W5L1

Review

曲面 $x:(u^1,u^2)\in D \longrightarrow x(D)=M\subset E^3$

$x=x(u^1,u^2)$

曲面第一基本形式

$\mathrm{I}=ds^2=|dx|^2=g_{\alpha\beta}du^\alpha du^\beta$ — $\boxed{\text{内蕴几何}}$

$\mathrm{II}=h_{\alpha\beta}du^\alpha du^\beta=(W(dx),dx)$ 外在几何

沿方向 $T=\frac{du^1}{ds}:\frac{du^2}{ds}$ "=" du^1/du^2

$=x_1\frac{du^1}{ds}+x_2\frac{du^2}{ds}$

法曲率 $k_n(T)=\dfrac{\mathrm{II}(du^1,du^2)}{\mathrm{I}(du^1,du^2)}$

$W: T_xM \longrightarrow T_xM$ W-变换 自共轭 ⇒ 两个特征方向

主曲率 k_1 k_2 特征值 $k_n(\theta)$ 的最大最小值

主方向 $e_1 \cdot e_2$ 特征方向 $0\le\theta\le 2\pi$

$W(x_\alpha)=h_\alpha^\beta x_\beta$

$W(e_\alpha)=k_\alpha e_\alpha$

$$H = \frac{k_1+k_2}{2} = \frac{1}{2}\frac{g_{11}h_{22}-2g_{12}h_{12}+g_{22}h_{11}}{g_{11}g_{22}-g_{12}^2}$$ 平均曲率

$$K = k_1k_2 = \frac{\det(h_{\alpha\beta})}{\det(g_{\alpha\beta})}$$ 总曲率. Gauss曲率

曲率线 C. $x(s) = (u^1(s), u^2(s))$

曲面上曲率线 $\dot{x}(s)$ 为主方向

$$W(\dot{x}(s)) = \lambda(s)\dot{x}(s).$$ C 为曲率线

Rodriques Thm

C是曲率线 $\Longleftrightarrow$ $dn = -\lambda(s)\, dx(s)$

use Weingarten 公式 $n_\alpha = -h_\alpha^\beta x_\beta$

设

$\left.\begin{array}{l} u^1\text{-线} \\ u^2\text{-线} \end{array}\right\}$ 曲率线 $\Rightarrow$ 曲率线网.

$$\begin{cases} W(x_1) = k_1 x_1 \\ W(x_2) = k_2 x_2 \end{cases}$$

why?

$$\langle x_1, x_2\rangle = 0 = g_{12}$$

$$\Rightarrow \mathrm{I} = g_{11}(du^1)^2 + g_{22}(du^2)^2$$

$$h_{12} = (W(x_1), x_2) = (k_1 x_1, x_2) = 0$$

$$h_{11} = (W(x_1), x_1) = k_1 g_{11}$$

$$h_{22} = (W(x_2), x_2) = k_2 g_{22}$$

$$\Rightarrow \mathrm{II} = k_1 g_{11}(du^1)^2 + k_2 g_{22}(du^2)^2.$$

If we have. $\begin{cases} \mathrm{I} = g_{11}(du^1)^2 + g_{22}(du^2)^2 \\ \mathrm{II} = h_{11}(du^1)^2 + h_{22}(du^2)^2 \end{cases}$

$$\Rightarrow k_1 = \frac{h_{11}}{g_{11}} \quad k_2 = \frac{h_{22}}{g_{22}}.$$

$g_{12} = h_{12} = 0 \iff (u^1, u^2)$ 是曲率线网

(无脐点邻域中).

P21. T2. ☆

Now. $x_1 \longrightarrow e_1 = \frac{x_1}{|x_1|} = \frac{x_1}{\sqrt{g_{11}}}$

$x_2 \longrightarrow e_2 = \frac{x_2}{|x_2|} = \frac{x_2}{\sqrt{g_{22}}}$

$$dx = x_\alpha du^\alpha = e_\alpha \boxed{\sqrt{g_{\alpha\alpha}}\, du^\alpha}$$

$$= e_\alpha \omega^\alpha$$

$$= e_1 \omega^1 + e_2 \omega^2$$

$$ds^2 = |dx|^2 = (\omega^1)^2 + (\omega^2)^2$$

$$\mathrm{II} = h_{\alpha\beta}\, du^\alpha du^\beta$$

$$= k_1 g_{11} (du^1)^2 + k_2 g_{22} (du^2)^2$$

$$= k_1 (\omega^1)^2 + k_2 (\omega^2)^2$$

$$dn = -\lambda(s)\, dx$$

这里两个α不同！由于是两项加起来，故都用α无所谓

$$n_\alpha du^\alpha = -\lambda(s)\, x_\alpha du^\alpha \qquad \cdot\, x_\beta$$

$$-h_{\alpha\beta} du^\alpha = -\lambda(s)\, g_{\alpha\beta} du^\alpha$$

$$-h_{\alpha\beta} du^\alpha + \lambda(s)\, g_{\alpha\beta} du^\alpha = 0.$$

$\beta = 1.$ $\quad (-1).\,(h_{11} du^1 + h_{21} du^2) + \lambda(s)\,(g_{11} du^1 + g_{21} du^2) = 0$

$\beta = 2$ $\quad (-1)\,(h_{12} du^1 + h_{22} du^2) + \lambda(s)\,(g_{12} du^1 + g_{22} du^2) = 0$

看成齐次线性方程组. $(-1, \lambda(s))$ 为非零解.

$$\begin{vmatrix} h_{11}du^1 + h_{21}du^2 & g_{11}du^1 + g_{21}du^2 \\ h_{12}du^1 + h_{22}du^2 & g_{12}du^1 + g_{22}du^2 \end{vmatrix} = 0.$$

$$\begin{vmatrix} (du^2)^2 & -du^1du^2 & (du^1)^2 \\ g_{11} & g_{12} & g_{22} \\ h_{11} & h_{12} & h_{22} \end{vmatrix} = 0$$

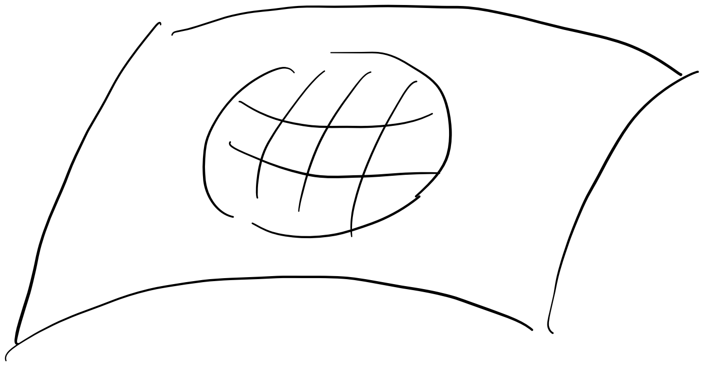
取 $g_{12} = h_{12} = 0. \Rightarrow du^1du^2(h_{11}g_{22} - h_{22}g_{11}) = 0$

$$\Downarrow$$

$\dfrac{h_{11}}{g_{11}} = \dfrac{h_{22}}{g_{22}} \Rightarrow$ 脐点

$\Rightarrow du^1du^2 = 0$ 曲率线网

"无脐点邻域" 选取曲率线网为参数网. (u^1, u^2)



$\{x; e_1, e_2, n\}$

$e_1 = \dfrac{x_1}{\sqrt{g_{11}}}$ $e_2 = \dfrac{x_2}{\sqrt{g_{22}}}$

$$P(x(u^1,u^2)) \quad P'(x(u^1+\Delta u^1), x(u^2+\Delta u^2))$$

$$x(u^1+\Delta u^1, u^2+\Delta u^2) - x(u^1,u^2)$$

$$= x_\alpha(u^1,u^2)\Delta u^\alpha + \frac{1}{2} x_{\alpha\beta}(u^1,u^2)\Delta u^\alpha \Delta u^\beta + \cdots$$

$$x_\alpha = e_\alpha \sqrt{g_{\alpha\alpha}} \qquad x_{\alpha\beta} = \boxed{\Gamma^\gamma_{\alpha\beta}} e_\gamma + h_{\alpha\beta} \cdot n$$

$$= e_1 \sqrt{g_{11}} \Delta u^1 + e_2 \sqrt{g_{22}} du^2$$

$$+ \frac{1}{2} (\Gamma^\gamma_{\alpha\beta} e_\gamma) \Delta u^\alpha \Delta u^\beta + \frac{1}{2} h_{\alpha\beta} n \cdot \Delta u^\alpha \Delta u^\beta + \cdots$$

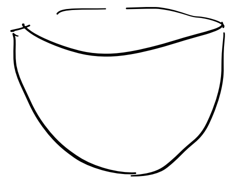
$$= (\sqrt{g_{11}} \Delta u^1 + o(\rho)) e_1 + (\sqrt{g_{22}} \Delta u^2 + o(\rho)) e_2$$

$$+ \frac{1}{2} n \cdot (h_{11} (\Delta u^1)^2 + h_{22} (\Delta u^2)^2 + o(\rho))$$

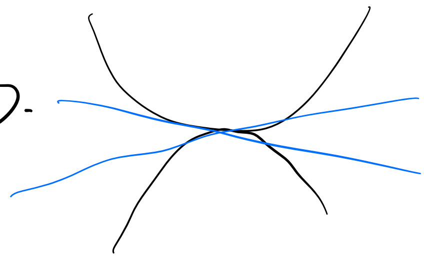
$$\begin{cases} y^1 = \sqrt{g_{11}} \Delta u^1 + o(\rho) \\ y^2 = \sqrt{g_{12}} \Delta u^2 + o(\rho). \\ y^3 = \frac{1}{2} (h_{11} (\Delta u^1)^2 + h_{22} (\Delta u^2)^2 + o(\rho)) \end{cases} \qquad \rho = \sqrt{(\Delta u^1)^2 + (\Delta u^2)^2}.$$

$$2y^3 = k_1\cdot(y^1)^2 + k_2(y^2)^2 .$$

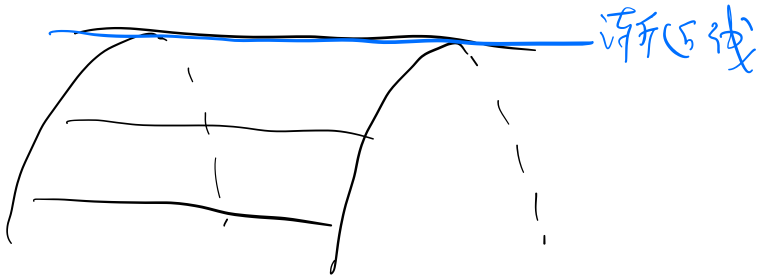
椭圆点 $k_1\cdot k_2>0$. 抛物面



双曲点 $k_1\cdot k_2<0$. 双曲抛物面.



抛物点 $k_1\cdot k_2=0$



渐近线 $C: x(s)=x(u^1(s),u^2(s))$, 每一点处 $\dot{x}(s)$ 为渐近方向. $\Longrightarrow$ 渐近线

渐近方向. 在一点处的某一方向. du^1/du^2

$k_n(\theta)=0 \Leftarrow \mathrm{II}(du^1,du^2)=0$ 该方向为渐近方向

$$h_{11}(du^1)^2+2h_{12}\,du^1du^2+h_{22}(du^2)^2=0$$

$$h_{11}\left(\frac{du^1}{du^2}\right)^2+2h_{12}\left(\frac{du^1}{du^2}\right)+h_{22}=0.$$

$$\Delta = 4(h_{12})^2-4h_{11}h_{22} = -4\det(h_{\alpha\beta}) > 0$$

$$\Longleftrightarrow K<0$$

$\Rightarrow$ 双曲点　有两个渐近方向

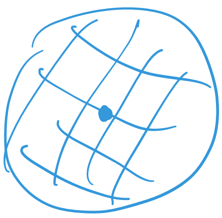
抛物点　一个渐近方向

椭圆点　无渐近方向

- 不是所有曲面上都存在渐近线
- 若能选取渐近线为参数曲线网

$\Rightarrow$ 渐近线网.

$K<0$



邻域内

u^1线 $\Leftrightarrow$ $u^2=$常数 $\Leftrightarrow$ $du^2=0$　$\mathrm{II}(du^1,0)=h_{11}(du^1)^2=0$

u^2线 $\Leftrightarrow$ $u^1=$常数 $\Leftrightarrow$ $du^1=0$　$\mathrm{II}(0,du^2)=h_{22}(du^2)^2=0.$

$\Rightarrow$　u^1线　$h_{11}=0$

　　u^2线　$h_{22}=0$

$$\boxed{\mathrm{II}(du^1,du^2)=2h_{12}\,du^1du^2}$$

例8. 如果曲面上的点全是圆点，则该曲面一定为球面的一部分.

Pf. 圆点 (u^1,u^2) 下 $\dfrac{h_{\alpha\beta}}{g_{\alpha\beta}} = \lambda(u^1,u^2)$

$$\lambda(u^1,u^2)=0$$

需证. $\lambda(u^1,u^2)$ 为常数.

由 $n_\alpha = -h_\alpha^\beta x_\beta$.

$$h_\alpha^\beta = h_{\alpha\gamma} g^{\gamma\beta} = \lambda g_{\alpha\gamma} g^{\gamma\beta} = \lambda(p)\,\delta_\alpha^\beta$$

$$\Rightarrow n_\alpha = -\lambda(p)\,\delta_\alpha^\beta x_\beta = -\lambda(p)\,x_\alpha$$

$$\Rightarrow n_1 = -\lambda(p)\,x_1 \qquad n_2 = -\lambda(p)\,x_2$$

$$n_{12} = -\lambda_2 x_1 - \lambda x_{12}. \qquad n_{21} = -\lambda_1 x_2 - \lambda x_{21}$$

$$\to n_{12}+\lambda x_{12} = -\lambda_2 x_1 \qquad n_{21}+\lambda x_{21} = -\lambda_1 x_2$$

$$\lambda_2 x_1 - \lambda_1 x_2 = 0 \Rightarrow \lambda_1, \lambda_2 = 0.$$

$\lambda =$ 常数.

$h_{\alpha\beta} = \lambda g_{\alpha\beta}.$

$$d(n+\lambda x) = dn + \lambda dx$$

$$= -h_\alpha^\beta x_\beta du^\alpha + \lambda dx$$

$$= -\lambda \delta_\alpha^\beta x_\beta du^\alpha + \lambda dx$$

$$= -\lambda x_\alpha du^\alpha + \lambda dx = 0$$

$\Rightarrow n + \lambda x =$ 常向量. $= x_0$

$$\left| x - \frac{x_0}{\lambda} \right| = \left| \frac{n}{\lambda} \right| = \frac{1}{|\lambda|}$$