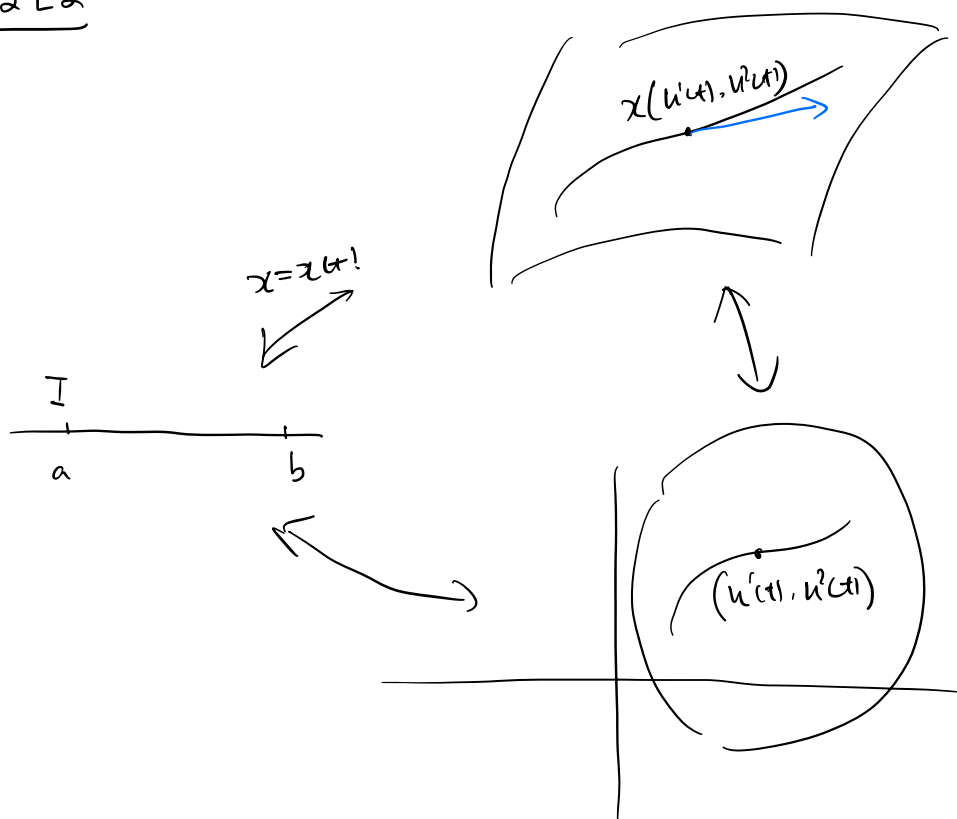


W2 L2



$$L = \int_a^b ds = \int_a^b |dx| = \int_a^b \sqrt{(x_\alpha du^\alpha)(x_\beta du^\beta)} \\ = \int_a^b \sqrt{g_{\alpha\beta} \frac{du^\alpha}{dt} \frac{du^\beta}{dt}} dt$$

t) 1)  $\frac{dx}{dt} = \frac{\partial x}{\partial u^1} \frac{du^1}{dt} + \frac{\partial x}{\partial u^2} \frac{du^2}{dt}$

$$= x_1 \frac{du^1}{dt} + x_2 \frac{du^2}{dt}$$

$$= a^1 x_1 + a^2 x_2$$

$$a^\alpha = \frac{du^\alpha}{dt}$$

t) 2) 中.  $\vec{a} = a^1 x_1 + a^2 x_2 = a^\alpha x_\alpha$

$$\vec{b} = b^1 x_1 + b^2 x_2 = b^\alpha x_\alpha$$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{(a^\alpha x_\alpha)(b^\beta x_\beta)}{|a^\alpha x_\alpha| |b^\beta x_\beta|} = \frac{g_{\alpha\beta} a^\alpha b^\beta}{\sqrt{g_{\alpha\beta} a^\alpha a^\beta} \sqrt{g_{\gamma\delta} b^\gamma b^\delta}}$$

$$\cos \langle x_1, x_2 \rangle = \frac{g_{12}}{\sqrt{g_{11}} \sqrt{g_{22}}} \quad \text{正交} \quad \cos 0 = 0 \Leftrightarrow g_{12} = 0$$

$\Rightarrow$  正交参数网

$$I = g_{11}(du^1)^2 + g_{22}(du^2)^2$$

"平面"  $x = (u^1, u^2, 0)$

$$x_1 = (1, 0, 0)$$

$$x_2 = (0, 1, 0)$$

$$ds^2 = (du^1)^2 + (du^2)^2$$

✓ 曲面  $x = x(u^1, u^2)$  的参数网的平方线的微分方程,

解: 满足条件的曲线为  $C$ .

$$\vec{r}(s) = x(u^1(s), u^2(s))$$

$$s \text{ 弧长参数. } T(s) = \frac{dr}{ds} = x_1 \frac{du^1}{ds} + x_2 \frac{du^2}{ds}$$

$T$  与  $x_1, x_2$  或与  $x_1, -x_2$  的夹角相等.

$$\varepsilon \frac{T \cdot x_1}{|x_1|} = \frac{T \cdot x_2}{|x_2|} \quad \varepsilon = \pm 1$$

$$\frac{\varepsilon}{\sqrt{g_{11}}} \left( g_{11} \frac{du^1}{ds} + g_{12} \frac{du^2}{ds} \right) = \frac{1}{\sqrt{g_{22}}} \left( g_{12} \frac{du^1}{ds} + g_{22} \frac{du^2}{ds} \right)$$

$$\left( \varepsilon \sqrt{g_{11}} - \frac{g_{12}}{\sqrt{g_{22}}} \right) \frac{du^1}{ds} = \left( \sqrt{g_{22}} - \frac{\varepsilon g_{12}}{\sqrt{g_{11}}} \right) \frac{du^2}{ds}$$

$$\cancel{\varepsilon \sqrt{g_{11}}} \left( 1 - \frac{\cancel{\varepsilon g_{12}}}{\cancel{\sqrt{g_{11}}} \sqrt{g_{22}}} \right) \frac{du^1}{ds} = \sqrt{g_{22}} \left( 1 - \frac{\cancel{\varepsilon g_{12}}}{\cancel{\sqrt{g_{11}}} \sqrt{g_{22}}} \right) \frac{du^2}{ds}$$

$\downarrow$   
 $-\frac{1}{2} > 0. ! \quad (\text{证})$

$$\Rightarrow \varepsilon \sqrt{g_{11}} \frac{du^1}{ds} = \sqrt{g_{22}} \frac{du^2}{ds}$$

$$\Rightarrow \varepsilon \sqrt{g_{11}} du^1 = \sqrt{g_{22}} du^2$$

$$u^1 \text{ 线的方程? } du^2 = 0. \quad (u^1 = u_0^1)$$

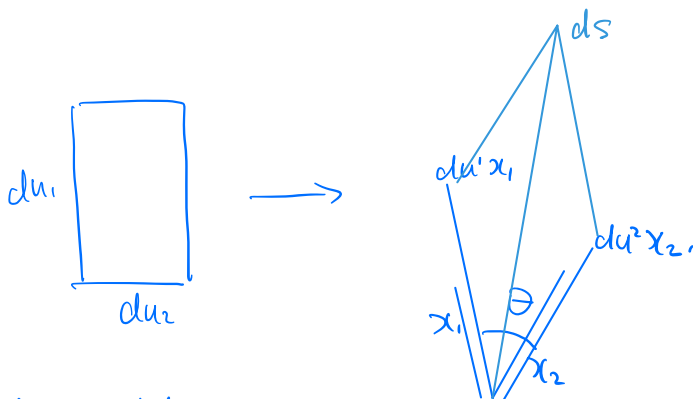
$$u^2 \text{ 线的方程? } du^1 = 0 \quad (u^2 = u_0^2)$$

Ex. 设曲面上含  $du^1, du^2$  的二次方程  $P(du^1)^2 + 2Q du^1 du^2 + R(du^2)^2 = 0$ .

$\Rightarrow$  确定了两个方向, 求证运动方向正交  $\Leftrightarrow ER - 2FQ + GP = 0$

Pf.  $P(du^1)^2 + 2Q du^1 du^2 + R(du^2)^2 = 0$   $P, Q, R$  are func of  $u^1, u^2$

$$P \left( \frac{du^1}{du^2} \right)^2 + 2Q \frac{du^1}{du^2} + R = 0 \quad \frac{du^1}{du^2} \cdot \frac{du^1}{du^2} = \frac{R}{P} \quad \frac{du^1}{du^2} + \frac{du^1}{du^2} = -\frac{2Q}{P}$$



$$\frac{du^1 du^1}{du^2 du^2} = \frac{R}{P} \quad \frac{du^1 du^1 + du^1 du^2 + du^2 du^1 + du^2 du^2}{du^2 du^2} = -\frac{2Q}{P}$$

$$ds_p = du_p^\alpha x_\alpha$$

$$0 = ds_1 \cdot ds_2 = (du_1^\alpha x_\alpha) (du_2^\beta x_\beta)$$

$$= du_1^\alpha x_\alpha du_2^\beta x_\beta$$

$$= du_1^\alpha du_2^\beta x_\alpha x_\beta$$

$$= du_1^\alpha du_2^\beta g_{\alpha\beta} = (du_1^1 du_2^2 + du_1^2 du_2^1) \cdot F + du_1^1 du_2^1 E + du_1^2 du_2^2 G$$

$$\Rightarrow 0 = -\frac{2Q}{P} F + \frac{R}{P} E + G$$

$$0 = RE - 2QF + PG$$



面积  $dA = |(x_1 du^1) \times (x_2 du^2)|$

$$= |x_1 \times x_2| du^1 du^2$$

$$|x_1 \times x_2|^2 = (x_1 \times x_2) \cdot (x_1 \times x_2) = (x_1 \cdot x_1)(x_2 \cdot x_2) - (x_1 \cdot x_2)^2$$

$$= g_{11}g_{22} - g_{12}^2$$

$$= \det(g_{\alpha\beta})$$

$$\Rightarrow dA = \sqrt{\det(g_{\alpha\beta})} du^1 du^2$$

M 面积  $A(M) = \iint_M dA = \iint_D \sqrt{\det g_{\alpha\beta}} du^1 du^2$

面积与参数无关?

$$x: D \rightarrow x(D) = M \in \mathbb{R}^3$$

则  $D, \bar{D}$  也同胚

$$\bar{x}: \bar{D} \rightarrow \bar{x}(\bar{D}) = M \in \mathbb{R}^3$$

$$\begin{cases} u^1 = u^1(\bar{u}^1, \bar{u}^2) \\ u^2 = u^2(\bar{u}^1, \bar{u}^2) \end{cases}$$

$$\frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \neq 0$$

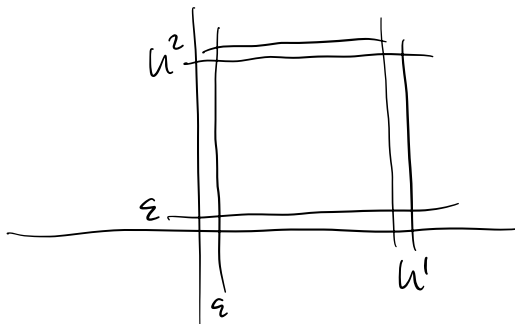
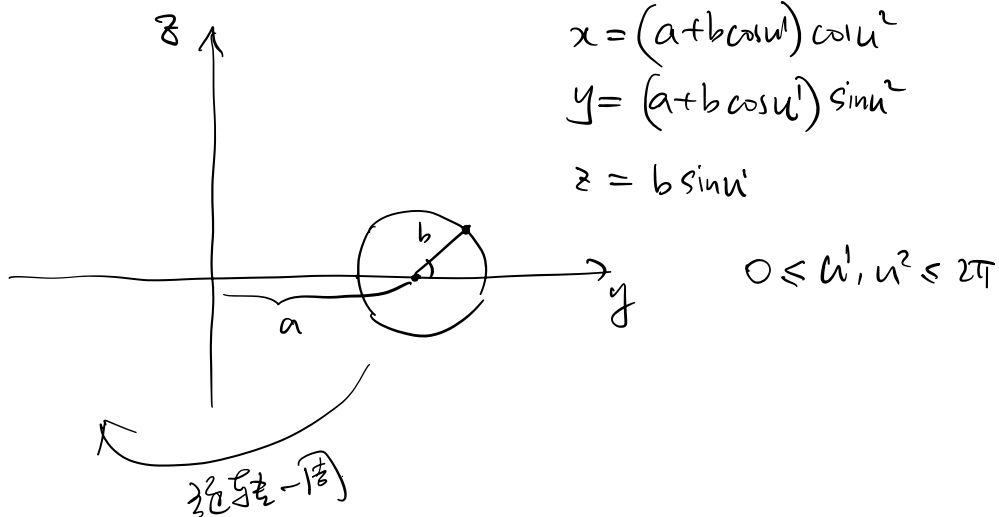
$$\iint_{\bar{D}} |\bar{x}_1 \times \bar{x}_2| d\bar{u}^1 d\bar{u}^2$$

$$\frac{\partial x}{\partial \bar{u}}$$

$$\begin{cases} \bar{x}_1 = \bar{x}_1 \frac{\partial u^1}{\partial \bar{u}^1} + \bar{x}_1 \frac{\partial u^2}{\partial \bar{u}^1} \\ \bar{x}_2 = \bar{x}_2 \frac{\partial u^1}{\partial \bar{u}^2} + \bar{x}_2 \frac{\partial u^2}{\partial \bar{u}^2} \end{cases}$$

$$|\bar{x}_1 \times \bar{x}_2| = |x_1 \times x_2| \left| \frac{\partial(u^1, u^2)}{\partial(\bar{u}^1, \bar{u}^2)} \right|$$

$$\Rightarrow |\vec{x}_1 \times \vec{x}_2| du^1 du^2 = |\vec{x}_1 \times \vec{x}_2| du^1 du^2$$



$$A_\epsilon = \iint_{D_\epsilon} \sqrt{\det g_{\alpha\beta}} du^1 du^2$$

$$= \int_\epsilon^{2\pi-\epsilon} du^1 \int_\epsilon^{2\pi-\epsilon} \sqrt{\det g_{\alpha\beta}} du^2$$

$$\vec{x}_1 = (-b \sin u^1 \cos u^2, -b \sin u^1 \sin u^2, b \cos u^1)$$

$$\vec{x}_2 = (-(a + b \cos u^1) \sin u^2, (a + b \cos u^1) \cos u^2, 0)$$

$$g_{11} = b^2 \quad g_{12} = 0 \quad g_{22} = (a + b \cos u^1)^2$$

$$\det g_{\alpha\beta} = b^2 (a + b \cos u^1)^2$$

$$\Rightarrow A_\epsilon = \int_\epsilon^{2\pi-\epsilon} du^1 \int_\epsilon^{2\pi-\epsilon} b (a + b \cos u^1) du^2$$

$$= (2\pi - 2\epsilon) \left[ ab(2\pi - 2\epsilon) + b^2 (\sin(2\pi - \epsilon) - \sin(\epsilon)) \right]$$

$$(\epsilon \rightarrow 0) = 4\pi^2 ab$$

内蕴的几何量 (第一基本形式所决定的).

$$M_1: x = x(u^1, u^2)$$

若两曲面的参数之间存在有 1-1 对应.

$$M_2: x = \bar{x}(\bar{u}^1, \bar{u}^2)$$

$$(u^1, u^2) \mapsto (\bar{u}^1, \bar{u}^2)$$

这里不是换参.

而是两个曲面.

$$\bar{u}^1 = \bar{u}^1(u^1, u^2)$$

$$\bar{u}^2 = \bar{u}^2(u^1, u^2)$$

$$\text{诱导 } \sigma: M_1 \rightarrow M_2 \quad 1-1 \text{ 对应}$$

$$\Rightarrow \text{它们第一基本形式 } I, \bar{I}$$

$$\text{满足 } \bar{I} = \varphi^2(u^1, u^2) I, \quad \varphi \neq 0$$

则称  $M_1, M_2$  之间的对应是 共形的 (conformal 保角).

$$\bar{I} = \bar{g}_{\bar{\alpha}\bar{\beta}} d\bar{u}^{\bar{\alpha}} d\bar{u}^{\bar{\beta}}$$

$$I = g_{\alpha\beta} du^{\alpha} du^{\beta}$$

$$\bar{g}_{\bar{\alpha}\bar{\beta}} d\bar{u}^{\bar{\alpha}} d\bar{u}^{\bar{\beta}} = \varphi^2(u^1, u^2) g_{\alpha\beta} du^{\alpha} du^{\beta}$$

$$\text{由于 } d\bar{u}^{\bar{\alpha}} \neq du^{\alpha} \quad \text{故 } \bar{g}_{\bar{\alpha}\bar{\beta}} \neq \varphi^2 g_{\alpha\beta}$$

但, 可将  $\bar{g}_{\bar{\alpha}\bar{\beta}} d\bar{u}^{\bar{\alpha}} d\bar{u}^{\bar{\beta}}$  写成  $g_{\alpha\beta} du^{\alpha} du^{\beta}$

$$\Rightarrow \bar{g}_{\bar{\alpha}\bar{\beta}} = \varphi^2 g_{\alpha\beta} \quad \checkmark$$

$$\bar{g}_{\bar{\alpha}\bar{\beta}} = \bar{x}_{\bar{\mu}} \bar{x}_{\bar{\mu}^c}$$

$$= \left( \bar{x}_\sigma \frac{\partial u^\sigma}{\partial \bar{u}^\alpha} \right) \left( \bar{x}_\sigma \frac{\partial u^\sigma}{\partial \bar{u}^\beta} \right)$$

$$= \bar{g}_{\sigma\sigma} \frac{\partial u^\sigma}{\partial \bar{u}^\alpha} \frac{\partial u^\sigma}{\partial \bar{u}^\beta}$$

$$\Rightarrow \bar{I} = \bar{g}_{\bar{\alpha}\bar{\beta}} d\bar{u}^\alpha d\bar{u}^\beta = \bar{g}_{\sigma\sigma} \frac{\partial u^\sigma}{\partial \bar{u}^\alpha} \frac{\partial u^\sigma}{\partial \bar{u}^\beta} d\bar{u}^\alpha d\bar{u}^\beta$$

$$= \bar{g}_{\sigma\sigma} du^\sigma du^\sigma$$

若  $\varphi \equiv 1$ , 则称为等距.  $\Rightarrow$  保角.

