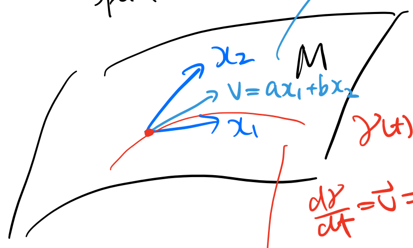
W3L1

$T_{x(u^1,u^2)}M \xrightarrow{\sigma^*} T_{\bar{x}(\bar{u}^1,\bar{u}^2)}\bar{M} \Rightarrow$ 线性的

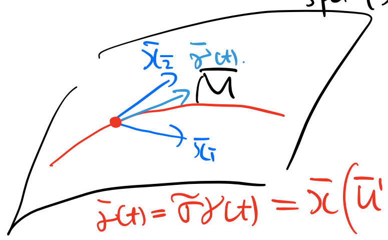
$T_{x(u^1,u^2)}M = \mathrm{span}(x_1, x_2)$

$T_{\bar{x}(\bar{u}^1,\bar{u}^2)}\bar{M} = \mathrm{span}(\bar{x}_1, \bar{x}_2)$



当成某条曲线的切向量.

$\xrightarrow{\tilde{\sigma}}$



$\gamma(t) = x(u^1(t), u^2(t))$

$\frac{d\gamma}{dt} = \vec{v} = \underbrace{\frac{du^1}{dt}}_{=a} x_1 + \underbrace{\frac{du^2}{dt}}_{=b} x_2$

$\tilde{\gamma}(t) = \tilde{\sigma}\gamma(t) = \bar{x}\left(\bar{u}^1(u^1(t), u^2(t)), \bar{u}^2(u^1(t), u^2(t))\right)$

$\left.\frac{d\tilde{\gamma}(t)}{dt}\right|_{t=0} \triangleq \sigma^*(v)$

$= \bar{x}_{\bar{u}^1}\frac{d\bar{u}^1}{dt}(0) + \bar{x}_{\bar{u}^2}\frac{d\bar{u}^2}{dt}(0)$

$= \bar{x}_1\left(\frac{\partial\bar{u}^1}{\partial u^1}\frac{du^1}{dt}(0) + \frac{\partial\bar{u}^1}{\partial u^2}\frac{du^2}{dt}(0)\right)$

$+ \bar{x}_2\left(\frac{\partial\bar{u}^2}{\partial u^1}\frac{du^1}{dt}(0) + \frac{\partial\bar{u}^2}{\partial u^2}\frac{du^2}{dt}(0)\right) = a\bar{x}_1 + b\bar{x}_2$

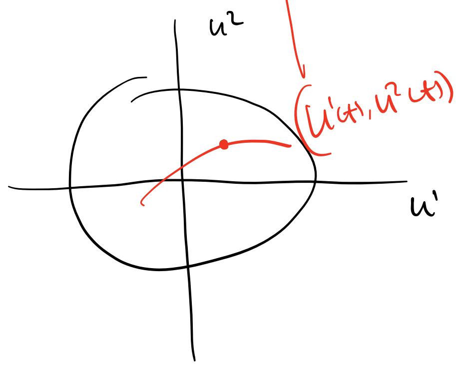
$\uparrow x \qquad x = x(u, v)$

$\uparrow \bar{x}$

$\sigma(u^1, u^2) = (\bar{u}^1, \bar{u}^2)$

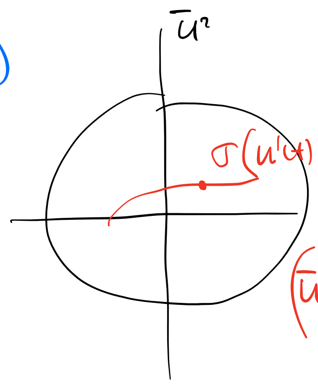
$\begin{cases} \bar{u}^1 = \bar{u}^1(u^1, u^2) \\ \bar{u}^2 = \bar{u}^2(u^1, u^2) \end{cases}$

$\left|\frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(u^1, u^2)}\right| \neq 0$



$(u^1(t), u^2(t))$

$\xrightarrow{\sigma}$



$\sigma(u^1(t), u^2(t)) = \left(\bar{u}^1(u^1(t), u^2(t)), \bar{u}^2(u^1(t), u^2(t))\right)$

$\mathrm{I} = ds^2 = g_{\alpha\beta}\,du^\alpha du^\beta \qquad \bar{\mathrm{I}} = g_{\bar{\alpha}\bar{\beta}}\,d\bar{u}^\alpha d\bar{u}^\beta$

若满足 $\bar{\mathrm{I}} = \varphi^2(u^1, u^2)\mathrm{I}$. $\Rightarrow \sigma$. 共形的 Conformal

(保角的)

$\sigma^*: T_xM \to T_{\bar{x}=\tilde{\sigma}(x)}\bar{M}$ 切映射

$\sigma_*(ax_1 + bx_2) = a\bar{x}_1 + b\bar{x}_2 = a\sigma_*(x_1) + b\sigma_*(x_2)$.

下面讨论保角性

$\begin{cases} \vec{a} = a^\alpha x_\alpha \\ \vec{b} = b^\alpha x_\beta \end{cases} \in T_xM \Rightarrow \begin{array}{l} \sigma^*(\vec{a}) = a^\alpha\sigma^*(x_\alpha) = a^\alpha\bar{x}_\alpha \\ \sigma^*(\vec{b}) = b^\alpha\sigma^*(x_\alpha) = b^\alpha\bar{x}_\alpha \end{array}$

$$\cos\angle(\sigma^*(\vec{a}),\sigma^*(\vec{b})) = \frac{(\sigma^*(\vec{a}),\sigma^*(\vec{b}))}{\|\sigma^*(\vec{a})\|\,\|\sigma^*(\vec{b})\|}$$

$$= \frac{a^\alpha b^\beta \bar{x}_\alpha \bar{x}_\beta}{(a^\sigma a^\tau \bar{x}_\sigma \bar{x}_\tau)^{\frac{1}{2}}(b^\gamma b^\delta \bar{x}_\gamma \bar{x}_\delta)^{\frac{1}{2}}}$$

$$= \frac{a^\alpha b^\beta \bar{g}_{\alpha\beta}}{\cdots} = \frac{\varphi^2 a^\alpha b^\beta g_{\alpha\beta}}{\cdots}$$

$$= \cos\angle(\vec{a},\vec{b})$$

Thm. 任何曲面,必在 局部与平面共形对应

即. 任何曲面的第一基本形式,都可以写成

$$I = \varphi^2(u^1,u^2)\left((du^1)^2+(du^2)^2\right)$$

参数网 (u^1,u^2) 称为 等温参数网.

S.-S. Chern. 论文, 现代几何学之部曲.

$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2$$

$$= \left(\sqrt{E}\,dp + \frac{F+i\sqrt{g}}{\sqrt{E}}\,dq\right)\left(\sqrt{E}\,dp + \frac{F-i\sqrt{g}}{\sqrt{E}}\,dq\right)$$

det

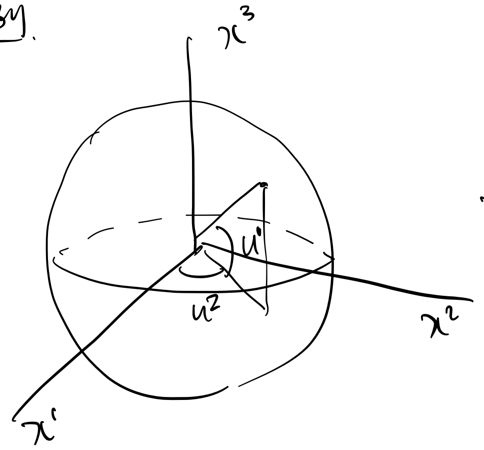
找 λ?

$$\lambda\left(\qquad\right) = du+idv$$

$$\bar{\lambda}\left(\qquad\right) = du-idv$$

$$\Rightarrow |\lambda|^2\,ds^2 = du^2+dv^2 \quad \cdots \quad \text{PDE}$$

例.



$|x|^2 = r^2$. i.e. $\sum_{i=1}^{3} (x^i)^2 = r^2$

$$\begin{cases} x^1 = r\cos u^1 \cos u^2 \\ x^2 = r\cos u^1 \sin u^2 \\ x^3 = r\sin u^1 \end{cases} \qquad \begin{cases} -\frac{\pi}{2} < u^1 < \frac{\pi}{2} \\ 0 < u^2 < 2\pi \end{cases}$$

$$I = r^2 (du^1)^2 + r^2\cos^2 u^1 (du^2)^2$$

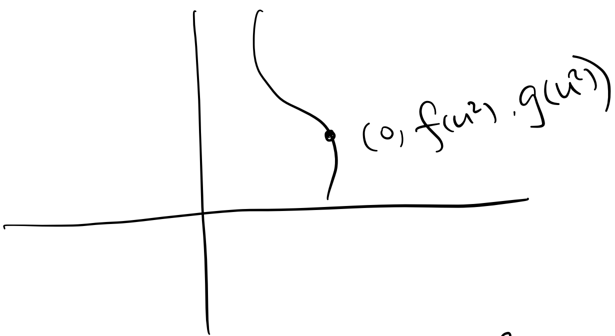
球极投影. $x^1 = \dfrac{2r^2\bar{u}^1}{r^2+(\bar{u}^1)^2+(\bar{u}^2)^2}$ $x^2 = \dfrac{2r^2\bar{u}^2}{r^2+(\bar{u}^1)^2+(\bar{u}^2)^2}$ $x^3 = r\dfrac{(\bar{u}^1)^2+(\bar{u}^2)^2-r^2}{(\bar{u}^1)^2+(\bar{u}^2)^2+r^2}$

$(\bar{u}^1, \bar{u}^2, 0)$ 在平面上

$$\Rightarrow I = \frac{4}{[1+a((\bar{u}^1)^2+(\bar{u}^2)^2)]^2}\left[(d\bar{u}^1)^2+(d\bar{u}^2)^2\right], \quad a = \frac{1}{r^2}$$

$\Rightarrow$ $\bar{u}^1, \bar{u}^2$ 为等温参数

例.

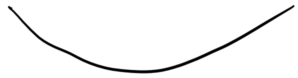
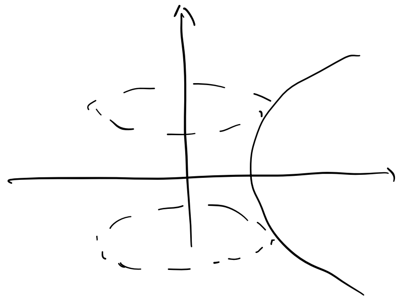


$$\Rightarrow x(u^1, u^2) = (f(u^2)\cos u^1, f(u^2)\sin u^1, g(u^2))$$

$$x_1 = (-f\sin u^1, f\cos u^1, g)$$

$$x_2 = (f'\cos u^1, f'\sin u^1, g')$$

$$\Rightarrow I = f^2 (du^1)^2 + [(f')^2+(g')^2](du^2)^2$$

悬链线  $\Rightarrow$ 

$$\Rightarrow \begin{cases} f(t)=f(u^2)= a\,\mathrm{ch}\frac{t}{a} \\ g(t)=g(u^2)=u^2 \end{cases}$$

$$\Rightarrow \begin{cases} f' = \mathrm{sh}\frac{u^2}{a} \\ g'=1 \end{cases} \Rightarrow I' = \mathrm{ch}^2\frac{u^2}{a}\left[a(du^1)^2+(du^2)^2\right]$$

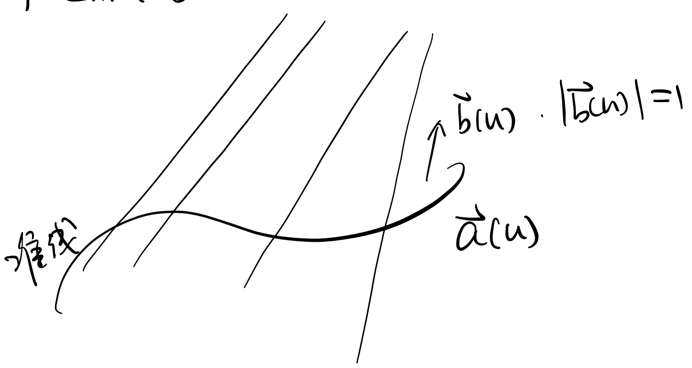
螺旋面 $x(\bar{u}^1,\bar{u}^2)=\left(f(\bar{u}^2)\cos\bar{u}^1,\ f(\bar{u}^2)\sin\bar{u}^1,\ g(\bar{u}^2)+a\bar{u}^1\right)$.

$$\bar{I} = (f^2+a^2)(d\bar{u}^1)^2+2ag'd\bar{u}^1d\bar{u}^2+(f'^2+g'^2)(du^2)^2$$

令 $f(\bar{u}^2)=\bar{u}^2$, $g(\bar{u}^2)=0$ $\Rightarrow$ 正螺面 $\bar{I}_1=(a^2+(\bar{u}^2)^2)(d\bar{u}^1)^2+(d\bar{u}^2)^2$

若令 $\bar{u}^1=u^1$, $\bar{u}^2=a\,\mathrm{sh}\frac{u^2}{a}$ $\Rightarrow$ $\bar{I}_1=I_1$. 故 正螺面与悬链面成等距对应.

例. 直纹面



$$x(u,v)=a(u)+v\,b(u)$$

$bb=1$
$\Rightarrow \underline{bb'=0}$

$x_u=x_1=a'+vb'$
$x_v=x_2=b$

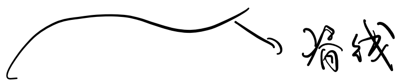
$$\begin{cases} g_{11}=|a'+vb'| \\ g_{12}=a'b \\ g_{22}=1 \end{cases}$$

$\Rightarrow I = |a'+vb'|^2 (du)^2 + 2a'b\,du\,dv + (dv)^2$

<u>In particular</u>. 1° $b(u)$ is constant $\Rightarrow$ 柱面.

2° $a(u)$ is constant $\Rightarrow$ 锥面

3° $\vec{b} \parallel \vec{a}'$ 切线面.



4° define. 可展曲面. 沿着任意一条直母线每一点处的曲面的切平面重合

"剪开"

<u>Thm</u>. 直纹面 $x(u,v) = a(u) + v b(u)$, $|\vec{b}|=1$ 是可展曲面 $\Leftrightarrow$ $(a', b, b') = 0$

<u>Pf</u>. 任取直母线 (v-线) $u = u^0$

$P(u^0, v^0)$. $Q(u^0, v^0+\Delta v)$ $\qquad \Delta v \neq 0$

P, Q 两点处的法向量

$(a'+vb')\times b \qquad (a'+(v+\Delta v)b')\times b$

故 可展 $\Leftrightarrow$ $0 = [(a'+vb')\times b] \times [(a'+(v+\Delta v)b')\times b]$

$\triangle = \Delta v^2 (a'\times b)(b'\times b)$

$= -\Delta v^2 (a', b, b')\, b$.