W2.L1

曲线论基本定理. $\kappa(s)$ 可微, $\tau(s)$ 连续

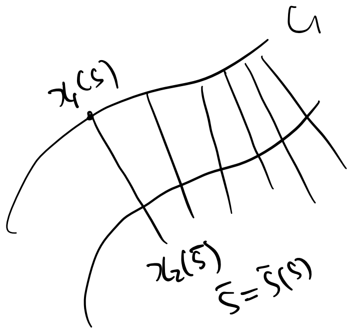
ODE.
$$\begin{cases} x = x(s) \\ \dot{x}(s) = T(s) \\ \dot{T}(s) = \kappa(s)N(s) \\ \dot{N}(s) = -\kappa(s)T(s) + \tau(s)B(s) \\ \dot{B}(s) = -\tau(s)N(s). \end{cases}$$

$\{x(s); T(s), N(s), B(s)\}$

初值条件. $\begin{cases} x(0)=x_0 \\ T(0)=T_0 \\ N(0)=N_0 \\ B(0)=B_0 \end{cases}$

$\Rightarrow$ 有解.

Bertrand. 曲线



$x(s)$ 与 $x_2(\bar{s})$ 处 相同 N

(法线) 证. 对应点间距离相同

对应点切向呈定角

Pf. $C_1: x = x_1(s)$, s 为弧长 $\qquad N_1(s)$

$C_2: x = x_2(\bar{s})$ $\quad \bar{s}$ 为弧长 $\qquad N_2(\bar{s}(s))$

对应关系 $\bar{s} = \bar{s}(s)$ $\quad$ 有 $N_2(\bar{s}(s)) = \varepsilon N_1(s)$

$$x_2(\bar{s}(s)) = x_1(s) + \lambda(s) N_1(s)$$

对 s 求导. $\quad T_2(\bar{s})\dfrac{d\bar{s}}{ds} = T_1(s) + \dot{\lambda} N_1 + \lambda(-\kappa_1(s)T_1(s) + \tau(s)B_1(s))$

点乘 $N_1(s)$ $\Rightarrow$ $0 = 0 + \dot{\lambda} + 0$ $\Rightarrow$ $\dot{\lambda} = 0$ $\Rightarrow$ $\lambda(s)$ 为常数.

$$\frac{d}{ds}\left(T_1(s)\cdot T_2(\bar{s}(s))\right) = k(s)\underline{N_1(s)\cdot T_2(\bar{s}(s))} + \underline{T_1(s)\cdot N_2(\bar{s})}\frac{d\bar{s}}{ds} = 0$$

$\Rightarrow T_1(s)\cdot T_2(\bar{s}(s)) = \cos\theta$ 为常数

曲面论

$D \subset E^2$. D为连通区域 坐标为 u^1, u^2

$x: D \longrightarrow E^3$ 同胚

$$x(u,v) = \left(x^1(u^1,u^2),\ x^2(u^1,u^2),\ x^3(u^1,u^2)\right)$$

C^k 曲面 M. $(k\geq 2)$：满足 ① $x^i(u^1,u^2)$ 具有k阶连续偏导.

② Jocobi 矩阵

$$\begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^2}{\partial u^1} & \frac{\partial x^3}{\partial u^1} \\ \frac{\partial x^1}{\partial u^2} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^3}{\partial u^2} \end{pmatrix}_{2\times 3} \triangleq \left(\frac{\partial x^i}{\partial u^\alpha}\right)_{\substack{1\leq \alpha\leq 2\\ 1\leq i\leq 3}}$$

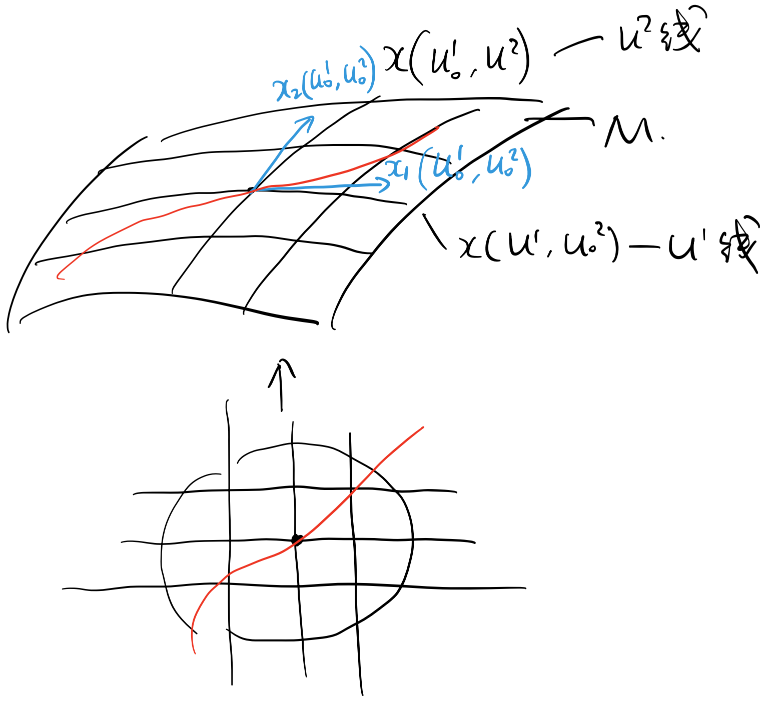
$x_1 = \frac{\partial x}{\partial u^1}$

$x_2 = \frac{\partial x}{\partial u^2}$

的秩处处为2

$\Leftrightarrow$ x_1 与 x_2 线性无关

$\Leftrightarrow$ $x_1 \times x_2 \neq 0$



构成了参数曲线网

span$\{x_1, x_2\}$

切平面. $T_{x(u_0^1,u_0^2)}M$

单位法向量. $n(u^1,u^2) = \dfrac{x_1 \times x_2}{|x_1 \times x_2|}(u^1,u^2)$ $\Rightarrow$ 构成向量场(光滑)

参数表示不唯一.

参数变换. $\bar{u}^\alpha = \bar{u}^\alpha(u^1,u^2)$

$\dfrac{\partial(\bar{u}^1,\bar{u}^2)}{\partial(u^1,u^2)} \neq 0.$ 则切平面不变. 单位法向量最多差一个符号.

若 $\dfrac{\partial(\bar{u}^1,\bar{u}^2)}{\partial(u^1,u^2)} > 0 \Rightarrow$ 单位法向量不变

$$x = \underbrace{x(u^1,u^2) = \bar{x}(\bar{u}^1,\bar{u}^2)}\ = \bar{x}(\bar{u}^1(u^1,u^2),\bar{u}^2(u^1,u^2))$$

$$\left\{\begin{aligned} \frac{\partial x}{\partial u^1} &= \frac{\partial \bar{x}}{\partial \bar{u}^1}\frac{\partial \bar{u}^1}{\partial u^1} + \frac{\partial \bar{x}}{\partial \bar{u}^2}\frac{\partial \bar{u}^2}{\partial u^1} \\ \frac{\partial x}{\partial u^2} &= \frac{\partial \bar{x}}{\partial \bar{u}^1}\frac{\partial \bar{u}^1}{\partial u^2} + \frac{\partial \bar{x}}{\partial \bar{u}^2}\frac{\partial \bar{u}^2}{\partial u^2} \end{aligned}\right.$$

$\underset{\left(\frac{\partial x}{\partial u^1},\frac{\partial x}{\partial u^2}\right)}{(x_1,x_2)}$ 与 $\underset{\left(\frac{\partial \bar{x}}{\partial \bar{u}^1},\frac{\partial \bar{x}}{\partial \bar{u}^2}\right)}{(\bar{x}_1,\bar{x}_2)}$ 相互线性表示

$\Rightarrow \operatorname{span}\{x_1,x_2\} = \operatorname{span}\{\bar{x}_1,\bar{x}_2\}$

$$x_1 \times x_2 = (\bar{x}_1 \times \bar{x}_2)\frac{\partial(\bar{u}^1,\bar{u}^2)}{\partial(u^1,u^2)}$$

曲面 M 上的一条曲线 C.

$$I \longrightarrow x(I) \subset M.$$

$$t \longmapsto (u^1(t), u^2(t)) \xrightarrow{x} x(u^1(t),u^2(t))$$

<u>C的弧长.</u> $s(t) = \int_0^t \left|\dfrac{dx}{dt}\right| dt$

$$\Rightarrow \frac{ds}{dt} = \left|\frac{dx}{dt}\right|$$

即, 在 u^1, u^2 平面中. $du^1, du^2 \Rightarrow M$ 中的 ds 大小

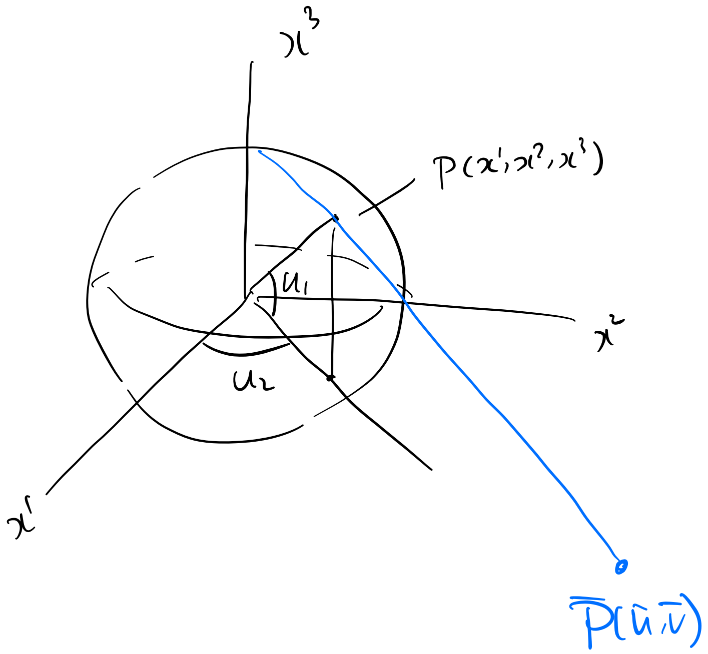
$$\mathrm{I} = ds^2 = |dx|^2 = |x_1 du^1 + x_2 du^2|^2 = x_1 \cdot x_1 \, du^{1^2} + 2 x_1 x_2 \, du^1 du^2 + x_2 \cdot x_2 \, du^{2^2}$$

令 $x_\alpha x_\beta = g_{\alpha\beta}$ $\qquad = g_{11} du^{1^2} + 2 g_{12} du^1 du^2 + g_{22} du^{2^2}$

(第一基本形式) $\qquad = g_{\alpha\beta} \, du^\alpha du^\beta$ 求和指标 △

$$g_{11} = E. \quad g_{12} = F \quad g_{22} = G$$

球面.



$$-\frac{\pi}{2} \leqslant u^1 \leqslant \frac{\pi}{2}$$

$$0 \leqslant u^2 \leqslant 2\pi$$

$$x^3 = r \cdot \sin u^1$$

$$x^1 = r \cos u^1 \cos u^2$$

$$x^2 = r \cos u^1 \sin u^2$$

$$x_1 = (-r \sin u^1 \cos u^2, \ -r \sin u^1 \sin u^2, \ r \cos u^1)$$

$$x_2 = (-r \cos u^1 \sin u^2, \ r \cos u^1 \cos u^2, \ 0)$$

$$E = r^2. \ F = 0. \ G = r^2 \cos^2 u^1 .$$

$$\mathrm{I} = r^2 (du^1)^2 + r^2 \cos^2 u^1 (du^2)^2 = \mathrm{I}((du^1, du^2), (du^1, du^2))$$

$$= \mathrm{I}(du^1, du^2)$$

球极投影.

$$\begin{cases} x^1 = \dfrac{2r^2\bar{u}_1}{r^2+(\bar{u}^1)^2+(\bar{u}^2)^2} \\ x^2 = \dfrac{2r^2\bar{u}_2}{r^2+(\bar{u}^1)^2+(\bar{u}^2)^2} \\ x^3 = \dfrac{r\left((\bar{u}^1)^2+(\bar{u}^2)^2-r^2\right)}{r^2+(\bar{u}^1)^2+(\bar{u}^2)^2} \end{cases}$$

$$x_1 = \left(\frac{2r^2\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right) - 2\bar{u}^1(2r^2\bar{u}^1)}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2},\ \frac{-2\bar{u}^1(2r^2\bar{u}_2)}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2},\ \frac{2r\bar{u}^1\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right) - 2\bar{u}^1 r\left((\bar{u}^1)^2+(\bar{u}^2)^2-r^2\right)}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2} \right)$$

$$= \left(\frac{2r^2\left(r^2+(\bar{u}^2)^2-(\bar{u}^1)^2\right)}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2},\ \frac{-4r^2(\bar{u}^1\bar{u}^2)}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2},\ \frac{2r\bar{u}^1(2r^2)}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2} \right)$$

$$x_2 = \left(\frac{-4r^2(\bar{u}^1\bar{u}^2)}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2},\ \frac{2r^2\left(r^2+(\bar{u}^1)^2-(\bar{u}^2)^2\right)}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2},\ \frac{2r\bar{u}^2(2r^2)}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2} \right)$$

$$g_{11} = \frac{4r^4}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^4}\left(\left(r^2+(\bar{u}^2)^2-(\bar{u}^1)^2\right)^2 + 4(\bar{u}^1\bar{u}^2)^2 + 4r^2(\bar{u}^1)^2 \right)$$

$$= \frac{4r^4}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^4}\left(r^2+(\bar{u}^2)^2+(\bar{u}^1)^2\right)^2 = \frac{4r^4}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2}$$

同理 $g_{22} = \dfrac{4r^4}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2}$

$$g_{12} = x_1\cdot x_2 = \frac{4r^4}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2}\left(\left(r^2+(\bar{u}^2)^2-(\bar{u}^1)^2\right)(-2\bar{u}^1\bar{u}^2) + \left(r^2+(\bar{u}^1)^2-(\bar{u}^2)^2\right)(-2\bar{u}^1\bar{u}^2) + 4r^2\bar{u}^1\bar{u}^2 \right)$$

$$= \frac{4r^4}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2} \times 0 = 0.$$

$$\Rightarrow \mathrm{I} = \frac{4r^4}{\left(r^2+(\bar{u}^1)^2+(\bar{u}^2)^2\right)^2}\left((d\bar{u}^1)^2+(d\bar{u}^2)^2\right).$$

$\Rightarrow$ 等温参数.

第一基本形式在参数变换下变化？

$$\bar{x}(\bar{u}^1,\bar{u}^2) = \bar{x}(\bar{u}^1(u^1,u^2),\bar{u}^2(u^1,u^2)) = x(u^1(\bar{u}^1,\bar{u}^2),u^2(\bar{u}^1,\bar{u}^2)).$$

$$\bar{u}^\alpha = \bar{u}^\alpha(u^1,u^2)$$

第一基本形式系数

$$I = g_{\alpha\beta}\,du^\alpha du^\beta \overset{?}{=} \bar{g}_{\alpha\beta}\,d\bar{u}^\alpha d\bar{u}^\beta$$

张量

$$\bar{g}_{\alpha\beta} = \bar{x}_\alpha \bar{x}_\beta = x_\gamma \frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\, x_\delta \frac{\partial u^\delta}{\partial \bar{u}^\beta}$$

$$= x_\gamma x_\delta \cdot \left(\frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\frac{\partial u^\delta}{\partial \bar{u}^\beta}\right) = g_{\gamma\delta}\frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\frac{\partial u^\delta}{\partial \bar{u}^\beta}$$

$$\frac{\partial \bar{x}}{\partial \bar{u}^\alpha} = \frac{\partial x}{\partial u^1}\frac{\partial u^1}{\partial \bar{u}^\alpha} + \frac{\partial x}{\partial u^2}\frac{\partial u^2}{\partial \bar{u}^\alpha}$$

$$\Rightarrow \bar{g}_{\alpha\beta}\,d\bar{u}^\alpha d\bar{u}^\beta = g_{\gamma\delta}\frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\frac{\partial u^\delta}{\partial \bar{u}^\beta}\,d\bar{u}^\alpha d\bar{u}^\beta$$

$$= g_{\gamma\delta}\,du^\gamma du^\delta$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$ 正定.

$g_{11}, g_{22} > 0$

Lagrang 恒等式 $(x_1\times x_2)(x_1\times x_2) = (x_1\cdot x_1)(x_2\cdot x_2) - (x_1\cdot x_2)(x_1\cdot x_2)$

$$\det\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (x_1\cdot x_1)(x_2\cdot x_2) - (x_1\cdot x_2)(x_1\cdot x_2) = |x_1\times x_2|^2 > 0$$

活动标架.

$$\{x(u^1,u^2);\ x_1(u^1,u^2),\ x_2(u^1,u^2),\ n(u^1,u^2)\}.$$

自然标架.

Schmidt 正交化

$$\begin{cases} e_1 = \dfrac{x_1}{|x_1|} = \dfrac{x_1}{|g_{11}|} \\ e_2 = \dfrac{x_2 - (x_2\cdot e_1)e_1}{|x_2 - (x_2\cdot e_1)e_1|} \end{cases}$$

$$x_1 = \sqrt{g_{11}}\, e_1$$

$$x_2 = \frac{g_{12}}{\sqrt{g_{11}}} e_1 + \frac{1}{\sqrt{g_{11}}}\sqrt{g_{11}g_{22} - g_{12}^2}\; e_2$$

实际随意正交标架:

$$\begin{cases} x_1 = a_1^1 e_1 + a_1^2 e_2 \\ x_2 = a_2^1 e_1 + a_2^2 e_2 \end{cases}$$

基底用下标. 坐标用上标.
矩阵, 上标对应基底
下标对应结果

$x_\alpha = a_\alpha^\beta e_\beta$ (a_α^β) 非退化的系数矩阵. $\{e_1, e_2\}$ 幺正

$$dx = x_\alpha du^\alpha = (a_\alpha^\beta e_\beta) du^\alpha$$

$$= \underbrace{a_\alpha^\beta du^\alpha}_{=\,\omega^\beta} e_\beta = \omega^\beta e_\beta.$$

ω^β → 一次微分形式

$$ds^2 = |dx|^2 = {\omega^1}^2 + {\omega^2}^2$$