

Week 3.

Measure.

Let (X, d) metric space. $2^X = \{\text{all subsets of } X\}$

Def. (Outer Measure) A map $\mu: 2^X \rightarrow [0, +\infty]$ is an outer measure.

ff. 1) $\mu(\emptyset) = 0$

2) $\mu(A) \leq \mu(B) \quad \forall A \subset B$

3) $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j) \quad \forall A_j \subset X.$

We say a subset $A \subset X$ is μ -measurable if
$$\mu(S) = \mu(S \setminus A) + \mu(S \cap A) \quad \forall S \subset X$$
$$\parallel$$
$$\mu(S \cap A^c)$$

Rmk. ① If A is μ -measurable $\Leftrightarrow A^c$ is μ -measurable

② A is μ -measurable $\Leftrightarrow \mu(S) \geq \mu(S \setminus A) + \mu(S \cap A) \quad \forall S \subset X.$

Def. A collection $\mathcal{S} \subset 2^X$ is a σ -algebra, if

1) $\emptyset, X \in \mathcal{S}$

2) If $A \in \mathcal{S}$, then $A^c \in \mathcal{S}$

3) If $A_1, A_2, \dots \in \mathcal{S}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$

Rmk. If $A_1, A_2, \dots \in \mathcal{S}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{S}.$

$$\text{If } \left(\bigcap_{j=1}^{\infty} A_j = X \setminus \bigcup_{j=1}^{\infty} A_j^c, \quad A_j^c \in \mathcal{S}, \quad \bigcup_{j=1}^{\infty} A_j^c \in \mathcal{S} \right)$$

Lemma. The collection \mathcal{M} of all μ -measurable subsets is a σ -algebra.

Furthermore. 1) $A_j \in \mathcal{M}$, $A_j \cap A_i = \emptyset$ then

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$

2) Increasing sequence: $A_1 \subset A_2 \subset \dots \in \mathcal{M}$, then

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j \rightarrow \infty} \mu(A_j)$$

3). Decreasing sequence: $A_1 \supset A_2 \supset \dots \in \mathcal{M}$ and $\mu(A_i) < +\infty$

then

$$\mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j \rightarrow \infty} \mu(A_j)$$

pf σ -algebra is clear ...

1). $A_i \cap A_j = \emptyset$ set $S = A_i \cup A_j$

$$\begin{aligned} \text{Since } A_i \in \mathcal{M} \Rightarrow \mu(A_i \cup A_j) &= \mu(S) = \mu(S \cap A_i) + \mu(S \setminus A_i) \\ &= \mu(A_i) + \mu(A_j) \end{aligned}$$

By induction

$$\mu\left(\bigcup_{i=1}^N A_i\right) = \sum_{i=1}^N \mu(A_i)$$

$$\begin{aligned} \Rightarrow \mu\left(\bigcup_{i=1}^{+\infty} A_i\right) &\geq \lim_{N \rightarrow \infty} \mu\left(\bigcup_{i=1}^N A_i\right) = \lim_{N \rightarrow \infty} \sum_{i=1}^N \mu(A_i) \\ &= \sum_{i=1}^{\infty} \mu(A_i) \end{aligned}$$

By induction of outer measure

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i) \quad \square$$

2). \Rightarrow 互不相交.

set $A_0 = \emptyset$ by 1)

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} (A_i \setminus A_{i-1})\right)$$

$$\stackrel{1)}{=} \sum_{i=1}^{\infty} \mu(A_i \setminus A_{i-1})$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A_i \setminus A_{i-1})$$

$$= \lim_{n \rightarrow \infty} \mu(A_n) \quad \square$$

3). Set $E_j = A_1 \setminus A_j$ then

$$E_1 \subset E_2 \subset E_3 \subset \dots \in \mathcal{M}$$

and $\mu(A_i) = \mu(A_j) + \mu(E_j) \quad \forall j \geq 1$ by (1)

$$\text{and } \bigcup_{j=1}^{\infty} E_j = A_1 \setminus \left(\bigcap_{j=1}^{\infty} A_j \right) \quad (*)$$

$$\stackrel{(*)}{\Rightarrow} \mu(A_1) = \mu\left(\bigcup_{j=1}^{\infty} E_j\right) + \mu\left(\bigcap_{j=1}^{\infty} A_j\right) \quad \text{by (1)}$$

$$= \lim_{j \rightarrow \infty} \mu(E_j) + \mu\left(\bigcap_{j=1}^{\infty} A_j\right) \quad \text{by (2)}$$

$$= \lim_{j \rightarrow \infty} (\mu(A_1) - \mu(A_j)) + \mu\left(\bigcap_{j=1}^{\infty} A_j\right)$$

$$= \mu(A_1) - \lim_{j \rightarrow \infty} \mu(A_j) + \mu\left(\bigcap_{j=1}^{\infty} A_j\right)$$

$$\Rightarrow \mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j \rightarrow \infty} \mu(A_j) \quad \square$$

Def (Borel set) We say the smallest σ -algebra containing all open subsets are Borel sets.

Thm (Caratheodory's criterion) Let (X, d) metric space, μ be an outer measure on X ..

$$\text{if } \mu(A \cup B) = \mu(A) + \mu(B)$$

$$\text{for all } A, B \subset X \text{ with } d(A, B) = \inf \{d(a, b) : a \in A, b \in B\} > 0$$

then all Borel sets are μ -measurable

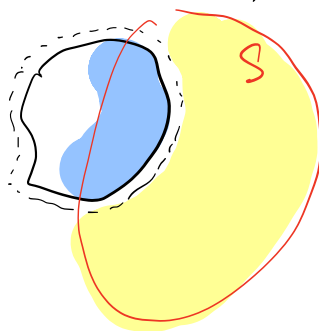
→ by Borel sets's definition
 μ is a σ -algebra

Pf. It suffices to show all closed set C is measurable

i.e. to prove

$$\mu(S) \geq \mu(S \setminus C) + \mu(S \cap C) \quad , \quad \forall S \subset X \quad (\mu(S) < +\infty)$$

$$\text{Set } C_j = \left\{ x \in X : d(x, C) \leq \frac{1}{j} \right\}.$$



$$d(S \setminus C_j, S \cap C) > 0.$$

$$\Rightarrow \mu(S) \geq \mu((S \setminus C_j) \cup (S \cap C)) = \mu(S \setminus C_j) + \mu(S \cap C)$$

need to prove. $\lim_{j \rightarrow \infty} \mu(S \setminus C_j) = \mu(S \setminus C)$

Since C is closed.

$$S \setminus C = \{x \in S : d(x, C) > 0\}$$

$$\Rightarrow S \setminus C = (S \setminus C_j) \cup \left(\bigcup_{k=j}^{\infty} R_k \right), \text{ where } R_k = \left\{ x \in S : \frac{1}{k+1} < d(x, C) \leq \frac{1}{k} \right\}$$

$$\Rightarrow \mu(S \setminus C_j) \leq \mu(S \setminus C) \leq \mu(S \setminus C_j) + \mu\left(\bigcup_{k=j}^{\infty} R_k\right)$$

only need to show. $\lim_{j \rightarrow \infty} \mu\left(\bigcup_{k=j}^{\infty} R_k\right) = 0$

$$\text{Note that } \mu\left(\bigcup_{k=j}^{\infty} R_k\right) \leq \sum_{k=j}^{\infty} \mu(R_k)$$

only need to show $\sum_{k=1}^{\infty} \mu(R_k) < +\infty$

Noting. $d(R_k, R_{k+2}) > 0, \forall k$

$$\sum_{k=1}^N \mu(R_k) = \mu\left(\bigcup_{k=1}^N R_k\right) \leq \mu(S) < +\infty$$

$$\sum_{k=1}^N \mu(R_{2k-1}) = \mu\left(\bigcup_{k=1}^N R_{2k-1}\right) \leq \mu(S) < +\infty$$

$$\Rightarrow \sum_{k=1}^{\infty} \mu(R_k) < +\infty$$

Def. We say a outer measure μ is Borel, if any Borel set is μ -measurable

We say a outer measure μ is Borel-regular if μ is Borel, and

$$\forall A \subset X, \exists \text{ Borel set } B \supset A, \text{ s.t. } \mu(A) = \mu(B)$$

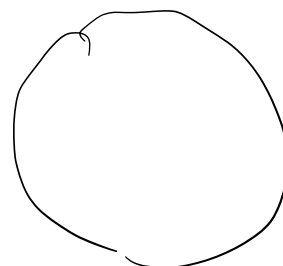
Def. (Hausdorff measure)

Let (X, d) be a metric space, $A \subset X, t \geq 0$.

the t -Hausdorff measure of A :

$$\mathcal{H}^t(A) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^t(A)$$

$$\text{and } \mathcal{H}_\delta^t(A) := \inf \left\{ \sum w_i r_i^t : A \subset \bigcup B_{r_i}(x_i), r_i \leq \delta \right\}$$



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where $w_t = \frac{\pi^{n/2}}{\Gamma(\frac{n+1}{2})} > 0$ $\left(w_n = \text{Vol}(B_1(0)), B_1(0) \subseteq \mathbb{R}^n \right)$

Rmk 1) $\mathcal{H}_s^+(A) \leq \mathcal{H}_{s'}^+(A) \quad \forall s' \leq s$

2) $\mathcal{H}_s^+, \mathcal{H}^+$ are outer measure.

Lemma The Hausdorff measure \mathcal{H}^t , is Borel regular, $\forall t \geq 0$

pf. (use criterion)

① $\forall A, B \subset X$ if $d(A, B) > 2\delta$

then $\mathcal{H}_\delta^+(A \cup B) = \mathcal{H}_\delta^+(A) + \mathcal{H}_\delta^+(B)$

\Downarrow

$\mathcal{H}^+(A \cup B) = \mathcal{H}^+(A) + \mathcal{H}^+(B) \quad \forall d(A, B) > 2\delta$

$\Rightarrow \forall A, B \subset X$ if $d(A, B) > 0$, then

$\mathcal{H}^+(A \cup B) = \mathcal{H}^+(A) + \mathcal{H}^+(B)$

\Rightarrow by criterion. \mathcal{H}^+ is Borel.

② For $A \subset X$. $\forall \delta > 0$. \exists covering $\mathcal{B} = \{B_{r_i}(x_i), r_i \leq \delta\}$ of A

St. $\sum w_t r_i^t - \delta \leq \mathcal{H}^t(A) \leq \sum w_t r_i^t + \delta$

set $U_\delta = \bigcup B_{r_i}(x_i)$ is open.

$\Rightarrow \mathcal{H}_\delta^+(A) \leq \mathcal{H}_\delta^+(U_\delta) \leq \sum_i \mathcal{H}_\delta^+(B_{r_i}(x_i)) \leq \sum w_t r_i^t$

$\Rightarrow \mathcal{H}^+(A) \geq \mathcal{H}_\delta^+(U_\delta) - \delta \quad (*)$

Let $U = \bigcap_{0 < \delta < 1} U_\delta$ is Borel. and $A \subset U \Rightarrow \mu(A) \leq \mu(U)$

by (*). $\mathcal{H}^+(A) \geq \mathcal{H}_\delta^+(U) - \delta$

$\Rightarrow \mathcal{H}^+(A) \geq \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^+(U) = \mathcal{H}^+(U) \quad \square$

Def (Hausdorff dimension)

$A \subset X$.

$\dim_H A := \dim A := \inf \{t \geq 0 : \mathcal{H}^t(A) = 0\}$

$$= \sup \{t > 0: \mathcal{H}^t(A) = +\infty\}$$

Ex. 1). Let (\mathbb{Q}, d) . rational points in \mathbb{R} , $d(x, y) = |x - y|$

then $\dim_H \mathbb{Q} = 0$

$$\Rightarrow \dim_H \mathbb{R}^n = n$$

pf. 1) $\forall t > 0$. show $\mathcal{H}^t(\mathbb{Q}) = 0$

since \mathbb{Q} is countable, a_1, a_2, a_3, \dots

$$\forall \delta \quad \mathbb{Q} \subseteq \bigcup_i B_{\delta/2^i}(a_i)$$

$$\Rightarrow \mathcal{H}_\delta^t(\mathbb{Q}) \leq \sum_i \omega_i (\delta/2^i)^t = \omega_1 \delta^t \sum_i 2^{-it} \leq C_t \omega_1 \delta^t$$

$$\Rightarrow \mathcal{H}^t(\mathbb{Q}) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^t(\mathbb{Q}) \leq 0$$

$$\Rightarrow \dim_H \mathbb{Q} = 0$$

2) n 个圆及论 $\Rightarrow H$ 测度 = L 测度.

$$\forall B_R(0) \subseteq \mathbb{R}^n$$

$$\forall B_{r_i}(x_i) \supset B_R(0) \quad r_i < \delta$$

$$\Rightarrow \sum \omega_i r_i^t = \sum \omega_i r_i^n \cdot \frac{\omega_i}{\omega_n} r_i^{t-n} \leq \delta^{t-n} \frac{\omega_1}{\omega_n} \sum \omega_n r_i^n \rightarrow 0$$

$$\Rightarrow \dim B_R(0) \leq n, \quad \mathcal{H}^n(B_R(0)) > 0 \Rightarrow \dim_H B_R(0) = n$$

$$\text{Since } \mathbb{R}^n = \bigcup_{i=1}^{\infty} B_i(0)$$

$$\text{Since } \mathcal{H}^t(B_i(0)) = 0 \quad \forall t > n$$

$$\Rightarrow \mathcal{H}^t(\mathbb{R}^n) = \lim_{t \rightarrow \infty} \mathcal{H}^t(B_i(0)) = 0$$

$$\Rightarrow \dim_H \mathbb{R}^n \leq n \quad \text{since } \dim_H B_1(0) = n \text{ and } B_1(0) \subset \mathbb{R}^n$$

$$\Rightarrow \dim_H \mathbb{R}^n = n$$

Def. (Minkowski dimension) $A \subset X$

$$\overline{\dim}_M A = \overline{\lim}_{\varepsilon \rightarrow 0} \frac{\log N(A, \varepsilon)}{\log(\frac{1}{\varepsilon})}$$

$$\underline{\dim}_M A = \underline{\lim}_{\varepsilon \rightarrow 0} \frac{\log N(A, \varepsilon)}{\log(\frac{1}{\varepsilon})}$$

where $N(A, \varepsilon) = \inf \# \{k: \exists B_2(x_1), \dots, B_2(x_k) \text{ s.t. } A \subset \bigcup_{i=1}^k B_2(x_i)\}$

Rmk. In \mathbb{R}^n . $N(B_1(0), \varepsilon) \simeq \frac{\text{Vol}(B_1(0))}{\text{Vol}(B_\varepsilon(0))} \simeq C(n) \varepsilon^{-n}$

$$\frac{\log N(B_1(0), \varepsilon)}{\log(\frac{1}{\varepsilon})} \simeq \frac{\log C_n + n \log(\frac{1}{\varepsilon})}{\log(\frac{1}{\varepsilon})} \rightarrow n$$

If $\overline{\dim}_M A = \underline{\dim}_M A$, we call $\dim_M A = \overline{\dim}_M A = \underline{\dim}_M A$

Rmk. $\dim_M(\mathbb{Q}) = 1$ \nearrow in \mathbb{R}

$\dim_M(\mathbb{Q}^n) = n$
 \hookrightarrow in \mathbb{R}^n