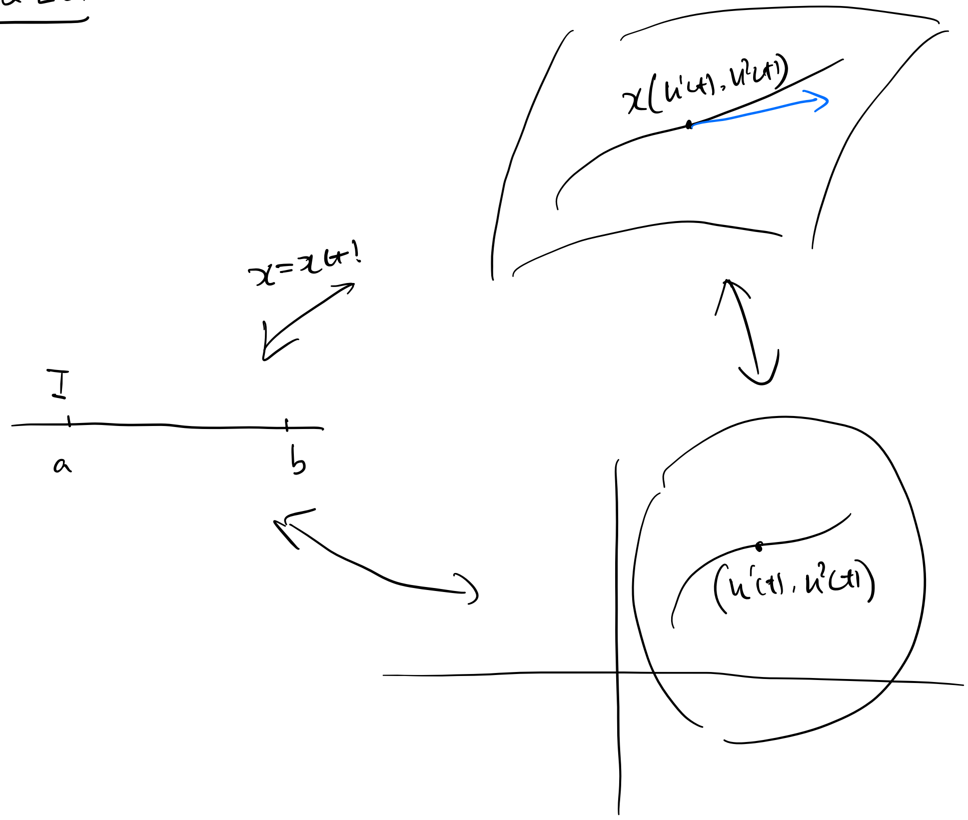
W2L2



$$L = \int_a^b ds = \int_a^b |dx| = \int_a^b \sqrt{(x_\alpha du^\alpha)(x_\beta du^\beta)}$$

$$= \int_a^b \sqrt{g_{\alpha\beta} \frac{du^\alpha}{dt} \frac{du^\beta}{dt}}\, dt$$

切向量 $$\frac{dx}{dt} = \frac{\partial x}{\partial u^1}\frac{du^1}{dt} + \frac{\partial x}{\partial u^2}\frac{du^2}{dt}$$

$$= x_1 \frac{du^1}{dt} + x_2 \frac{du^2}{dt}$$

$$= a^1 x_1 + a^2 x_2 \qquad a^\alpha = \frac{du^\alpha}{dt}$$

切平面中, $\vec{a} = a^1 x_1 + a^2 x_2 = a^\alpha x_\alpha$

$\vec{b} = b^1 x_1 + b^2 x_2 = b^\alpha x_\alpha$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(a^\alpha x_\alpha)(b^\beta x_\beta)}{|a^\alpha x_\alpha|\,|b^\beta x_\beta|} = \frac{g_{\alpha\beta} a^\alpha b^\beta}{\sqrt{g_{\gamma\sigma} a^\gamma a^\sigma}\sqrt{g_{\tau\delta} a^\tau a^\delta}}$$

$$\cos\langle x_1, x_2\rangle = \frac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}$$

正交. $\cos\theta = 0 \iff g_{12} = 0$

$\Rightarrow$ 正交参数网

$$\mathrm{I} = g_{11}(du^1)^2 + g_{22}(du^2)^2$$

"平面" $x = (u^1, u^2, 0)$

$x_1 = (1, 0, 0)$

$x_2 = (0, 1, 0)$

$$ds^2 = (du^1)^2 + (du^2)^2$$

求曲面 $x = x(u^1, u^2)$ 的参数网的平分线的微分方程.

解: 满足条件的曲线为 C.

$$\vec{r}(s) = x(u^1(s), u^2(s))$$

s 弧长参数. $T(s) = \frac{dr}{ds} = x_1\frac{du^1}{ds} + x_2\frac{du^2}{ds}$

T 与 x_1, x_2 或与 x_1, $-x_2$ 的夹角相等.

$$\varepsilon\frac{T\cdot x_1}{|x_1|} = \frac{T\cdot x_2}{|x_2|} \qquad \varepsilon = \pm 1$$

$$\frac{\varepsilon}{\sqrt{g_{11}}}\left(g_{11}\frac{du^1}{ds} + g_{12}\frac{du^2}{ds}\right) = \frac{1}{\sqrt{g_{22}}}\left(g_{12}\frac{du^1}{ds} + g_{22}\frac{du^2}{ds}\right)$$

$$\left(\varepsilon\sqrt{g_{11}} - \frac{g_{12}}{\sqrt{g_{22}}}\right)\frac{du^1}{ds} = \left(\sqrt{g_{22}} - \frac{\varepsilon g_{12}}{\sqrt{g_{11}}}\right)\frac{du^2}{ds}$$

$$\varepsilon\sqrt{g_{11}}\left(1-\cancel{\frac{\varepsilon g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}}\right)\frac{du^1}{ds} = \sqrt{g_{22}}\left(1-\cancel{\frac{\varepsilon g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}}\right)\frac{du^2}{ds}$$

一定 >0. !（正定）

$$\Rightarrow \varepsilon\sqrt{g_{11}}\frac{du^1}{ds} = \sqrt{g_{22}}\frac{du^2}{ds}$$

$$\Rightarrow \varepsilon\sqrt{g_{11}}\,du^1 = \sqrt{g_{22}}\,du^2$$

u^1线的方程？ $du^2=0$. $(u^1=u^1_0)$

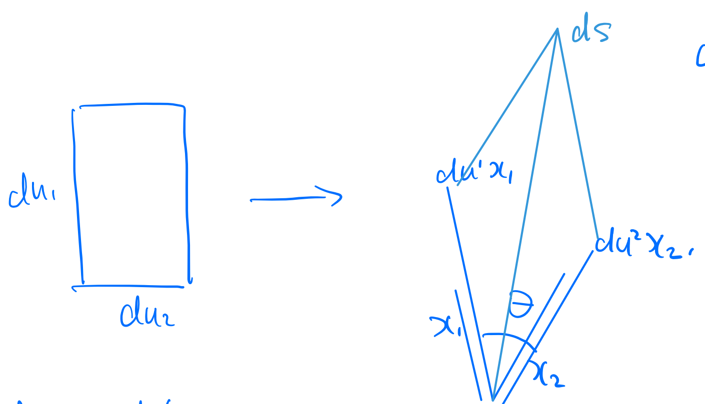
u^2线的方程？ $du^1=0$ $(u^2=u^2_0)$

Ex. 设曲面上含 du^1, du^2 的二次方程 $P(du^1)^2+2Q\,du^1du^2+R(du^2)^2=0$.

$\Rightarrow$确定了两个方向，求证这两方向正交 $\Leftrightarrow$ $ER-2FQ+GP=0$

Pf. $P(du^1)^2+2Q\,du^1du^2+R(du^2)^2=0$. P, Q, R are func of u^1, u^2

$$P\left(\frac{du^1}{du^2}\right)^2+2Q\frac{du^1}{du^2}+R=0 \qquad \frac{du_1^1}{du_1^2}\cdot\frac{du_2^1}{du_2^2}=\frac{R}{P} \qquad \frac{du_1^1}{du_1^2}+\frac{du_2^1}{du_2^2}=-\frac{2Q}{P}$$



$$\frac{du_1^1du_2^1}{du_1^2du_2^2}=\frac{R}{P} \qquad \frac{du_1^1du_2^2+du_2^1du_1^2}{du_1^2du_2^2}=-\frac{2Q}{P}$$

$$ds_\beta = du_\beta^\alpha x_\alpha$$

$$0=ds_1\cdot ds_2=(du_1^\alpha x_\alpha)(du_2^\beta x_\beta)$$

$$= du_1^\alpha x_\alpha \, du_2^\beta x_\beta$$

$$= du_1^\alpha du_2^\beta x_\alpha x_\beta$$

$$= du_1^\alpha du_2^\beta g_{\alpha\beta} = (du_1^1 du_2^2 + du_1^2 du_2^1) \cdot F + du_1^1 du_2^1 E + du_1^2 du_2^2 G$$

$$\Rightarrow 0 = -\frac{2Q}{P} F + \frac{R}{P} E + G$$

$$0 = RE - 2QF + PG \qquad \square$$

面积 $dA = \left|(x_1 du^1) \times (x_2 du^2)\right|$

$$= |x_1 \times x_2| \, du^1 du^2$$

$$|x_1 \times x_2|^2 = (x_1 \times x_2)(x_1 \times x_2) = (x_1 \cdot x_1)(x_2 \cdot x_2) - (x_1 \cdot x_2)^2$$

$$= g_{11} g_{22} - g_{12}^2$$

$$= \det(g_{\alpha\beta})$$

$$\Rightarrow \quad dA = \sqrt{\det(g_{\alpha\beta})} \, du^1 du^2$$

M 面积 $\quad A(M) = \iint_M dA = \iint_D \sqrt{\det g_{\alpha\beta}} \, du^1 du^2$

面积与参数无关？

$$x : D \longrightarrow x(D) = M \in \mathbb{R}^3$$

$$\bar{x} : \bar{D} \longrightarrow \bar{x}(\bar{D}) = M \in \mathbb{R}^3$$

则 D, $\bar{D}$ 也同胚

$$\begin{cases} u^1 = u^1(\bar{u}^1, \bar{u}^2) \\ u^2 = u^2(\bar{u}^2, \bar{u}^2) \end{cases}$$

$$\frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \neq 0$$

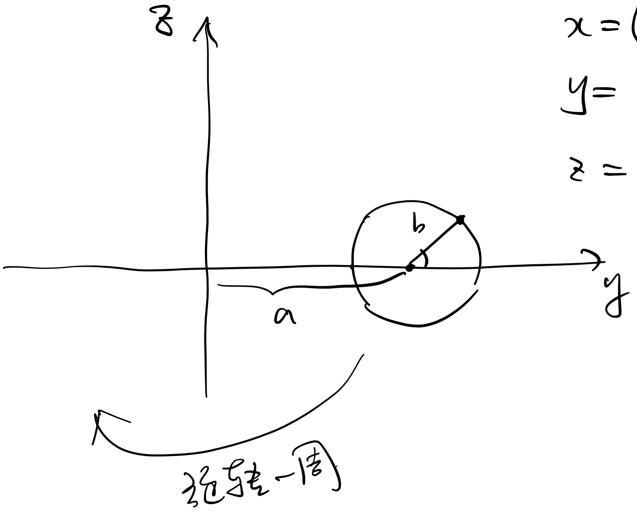
$$\iint_{\bar{D}} |\bar{x}_{\bar{1}} \times \bar{x}_{\bar{2}}| \, d\bar{u}^1 d\bar{u}^2$$

$$\frac{\partial \bar{x}}{\partial \bar{u}}$$

$$\begin{cases} \bar{x}_{\bar{1}} = \bar{x}_1 \dfrac{\partial u^1}{\partial \bar{u}^1} + \bar{x}_1 \dfrac{\partial u^2}{\partial \bar{u}^1} \\ \bar{x}_{\bar{2}} = \bar{x}_2 \dfrac{\partial u^1}{\partial \bar{u}^2} + \bar{x}_2 \dfrac{\partial u^2}{\partial \bar{u}^2} \end{cases}$$

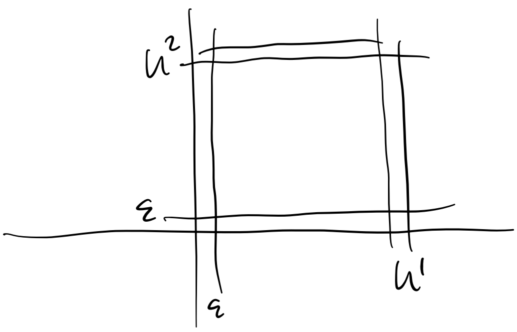
$$|\bar{x}_1 \times \bar{x}_2| = |x_1 \times x_2| \left| \frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \right|$$

$$\Rightarrow |\bar{x}_1 \times \bar{x}_2|\, d\bar{u}^1 d\bar{u}^2 = |x_1 \times x_2|\, du^1 du^2$$



$$x = (a + b\cos u^1)\cos u^2$$
$$y = (a + b\cos u^1)\sin u^2$$
$$z = b\sin u^1$$

$$0 \leq u^1, u^2 \leq 2\pi$$



$$A_\varepsilon = \iint_{D_\varepsilon} \sqrt{\det g_{\alpha\beta}}\, du^1 du^2$$

$$= \int_\varepsilon^{2\pi-\varepsilon} du^1 \int_\varepsilon^{2\pi-\varepsilon} \sqrt{\det g_{\alpha\beta}}\, du^2$$

$$x_1 = (-b\sin u^1\cos u^2,\ -b\sin u^1\sin u^2,\ b\cos u^1)$$
$$x_2 = (-(a+b\cos u^1)\sin u^2,\ (a+b\cos u^1)\cos u^2,\ 0)$$

$$g_{11} = b^2 \quad g_{12} = 0 \quad g_{22} = (a+b\cos u^1)^2.$$

$$\det g_{\alpha\beta} = b^2(a+b\cos u^1)^2$$

$$\Rightarrow A_\varepsilon = \int_\varepsilon^{2\pi-\varepsilon} du^1 \int_\varepsilon^{2\pi-\varepsilon} b(a+b\cos u^1)\, du^1$$

$$= (2\pi - 2\varepsilon)\left[ab(2\pi-2\varepsilon) + b^2(\sin(2\pi-\varepsilon) - \sin(\varepsilon))\right]$$

$$(\varepsilon \to 0) = 4\pi^2 ab$$

内蕴的几何量（第一基本形式所决定的）

$M_1 \quad x = x(u^1, u^2)$

$M_2 \quad x = \bar{x}(\bar{u}^1, \bar{u}^2)$

这里不是换参

而是两个曲面

若两曲面的参数之间存在有 1-1 对应，

$(u^1, u^2) \longmapsto (\bar{u}^1, \bar{u}^2)$

$\bar{u}^1 = \bar{u}^1(u^1, u^2)$

$\bar{u}^2 = \bar{u}^2(u^1, u^2)$

诱导 $\sigma: M_1 \longrightarrow M_2$. 1-1 对应

$\Rightarrow$ 它们第一基本形式 $\mathrm{I}, \bar{\mathrm{I}}$

满足 $\bar{\mathrm{I}} = \varphi^2(u^1, u^2)\mathrm{I}$. $\varphi \neq 0$

则称 M_1, M_2 之间的对应是共形的（conformal 保角）

$\bar{\mathrm{I}} = \bar{g}_{\bar{\alpha}\bar{\beta}}\, d\bar{u}^\alpha d\bar{u}^\beta$

$\mathrm{I} = g_{\alpha\beta}\, du^\alpha du^\beta$

$\bar{g}_{\bar{\alpha}\bar{\beta}}\, d\bar{u}^\alpha d\bar{u}^\beta = \varphi^2(u^1, u^2) g_{\alpha\beta}\, du^\alpha du^\beta$

由于 $du^\alpha \neq d\bar{u}^\alpha$ 故 $\bar{g}_{\alpha\beta} \neq \varphi^2 g_{\alpha\beta}$.

但可将 $\bar{g}_{\bar{\alpha}\bar{\beta}}\, d\bar{u}^\alpha d\bar{u}^\beta$ 写成 $\bar{g}_{\alpha\beta}\, du^\alpha du^\beta$

$\Rightarrow \bar{g}_{\alpha\beta} = \varphi^2 g_{\alpha\beta}$ ✓

$$\bar{g}_{\bar\alpha\bar\beta} = \bar{x}_{\bar{u}^\alpha}\,\bar{x}_{\bar{u}^\beta}$$

$$= \left(\bar{x}_\gamma \frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\right)\left(\bar{x}_\sigma \frac{\partial u^\sigma}{\partial \bar{u}^\beta}\right)$$

$$= \bar{g}_{\gamma\sigma} \frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\frac{\partial u^\sigma}{\partial \bar{u}^\beta}$$

$$\Rightarrow \bar{\mathrm{I}} = \bar{g}_{\bar\alpha\bar\beta}\, d\bar{u}^\alpha d\bar{u}^\beta = \bar{g}_{\gamma\sigma} \frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\frac{\partial u^\sigma}{\partial \bar{u}^\beta}\, d\bar{u}^\alpha d\bar{u}^\beta$$

$$= \bar{g}_{\gamma\sigma}\, du^\gamma du^\sigma$$

若 $\varphi \equiv 1$. 则称为等距. $\Rightarrow$ 保角.