Algebra Lec 6. Group actions, simpleness of An.

[L] § 5 Operations of a group on a set.

G. group. S. set.

An operation or an action of G on S.

$$\pi: G \longrightarrow \mathrm{Perm}(S), \quad \text{hom.}$$

$$x \longmapsto \pi_x : (S \to S)$$

hom $\Rightarrow$ $\begin{cases} \pi_{xy}(s) = \pi_x \pi_y(s) \\ \pi_e(s) = s. \end{cases}$ $\quad \pi_x(s) \overset{\text{denote}}{=\!=} x \cdot s = xs.$

$$\Downarrow$$

$$\begin{cases} (xy)\cdot s = x \cdot (y \cdot s) \\ e \cdot s = s \end{cases}$$

(Another definition).

Conversly, given a map

$$G \times S \longrightarrow S$$

$$(x, s) \longmapsto \underset{\pi_x(s)}{\overset{x \cdot s.}{\|}} \quad \text{s.t.} \quad \begin{cases} (xy)s = x(ys) \\ es = s \end{cases}$$

claim π_x is a permutation. & $x \to \pi_x$: hom.

$\pi_x : S \longrightarrow S.$

pf. of claim.

① permutation.

$\pi_g \pi_{g^{-1}} = id_S = \pi_{g^{-1}} \pi_g$

$\Rightarrow$ bijective

② hom is clear.

eg. ① conjugation. G. group. $x \in G$.

let $\begin{aligned} \gamma_x : G &\to G \\ y &\mapsto xyx^{-1} \end{aligned}$) conjugation of G by $x \in G$

1st. method.

$\begin{aligned} G &\longrightarrow Perm(G) \\ x &\longmapsto \gamma_x. \end{aligned}$

hom? $\gamma_{xy} = \gamma_x \cdot \gamma_y$ clear

2nd. method.

$$\begin{array}{ccc} G\times S & \longrightarrow & S \\ \shortparallel & & \shortparallel \\ G & & G \\ (x,y) & \longmapsto & xyx^{-1} \end{array}$$

Rmk. G. group S = the set of subsets of G

$G \times S \longrightarrow S$

$(x, A) \mapsto xAx^{-1}$

② $G \times \underset{}{S}^{=G} \longrightarrow S^{=G}$

$(x, y) \longmapsto xy = T_x(y)$ Translation by x.

left mutiplication by x.

$\dim V < \infty$.

③ $G = GL(V) = \{$ inv lin tranf. $V \to V \}$

$G \times V \longrightarrow V$

$(A, v) \longrightarrow Av$

Def.

We call S is a G-set if there is a group action of G on S

S, S' are G-sets

$f: S \to S'$ s.t. $f(gs) = g f(s)$ $\forall g \in G, s \in S$.

G-map.

$$
\begin{tikzcd}
(g,s) \in G \times S \arrow[r, "id \times f"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & G \times S' \ni (g, f(s)) \arrow[d] \\
(g \cdot s) \in S \arrow[r, "f"] & S'
\end{tikzcd}
\Longrightarrow \text{G-map.}
$$

$(g, s) \mapsto (g, f(s))$

$(g \cdot s) \mapsto f(gs) = g \cdot f(s)$

Def G group action on S.

fix $s \in S$

$\{x \in G \mid x \cdot s = s\}$ $\quad (x^{-1}s) = x^{-1}(xs) = es = s.$

$\Rightarrow$ this is a group. $\triangleq G_s \leq G$

(isotropy group of s in G).

(Stabilizer of s).

eg. · G acts on $S = G$ by conjugation.

let $s \in S = G$

(the isotropy group of $s \in G$) $G_s = $ (the normalizer of $s \in G$) $\leq G$

· G acts on S. $s, s' \in S$. $y \in G$. $ys = s'$

$\Rightarrow G_{s'} = yG_{s'}y^{-1}$

pf. $G_{s'} = \{g \in G \mid gs' = s'\}$

$g(ys) = ys \quad g \in G_{s'}$

$(y^{-1}gy) \cdot s = s$

$\Rightarrow y^{-1}gy \in G_s \quad \Rightarrow \quad y^{-1}G_{s'}y \subseteq G_s$

Ex. the other containment.

- $G \longrightarrow \mathrm{Perm}(S)$

$K = \text{kernal} = \bigcap_{s \in S} G_s$

∘ Def. • A G-action is called "faithful" if $K = \{e\}$

• A faithful point of G is an element $s \in S$,

s.t. $gs = s$. $\forall g \in G$

i.e. $G_s = G$

Def.

G acts on S. $s \in S$

$Gs \triangleq \{gs \in S \mid g \in G\} = \left(\text{the orbit of } s \in S \text{ under } G \text{ action}\right)$

the orbit of $s \in S$

the orbit containing $s \in S$

Rmk.

G acts on S, $s \in S$.

G_s isotropy of $s \in S$. $(G_s \leq G$, not nec. normal$)$

- if x, y is in the same left coset of the subgroup $H = G_s$

i.e. $xH = yH \Rightarrow xs = ys$

$xh_1 = yh_2$

$(xh_1)s = (yh_2)s$

- if $xs = ys$

$\Rightarrow (y^{-1}x)s = s \quad \Rightarrow y^{-1}x \in H. \quad \Rightarrow xH = yH$

$\Downarrow$ well define. f.

<u>Prop</u>.

G. group. $s \in S$. fixed. $H = G_s$)

$f: G/H \longrightarrow S.$

$xH \longmapsto x.s$

$\downarrow$

$f: G/H \twoheadrightarrow G.s$

$G.s$ ↳ orbit containing $s \in S$.

<u>f: bijection</u>.

$\Rightarrow$. $G \circlearrowright S. \ni s,$

$\Rightarrow (G : G_s) = |G.s|$

<u>Rmk</u>. G. group. $H \le G$

(The number of subgroups conjugate to H)

$= |G| / |N_H|$

$$\overset{S}{\overbrace{G \times \{\text{subgroups of } G\}}} \longrightarrow S$$

Def. G acts on S.

If there is only one orbit, then the action is called

transitive.

Thm. The orbit decomp formula

G acts on S.

- 2 orbits of G are either disjoint or are equal

  Pf. $Gs_1 \cap Gs_2 \ni s$.

  $s = x s_1$ for some $x \in G$

  $$Gs = G(xs_1) = Gs_1$$

  $$\Rightarrow Gs_1 = Gs_2$$

$$\therefore S = \bigsqcup_{i \in I} G \cdot s_i$$

If $|S| < \infty$

The orbit decomp formulat

$$\Rightarrow |S| = \sum_{i \in I} |G \cdot s_i| = \sum_{i \in I} (G : G_{s_i})$$

- specialize to G acts on G by conjugation

$$G \times S^{=G} \longrightarrow S^{=G}$$

$$(x, s) \longmapsto x s x^{-1}$$

$$\Rightarrow |G| = \sum_{x\in C}(G:G_x) = \sum_{x\in C_0}(G:G_x) + \sum_{x\in C'}(G:G_x)$$

a set of representatives for distinct orbits → conjugacy class

orbits with only 1 elem ⇒ $C_0 = Z(G)$ center

$$= |Z| + \sum_{x\in C'}(G:G_x)$$

class equation

HW.

[L] Chap I ex19 ex15

Let $G = GL(2,\mathbb{F}_p)$, $S = M_{2\times 2}(\mathbb{F}_p)$.

Consider the group action of G on G and S by conjugation respectively

Clearify all the orbits and compute the orders of orbits

- A_n. $n \geq 5$. simple

$$\Delta(x_1, \ldots, x_n) = \prod_{i<j} (x_j - x_i)$$

$\tau \in S_n$. $\tau\Delta(x_1, \ldots, x_n) = \prod_{i<j} (x_{\tau(j)} - x_{\tau(i)})$

- if τ is a 2-cycle, $(s\,r)$ transposition

$\tau\Delta = -\Delta$

- $\sigma \in S_n$, $\sigma\Delta = \varepsilon(\sigma)\Delta$

verify $\varepsilon(\sigma\tau) = \varepsilon(\sigma)\varepsilon(\tau)$

$\Rightarrow$ $\varepsilon : S_n \longrightarrow \{\pm 1\}$ hom

$A_n \triangleq \ker(\varepsilon)$ $\Rightarrow$ $S_n/A_n \cong \{\pm 1\}$

$\Rightarrow$ $|S_n| = 2|A_n|$ $A_n \trianglelefteq S_n$ verify.

Thm. $n \geq 5$. A_n: simple. (non-abelian)

Rmk. if the thm holds

S_n: not solvable $n \geq 5$

$$G = S_n \supseteq A_n \supseteq \{e\}$$

$S_n / A_n = \{\pm 1\}$ simple. abelian

simple not abelian.

- A_n is generated by 3-cycles

$$(1\,2)(1\,2) = e$$

$$(1\,2)(2\,3) = (1\,2\,3)$$

$$(1\,2)(3\,4) = (1\,2\,3)(2\,3\,4)$$

$A_n =$ products of 2. 2-cycle $=$ product of 3-cycle

- $n \geq 5$. all 3-cycles are conjugate in A_n.

$\exists\, \gamma \in A_n$. s.t. $\gamma (i\; j\; k) \gamma^{-1} = (i'\; j'\; k')$

$\gamma \in S_n$

$$\gamma \underbrace{(i_1\; i_2\; \ldots\; i_m)}_{m\text{-cycle}} \gamma^{-1} = (\gamma(i_1)\;\; \gamma(i_2)\; \ldots\; \gamma(i_m))$$

Given 2 3-cycle $\quad (i\; j\; k) \quad (i'\; j'\; k')$

let $\gamma(i) = i' \quad \gamma(j) = j' \quad \gamma(k) = k'$

$$\gamma (i\; j\; k) \gamma^{-1} = (i'\; j'\; k')$$

γ: even. i.e. $\gamma \in A_n$. done!

γ. odd $\quad \gamma(\cancel{r}s)(ijk)(r\cancel{s})^{-1}\gamma^{-1} = (i'\ j'\ k')$

all distinct

Pf of thm.

(Lemma) meant to prove

$\{e\} \neq N \trianglelefteq A_n$. if N contains a 3-cycle

$\Rightarrow$ N contains all 3-cycles $\Rightarrow$ $N = A_n$

let $\sigma \in N$. $\sigma \neq e$.

σ fixing the maximal numbers in $\{1, 2, \ldots, n\}$

claim. σ. must fix something.

Pf of the claim. write σ in terms of cycle decomp

$\sigma \neq (p\text{-cycle}).(q\text{-cycle})$

$p \neq q$. say $p < q \Rightarrow \sigma^p = (p\text{-cycle})^p (q\text{-cycle})^q$

$\sigma^p = e$, violate the max assumption

Assume σ fixes nothing

- $\sigma = (1\ 2)(3\ 4)(5\ 6)(7\ 8)$ in A_8

let $\tau = (6\,7\,8) \in A_n$

$$\underset{N}{\underset{\rotatebox{90}{$\in$}}{(\tau\sigma\tau^{-1})}}\ \underset{N}{\underset{\rotatebox{90}{$\in$}}{\sigma^{-1}}} = \underline{(1\,2)(3\,4)}(5\,7)(8\,6)\ \underline{(1\,2)(3\,4)}(5\,6)(7\,8)$$

for N is normal

$$\Downarrow$$

fix $1\ 2\ 3\ 4$ $\quad\times$

- $\sigma = (1\,2\,3)(3\,4\,5) \in A_6$

let $\tau = (3\,5\,6)$

$$(\tau\sigma\tau^{-1})\sigma^{-1} = (1\,2\,5)(4\,6\,3)(3\,2\,1)(6\,5\,4)$$

$$\Downarrow$$

fix 2 $\quad\times$

- $\sigma = (1\,2\,3\,4)(5\,6\,7\,8)$

$\tau = (4\,7\,8)$

- $\sigma = (1\,2\,3\,4\,5)$

$\tau = (3\,4\,5)$ $\quad\square$

$\sigma = (\text{cyclic decomp}) = (\underline{\text{disjoint orbits of } \langle\sigma\rangle})$

考虑 $\langle\sigma\rangle$ 作用在 $\{1,\ldots,n\}$ 上的 group action.

• all orbits have 2 elements (i.e. 2-cycles).

$\sigma = (1\ 2)(3\ 4)$. fixing 5.

let $\tau = (3\ 4\ 5) \in A_n$.

$(\tau\sigma\tau^{-1})\sigma^{-1} = (\cancel{1\ 2})(4\ 5)(\cancel{1\ 2})(3\ 4)$.

fixing 1, 2

• if σ is 3-cycle done

if not. $\sigma = (1\ 2\ 3)(4\ 5\ 6)$. fixing 7

let $\tau = (5\ 6\ 7)$

$(\tau\sigma\tau^{-1})\sigma^{-1} \Rightarrow$ fixing 1, 2, 3. ✗

not fixing 7

• if $\sigma = (1\ 2\ 3\ 4\ 5)$. fixing 6

$\tau = (4\ 5\ 6)$ $(\tau\sigma\tau^{-1})\sigma^{-1}$ fixing 2 3, not fixing 6.

. . . . . $\Rightarrow$ σ is 3-cycle $\Rightarrow$ prove the Lemma $\square$