Vol 为什么是测度?

$\forall f \in C_0^\infty(\mathbb{R}^n) \quad \forall x \in \mathbb{R}^n$

$$f(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\langle x-y, \nabla f(y) \rangle}{|y-x|^n} \, dy$$

$$|f(x)| \leq \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|\nabla f(y)|}{|y-x|^{n-1}} \, dy$$

在流形中也有对应.

This estimate also holds for Lipschitcz.

$f \in Lip_0(\mathbb{R}^n)$.

光滑函数逼近?

i.e. $\begin{cases} f \equiv 0, \ \forall x. \ |x| \gg 1 \\ \dfrac{|f(x) - f(y)|}{|x-y|} \leq \Lambda < +\infty \end{cases}$

用来prove这个.

(i.e. $\exists f_i \in C_0^\infty(\mathbb{R}^n)$. s.t. $\|f_i - f\|_{L^\infty} \to 0$ and

$\|\nabla f_i - \nabla f\|_{L^p} \to 0 \quad \forall\, 0 < p < +\infty$)

New proof of Sobelev inequality

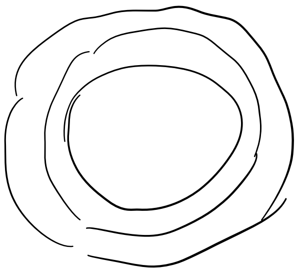
$$\|f\|_{L^{\frac{np}{n-p}}} \leq C(n,p) \|\nabla f\|_{L^p}$$

Case $p=1$: $\forall s>0$ Consider

$$\int_{\mathbb{R}^n} \frac{|\nabla f(y)|}{|y-x|^{n-1}}\,dy = \int_{|y-x|\le s} \frac{|\nabla f(y)|}{|y-x|^{n-1}}\,dy$$

$$+ \int_{|y-x|\ge s} \frac{|\nabla f(y)|}{|y-x|^{n-1}}\,dy$$

$$\le \sum_{0<2^{-i}\le s} \int_{2^{-i}\le |y-x|\le 2^{-i+1}} \frac{|\nabla f(y)|}{|y-x|}\,dy$$



$$+ s^{1-n}\int_{\mathbb{R}^n} |\nabla f(y)|\,dy$$

$$\le \sum_{0<2^{-i}\le s} 2^{i(n-1)} \int_{|y-x|\le 2^{-i+1}} |\nabla f(y)|\,dy$$

$$+ s^{1-n}\int_{\mathbb{R}^n} |\nabla f(y)|\,dy$$

用极大函数

$$\le \sum_{0<2^{-i}\le s} 2^{-i}\,\overbrace{C(n)}^{\text{关于n的常数}} \int_{|y-x|\le 2^{-i+1}} |\nabla f|(y)\,dy$$

$$+ s^{1-n}\int_{\mathbb{R}^n} |\nabla f|(y)\,dy$$

极大函数定义

$$\leq \sum_{0<2^{i}\leq s} 2^{i}\cdot C(n)\, M(|\nabla f|)(x)$$

$$+\ s^{1-n}\int_{\mathbb{R}^n} |\nabla f|(y)\,dy$$

$$\leq C(n)\cdot s\, M(|\nabla f|)(x) + s^{1-n}\int_{\mathbb{R}^n} |\nabla f|(y)\,dy$$

常数不写了

choose s. s.t. $s^{-n}\int_{\mathbb{R}^n} |\nabla f|(y)\,dy = M(|\nabla f|)(x)$

$$\Rightarrow \int_{\mathbb{R}^n} \frac{|\nabla f|(y)}{|y-x|^{n-1}}\,dy \leq C(n)\cdot M(|\nabla f|)^{1-\frac{1}{n}}(x)\left(\int_{\mathbb{R}^n} |\nabla f|(y)\,dy\right)^{1/n}$$

Since $|f(x)| \leq \int_{\mathbb{R}^n} \frac{|\nabla f|(y)}{|y-x|^{n-1}}\,dy$

$\Rightarrow \forall t>0$

$$\mathrm{Vol}\left(\{x : |f(x)|>t\}\right)$$

$$\leq \mathrm{Vol}\left(\left\{x : C(n)\cdot M(|\nabla f|)^{1-\frac{1}{n}}(x)\left(\int_{\mathbb{R}^n} |\nabla f|(y)\,dy\right)^{1/n} > t\right\}\right)$$

$$\leq \mathrm{Vol}\left(\left\{x : M(|\nabla f|) > C(n)\, t^{\frac{n}{n-1}}\left(\int_{\mathbb{R}^n} |\nabla f(y)|\,dy\right)^{-\frac{1}{n-1}}\right\}\right)$$

极大函数定理

$$\leqslant C(n)\, t^{-\frac{n}{n-1}} \left(\int_{\mathbb{R}^n} |\nabla f|(y)\, dy \right)^{\frac{1}{n-1}} \int_{\mathbb{R}^n} |\nabla f|(y)\, dy$$

$$= C(n)\, t^{-\frac{n}{n-1}} \left(\int_{\mathbb{R}^n} |\nabla f(y)|\, dy \right)^{\frac{n}{n-1}}$$

For any integer k define

$$f_k(x) = \begin{cases} 2^{k-1}, & \text{if } |f(x)| \geq 2^k \\ |f(x)| - 2^{k-1} & \text{if } 2^{k-1} \leq |f(x)| < 2^k \\ 0, & \text{if } |f(x)| < 2^{k-1} \end{cases}$$

$\Rightarrow$ f_k is Lipschitz and

$$\nabla f_k(x) = \begin{cases} 0 & \text{if } |f(x)| \geq 2^k \\ |\nabla f(x)| & \text{if } 2^{k-1} \leq |f(x)| < 2^k \\ 0 & \text{if } |f(x)| < 2^{k-1} \end{cases}$$

$$\mathrm{Vol}\left(\{x : |f(x)| \geq 2^k\} \right) \underset{\Delta}{\leq} \mathrm{Vol}\left(\{x : |f_k(x)| \geq 2^{k-1}\} \right)$$

$$\leq C(n)\, 2^{-(k-1)\frac{n}{n-1}} \left(\int_{\mathbb{R}^n} |\nabla f_k|\, dy\right)^{\frac{n}{n-1}}$$

$$\leq C(n)\, 2^{-k\frac{n}{n-1}} \left(\int_{2^{k-1} \leq |f(x)| < 2^k} |\nabla f|(y)\, dy\right)^{\frac{n}{n-1}}$$

Recall. $\int_{\mathbb{R}^n} |f|^p\, d\mu = \int_0^\infty p t^{p-1} \mu(\{x : |f|(x) > t\})\, dt$

$$\simeq \sum_{-\infty \leq i \leq +\infty} 2^{ip} \mu(\{x : |f(x)| > 2^i\})$$

last :

$$\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \leq C(n) \sum_{-\infty < k < +\infty} 2^{\frac{kn}{n-1}} \mathrm{Vol}(\{x : |f(x)| > 2^k\})$$

$$\leq C(n) \sum_{-\infty < k < +\infty} \left(\int_{2^{k-1} \leq |f| < 2^k} |\nabla f|(y)\, dy\right)^{\frac{n}{n-1}}$$

Recall. $a_1^p + a_2^p + \cdots + a_k^p \leq (a_1 + \cdots + a_k)^p$. if $p > 1$

$$\Rightarrow \int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \leq C(n) \left(\int_{\mathbb{R}^n} |\nabla f|(y)\, dy\right)^{\frac{n}{n-1}} \quad \square$$

$\|f\|_{L^{\frac{np}{n-p}}} \leq C(n,p) \|\nabla f\|_{L^p}$ $\quad$ $p=1$.

Newman Sobolev inequality

Thm. Let $B \subset \mathbb{R}^n$ be a ball of radius $r>0$.

$\forall\ 1 \le p < n$. then.

$$\|f - f_B\|_{L^q} \le C(n,p)\, r\, \|Df\|_{L^p}$$

$$\forall f \in C^\infty(\overline{B})$$

where $q = \dfrac{np}{n-p}$. $\quad f_B = \dfrac{1}{\mathrm{Vol}(B)} \int_B f(x)\,dx$

Lemma. Let $B \subset \mathbb{R}^n$ be a ball of radius $r>0$

Then. $\displaystyle |f(x) - f_B(x)| \le C(n) \int_B \frac{|Df|(y)}{|x-y|^{n-1}}\,dy$

$$\forall f \in C^\infty(B),\ \forall x \in B$$

and $\forall x, y \in B$

$$|f(x) - f(y)| \le C(n)\, r \Big(M(|Df|)(x) + M(|Df|)(y) \Big)$$

pf. (1) $\forall x, y \in B$

$$f(x) - f(y) = -\int_0^{|x-y|} \partial_\rho f\Big(x + \rho \frac{y-x}{|y-x|}\Big)\,d\rho$$

Define $F(z) = \begin{cases} |\nabla f|(z) & \text{if } z \in B \\ 0, & z \notin B \end{cases}$

梯度的
取max

$$\Rightarrow |f(x) - f(y)| \leq \int_0^{+\infty} F\left(x + \rho \frac{y-x}{|y-x|}\right) d\rho$$

Integrating. with respect to $y \in B$

$$|f(x) - f_B| \leq \left| f(x) - \frac{1}{\mathrm{Vol}(B)} \int_B f(y)\, dy \right|$$

$$\leq \frac{1}{\mathrm{Vol}(B)} \int_B |f(x) - f(y)|\, dy$$

$$\leq \frac{1}{\mathrm{Vol}(B)} \int_B \int_0^{+\infty} F\left(x + \rho \frac{y-x}{|y-x|}\right) d\rho\, dy$$

$$\leq C(n) r^{-n} \int_{\{y : |x-y| \leq 2r\}} \int_0^{+\infty} F\left(x + \rho \frac{y-x}{|y-x|}\right) d\rho\, dy$$

$$\leq C(n) r^{-n} \int_0^{2r} \int_{S^{n-1}} \int_0^{+\infty} F(x + \rho\theta)\, s^{n-1}\, d\rho\, ds\, d\theta$$

$$\leq C(n) \int_{S^{n-1}} \int_0^{+\infty} F(x + \rho\theta)\, d\rho\, d\theta.$$

$$\leq C(n)\int_{S^{n-1}}\int_0^{+\infty} \frac{F(x+\rho\theta)}{\rho^{n-1}}\,\rho^{n-1}\,d\rho\, d\theta$$

$$\leq C(n)\int_B \frac{|\nabla f|(y)}{|x-y|^{n-1}}\,dy$$

超出部分默认为0

② $$\int_B \frac{|\nabla f|(y)}{|x-y|^{n-1}}\,dy \leq \sum_{2^i\leq 2r}\int_{\{2^{i-1}\leq|y-x|\leq 2^i\}\cap B}\frac{|\nabla f|(y)}{|y-x|^{n-1}}\,dy$$

$$\leq \sum_{2^i\leq 2r} 2^{-(i-1)(n-1)}\int_{\{|y-x|\leq 2^i\}\cap B}|\nabla f|(y)\,dy$$

$$\leq \sum_{2^i\leq 2r} C(n)\cdot 2^i ⨍_{\{|y-x|\leq 2^i\}\cap B}|\nabla f|(y)\,dy$$

用极大函数

$$\leq \sum_{2^i\leq 2r} C(n)\, 2^i\, M(|\nabla f|)(x)$$

$$\leq C(n)\cdot r\cdot M(|\nabla f|)(x)$$

$$\Rightarrow |f(x)-f_B|\leq C(n)\, r\, M(|\nabla f|)(x)$$

$$\Rightarrow |f(x)-f(y)|\leq C(n)\, r\big(M(|\nabla f|)(x)+M(|\nabla f|)(y)\big)$$

$\square$

Poincaré inequality $\forall f \in C^\infty(B)$. radius $(B) = r$.

then $\fint_B |f(x) - \fint_B f(y)dy|^2 dy \leq C(n)\, r^2 \fint_B |\nabla f|^2(y)\,dy$

Pf. By $|f(x)-f(y)| \leq C(n)\, r \left(M(|\nabla f|)(x) + M(|\nabla f|)(y) \right)$

$$\left| f(x) - \fint_B f(y)dy \right| \leq \fint_B |f(x) - f(y)|\, dy$$

$$\leq \fint_B C(n)\, r \left(M(|\nabla f|)(x) + M(|\nabla f|)(y) \right) dy$$

$$\leq C_n r\, M(|\nabla f|)(x)$$
$$+ C(n)\, r \left(\fint_B \left(M(|\nabla f|)(y) \right)^2 dy \right)^{1/2}$$

$$\leq C_n r\, M(|\nabla f|)(x)$$
$$+ C(n)\, r \left(\fint_B |\nabla f|^2(y)\, dy \right)^{1/2}$$

$$\Rightarrow \fint_B \left| f(x) - \fint_B f(y)dy \right|^2 dx$$

$$\leq C(n)\, r^2 \fint_B |\nabla f|^2 dy$$

Sobolev 空间

$\forall \Omega \subseteq \mathbb{R}^n.\ f \in C^\infty(\Omega).$ $\quad p \geq 1$

Define $\|f\|_{W^{1,p}} = \|f\|_{L^p} + \|\nabla f\|_{L^p}$ 为一个范数

$\forall f, g \in C^\infty(\Omega)$

$$0 \leq \|f-g\|_{W^{1,p}} \leq \|f\|_{W^{1,p}} + \|g\|_{W^{1,p}}.$$

Denote $W^{1,p}(\Omega)$ is the completion of $C^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{1,p}}$

Sobolev 空间

$\|\ \| \Rightarrow$ 度量 $\Rightarrow$ 完备化.

Fact. $f \in W^{1,p}(\Omega)$ iff $f \in L^p(\Omega)$. and $\exists f_i \in C^\infty(\Omega)$ s.t.

$$\|f_i - f_j\|_{W^{1,p}} \leq \varepsilon_j \to 0 \quad i \geq j$$

and $\|f_i - f\|_{L^p} \to 0$

Define. $\nabla f \triangleq L^p\text{-limit of } \nabla f_i$

与f本身有关

☆ 光滑函数逼近

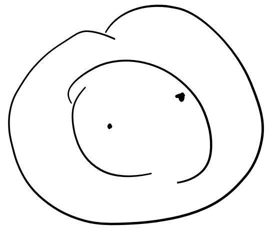
Sobolev Space in metric measure Space?

$\mathbb{R}^n$:

If $f \in C^\infty(\Omega)$: $\Omega = B \subset \mathbb{R}^n$ radius $(B) = r$.

$$|f(x) - f(y)| \leq C(n)\, r \left(M(|\nabla f|)(x) + M(|\nabla f|(y)) \right).$$

$\forall x, y \in B$. choose $\tilde{B} = B_{|x-y|}(z) \subseteq B$.



$$\Rightarrow |f(x) - f(y)| \leq C(n)\, |x-y| \left(M(|\nabla f|)(x) + M(|\nabla f|)(y) \right)$$

Fact: If $f \in W^{1,p}$ $p > 1$ $\Rightarrow$

$$M(|\nabla f|)(x) \in L^p. \quad M(|\nabla f|)(y) \in L^p$$

△ **Fact** $f \in W^{1,p} \overset{\Delta}{\Longleftrightarrow} f \in L^p$ $\exists g \in L^p$ s.t. a.e. $x, y \in B$

$$|f(x) - f(y)| \leq |x-y| \left(g(x) + g(y) \right)$$

另一种~~弱~~定义"梯度"

上梯度. 线积分

Def. (Hajłasz)

Let (X, d, μ). m.m.s with bounded diameter.

μ. finite Borel measure. Given $p>1$ we say

$f \in L^{1,p}(X,\mu)$ iff.

(1) f is measurable

(2) $\exists E \subset X$. and nonnegative $g \in L^p(X,\mu)$

s.t $\mu(E)=0$, $\forall x, y \in X \setminus E$. the following holds

$$|f(x)-f(y)| \leq d(x,y)\,(g(x)+g(y))$$

and $W^{1,p}(X,\mu) = \{ f \in L^{1,p} ; f \in L^p \}$

Def. $\|f\|_{L^{1,p}} = \inf_g \|g\|_{L^p}$ for $f \in L^{1,p}(X,\mu)$

$\|f\|_{W^{1,p}} = \|f\|_{L^{1,p}} + \|f\|_{L^p}$

Rmk If $f \in W^{1,p}$ then

(1) $\exists$ minimizer g_0. s.t. $\|g_0\|_{L^p} = \inf_g \|g\|_{L^p}$

(2). $W^{1,p}(X, \mu)$ is a Banach space

(3). $\exists$ Lipschitz $\underline{f_i}$. s.t. $\|f - f_i\|_{W^{1,p}} \to 0$.

$$\frac{|f_i(x) - f_i(y)|}{d(x,y)} \leq \Lambda < +\infty \quad \forall x, y \in X.$$