Algebra. Lec 4. Butterfly lemma, Jordan–Hölder theorem.

<u>iso theorem 2nd.</u>   motivation ?

$H \trianglelefteq G$

$$\begin{array}{rccc} \varphi : & G & \longrightarrow & G/H \\ & \text{\rotatebox{90}{$\geq$}} & & \\ \varphi|_{G'} : & G' & \longrightarrow & G/H. \\ & \text{\rotatebox{90}{$\in$}} & & \\ & g' & \longmapsto & g'H. \end{array}$$

H. 本与 G' 无关.

$\varphi|G'$ 是一个 can. proj 的 限制. 而 H 的 normal.

$\Rightarrow G' \subseteq N_H.$

(1st. iso)

$$G'/\ker(\varphi) \simeq \operatorname{im}(\varphi|_{G'}) = \{g'H \mid g' \in G'\} = G'H/H$$

$$\ker(\varphi) = G' \cap H.$$

$$\bigcup_{g' \in G'} g'H = G'H.$$

$$\left(\begin{array}{l} G', H \leq G \\ G' \subseteq N_H = G \end{array} \right) \Rightarrow \begin{array}{l} G'H = HG' \leq G \\ H \trianglelefteq G'H \end{array}$$

$$GL(n,k) \supset \underbrace{T \supset U}_{} = U_1 \supset U_2 \supset \cdots \supset U_n = \{I\}$$ abelian normal tower.

$$U_r/U_{r+1} \simeq (k^{n-r}, +)$$

$$T \xrightarrow{\phi} D \qquad \ker(\phi) = U. \quad D \simeq T/U \simeq (k - \{0\})^n.$$

$$A \longmapsto \operatorname{diag}(A). \qquad U \trianglelefteq T.$$

[L] Theorem 3.5. (Jordan-Hölder)

G : group $|G| < \infty$

$\Rightarrow$ G has a normal tower, ending in $\{e\}$.

s.t. G_i / G_{i+1} : simple group

Such a tower is "unique" up to "equivalence"

Def A group G is called "simple"

if $\underset{\{e\}}{G \neq}$ have no normal subgroup other than $\{e\}$ and G

$$\left(\begin{array}{l} 0 \longrightarrow G_{i+1} \hookrightarrow G_i \twoheadrightarrow G_i / G_{i+1} \rightarrow 0. \quad \text{exact seq} \\ \quad \text{we say } G_i \text{ is an extension of } G_i/G_{i+1} \text{ by } G_{i+1} \end{array} \right).$$

Classification of G

brief intro.

① classify all simple groups

② classify all extensions of simple groups.

$\rightsquigarrow$ group action sylow theorem. group cohomology. (引入作为 tools)

(finite)
<u>Simple groups</u>

① 18 infinite families of simple groups

eg. • $\mathbb{Z}/p\mathbb{Z}$: p: prime.

$GL_n(F)$. det = 1.

$\mathbin{\rotatebox[origin=c]{90}{$\leq$}}$

• $PSL_n(F) = \{ SL_n(F) / Z(SL_n(F)) \}$

↓

projective special liear group.

• $A_n = \{ \sigma \in S_n \mid \sigma : \text{even permutation} \}$

alternating group $n \geq 5$

② 26 sporadic simple groups.

$\begin{cases} \text{20 quotiant groups of "Monster group"} \\ \text{6. pariahs.} \end{cases}$

<u>Prop</u> G: finite group.

An abelian tower of G admites a cyclic refinement

In particular, G: finite solvable. ⇒ G admites a cyclic tower ending in $\{e\}$.

Pf. abelian tower.

$$G = G_0 \supset G_1 \supset \cdots \supset G_m$$

look at the 1st step.

$$\begin{tikzcd}
G_0 \arrow[r, phantom, "\supset"] \arrow[d, "f"'] & G_1 \arrow[d] \\
G_1/G_0 \supset \cdots \supset \cdots \supset \cdots \arrow[r, phantom] & \{G_1\}
\end{tikzcd}$$

preimage

cyclic abelian

if (*) is true.

(*) (if G: finite abelian. $\Rightarrow$ G admits a cyclic tower ending in $\{e\}$)

now prove this.

We apply induction to the order of G, $\underbrace{\text{finite abelian}}_{}$.

$e \neq x \in G$. $X = \langle x \rangle$ normal.

$$\begin{tikzcd}
G \arrow[r, dashed] \arrow[d, "f"'] & X \supset \{e\} \\
G/X \supset \supset \supset \arrow[r, phantom] & \{x\} \arrow[u, phantom, "\text{pre-image}"]
\end{tikzcd}$$

$\exists$ a cyclic tower.

Rmk. abelian tower $\xrightarrow{\text{refinement}}$ $\simeq$ cyclic tower.

Thm. 3.2 G : group $H \trianglelefteq G$

G is solvable $\Longleftrightarrow$ G/H, H solvable.

Pf. "$\Leftarrow$"

$$\begin{array}{ccccc} G & \supset & H & \supset & \{e\} \\ f \downarrow & & \text{pre-image} \downarrow & & \\ G/H & \supset \cdots \supset & \cdots & \{H\} & \end{array}$$

"$\Rightarrow$" G : solvable

$$G = G_0 \supset G_1 \supset \cdots \supset G_m = \{e\}$$

$$(H = H_0) \supset (H_1 = H \cap G_1) \supset \cdots \supset \{e\}$$

$$(G_0H/H = G/H) \supset (G_1H/H) \supset (G_2H/H) \supset \cdots \supset (eH/H) = H.$$

image of the tower of G under can proj. to G/H.

HW 04

- finish the proof of [L] Thm 3.2

  [DF] Sec 3.4 ex 3. ex 8.

  (Composition series in [DF] = (normal tower, G_i/G_{i+1} simple))

- reading

  [L] Chap I Thm 5.4 Thm 5.5

Lemma. 3.3. (Butterfly lemma).

G. group. $U, V \leq G$.

$u \trianglelefteq U$. $v \trianglelefteq V$.

$\Rightarrow$ • $u(U\cap v) \trianglelefteq u(U\cap V)$

• $(u\cap V)v \trianglelefteq (U\cap V)v$ and.

$$u(U\cap V)/u(U\cap v) \cong (U\cap V)v/(u\cap V)v$$

Pf. claim : $(u\cap V) \trianglelefteq (U\cap V)$

$\forall g \in u\cap V$, $x \in U\cap V$

$\Rightarrow xgx^{-1} \in u, V \Rightarrow xgx^{-1} \in u\cap V$.

similarly $(U\cap v) \trianglelefteq (U\cap V)$

$D \triangleq (u\cap V)(U\cap v)$. is the smallest normal subgroup of $(U\cap V)$

containing $(u\cap V)$ and $(U\cap v)$

$D \trianglelefteq (U\cap V)$

we will show:

$$\begin{tikzcd}
\dfrac{u(U\cap V)}{u(U\cap v)} \arrow[r, "\simeq"] \arrow[rr, bend left, "\simeq"] & \dfrac{U\cap V}{D} \arrow[r, "\simeq"] & \dfrac{(U\cap V)v}{(u\cap V)v}
\end{tikzcd}$$

只需证.　　　　　by symmetry

$$\phi: u(U\cap V) \longrightarrow \frac{U\cap V}{D}$$

$$ax \longmapsto xD$$

$$\begin{cases} a\in u \\ x\in U\cap V \end{cases}$$

ϕ: well-defined.

if $ax = a'x'$

$(a')^{-1}a = x'x^{-1} \in u\cap(U\cap V)$

$= (u\cap V) \subseteq D$

$\Rightarrow xD = x'D$

- hom $\quad axa'x' = (axa'x^{-1})xx'$

$$= \underset{\substack{\cap\\ u}}{a}\,\underset{\substack{\cap\\ u}}{(xa'x^{-1})}\,\underset{\substack{\cap\\ U\cap V}}{xx'} \in u(U\cap V)$$

$$\phi(axa'x') = (xx')D = (xD)(x'D) = \phi(ax)\,\phi(a'x')$$

- surjective.
- $\ker(\varphi) = \{ax \in u(U\cap V) \mid \overset{xD=D}{x\in D}\} = u(u\cap V)(U\cap v)$

$= u(U\cap v)$

$$\Rightarrow \frac{u(U\cap V)}{u(U\cap v)} \simeq \frac{U\cap V}{D} \simeq \frac{(U\cap V)v}{(u\cap V)v} \qquad \square$$

Equivalent towers

$$G = G_1 \supset G_2 \supset \cdots \supset G_r = \{e\} \qquad G_{i+1} \trianglelefteq G_i \quad (r-1 \uparrow)$$

$$G = H_1 \supset H_2 \supset \cdots \supset H_s = \{e\} \qquad H_{j+1} \trianglelefteq H_j \quad (r-1 \uparrow).$$

we say they are equivalent

if ① $r = s$

② $\exists$ permutation of indixes

$i = 1, 2, \ldots, r-1$, written as $i \to i'$

s.t. $G_i / G_{i+1} \simeq H_{i'} / H_{i'+1}$

Rmk. They have isomorphic quotient factors, up to permutation.

Theorem 3.4 (Schreier) Let G be a group. Two normal towers of subgroups ending with $\{e\}$. have equivalent refinements

Pf. $G = G_1 \supset G_2 \supset \cdots \supset G_r = \{e\}$

$G = H_1 \supset H_2 \supset \cdots \supset H_s = \{e\}.$

For $i = 1, 2, \ldots, r-1$. $j = 1, 2, \ldots, s$

we define $\quad G_{i,j} = G_{i+1}(H_j \cap G_i)$

$G_{i,s} = G_{i+1}$.

$$G = G_{1,1} \supset G_{1,2} \supset \cdots \supset G_{1,s-1} \supset \underset{\underset{G_2 = G_{2,1}}{\parallel}}{G_{1,s}} \supset G_{2,2} \supset \cdots \supset \underset{\underset{G_3 = G_{3,1}}{\parallel}}{G_{2,s}}$$

$$\cdots \supset G_{r-1,s-1} \supset G_{r-1,s} = G_r = \{e\}$$

Use butterfly lemma

$$\underbrace{G_{i+1}(H_{j+1} \cap G_i)}_{G_{i,j+1}} \trianglelefteq \underbrace{G_{i+1}(H_j \cap G_i)}_{G_{i,j}}$$

normal tower.

similarly define $\quad H_{j,i} = H_{j+1}(G_i \cap H_j) \qquad j = 1, \ldots, s-1, \; i = 1, \ldots, r.$

$\Rightarrow$ obtain a refinement of 2nd tower

$$\frac{G_{i,j}}{G_{i,j+1}} = \frac{G_{i+1}(H_j \cap G_i)}{G_{i+1}(H_{j+1} \cap G_i)} \simeq \frac{H_{j+1}(G_i \cap H_j)}{H_{j+1}(G_{i+1} \cap H_j)} = \frac{H_{j,i}}{H_{j,i+1}} \qquad \square$$

Theorem 3.5. (Jordan-Hölder) let G be a group and let

$$G = G_1 \supset G_2 \supset G_3 \supset \cdots \supset G_r = \{e\}$$

be a normal tower such that each group G_i/G_{i+1} is simple. and $G_i \neq G_{i+1}$ for $i = 1, \ldots, r-1$, Then any other normal tower of G

having the same props is equivalent to this one.

Pf. $G = G_1 \supset G_2 \supset \cdots \supset G_r = \{e\}$

$G = H_1 \supset H_2 \supset \cdots \supset H_s = \{e\}$.

$G_{ij} = G_{i+1}(H_j \cap G_i)$

$$\begin{tikzcd}
G_{i,1} \arrow[r, phantom, "\supsetneq"] \arrow[d, phantom, "\parallel"] & G_{i,2} \arrow[r, phantom, "\cdots"] & \cdots \supsetneq G_{i,s} \arrow[d, phantom, "\parallel"] \\
G_i \arrow[d, "\text{can.}"'] & & G_{i+1} \arrow[d] \\
G_i/G_{i+1} \arrow[r, phantom, "\trianglerighteq"] & G_{i,2}/G_{i+1} \arrow[r, phantom, "\trianglerighteq"] & \{G_{i+1}\}
\end{tikzcd}$$

third iso thm.

$\Rightarrow$ contradicting to the fact G_i/G_{i+1} is simple.

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how to find.

$G \trianglerighteq H_1 \trianglerighteq \{0\}$.

$G/H_1 \cdots \{H_1\}$

by induction

若 simple 则 ✓

不 simple. can proj 再细分. ✓