Chap 6

复积分

$\gamma:[a,b]\to\mathbb{C}$. $\qquad P=\{t_0,\dots,t_n\}$.

$\qquad\qquad\qquad\qquad z_k=\gamma(t_k)$.

$$S(P)=\sum_{k=1}^{n} f(\zeta_k)(z_k-z_{k-1}) \quad |P|\to 0. \Rightarrow \int_\gamma f(z)\,dz=\lim_{|P|\to 0} S(P)$$

$$\int_\gamma f(z)\,dz=\int_\gamma u\,dx-v\,dy+i\int_\gamma u\,dx+u\,dy$$

$$\int_\gamma f(z)\,dz=\int_a^b f(\gamma(t))\gamma'(t)\,dt$$

$$\left\{\begin{aligned}&\int_\gamma |f(z)dz|=\int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt\\&\int_\gamma f(z)\,|dz|=\int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt\\&\int_\gamma f(z)\,d\bar z=\int_a^b f(\gamma(t))\,\overline{\gamma'(t)}\,dt\end{aligned}\right.$$

性质.

$$\int_{\gamma^-} f(z)\,dz=-\int_\gamma f(z)\,dz.$$

$$\int_{\tilde\gamma} f(z)\,dz=\int_\gamma f(z)\,dz$$

$$\int_\gamma(\alpha f(z)+\beta g(z))\,dz=\alpha\int_\gamma f(z)\,dz+\beta\int_\gamma g(z)\,dz$$

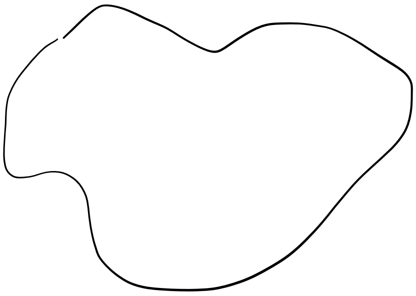
$$\int_{\gamma_1\cup\gamma_2} f(z)\,dz=\int_{\gamma_1} f(z)\,dz+\int_{\gamma_2} f(z)\,dz$$

☆ 复积分基本不等式

$$\left|\int_\gamma f(z)\,dz\right|\leq\int_\gamma |f(z)||dz|\leq\max_{z\in\gamma}|f(z)|\,L(\gamma)$$

Ex. $\int_\gamma z^n dz$.

$\gamma(t) = e^{it}$, $0 \le t \le 2\pi$



→ 原函数

Thm 6.1 $\Longrightarrow$ 判断原函数，排除一些函数无原函数.

e.g. $f(z) = \bar{z}$ $\quad \int_\gamma \bar{z}\, dt \overset{e^{it}}{=\!=} \int_0^{2\pi} e^{-it}\, i e^{it}\, dt = 2\pi i \ne 0$.

$f(z) = \frac{1}{z}$ $\quad \int_\gamma \frac{1}{z}\, dt \overset{e^{it}}{=\!=} \int_0^{2\pi} e^{-it}\, i e^{it}\, dt = 2\pi i \ne 0$

$\frac{1}{z}$. 上半平面 (任何不包括0的单连通区域上有原函数)

$\log z$.

$\gamma_1(t) = t$

$\gamma_2(t) =$

6.4 习题

1、 设 $F_1(z)$, $F_2(z)$ 为 $f(z)$ 的原函数.

$$F_1'(z) = F_2'(z) = f(z)$$

$$\Rightarrow (F_1(z) - F_2(z))' = F_1'(z) - F_2'(z) = 0$$

$$\Rightarrow F_1(z) - F_2(z) = c \in \mathbb{C}.$$

$\Rightarrow$ $F(x)$ 在差一个常数意义下唯一.

2. $f(z) = \frac{1}{z}$ 在 $\mathbb{C} \setminus \{0\}$ 上

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} e^{-it} \cdot i e^{it}\,dt = 2\pi i \neq 0$$

$$\gamma(t) = e^{it}$$

$f(z) = \bar{z}$ 在 $\mathbb{C}$ 上

$$\int_\gamma f(z)\,dz \underset{\gamma(t)=e^{it}}{=} \int_0^{2\pi} e^{-it}\, i e^{it}\,dt = 2\pi i \neq 0$$

$\Rightarrow$ 均不存在原函数

4. $\gamma(t) = z_0(1-t) + t z_1$. $t \in [0,1]$ 线路 $[z_0, z_1]$

$$\int_\gamma \mathrm{Re}(z)\,dz = \int_0^1 \mathrm{Re}(z_0(1-t) + t z_1)(z_1 - z_0)\,dt$$

$$= \int_0^1 \left[\mathrm{Re}(z_0)(1-t) + \mathrm{Re}(z_1)t\right](z_1 - z_0)\,dt$$

$$= \mathrm{Re}(z_0)(z_1 - z_0) + \frac{(\mathrm{Re}(z_1) - \mathrm{Re}(z_0))(z_1 - z_0)}{2}$$

$$= \frac{(z_1 - z_0)(\mathrm{Re}(z_1) + \mathrm{Re}(z_0))}{2}$$

$$\int_\gamma \bar{z}\,dz = \int_0^1 (\bar{z}_0(1-t) + t\bar{z}_1)(z_1 - z_0)\,dt$$

$$= \int_0^1 (\bar{z}_0 z_1 - |z_0|^2) + |z_1 - z_0|^2 t\,dt$$

$$= \bar{z}_0 z_1 - |z_0|^2 + \frac{1}{2}|z_1 - z_0|^2$$

6. f · z_0 连续

①
$$f(z_0) = \lim_{r\to 0^+} \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z-z_0}\,dz.$$

$$\gamma(t) = z_0 + re^{i\theta}$$

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{\cancel{re^{i\theta}}}\,\cancel{r}\cdot i\cancel{e^{i\theta}}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta.$$

$$\lim_{r\to 0^+}\left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta - f(z_0)\right|$$

$$\le \lim_{r\to 0^+} \frac{1}{2\pi}\int_0^{2\pi}\left|f(z_0+re^{i\theta})-f(z_0)\right|d\theta$$ 有 $\forall \varepsilon>0\ \exists\delta>0.$ $r<\delta$ 时 有 $\left|f(z_0+re^{i\theta})-f(z_0)\right|<\varepsilon$

$$\le \lim_{r\to 0^+} \frac{1}{2\pi}\,2\pi\,\varepsilon = 0.$$

②
$$\lim_{r\to 0^+} \frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(z)}{z-z_0}\,d\bar{z}$$

$$\gamma(t) = z_0 + re^{i\theta} \qquad d\ z_0+re^{-i\theta}.$$

$$\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{re^{i\theta}}\left(-r\cdot i\,e^{-i\theta}\right)d\theta$$

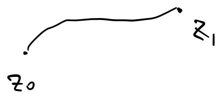
$$= \frac{-1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,e^{-2i\theta}\,d\theta \longrightarrow 0$$

③
$$\lim_{r\to 0^+} \frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(z)}{z-z_0}\,|dz| = 0$$

$$\frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(z)}{z-z_0}\,|dz| = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{re^{i\theta}}\,r\,d\theta.$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} f(z_0+re^{i\theta})\,e^{-i\theta}\,d\theta \to 0.$$

7. f. Ω. C^1 实可微函数



1). $\int_\gamma \left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z\right) = f(z_1) - f(z_0)$

$$\frac{df(\gamma(t))}{dt} = \frac{\partial f}{\partial z}(\gamma(t))\,\gamma'(t) \quad + \frac{\partial f}{\partial \bar z}(\gamma(t))\,\bar\gamma'(t)$$

$$\Rightarrow df(\gamma(t)) = \frac{\partial f}{\partial z}(\gamma(t))\,\gamma'(t)\,dt + \frac{\partial f}{\partial \bar z}(\gamma(t))\,\bar\gamma'(t)\,dt.$$

$$\Rightarrow \int_\gamma \left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar z}d\bar z\right) = \int_\gamma d f(\gamma(t)) = f(z_1) - f(z_0)$$

2). f 取实数值

$$\mathrm{Re}\left(\int_\gamma \frac{\partial f}{\partial z}dz\right) = \frac{1}{2}\left(f(z_1) - f(z_0)\right)$$

下证. $\int_\gamma \frac{\partial f}{\partial z}dz = \int_\gamma \frac{\partial f}{\partial \bar z}d\bar z$

$$\mathrm{RHS} = \int_\gamma \frac{\partial f}{\partial \bar z}(\gamma(t))\,\bar\gamma'(t)\,dt$$

$$= \int_\gamma \overline{\frac{\partial f}{\partial z}(\gamma(t))\,\gamma'(t)\,dt}$$

$$= \overline{\int_\gamma \frac{\partial f}{\partial z}(\gamma(t))\gamma'(t)dt} = \int_\gamma \frac{\partial f}{\partial z}(\gamma(t))\gamma'(t)\,dt = \int_\gamma \frac{\partial f}{\partial z}dz.$$

8. f 在 a 处实可微.

$$\lim_{r\to 0} \frac{1}{r^2}\int_{|z-a|=r} f(z)dz = 2\pi i \frac{\partial f}{\partial \bar z}(a).$$

$$\lim_{r\to 0}\frac{1}{r^2}\int_{|z-a|=r} f(z)\,dz = \lim_{r\to 0}\frac{1}{r^2}\int_{|z-a|=r} f(a)+\frac{\partial f}{\partial z}(a)(z-a)+\frac{\partial f}{\partial \bar{z}}(a)(\bar{z}-\bar{a})\ dz$$

$$= \lim_{r\to 0}\frac{1}{r^2}\int_0^{2\pi}\left(f(a)+\frac{\partial f}{\partial z}(a)(re^{i\theta})+\frac{\partial f}{\partial \bar{z}}(a)(re^{-i\theta})\right) ri e^{i\theta}\,d\theta$$

$$= \lim_{r\to 0}\int_0^{2\pi} 0+0+\frac{\partial f}{\partial \bar{z}}(a)\cdot i\,d\theta$$

$$= 2\pi i\,\frac{\partial f}{\partial \bar{z}}(a)$$

11. f 为平面区域 Ω 上连续复值函数.

$$\left|\int_\Omega f(x)\,dxdy\right|$$

$$= e^{-i\theta}\int_\Omega f(z)\,dz = \iint e^{-i\theta} f(z)\,dxdy$$

$$= \iint \mathrm{Re}\left(e^{-i\theta} f(z)\right) dxdy$$

$$\leq \iint |f(z)|\,dxdy$$

$$= \int_\Omega |f(z)|\,dxdy$$

$$\mathrm{Im}(e^{i\theta} f(z)) = 0$$

$f(z) = re^{i\theta}$ $\quad \forall z \quad f(z)$ 辐角相同