Thm Let $\mathcal{X} = \{$all bdd closed subset $Z \subset X\}$ Then

$\Leftarrow$ 也有

1) If (X, d) is complete, then $(\mathcal{X}, d_H)$ is complete metric space

2). (X, d) is compact $\Leftrightarrow$ $(\mathcal{X}, d_H)$ is compact

where d_H is "Hausdorff distance"

$$d_H(Z_1, Z_2) \triangleq \inf\{\varepsilon : Z_1 \subset B_\varepsilon(Z_2),\ Z_2 \subset B_\varepsilon(Z_1)\}$$

pf. 1). Check. $(\mathcal{X}, d_H)$ is a metric space.

a) $d_H(Z_1, Z_2) = d_H(Z_2, Z_1) \geqslant 0$

b) $d_H(Z_1, Z_2) = 0 \Leftrightarrow Z_1 = Z_2$

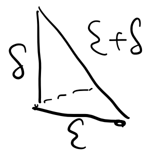
pf. If $d_H(Z_1, Z_2) = 0$, assume $Z_1 \neq Z_2$ $\exists x \in Z_1 \setminus Z_2$

$r_0 := d(x, Z_2) = \inf\{d(x, y) : y \in Z_2\} > 0$. since Z_2 is closed.

$\Rightarrow B_{\frac{r_0}{2}}(Z_2) \not\ni x$. 与 $d_H(Z_1, Z_2) = 0$ ※.

c).

$Z_1 \subset B_\varepsilon(Z_2)$. $Z_2 \subset B_\delta(Z_3)$



$\Rightarrow Z_1 \subset B_{\varepsilon+\delta}(Z_3)$

$\Rightarrow d_H(Z_1, Z_2) \leqslant d_H(Z_1, Z_3) + d_H(Z_2, Z_3)$

- If (X, d) is complete

Given $\{Z_1, Z_2, \dots, Z_n, \dots\} \subseteq \mathcal{X}$. Cauchy seq.

$d_H(Z_i, Z_j) \leqslant \varepsilon_i \to 0$. $j \geqslant i$

Construction $\left(\forall Z_i \subseteq B_{\varepsilon_i}(Z_i)\right)$

Define $\tilde{Z}_i = \bigcup_{j \geq i} Z_j$. Let $\hat{Z}_i$ = closure of $\tilde{Z}_i$

$$\hat{Z}_i \supset \hat{Z}_{i+1} \supset \hat{Z}_{i+2} \supset \cdots$$

Let $Z_\infty \triangleq \bigcap_{i \geq 1} \hat{Z}_i$ (完备)

<u>claim</u>. $d_H(Z_i, Z_\infty) \to 0$. Z_∞ closed bounded $\in \mathcal{X}$

Pf. we have

$$d_H(Z_i, \tilde{Z}_i) \leq \varepsilon_i$$

$$d_H(Z_i, \tilde{Z}_j) \leq \varepsilon_i \quad j \geq i$$

$$d_H(\hat{Z}_i, \hat{Z}_j) \leq \varepsilon_i, \quad j \geq i$$

$$\Rightarrow \quad Z_\infty \subseteq B_{\varepsilon_i}(Z_i)$$

<u>第一项</u> want to prove $Z_i \subseteq B_{10\varepsilon_i}(Z_\infty)$

<u>Argue by construction</u> Assume $\exists\, z_i \in Z_i \setminus B_{10\varepsilon_i}(Z_\infty)$

$z_i \in Z_i$, $\quad \forall j \geq i.\ \exists\, z_j \in \tilde{Z}_j$. s.t. $d(z_i, z_j) \leq 2\varepsilon_i$

\begin{tikzcd}
Z_i \arrow[d] \arrow[dr] & \\
Z_j & \tilde{Z}_j \quad j \geq i
\end{tikzcd}

let $i_0 = i$, $i_1 > i_0$. s.t. $\varepsilon_{i_1} \leq \frac{\varepsilon_{i_0}}{10}$

$\Rightarrow$找 $i_0 < i_1 < i_2 < \cdots$ s.t. $\varepsilon_{i_k} \leq \frac{1}{10} \varepsilon_{i_{k-1}}$

\begin{tikzcd}
i_0 \arrow[d] & i_1 \arrow[d, dashed] & i_2 \arrow[d, dashed] \\
z_{i_0} \arrow[d, phantom, "\parallel"] & z_{i_1} & z_{i_2} \\
z_i & &
\end{tikzcd}

s.t. $d(z_{i_k}, z_{i_{k-1}}) \leq 2\varepsilon_{i_{k-1}} \leq 2\left(\frac{1}{10}\right)^{k-1} \varepsilon_{i_0}$

可推.

$\Rightarrow \{z_{i_0}, z_{i_1}, z_{i_2}, \dots\}$ is Cauchy seq let $z_\infty = \lim_{i_k \to \infty} z_{i_k} \Rightarrow z_\infty \in \hat{Z}_k \ \forall k. \Rightarrow z_\infty \in Z_\infty$

$\Rightarrow d(z_i, z_\infty) \leq \sum_k d(z_{i_k}, z_{i_{k-1}}) < 6 \varepsilon_{i_0}$ $\times$

2). ① Assume (X, d) compact.

want to prove $(\mathcal{X}, d_H)$ is compact

It suffices to show $(\mathcal{X}, d_H)$ is totally bounded. ☆

Since (X, d) is totally bounded. $\forall \varepsilon > 0$. $\exists$ finite ε-net.

$T = \{x_1, x_2, \ldots, x_k\}$. s.t. $X \subseteq \bigcup_{i=1}^{k} B_\varepsilon(x_i)$

$\forall Z \in \mathcal{X}$. define $T_Z = \{x_i \in T : d(x_i, Z) < \varepsilon\} \neq \phi$.
$\subseteq T$.

and $T_Z \in \mathcal{X}$.

and $d_H(Z, T_Z) < \varepsilon$

let $\{T_1, T_2, \ldots, T_\ell\} \subset \mathcal{X}$ be all subset of T (finite)

$\Rightarrow \mathcal{X} \subseteq \bigcup_{i=1}^{\ell} B_\varepsilon(T_i). \Rightarrow \mathcal{X}$ is compact

② Assume $(\mathcal{X}, d_H)$ compact.

want to prove. (X, d) is compact

It suffices to show $(\mathcal{X}, d_H)$ is totally bounded. ☆

choose $\{x_1, x_2, x_3, \ldots\} \subseteq X$. s.t. $d(x_i, x_j) > \varepsilon$, $i \neq j$

$\Rightarrow \{\{x_1\}, \{x_2\}, \{x_3\}, \dots\} \subseteq \mathcal{X}$ $\quad d_H(\{x_i\}, \{x_j\}) > \varepsilon$

Since $(\mathcal{X}, d_H)$ is compact $\Rightarrow$ finite $\Rightarrow$ X is compact. $\square$

<u>Rmk.</u> $Z_1 = (0,1)$, $Z_2 = [0,1] \subseteq \mathbb{R}$

$d_H(Z_1, Z_2) = 0$ $\quad$ so closed is needed !!

<u>Def.</u> let (X, d_X), (Y, d_Y) metric space. we say map $f: X \to Y$ is conti. iff

$$d_X(x, x_i) \to 0 \Rightarrow d_Y(f(x_i), f(x)) \to 0$$

Let $\gamma: [0,1] \to X$ be a continue map, call it a curve

<u>Def.</u> $L[\gamma] = \sup \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1}))$ $\quad$ over all division of $[0,1]$

$0 = t_0 < t_1 < \dots < t_n = 1$

<u>Rmk.</u> $L[\gamma] \geq d(\gamma(0), \gamma(1))$. $L[\gamma] \geq d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(1))$.

<u>Def</u> (length space) We say a metric space (X, d) is a length space or geodesic space if.

1). X is path connected. i.e. $\forall x, y \in X$ $\exists$ curve $\gamma: [0,1] \to X$. s.t.

$\gamma(0)=x,\ \gamma(1)=y$

2) $\forall\, x, y \in X$. $\exists\, \gamma : [0,1] \to X$, connecting x, y. s.t. $L[\gamma] = d(x,y)$

Ex. 1). $(\mathbb{R}^n, d)$. length space

黎曼度量?

2) (S^n, d) length space. , surface with induced by metric

Thm. Let (Y, d) be complete, then the following are equivalent.

1). Y is a length space

2). $\forall\, y_1, y_2 \in Y$. $\exists$ midpoint. $y_3 \in Y$. of y_1, y_2 i.e.

$$d(y_1, y_3) = d(y_2, y_3) = \frac{1}{2} d(y_1, y_2)$$

Pf

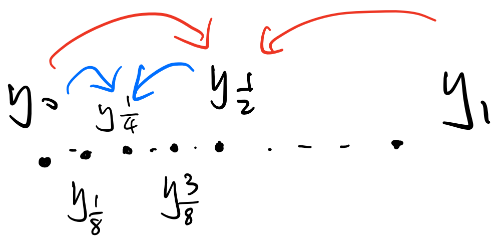
(1) $\Rightarrow$ (2) $\quad \forall\, y_1, y_2 \in Y$. $\exists\, \gamma$. $L[\gamma] = d(y_1, y_2)$

Chose $y_3 \in \gamma$ s.t. $d(y_1, y_3) = \frac{1}{2} d(y_1, y_2)$.

$$d(y_1,y_2) \le d(y_1, y_3) + d(y_2, y_3) \le L[\gamma] = d(y_1, y_2)$$

$\Rightarrow y_3$ is midpoint

(2) $\Rightarrow$ (1) $\quad \forall\, y_0, y_1 \in Y$. find a curve γ. s.t. $L[\gamma] = d(y_0, y_1)$



Define $T=\left\{\frac{i}{2^k},\ 0\leqslant i\leqslant 2^k,\ k\geqslant 1\right\}$ dense in $[0,1]$

one can check $\forall t,s\in T$. we have y_t, y_s. (use 三角不等式)

s.t. $d(y_t,y_s)=|t-s|\,d(y_0,y_1)$

In particular. $\forall t_0<t_1<\cdots<t_N$. $t_i\in T$

$$d(y_{t_0},y_{t_N})=\sum_{i=0}^{N-1} d(y_{t_i},y_{t_{i+1}})$$

(完备性)

Def $\gamma: T\to X$. $\gamma(t)=y_t$. then γ is continue $\triangle$

since X is complete. we can extend γ to $[0,1]$

$\Rightarrow L[\gamma]=d(y_0,y_1) \Rightarrow Y$ is length space $\square$

Def (boundedly compact) (X,d) is boundly compact if any bold closed subset of Y is compact

Ex. $\left(C([0,1]),\ d_{max}\right)$ not locally compact

not boldly compact

Thm (??). If (X,d) is locally compact, complete and

$$\overline{B_R(x)} = \overline{B_R(x)} \quad \forall R>0,\ x\in X$$

then (X,d) is bddly compact.

Thm. Let (X,d) be a locally compact, complete, length space. then (X,d) is bddly compact

$\Downarrow$ $\overline{B_R(x)} = \overline{B_R(x)}$

Pf. let $x\in X$. since locally compact

$$\Rightarrow \exists\ r_0>0 \text{ s.t. } \overline{B_{r_0}(x)} \text{ is compact}$$

$$\Rightarrow \forall\ 0<\rho<r_0,\ \overline{B_\rho(x)} \text{ is compact. (紧闭子集)}$$

Define. $R = \sup\{r : \overline{B_r(x)} \text{ is compact}\}$

$\boxed{\text{Need to prove } R=+\infty}$ now we only have $R \geq r_0$.

By contradiction, assume $R<+\infty$

Claim 1 $\overline{B_R(x)}$ is compact

pf. $\forall \varepsilon>0$. find finite ε-net

Since $\overline{B_{R-\frac{\varepsilon}{4}}(x)}$ is compact.

$$\Rightarrow \exists \text{ finite } \tfrac{\varepsilon}{3}\text{-net of } \overline{B_{R-\frac{\varepsilon}{4}}(x)}$$

$\{x_1, x_2, \ldots, x_k\}$

$\Rightarrow$

$$\overline{B_{R-\frac{\varepsilon}{4}}(x)} \subseteq \bigcup_{i=1}^{k} B_{\frac{\varepsilon}{3}}(x_i).$$

$$\overline{B_R}(x) \subseteq B_{\frac{\varepsilon}{3}}\left(\overline{B_{R-\frac{\varepsilon}{4}}}(x)\right) \subseteq \bigcup_{i=1}^{k} B_\varepsilon(x_i) \Rightarrow \overline{B_R}(x) \text{ is compact}$$

$\square$

$\forall y \in \overline{B_R}(x) \Rightarrow \exists\ \overline{B_{r_y}}(y)$ is compact. (locally compact)

$\Rightarrow \overline{B_R}(x) \subseteq \bigcup_{y \in \overline{B_R}(x)} B_{r_y}(y)$. (open covering)

$\Rightarrow \exists$ finite $\overline{B_R}(x) \subseteq \bigcup_{j=1}^{k} B_{r_{y_j}}(y_j) =: U$

$\Rightarrow \overline{U}$ is compact. (由 $\overline{U}$ 为有限个闭集的并)

$\Rightarrow$ any closed subset of U is compact

<u>Claim 2</u> $\exists\ \varepsilon > 0$, s.t. $B_\varepsilon(\overline{B_R}(x)) \subset U$

<u>pf.</u> <u>by contradiction.</u>

$\exists\ \{x_i \notin U\}$. s.t. $d(x_i, \overline{B_R}(x)) \leq 2^{-i}$

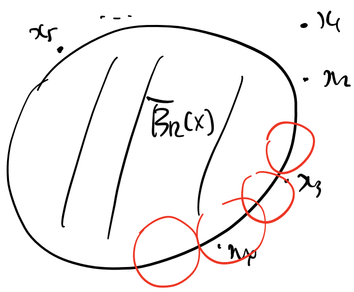
Since. $\overline{B_R}(x)$ is compact. $\Rightarrow \exists$ subseq $\{x_{i'}\} \subset \{x_i\}$

s.t. $x_{i'} \longrightarrow x_\infty$. Cauchy seq

$\exists\ r > 0$

$x_\infty \in \overline{B_R}(x)$ and $B_r(x_\infty) \subset U$. $\times$

pf.



$\exists\ y_\alpha, 1 \leq \alpha \leq k_i$

$\forall i, \bigcup_{\alpha=1}^{k_i} B_{2^{-i}}(y_\alpha) \supset \overline{B_R}(x)$

$\Rightarrow \bigcup_{\alpha=1}^{k_i} B_{2 \cdot 2^{-i}}(y_\alpha) \supset B_{2^{-i}}(\overline{B_R}(x))$

$\Rightarrow \exists$ 一个 $B_{2\cdot\varepsilon^{-1}}(y_\alpha)$ 中无限个 x_i

by claim 2. $\quad \overline{B_{R+\frac{\varepsilon}{2}}}(x) \subseteq B_\varepsilon(\overline{B_R(x)}) \subset U$

compact.

$\Rightarrow$ $\divideontimes$ $\Rightarrow R = +\infty$. $\square$