Algebra. Lec 17. Normal extensions.

k. field. $f \in k[x]$. $\deg f \geq 1$.

K is called splitting field of $f \in k[x]$

$\Rightarrow$ alg $/k$.

if it's an ext of k. s.t. f splits into linear factors in K
(smallest)

Thm. 3.1 K. a splitting field of $f \in k[x]$

E: another splitting field of f

$\Rightarrow \exists \sigma : E \xrightarrow{\sim} K$. $\sigma|_k = id_k$

<u>pf.</u> K^a . alg. closure of K

(alg closed. alg $/K$)

K: alg $/k$.

$\Rightarrow$. K^a : alg$/k$ $\quad$ $\therefore$ K^a is also an alg closure of k

(i.e.) $K^a \simeq k^a$

$\exists \sigma : E \hookrightarrow K^a$ (By Thm 2.8).
(inducing identity on k)

<u>in E</u> $\quad f(x) = c(x-\beta_1)(x-\beta_2)\cdots(x-\beta_n)$ $\quad \beta_i \in E$. $c \in k$

<u>then</u> $E = k(\beta_1, \ldots, \beta_n)$

$f(x) = f^\sigma(x)$ $\quad$ (for $f(x) \in k[x]$)

$$f^\sigma(x) = c(x-\sigma\beta_1)\cdots(x-\sigma\beta_n).$$

$$\underset{k^a}{\rotatebox{90}{$\in$}} \qquad \underset{k^a}{\rotatebox{90}{$\in$}}$$

in $k^a[x]$ $\quad (K \subseteq k^a)$

$f(x) = c(x-\alpha_1)\cdots(x-\alpha_n)$ $\quad \alpha_i \in K.$ $\quad$ for K is a spliting field.

we have $k^a[x]$ is a UFD

then $(\sigma\beta_1, \dots, \sigma\beta_n) = (\alpha_1, \dots, \alpha_n)$ $\quad \therefore \sigma\beta_i \in K$

up to permutation.

$\Rightarrow \sigma E \subset K$

$\Rightarrow \sigma E = K$

$\Rightarrow \quad \sigma : E \xrightarrow{\sim} K \subseteq k^a$ $\quad \square$

Thm 3.3. K alg ext$/k$

$k \subset K \subset k^a.$

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, dash] & k^a \arrow[d, dash] \\ k \arrow[r, dash] & k \end{tikzcd}$$

TFAE. NOR1. every embedding $K \hookrightarrow k^a$ induce an automorphism of K

NOR2 K is the splitting field of a family $\{f_i(x)\}_{i \in I}$

$f_i(x) \in k[x]$

NOR 3. $f(x) \in k[x]$ $f(x)$. irred. $f(x)$ has a root in K.

$\Rightarrow f(x)$ splites into linear factors in K

Pf. **NOR1 $\Rightarrow$ NOR2.**

Let $\alpha \in K$. $P_\alpha(x) \in k[x]$ irred poly of α. $\to \mathrm{Irr}(\alpha, k, x)$

let β be a root of $P_\alpha(x)$ in k^a.

$\therefore \; k(\alpha) \xrightarrow{\sim} k(\beta)$

$\alpha \longmapsto \beta$

diagram

$$\begin{tikzcd} K \arrow[r, hook, "\exists\sigma"', "\text{by Thm 2.8}"] & k^a \arrow[d, dash] \\ k(\alpha) \arrow[u, "\text{alg}"] \arrow[r, phantom, "\simeq"] \arrow[d, dash] & k(\beta) \\ k \arrow[r, equal] & k \end{tikzcd}$$

alg

by NOR 1. the extension σ induces an automorphism of K

$\therefore \; \sigma(\alpha) = \beta \in K$ (σ is the extension of $\swarrow$)

$\therefore$ every root of $P_\alpha(x)$ is in K

$\therefore$ K is the spliting field of $\{P_\alpha(x)\}_{\alpha \in K}$.

NOR2 $\Rightarrow$ NOR1.

$\{f_i\}_{i \in I}$ a family of polynomials in $k[x]$.

K: spliting field.

let α be a root of some f_i
$\alpha \in K$

$$\begin{tikzcd} K \arrow[r, hook, "\exists\sigma\ (\text{Thm 2.8})"] & k^a \\ k \arrow[u, dash, "\text{alg}"] \arrow[r, equal] & k \arrow[u, dash] \end{tikzcd}$$

$0 = f_i(\alpha) \xrightarrow{\sigma} f_i^\sigma(\sigma\alpha) = f_i(\sigma\alpha) \Rightarrow \sigma\alpha$ is also a root of f_i

i.e. $\sigma: \{\text{roots of } f_i\} \longrightarrow \{\text{roots of } f_i\}$.

$$
\therefore \ \sigma.\quad
\begin{tikzcd}
K \arrow[r, hook, "\text{emb.}"] \arrow[d, dash, "\text{alg}"'] & K \arrow[d, dash, "\text{alg}"] \\
k \arrow[r, equal] & k
\end{tikzcd}
\quad \Rightarrow \ \text{by Lem 2.1.} \quad \sigma \text{ is an iso.}
$$

NOR1 $\Rightarrow$ NOR3

$f(x) \in k[x]$. irred.

if $\exists \alpha \in K$. s.t. $f(\alpha)=0$ let β be another root of $f(x)$.

$$
\begin{tikzcd}
K \arrow[r, hook, "\exists\text{emb}", "\text{by thm 2.8}"'] \arrow[d, dash, "\text{alg}"'] & k^a \arrow[d, dash] \\
k(\alpha) \arrow[r, "\sim"] \arrow[d, dash] & k(\beta) \arrow[d, dash] \\
k \arrow[r, equal] & k
\end{tikzcd}
\qquad \alpha \in k(\alpha),\ \alpha \mapsto \beta
$$

$\Rightarrow \exists \sigma.\ K \hookrightarrow k^a$. (Thm 2.8)

$\therefore \sigma: K \xrightarrow{\sim} K$ by NOR1

we have $\beta = \sigma(\alpha) \in K$.

i.e. all roots of f are in K

NOR3 $\Rightarrow$ NOR1

$$
\begin{tikzcd}
K \arrow[d, dash, "\text{alg}"'] & k^a \arrow[d, dash] \\
k \arrow[r, equal] & k
\end{tikzcd}
$$

$\sigma: K \hookrightarrow k^a$

let $\alpha \in K$. $p(x) = \mathrm{Irr}(\alpha, k, x)$ $\quad p(\alpha)=0$.

$0 = \sigma(p(\alpha)) = p(\sigma\alpha)$ $\quad \therefore \sigma\alpha$ is another root

by NOR3. $\sigma\alpha \in K$.

$$
\sigma: \begin{tikzcd}
K \arrow[rr, hook] \arrow[dr, dash] & & K \arrow[dl, dash] \\
& k &
\end{tikzcd}
\quad \text{by lem 2.1} \Rightarrow \sigma: K \xrightarrow{\sim} K. \ \square
$$

Rmk. NOR1., NOR2. , NOR3. normal extension.

Rmk. deg $=2$ extension is normal.

(i.e. $[k(\alpha):k]=2$

$p(x) = Irr(\alpha, k, x)$. $\deg p(x) = 2$.

roots α, β $\Rightarrow$ $\alpha+\beta$, $\alpha\beta \in k$ $\Rightarrow$ $\beta \in k(\alpha)$

(coff of $p(x)$).

$\therefore$ $k(\alpha)$: splitting field of $p(x)$.

eg.

$$\overbrace{E = \mathbb{Q}(\sqrt[4]{2}) \supset F = \mathbb{Q}(\sqrt{2}) \supset \mathbb{Q}}^{\text{degree } 4}$$

$E \supset F$: degree 2, $F \supset \mathbb{Q}$: degree 2.

$Irr(\sqrt[4]{2}, \mathbb{Q}, x) = x^4 - 2$ (Eisenstein)

E: normal ext of F

F: normal ext of $\mathbb{Q}$

But E is not normal ext of $\mathbb{Q}$.

$\therefore$ normal ext is not distinguished.

Thm. 3.4.

- normal extension remains normal under lifting
- if $K \supset E \supset k$. K. normal $/k$ $\Rightarrow$ K normal$/E$
- K_1, K_2 normal $/k$. contained in L

$\Rightarrow$ K_1K_2 $\quad K_1 \cap K_2$: normal $/k$.

pf.

①

$$\begin{tikzcd}[column sep=small] & KF \arrow[dr, dash, "\text{alg}"] & \\ K \arrow[ur, dash] \arrow[dr, dash, "\text{normal} \Downarrow \text{alg}"'] & & F \\ & k \arrow[ur, dash] & \end{tikzcd}$$

$\underline{K.F \subset L}$

$$\begin{tikzcd} KF \arrow[r, hook, "\exists.\,\text{emb.}\,\sigma\ (\text{Thm } 2.8)"] \arrow[d, dash, "\text{alg}"'] & L^a \arrow[d, dash] \\ F \arrow[r, equal] & F \end{tikzcd}$$

$$\sigma : KF \hookrightarrow L^a$$

$$\sigma|_F = id_F \qquad \sigma|_K : K \xrightarrow{\sim} K \quad (\text{NOR1}).$$

$$(KF)^\sigma = K^\sigma F^\sigma = KF \quad \Rightarrow \quad \text{NOR1} \quad \Rightarrow \quad KF : \text{normal over } F.$$

② K : normal $/k$. $\qquad K \supset E \supset k$ (alg; normal $\Rightarrow$ alg)

$$\begin{tikzcd} K \arrow[r, hook, "\exists\,\text{emb.}\,\sigma\ \text{Thm}2.8"] \arrow[d, dash, "\text{alg}"'] & E^a \arrow[d, dash] \\ E \arrow[r, equal] \arrow[d, dash] & E \arrow[d, dash] \\ k \arrow[r, equal] & k \end{tikzcd}$$

$\sigma : K \hookrightarrow E^a$

$\Downarrow$ also emb $/k$.

$$\Rightarrow \sigma|_K : K \xrightarrow{\sim} K \quad \Rightarrow K \text{ normal}/E$$

③.

$$\begin{tikzcd} K_1K_2 \arrow[r, hook, "\exists.\sigma.\,\text{emb.}"] \arrow[d, dash, "\text{alg}"'] & L^a \arrow[d, dash] \\ K_1 \arrow[r, equal] \arrow[d, dash] & K_1 \arrow[d, dash] \\ k \arrow[r, equal] & k \end{tikzcd}$$

$$\sigma(K_1K_2) = \sigma(K_1)\,\sigma(K_2) = K_1K_2$$

$$\sigma(K_1 \cap K_2) = K_1^\sigma \cap K_2^\sigma = K_1 \cap K_2 \qquad \square$$

HW. Lang. Chap V.

ex. 7. 8. 10. 21