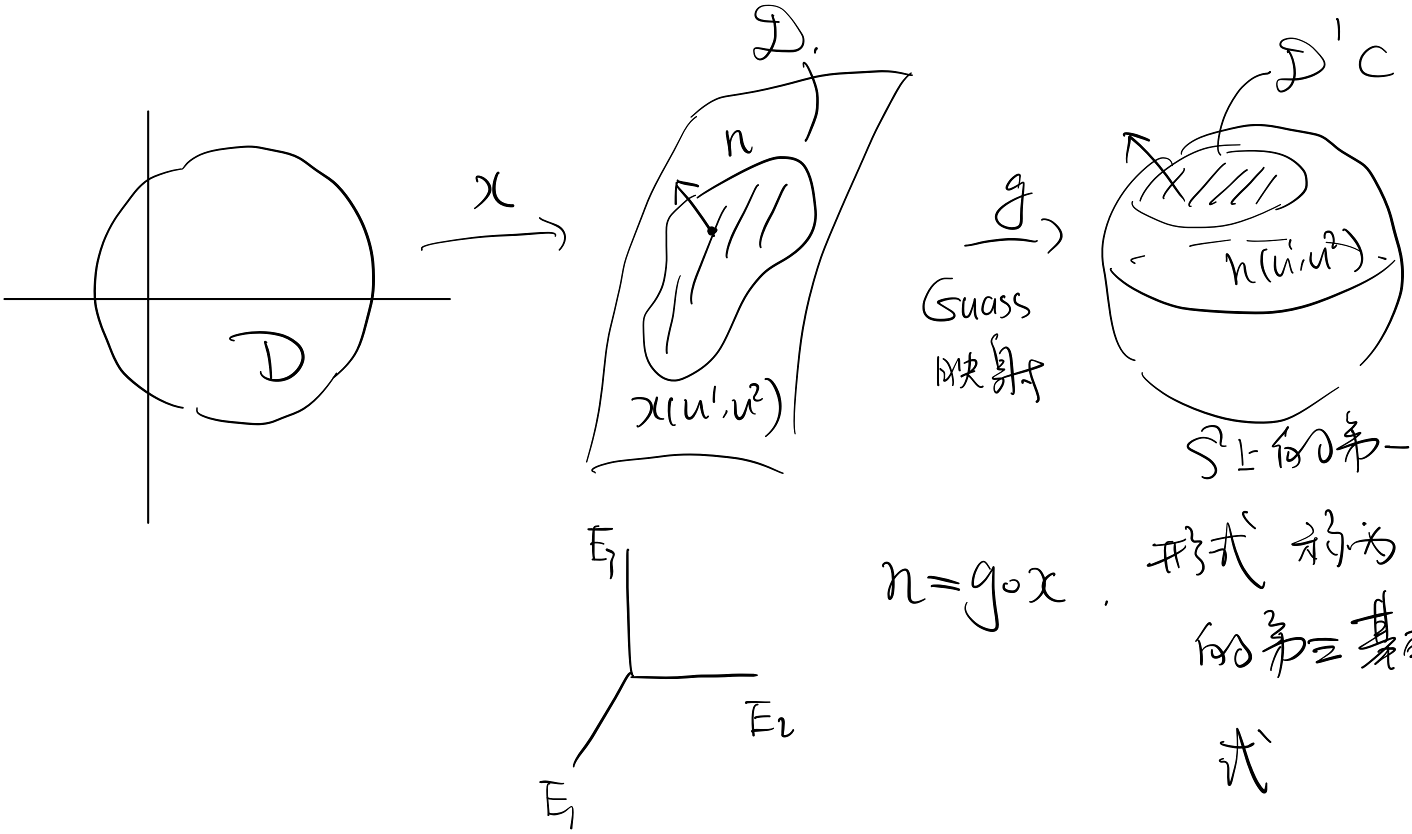
$x: D \longrightarrow x(D) \subset \mathbb{R}^3$

$x_1 \times x_2 \neq 0$ 正则



$n = g \circ x$. S^2上的第一基本形式称为M上的第三基本形式

$$\mathrm{III} = |dn|^2 = dn \cdot dn.$$

$$= n_\alpha n_\beta \, du^\alpha du^\beta \triangleq f_{\alpha\beta} \, du^\alpha du^\beta$$

$$f_{\alpha\beta} = n_\alpha n_\beta$$

Thm. $\mathrm{III} - 2H\,\mathrm{II} + K\,\mathrm{I} = 0$

Pf. 先考虑无脐点

取曲率线网为参数网

x_1, x_2 主方向 k_1, k_2 主曲率

$h_{11} = k_1 g_{11} \qquad h_{22} = k_2 g_{22} \qquad g_{12} = h_{12} = 0$

$n_\alpha = -h_\alpha^\beta x_\beta = -k_\alpha x_\alpha$.

$\mathrm{III} = n_\alpha n_\beta du^\alpha du^\beta$.

$= (k_\alpha x_\alpha)(k_\beta x_\beta) du^\alpha du^\beta$

$= k_\alpha k_\beta g_{\alpha\beta} du^\alpha du^\beta = k_1^2 g_{11} (du^1)^2 + k_2^2 g_{22} (du^2)^2$

$\mathrm{II} = k_1 g_{11} (du^1)^2 + k_2 g_{22} (du^2)^2$

$2H\,\mathrm{II} = (k_1 + k_2)(k_1 g_{11} (du^1)^2 + k_2 g_{22} (du^2)^2)$

$= \mathrm{III} + K\,\mathrm{I}$

② x_0为脐点 在x_0处任取幺正标架.

$k_1 = k_2 = H = \lambda$. $K = \lambda^2 \Rightarrow$ 代入验证.

Gauss 曲率的另一种表示.

$K = k_1 k_2$

用 Gauss 映射 ⇒ Minkowski 问题

Given f. 反问题 $\underline{K = f}$?

$k_1 k_2$ - $\underline{\text{种面积变换}}$

$\mathcal{D}'$ 的面积 $dA_g = |n_1 \times n_2|\, du^1 du^2$

$$= \left|(h_1^1 x_1 + h_1^2 x_2) \times (h_2^1 x_1 + h_2^2 x_2)\right| du^1 du^2$$

$$= |h_1^1 h_2^2 - h_1^2 h_2^1|\, |x_1 \times x_2|\, du^1 du^2$$

$$= |\det(h_\alpha^\beta)|\, dA$$

M.

$$= |K|\, dA.$$

$$\Rightarrow \quad |K(x_0)| = \frac{dA_g}{dA}$$ 不够准确

$$A(\mathcal{D}') = \int_{\mathcal{D}'} dA_g$$

$$= \iint_D |n_1 \times n_2|\, du^1 du^2 = \iint_D |K(x)|\, |x_1 \times x_2|\, du^1 du^2$$

$$A(\mathcal{D}) = \int_{\mathcal{D}} dA = \iint_{D} |x_1 \times x_2| \, du^1 du^2$$

中值定理.

$$= |K(x')| \iint_{D} |x_1 \times x_2| \, du^1 du^2$$

$$= |K(x')| A(\mathcal{D})$$

先 $\downarrow$ $\mathcal{D}$

$$|K(x_0)| = \lim_{D \to x_0} \frac{A(\mathcal{D}')}{A(\mathcal{D})}$$

or $\mathcal{D} / \mathcal{D}'$

总之缩到某一点.

actualy K 与 II 无关

~~内积~~量

可以用 $g_{\alpha\beta}$ 算出来!

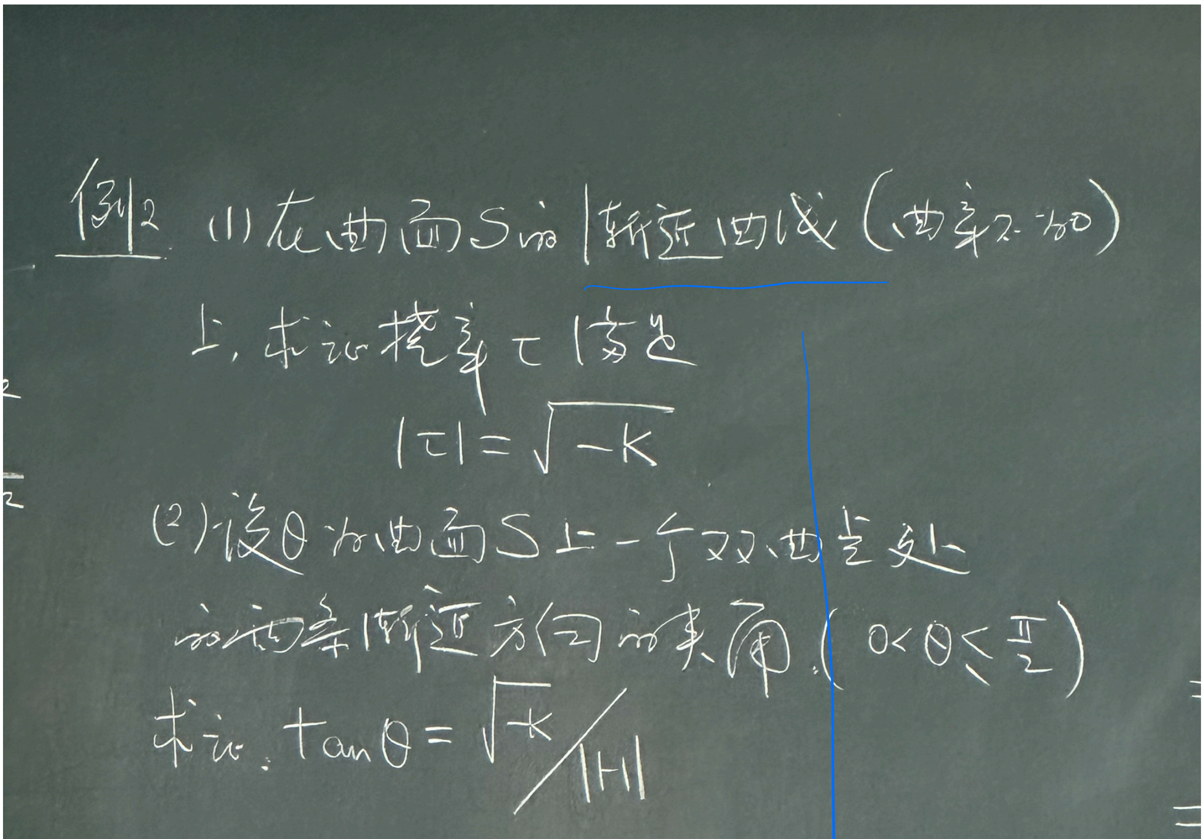
直纹面

$$x(u^1,u^2) = a(u^1) + u^2 b(u^1) \qquad |b| = 1$$

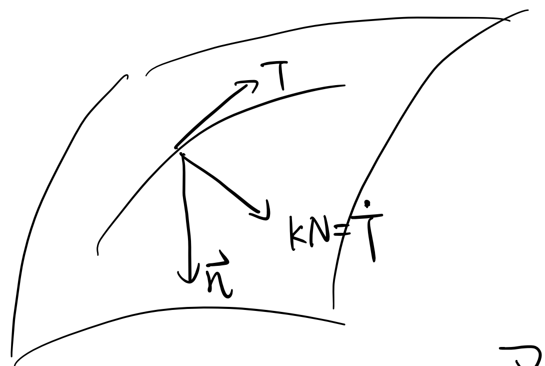
Gauss 曲率. $$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{(a', b, b')^2 / |(a'+u^2 b')\times b|^2}{|a'+u^2b'|^2 - (a'\cdot b)^2}$$

可展 $\Longleftrightarrow K \equiv 0$

$\Longleftrightarrow$ 下一章



(1).



$$k_n(T) = \frac{\mathrm{II}(T,T)}{\mathrm{I}(T,T)} = 0$$

$$\| $$

$$kN\cdot n \Longrightarrow N\cdot n = 0$$

又 $T\cdot n = 0$

$$\Rightarrow \quad B = \pm n \qquad \dot{B} = \pm \dot{n}$$
$$\dot{B} = -\tau N$$

$$dB = \pm dn = -\tau N\, ds.$$

由 $\mathrm{III} - 2H\underset{\overset{\|}{0}}{\mathrm{II}} + K\mathrm{I} = 0,$

$$\Rightarrow K = -\frac{\mathrm{III}}{\mathrm{I}} = -\frac{|dn|^2}{ds^2}$$

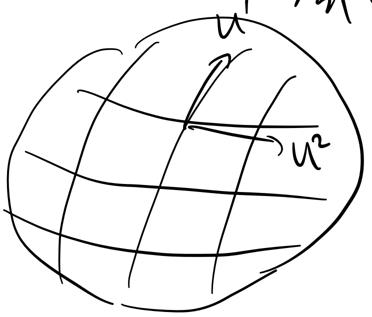
$$|\pm dn|^2 = |-\tau N|^2 ds^2 = |\tau|^2 ds^2$$

$$-K = |\tau|^2$$

$$|\tau| = \sqrt{-K}$$

(2). $K<0$

邻域内 $K<0$.



渐近线为参数网

x_1 — 渐近方向

u^2 常数. $du^1 = 0$

$$\mathrm{II}(du^1 = 0) = h_{11}(du^1)^2 = 0 \Rightarrow h_{11} = 0$$

同理 $h_{22}=0$

$$\mathrm{II} = 2h_{12}\,du^1 du^2$$

$$\cos\theta = \frac{x_1\cdot x_2}{\sqrt{g_{11}}\sqrt{g_{22}}} = \frac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{\sqrt{\det(g_{\alpha\beta})}}{\sqrt{g_{11}}\sqrt{g_{22}}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{\det(g_{\alpha\beta})}}{g_{12}}$$

$$K = \frac{\det(h_{\alpha\beta})}{\det(g_{\alpha\beta})} = \frac{-h_{12}^2}{\det(g_{\alpha\beta})}$$

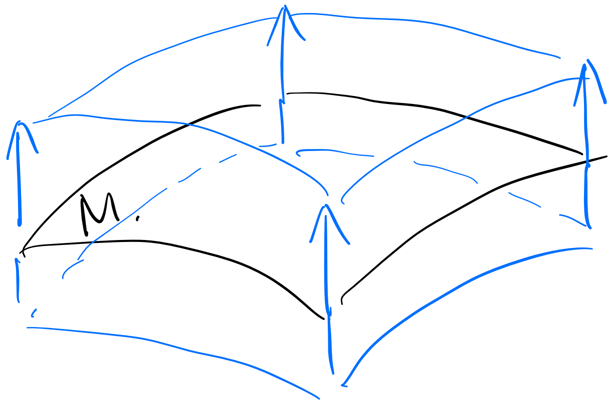
$$H = \frac{1}{2}\,\frac{h_{11}g_{22}-2h_{12}g_{12}+h_{22}g_{11}}{\det(g_{\alpha\beta})} = \frac{-2h_{12}\,g_{12}}{\det(g_{\alpha\beta})}$$

$$\Rightarrow \quad \frac{\sqrt{-K}}{|H|} = \tan\theta \qquad \square$$

题意各种公式、各种网

极小曲面

$H=0$



变化一下

仅在M上面积 min

<u>变分法</u>

M. $x = x(u^1,u^2)$. $(u^1,u^2)\in D$.

$M^t: x^t = x(u^1,u^2) + t\varphi(u^1,u^2)\, n(u^1,u^2)$.

一族变分曲面　　　　任意函数.

$M^0 = M$. $-\varepsilon < t < \varepsilon$

对于M. $h_{11} = -x_1 n_1$　$h_{12} = -x_1 n_2 = -x_2 n_1$　$h_{22} = -x_2 n_2$

$$\begin{cases} x_1^t = x_1 + t\varphi_1 n + t\varphi n_1 \\ x_2^t = x_2 + t\varphi_2 n + t\varphi n_2 \end{cases}$$

$$g_{11}^t = x_1^t x_1^t = (x_1 + t\varphi_1 n + t\varphi n_1)^2$$

$$= g_{11} - 2t\varphi h_{11} + o(t)$$

$$g_{12}^{t} = (\quad)(\quad)$$

$$= g_{12} - 2t\varphi h_{12} + o(t)$$

$$g_{22}^{t} = g_{22} - 2t\varphi h_{22} + o(t)$$

$$g_{11}^{t} g_{22}^{t} - (g_{12}^{t})^2 = (g_{11} g_{22} - g_{12}^2) - 2t\varphi (g_{11} h_{22} + g_{22} h_{11} - 2h_{12} g_{12})$$

$$+ o(t)$$

$$= \det(g_{\alpha\beta}) \left(1 - 4t\varphi H + o(t)\right)$$

泛函，函数的函数，曲面族上的函数.

$$A(t) = \iint_D \sqrt{g_{11}^{t} g_{22}^{t} - (g_{12}^{t})^2}\, du^1 du^2.$$

$$= \iint_D \sqrt{\det g_{\alpha\beta}} \sqrt{1 - 4t\varphi H + o(t)}\, du^1 du^2$$

$$\frac{d}{dt} A(t) \Big|_{t=0} = \iint_D \sqrt{\det g_{\alpha\beta}}\, \frac{1}{2} \frac{-4\varphi H}{\sqrt{1}}\, du^1 du^2$$

$$= -\iint_D 2\varphi H \sqrt{\det(g_{\alpha\beta})}\, du^1 du^2$$

$\neq 0$

Thm M 为极小曲面 $(H=0)$

$\iff$ M 的面积达到变分的临界点

Pf $\Rightarrow$ 显然

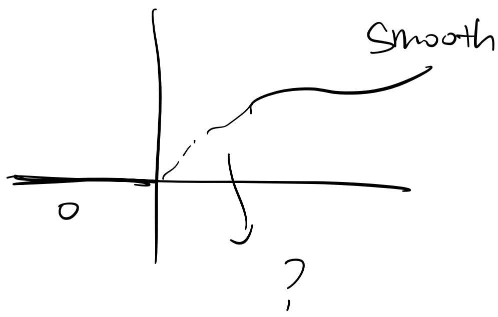
$\Leftarrow$ 若 $H\not\equiv 0$ $\exists x_0.$ $H(x_0)\neq 0$

不妨 $H(x_0)>0.$ $\Rightarrow$ $\exists U(x_0)$

$H|_{U(x_0)}>0.$

磨光函数

令 $\varphi|_{U(x_0)}>0$



$\varphi|_{M-U(x_0)}=0$ 矛盾.

只是临界点 不一定最大值 最小值