Algebra Lec 9. Semidirect products

Recall: $H, K \le G$

$HK = G$; $H \cap K = \{e\}$; $xy = yx \quad \forall x \in H, y \in K$

$\Rightarrow G \cong H \times K$ in Lec 1.

generalization
- $H_1, \ldots, H_n$ commute with each other
- $H_1 H_2 \cdots H_n = G$
- $H_{i+1} \cap (H_1 \cdots H_i) = \{e\}$

$\Rightarrow G \cong H_1 \times H_2 \times \cdots \times H_n$ (by induction)

Rmk 1. $H, N \le G$

$HN = \{hn \mid h \in H, n \in N\}$

The number of distinct ways of writing elements of HN in the form hn. $= |H \cap K|$

pf. $HN = \bigcup_{h \in H} hN$ $\qquad h_1 N = h_2 N$

$\Leftrightarrow h_2^{-1} h_1 \in N$ when h_1 is fixed

h_2 have $|H \cap N|$ options

Rmk 2. $H \leq G$, $N \trianglelefteq G$. $(H \leq N_N)$

s.t. $H \cap N = \{e\}$.

2nd iso thm
$\Rightarrow$ $HN = HN \leq G$

by last Rmk $\Rightarrow$ $|HN| = |H||N| / |H \cap N| = |H||N|$

and every elem in NH can be written as nh

$n \in N$, $h \in H$ uniquely

$$(n_1 h_1)(n_2 h_2) = (n_1 h_1 n_2 h_1^{-1})(h_1 h_2)$$

$$\begin{array}{cc} \| & \| \\ n_3 & h_3 \\ \cap & \\ N & \end{array}$$

Motivation. Start form 2 groups H. N

Construct a group G. s.t. $\begin{cases} N \trianglelefteq G.\ H \leq G \\ H \cap N = \{e\} \end{cases}$

(后面构造) 这里的 G 不是上面的 G. 而是 HN 这个群.

Given.

Def. $\varphi: H \longrightarrow \mathrm{Aut}(N)$. group hom

$h \longmapsto \varphi(h)$

$\Downarrow$

$h \cdot n \triangleq \varphi(h)(n)$. $\Rightarrow$ group action $h_1 \cdot h_2 \cdot n = h_1(\varphi(h_2)(n))$

$$= \varphi(h_1)\left(\varphi(h_2)(n)\right)$$

$$= \varphi(h_1 h_2)(n)$$

$$= (h_1 \cdot h_2) \cdot n$$

Let $G \triangleq \{ (n, h) \mid n \in N, h \in H \}$

$$(n_1, h_1) \cdot (n_2, h_2) \triangleq \left(n_1 \cdot (h_1 \cdot n_2) \, , \, h_1 h_2 \right)$$

$$\Rightarrow (n, h)^{-1} = (h^{-1} n^{-1}, h^{-1})$$

$\Rightarrow$ ① The multiplication makes G into a group of order $|N||H|$

② $N \simeq \underbrace{\{(n, 1) \mid n \in N\}}_{= \tilde{N}} \trianglelefteq G$

$H \simeq \underbrace{\{(1, h) \mid h \in H\}}_{= \tilde{H}} \leq G$

③ $\tilde{N} \cap \tilde{H} = \{(e, e)\}$

④ $\forall n \in N, h \in H$

$$(1, h) \cdot (n, 1)(1, h)^{-1} = (h \cdot n, 1)$$

pf. ① • check associativity

• $(n, h)^{-1} = (h^{-1} \cdot n^{-1}, h^{-1})$

• $|G| = |H| \cdot |N|$

② $\{(n, 1) \mid n \in N\} \leq G$

$\{(1,h) \mid h \in H\} \leq G$ check the inverse

③ $\tilde{N} \cap \tilde{H} = \{(e,e)\}$

④ $(1,h)(n,1)(1,h)^{-1} = (h\cdot n, 1)$

$\Rightarrow H \leq N_H$

with $N \leq N_H$

$\Rightarrow NH \leq N_H \Rightarrow G = N_H$

$\Rightarrow N \trianglelefteq G$

Def. $\varphi: H \to \mathrm{Aut}(N)$ hom.

Construct G. s.t.
- $H \leq G$
- $N \trianglelefteq G$
- $N \cap H = \{e\}$
- $|G| = |N||H|$

$N \rtimes_\varphi H$

the semidirect product of H and N with $\varphi: H \to \mathrm{Aut}(N)$

Prop. N, H : group.

$\varphi: H \to \mathrm{Aut}(N)$ hom

when semi-direct product $\Rightarrow$ direct product

TFAE

① $N \rtimes_\varphi H \xrightarrow{\phi} N \times H$ group hom

$(n,h) \longmapsto (n,h)$

② φ trivial (i.e. $\varphi(h) = $ identity map on N)

③ $H \trianglelefteq (N \rtimes_\varphi H)$

pf. ① $\Rightarrow$ ②

$$\phi((n_1, h_1), (n_2, h_2)) = \phi((n_1, h_1))\,\phi((n_2, h_2))$$

$$= (n_1 n_2, h_1 h_2)$$

$$\|$$

$$\phi((n_1(h_1 n_2), h_1 h_2)) = (n_1 h_1 n_2, h_1 h_2)$$

$$\Rightarrow h_1 n_2 = n_2 \quad . \quad \forall h_1 \in H.\ n_2 \in N$$

② $\Rightarrow$ ③

in G

$$(\tilde{n}, \tilde{h})(1, h)(\tilde{n}, \tilde{h})^{-1}$$

$$\underset{}{\overset{(\text{use } \varphi \text{ trivial})}{=}} \cdots = (1, \tilde{h} h \tilde{h}^{-1})$$

$\therefore H \trianglelefteq (N \rtimes_\varphi H)$

③ $\Rightarrow$ ②

$H \trianglelefteq (N \rtimes_\varphi H)$

(in G)

then $n h n^{-1} h^{-1} \in N \cap H = \{e\}$

$$\Rightarrow nh = hn \Rightarrow h n h^{-1} = n$$

✓

then action of H on N is trivial

why ?

$$(1,h)(n,1)(1,h^{-1}) = (hn,1) = (n,1) \checkmark$$

② $\Rightarrow$ ① clear

Semi-direct product recognition theorem

Thm. G: group . $N, H \leq G$

s.t. ① $N \trianglelefteq G$

② $H \cap N = \{e\}$

Let. $\varphi: H \longrightarrow \mathrm{Aut}(N)$ group hom.

$$h \longmapsto \left(\begin{array}{l} N \longrightarrow N \\ n \longmapsto hnh^{-1} \end{array}\right)$$

$\Rightarrow NH \cong N \rtimes_\varphi H$

Moreover if $G = NH \Rightarrow G \cong N \rtimes_\varphi H$.

HW.

[L]. Chap I ex 29 ex 32 ex 40 (a) (b)

ex 41 (a) (b) (c)

ex 43.

eg. $|G| = pq$, $q > p$. $p \mid q-1$

$n_p = 1, q$ $\quad n_p \mid q$

$n_q = 1$,

$|Q| = q$. $Q \trianglelefteq G$ $(n_q = 1)$

$\left\{\begin{array}{l} |P| = p \cdot P \leq G \quad (n_p = 1, q) \\ P \cap Q = \{e\} \quad (\text{for } (p,q) = 1) \end{array}\right.$ $\Rightarrow G = PQ$ for

$$|G| = \underline{pq} = \frac{|P||Q|}{|P \cap Q|} = |P||Q|$$

$\Rightarrow G \cong Q \rtimes_\varphi P$ $\qquad P = \langle x \rangle$. $Q = \langle y \rangle$

$$\varphi: P \longrightarrow \mathrm{Aut}(Q) = \mathrm{Aut}(\langle y \rangle) \cong \mathrm{Aut}(\mathbb{Z}/q\mathbb{Z}) \cong (\mathbb{Z}/q\mathbb{Z})^\times \cong (\mathbb{Z}/(q-1)\mathbb{Z})$$

$$x \longmapsto \begin{pmatrix} Q \to Q \\ y \mapsto xyx^{-1} \end{pmatrix}$$

for $|x| = p \Rightarrow x$ is sent to an element in $\mathbb{Z}/(q-1)\mathbb{Z}$ of order 1 or p.

then. $\langle \gamma \rangle \leq \mathbb{Z}/(q-1)\mathbb{Z}$ $\quad \langle \gamma \rangle$ is the unique subgroup of order p

①. φ is trivial $\Rightarrow Q \rtimes_{\varphi} P = Q \times P$

②. $\mathrm{im}(\varphi) = \langle \gamma \rangle \leq \mathbb{Z}/(q-1)\mathbb{Z}$.

i.e. $\varphi_i(x) = \gamma^i \quad i = 1, \ldots, p-1$ ($p-1$ 种 φ)

$\Downarrow$

$\Rightarrow \exists n_i$. s.t. $\varphi_i(x^{n_i}) = \gamma$

but actually $Q \rtimes_{\varphi_1} P \cong Q \rtimes_{\varphi_i} P$

$(y, x) \longmapsto (y, x^{n_i})$

$\Rightarrow G$ 两种 , $\mathbb{Z}_p \times \mathbb{Z}_q$, and $Q \rtimes_{\varphi_1} P$ (S_3)