更多的例子.

$\mathbb{RP}^n$ 的齐次坐标

Def $\mathbb{RP}^n = \left(\mathbb{R}^{n+1}\setminus\{0\} \big/ \overbrace{\mathbb{R}^*}^{\mathbb{R}-\{0\}}\right) \iff \vec{x}\sim\vec{y}$

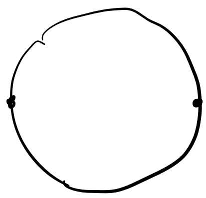
i.e. $\vec{x} = k\vec{y}$, $k\in\mathbb{R}^*$

任意 $p\in\mathbb{RP}^n$ 有代表元 $(x_0, x_1, \dots, x_n)$

这个等价类写成 $[x_0, x_1, \dots, x_n]$ (齐次坐标)

eg $\mathbb{RP}^1 = (\mathbb{R}\setminus\{0\})/\mathbb{R}^*$

$[x,y] = [2x,2y] = [kx,ky]$



$\simeq S^1$

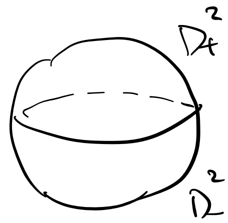
$\mathbb{RP}^n$ 之间的关系

1). 在 $\mathbb{RP}^2 = \{[x_0,x_1,x_2] \mid x_0, x_1, x_2\in\mathbb{R}\}$

取 $\{x_2=0\}\subset\mathbb{RP}^2$

$\Downarrow$

$\mathbb{RP}^1 \nearrow$



2) $\mathbb{RP}^n \supset \{x_n=0\} \cong \mathbb{RP}^{n-1}$

3) $\mathbb{RP}^n\setminus\mathbb{RP}^{n-1} = \mathring{D}^n \cong \mathbb{R}^n$

$\mathbb{RP}^n = \mathbb{RP}^{n-1}\sqcup\mathring{D}^n$.

$\hookrightarrow$ 我们希望去理解是如何粘的.

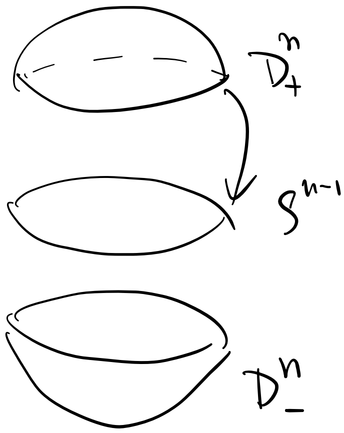
eg (1) $S^n = \{N\} \sqcup \mathring{D}^n$

$= \{N\} \sqcup D^n / \sim$

边界全部粘到一起

$f: \partial D^n = S^{n-1} \xrightarrow{c} N$ $\quad$ $x \in \partial D^n \sim f(x)$

(2) $S^n = S^{n-1} \sqcup D_+^n \sqcup D_-^n / \sim$



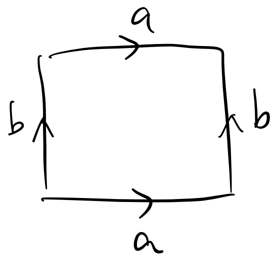
$f: \partial(D_+^n \sqcup D_-^n) \simeq S_+^{n-1} \sqcup S_-^{n-1} \to S^{n-1}$

$f(x) \sim x$

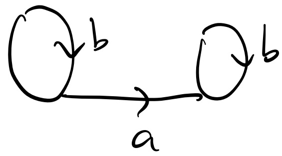
考虑 $\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup D^n / \sim$

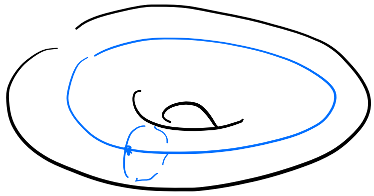
$f: \partial D^n \cong S^{n-1} \xrightarrow{c.p.} \mathbb{RP}^{n-1}$ $\quad$ 看  来理解.

闭曲面的多边形表示.

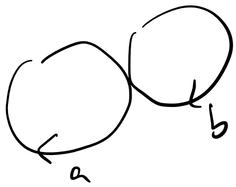


只看边界 $\Longrightarrow$





$\Longrightarrow$



$= S^1\wedge S^1$ (单点并)

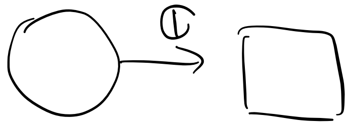
Def. $X\wedge Y := X\sqcup Y/\sim$ 给定 $x\in X,\ y\in Y$.

$x\sim y$

$T^2 = D^2 \sqcup (S^1\wedge S^1)/\sim_1$ 骨架+肉

①

存在 $\varphi: D^2 \xrightarrow{\sim} C$ 正方形



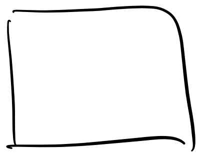
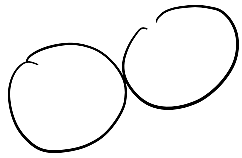
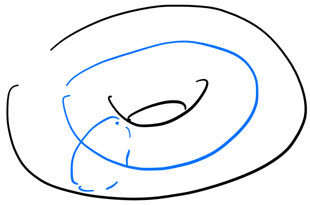
$\xrightarrow[\text{再加骨架}]{②}$

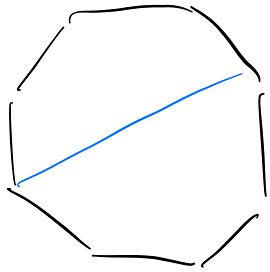
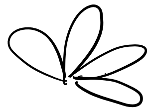
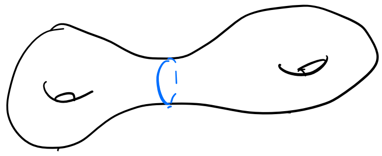
$\varphi|_{\partial D^2} \xrightarrow{\sim} \partial C \Rightarrow C = D^2 \sqcup \partial C/\sim_0 \quad \varphi|_{\partial D^2}(x) \sim_0 x$

② $f: \partial D^2 \to S^1\wedge S^2$ 可分解为 $\varphi|_{\partial D^2}: \partial D^2 \to \partial C$

$P: \partial C \xrightarrow{c.p.} S^1\wedge S^1$

$f: P\cdot\varphi|_{\partial D^2}$

 $\longrightarrow$  在  中.

 $\xrightarrow{?}$  在  中?

胞腔复形

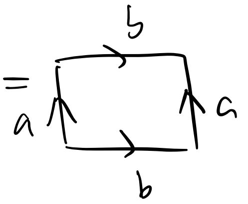
Def. X 是一个胞腔复形，若 $X=\bigcup_{n\in\mathbb{N}} X^n$，且

1). X^0 是(有限)离散点集 (零维胞腔)

2). X^n 为一个商空间，$X^{n-1}\sqcup\{e^n_\alpha\}_{\alpha\in A}/\sim$. $e^n_\alpha\simeq D^n$ (n维胞腔).

$f:\partial e^n_\alpha\longrightarrow X^{n-1}$ 为粘贴映射. $f(x)\sim x$ $|A|<\infty$ (局部有限)

3). $U\subseteq X$ 开 $\iff$ $U\cap X^n$ 开对所有 X^n

e.g. (1). $\mathbb{T}^2=$  $=$ ① $\cdot\, e^0$ X^0

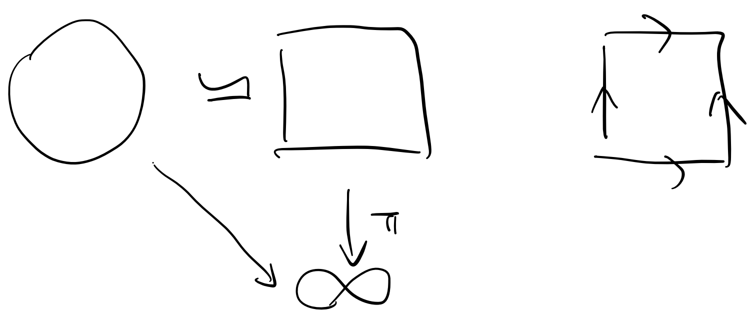
② $(\cdot\sqcup {}^{p}\!\text{—}^{q}\sqcup {}^{r}\!\text{—}^{s})/\sim$ X^1

$f:\{p,q,r,s\}\to e^0$.

$\Downarrow$

∞ (1-skeleton)

③ $(\infty\sqcup D^2)/\sim$ X^2



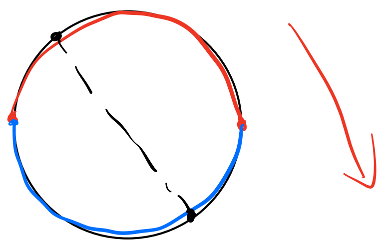
(2). $\mathbb{RP}^2$ X^0 : $\cdot$ e^0

X^1 : $\bigcirc = (\cdot \amalg \overset{a}{\vdash}\!\!-\!\!\overset{b}{\dashv}) / f: \{a,b\} \to e^0$

X^2 : $\mathbb{RP}^2 = (\bigcirc \amalg D^2)/\sim$

$S^1 = \{|z|=1\} = \{e^{i\theta} : \theta \in [0, 2\pi]\}$.

$f: z \longmapsto z^2$ 一圈映到两圈. $\Rightarrow$ 对径点相连.



$x \sim y \Leftrightarrow f(x) = f(y)$

且 $x \sim f(x)$

$\Rightarrow$ 全粘起来.

(3)

: $X^0 = \cdot$

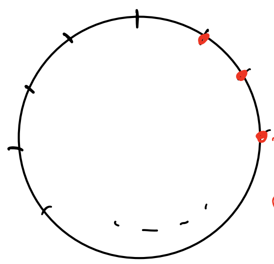
$X^1 = \infty$

$X^2 = (\infty \amalg D^2)/\sim$.

(4). $X^0 = \cdot$

$X' = X^1 \amalg D^2/\sim$. $f: S^1 \longrightarrow S^1$

$z \longmapsto z^n$



每一块粘起来 2+3 笨帽.

$n \neq 2$ 时 不是流形? 看这种点.

类似 Y $\times[0,1]$ 这种.

不 $\simeq \mathbb{R}^n$.

(5) $\mathbb{RP}^n$ $\qquad \mathbb{RP}^n \setminus \mathbb{RP}^{n-1} = \mathring{D}^n$

$$\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup D^n / \sim$$

$$f: \partial D^n \longrightarrow \mathbb{RP}^{n-1}, \quad (z_0, \dots, z_{n-1}) \longmapsto [z_0, \dots, z_{n-1}]$$

$$\partial D^n \ni x \sim y \Leftrightarrow f(x) = f(y)$$

$\Longrightarrow$ $\mathbb{RP}^\infty$