第二基本形式

两种方式得到

- "局部展开"
- 曲率在法向量上的投影

W 变换 即为一个线性变换. 描述一个法向量变化如何由切向变化决定.

例. 若曲面上的所有点都为平点. 则该曲面为平面

Pf. (u^1,u^2) 下. $h_{\alpha\beta}=\lambda g_{\alpha\beta}$ $\quad \lambda\equiv 0$

$h_{\alpha\beta}\equiv 0$

$-x_\alpha n_\beta \equiv 0$ "一边算一边看意义"

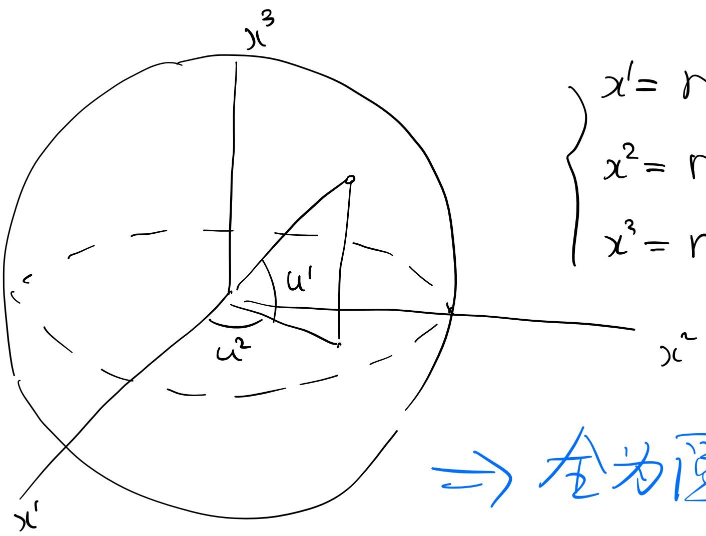
由 $|n|=1$ $\quad n_\alpha n=0$

$\Rightarrow n_\alpha \perp n$ 且 $n_\alpha \perp x_\beta \Rightarrow n$ 为常向量

任取 $x_0\in M$. 对于M上任一点, $\frac{\partial}{\partial u^\alpha}\big((x-x_0)\cdot n\big)=x_\alpha\cdot n+(x-x_0)\cdot n_\alpha=0$

$\Rightarrow (x-x_0)\cdot n=$ 常数 $=0$

$\Rightarrow (x-x_0)\cdot n=0$ —— 平面



$$\begin{cases} x^1=r\cos u^1\cos u^2 \\ x^2=r\cos u^1\sin u^2 \\ x^3=r\sin u^1 \end{cases}$$

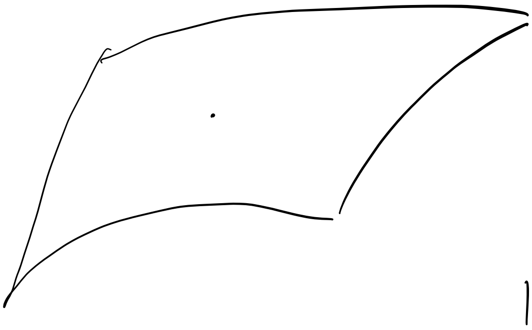
$g_{11}=r^2$ $\quad h_{11}=-r$

$g_{12}=0$ $\quad h_{12}=0$

$g_{22}=r^2\cos^2u^1$ $\quad h_{22}=-r\cos^2u^1$

$h_{\alpha\beta}=\underbrace{-\frac{1}{r}}_{\lambda}g_{\alpha\beta}$

$\Rightarrow$ 全为圆点



(u^1,u^2)

$$W(x_\alpha) = h_\alpha^\beta x_\beta \quad . \quad \{h_\alpha^\beta\} = A$$

$$|A-\lambda I| = 0 \qquad \Longleftrightarrow \qquad |h_{\alpha\gamma}-\lambda g_{\alpha\gamma}| = 0$$

$$= |h_\alpha^\beta - \lambda \delta_\alpha^\beta|$$

$$= |h_{\alpha\gamma} g^{\gamma\beta} - \lambda g_{\alpha\gamma} g^{\gamma\beta}|$$

$$= |h_{\alpha\gamma}-\lambda g_{\alpha\gamma}| \, \underbrace{|g^{\gamma\beta}|}_{\neq 0}$$

$$\begin{vmatrix} h_{11}-\lambda g_{11} & h_{12}-\lambda g_{12} \\ h_{12}-\lambda g_{12} & h_{22}-\lambda g_{22} \end{vmatrix} = 0$$

$$\lambda^2 (g_{11}g_{22}-g_{12}^2) - \lambda (h_{11}g_{22}+h_{22}g_{11} - 2h_{12}g_{12}) + (h_{11}h_{22}-h_{12}^2) = 0$$

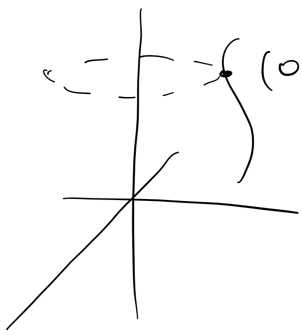
$$k_1+k_2 = \frac{h_{11}g_{22}-2h_{12}g_{12}+h_{22}g_{11}}{g_{11}g_{22}-g_{12}^2}$$

平均曲率 $H = \frac{k_1+k_2}{2} = \frac{\overset{LG+NE}{h_{11}g_{22}-2h_{12}g_{12}+h_{22}g_{11}}}{2(g_{11}g_{22}-g_{12}^2)}$

高斯曲率（总曲率） $K = k_1k_2 = \frac{h_{11}h_{22}-h_{12}^2}{g_{11}g_{22}-g_{12}^2} = \frac{\det(h_{\alpha\beta})}{\det(g_{\alpha\beta})}$

若 $H \equiv 0$. 称曲面 M 为极小曲面

球面上. $\begin{cases} H = -\frac{1}{r} \\ K = \frac{1}{r^2} \end{cases}$



旋转面. $x(u^1,u^2) = (f(u^2)\cos u^1, f(u^2)\sin u^1, g(u^2))$

$g_{11} = f^2$. $g_{12} = 0$ $g_{22} = f'^2 + g'^2$

$h_{11} = \frac{-fg'}{\sqrt{(f')^2+(g')^2}}$ $h_{12} = 0$ $h_{22} = \frac{f''g' - f'g''}{\sqrt{(f')^2+(g')^2}}$

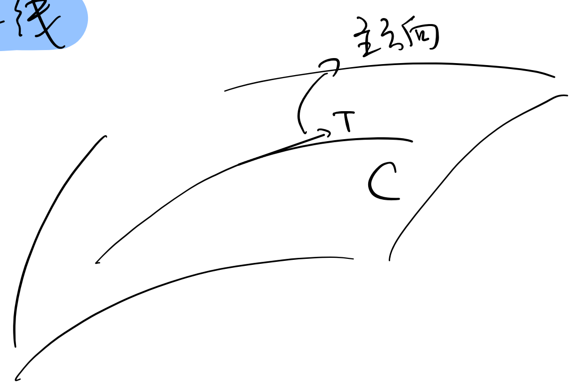
$$\Rightarrow \quad H = \boxed{} \left(f(f''g' - f'g'') - g'((f')^2 + (g')^2) \right) = 0$$ → 极小曲率

then 令 $g(u^2)=u^2$ 解 $f(u^2)$

$\Rightarrow f = a\,\mathrm{ch}\frac{u^2}{a} \Rightarrow$ 悬链线

正螺面. $\Rightarrow$ 也是极小的

曲率线



始终沿着主方向.

$x = x(s) = x(u^1(s), u^2(s))$.

C 称为曲面 M 上的曲率线

Thm. (Rodrigues).

曲面上曲线 C: $x(s)=x(u^\alpha(s))$ 为曲率线的充要条件为 $\exists \lambda(s)$. s.t.

$$dn(s) = -\lambda(s)\,dx(s)$$

这时 $\lambda(s)$ 正是曲面沿 $x(s)$ 的主曲率.

Pf. C 为曲率线. $\Longleftrightarrow$ $dx(s)$ 为主方向.

$\Longleftrightarrow W(d(x(s))) = \lambda(s)\,dx(s)$

推个公式 $|n|=1 \Rightarrow n_\alpha \cdot n = 0 \Rightarrow n_\alpha = b_\alpha^\beta x_\beta$. $b_\alpha^\beta = ?$

$$\boxed{dn = -W(dx)}$$

$-h_{\gamma\alpha} = x_\gamma \cdot n_\alpha = b_\alpha^\beta x_\gamma x_\beta = b_\alpha^\beta g_{\gamma\beta}$

$-h_{\gamma\alpha} g^{\gamma\sigma} = b_\alpha^\beta g_{\gamma\beta} g^{\gamma\sigma} = b_\alpha^\beta \delta_\beta^\sigma = b_\alpha^\sigma$

$-h_\alpha^\sigma = b_\alpha^\sigma \Rightarrow b_\alpha^\sigma = -h_\alpha^\sigma$

$\Rightarrow \boxed{n_\alpha = -h_\alpha^\beta x_\beta}$ — Weingarten 变换 ☆'

$$\Rightarrow \quad dn = n_\alpha du^\alpha = -h_\alpha^\beta x_\beta du^\alpha = -W(x_\alpha) du^\alpha = -W(dx)$$

then. Rodrigues 定理 ✓

若 u^1. u^2 线都为曲率线，称 (u^1, u^2) 为曲率线网.

$$\Rightarrow \quad I = g_{11}(du^1)^2 + g_{22}(du^2)^2$$

x_1. x_2 为主方向.

k_1 k_2 主曲率.

$$W(x_\alpha) = k_\alpha x_\alpha \Rightarrow h_{\alpha\beta} = W(x_\alpha)\cdot x_\beta = k_\alpha g_{\alpha\beta}$$

$$\Rightarrow \begin{cases} h_{11} = k_1 g_{11} \\ h_{12} = 0 \\ h_{22} = k_2 g_{22} \end{cases}$$

$$II = k_1 g_{11}(du^1)^2 + k_2 g_{22}(du^2)^2$$

$$\Rightarrow k_1 = \frac{h_{11}}{g_{11}}, \quad k_2 = \frac{h_{22}}{g_{22}}.$$