$$\sigma(s_i + q_j) = p(i \text{ взял } j)$$

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

$$s_i + q_j \sim \sigma^{-1}(p(\cdot))$$

t : $s_1 \; s_2 \ldots s_k$

$$z_{ij} = [i \text{ взял } j] \qquad s_i \in t_e$$

$$\underline{t_{e_j} = 0} \Rightarrow s_{ij} = 0 \quad \forall i \in t_e$$

$$\underline{t_{e_j} = 1} \Rightarrow \exists i : s_{ij} = 1$$

E-шаг

$$\mathbb{E} z_{ij} = p(i \text{ взял } j \mid s_i, q_j) =$$

$$= p(\underline{i \text{ взял } j} \mid \text{кто-то в } t_e \text{ взял } j) =$$

$$= \frac{\sigma(s_i + q_j)}{1 - \prod\limits_{i' \in t_e} (1 - \sigma(s_{i'} + q_j))}$$

$$H[q] = \int q \ln q \, dz$$

$$H[q + \delta q] = \ldots$$

$$p(X \mid \theta) \xrightarrow[\theta]{} \max \qquad p(X \mid \theta) = \int p(X, Z \mid \theta) \, dZ$$

$$p(X, Z \mid \theta) = p(X \mid \theta) \, p(Z \mid X, \theta)$$

$$\ln p(X \mid \theta) = \ln p(X, Z \mid \theta) - \ln p(Z \mid X, \theta)$$

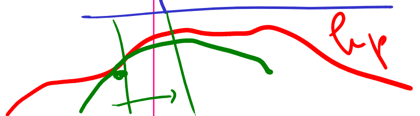
$$\ln p(X \mid \theta) = \mathbb{E}_{q(Z)} [\ln p(X, Z \mid \theta) - \ln p(Z \mid X, \theta)] =$$

$$= \int q(Z) \ln p(X, Z \mid \theta) \, dZ - \int \ln p(Z \mid X, \theta) \, q(Z) \, dZ$$

$$- \int \ln q(Z) \, q(Z) \, dZ \qquad + \int \ln q(Z) \, q(Z) \, dZ$$

$$\ln p(X \mid \theta) = \underbrace{\int \ln \frac{p(X, Z \mid \theta)}{q(Z)} q(Z) \, dZ}_{\mathcal{L}(q, \theta) \xrightarrow[\theta]{} \max} - \underbrace{\int \ln \frac{p(Z \mid X, \theta)}{q(Z)} q(Z) \, dZ}_{=\, KL(q(Z) \,\|\, p(Z \mid X, \theta))}$$

$$\underline{\ln p(X \mid \theta)} = \underline{\mathcal{L}(q, \theta)} + \underline{KL(q(Z) \,\|\, p(Z \mid X, \theta))}$$



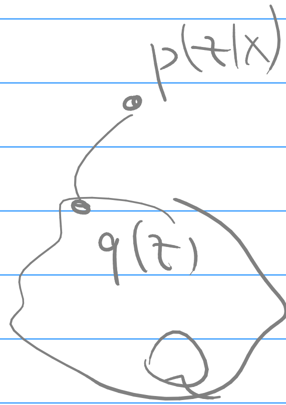
$$\boxed{q(Z) = p(Z \mid X, \underline{\theta}^{(m)})} \Rightarrow \mathcal{L}(q, \theta^{(m)}) = \ln p(X \mid \theta^{(m)})$$

$$p(x,z) \qquad p(z|x) = ? \qquad p(x)$$

$$\ln p(x) = \mathbb{E}_{q(z)}\left[\ln p(x,z) - \ln p(z|x)\right]$$

$$\ln p(x) = \int \ln \frac{p(x,z)}{q(z)}\, q(z)\, dz - \int \ln \frac{p(z|x)}{q(z)}\, q(z)\, dz$$

$$\ln p(x) = \underbrace{L(q)}_{q(z)\ \max} + \underbrace{KL(q \,\|\, p(z|x))}_{q(z) \to \min} \qquad \Longleftrightarrow$$



$$\boxed{q(Z) = \prod_{i=1}^{M} q_i(Z_i)}, \quad Z_i \cap Z_j = \emptyset$$

$$Z = \{z_1, z_2, \ldots z_k\} \qquad Z_i \subset Z \qquad q(z_1, \ldots z_4) = q(z_1, z_2) \cdot q(z_3, z_4)$$

$$L(q) = \int \ln \frac{p(x,z)}{q(z)}\, q(z)\, dz =$$

$$= \int \ln p(x,z)\, q(z)\, dz - \int \ln q(z)\, q(z)\, dz =$$

$$= \int \ln p(x,z) \cdot \prod_i q_i(z_i)\, dz - \sum_j \int \ln q_j(z_j) \cdot \prod_i q_i(z_i)\, dz$$

$$\int \ln q_j(z_j) \prod q_i\, dz = \int \ln q_j \Big(\underbrace{\int \prod_{i \neq j} q_i\, dz_{-j}}_{=1}\Big) \cdot q_j\, dz_j = \int \ln q_j\, q_j\, dz_j$$

$$L(q_j) = \int \ln p(x,z) \prod_i q_i(z_i)\, dz - \int \ln q_j(z_j)\, q_j(z_j)\, dz_j + const$$

$$\int \ln p(x,z) \prod q_i\, dz_1 \ldots dz_M = \int \underbrace{\Big(\int \ln p(x,z) \prod_{i \neq j} q_i\, dz_{-j}\Big)}_{\ln \tilde{p}(x, z_j)} q_j(z_j)\, dz_j$$

$$\ln \tilde{p}(X, z_j) = \mathbb{E}_{\prod_{i \neq j} q_i(z_i)} \left[\ln p(X, Z) \right] + \text{const}$$

$$\mathcal{L}(q_j) = \int \ln \tilde{p}(X, z_j)\, q(z_j)\, dz_j - \int \ln q_j \, q_j \, dz_j + \text{const} =$$

$$= \int \ln \frac{\tilde{p}(X, z_j)}{q_j} \, q_j \, dz_j = -KL(q_j \| \tilde{p}) + \text{const}$$

$$\mathcal{L}(q_j) \xrightarrow{q_j} \max \iff \underline{q_j^*(z_j) = \tilde{p}(X, z_j)}$$

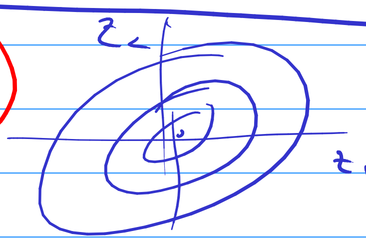
Если $q(Z) = \prod_{i=1}^{M} q_i(z_i)$ для $z_i \wedge z_j = \emptyset$, то

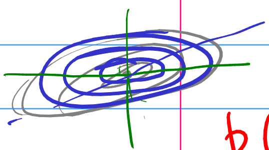
$$\forall j \quad \boxed{\ln q_j^*(z_j) = \mathbb{E}_{i \neq j} \left[\ln p(X, Z) \right] + \text{const}}$$

где $q^* = \prod_i q_i^*$ — это опт. аппр. $p(Z|X)$

(1) $p(\bar{z}) = \mathcal{N}(\bar{z} \mid \bar{\mu}, \Lambda^{-1})$ $\quad \bar{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$\bar{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, $\quad \Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12} & \Lambda_{22} \end{pmatrix}$





$$p(\bar{z}) \approx q(\bar{z}) = q_1(z_1)\, q_2(z_2)$$

$$\ln q_1^*(z_1) = \mathbb{E}_{q_2^*} \left[\ln p(z_1, z_2) \right] + \text{const}$$

$$\ln q_2^*(z_2) = \mathbb{E}_{q_1^*} \left[\ln p(z_1, z_2) \right] + \text{const}$$

$$\ln p(z_1, z_2) = -\ln 2\pi - \frac{1}{2} \ln \det \Lambda - \frac{1}{2} (\bar{z} - \bar{\mu})^T \Lambda (\bar{z} - \bar{\mu}) =$$

$$= \text{const} - \frac{1}{2} \left((z_1 - \mu_1)^2 \Lambda_{11} + 2 (z_1 - \cancel{\mu_1})(z_2 - \mu_2) \Lambda_{12} + (z_2 - \cancel{\mu_2})^2 \Lambda_{22} \right)$$

$$\ln q_1^*(z_1) = \mathbb{E}_{q_2^*(z_2)} \left[-\frac{1}{2} (z_1 - \mu_1)^2 \Lambda_{11} - z_1 \Lambda_{12} (z_2 - \mu_2) \right] + \text{const}$$

$$\ln q_1^*(z_1) = -\frac{1}{2} \Lambda_{11} z_1^2 + z_1 \mu_1 \Lambda_{11} - z_1 \Lambda_{12} \left(\mathbb{E}_{q_2^*}[z_2] - \mu_2 \right) + \text{const}$$

$$\ln q_1^*(z_1) = -\frac{1}{2}\underline{\underline{\Lambda_{11}}} z_1^2 + \left(\mu_1\Lambda_{11} - \Lambda_{12}(\mathbb{E}[z_2]-\mu_2)\right)\underline{\underline{z_1}} + \text{const}$$

$$q_1^*(z_1) = \mathcal{N}(z_1 \mid m_1, \alpha_1^{-1})$$

$$\underline{\alpha_1 = \Lambda_{11}}$$

$$m_1 = \frac{1}{\Lambda_{11}}\left(\mu_1\Lambda_{11} - \Lambda_{12}(\ldots)\right)$$

$$m_1 = \mu_1 - \frac{\Lambda_{12}}{\Lambda_{11}}\left(\mathbb{E}_{q_2^*}[z_2] - \mu_2\right)$$

$$\ln q_2^*(z_2) = \mathbb{E}_{q_1^*(z_1)}\left[-\frac{1}{2}(z_2-\mu_2)^2\Lambda_{22} - z_2\Lambda_{12}(z_1-\mu_1)\right] + \text{const}$$

$$\ln q_2^*(z_2) = -\frac{1}{2}\Lambda_{22}z_2^2 + \left(\mu_2\Lambda_{22} - \Lambda_{12}(\mathbb{E}[z_1]-\mu_1)\right)$$

$$q_2^*(z_2) = \mathcal{N}(z_2 \mid m_2, \alpha_2^{-1}), \qquad \alpha_2 = \Lambda_{22}$$

$$m_2 = \mu_2 - \frac{\Lambda_{12}}{\Lambda_{22}}\Big(\underbrace{\mathbb{E}_{q_1^*}[z_1]}_{m_2} - \mu_1\Big)$$

$$\begin{cases} m_1 = \mu_1 - \dfrac{\Lambda_{12}}{\Lambda_{11}}\left(\underline{m_2 - \mu_2}\right) \\ m_2 = \mu_2 - \dfrac{\Lambda_{12}}{\Lambda_{22}}\left(\underline{m_1 - \mu_1}\right) \end{cases} \qquad \begin{aligned} m_1 &= \mu_1 \\ m_2 &= \mu_2 \end{aligned}$$

$$\underline{KL(q\|p) \xrightarrow{q} \min}$$

$$p(\bar{z}) = \mathcal{N}\left(\bar{z} \,\middle|\, \bar{\mu}, \begin{pmatrix}\Lambda_{11} & \Lambda_{12} \\ \Lambda_{12} & \Lambda_{22}\end{pmatrix}^{-1}\right) \approx q_1(z_1)q_2(z_2)$$

$$\begin{cases} q_1(z_1) = \mathcal{N}(z_1 \mid \mu_1, \Lambda_{11}^{-1}) \\ q_2(z_2) = \mathcal{N}(z_2 \mid \mu_2, \Lambda_{22}^{-1}) \end{cases}$$

$$KL(p\|q) \to \min$$

$$\| \quad \int p(z)\ln\frac{p(z)}{q(z)}dz = \int p\ln p\,dz - \underbrace{\int p(z)\ln q(z)\,dz}_{\to \max} \qquad q \to \min$$

$q = \prod q_i \qquad \sum_i \int p(z) \ln q_i(z_i)\, dz$

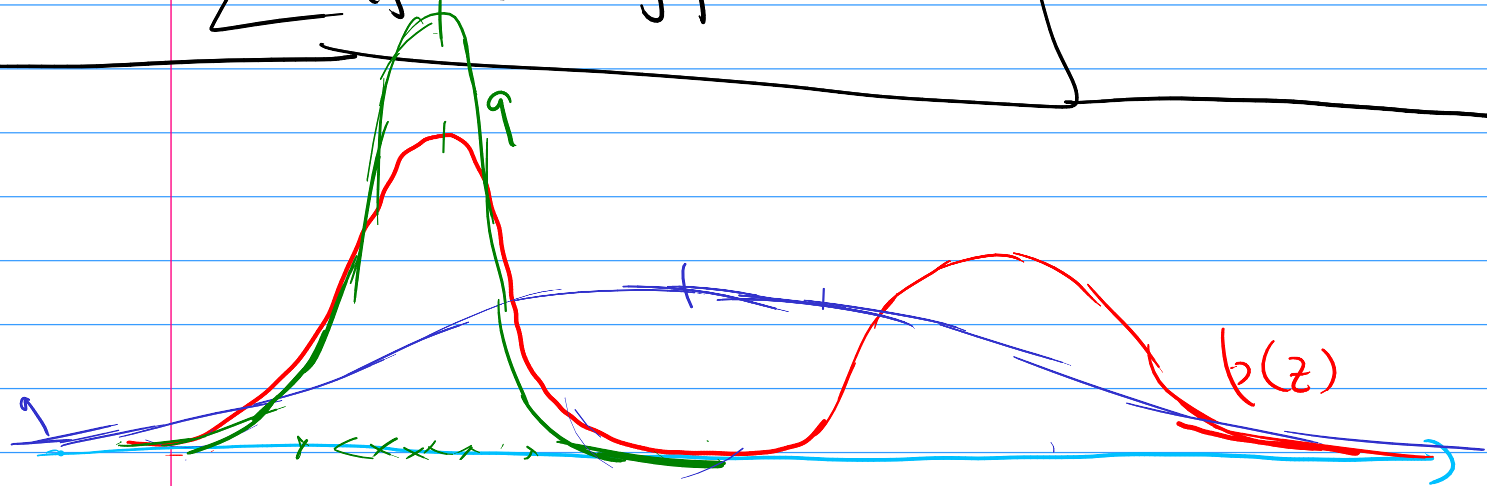
$j: \quad \underline{\int p(z) \ln q_j(z_j)\, dz} \to \max$

$= \int \underline{\tilde{p}(z_j)} \ln q_j(z_j)\, dz_j$

$\tilde{p}(z_j) = \int p(z)\, dz_{-j}$

$$\boxed{q_j^*(z_j) = \int p(z)\, dz_{-j}}$$

$KL(p\|q)$



$KL(p\|q) = \int \ln \frac{p}{q}\, p\, dz \to \min$

$KL(q\|p) = \int \ln \frac{q}{p}\, q\, dz \to \min$

② $D = \{x_1, x_2, \dots, x_N\}$

$$p(D \mid \mu, \tau) = \prod_{n=1}^{N} \sqrt{\frac{\tau}{2\pi}}\, e^{-\frac{\tau}{2}(x_n - \mu)^2}$$

$p(\tau) = Gam(\tau \mid a_0, b_0) \propto \tau^{a_0 - 1} e^{-b_0 \tau}$

$p(\mu \mid \tau) = \mathcal{N}(\mu \mid \mu_0, (\lambda_0 \tau)^{-1})$

$$\underline{p(\mu, \tau \mid D)} \propto \overbrace{p(\tau)\, p(\mu \mid \tau)}^{\text{prior}}\, \underbrace{p(D \mid \mu, \tau)}_{\text{likelihood}}$$

$$\underline{p(\mu,\tau|D) \approx q(\mu,\tau) = q_\mu(\mu)\, q_\tau(\tau)}$$

$$p(Z,X) = p(\mu,\tau,D) = p(\tau)\,p(\mu|\tau)\,p(D|\mu,\tau)$$

$$\ln p(\mu,\tau,D) = \text{const} + (a_0-1)\ln\tau - b_0\tau + \tfrac{1}{2}\ln\tau - \frac{\lambda_0\tau}{2}(\mu-\mu_0)^2$$
$$+ \frac{N}{2}\ln\tau - \frac{\tau}{2}\sum_{n=1}^{N}(x_n-\mu)^2$$

$$\ln q_\tau^*(\tau) = \mathbb{E}_{q_\mu^*}\left[\ln p(\mu,\tau,D)\right] + \text{const} =$$

$$= (a_0-1)\ln\tau - b_0\tau + \tfrac{1}{2}\ln\tau + \frac{N}{2}\ln\tau - \frac{\tau}{2}\mathbb{E}_\mu\left[\lambda_0(\mu-\mu_0)^2 + \sum_n(x_n-\mu)^2\right] + \text{const}$$

$$q_\tau^*(\tau) = \mathrm{Gam}(\tau|a,b)$$

$$a = a_0 + \frac{N+1}{2}; \quad b = b_0 + \frac{1}{2}\mathbb{E}_\mu\left[\lambda_0(\mu-\mu_0)^2 + \sum_n(x_n-\mu)^2\right]$$

$$\ln q_\mu^*(\mu) = \mathbb{E}_{q_\tau^*}\left[\ln p(\mu,\tau,D)\right] + \text{const} =$$

$$= \mathbb{E}_{q_\tau^*}\left[-\frac{\lambda_0\tau}{2}(\mu-\mu_0)^2 - \frac{\tau}{2}\sum_n(x_n-\mu)^2\right] + \text{const}$$

$$= -\frac{\lambda_0\mathbb{E}\tau}{2}(\mu-\mu_0)^2 - \frac{\mathbb{E}\tau}{2}\sum_n(x_n-\mu)^2 + \text{const}$$

$$= -\frac{\mathbb{E}\tau}{2}\left(\lambda_0\mu^2 - 2\lambda_0\mu_0\cdot\mu + N\cdot\mu^2 - 2\mu\cdot\sum_n x_n\right) + \text{const}$$

$$= -\frac{\mathbb{E}\tau(\lambda_0+N)}{2}\left(\mu - \frac{\lambda_0\mu_0 + \sum_n x_n}{\lambda_0+N}\right)^2 + \text{const}$$

$$q_\mu^*(\mu) = \mathcal{N}(\mu|m,\alpha^{-1}), \quad \alpha = (\lambda_0+N)\cdot\mathbb{E}\tau$$

$$m = \frac{\lambda_0\mu_0 + \sum_n x_n}{\lambda_0+N} = \frac{1}{N}\sum_n x_n$$

$$\mathbb{E}[\tau] = \frac{a}{b} = \frac{a_0 + \frac{N+1}{2}}{b_0 + \frac{1}{2}\mathbb{E}[\ldots]}$$

$$\mathbb{E}_\mu\left[\lambda_0(\mu-\mu_0)^2 + \sum_n (x_n-\mu)^2\right] =$$

$$= \mathbb{E}_\mu\left[\mu^2(\lambda_0+N) - 2\mu\left(\lambda_0\mu_0 + \sum_n x_n\right) + \lambda_0\mu_0^2 + \sum_n x_n^2\right] \circledast$$

$$\mathbb{E}_\mu[\mu] = \frac{\lambda_0\mu_0 + \sum_n x_n}{\lambda_0+N} \;; \quad \mathbb{E}_\mu[\mu^2] = [\mathbb{E}\mu]^2 + \mathrm{Var}[\mu] \overset{\frac{1}{\alpha}}{=}$$

$$= \left(\frac{\quad}{\quad}\right)^2 + \frac{1}{(\lambda_0+N)\mathbb{E}\tau}$$

$$\circledast = \cancel{(\lambda_0+N)}\cdot\left(\frac{(\lambda_0\mu_0+\sum_n x_n)^2}{(\lambda_0+N)^{\cancel{2}}} + \frac{1}{\cancel{(\lambda_0+N)}\mathbb{E}\tau}\right) - \frac{2(\lambda_0\mu_0+\sum_n x_n)^2}{\lambda_0+N} + \lambda_0\mu_0 + \sum x_n^2$$

$$= \frac{1}{\mathbb{E}\tau} - \frac{(\lambda_0\mu_0 + \sum_n x_n)^2}{\lambda_0+N} + \lambda_0\mu_0 + \sum x_n^2 =$$

$$= \frac{1}{\mathbb{E}\tau} - \frac{(\sum x_n)^2}{N} + \sum x_n^2$$

$$\mathbb{E}\tau = \frac{a}{b} = \frac{a_0 + \frac{N+1}{2}}{b_0 + \frac{1}{2}\left(\frac{1}{\mathbb{E}\tau} - \ldots - + \ldots\right)} = \frac{N+1}{\frac{1}{\mathbb{E}\tau} - \ldots -}$$

$$\frac{1}{\mathbb{E}\tau} = \frac{1}{a_0 + \frac{N+1}{2}}\cdot\left(b_0 + \frac{1}{2}\cdot\frac{1}{\mathbb{E}\tau} - \frac{1}{2}\frac{(\;)^2}{(\;)} + \frac{1}{2}(\,\_\_\,)\right)$$

non-informative priors $\mu_0 = 0 = a_0 = b_0 = \lambda_0$

$$\frac{1}{\mathbb{E}\tau} = \frac{1}{N+1}\left(\frac{1}{\mathbb{E}\tau} - \frac{(\sum x_n)^2}{N} + \sum x_n^2\right)$$

$$\frac{N}{N+1}\frac{1}{\mathbb{E}\tau} = \frac{1}{N+1}\cdot\left(\sum x_n^2 - \frac{1}{N}\left(\sum x_n\right)^2\right)$$

$$\boxed{\frac{1}{E\tau} = \frac{1}{N}\cdot\left(\sum x_n^2 - \frac{1}{N}\left(\sum x_n\right)^2\right)}$$

$$\bar{x} = \frac{1}{N}\cdot\sum x_n$$

$$\sigma^2 = \frac{1}{N}\cdot\sum_n (x_n-\bar{x})^2 = \frac{1}{N}\sum_n\left(x_n^2 - 2x_n\cdot\bar{x} + \bar{x}^2\right) =$$

$$= \frac{1}{N}\left(\sum_n x_n^2 + N\bar{x}^2 - 2\bar{x}\cdot\sum x_n\right) =$$

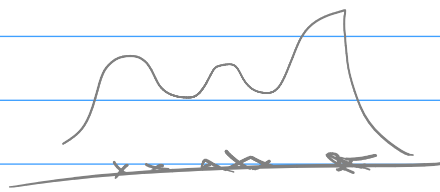
$$= \frac{1}{N}\left(\sum_n x_n^2 + \frac{1}{N}\left(\sum x_n\right)^2 - 2\cdot\frac{1}{N}\cdot\left(\sum x_n\right)^2\right)$$

---

③ Смесь гауссианов $X = \{x_1 \ldots x_N\}$ $\quad \pi_k,\ \sum \pi_k = 1$

кластеры $\mathcal{N}(\bar{x}\mid\bar{\mu}_k, \lambda_k^{-1})$, $\quad k = 1..K$

$$p(x_n\mid\ldots) = \sum_{k=1}^{K} \pi_k \mathcal{N}(x_n\mid\mu_k, \lambda_k^{-1})$$



$$p(\bar{\mu}, \bar{\lambda}, \bar{\pi}) = p(\bar{\pi})\prod_{k=1}^{K} p(\mu_k, \lambda_k)$$

$$p(\bar{\pi}) = \mathrm{Dir}(\bar{\pi}\mid\alpha_0) \propto \pi_1^{\alpha_0-1}\,\pi_2^{\alpha_0-1}\ldots\pi_K^{\alpha_0-1}$$

$$p(\mu_k, \lambda_k) = \mathrm{Gam}(\lambda_k\mid a_0, b_0)\,\mathcal{N}\left(\mu_k\mid\mu_0, (\beta_0\lambda_k)^{-1}\right)$$

$$z_{nk} = [x_n \text{ порожд. } \mathcal{N}(\mu_k, \lambda_k^{-1})]$$

$$\ln p(X, Z, \bar{\pi}, \bar{\mu}, \bar{\lambda}) = \ln\left[p(\bar{\pi})\prod_{k=1}^{K} p(\mu_k, \lambda_k)\prod_{n,k}\left(\pi_{nk}\,\mathcal{N}(x_n\mid\mu_k, \lambda_k^{-1})\right)^{z_{nk}}\right]$$

$$= \ln p(\bar{\pi}) + \sum_k\left[\ln p(\lambda_k) + \ln p(\mu_k\mid\lambda_k)\right] +$$

$$+ \sum_n\sum_k z_{nk}\left(\ln\pi_{nk} + \ln\mathcal{N}(x_n\mid\mu_k, \lambda_k^{-1})\right)$$

$p(x|X)$

$$p(Z, \bar{\pi}, \bar{\mu}, \bar{\lambda} | X) \approx q(Z, \bar{\pi}, \bar{\mu}, \bar{\lambda})$$

$$= \quad \text{единств. предполож.}$$

$$(\bar{\pi}, \bar{\mu}, \bar{\lambda})_{ML} \qquad q(Z)\, q(\bar{\pi}, \bar{\mu}, \bar{\lambda})$$

$$\ln q^*(Z) = \mathbb{E}_{\pi,\mu,\lambda}\left[\ln p(X, Z, \bar{\pi}, \bar{\mu}, \bar{\lambda})\right] + \text{const} =$$

$$= \mathbb{E}_{\pi,\mu,\lambda}\left[\ln p(Z|\pi) + \ln p(X|Z,\mu,\Lambda)\right] + \text{const} =$$

$$= \mathbb{E}_{\pi,\mu,\lambda}\Big[\sum_{n,k} z_{nk}\big(\ln \pi_k + \underbrace{\ln \mathcal{N}(x_n | \mu_k, \lambda_k^{-1})}_{= -\frac{1}{2}\ln 2\pi + \frac{1}{2}\ln\lambda_k - \frac{\lambda_k}{2}(x_n-\mu_k)^2}\big)\Big] + \text{const} =$$

$$= \sum_{n,k} z_{nk}\left(\mathbb{E}[\ln \pi_k] - \frac{1}{2}\ln 2\pi + \frac{1}{2}\mathbb{E}[\ln\lambda_k] - \frac{1}{2}\mathbb{E}[\lambda_k(x_n-\mu_k)^2]\right) + \text{const}$$

$$q^*(Z) \propto \prod_n \prod_k \rho_{nk}^{z_{nk}}, \quad \text{где}$$

$$\ln \rho_{nk} = \mathbb{E}[\ln \pi_k] - \_\_\_\_$$

$$\mathbb{E}[z_{nk}] = p(z_{nk}=1) = \frac{\rho_{nk}}{\sum_l \rho_{nl}} = r_{nk},$$

$$\text{где} \quad \ln r_{nk} = \_\_\_\_ + \text{const}$$

$$\ln q^*(\bar{\pi}, \bar{\mu}, \bar{\lambda}) = \mathbb{E}_Z\left[\ln p(X, Z, \bar{\pi}, \mu, \lambda)\right] + \text{const} =$$

$$= \ln p(\pi) + \sum_k \left(\ln p(\lambda_k) + \ln p(\mu_k | \lambda_k) \right) +$$

$$+ \mathbb{E}_z \left[\ln p(z | \pi) \right] + \mathbb{E}_z \left[\ln p(x | z, \pi, \mu, \Lambda) \right] =$$

$$= (\alpha_0 - 1) \sum_k \ln \pi_k + \sum_k \left((a_0 - 1) \ln \lambda_k - b_0 \lambda_k + \frac{1}{2} \ln \lambda_k - \frac{\lambda_k \beta_0}{2} (\mu_k - \mu_0)^2 \right)$$

$$+ \sum_{n,k} \mathbb{E}_z [z_{nk}] \ln \pi_k + \sum_{n,k} \mathbb{E}[z_{nk}] (\ldots)$$

$$q^*(\pi, \mu, \lambda) = q^*(\pi) \cdot \prod_k q^*(\lambda_k, \mu_k)$$

$$\ln q^*(\pi) = \sum_k \ln \pi_k \cdot \left(\alpha_0 - 1 + \sum_n z_{nk} \right)$$

$$q^*(\pi) = \mathrm{Dir}(\pi | \alpha), \quad \alpha_k = \alpha_0 + \sum_n z_{nk}$$

$$\ln q^*(\lambda_k, \mu_k) = (a_0 - 1) \ln \lambda_k - b_0 \lambda_k + \frac{1}{2} \ln \lambda_k - \frac{\lambda_k \beta_0}{2} (\mu_k - \mu_0)^2$$

$$+ \sum_n z_{nk} \left(\frac{1}{2} \ln \lambda_k - \frac{\lambda_k}{2} (x_n - \mu_k)^2 \right)$$

$$- \frac{\lambda_k}{2} \left(\beta_0 + \sum_n z_{nk} \right) \mu_k^2$$

$$\lambda_k \left(\beta_0 \mu_0 + \sum_n z_{nk} x_n \right) \mu_k$$

$$q^*(\lambda_k, \mu_k) = \mathrm{Gam}(\lambda_k | a, b) \cdot \mathcal{N}\left(\mu_k | m, (\lambda_k \beta)^{-1} \right)$$

$$\beta = \beta_0 + \sum_n z_{nk} \qquad a = a_0 + \frac{1}{2} \sum_n z_{nk}$$

$$m = \frac{\beta_0 \mu_0 + \sum_n z_{nk} x_n}{\beta_0 + \sum_n z_{nk}} \qquad b = b_0 + \ldots$$

$$p(d \mid \ldots) = \prod p(w \mid \ldots)$$