(E) The argument principle.

(Discussion) Let $D \underset{\text{open}}{\subseteq} \mathbb{C}$ be a Greendomain. and f is a holo. func s.t. $\partial D \subseteq D_f$ and f has an inessential iso. sing. at some $c \in D$

then c is also an inessential sing. of $\frac{f'}{f}$

$$f(z) = \overset{\overset{0}{\neq}}{a_m}(z-c)^m + \cdots \quad (m = \mathrm{ord}_c f \in \mathbb{Z})$$

$\uparrow$

$z (\neq c)$ near c

can be defined on $D_f \cup \{c\}$

$= (z-c)^m g(z)$ where $\underline{g(z)} = \frac{f(z)}{(z-c)^m}$, which has a removable sing. at $z = c$. and $g(c) (= a_m) \neq 0$

$$\frac{f'(z)}{f(z)} = \frac{m}{z-c} + \boxed{\frac{g'(z)}{g(z)}}$$

$\searrow$ holo near c.

(取 log 再微分)

$\Rightarrow$ 只用看 $\frac{m}{z-c}$

Conclusion. If $\mathrm{ord}_c f \in \mathbb{Z}$, then $\mathrm{Res}_c \frac{f'}{f} = \mathrm{ord}_c f$

Suppose, now that $\forall c \in D$ $\mathrm{ord}_c f \in \mathbb{Z}$. (not essential)

(phrased as
f is meromorphic
on D)

By the theorem of residue applied to $\frac{f'}{f}$ on D.

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}\,dz = \sum_{C\in D}\operatorname{Res}_C\frac{f'}{f} = \sum_{C\in D}\operatorname{ord}_C f.$$

$$= \underset{\text{零点个数}}{\underset{\swarrow}{Z_f}} - \underset{\text{pole 个数.}}{\underset{\searrow}{P_f}}$$

"Z_f" = the number of zeros of f in D. $\left(\text{counted with } \underset{\text{their order}}{\underset{\downarrow}{\text{multiplicaties}}}\right)$

"P_f." = . — — poles . — -

More genally, we have

The argument principle Let f and D are as above (f 只有非本质) and ϕ is a holo func s.t. $\overline{D}\subseteq D_\phi$. Then.

$$\frac{1}{2\pi i}\int_{\partial D}\phi(z)\frac{f'(z)}{f(z)}\,dz = \sum_{C\in D}(\operatorname{ord}_C f)\,\phi(C)$$

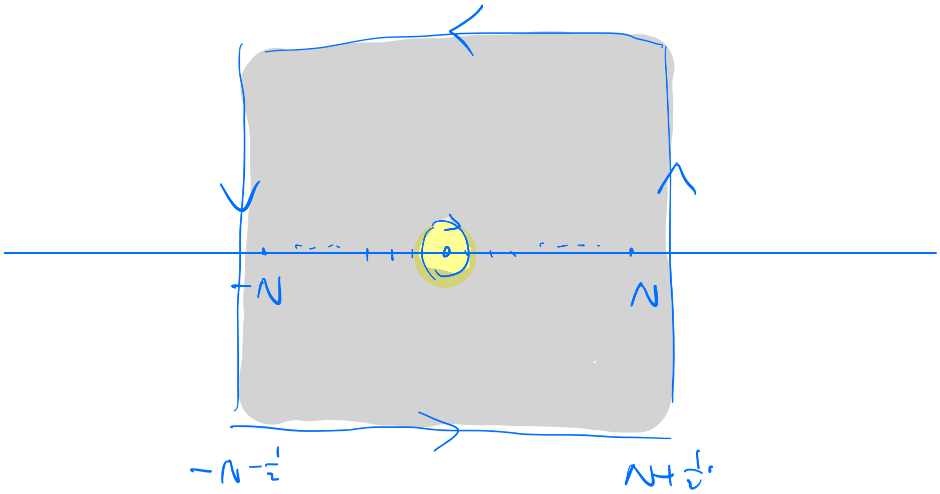
Compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$ via argument principle

$$\sum (\mathrm{ord}_c f) \frac{1}{c^2}$$

$\downarrow$

整数上 $\mathrm{ord}_c f = 1$. 且在整点有零点.

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$$



$$\frac{1}{2\pi i}\int_{\partial C_N} \frac{1}{z^2} \frac{\pi \cos \pi z}{\sin \pi z} dz - \frac{1}{2\pi i}\int_{\partial B_{\frac{1}{2}}} \frac{1}{z^2} \frac{\pi \cos \pi z}{\sin \pi z} dz = 2\sum_{n=1}^{N} \frac{1}{n^2}$$

$\downarrow$ 0

$\downarrow$ 在里面用留数定理

$\downarrow$ $2\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\frac{1}{2\pi i}\int_{\partial B_{\frac{1}{2}}} f(z) dz = \mathrm{Res}(f, 0)$$

$$f(z) = \frac{1}{z^2} \frac{\pi \cos \pi z}{\sin \pi z}$$

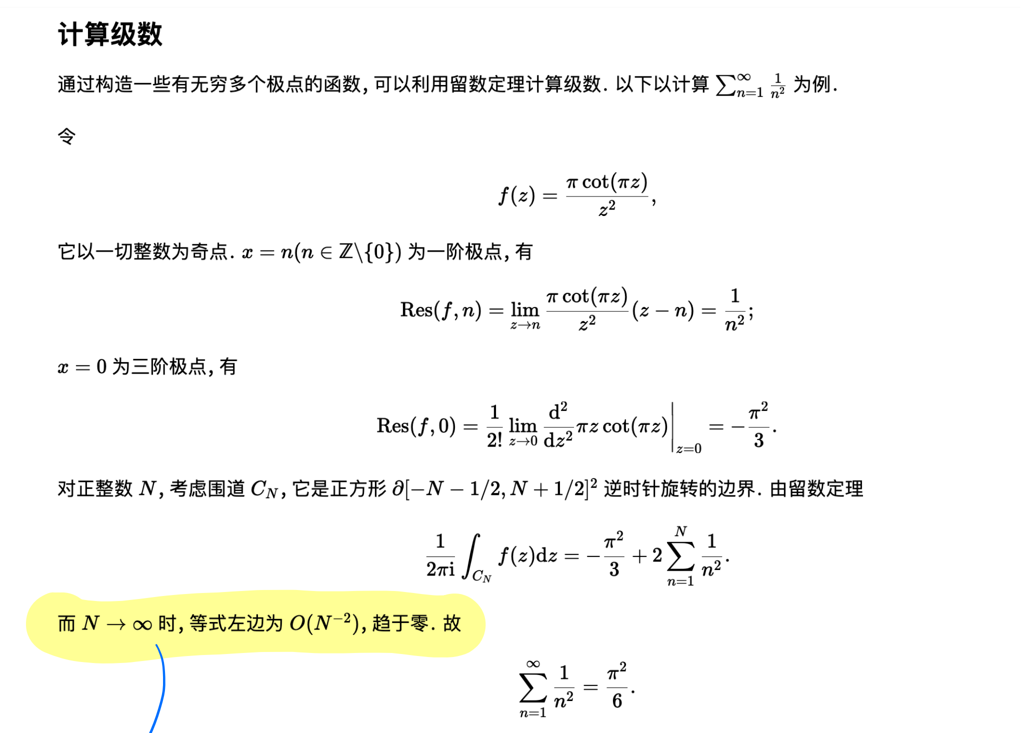
0处 $\frac{1}{z^2} \frac{\pi}{(\pi z) + \frac{1}{3!}(\pi z^3) + \cdots}$

$\boxed{\text{三阶极点.}}$

$$\Longrightarrow \mathrm{Res}(f, 0) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \underset{\uparrow \atop f(z) \cdot z^3}{\pi z \cot(\pi z)} \Big|_{z=0} = -\frac{\pi^2}{3}.$$

$$\Rightarrow 0-\frac{\pi^2}{3}=2\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

直接用留数定理也可以，但不好找函数. 用幅角原理更自然.



计算级数

通过构造一些有无穷多个极点的函数, 可以利用留数定理计算级数. 以下以计算 $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 为例.

令

$$f(z)=\frac{\pi\cot(\pi z)}{z^2},$$

它以一切整数为奇点. $x=n(n\in\mathbb{Z}\backslash\{0\})$ 为一阶极点, 有

$$\operatorname{Res}(f,n)=\lim_{z\to n}\frac{\pi\cot(\pi z)}{z^2}(z-n)=\frac{1}{n^2};$$

$x=0$ 为三阶极点, 有

$$\operatorname{Res}(f,0)=\frac{1}{2!}\lim_{z\to 0}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\pi z\cot(\pi z)\bigg|_{z=0}=-\frac{\pi^2}{3}.$$

对正整数 N, 考虑围道 C_N, 它是正方形 $\partial[-N-1/2,N+1/2]^2$ 逆时针旋转的边界. 由留数定理

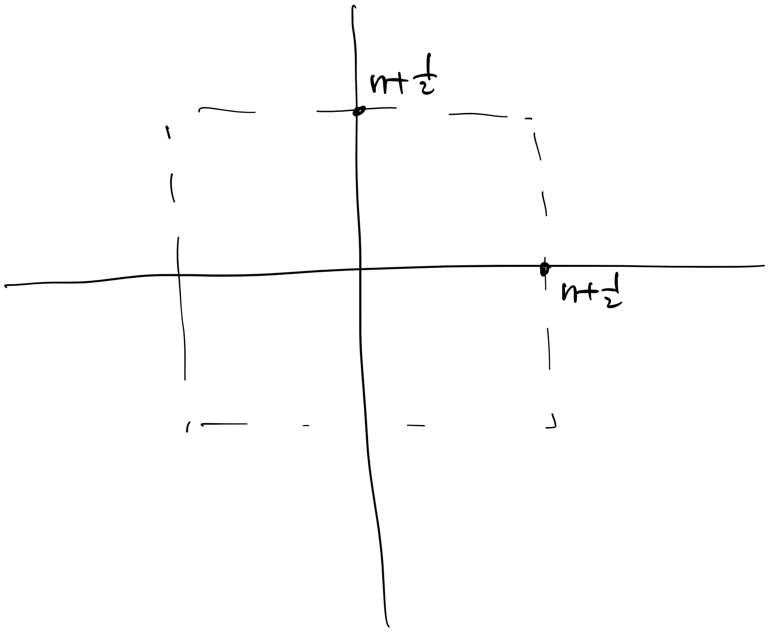
$$\frac{1}{2\pi\mathrm{i}}\int_{C_N}f(z)\mathrm{d}z=-\frac{\pi^2}{3}+2\sum_{n=1}^{N}\frac{1}{n^2}.$$

而 $N\to\infty$ 时, 等式左边为 $O(N^{-2})$, 趋于零. 故

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

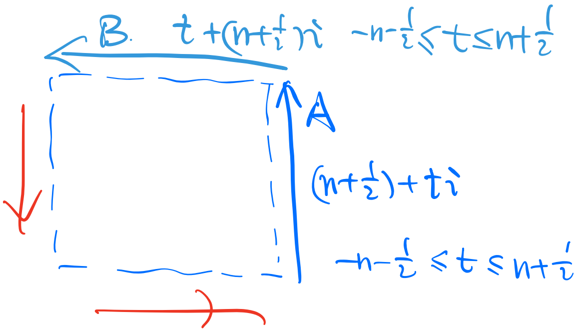
$$\left|\frac{\cos\pi z}{\sin\pi z}\right|\leq ? \quad \text{for } z\in\partial D_n \text{ . where } D_n:=\left\{z\in\mathbb{C}\;\middle|\;\begin{array}{l}|\operatorname{Re}z|<n+\frac{1}{2},\\ |\operatorname{Im}z|<n+\frac{1}{2}\end{array}\right\}$$

$n\to+\infty$



$$\frac{\cos\pi z}{\sin\pi z} = i\,\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}} = i\,\frac{e^{2i\pi z}+1}{e^{2i\pi z}-1}$$

考虑 $e^{2i\pi z}$



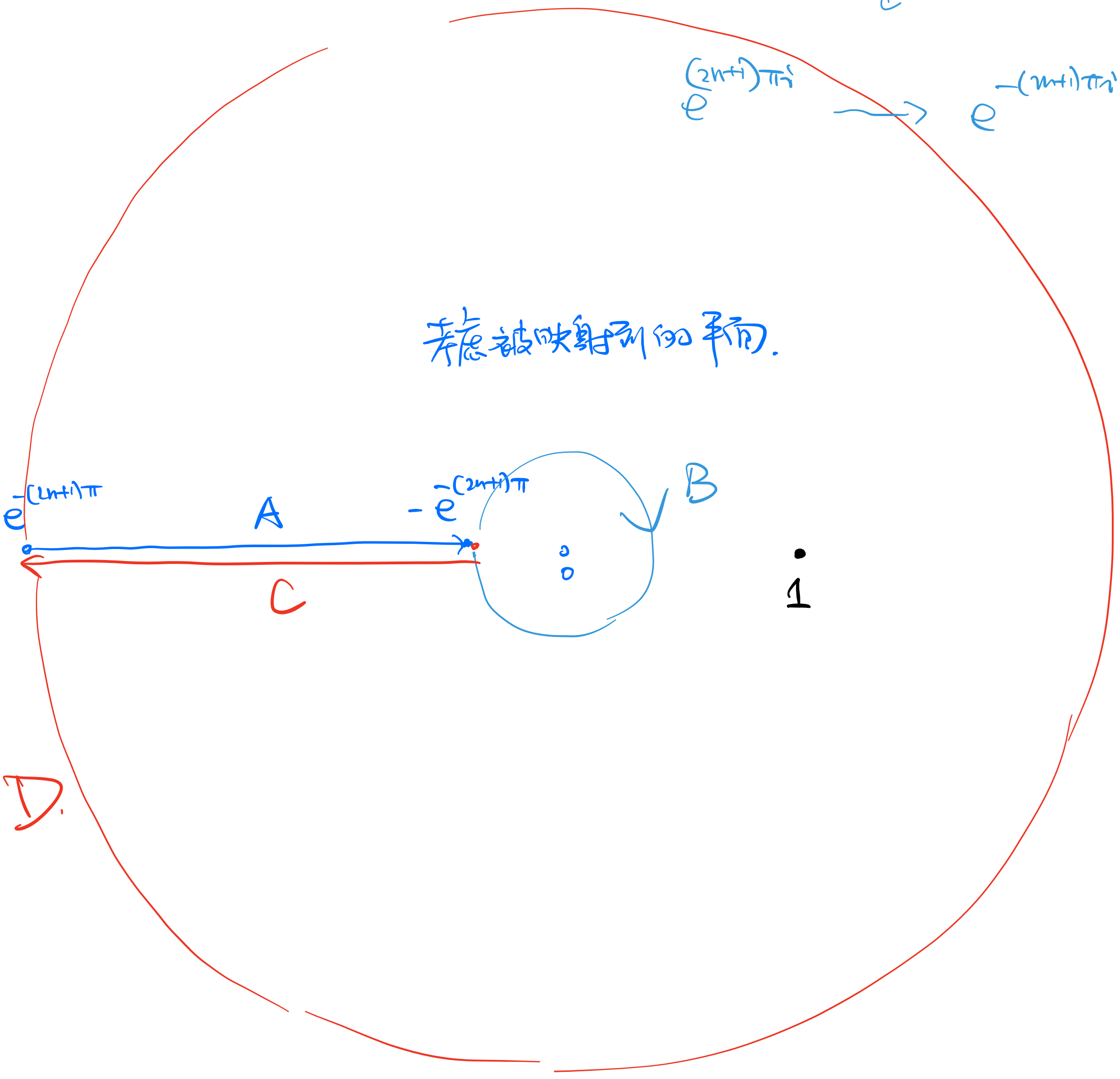
$$A:\ e^{2i\pi z} = e^{2i\pi(n+\frac{1}{2}+ti)} = e^{(2n+1)\pi i}\,e^{-2\pi t} = -e^{-2\pi t}$$

$$-e^{(2n+1)\pi} \longrightarrow -e^{-(2n+1)\pi}$$

$$B:\ e^{2i\pi z} = e^{2i\pi(t+(n+\frac{1}{2})i)} = e^{-(2n+1)\pi}\cdot e^{2i\pi t}$$

$$e^{(2n+1)\pi i} \longrightarrow e^{-(2n+1)\pi i}$$

考虑被映射到的平面.



$$\Rightarrow \left|\frac{\cos\pi z}{\sin\pi z}\right| = \left|1+\frac{2}{e^{2i\pi z}-1}\right| \leqslant 1+\left|\frac{2}{e^{2i\pi z}-1}\right|$$ 有上界.