Def. (Doubling measure) (X, d, μ). we say μ is a doubling measure w. doubling constant $K > 0$, if.

$$\forall\, 0 \le r \le \operatorname{diam} X. \quad \forall x \in X$$

$$\mu(B_r(x)) \le K\, \mu(B_{\frac{r}{2}}(x)).$$

用小的控制大的. $\forall r > 0$ 也行 r很大左右两边都是X.

Def. (maximal function). (X, d, μ). m. m.s (度量 测度空间)

$u: X \to \mathbb{R}_+$. define maximal function of u:

$$M(u)(x) := \sup_{0 < r \le \operatorname{diam} X} \fint_{B_r(x)} u \, d\mu.$$

where. $\fint_{B_r(x)} = \frac{1}{\mu(B_r(x))} \int$

Prop (maximal-function. L^1-version)

Let (X, d, μ) m.ms. $u: X \to \mathbb{R}_+$. $u \in L^1(X, \mu)$.

Assume μ doubling w. constant $K > 0$. then.

$\forall t>0 \quad \mu(\{x : M(u)(x) > t\}) \leq C(K)\, t^{-1} \int_X u \, d\mu.$

Rmk. $M(u) \in L^1_{weak}$. $\triangle$

Pf. Denote $A_t = \{x : M(u)(x) > t\}$

$\forall x \in A_t. \ \exists r_x > 0. \ s.t.$

$$\fint_{B_{\frac{r_x}{5}}(x)} u \, d\mu > t$$

then $A_t \subseteq \bigcup_{x \in A_t} B_{\frac{r_x}{5}}(x)$ is a covering of A_t

Vitali covering lemma

$$\Longrightarrow \exists \{B_{r_{x_i}/5}(x_i) \ . \ x_i \in A_t\}$$

s.t. $B_{r_{x_i}/5}(x_i) \cap B_{r_{x_j}/5}(x_j) = \phi$. and

$A_t \subseteq \bigcup_i B_{r_{x_i}}(x_i)$

$$\Longrightarrow \mu(A_t) \leq \sum_i \mu(B_{r_{x_i}}(x_i)) \leq C(K) \sum_i \mu(B_{r_{x_i}/5}(x_i)).$$

$$\leq C(K)\, t^{-1} \sum_i \int_{B_{x_i/5}(x_i)} u \, d\mu.$$

$$\leq C(K)\, t^{-1} \int_X u \, d\mu$$

now see why need disjointed

Recall

$$\int_X |u|^p \, d\mu = \int_0^\infty p \, \mu(\{x : |u(x)| > t\}) \, t^{p-1} \, dt$$

$$= \sum_{i=-\infty}^{+\infty} \int_{2^i}^{2^{i+1}} p \, \mu(\{x : |u(x)| > t\}) \, t^{p-1} \, dt$$

$$\int_X |u|^p \, d\mu \leq \sum_{i=-\infty}^{+\infty} \int_{2^i}^{2^{i+1}} p \, \mu(\{x : |u(x)| > 2^i\}) \, 2^{(i+1)(p-1)} \, dt$$

$$= \sum_{i=-\infty}^{+\infty} p \cdot 2^{(p-1)} \, 2^{ip} \, \mu(\{x : |u(x)| > 2^i\})$$

same

$$\geq \sum_{i=-\infty}^{+\infty} \int_{2^i}^{2^{i+1}} p \, \mu(\{x : |u(x)| > 2^{i+1}\}) \, 2^{i(p-1)} \, dt$$

$$= \sum_{i=-\infty}^{+\infty} p \, 2^{ip} \, \mu(\{x : |u(x)| > 2^{i+1}\})$$

$$\overset{i+1=j}{=} \sum_{j=-\infty}^{+\infty} p \, 2^{(j-1)p} \, \mu(\{x : |u(x)| > 2^j\})$$

$$= \sum_{i=-\infty}^{+\infty} p \, 2^{-p} \, 2^{ip} \, \mu(\{x : |u(x)| > 2^i\}).$$

Prop (Maximal function L^p-version. $p>1$).

Let (X, d, μ) m. m.s $u: X \to \mathbb{R}_+$ $u \in L^p$, $p>1$

Assume μ doubling w. constant $K>0$. then

$$\int_X |M(u)|^p \, d\mu \leq C(K,p) \int_X |u|^p \, d\mu$$

Pf. Estimate $\{x : M(u)(x) > t\}$.

$\forall t>0$. write

$$u(x) = u(x)\chi_{\{|u| \leq \frac{t}{2}\}}(x) + u(x)\chi_{\{|u| > \frac{t}{2}\}}(x)$$

$$:= \hat{u}(x) + \tilde{u}(x).$$

$$\Rightarrow M(u) \leq M(\hat{u}(x)) + M(\tilde{u}(x))$$

极大函数定义

$$\leq \frac{t}{2} + M(\tilde{u})$$

$$\Rightarrow \{x : M(u)(x) > t\} \subset \{x : M(\tilde{u}) > \frac{t}{2}\}$$

$\forall s > 0.$

$$\{x : \quad \hat{u} > s\}$$

$$= \begin{cases} \{x : |u| > \frac{t}{2}\}, & \text{if } s < \frac{t}{2} \\ \{x : |u| > s\} & \text{if } s \geq \frac{t}{2} \end{cases}$$

$$\int_X |M(u)|^p \, d\mu = \int_0^{+\infty} p\,\mu(\{x : M(u)(x) > t\})\, t^{p-1}\, dt$$

$$\leq \int_0^{+\infty} p\,\mu(\{x : M(\hat{u})(x) > \tfrac{t}{2}\})\, t^{p-1}\, dt$$

L^1-version.

$$\leq \int_0^{+\infty} \left(C(K)\, p \cdot \left(\tfrac{t}{2}\right)^{-1} \int_X \hat{u}\, d\mu \right) t^{p-1}\, dt$$

$$= \int_0^{+\infty} \mu(\{x : |\hat{u}| > s\})\, ds.$$

$$= \int_0^{\frac{t}{2}} \mu(\{x : u > \tfrac{t}{2}\})\, ds.$$

$$+ \int_{\frac{t}{2}}^{+\infty} \mu(\{x : u > s\})\, ds$$

$$= \frac{t}{2}\cdot \mu\left(x: u > \frac{t}{2}\right)$$
$$+ \int_{\frac{t}{2}}^{+\infty} \mu(\{x: u > s\})\, ds$$

$$\Rightarrow \int_X |M(u)|^p\, d\mu \le \int_0^{+\infty} C(k,p)\left[\left(\mu\left(\{x: u > \tfrac{t}{2}\}\right) t^{p-1} + \int_{\frac{t}{2}}^{+\infty} \mu(\{x: u > s\})\, ds \cdot t^{p-2}\right]dt\right.$$

$$= C(k,p)\int_0^{+\infty} t^{p-1}\mu\left(\{x: u > \tfrac{t}{2}\}\right) dt$$
$$+ C(k,p)\int_0^{+\infty}\int_{\frac{t}{2}}^{+\infty} \mu(\{x: u > s\})\, t^{p-2}\, ds \cdot dt.$$

$$\le C(k,p)\int_X |u|^p\, d\mu.$$

用示性函数记号

$$+ C(k,p)\int_0^{+\infty}\int_0^{+\infty} \mu(\{x: u > s\})\, \chi_{\{s > \frac{t}{2}\}}(s,t)\, t^{p-2}\, ds\, dt$$

Fubini.

$$= C(k,p)\int_X |u|^p\, d\mu + C(k,p)\int_0^{+\infty} \mu(\{x: u > s\})\int_0^{2s} t^{p-2}\, dt\, ds$$

$$\le C(k,p)\int_X |u|^p\, d\mu.$$

结由写了.

Sobolev inequality:

<u>Thm</u> Given integer $n \geq 2$. $\forall\ 1 \leq p < n$.

then.

$$\|f\|_{L^q} \leq C(n,p)\, \|\nabla f\|_{L^p} \quad \forall f \in C_0^\infty(\mathbb{R}^n)$$

where $q = \dfrac{np}{n-p}$. $\nabla f = (\partial_{x_1} f, \partial_{x_2} f, \dots, \partial_{x_n} f)$

<u>pf.</u> Since $f \in C_0^\infty(\mathbb{R}^n)$, $\forall x \in \mathbb{R}^n$. $x = (x_1, x_2, \dots x_n)$.

$$f(x) = -\int_{x_i}^{+\infty} \partial_i f(x_1, x_2, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\, dt$$

$$\partial_{x_i} f \triangleq \partial_i f$$

$$f(x) = \int_{-\infty}^{x_i} \partial_i f(x_1, x_2, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\, dt$$

$$\Rightarrow \forall i \quad f(x) \leq \int_{-\infty}^{+\infty} |\partial_i f|(x_1, x_2, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\, dt$$

Set $F_i(x) = \int_{-\infty}^{+\infty} |\partial_i f|(x_1, x_2, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\, dt$

$\forall i = 1, \dots, n$

$\downarrow$

$(n-1)$-variable

$$F_{i,m}(x) = \begin{cases} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |\partial_i f|(x)\, dx_1\, dx_2 \cdots dx_m & \text{if } i \le m \\ \\ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F_i(x)\, dx_1 \cdots dx_m & \text{if } i > m. \end{cases}$$

Note $F_{i,n}(x) \equiv$ constant

Since. $|f(x)| \le \frac{1}{2} F_i(x) \qquad \forall i = 1, \dots, n$

$$\Rightarrow |f(x)|^n \le \left(\tfrac{1}{2}\right)^n F_1(x) F_2(x) \cdots F_n(x)$$

$p=1$

$$\Rightarrow \quad |f(x)|^{\frac{n}{n-1}} \le \left(\tfrac{1}{2}\right)^{\frac{n}{n-1}} \left(F_1(x) F_2(x) \cdots F_n(x)\right)^{\frac{1}{n-1}}$$

Recall Hölder inequality $\forall f, g$

$$\left(\int f \cdot g \, d\mu\right) \le \|f\|_{L^p} \|g\|_{L^q}. \quad \frac{1}{p} + \frac{1}{q} = 1$$

$\forall f_1, \dots, f_k$

$$\int f_1 f_2 \cdots f_k \, d\mu \le \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \cdots \|f_k\|_{L^{p_k}}$$

$$\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_n}=1$$

<u>claim</u>

$$\int\cdots\int |f(x)|^{\frac{n}{n-1}} \, dx_1\cdots dx_m \le \left(\frac{1}{2}\right)^{n/(n-1)} \left(F_{1,m}(x)\cdots F_{n,m}(x)\right)^{1/(n-1)}$$

Pf $m=1$

$$\int |f(x)|^{\frac{n}{n-1}} \, dx_1 \le \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \int_{-\infty}^{+\infty} \left(F_1(x) F_2(x) \cdots F_n(x)\right)^{\frac{1}{n-1}} dx_1$$

无 x_1 分量

$$= \left(\frac{1}{2}\right)^{\frac{n}{n-1}} F_1^{\frac{1}{n-1}}(x) \int_{-\infty}^{+\infty} \left(F_2(x)\cdots F_n(x)\right)^{\frac{1}{n-1}} dx_1$$

Hölder

$$\le \left(\frac{1}{2}\right)^{\frac{n}{n-1}} F_1^{\frac{1}{n-1}}(x) \left(\int_{-\infty}^{+\infty} F_2(x)\, dx_1\right)^{\frac{1}{n-1}} \cdots$$

$$\cdots \left(\int_{-\infty}^{+\infty} F_n(x)\, dx_1\right)^{\frac{1}{n-1}}$$

use denote

$$= \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \left(F_{1,1}(x) \cdot F_{2,1}(x) \, F_{3,1}(x) \cdots F_{n,1}(x)\right)^{\frac{1}{n-1}}$$

Assume claim holds for $m=k$, we will prove $m=k+1$

$$\int \cdots \int |f(x)|^{\frac{n}{n-1}} \, dx_1 \cdots dx_k \, dx_{k+1}$$

$$\leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \int_{-\infty}^{+\infty} \left(F_{1,k} F_{2,k} \cdots F_{n,k}\right)^{\frac{1}{n-1}} dx_{k+1}$$

$$= \left(\frac{1}{2}\right)^{\frac{n}{n-1}} F_{k+1,k}^{\frac{1}{n-1}} \int_{-\infty}^{+\infty} \left(F_{1,k} F_{2,k} \cdots F_{k,k} F_{k+2,k} \cdots F_{n,k}\right)^{\frac{1}{n-1}} dx_{k+1}$$

Hölder

$$\leq \left(\frac{1}{2}\right)^{\frac{n}{n-1}} F_{k+1,k}^{\frac{1}{n-1}} \left(\int_{-\infty}^{+\infty} F_{1,k} \, dx_{k+1}\right)^{\frac{1}{n-1}} \cdots \left(\int_{-\infty}^{+\infty} F_{n,k} \, dx_{k+1}\right)^{\frac{1}{n-1}}$$

看??[illegible]

$$= \left(\frac{1}{2}\right)^{\frac{n}{n-1}} \left(F_{1,k+1} F_{2,k+1} \cdots F_{n,k+1}\right)^{\frac{1}{n-1}} \qquad \square$$

By claim. $m=n$. $\overset{\text{积完了}}{\Longrightarrow}$

$$\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \leq \left(\frac{1}{2}\right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}^n} |\partial_1 f| dx \cdot \int_{\mathbb{R}^n} |\partial_2 f| dx \cdots \int_{\mathbb{R}^n} |\partial_n f|(x) dx\right)^{\frac{1}{n-1}}$$

$$\Rightarrow \|f\|_{L^{\frac{n}{n-1}}} \le \frac{1}{2}\left(\|\partial_1 f\|_{L^1} \cdots \|\partial_n f\|_{L^1}\right)^{\frac{1}{n}}$$

$$\le \frac{1}{2}\,\frac{1}{n}\left(\sum_{i=1}^{n} \|\partial_i f\|_{L^1}\right)$$

$$\le C(n)\,\|\nabla f\|_{L^1} \quad (\, p=1 \quad \checkmark\,)$$

<u>For $p>1$</u> $\quad \forall \alpha > 1$

$$|\nabla |f|^{\alpha}| = \alpha\, |f|^{\alpha-1}\, |\nabla f|$$

$$\Rightarrow g = |f|^{\alpha}. \quad \|g\|_{L^{\frac{n}{n-1}}} \le C(n)\,\|\nabla g\|_{L^1}.$$

$$\left(\int_{\mathbb{R}^n} |f|^{\alpha\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \le C(n)\int_{\mathbb{R}^n} \alpha |f|^{\alpha-1} |\nabla f|\, dx$$

$$\le C(n)\left(\int_{\mathbb{R}^n} |\nabla f|^p\, dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n} |f|^{(\alpha-1)q}\, dx\right)^{\frac{1}{q}}$$

$$\frac{1}{q} + \frac{1}{p} = 1. \quad q = \frac{p}{p-1}.$$

Choose $\alpha = \dfrac{(n-1)p}{n-p}$

$$\Rightarrow \quad \alpha = \frac{(n-1)p}{n-p} \Rightarrow \alpha\frac{n}{n-1} = \frac{np}{n-p}$$

$$(\alpha-1)q = \cdots = \frac{np}{n-p}$$

$$\left(\int_{\mathbb{R}^n} |f|^{\frac{np}{n-p}} dx\right)^{\frac{n-1}{n}} \leq C(n,p) \left(\int_{\mathbb{R}^n} |\nabla f|^p\right)^{\frac{1}{p}} \cdot \left(\int_{\mathbb{R}^n} |f|^{\frac{np}{n-p}} dx\right)^{\frac{p-1}{p}}$$

Since $\frac{n-1}{n} - \frac{p-1}{p} = \frac{n-p}{np}$

$$\Longrightarrow \quad \|f\|_{L^{\frac{np}{n-p}}} \leq C(n,p) \|\nabla f\|_{L^p} \qquad \square$$

Another approach:

$\forall x \in \mathbb{R}^n$. let $\theta = \frac{y-x}{|y-x|} \in S^{n-1}$, $= (\theta_1, \dots, \theta_n)$. $r = |y-x|$

$\Rightarrow y = x + r\theta$ "极坐标."

$$\Rightarrow f(x) = -\int_0^{+\infty} \partial_r f(x+r\theta)\, dr, \quad \forall \theta \in S^{n-1}$$

$$\partial_r f(x+r\theta) = \lim_{s\to 0} \frac{f(x+(r+s)\theta) - f(x+r\theta)}{s}$$

$$= \lim_{s\to 0} \frac{f(x_1+(r+s)\theta_1, x_2+(r+s)\theta_2, \dots, x_n+(r+s)\theta_n) - f(x_1+r\theta_1, \dots, x_n+r\theta_n)}{s}$$

$$= \partial_1 f_1\, \theta_1 + \partial_2 f\, \theta_2 + \partial_n f\, \theta_n$$

$$= \langle \nabla f, \theta \rangle = \langle \nabla f, \frac{y-x}{|y-x|} \rangle$$

$$f(x) = -\int_0^{+\infty} \langle \nabla f, \frac{y-x}{|y-x|} \rangle\, dr$$

y仍记变量

$$= -\frac{1}{\omega_{n-1}} \int_{S^{n-1}} \int_0^{+\infty} \langle \nabla f, \frac{y-x}{|y-x|} \rangle\, dr\, d\theta.$$

$$= -\frac{1}{\omega_{n-1}} \int_{S^{n-1}} \int_0^{+\infty} \left\langle \nabla f, \frac{y-x}{|y-x|} \right\rangle r^{1-n} r^{n-1} \, dr \, d\theta$$

平移

$$= -\frac{1}{\omega_{n-1}} \int_{S^{n-1}} \int_0^{+\infty} \frac{\langle \nabla f, y-x \rangle}{|y-x|^{n-}} r^{n-1} \, \underline{dr \, d\theta}$$

$$= \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\langle \nabla f, x-y \rangle}{|y-x|^n} \, dy$$

$$f(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\langle \nabla f, x-y \rangle}{|y-x|^n} \, dy$$