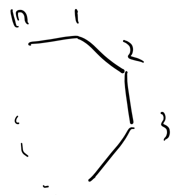
Algebra. Lec 2. Cosets, Normal subgroup, Quotient group. Isomorphism Theorem. 1.

eg. dihedral group. D_{2n}. $n \geqslant 3$

D_{2n} = the set of symmetries of a. regular n-gon.



r. clockwise rotation. of $\frac{2\pi}{n}$.

s. reflection. about the line through. vertex 1 and origion

① $1, r, r^2 \cdots r^{n-1}$ all distinct. $|r| = n$

② $s^2 = 1$. $|s| = 2$.

③ $s \neq r^i \quad \forall i$

④ $sr^i \neq sr^j \quad i \neq j$

⑤ $rs = sr^{-1}$.

⑥ $r^i s = s r^{-i}$

$D_{2n} = \{1, r, r^2, \cdots, r^{n-1}, s, sr, sr^2, \cdots, sr^{n-1}\}$.

Generators and relations.

G : group. $S \subseteq G$.

If every element of G can be expressed as a finite product of elements in S and their inverses.

S is called a set of generators.

$G = \langle S \rangle$

e.g. for D_{2n} $S = \{r, s\}$

$$G = \langle r, s \mid r^n = 1 = s^2, rs = sr^{-1} \rangle$$

Coset and Quotiant groups

G · group. $H \leq G$. subgroup.

a left coset of H in G is a subset of the form aH for some $a \in G$

$$\cdot \quad H \to aH \qquad \text{(bijection). Ex}$$
$$x \mapsto ax$$

$|H| = |aH|$.

Prop. if aH. bH have at least one element in common.

$\Rightarrow aH = bH$.

Pf $ax = by$ for some $x, y \in H$.

$aH = (byx^{-1})H$.

$yx^{-1} \in H \Rightarrow (yx^{-1})H = H$

$\Rightarrow aH = bH$.

Prop. $G =$ disjoint union of left cosets of H.

$(G:H)$ = the number of left cosets of H in G.

finite : $(G:H) = |G|/|H|$ if $|G| < \infty$.

Ex. $K \leq H \leq G \Rightarrow (G:K) = (G:H)(H:K)$

Normal subgroup.

$f : G \to G'$ group hom.

$\ker(f) = \{ y \in G \mid f(y) = e' \} = H \leq G$

let $x \in G$. • $xHx^{-1} \triangleq \{ xhx^{-1} \mid h \in H \} \subseteq H \Rightarrow xH \subseteq Hx$.

for $f(xhx^{-1}) = f(x)f(h)f(x^{-1}) = e'$

• $x^{-1}Hx \subseteq H \Rightarrow Hx \subseteq xH$.

$\Rightarrow xH = Hx$

def. G : group. $H \leq G$

H is called normal subgroup. if $xH \subseteq Hx$, $\forall x \in G$

$$\left(\begin{array}{ll} xHx^{-1} \subseteq H. & \forall x \in G \\ xHx^{-1} = H & \forall x \in G \\ xhx^{-1} \in H & \forall h \in H \\ & \forall x \in G \end{array} \right)$$

notation. $H \trianglelefteq G$.

Rmk. will see a normal subgroup of G

is the kernel of a group hom.

Quotient group & pf of Rmk

G : group. $H \trianglelefteq G$. let G' = the set of cosets of H. (left / right. the same)

define $(xH)\cdot(yH) \triangleq (xy)H$ in G' (well defined ?)

$xH = x'H$

$yH = y'H$

$(xy)H = (x'y')H$?

[$H \leq G$.
$aH = bH$
$\Leftrightarrow a^{-1}b \in H$.
$b^{-1}a \in H$].

$\Rightarrow x^{-1}x'$, $x'^{-1}x$ and normal subgroup
$y^{-1}y'$, $y'^{-1}y$ $\in H$ $\Rightarrow$ $x'x^{-1}$, xx'^{-1}
$y'y^{-1}$, yy'^{-1} $\in H$

$\underbrace{x^{-1}x'}_{\in H}\,\underbrace{y'y^{-1}}_{\in H} \in H$.

$\Rightarrow x^{-1}x'y'y^{-1}H = H$.

$\Rightarrow x'y'y^{-1}H = xH$

$\parallel$

$x'y'Hy^{-1} = xH$.

$x'y'H = xHy$

$x'y'H = xyH$.

We define a law of composition on G'.

$H \in G'$ the unit element.

$(xH)^{-1} = (x^{-1})H$. inverse.

$\Rightarrow$ G' = the set of cosets a group.

$f: G \longrightarrow G'$

$x \longmapsto xH$.

$f(xy) = f(x)f(y)$? hom.

$xyH = (xH)(yH)$

f surjective.

$\ker(f) = \{g \in G \mid f(g) = gH = H\} = \{g \in G \mid g \in H\}$
$= H$.

Notation. $G' = G/H$.

If $H \leq G$. we can have a set $G'' = G/H$.
but G'' is not a group
if H is not a normal subgroup

the quotient group of G by H.

(the factor group).

e.g. $G = S_3 = \{1, (1\,2), (2\,3)\ (3\,1)\ (1\,2\,3)\ (1\,3\,2)\}$.

$H = \{1, (1\,2)\} \leq G$

$x = (2\,3) \in G$.

$xH = \{(2\,3), (1\,3\,2)\}$

$Hx = \{(2\,3), (1\,2\,3)\}$

$\Rightarrow$ H is not normal in G.

Rmk. $\{H_i\}_{i\in I}$. a family of normal subgroups of G

$$\Rightarrow H = \bigcap_{i\in I} H_i \trianglelefteq G$$

Check. $ghg^{-1} \in H \quad \forall h\in H \; g\in G$.

$S \subseteq G$. subset.

- N_S. = the normalizer of S in G

$$\triangleq \{x\in G \mid xS = Sx\} \quad (xSx^{-1} = S).$$

$\Rightarrow N_S \leqslant G$ Ex

- Z_S = the centralizer of S in G

$$= \{g\in G \mid gs = sg, \forall s\in S\}$$

$\Rightarrow Z_S \leqslant G$. Ex.

- $Z_G = \{g\in G \mid g\hat{g} = \hat{g}g, \forall \hat{g}\in G\}$ = the center of G.

<u>Rmk</u>. • $H \leq G$

$$H \leq N_H \leq G$$

$\Rightarrow$ $H \trianglelefteq N_H$. Ex.

• N_H is the largest subgroup of G in which H is normal.

• $G' \xrightarrow{f} G \xrightarrow{g} G''$ f. g. group. hom.

We say the sequence is exact if $\mathrm{im}(f) = \ker(g)$.

e.g. $H \trianglelefteq G$. $\{0\} \to H \hookrightarrow G \to G/H \to \{0\}$. exact.

$x \mapsto xH$

Isomorphism theorem

① $f: G \to G'$ group hom. $\ker(f) = H \trianglelefteq G$

$\varphi: G \to G'/H$.

$x \longmapsto xH$ Canonical quotient map

☆理解

claim.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & G' \\ G/H \arrow[ur, "f_*"'] & \end{tikzcd}$$

$x \mapsto f(x)$, $x \mapsto xH$

$\exists!$ hom. $f_*: G/H \longrightarrow G'$

s.t. $f = f_* \circ \varphi$. and. f_* is injective.

use diagram to define f_*

Check $\begin{cases} \text{well-defined} \\ \text{hom.} \\ \text{injective.} \end{cases}$ Ex.

$\exists!$

$\triangle$

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f_* induces an isomorphism.

$\lambda: G/H \xrightarrow{\sim} \mathrm{im}(f_*)$. (1st. isomorphism theorem)

$\qquad\qquad\qquad\quad \| $

$\qquad\qquad\quad \mathrm{im}(f)$.

$G/\ker(f) \simeq \mathrm{im}(f)$

$$\begin{tikzcd} G \arrow[r, "\varphi", "\text{surjective}"'] \arrow[rrr, bend right, "f"'] & G/H \arrow[r, "\lambda", "\sim"'] & \mathrm{im}(f) \arrow[r, hook, "\text{injective.}"] & G' \end{tikzcd}$$

e.g. Any finite group is isomorphism to a subgroup of a symmetry group.

$|G| = n \quad G = \{g_1, \dots, g_n\}$

$\varphi : G \longrightarrow S_n \qquad S_n =$ symmetry group of $\{g_1, \dots, g_n\}$.

$x \longmapsto \sigma_x : \{g_1, \dots, g_n\} \to \{g_1, \dots, g_n\}$

$g_i \longmapsto x g_i$

$\varphi(xy) = \sigma_{xy}$

$\varphi(x)\varphi(y) = \sigma_x \sigma_y$ $\Rightarrow$ φ is hom.

$\ker(\varphi) = \{e\}$

$G = G/\ker(\varphi) \simeq \mathrm{Im}(\varphi) \leqslant S_n$

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②.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\text{canonical } \varphi"'] & G' \\ G/N \arrow[ur, "\exists! f_*"'] & \end{tikzcd}$$

$H \leqslant \ker(f)$.

$N =$ intersection of all normal subgroup of G, containing H.

$=$ smallest normal subgroup of G contaning H.

and. $N \leqslant \ker(f)$

or we just let $\boxed{\begin{matrix} N \trianglelefteq G \\ N \leqslant \ker(f). \end{matrix}}$

$f_*(xN) \triangleq f(x)$.

$\left\{ \begin{array}{l} \text{well-defined?} \\ \text{hom.} \\ \text{unique.} \\ \cancel{\text{injective}} \end{array} \right.$

Ex

$\ker(f_*) = \{N\}$. ? ✗

$\forall x \in \ker(f)$. $xN = N = eN$ not always

$f_*(xN) = f(x) = e'$

Rmk. • (universal property of quotient group).

G: group. $H \trianglelefteq G$

let $\varphi : G \to \tilde{G}$ be a group hom. $H \leq \ker(\varphi)$.

s.t. ($\forall\, f : G \to \tilde{\tilde{G}}$ with $H \leq \ker(f)$ & hom.

we have a unique $f_* : \tilde{G} \to \tilde{\tilde{G}}$ satisfying $f = f_* \circ \varphi$)

$\Rightarrow \tilde{G} \cong G/H$.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\varphi"'] & \tilde{\tilde{G}} \\ \tilde{G} \arrow[ur, "\exists!\, f_*"'] & \end{tikzcd}$$

$\to H \leq \ker(f)$.

Ex.

Hw 02.

[L] chap I. ex 3. ex 4. ex 5. ex 8. ex 12.