Algebra. Lec 18 Seperable extension.

E. alg ext of F

$\sigma : F \overset{\text{emb.}}{\hookrightarrow} L$: alg closed.

Study the extension of σ to E

S_σ : $\{$ the set of all extension of σ to $E \hookrightarrow L \}$.

S_τ : $\{$ the set of all extension of τ to $E \hookrightarrow L' \}$

$$\begin{tikzcd}
\text{alg closed. } L' \arrow[d, dash, "\text{alg}"'] & & L \text{ : alg closed.} \arrow[ll, "\exists \lambda\ (\text{Thm } 2.8)", "\sim"'] \arrow[d, dash, "\text{alg}"] \\
 & E \arrow[ul, "\tau^*"'] \arrow[ur, "\sigma^*"] \arrow[d, dash] & \\
\tau F & F \arrow[l, "\sim", "\tau"'] \arrow[r, "\sim"', "\sigma"] & \sigma F \arrow[ll, bend left, "\text{Compose} \cdot \tau \circ \sigma^{-1}"]
\end{tikzcd}$$

i.e. L. alg closure of σF

L' alg closure of τF

if $\sigma^* \in S_\sigma$

then. $\lambda \circ \sigma^* \in S_\tau$. $\Longrightarrow$ S_σ, S_τ. bijective as set.

$|S_\sigma| = |S_\tau|$. $\left(\text{indep of the choice of } \sigma, \tau, L, L'\right)$.

when $|S_\sigma| = |S_\tau| < \infty$. $|S_\sigma| = |S_\tau| \triangleq [E:F]_s$. separable degree of E over F

Thm 4.1 $E \supset F \supset k$

$\Rightarrow [E:k]_s = [E:F]_s [F:k]_s$.

Moreover if E : finite/k (i.e. $[E:k] < \infty$)

then. $[E:k]_s < \infty$. $[E:k]_s \leq [E:k]$

Pf. ① $\sigma : k \hookrightarrow L$: alg. closed

$\{\sigma_i\}_{i \in I}$: the family of ext of σ to $F \hookrightarrow L$

$(\sigma_i : F \hookrightarrow L)$

for each $i \in I$. $\{\tau_{ij}\}$: the family of ext of σ_i to E

$$\begin{array}{c} \sigma_i : F \hookrightarrow L \\ \downarrow \\ \tau_{ij} : E \hookrightarrow L \end{array}$$

claim $\{\tau_{ij}\}$: all embeddings. $E \hookrightarrow L$ over k.

then $|\{\tau_{ij}\}| = [E:F]_s [F:k]_s$

any embedding $E \hookrightarrow L$. must be one of τ_{ij} $\square$

② $[E:k] < \infty$.

∴ we can obtain E by

$$k \subset k(\alpha_1) \subset k(\alpha_1, \alpha_2) \subset \cdots \subset k(\alpha_1, \dots, \alpha_r) = E$$

$$\begin{matrix} \parallel & \parallel & & \parallel \\ F_0 & F_1 & \cdots & F_r. \end{matrix}$$

$F_{\nu+1} = F_\nu(\alpha_{\nu+1})$: $\alpha_{\nu+1}$ alg $/F_\nu$ By prop 2.7.

$$[F_\nu(\alpha_{\nu+1}) : F_\nu]_s \leq [F_\nu(\alpha_{\nu+1}) : F_\nu]$$

$$\therefore [E:k]_s \leq [E:k]$$

Rmk. (Cor 4.2). E. field over k.

$$E \supset F \supset k.$$

$$[E:k]_s = [E:k] \iff \begin{matrix} [E:F]_s = [E:F] \\ [F:k]_s = [F:k] \end{matrix}$$

Rmk. Later

$$[E:k]_s \mid [E:k]$$

$\therefore$ can define $[E:k] = [E:k]_s [E:k]_i$

$\rightarrow$ inseparable degree

def. E. finite $/k$.

- We say E is a separable extension over k.

  if $[E:k] = [E:k]_s$

- α. alg $/k$. is said to be separable over k.

  if $k(\alpha)$ is separable extension over k. (ie. $\mathrm{Irr}(\alpha, k, X)$ has no

- $f(x) \in k[x]$ is called separable if it has no multiple root in $\bar{k}^a$. (multiple root $\to$ $Irr(\alpha, k, x)$ is sep.)

Rmk. • if α is a root of a sep polynomial $g(x) \in k[x]$

$\Rightarrow Irr(\alpha, k, x) \mid g(x)$ and $Irr(\alpha, k, x)$ separable.

$\Downarrow$

α is sep

• if $k \subset F \subset K$, $\alpha \in K$, α: separable $/k$.

$\therefore k(\alpha)$: sep $/k$. $Irr(\alpha, k, x)$. sep polynomial

$[k(\alpha) : k]_s = [k(\alpha) : k] \xLeftarrow{\text{prop 2.7}}$ no multiple roots

$Irr(\alpha, F, x) \mid Irr(\alpha, k, x) \Rightarrow Irr(\alpha, F, x)$. separable.

$\therefore [k(\alpha) : F]_s = [k(\alpha) : F]$

$\therefore \alpha$: separable $/F$.

Thm. 4.3 E. finite ext $/k$.

E: separable $/k$. (i.e. $[E : k]_s = [E : k]$)

$\Leftrightarrow$ each elem of E is sep $/k$.

pf. $\Rightarrow$). Assume. E: sep $/k$.

Let $\alpha \in E$

Consider the tower $k \subset k(\alpha) \subset E$.

$\therefore$ we have $[k(\alpha) : k]_s = [k(\alpha) : k]$

$$[E : k(\alpha)]_s = [E : k(\alpha)]$$

$\therefore$ α is sep.

$\Leftarrow$) assume $\forall \alpha \in E$. α: separable $/k$

Write: $E = k(\alpha_1, \dots, \alpha_n)$. α_i. alg$/k$. $\cdot$ α_i sep$/k$

Consider

$$\underbrace{k \subset k(\alpha_1)}_{\alpha_1 \text{ sep}/k} \subset \underbrace{k(\alpha_1, \alpha_2)}_{\alpha_2 \text{ sep}/k} \subset \cdots \subset \underbrace{k(\alpha_1, \dots, \alpha_n)}_{\alpha_n \text{ sep}/k(\alpha_1, \dots, \alpha_{n-1})} = E$$

$\Downarrow$

α_2. sep$/k(\alpha)$ by Remark.

$$[E : k]_s = [k(\alpha_1, \dots, \alpha_n) : k(\alpha_1, \dots, \alpha_{n-1})]_s \cdots [k(\alpha_1) : k]_s$$

$$= [k(\alpha_1, \dots, \alpha_n) : k(\alpha_1, \dots, \alpha_{n-1})] \cdots [k(\alpha_1) : k]$$

$$= [E : k]$$

$\square$

<u>Rmk</u>. E. arbitrary alg ext of k.

we define E to be separable$/k$.

if every ext $k(\alpha_1, \dots, \alpha_n)$: separable$/k$
$\alpha_i \in E$

why finite? prop 1.6.

Rmk (Thm 4.4). E: alg ext $/k$.

generated by $\{\alpha_i\}_{i\in I}$

if each α_i is sep$/k$

$\Rightarrow E$ sep$/k$

pf. by definition. is clear

Thm 4.5 Separable extensions

form a distinguished class of exts

pf ① $E \supset F \supset k$

Assume E: sep$/k$ $\Rightarrow$ $\begin{cases} \text{every elem of } E. \text{ sep}/F \\ \text{every elem of } F. \text{ sep}/k \end{cases}$

Conversely E/F sep. F/k. sep.

- if $[E:k] < \infty$

$$[E:k]_s = [E:F]_s [F:k]_s$$

$$= [E:F][F:k] = [E:k] \Rightarrow E \text{ sep}/k$$

- if $[E:k]$ not finite $\alpha \in E$. α is a root of a sep polynomial $f(x) \in F[x]$ $(E/F$ sep$)$

$f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $a_i \in F$

a_i sep$/k$. alg$/k$

$F_0 \triangleq k(a_0, a_1, \ldots, a_n)$.

also. F_0 coff.

Consider

$$\underbrace{k \subset F_0}_{\substack{\text{sep} \\ \text{finite}}} \underbrace{\subset F_0(\alpha)}_{\substack{\text{sep} \\ \text{finite}}} \Rightarrow F_0(\alpha) : \text{sep}/k$$

Why !!!

prop. 1.6.

$\alpha \in F_0(\alpha)$: sep$/k$

②

$$\begin{tikzcd} & EF \arrow[dl, dash] \arrow[dr, dash] & \\ E \arrow[dr, dash] & & F \arrow[dl, dash] \\ & k & \end{tikzcd}$$

E/k. sep. $E \cdot F \subset L$

$\forall \alpha \in E$. α : sep$/k$. $\Rightarrow$ α : sep$/F$
$\Uparrow$
$E \subset L$

$\because$ EF : generated by all elements of E over F

$\xRightarrow{\text{Thm 4.4}}$ EF sep$/F$

alg and . $[E:k]_s \le [E:k] < \infty$

Rmk. $\cdot$ E finite ext of k. $\underbrace{k \subset E}_{\text{alg}} \subset E^a$

$$K \triangleq \bigcap_{\substack{k \subset E \subset \tilde{k} \subset E^a \\ \tilde{k} \text{ normal}/k}} \tilde{k} = \left(\begin{array}{c} \text{smallest} \\ \text{normal extension of } k, \text{ in } E^a \\ \text{containing } E \end{array} \right)$$

How to prove ?

Lem. $\{K_i\}_{i\in I}$, K_i normal$/k$ $\Longrightarrow$ $\bigcap_{i\in I} K_i$: normal$/k$

let. $f(x) \in k[x]$. irred $\alpha \in \bigcap_{i\in I} K_i$ $f(\alpha)=0$

i.e. $f(x)$ has a root in K_i, $i \in I$

$\Longrightarrow$ $f(x)$ has all roots in K_i $\forall i \in I$

$\Longrightarrow$ all roots $\in \bigcap_{i\in I} K_i$

<u>diag</u>.

$$\begin{tikzcd}
 & E^a \arrow[d, dash] \\
K \arrow[ur] \arrow[r, hook, "\sim"] \arrow[dd, dash] & K \arrow[d, dash] \\
 & E \arrow[d, dash] \\
k \arrow[r, equal] & k
\end{tikzcd}$$

· $\{\sigma_1, \dots, \sigma_n\}$ all distinct embedding of E in E^a (finite)

$\sigma_i : E \hookrightarrow E^a$ $(\sigma_1 E)(\sigma_2 E)\cdots(\sigma_n E) \triangleq K' = K$

① K' is normal ext $/ k$

Pf. $\tau : K' \hookrightarrow E^a$

$$\begin{tikzcd}
E^a \arrow[d, dash] \\
E \arrow[d, dash] \\
k
\end{tikzcd}$$

$\Longrightarrow$ $(\tau\sigma_1, \dots, \tau\sigma_n) = (\sigma_1, \dots, \sigma_n)$

permutation.

- $\tau\sigma_i : E \to \sigma_i E \to \tau\sigma_i E$. $\because$ an embedding $E \hookrightarrow E^a$.

  $\therefore \tau\sigma_i = \sigma_j$ for some j

- if $\tau\sigma_i = \tau\sigma_j$. i.e. $\tau\sigma_i(x) = \tau\sigma_j(x) \quad \forall x \in E$

  $\therefore \sigma_i(x) = \sigma_j(x). \quad \forall x \in E$

  $\therefore \sigma_i = \sigma_j$.

$\therefore \tau\tilde{K} = \tau\big((\sigma_1 E)(\sigma_2 E)\cdots(\sigma_n E)\big) = K'$ $\qquad \therefore K'$ normal

② Any normal extension $N \subseteq E^a$, containing E, must contain $\sigma_i E$, $\forall i$

$\Longrightarrow N \supseteq K'$ $\therefore$ the K' is the smallest

$$\begin{tikzcd}
 & E^a \arrow[dd, dash] \\
N \arrow[ur, "\exists\tilde{\sigma}_i\ (\text{Thm } 2.8)"] \arrow[d, dash, "\text{normal}"] & \\
E \arrow[r, "\sim"] \arrow[d, dash] & \sigma_i E \\
k \arrow[r, equal] \arrow[uu, bend left, "\text{normal}"] & k
\end{tikzcd}$$

$\text{im}(\tilde{\sigma}_i) = N \supseteq \sigma_i E$

$\square$

Rmk. E. sep ext $/k$ $\quad \{\sigma_1, \dots, \sigma_n\}$ all distinct embedding of $E \hookrightarrow E^a$

$\therefore$ $\sigma_i E$. sep $/k$

$$
\begin{tikzcd}
 & (\sigma_i E)(\sigma_j E) & \\
\sigma_i E \arrow[ur, dash] & & \sigma_j E \arrow[ul, dash] \\
 & k \arrow[ul, dash, "\text{sep}"] \arrow[ur, dash, "\text{sep}"'] \arrow[uu, dash, "\text{sep}"'] &
\end{tikzcd}
$$

$\Longrightarrow$ $K = (\sigma_1 E) \cdots (\sigma_n E)$. smallest normal ext $/k$ containing E and sep.