

MAP

$$\theta_1 \xrightarrow{\text{Sample}} X_{1,r}$$

$$\theta_2 \dots$$

$$X \sim p(X \mid \theta, y)$$

$x_i^{(1)} \dots x_i^{(T)}$, $i = 1 \dots N$

Sampling

$X \sim p(X|\theta, y)$

$= \{\bar X_1, \bar X_2, \dots \bar X_N\}$

$= (x_1^{(1)}, x_1^{(2)} \dots x_1^{(T)})$

Gibbs sampling

$j=1 \quad \dots \quad \bar X_1 \quad \dots$

$j=2 \quad \dots \quad \bar X_2 \quad \dots$

$j=N \quad \dots \quad \bar X_N \quad \dots$

$$\bar X_j \sim p(\bar X_j \,|\, y, \theta, X_{-j})$$

$x_j^{(t)} \in \{S, I, R\}$

Hidden Markov Model

$$\begin{tikzcd}
x_j^{(1)} \arrow[r] \arrow[d] & x_j^{(2)} \arrow[r] \arrow[d] & x_j^{(3)} \arrow[r] \arrow[d] & \dots \arrow[r] & x_j^{(t)} \arrow[r] \arrow[d] & x_j^{(t+1)} \arrow[r] \arrow[d] & \dots \arrow[r] & x_j^{(T)} \arrow[d] \\
y^{(1)} & y^{(2)} & y^{(3)} & \dots & y^{(t)} & y^{(t+1)} & \dots & y^{(T)}
\end{tikzcd}$$

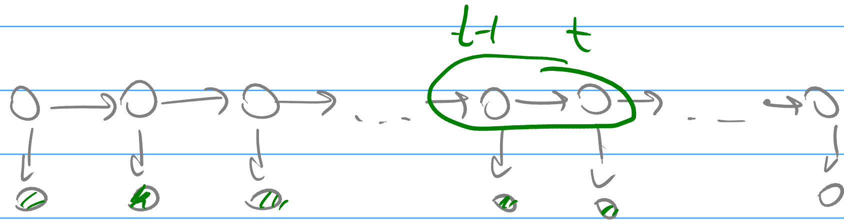
Stochastic Viterbi algorithm

$t: \; S_{-j}^{(t)}, \; I_{-j}^{(t)}, \; R_{-j}^{(t)}, \; y^{(t)} \sim Binom(y^{(t)} \,|\, I^{(t)}, p)$

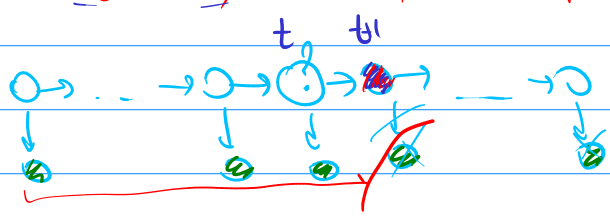
$$I^{(t)} = \begin{cases} I_{-j}^{(t)}, & x_j^{(t)} \in \{S, R\} \\ I_{-j}^{(t)} + 1, & x_j^{(t)} = I \end{cases}$$

$$p(\bar X_j \,|\, X_{-j}, y, \theta) = p(x_j^{(T)} \,|\, \_) \, p(x_j^{(T-1)} \,|\, x_j^{(T)}, \_) \, p(x_j^{(T-2)} \,|\, x_j^{(T-1)}, \cancel{x_j^{(T)}}, \dots) \dots$$

$$q_{j,s',s}^{(t)} = p(x_j^{(t)} = s, \; x_j^{(t-1)} = s' \,|\, y^{(1)}, \dots, y^{(t)}, X_{-j}, \theta)$$

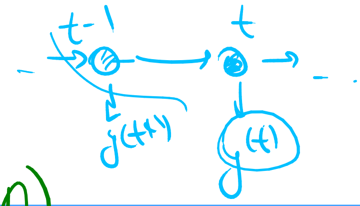


$$p(x_j^{(t)} = s \,|\, x_j^{(t+1)} = s', y, X_{-j}, \theta) = p(x_j^{(t)} = s \,|\, x_j^{(t+1)} = s', y^{(1)}, \dots, y^{(t)}, X_{-j}, \theta)$$



$\propto q_{j,s,s'}^{(t+1)}$

$$q^{(t)}_{j,s's} = p(x_j^{(t)}=s,\ x_j^{(t-1)}=s' \mid y^{(1)}\dots y^{(t)}, X_{-j}, \theta) \propto$$



$$\propto p(x_j^{(t)}=s,\ x_j^{(t-1)}=s',\ y^{(t)} \mid y^{(1)}\dots y^{(t-1)}, X_{-j}, \theta) =$$

$$= p(x_j^{(t)}=s \mid x_j^{(t-1)}=s', \cancel{y^{(1..t-1)}}, X_{-j}, \theta)\cdot p(y^{(t)} \mid \cancel{x_j^{(t-1)}=s'}, x_j^{(t)}=s, \cancel{y^{(1..t-1)}}, X_{-j}, \theta)\cdot$$

$$\underline{p(x_j^{(t-1)}=s' \mid y^{(1..t-1)}, X_{-j}, \theta)} =$$

$$= \underbrace{p(x_j^{(t)}=s \mid x_j^{(t-1)}=s', X_{-j}, \theta)}\cdot \underbrace{\mathrm{Binom}(y^{(t)} \mid I_{-j}^{(t)} + [x_j^{(t)}=I], p)} \times ⊛$$

$$p(x_j^{(t-1)}=s' \mid y^{(1..t-1)}, X_{-j}, \theta) = \sum_{s''} p(x_j^{(t-1)}=s', x_j^{(t-2)}=s'' \mid y^{(1..t-1)}, X_{-j}, \theta)$$

$$⊛ \times \underbrace{\sum_{s''} q^{(t-1)}_{j,s'',s'}}$$

Алгоритм stoch_Vit:

– слева направо

$$Q_j^{(1)}, Q_j^{(2)}, \dots, Q_j^{(T)}$$

– справа налево

$$p(x_j^{(t)}=s \mid x_j^{(t+1)}=s', y, X_{-j}, \theta) \propto q^{(t+1)}_{j,s,s'}$$

$$Q_j^{(t)} = \begin{array}{cc} & s \\ s' & \left(\boxed{q^{(t)}_{j,s',s}} \right) \end{array}$$

Gibbs sampling:

– loop
– for $j=1..N$
– $\bar{x}_j \sim \text{stoch.Viterbi}(X_{-j})$

$$p(\theta \mid y, X) \propto p(\theta)\, p(y, X \mid \theta) = \underbrace{p(\rho)}_{\mathrm{Beta}(\rho \mid a_\rho, b_\rho)} p(\mu) \underbrace{p(\beta)}_{\mathrm{Beta}(\beta \mid a_\beta, b_\beta)} p(\pi) \boxed{p(y, X \mid \theta)}$$

$$\log p(\theta \mid y, X) = \mathrm{const} + (a_\rho - 1)\log \rho + (b_\rho - 1)\log(1-\rho) + (a_\mu - 1)\log \mu +$$
$$+ (b_\mu - 1)\log(1-\mu) + (a_\beta - 1)\log \beta + (b_\beta - 1)\log(1-\beta) + (a_\pi - 1)\log \pi + (b_\pi - 1)\log(1-\pi)$$
$$+ \sum_{i=1}^{N} \left([x_i^{(1)} = I] \log \pi + [x_i^{(1)} = S] \log(1-\pi) \right) +$$
$$+ \sum_{t=1}^{T} \left(y^{(t)} \log \rho + \left(I^{(t)} - y^{(t)}\right) \log(1-\rho) \right) +$$
$$+ \sum_{i=1}^{N} \sum_{t=1}^{T} \left([x_i^{(t)} = I, x_i^{(t+1)} = R] \log \mu + [x_i^{(t)} = I, x_i^{(t+1)} = I] \log(1-\mu) \right)$$
$$+ \sum_{i=1}^{N} \sum_{t=1}^{T} [x_i^{(t)} = S] \cdot \left(\underbrace{P_i^{(t)}}_{\text{\#конт., при к-рых } x_i \text{ заразился}} \log \beta + \underbrace{N_i^{(t)}}_{\text{\#конт. не заразился}} \log(1-\beta) \right)$$

$$I \xrightarrow{1-\mu} I \xrightarrow{1-\mu} I \xrightarrow{\mu} R$$

$\sum_{i,t:\, x_i^{(t)} = I}$

$\sum_{i,t} x_i^{(t)} = R \; \text{"}0\text{"}$

$$\frac{\sum_{i,t} x_i^{(t)} = S}{x_i^{(t+1)} = \begin{smallmatrix} S \\ I \end{smallmatrix}}$$

$\underbrace{I^{(t)}}$

$P_i^{(t)} + N_i^{(t)}$

$$P_i^{(t)} + N_i^{(t)} = I^{(t)}$$

- если $x_i^{(t+1)} = S$, то $P_i^{(t)} = 0$, $N_i^{(t)} = I^{(t)}$
- если $x_i^{(t+1)} = I$, то $P_i^{(t)} \geq 1$

presence-only data

$$\mathbb{E}\left[P_i^{(t)} \mid x_i^{(t+1)} = I\right] = I^{(t)} \cdot p\left(\begin{smallmatrix}\text{зараж. от} \\ 1 \text{ конт.}\end{smallmatrix} \,\middle|\, \begin{smallmatrix}\geq 1 \\ \text{зараж.}\end{smallmatrix}\right) = I^{(t)} \cdot \frac{\beta}{1-(1-\beta)^{I^{(t)}}}$$

$$P_i^{(t)} = \begin{cases} 0, & \text{если } x_i^{(t+1)} = S \\ I^{(t)} \cdot \dfrac{\beta}{1-(1-\beta)^{I^{(t)}}}, & x_i^{(t+1)} = I \end{cases} \qquad N_i^{(t)} = I^{(t)} - P_i^{(t)}$$

$$a'_\beta = a_\beta + \sum_{t=1}^{T} \sum_{i:\, x_i^{(t)} = S} p_i^{(t)}, \quad b'_\beta = b_\beta + \sum_{t=1}^{T} \sum_{i:\, x_i^{(t)} = S} N_i^{(t)}$$

$$a'_\mu = a_\mu + \sum_{t=1}^{T-1} \sum_{i=1}^{N} [x_i^{(t)} = I, x_i^{(t+1)} = R], \quad b'_\mu = b_\mu + \sum_t \sum_i [x_i^{(t)} = I, x_i^{(t+1)} = I]$$

$$a'_p = a_p + \sum_{t=1}^{T} y^{(t)} \qquad b'_p = b_p + \sum_{t=1}^{T} (I^{(t)} - y^{(t)})$$

$$a'_\pi = a_\pi + \sum_{i=1}^{N} [x_i^{(1)} = I] \qquad b'_\pi = b_\pi + \sum_{i=1}^{N} [x_i^{(1)} = S]$$

EM-алгоритм:

- loop over k:
  - E-шаг: Gibbs sampling при условии θ_k
    - loop $j = 1..N, 1..N, ...$:
      - $\bar{x}_j \sim p(\bar{x}_j \mid X_{-j}, y, \theta_k)$:

        stochastic viterbi [
        - for $t = 1..T$ compute $Q_j^{(t)}$
        - for $t = T..1$ $x_j^{(t)} \sim p(x_j^{(t)} \mid x_j^{(t+1)}, y, X_{-j}, \theta_k)$
    - время от времени сохр. X
  - M-шаг:
    - $\theta_{k+1} := \arg\max p(\theta_{k+1} \mid X, y)$
      - presence-only data