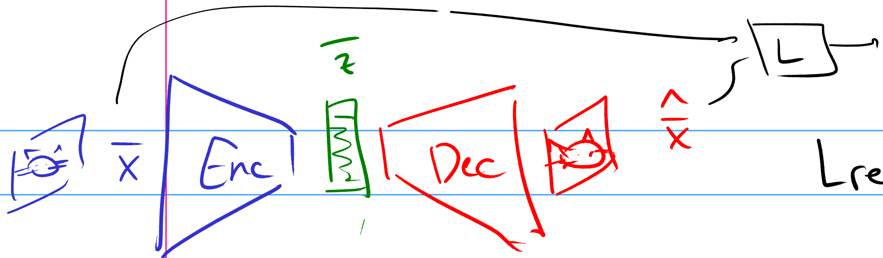
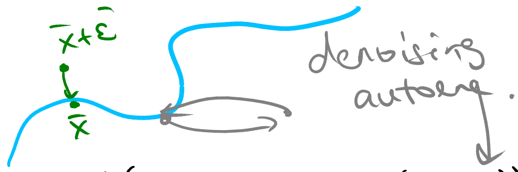
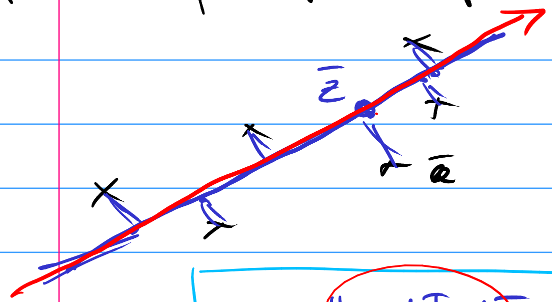




$$L_{rec} = d(\bar{x}, Dec(Enc(\bar{x}+\bar{\varepsilon}))) \to \min$$

~~$y \sim W^T \bar{x}$~~

PCA – principal comp. analysis

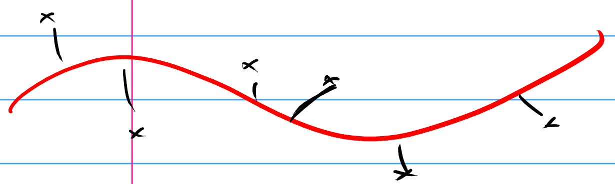


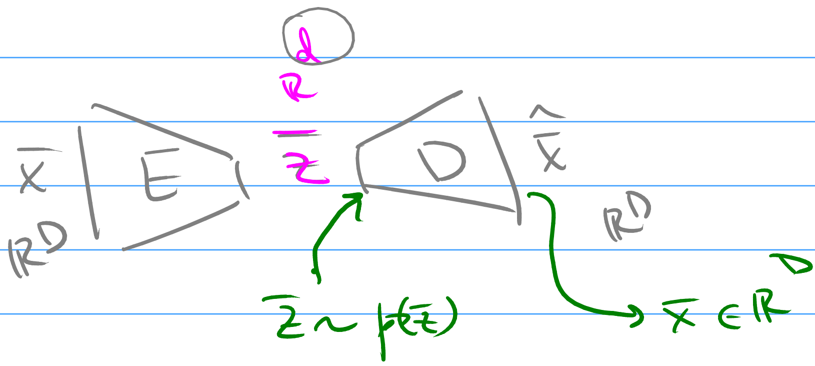
$$\underline{\bar{z}} = \underline{W}\bar{x}$$

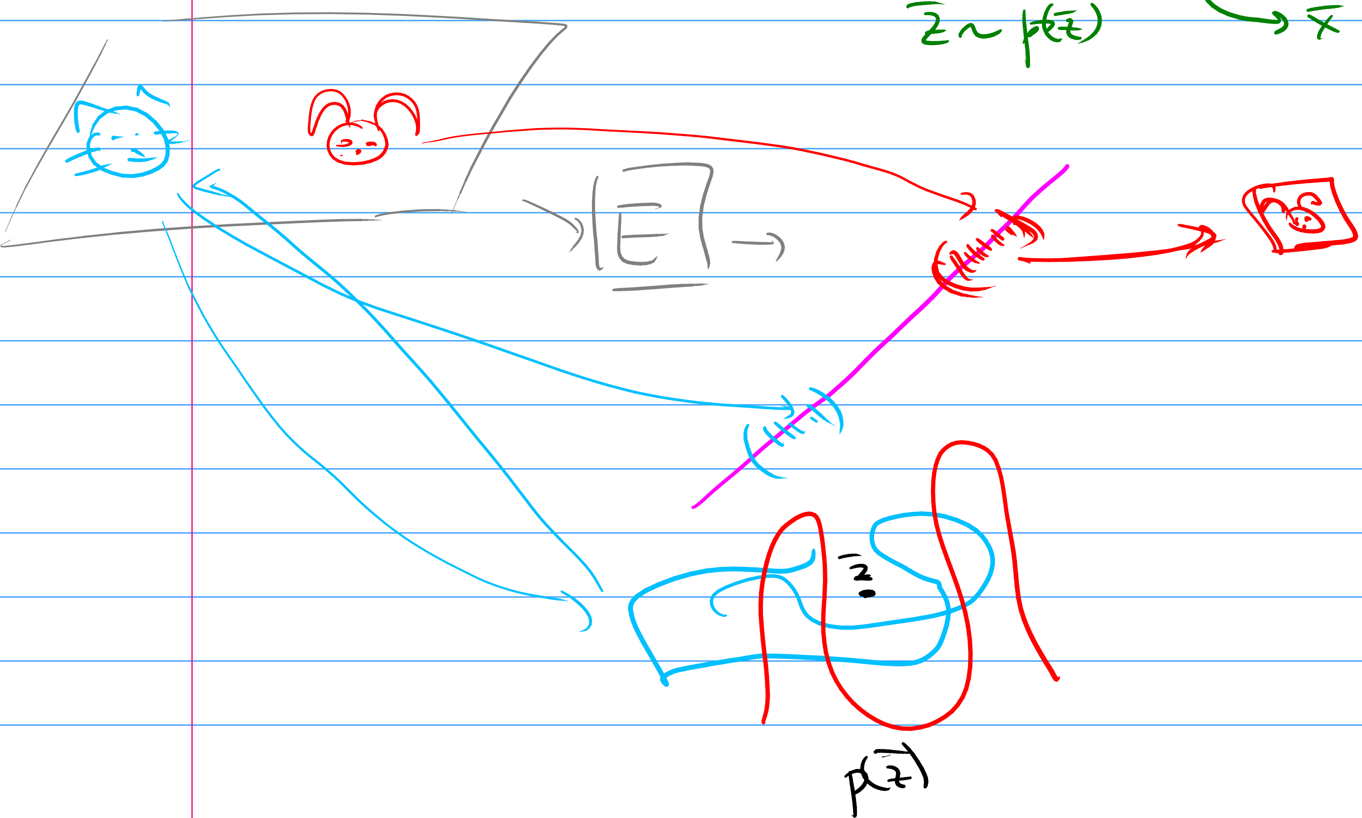
$$\bar{x} \to \bar{z} = W\bar{x} \to \hat{x} = W^T\bar{z} \to \boxed{L_2}$$

$$\boxed{\sum_n \|W^T W \bar{x}_n - \bar{x}_n\|_2^2 \xrightarrow[W]{} \min}$$

$$\sum_n \|Dec(Enc(\bar{x})) - \bar{x}\|_2^2 \xrightarrow[\theta]{} \min$$







AAE

$\bar{x} - \boxed{E} - \bar{z} - \boxed{D} - \hat{\bar{x}}$

$\bar{z} \to \boxed{\text{Discr}} - 0/1$

$\bar{z} \sim p(\bar{z}) \to \boxed{\text{Discr}}$

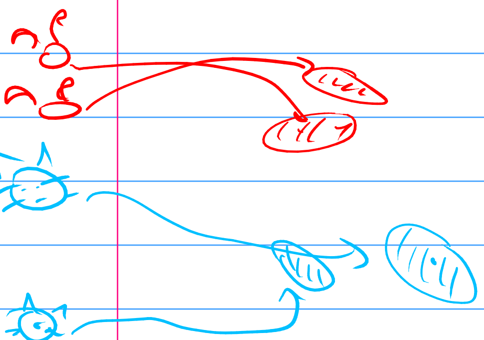
VAE – интуиция

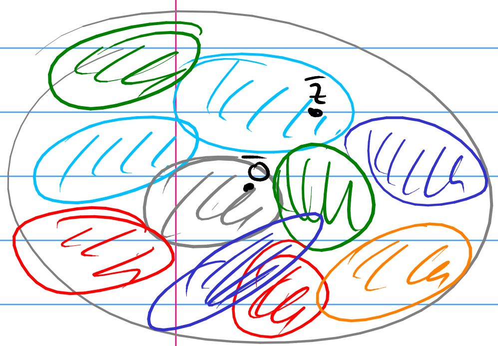
$\bar{x} - \boxed{\text{Enc}} - p(\bar{z}|\bar{x}) \xrightarrow{\text{sample}} \bar{z} - \boxed{\text{Dec}} - \hat{\bar{x}}$

$\bar{x} - \boxed{\text{Enc}} - \bar{\theta} = (\bar{\mu}_{\bar{x}}, \bar{\sigma}_{\bar{x}})$

$\bar{z} \sim p(\bar{z}|\bar{\theta}) \to \boxed{\text{Dec}} - \hat{\bar{x}}$

$= \mathcal{N}(\bar{z}|\bar{\mu}_{\bar{x}}, \bar{\sigma}_{\bar{x}})$





$$L = \underline{L_{rec}} + L_{reg}$$

$$L(\bar{x}) = \|\bar{x} - \hat{\bar{x}}\|^2 + KL\left(\mathcal{N}(\bar{z}|\bar{\mu}_x, \bar{\sigma}_x) \,\|\, \mathcal{N}(\bar{z}|\bar{0}, I)\right)$$

VAE

$\bar{z}$

Encoder ↑ ↓ Decoder

$\bar{x}$

$$\underbrace{p(\bar{z})}_{\mathcal{N}(\bar{0}, I)} \underbrace{p(\bar{x}|\bar{z})}_{\text{Decoder}} = p(\bar{x}, \bar{z})$$

$$\mathcal{N}\left(\bar{z}|\bar{\mu}_{\bar{x}}, \overbrace{\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_d \end{pmatrix}}^{\bar{\sigma}_x}\right)$$

$$p(\bar{z}|\bar{x}) = \frac{p(\bar{z})\, p(\bar{x}|\bar{z})}{p(\bar{x})} \approx \underline{q(\bar{z})}$$

Variational approximation:

$$p(x,z) = p(x)\, \underline{p(z|x)}$$

$$\ln p(x,z) = \ln p(x) + \ln p(z|x)$$

$$\mathbb{E}_{q(z)}\left[\ln p(x)\right] = \left[\ln p(x,z) - \ln p(z|x)\right]$$

$$\ln p(x) = \int q(z)\ln p(x,z)\,dz - \int q(z)\ln p(z|x)\,dz$$

$$- \int q(z)\ln q(z)\,dz \qquad + \int q \ln q\,dz$$

$$\underbrace{\ln p(x)}_{\text{const}} = \underbrace{\int q(z)\ln\frac{p(x,z)}{q(z)}\,dz}_{\mathcal{L}(q)\ \xrightarrow[q]{}\ \max} + \underbrace{\int q(z)\ln\frac{q(z)}{p(z|x)}\,dz}_{KL(q\,\|\,p(z|x))\ \xrightarrow[q]{}\ \min}$$

$$\Leftrightarrow$$

$$\mathcal{L}(q) = \int q(z)\ln\frac{p(x,z)}{q(z)}\,dz \xrightarrow[q]{} \max \qquad p(x,z) = p(z)\,p(x|z)$$

$$\bar{x} \to \boxed{E} \to p(\bar{z}|\bar{x}) \sim \bar{z} \to \boxed{D} \to \hat{\bar{x}} \sim p(\bar{x}|\bar{z})$$

$$p(\bar{x}|\bar{z}) = \mathcal{N}(\bar{x}\,|\,f(\bar{z}),\ c\mathcal{I})$$

$$p(\bar{z}) = \mathcal{N}(\bar{z}\,|\,\bar{0},\ \mathcal{I})$$

$$\mathcal{L}(q) = \int q(\bar{z})\ln\frac{p(\bar{z})\,p(\bar{x}|\bar{z})}{q(\bar{z})}\,d\bar{z} =$$

$$= \underbrace{\int q(\bar{z})\ln p(\bar{x}|\bar{z})\,d\bar{z}}_{\mathbb{E}_{q(\bar{z})}\left[-\frac{1}{2c}\cdot(\bar{x}-f(\bar{z}))^T(\bar{x}-f(\bar{z}))\right]} - \underbrace{\int q(\bar{z})\ln\frac{q(\bar{z})}{p(\bar{z})}\,d\bar{z}}_{KL(q(\bar{z})\,\|\,p(\bar{z}))} \to \max$$

$$\mathbb{E}_{\bar{z}\sim q_{\bar{x}}(\bar{z})}\left[\frac{1}{2c}\|\bar{x}-f(\bar{z})\|^2\right] + KL\big(\underbrace{q_{\bar{x}}(\bar{z})}_{\approx p(\bar{z}|\bar{x})}\,\|\,\overbrace{p(\bar{z})}^{\mathcal{N}(\bar{z}|\bar{0},\mathcal{I})}\big) \to \min$$

$$\bar{x} \sim \boxed{Enc} \to \bar{\mu}_x,\ \bar{\sigma}_x \to \boxed{KL(\mathcal{N}(\bar{z}|\bar{\mu}_x,\bar{\sigma}_x)\,||\,\mathcal{N}(\bar{z}|\bar{0}, I))} \to L_{reg}$$

$$\bar{z} \sim \mathcal{N}(\bar{z}|\bar{\mu}_x,\bar{\sigma}_x) \to \overset{p(\bar{x}|\bar{z})}{\boxed{Dec}} \to f(\bar{z})$$

gen: $\underline{z \sim p(\bar{z}|\theta)} \to \boxed{Dec} \to$

$$\boxed{\frac{1}{2c}||\bar{x} - f(\bar{z})||^2} \to L_{rec}$$

$$\underbrace{p(\bar{x})\,p(\bar{z}|\bar{x})}_{\sim q(\bar{z}) = Enc} = \underline{p(\bar{x},\bar{z})} = \underbrace{\overset{\mathcal{N}(\bar{z}|\bar{0},I)}{p(\bar{z})}\ \overset{\mathcal{N}(\bar{x}|f(\bar{z}),cI)}{p(\bar{x}|\bar{z})}}_{Dec,\ generation}$$

Reparamerization trick

$$\bar{a} \sim \mathcal{N}(\bar{a}|\bar{\mu},\bar{\sigma})$$

$$a_i \sim \mathcal{N}(a_i|\mu_i,\sigma_i) = \sigma_i \cdot \mathcal{N}(b_i|0,1) + \mu_i$$

$$\bar{x} - \boxed{Enc} - \bar{\mu}_x,\ \bar{\sigma}_x \odot \bar{u} \oplus \bar{\mu}_x - \bar{z} - \boxed{Dec} - \hat{x}$$

$$\bar{u} \sim \mathcal{N}(\bar{0}, I)$$

$$\bar{x},\ \bar{u} - \boxed{VAE} - \hat{x}$$

$$KL\left(\mathcal{N}\left(\bar{z}\,\middle|\,\bar{\mu}, \begin{pmatrix}\sigma_1 & 0 \\ 0 & \sigma_d\end{pmatrix}\right)\middle|\middle|\ \mathcal{N}(\bar{z}|\bar{0}, I)\right) = (*)$$

$$\text{''}\ \prod \mathcal{N}(z_i|\mu_i,\sigma_i) \qquad \text{''}\ \prod \mathcal{N}(z_i|0,1)$$

$$KL\left(\prod_i q_i(z_i) \,\|\, \prod_i p_i(z_i)\right) = \int \prod_i q_i \cdot \ln \frac{\prod q_i}{\prod p_i} dz =$$

$$= \int \prod_i q_i(z_i) \cdot \left(\ln \frac{q_1}{p_1} + \ln \frac{q_2}{p_2} + \dots + \ln \frac{q_d}{p_d}\right) dz$$

$$\int \prod_i q_i(z_i) \ln \frac{q_j(z_j)}{p_j(z_j)} dz = \underbrace{\left(\int q_j(z_j) \ln \frac{q_j}{p_j} dz_j\right)}_{KL(q_j \| p_j)} \cdot \cancel{\int \prod_{i \neq j} q_i \, dz_{-j}}$$

$$= \sum_i KL(q_i \| p_i)$$

$$(*) = \sum_i \underbrace{KL\left(\mathcal{N}(z_i \mid \mu_i, \sigma_i) \,\|\, \mathcal{N}(z_i \mid 0, 1)\right)}$$

$$\int \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{1}{2\sigma_i^2}(z_i - \mu_i)^2} \cdot \ln \frac{\mathcal{N}\dots\mu,\sigma}{\mathcal{N}\dots 0,1} dz_i =$$

$$= \mathbb{E}_{z_i \sim \mathcal{N}(z_i \mid \mu_i, \sigma_i)}\left[\cancel{-\frac{1}{2}\ln 2\pi} - \frac{1}{2}\ln\sigma_i^2 - \frac{1}{2\sigma_i^2}(z_i - \mu_i)^2 + \cancel{\frac{1}{2}\ln 2\pi} + \frac{1}{2} z_i^2\right] =$$

$\mathbb{E} z_i = \mu_i$

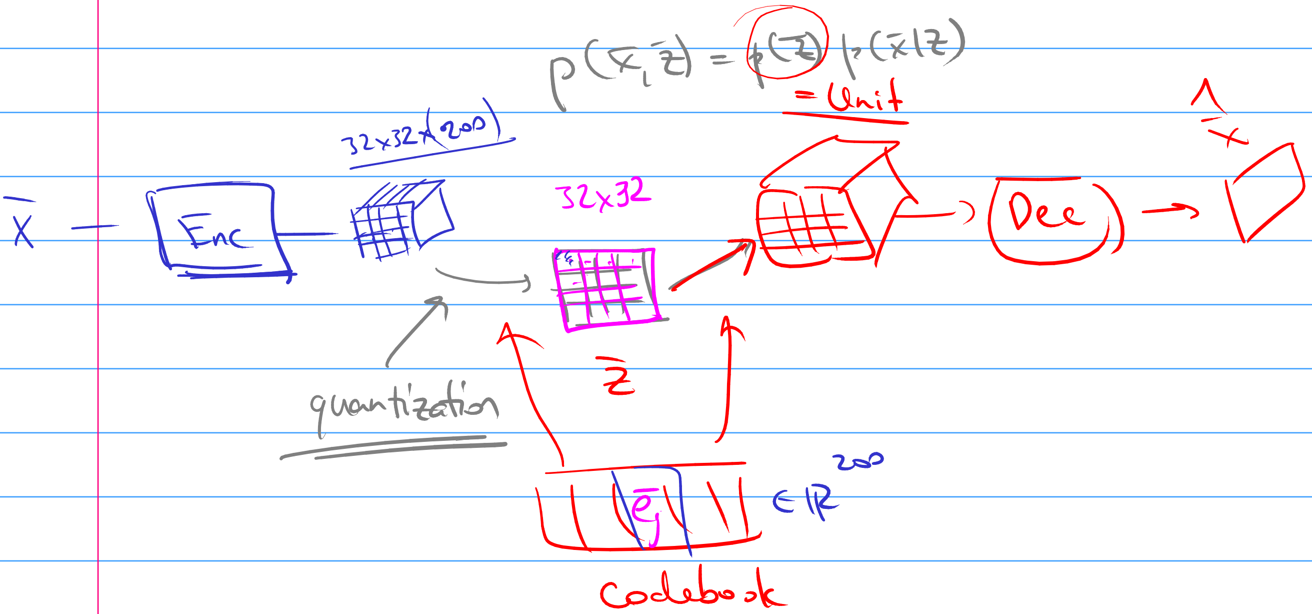
$\mathbb{E} z_i^2 = (\mathbb{E} z_i)^2 + \mathrm{Var}\, z_i = \mu_i^2 + \sigma_i^2$

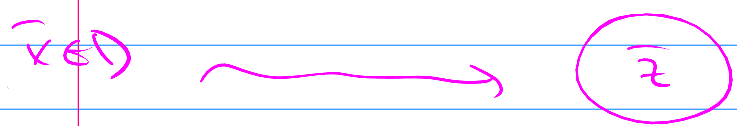
$$= -\frac{1}{2}\ln\sigma_i^2 - \frac{\mu_i^2}{2\sigma_i^2} + \mathbb{E}_{z_i \sim \mathcal{N}(\mu_i, \sigma_i)}\left[\frac{z_i^2}{2} - \frac{z_i^2}{2\sigma_i^2} + \frac{z_i \mu_i}{\sigma_i^2}\right]$$

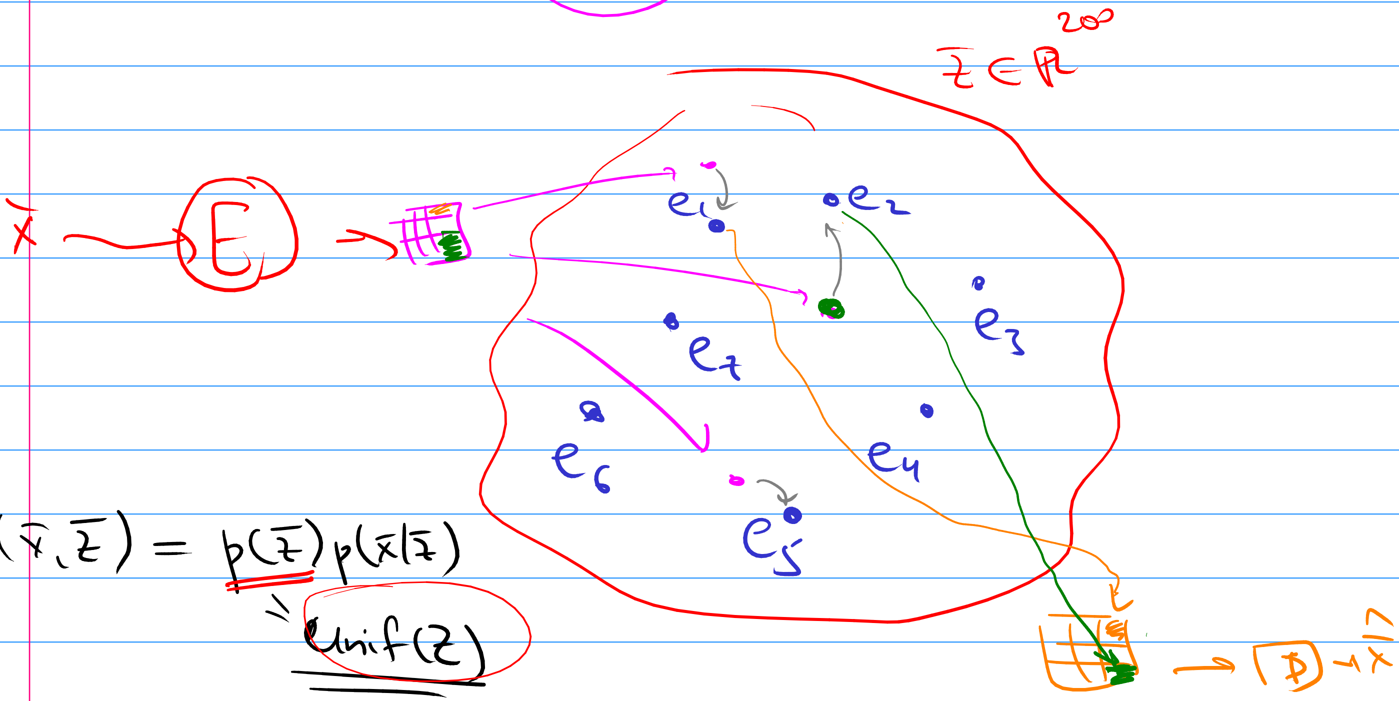
$$\frac{1}{2}\mu_i^2 + \frac{1}{2}\sigma_i^2 - \frac{1}{2}\frac{\mu_i^2}{\cancel{2\sigma_i^2}} + \frac{1}{2}$$

$$= -\frac{1}{2}\ln\sigma_i^2 - \cancel{\frac{\mu_i^2}{2\sigma_i^2}} + \cancel{\frac{\mu_i^2}{\sigma_i^2}} + \frac{1}{2}\left(1 - \frac{1}{\sigma_i^2}\right)(\mu_i^2 + \sigma_i^2)$$

$$KL = \sum_i \left(-\frac{1}{2}\ln\sigma_i^2 + \frac{1}{2}\mu_i^2 + \frac{1}{2}\sigma_i^2 + \cancel{\frac{1}{2}}\right)$$

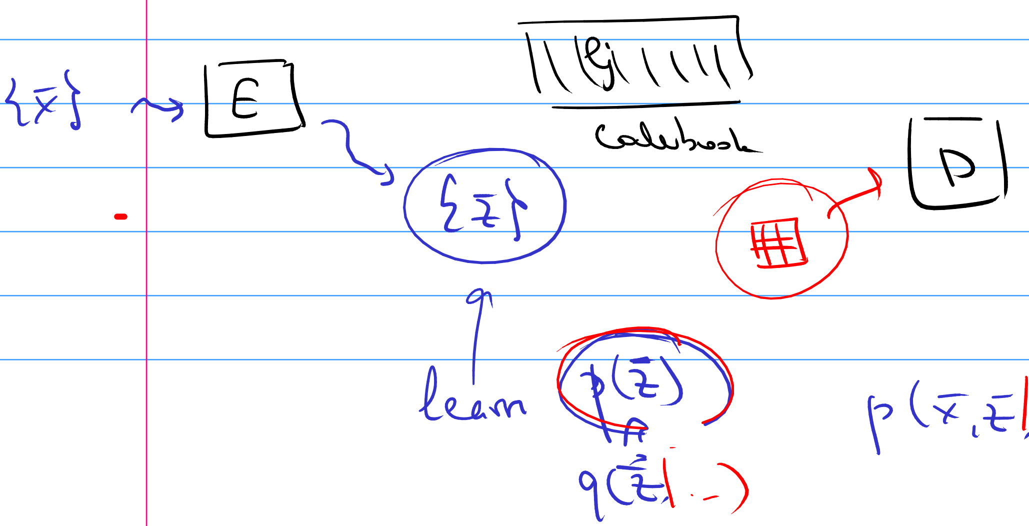






$$p(\bar{x}, \bar{z}) = p(\bar{z})\, p(\bar{x}|\bar{z})$$

$\searrow$ Unif(Z)



$$p(\bar{x}, \bar{z}|\ldots) = \overbrace{q(\bar{z})}^{\text{PixelCNN}}\ \overbrace{p(\bar{x}|\bar{z})}^{\text{Decoder}}$$