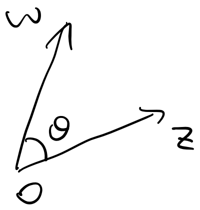
W1L2

Some Review.

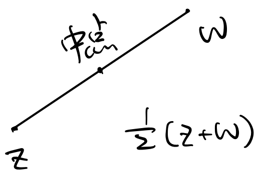
Complex plane

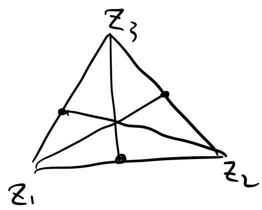


$$w = z\cdot e^{i\theta}$$

$$\Updownarrow$$

$\triangle OZW$ 等腰. $\angle O = \theta$

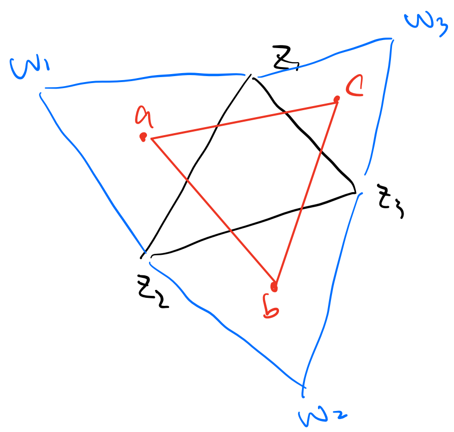




中点 $\frac{1}{3}(z_1+z_2+z_3)$

2. 复数与几何

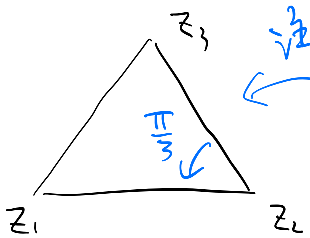
2.1. Napoleon 定理



正三角

在$\triangle z_1z_2z_3$ 每边外做一个正三角形

$\triangle w_1z_2z_1$ $\triangle z_2w_2z_3$. $\triangle z_1z_3w_3$ 中心连线成一个正三角



注意是逆时针的.

$\triangle z_1z_2z_3$ 正三角形 $\Leftrightarrow$ $z+\omega z_2+\omega^2 z_3 = 0$

$$\Updownarrow$$

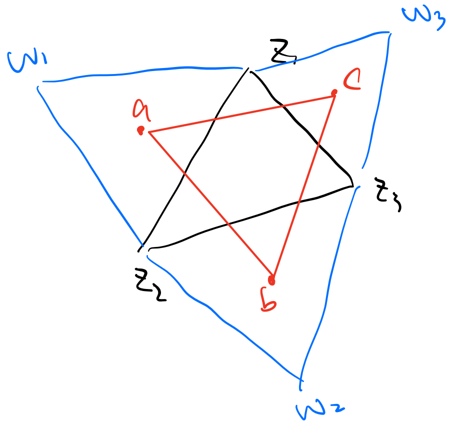
$$\frac{z_1-z_2}{z_3-z_2} = e^{\frac{\pi}{3}i} \qquad \omega = e^{\frac{2\pi i}{3}}$$

$$\Leftrightarrow z_1 - z_2 = (z_3-z_2)\,e^{\frac{\pi}{3}i}$$

$$\Longleftrightarrow \quad z_1 + \underbrace{(e^{\frac{\pi}{3}i}-1)}_{= e^{\frac{2\pi}{3}i} = \omega} z_2 \underbrace{- e^{\frac{\pi}{3}i}}_{= e^{\frac{4\pi}{3}i} = \omega^2} z_3 = 0$$

Pf.

$$\begin{cases} w_1 + \omega z_2 + \omega^2 z_1 = 0 \\ z_2 + \omega w_2 + \omega^2 z_3 = 0 \\ z_1 + \omega z_3 + \omega^2 w_3 = 0 \end{cases}$$

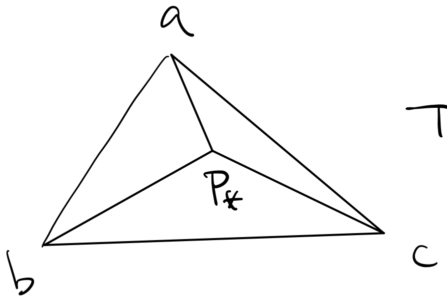


$$\overset{\sum}{\Longrightarrow} \quad (w_1 + z_2 + z_1) + (z_2 + w_2 + z_3)\omega + (z_1 + z_3 + w_3)\omega^2 = 0 \qquad \square$$

往内做三角也有类似结论

2.2. Fermat 问题

在△内部找一点，使其到三顶点距离和最小 (内角 $< \frac{2\pi}{3}$)



Torricelli

$\Rightarrow$ P_*, s.t. $P_*a \cdot P_*b \cdot P_*c$ 张角 120°，距离和最小

存在：两个圆交一下，且唯一.

define: $d(p) = |p-a| + |p-b| + |p-c| \quad p \in \mathbb{C}$.

定理 (Fermat-Torricelli) 对任意 $p \in \mathbb{C}$ $\quad d(p) \geq d(P_*) = |a + \omega^2 b + \omega c|$,

(设△abc内角 $< \frac{2\pi}{3}$) $\quad \omega = e^{\frac{2\pi}{3}i}$

"=" $\Longleftrightarrow p = P_*$

Pf. P_* $\quad a - P_*,\ \omega^2(b - P_*),\ \omega(c - P_*)$ 辐角相同

$d(p) = |p-a| + |p-b| + |p-c|$

$= |p-a| + |(p-b)\omega^2| + |(p-c)\omega|$

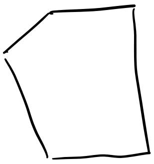
$\geq |p-a+\omega^2(p-b)+\omega(p-c)|$.

$= |a+\omega^2 b+\omega c| = d(p_*)$ □

内角>$\frac{2\pi}{3}$? 在顶点处 min

四边形情况? 对角线交点.（三角不等式即可）

五边形情况?

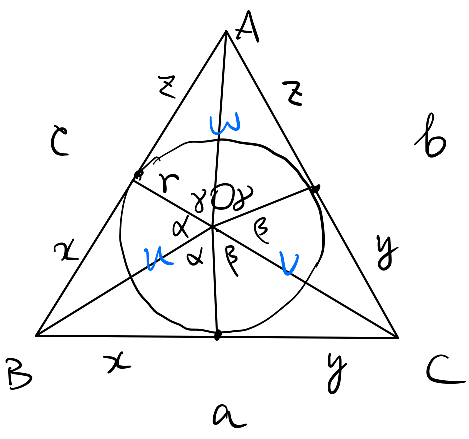


2.3 Heron 公式

定理. 设 $\triangle ABC$ 边长分别为 $a.b.c$ 记 $p=\frac{1}{2}(a+b+c)$. 则 $\triangle ABC$ 面积

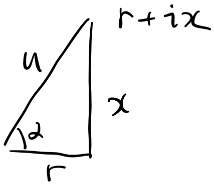
$$S=\sqrt{p(p-a)(p-b)(p-c)}$$

证.
$$\begin{cases} x+y=a \\ y+z=b \\ x+z=c \end{cases}$$



$$x+y+z=\frac{a+b+c}{2}$$

$$\begin{cases} z=\frac{b+c-a}{2} = p-a \\ x=\frac{a+c-b}{2} = p-b \\ y=\frac{a+b-c}{2} = p-c \end{cases}$$



构造复数,
$$\begin{cases} r+ix = ue^{i\alpha} \\ r+iy = ve^{i\beta} \\ r+iz = we^{i\gamma} \end{cases}$$

$$(r+ix)(r+iy)(r+iz) = uvw\, e^{i(\alpha+\beta+\gamma)} = -uvw$$

$$\parallel$$

$$r^3 + r^2(x+y+z)i + r(-xy-yz-xz) - i\,xyz$$

$$r^2(x+y+z) = xyz \qquad r = \sqrt{\frac{xyz}{x+y+z}}$$

$$\Rightarrow \quad S = \frac{1}{2} r \cdot 2p = \sqrt{pxyz} \qquad \square$$

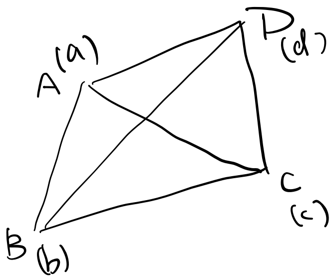
四边形面积公式？ . . .

注:圆内接四边形. a. b. c. d.. 面积 $S = \sqrt{(p-a)(p-b)(p-c)(p-d)}$

给定四边. 四边形不确定. 什么时候面积 max? (内接四边形时)

2.4. Ptolemy-Euler 定理.

设 ABCD 是平面凸四边形 顶点逆时针.



则 $AC\cdot BD \le AB\cdot CD + BC\cdot AD$.

"$=$" $\Longleftrightarrow$ 内接四边形

Pf. $LHS = |(a-c)(b-d)|$

$RHS = |(a-b)(c-d)| + |(b-c)(a-d)|$

而. $(a-c)(b-d) = ab+cd-ad-bc$

$(a-b)(c-d)+(b-c)(a-d) = \cancel{ac}+\cancel{bd}-bc-ad+ab-\cancel{ac}-\cancel{bd}+cd$

$= ab+cd-bc-ad.$

用三角不等式.

$$\text{"="} \Longleftrightarrow \frac{(a-d)(c-d)}{(b-c)(a-d)} \in \mathbb{R}_+$$

$$\parallel$$

$$-\frac{b-a}{d-a}\,\frac{d-c}{b-c} = \frac{AB\cdot CD}{AD\cdot BC}\cdot e^{i\pi}\cdot e^{-i\angle A}\, e^{-i\angle C}$$

$$= \frac{AB\cdot CD}{AD\cdot BC}\, e^{i(\pi-\angle A-\angle C)}$$

$$\cap\!\!\!\!\mid\quad \mathbb{R} \Longleftrightarrow \pi = \angle A + \angle C$$

$$\Longleftrightarrow$$ 圆内接. □

Conclusion

1. 角. $e^{i\theta}$, 正三角形.

2. 长度 模长. 为了用三角不等式. 转到同向.?

3. 图形中的 直角三角形 ⇒ 构造复数

4. 模长的乘积 = 乘积的模长.

// 长度的乘积

HW. $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ ☆.