外代数

V. $\mathbb{R}$上的 m维线性空间

$\{\sigma_1, \sigma_2, \dots, \sigma_m\}$. 基

$\xi \in V$, $\xi = \xi^\alpha \sigma_\alpha$

在 V 的元素之间定义一个运算. 外乘(外积"$\wedge$") (形式定义)

① 1° $\sigma_1, \dots, \sigma_m$

2° $\sigma_\alpha \wedge \sigma_\beta = -\sigma_\beta \wedge \sigma_\alpha \Rightarrow \sigma_\alpha \wedge \sigma_\alpha = 0$

$\alpha < \beta$ $\quad \sigma_1 \wedge \sigma_2 \quad \sigma_1 \wedge \sigma_3 \quad \cdots \quad \sigma_{m-1} \wedge \sigma_m$ 新元素 $\binom{n}{2}$

$\vdots$

k° $\sigma_{\alpha_1} \wedge \sigma_{\alpha_2} \wedge \cdots \wedge \sigma_{\alpha_k}$. $\quad 1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq m \quad \binom{m}{k}$

m°. $\sigma_1 \wedge \cdots \wedge \sigma_m$. $\quad \binom{m}{m} = 1$.

② 对于 固定的 k. $(k \geq 2)$

有如下性质

$$\sigma_{\alpha_1} \wedge \cdots \wedge \sigma_{\alpha_p} \wedge \cdots \wedge \sigma_{\alpha_q} \wedge \cdots \wedge \sigma_{\alpha_k}$$

$$= - \sigma_{\alpha_1} \wedge \cdots \wedge \sigma_{\alpha_q} \wedge \cdots \wedge \sigma_{\alpha_p} \wedge \cdots \wedge \sigma_{\alpha_k}.$$ 反对称性

if $\sigma_{\alpha_p} = \sigma_{\alpha_q}$ 则 $=0$

③ 利用线性扩张. 外乘可以推广到整个 V上

$\xi = \xi^\alpha \sigma_\alpha$. $\quad \eta = \eta^\beta \sigma_\beta \quad$ $$\xi \wedge \eta = (\xi^\alpha \sigma_\alpha) \wedge (\eta^\beta \sigma_\beta) = \sum_{\alpha, \beta} \xi^\alpha \eta^\beta \sigma_\alpha \wedge \sigma_\beta$$

$$= \sum_{\alpha<\beta} \xi^\alpha \eta^\beta \sigma_\alpha \wedge \sigma_\beta$$

$$+ \sum_{\alpha>\beta} \xi^\alpha \eta^\beta \sigma_\alpha \wedge \sigma_\beta$$

$$= \sum_{\alpha<\beta} \xi^\alpha \eta^\beta \sigma_\alpha \wedge \sigma_\beta$$

$$+ \sum_{\alpha<\beta} \xi^\beta \eta^\alpha \sigma_\beta \wedge \sigma_\alpha$$

换了α与β

$$= \sum_{\alpha<\beta} (\xi^\alpha \eta^\beta - \xi^\beta \eta^\alpha) \sigma_\alpha \wedge \sigma_\beta$$

$$= \sum_{\alpha<\beta} \begin{vmatrix} \xi^\alpha & \xi^\beta \\ \eta^\alpha & \eta^\beta \end{vmatrix} \sigma_\alpha \wedge \sigma_\beta$$

$\zeta = \zeta^\gamma \sigma_\gamma$ $\quad \xi \wedge \eta \wedge \zeta = \sum_{\alpha,\beta,\gamma} \xi^\alpha \eta^\beta \zeta^\gamma \sigma_\alpha \wedge \sigma_\beta \wedge \sigma_\gamma$

$$= \sum_{\alpha<\beta<\gamma} \begin{vmatrix} \xi^\alpha & \xi^\beta & \xi^\gamma \\ \eta^\alpha & \eta^\beta & \eta^\gamma \\ \zeta^\alpha & \zeta^\beta & \zeta^\gamma \end{vmatrix} \sigma_\alpha \wedge \sigma_\beta \wedge \sigma_\gamma.$$

V中任意k个元素作外积 得到的称为 k-元素 V_k

$V_k = \mathrm{span}\{\sigma_{\alpha_1} \wedge \cdots \wedge \sigma_{\alpha_k} ;\ 1 \le \alpha_1 < \cdots \alpha_k \le m\}$. k-空间

$\dim V_k = \binom{m}{k}$

$V_1 = V.$ $\quad V_0 = \mathbb{R}$ 基为$\{1\}$

$V_m = \{a\, \sigma_1 \wedge \cdots \wedge \sigma_m\}$ $\quad \dim V_m = 1$

$\Lambda(V) = \bigoplus_{k=0}^{m} V_k$ ← 分次代数. $\quad \dim \Lambda(V) = \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{m} = 2^m$

$G(V)$

set $\xi = \sigma_{\alpha_1} \wedge \cdots \wedge \sigma_{\alpha_k}$ k-元素

$\eta = \sigma_{\beta_1} \wedge \cdots \wedge \sigma_{\beta_s}$ s-元素

$$\xi \wedge \eta = (\sigma_{\alpha_1} \wedge \cdots \wedge \sigma_{\alpha_k}) \wedge (\sigma_{\beta_1} \wedge \cdots \wedge \sigma_{\beta_s})$$

$$:= \sigma_{\alpha_1} \wedge \cdots \wedge \sigma_{\alpha_k} \wedge \sigma_{\beta_1} \wedge \cdots \wedge \sigma_{\beta_s} \qquad (k+s)\text{-元素}$$

若 $k+s > m$. $\xi \wedge \eta = 0$

"$\wedge$" 有如下运算律 $\Lambda(V)$ 分层 Λ—结合代数 里面的元素

$\omega_0 + \omega_1 + \cdots + \omega_m$

$\omega_k \in V_k$.

1° 结合律 ξ-k元素, η-s元素, ζ-r元素

$$(\xi \wedge \eta) \wedge \zeta = \xi \wedge (\eta \wedge \zeta) = \xi \wedge \eta \wedge \zeta$$

2° 分配律 $a, b, c \in \mathbb{R}$

$$(a\xi + b\eta) \wedge \zeta = a\xi \wedge \zeta + b\eta \wedge \zeta$$

$$\xi \wedge (a\eta + b\zeta) = \xi \wedge a\eta + \xi \wedge b\zeta$$

命题 1). $\varphi \in V_p$. $\psi \in V_q$

$$\varphi \wedge \psi = (-1)^{pq} \psi \wedge \varphi$$

设 $\varphi = \sigma_{\alpha_1} \wedge \cdots \wedge \sigma_{\alpha_p}$

$\psi = \sigma_{\beta_1} \wedge \cdots \wedge \sigma_{\beta_q}$

$$\varphi \wedge \psi = (\sigma_{\alpha_1} \wedge \cdots \wedge \sigma_{\alpha_p}) \wedge (\sigma_{\beta_1} \wedge \cdots \wedge \sigma_{\beta_q})$$

$$= (-1)^{pq} \psi \wedge \varphi.$$

2) $\varphi_r = a_r^\alpha \sigma_\alpha \quad 1 \leq r \leq k \leq m$

$\varphi_r \in V_1$. 阶 $\quad 1 \leq \alpha \leq m.$

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_K = \sum_{\alpha_1 < \cdots < \alpha_k} \begin{vmatrix} a_1^{\alpha_1} & \cdots & a_1^{\alpha_k} \\ \cdots & \cdots & \cdots \\ a_k^{\alpha_1} & \cdots & a_k^{\alpha_k} \end{vmatrix} \sigma_{\alpha_1} \wedge \cdots \wedge \sigma_{\alpha_k}$$

外微分形式 $\quad \mathbb{R}^m = \{(u^1, \ldots, u^m) \mid u^\alpha \in \mathbb{R}.\ 1 \leq \alpha \leq m\}$

$u = (u^1, \ldots, u^m)$

以 $du^1, \ldots, du^m$ 为基. 生成 $\mathbb{R}$上的线性空间 $V(u)$

$V(u) = L[u]$

$\triangleq \operatorname{span}\{du^1, \ldots, du^m\}$

$\Lambda(L[u]) \qquad V_k = \Lambda^k(V) = \Lambda^k(L[u]).$ $\quad$ k-元素 $\quad$ k-空间

$\forall \varphi \in \Lambda^k(L[u]).$

$$\varphi = \sum_{\alpha_1 < \cdots < \alpha_k} a_{\alpha_1 \cdots \alpha_k} du^{\alpha_1} \wedge \cdots \wedge du^{\alpha_k}$$

当u变化时. $u \in U \subset \mathbb{R}^m$ $\quad \varphi(u) = \sum_{\alpha_1 < \cdots < \alpha_k} a_{\alpha_1 \cdots \alpha_k}(u)\, du^{\alpha_1} \wedge \cdots \wedge du^{\alpha_k}$

称为 U上的 k 次微分形式

k-形式

$$\varphi(u) = \sum_{1 \leq \alpha_1 \cdots \alpha_k \leq m} a_{\alpha_1 \cdots \alpha_k}(u)\, du^{\alpha_1} \wedge \cdots \wedge du^{\alpha_k}$$

$$= \sum_{1 \leq \alpha_1 \cdots \alpha_k \leq m} \tilde{a}_{\alpha_1 \cdots \alpha_k}(u)\, du^{\alpha_1} \wedge \cdots \wedge du^{\alpha_k}$$

其中 $\tilde{a}_{\alpha_1 \cdots \alpha_k}(u)$ 关于 $\alpha_1, \ldots, \alpha_k$ 是反对称的.

$a_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) + \frac{1}{2}(a_{ij}-a_{ji})$ 一对一对去看只有反对称的部分留下来

对称 反对称

$$= k! \sum_{1\le\alpha_1\le\cdots\le m} \tilde{a}_{\alpha_1\cdots\alpha_k}(u)\, du^{\alpha_1}\wedge\cdots\wedge du^{\alpha_k}$$