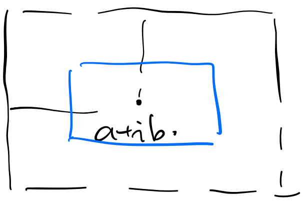
Thm. Let $\Omega \xrightarrow{f} \mathbb{C}$ be a function which is cpx. diff. every where on Ω. Then f is holo.

($\Omega \overset{\text{open}}{\subset} \mathbb{C}$)

Pf. We may let Ω be an open rectangle.

Ω



we should have

$$(*) \int_{\partial K} f(z)\,dz = 0 \text{ for every closed rectangle } K \subseteq \Omega$$

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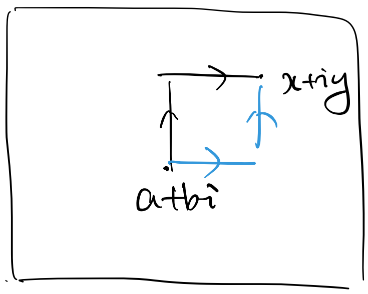
claim. $(*) \Rightarrow f$ is holo.

we have. f. cpx. diff. $\Rightarrow f$. conti

If. there exists $\Omega \xrightarrow{F} \mathbb{C}$. s.t. $F' = f$. then. F is holo,

and hence f is holo, too.



For example. we may try.

$$F(x+iy) = \int_b^y f(a+it)\,dt + \int_a^x f(s+iy)\,ds$$

$$\frac{\partial F}{\partial x}(x+iy) = \lim_{h \to 0} \left(\int_x^{x+h} f(s+iy)\,ds \right) \Big/ h$$

$$= f(x+iy)$$

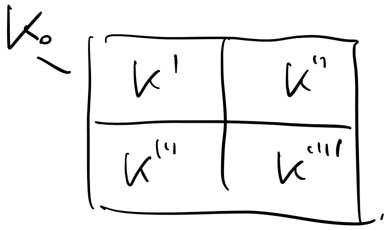
we have $(*)$ then $F(x+iy) = \int_a^x f(s+ib)\,ds + \int_b^y f(x+it)\,dt.$

and hence $\frac{dF}{dy}(x+iy) = i f(x+iy)$

$$\Rightarrow F(x+iy) = U(x,y) + iV(x,y) \Rightarrow \begin{cases} \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y} \end{cases} + F \; C^1 \Rightarrow F \text{ holo}$$

Check: $f = F' \Rightarrow f$ holo.

$\boxed{\text{Pf of (*)}}$: Let K be a closed rectangle in Ω



$$\int_{\partial K} f(z)\,dz = \int_{\partial K'} f(z)\,dz + \int_{\partial K''} + \int_{\partial K'''} + \int_{\partial K''''}$$

$$I(K) = I(K') + I(K'') + I(K''') + I(K'''')$$

$$\Rightarrow \quad |I(K)| \leq |I(K')| + |I(K'')| + |I(K''')| + |I(K'''')|.$$

Let K_1 be the one among K', K'', K''', K'''' which has the largest $|I(\cdot)|$. $\Rightarrow$ Then $\frac{I(K_0)}{4} \leq I(K_1)$

$\vdots$

Suppose that we have obtained K_j $(j = 0, 1, \dots, k)$ closed rectangle. s.t.

$$K_j \subseteq K_{j-1} \text{ and } \operatorname{diam}(K_j) = \tfrac{1}{2}\operatorname{diam}(K_{j-1}) \text{ and } \frac{I(K_{j-1})}{4} \leq I(K_j)$$

$$\Rightarrow \bigcap_{j=0}^{\infty} K_j = \{c\} \text{ for some } c \in \Omega$$

f is cpx. diff at c.

$$\Rightarrow f(z) = f(c) + f'(z)(z-c) + h(z)(z-c) \quad \text{for some func. } \Omega \xrightarrow{h} \mathbb{C}$$

s.t. $h(c) = 0 = \lim_{z \to c} h(z)$

$$\int_{\partial K_m} f(z)\,dz = \int_{\partial K_m} \Big[\underbrace{f(c) + f'(c)(z-c)}_{\text{by holo (or poly)}} + h(z)(z-c)\Big]\,dz = \int_{\partial K_m} h(z)(z-c)\,dz.$$

$$|I(K_m)| = \left|\int_{\partial K_m} h(z)(z-c)\,dz\right| \leqslant \int_{\partial K_m} |h(z)|\,|z-c|\,|dz|.$$

$h(c)=0, = \lim_{z\to c} h(z) \Longrightarrow \forall \varepsilon>0\ \exists \delta>0.\ \forall z\in B_\delta(c) \Longrightarrow |h(z)|<\varepsilon$

$K_m \subseteq B_\delta(c)$ if m is sufficient large.

周长与 diam 比例.

$$\leqslant \varepsilon\, \mathrm{diam}(K_m) \int_{\partial K_m} |dz| = \varepsilon\, C \cdot \mathrm{diam}^2(K_m)$$

In summary $\forall \varepsilon>0.\ \exists m\in \mathbb{N}.\ |I(K_m)| \leqslant \varepsilon C\, \mathrm{diam}(K_m)^2$

$$\begin{array}{ccc} |I(K_m)| & & \mathrm{diam}(K_m)^2 \\ \geqslant & & \| \\ \frac{1}{4}|I(K_{m-1})| & & \left(\frac{1}{2}\mathrm{diam}(K_{m-1})\right)^2 \\ \geqslant & & \| \\ \vdots & & \vdots \\ \geqslant & & \| \\ \frac{1}{4^m}|I(K_0)| & & \frac{1}{4^m}\mathrm{diam}(K_0)^2 \end{array}$$

$$\Rightarrow |I(K)| \leqslant \underset{\downarrow\ 0}{\varepsilon}\, C\, \mathrm{diam}(K)^2$$

$\square$