Integration

- (X,d) & 一般测度上
- 积分定义

Def. Let (X,d,μ) is a metric measure space $f: X \to [-\infty,+\infty]$ is measurable if $\forall t \in \mathbb{R}$. $f^{-1}((t,+\infty]) = \{x : f(x) > t\}$ is measurable

Facts: 1) f, g measurable, then $\max\{f,g\}$, $\min\{f,g\}$, $f+g$ are measurable

pf. $\{x: \max\{f,g\} > t\} = \{f>t\} \cup \{g>t\}$

$\{x: \min\{f,g\} > t\} = \{f>t\} \cap \{g>t\}$

$\{f+g > t\} = \bigcup\limits_{n=-\infty}^{+\infty} \bigcup\limits_{m=-\infty}^{+\infty} \{f > \frac{n}{m}\} \cap \{g > t - \frac{n}{m}\}$

measurable.

2) $\{f_i\}$ be a seq of measurable functions, then

$\inf f_n$, $\sup f_n$, $\liminf f_n$, $\limsup f_n$ are measurable

pf. $\{\limsup f_n > t\} = \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} \{f_k > t\}$

3) If $\liminf f_n = \limsup f_n \Rightarrow \lim f_n = f(x)$ measurable

4) Assume f is measurable, then for any interval $I \subset [-\infty,+\infty]$, the set $f^{-1}(I)$ is measurable

用 $>t$ 这样的区间逼近

Rmk. $f: X \to [0,+\infty]$ measurable

$\forall E \subset X$. measurable, $\forall t \geq 0$. $h(t) = \mu(\{x \in E : f(x) > t\})$.

$h: [0,+\infty] \longrightarrow [0,+\infty]$ monotone function.

$\Rightarrow h(t)$ measurable in $\mathbb{R}$.

Lebesgue

Def (Integration)

$f: X \to [0,+\infty]$. measurable $E \subset X$. measurable

we define $\int_E f \, d\mu = \int_0^{+\infty} \mu(\{x \in E : f(x) > t\}) \, dt$ 与实变中定义不同.

但等价且有好处

Rmk. $\int_E f \, d\mu \leq \int_E g \, d\mu$ when $f \leq g$

Q. $\int_E (f+g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu$. ? 难直接看出来

Def. (Simple function) let $\{A_i\}_{i=1}^n$ measurable, $\{a_i \geq 0\}_{i=1}^n$.

We call $f = \sum_{i=1}^n a_i \chi_{A_i}(x)$ is a simple function.

where $\chi_{A_i} = \begin{cases} 1, & x \in A_i \\ 0, & x \notin A_i \end{cases}$

Lemma. Let $f = \sum_{i=1}^N \overset{\geq 0}{a_i} \chi_{A_i}$ simple function. the $\int_E f d\mu = \sum_{i=1}^N a_i \mu(E \cap A_i)$

In particular, if $f = \sum_{i=1}^N a_i \chi_{A_i}$, $g = \sum_{i=1}^N b_i \chi_{B_i}$. then

$$\int_E (f+g) d\mu = \int_E f d\mu + \int_E g d\mu$$

Pf. Without loss of generality

we assume $0 \leq a_1 < a_2 < \cdots < a_N$ (If $a_i = a_j$. define $\hat{A_i} = A_i \cup A_j$, $\hat{a_i} = a_i$)

we assume $A_i \cap A_j = \phi$. $i \neq j$ (If $N=2$. $A_1 \cap A_2 \neq \phi$. define $\hat{A_1} = A_1 \backslash A_2$ $\hat{A_2} = A_2 \backslash A_1$ $\hat{A_3} = A_1 \cap A_2$. $\hat{a_1} = a_1$. $\hat{a_2} = a_2$ $\hat{a_3} = a_1 + a_2$)

We have.

(1) $\mu(\{x \in E : f(x) > t\}) = 0$. if $t \geq a_N$

(2) $\mu(\{x \in E : f(x) > t\}) = \sum_{i=1}^N \mu(E \cap A_i)$ if $0 \leq t < a_1$

(3). $\mu(\{x \in E \cdot f(x) > t\}) = \sum_{i=k+1}^N \mu(E \cap A_i)$ if $a_k \leq t < a_{k+1}$. $k = 1, 2, \cdots, N-1$

Set $a_0 = 0$. $\int_E f d\mu = \sum_{k=0}^{N-1} \int_{a_k}^{a_{k+1}} \mu(\{x \in E : f(x) > t\}) dt$

$$+ \int_{a_N}^{+\infty} \mu(\{x \in E : f(x) > t\}) dt$$

$$= \sum_{k=0}^{N-1} (a_{k+1} - a_k) \sum_{i=k+1}^N \mu(E \cap A_i).$$

$$= \sum_{k=0}^{N-1} \sum_{i=0}^N (a_{k+1} - a_k) \mu(E \cap A_i) \chi_{\{i \geq k+1\}}$$

$$= \sum_{i=1}^{N}\left(\sum_{k=0}^{N-1}(a_{k+1}-a_k)\,\chi_{\{i\geq k+1\}}\right)\mu(E\cap A_i)$$

$$= \sum_{i=1}^{N}\mu(E\cap A_i)\sum_{k+1\leq i}(a_{k+1}-a_k)$$

$$= \sum_{i=1}^{N} a_i\,\mu(E\cap A_i). \quad \blacksquare$$

Thm. (Monotone convergence) Let $\{f_i\}$. $f_i(x)\leq f_{i+1}(x)$ $\forall x\in X$

Assume $f = \lim_{i\to\infty} f_i(x)$. then $\lim_{i\to\infty}\int_E f_i\,d\mu = \int_E f\,d\mu$.

pf ① Since $f_i \leq f \Rightarrow \int_E f_i\,d\mu \leq \int_E f\,d\mu$

$$\Rightarrow \lim_{i\to\infty}\int_E f_i\,d\mu \leq \int_E f\,d\mu$$

Set $h_n(t) = \mu(\{x\in E : f_n(x)>t\})$. $h: [0,+\infty]\to[0,+\infty]$

$h_n(t) \leq h_{n+1}(t)$

By monotone convergence of Lebesgue integration

$$\Rightarrow \lim_{n\to\infty}\int_E f_n\,d\mu = \lim_{n\to\infty}\int_0^{+\infty} h_n(t)\,dt = \int_0^{+\infty}\lim_{n\to\infty} h_n(t)\,dt$$

$\forall \varepsilon>0$. $\{x\in E: f(x)>t\} \subset \bigcup_{n=k}^{\infty}\{x\in E: f_n(x)>t\}$ $\forall k$.

$$\mu(\{x\in E: f(x)>t\}) \leq \mu\Big(\underbrace{\bigcup_{n=1}^{\infty}\{x\in E: f_n(x)>t\}}_{\text{“单调”}}\Big). \ \forall k$$

$$= \lim_{n\to\infty}\mu(\{x\in E: f_n(x)>t\})$$

$$= \lim_{n\to+\infty} h_n(t)$$

$$\Rightarrow \lim_{n\to\infty}\int_E f_n\,d\mu = \int_0^{+\infty}\lim_{n\to\infty} h_n(t)\,dt \geq \int_0^{\infty}\mu(\{x\in E: f(x)>t\})\,dt$$

$$= \int_E f\,d\mu$$

$$\Rightarrow \lim_{n\to\infty} \int_E f\,d\mu = \int_E f\,d\mu \quad \blacksquare$$

Lemma (Approximation by simple functions)

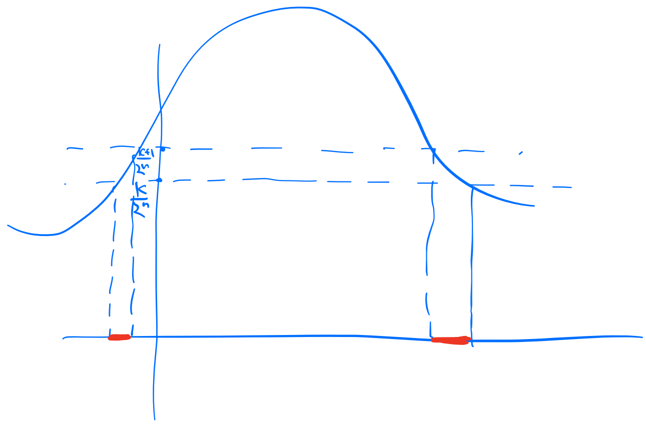
Let $f: X \to [0,+\infty]$, then $\exists$ a sequence of simple functions f_n s.t.

$$f_n \le f_{n+1},\ \forall x\in X \quad \text{and} \quad \lim_{n\to\infty} f_n(x) = f(x),\ \forall x\in X.$$

Pf $\forall n$. define

$$A_{n,k} = f^{-1}\left([k2^{-n}, (k+1)2^{-n}]\right) \qquad 0\le k\le n2^n-1$$

$$A_{n,n2^n} = f^{-1}\left([n,+\infty]\right).$$



Let $f_n(x) = \sum_{k=0}^{n\cdot 2^n} \frac{k}{2^n} \chi_{A,k}$

$$\Rightarrow f_n(x) \le f(x). \quad \forall n.\ \forall x\in X.$$

Can check: $\begin{cases} f_n(x) \to f(x). \\ f_n(x) \le f_{n+1}(x). \end{cases}$ $\blacksquare$

Thm. Let $f, g: X \to [0,+\infty]$. then

$$\int_E (f+g)\,d\mu = \int_E f\,d\mu + \int_E g\,d\mu.$$

Pf. $\exists \{f_n\}_{n\ge1}\ \{g_n\}_{n\ge1}$ s.t. $f_n \le f_{n+1}$ $g_n \le g_{n+1}$

$$\lim_{n\to\infty} f_n = f \quad \lim_{n\to\infty} g_n = f$$

and $\lim_{n\to\infty}(f_n+g_n) = f+g$ $\qquad f_n+g_n \le f+g$

Since $\int_E (f_n+g_n)\,d\mu = \int_E f_n\,d\mu + \int_E g_n\,d\mu$

$\Downarrow$ monotone convergence

$$\int_E (f+g)\,d\mu = \int_E f\,d\mu + \int_E g\,d\mu \qquad \blacksquare$$

Prop Let $\{f_n\}_{n\ge 1}$, $f_n \ge 0$. then

$$\int_E \sum_{n=1}^{\infty} f_n\,d\mu = \sum_{n=1}^{\infty} \int_E f_n\,d\mu.$$

Pf. set $g_N = \sum_{n=1}^{N} f_n$ $\quad g_n \le g_{n+1}$

$$\lim_{n\to\infty}\int_E g_n\,d\mu = \int_E \lim_{n\to\infty} g_n\,d\mu$$

$$\lim_{N\to\infty}\sum_{n=1}^{N}\int_E f_n\,d\mu = \int_E \left(\sum_{n=1}^{\infty} f_n\right) d\mu$$

$$= \sum_{n=1}^{\infty}\int_E f_n\,d\mu$$

Prop Let $f: X \to [0,+\infty]$. then

绕过课本的定义.

$$\int_E f\,d\mu = \sup\left\{\int_E \varphi\,d\mu : 0 \le \varphi \le f,\ \text{where } \varphi \text{ is simple}\right\}$$

Pf. ① Since $\varphi \le f \Rightarrow \int_E f\,d\mu \ge \int_E \varphi\,d\mu$

$\Rightarrow \sup\left\{\int_E \varphi\,d\mu\right\} \le \int_E f\,d\mu$

② by Approximation Lemma. $\exists f_n$ s.t. $f_n \le f_{n+1} \le f$.

$$\int_E f\,d\mu = \lim_{n\to\infty}\int_E f_n\,d\mu \le \sup\left\{\int_E \varphi\,d\mu\right\}. \qquad \blacksquare$$

For general $f: X \to [0, +\infty]$ define $f_+ = \max\{f, 0\} \geq 0$

$f_- = \max\{-f, 0\} \geq 0$.

and. $f = f_+ - f_-$

Def. $f: X \to [-\infty, +\infty]$. If $\int_E f_+ d\mu$ $\int_E f_- d\mu$. is finite.

Define $\int_E f \, d\mu = \int_E f_+ d\mu - \int_E f_- d\mu$

we say $f \in L^1(X, \mu)$

If $|f|^p \in L^1(X, \mu)$ we call $f \in L^p(X, \mu)$

Lemma Assume. $f \in L^p(X, \mu)$. $p > 0$. then

$$\int_X |f|^p d\mu = \int_0^{+\infty} p \, \mu(\{x : |f|(x) > t\}) \, t^{p-1} \, dt$$

pf. $\int_X |f|^p d\mu = \int_0^{+\infty} \mu(\{x : |f|^p(x) > t\}) \, dt$

$$= \int_0^{+\infty} \mu(\{x : |f|^p(x) > s^p\}) \, ds^p$$

$$= \int_0^{+\infty} p \, \mu(\{x : |f|(x) > s\}) \, s^{p-1} ds. \quad \blacksquare$$

Notes. $\int_X |f|^p d\mu = \int_X \left(\int_0^{|f|} p t^{p-1} dt \right) d\mu.$

$$= \int_X \int_0^{+\infty} p t^{p-1} \chi_{\{|f| > t\}}(x, t) \, dt \, d\mu$$

$$= \int_0^{\infty} p t^{p-1} dt \int_X \chi_{\{|f| > t\}}(x, t) \, d\mu$$

$$= \int_0^{+\infty} p t^{p-1} \mu(\{x : |f(x)| > t\}) \, dt \quad \blacksquare$$

Cor $\int_X |f|^p d\mu \simeq \sum_{i=-\infty}^{+\infty} p \, \mu(\{x : |f|(x) > 2^i\}) 2^{ip}$

where $A \simeq B$, $\exists\, C$. s.t. $\frac{1}{C}B \leq A \leq CB$

Vitali covering.

Lemma. Let $S \subset X$ $\{B_r(x_i),\ x_i \in S\}_{i=1}^{N}$ be a maximal collection of disjoint balls with $x_i \in S$. $r > 0$.

Then. $S \subseteq \bigcup_i B_{2r}(x_i)$.

Pf. Assume $x \in S \setminus \bigcup_i B_{2r}(x_i)$

$B_r(x) \cap B_r(x_i) = \phi$.

与极大矛盾. contradiction.

Lemma. (Vitali covering. 5-times covering)

Let $A \subset X$. Assume $\mathcal{B} = \{B_r(y)\}$ is a covering of A

and $\sup\{r : B_r(y) \in \mathcal{B}\} < +\infty$ then $\exists$ pairwise disjoint countable subcollection

$\mathcal{B}' \subset \mathcal{B}$ s.t. 两两不相交.

$$A \subset \bigcup_{B' \in \mathcal{B}'} 5B' \quad \text{where } 5B' = B_{5r}(y')$$

where $B' = B_r(y')$

Pf Let $\bar{r} = \sup\{r : B_r(y) \in \mathcal{B}\} < +\infty$.

Denote $\mathcal{B}_j = \{B \in \mathcal{B} : \bar{r}2^{-j-1} < \frac{diam(B)}{2} \leq \bar{r}2^{-j}\}$

$\Rightarrow \mathcal{B} = \bigcup_{j=0}^{\infty} \mathcal{B}_j$

let $\mathcal{B}_0' \subset \mathcal{B}_0$. be the maximal pairwise disjoint subcollection

Assume $\mathcal{B}_k'$ is defined. define $\mathcal{B}_{k+1}'$. be the maximal pairwise disjoint subcollection of

$\mathcal{B}_k' \cup \mathcal{B}_{k+1}$. and $\mathcal{B}_{k+1}' \supset \mathcal{B}_k'$

Let $\mathcal{B}' = \bigcup_{k\geq 0} \mathcal{B}_k$ check $\mathcal{B}'$:

$\forall x \in A$. define $r_x = \sup\{ \frac{\mathrm{diam}(B)}{2} : x \in B \subset \mathcal{B} \}$

$\Rightarrow r \cdot 2^{-j-1} < r_x \leq r 2^{-j}$, for some j

$\Rightarrow \exists B \subset \mathcal{B}_j$ s.t. $x \in B$

$\Rightarrow B \in \mathcal{B}_j'$. or $B \notin \mathcal{B}_j'$

if $B \notin \mathcal{B}_j'$. $B \cap B' \neq \emptyset$. $B' \in \mathcal{B}_j'$ $\frac{\mathrm{diam}(B')}{2} > r 2^{-j-1}$

$\Rightarrow \frac{\mathrm{diam}(B')}{2} > r 2^{-j-1} > \frac{1}{2} \frac{\mathrm{diam}(B)}{2}$

$\Rightarrow B \subset 5B'$