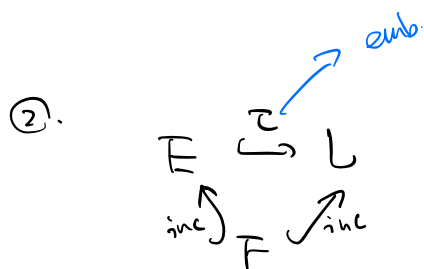


σ : injective field hom (not inclusion).

τ : injective field hom

We say $\tau: E \hookrightarrow L$ is an embedding of E over F if $\tau|_F = \sigma$

↓
embedding = injective field hom.



$\tau|_F = \text{inclusion}$.

Rmk $f(x) \in F[x]$ $\alpha \in E$. a root of $f(x)$

$$f(x) = a_0 + \dots + a_n x^n, \quad a_i \in F$$

$$\therefore 0 = a_0 + \dots + a_n \alpha^n \in E$$

if τ extends σ as above diagram.

$$0 = \tau(0) = \tau(f(\alpha)) = \sigma(a_0) + \sigma(a_1)(\tau(\alpha)) + \dots + \sigma(a_n)(\tau(\alpha))^n.$$

write as

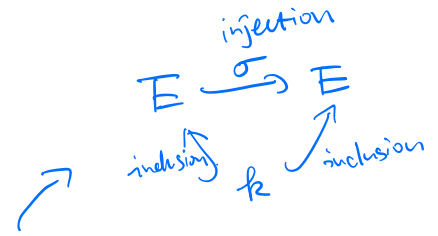
$$a_0^\sigma + a_1^\sigma(\tau\alpha) + \dots + a_n^\sigma(\tau\alpha)^n.$$

$$\therefore \tau(\alpha) \text{ is a root of } f^\sigma(x) = a_0^\sigma + a_1^\sigma x + \dots + a_n^\sigma x^n.$$

Lem. 2.1

$E : \text{alg ext.} / k$

(injective hom.)



$\sigma : E \rightarrow E$ an. embedding of E into E over k
 $(\sigma|_k = \text{id}_k)$

$\Rightarrow \sigma : \text{isomorphism.}$

pf. want to show $\sigma : \text{injective.}$

let $\alpha \in E$. $p(x) = \text{Irr}(\alpha, k, X)$

$E' = k(\text{all roots of } p(x) \text{ laying in } E) \subseteq E$

$\therefore E'$ fin. gen ^{why?} $/k$ and every generators is algebraic $/k$.

Lem. 1.6.

$\Rightarrow E' : \text{finite ext} / k$

$\Rightarrow f^\sigma(x) = a_0^\sigma + a_1^\sigma x + \dots + a_n^\sigma x^n$
 $= a_0 + a_1 x + \dots + a_n x^n$

$\sigma : (\text{a root of } p(x) \text{ in } E) \xrightarrow{\text{by def.}} (\text{a root of } p(x) \text{ in } E)$

$\Rightarrow \sigma : E' \xrightarrow{vs/k} E' \text{ injective}$

Now regard σ as a k -homomorphism of vector space
 (ie. linear transf $E' \rightarrow E'$ $E' : vs/k$)

$\therefore [\sigma(E'), k] = [E', k] \xrightarrow{\text{有限维}} \Rightarrow \sigma(E') = E' \ni \alpha$

$\therefore \sigma$ surjective □

Rmk - E, F ext/ k contained in some larger field L .

$E[F]$ = the ring gen. by F over $E = \{a_1 b_1 + \dots + a_n b_n \mid a_i \in E, b_j \in F\} = F[E]$

EF : quotient field of $E[F] = F[E]$ (or we can say

EF is the field of quotients of these elements).

Lem 2.2, E_1, E_2 : ext/ k contained in E

$$\sigma : E \xrightarrow{\text{emb.}} L$$

$$\Rightarrow \sigma(E_1 E_2) = \sigma(E_1) \sigma(E_2)$$

pf. $\sigma \left(\frac{a_1 b_1 + \dots + a_n b_n}{a_1' b_1' + \dots + a_m' b_m'} \right) = \frac{a_1^\sigma b_1^\sigma + \dots + a_n^\sigma b_n^\sigma}{a_1'^\sigma b_1'^\sigma + \dots + a_m'^\sigma b_m'^\sigma}$ is the elem of $\sigma(E_1) \sigma(E_2)$

and is surjective.

prop. 2.3, k field. $f \in k[x]$. $\deg f \geq 1$

$\Rightarrow \exists$ an ext E of k in which f have a root.

pf. may assume. $f(x) = p$: irred. in $k[x]$

$$\sigma : k[x] \rightarrow k[x] / (p(x))$$

$$x \mapsto \sigma(x) = \xi = x + (p(x))$$

$$p(x) \mapsto 0 = (p(x))^\sigma = p^\sigma(x^\sigma) = p^\sigma(\xi)$$

$(\sigma|_k) \sigma : k \rightarrow k[x] / (p(x))$ → PID → irred \Rightarrow max $\Rightarrow k[x] / (p(x))$ field. ↙ injective field hom

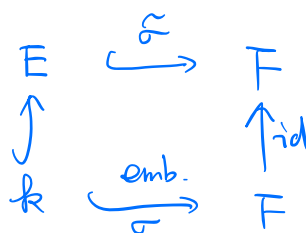
k : field.
 \rightarrow ideal of $f = p(x)$ or k

$\ker(\sigma) \neq k$
 $\ker(\sigma)$: ideal of k . $\Rightarrow \ker(\sigma) = \{0\}$.

i.e. $\sigma: k \xrightarrow{\text{emb.}} F \cong k[x]/(p(x))$

s.t. $p^\sigma(x)$ has a root ξ in F now, we still don't get a root for $p(x)$ but $p^\sigma(x)$. next we do the ext.

now $\tilde{\sigma}: (F - \sigma k) \cup k \rightarrow F$



$\tilde{\sigma}|_k = \sigma$
 $\tilde{\sigma}|_{F - \sigma k} = \text{id}$ $\left. \vphantom{\begin{matrix} \tilde{\sigma}|_k = \sigma \\ \tilde{\sigma}|_{F - \sigma k} = \text{id} \end{matrix}} \right\} \tilde{\sigma}: \text{bijection as sets}$

on E , define a field structure.

$$\begin{cases} xy \triangleq \tilde{\sigma}^{-1}(\tilde{\sigma}(x)\tilde{\sigma}(y)) \\ x+y \triangleq \tilde{\sigma}^{-1}(\tilde{\sigma}(x)+\tilde{\sigma}(y)) \end{cases} \Rightarrow E \text{ is a field.} \\ k \subseteq E.$$

$$\Rightarrow \tilde{\sigma}: E \xrightarrow{\sim} F.$$

$$\text{now: } p(\tilde{\sigma}^{-1}(\xi)) \rightarrow p^\sigma(\xi) = 0$$

$$\Rightarrow p(\tilde{\sigma}^{-1}(\xi)) = 0. \text{ by isomorphism}$$

□

Cor 2.4 k : field

$$f_1, \dots, f_n \in k[x]. \deg f_i \geq 1$$

$$\Rightarrow \exists \text{ an ext } E \text{ of } k. \text{ in which } f_i \text{ has a root in } E$$

def A field L is called algebraically closed.

if every polynomial in $L[x]$ has a root in L

Thm. 2.5 k field. \exists an alg closed field containing k .

pf. First, construct E_1/k , st. every polynomial in $k[x]$ has a root in E_1 .
 Δ tricky

(Artin) $\forall f \in k[x]$ $\deg f \geq 1$ $\xrightarrow{\text{associate a formal variable.}}$ x_f

S : the set of all x_f .

Form. the polynomial ring $k[S]$

Claim $I = (f(x_f) \mid x_f \in S)$: ideal of $k[S]$

\hookrightarrow I is not the unit ideal (proper, i.e. $\neq k[S]$)

pf of claim if $g_1 f_1(x_{f_1}) + \dots + g_n f_n(x_{f_n}) = 1$.

$g_i \in k[S]$. Write $x_{f_i} = x_i$

$$\Rightarrow \sum_{i=1}^n g_i(x_1, \dots, x_N) f_i(x_i) = 1$$

\downarrow
variables may include $x_1 \sim x_n$ and none
 \downarrow
write as x_1, \dots, x_N .

$N \geq n$.

let F : finite ext/ k . in which f_1, \dots, f_n have a root α_i

let $\alpha_i = 0$ for $i=n+1 \dots N$.

Substitute α_i for x_i in our relation

\Downarrow

$$\sum_{i=1}^n g_i(\alpha_1, \dots, \alpha_N) f_i(\alpha_i) = 0 = 1 \quad \star \quad \square \text{ of claim.}$$

$$\therefore I = (f(x_f) \mid x_f \in S) \subset \underline{m} \subset k[s]$$

\downarrow
some max.

$$\sigma: k[s] \longrightarrow k[s]/\underline{m}$$

$$(k \longrightarrow \sigma k)$$

$$f(x_f) \longmapsto \underbrace{f^\sigma(\sigma x_f)}_{\cap} = 0$$

$$I \subset \underline{m}.$$

\Rightarrow let $f \in k[x]$, $\deg f \geq 1$. f^σ has a root in $k[s]/\underline{m}$. which is ext of σk .

\therefore By similar argument as in prop 2.3

we have E_1 : ext/ k . st. every $f(x) \in k[x]$ has a root in E_1

Inductively, $k \subset E_1 \subset E_2 \dots \subset E_n \subset$

all $f \in E_1[x]$ has a root in E_2

$\sim E_2[x] \sim$ in $E_3 \dots$

Let $E = \text{union of all } E_n$

$\Rightarrow E$ is a field $\because x, y \in E \Rightarrow x, y \in E_n$ for some n

$\Rightarrow xy \in E_n \subset E, x+y \in E_n \subset E$

\because every polynomial in $E[x]$, has its coeff in E_n for some n .

\Rightarrow have a root $\in E_{n+1} \subset E$

$\Rightarrow E$ is alg closed \square

Cor 2.6. k : field.

Then \exists an ext k^a , which is algebraic over k and algebraically closed.

pf by Thm. 2.5. $\exists E$: ext of k E alg closed.

Let $k^a = \text{union of all subextension of } E \text{ alg}/k$

$\Rightarrow k^a$ alg over k .

If $\alpha \in E$. $\alpha: \text{alg}/k^a$ \checkmark we already have $k^a: \text{alg}/k$. then $\Rightarrow \alpha: \text{alg}/k$. (Prop 1.7).

$k^a = \bigcup_{k \subseteq F \subseteq E} F = \{ \alpha \in E \mid \alpha: \text{alg}/k \}$
 $F: \text{alg}/k$ alg/k
 \downarrow
 up to E ?

$f \in k^a[x]$ $\deg f \geq 1$ then f has a root $\alpha \in E$.
 \cap
 E

$\therefore \alpha: \text{alg}/k^a \Rightarrow \alpha: \text{alg}/k$

$\Rightarrow \alpha \in k^a$

$\Rightarrow k^a$ alg closed. \square

k . field.

$\sigma: k \xrightarrow{\text{emb.}} L$. L is alg closed.

E : ext of k .

Want to study the extension of σ to E/k

Let $E = k(\alpha)$, $\alpha: \text{alg}/k$. $p(x) = \text{Irr}(\alpha, k, x)$.

$$p(x) \longrightarrow p^\sigma(x) \in \sigma k[x] \quad \exists \beta \in L. p^\sigma(\beta) = 0$$

\uparrow
 L

define an extension of σ

$$\tilde{\sigma}: E = k(\alpha) = k[\alpha] \longrightarrow L$$

$$\begin{array}{ccc} \downarrow & & \\ f(\alpha) & \longmapsto & f^\sigma(\beta) \\ \circ = p(\alpha) & \longmapsto & p^\sigma(\beta) = 0 \end{array}$$

well-defined? $g(\alpha) = f(\alpha) \Rightarrow (g-f)(\alpha) = 0$

$$\Rightarrow p(x) \mid (g-f)(x)$$

$$p^\sigma(x) \mid (g^\sigma - f^\sigma)(x) \Rightarrow g^\sigma(\beta) = f^\sigma(\beta)$$

$\Rightarrow \tilde{\sigma}$. homomorphism. extending σ

Prop 2.7.

$$\sigma: k \xrightarrow{\text{emb.}} L \quad L. \text{ alg closed}$$

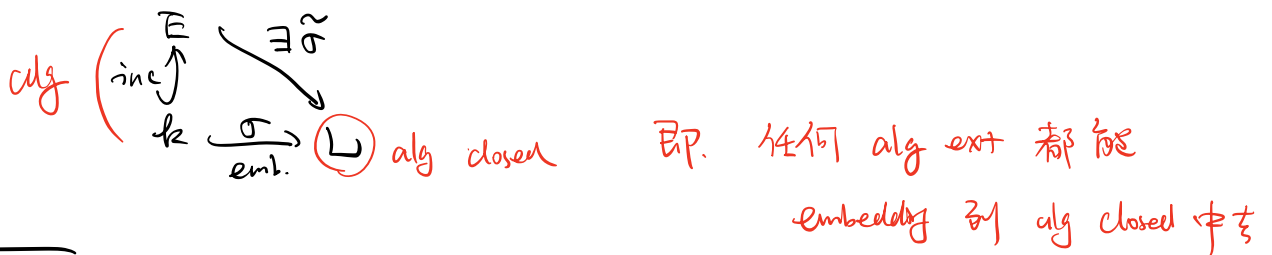
$$\downarrow$$
$$\tilde{\sigma}: E = k(\alpha) = k[\alpha] \longrightarrow L$$

$\alpha. \text{ alg}/k$.

$$\left(\begin{array}{l} \text{The number of extension} \\ \text{(distinct roots of} \\ \text{ } p(x) \text{ in } L) \end{array} \right) \leq \deg \text{Irr}(\alpha, k, x) \quad \parallel \quad \deg p(x).$$

Thm. 2.8 k : field. $\sigma: k \xrightarrow{\text{emb.}} L$. L alg closed.
 E : alg ext/ k .

$\Rightarrow \exists$ an extension of σ to embedding of E into L



If moreover E is alg closed. L : alg/ok.

\Rightarrow any such ext $\tilde{\sigma}: E \hookrightarrow L$ is an iso

Pf or thm 2.8

$$S = \{ (F, \tau) \mid E \supseteq F \supseteq k, \tau: F \xrightarrow{\text{emb.}} L, \tau|_k = \sigma \}.$$

S : nonempty for $(k, \sigma) \in S$.

$$(F, \tau), (F', \tau') \in S$$

we write $(F, \tau) \leq (F', \tau')$ iff $F \subseteq F'$ and $\tau'|_F = \tau$.

If we have a chain in S .

$$\text{i.e. } (F_1, \tau_1) \leq (F_2, \tau_2) \leq (F_3, \tau_3) \leq \dots$$

let $F = \bigcup F_i \Rightarrow$ a field contained in E

}

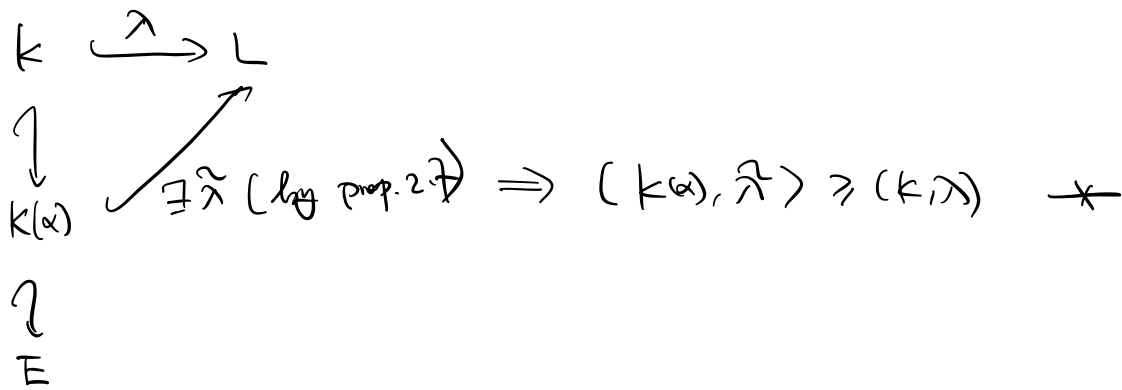
$\tau|_{F_i} = \tau_i$ (definition of τ).

Then, $(F, \tau) \in S$. $\therefore (F, \tau)$ is upper bound of the chain.

By Zorn's Lemma \exists a max elem in S . say $(K, \lambda) \in S$. $\lambda|_K = \sigma$

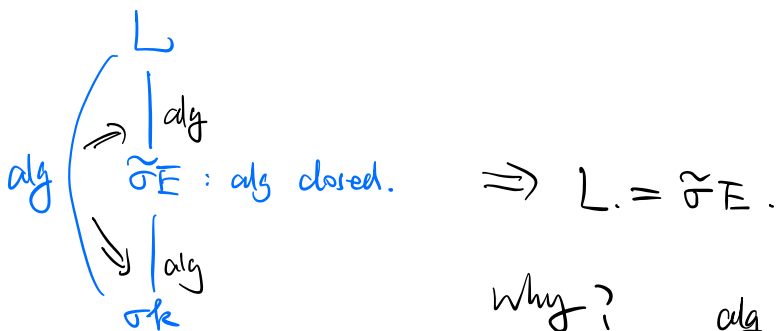
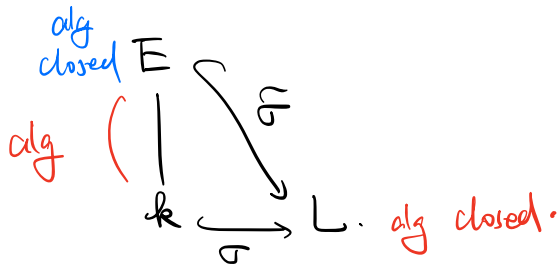
claim. $K = E$

pf of claim. if not. $K \subsetneq E \Rightarrow \exists \alpha \in E, \alpha \notin K$.



Moreover part

now:



why? alg ext of alg closed. is itself.

pf. if $L \neq \sigma E$

$\exists \alpha \in L - \sigma E$. α alg / σE

$\exists p(x) \in (\sigma E)[x]$ st. $p(\alpha) = 0 \Rightarrow \alpha \in \sigma E$. *

Cor 2.9 k . field. E, E' alg/ k

Assume E, E' : alg closed.

$\Rightarrow \exists \tau : E \xrightarrow{\sim} E'$ including id on k . $\Rightarrow k^a$ unique.

Rmk. k^a : algebraic closure of k . unique up to iso.

or \bar{k}

Rmk $\mathbb{Q} \subset \mathbb{Q}^a \subseteq \mathbb{C}$ alg closed.

$$\mathbb{Q}^a = \{ x \in \mathbb{C} \mid x : \text{alg}/\mathbb{Q} \}.$$

\mathbb{Q} : countable $\Rightarrow \mathbb{Q}^a$ countable

Pf. polynomial $\in \mathbb{Q}[x]$

$$\left(\begin{array}{c} \text{roots of} \\ a_1x + a_0 \end{array} \right) \left(\begin{array}{c} \text{roots of} \\ a_2x^2 + a_1x + a_0 \end{array} \right) \dots$$

all countable

Countable union of countable sets.