A brief introduction to complex analysis

Def. (1) A func $U \xrightarrow{f} \mathbb{C}$ is complex differentiable at a point $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \text{ exists}$$

而这里是 $\dfrac{f(z)-f(z_0)-Cz}{h} = 0$

and $z_0 \in \operatorname{int} U$. (or even $U \underset{\text{open}}{\subseteq} \mathbb{C}$)

$\mathbb{R}^2 \to \mathbb{R}^2$ 时，极限应该是

$$\lim_{h\to 0} \frac{\|\vec{f}(x,y) - \vec{f}(x_0,y_0) - Ah\|}{\|h\|} = 0$$

矩阵。

If f is cpx. diff. z_0, we call the limit the cpx. derivative of f at z_0 and denote it by $f'(z_0)$, or $\frac{df}{dz}(z_0)$

(2). If $U \xrightarrow{f} \mathbb{C}$ is cpx diff at every $z \in U$, and the func

$$\begin{aligned} U &\xrightarrow{f'} \mathbb{C} \\ z &\longmapsto f'(z) \end{aligned}$$

is continuous, then we say f is holomophic

Remark Goursat's theorem

$U \xrightarrow{f} \mathbb{C}$ is cpx. diff everywhere $\Longleftrightarrow$ f is holomophic on U.

(1). $U \xrightarrow{f} \mathbb{C}$. is cpx. diff at z_0. $\Longleftrightarrow$ $\exists C \in \mathbb{C}$. $f(z) = f(z_0) + C(z-z_0)$ as $z \to z_0$

$\Longleftrightarrow$ $\exists C \in \mathbb{C}$. and $U \xrightarrow{\eta} \mathbb{C}$. s.t. $\begin{cases} f(z) = f(z_0) + C(z-z_0) + \eta(z)|z-z_0| & \forall z \in U \\ \eta(z_0) = 0 = \lim\limits_{z\to z_0} \eta(z). \end{cases}$

Chain rule

$$\begin{array}{c} U \xrightarrow{f} V \xrightarrow{g} \mathbb{C} \\ \cup\!\!\shortmid \\ z_0 \end{array}$$

if $\begin{cases} f \text{ is cpx. diff at } z_0 \\ g \text{ is cpx. diff at } f(z_0). \end{cases}$ then.

$g \circ f$ is cpx. diff at z_0 and

$$(g\circ f)'(z_0) = g'(f(z_0)) f'(z_0)$$

(2) We may identify $U \subseteq \mathbb{C}$ with $U_{\mathbb{R}} = \{(x,y) \in \mathbb{R}^2 \mid x+iy \in U\} \subseteq \mathbb{R}^2$
and identify $U \xrightarrow{f} \mathbb{C}$ with $U_{\mathbb{R}} \xrightarrow{f_{\mathbb{R}}} \mathbb{R}^2$ in the manner that

$$(x,y) \longmapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

$$\begin{cases} u(x,y) = \operatorname{Re} f(x+iy) \\ v(x,y) = \operatorname{Im} f(x+iy) \end{cases}$$

$z_0 = x_0 + iy_0$.

- f is cpx. diff at z_0. $\Longleftrightarrow$ $f_{\mathbb{R}}$ is diff at (x_0,y_0) and. $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$ at (x_0, y_0)

Why?

$C = A + iB$ $\quad \eta(z) = \eta_1(x,y) + i\,\eta_2(x,y)$

$A, B \in \mathbb{R}$.

$$\Rightarrow \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + \begin{pmatrix} \eta_1(x,y) \\ \eta_2(x,y) \end{pmatrix} |(x-x_0, y-y_0)|$$

这个就是原来 Jacobi 矩阵.

$A = \frac{\partial u}{\partial x} \quad -B = \frac{\partial u}{\partial y}$

$B = \frac{\partial v}{\partial x} \quad A = \frac{\partial v}{\partial y}$.

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

C-R function.

主要用到柯西-黎曼方程.

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \quad \text{at } z_0 = x_0 + iy_0.$$

two ways to get ① we have. $f(z) = f(z_0) + C(z-z_0) + \eta(z)|z-z_0|$ $\eta(z) \to 0.$ (when $z \to z_0$)

write. $z = x+iy$. $C = A+Bi$. $\eta = \eta_1 + \eta_2 i$ $f(z) = u+iv$

then get.

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} |z-z_0|$$

(write as vector.)

and then. if we see $(x,y) \to (u(x,y), v(x,y))$ as a map : $\mathbb{R}^2 \to \mathbb{R}^2$

then we have $A = \frac{\partial u}{\partial x}$. . $-B = \frac{\partial u}{\partial y}$. $B = \frac{\partial v}{\partial x}$ $A = \frac{\partial v}{\partial y}$ $\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$ 証

②. $z \to z_0$ along x-axis $\lim\limits_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim\limits_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$

$$= \lim_{h\to 0} \frac{u(x_0+h, y_0) + iv(x_0+h, y_0) - (u(x_0,y_0) + iv(x_0,y_0))}{h}$$

$$= \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0)$$

along y-axis $\lim\limits_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim\limits_{h\to 0} \frac{f(z+h)-f(z_0)}{h}$

$$= \lim_{k\to 0} \frac{u(x_0, y_0+k) + iv(x_0, y_0+k) - (u(x_0,y_0) + iv(x_0,y_0))}{ik}$$

$$= \frac{1}{i}\left(\frac{\partial u}{\partial y}(z_0) + i\frac{\partial v}{\partial y}(z_0)\right) = \frac{\partial v}{\partial y}(z_0) - i\frac{\partial u}{\partial y}(z_0)$$

$$\Longrightarrow \quad \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) - i\frac{\partial u}{\partial y}(z_0) = f'(z_0) \qquad (*)$$

we can get $\begin{cases} \frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \\ \frac{\partial v}{\partial x}(z_0) = -\frac{\partial u}{\partial y}(z_0) \end{cases}$

and $(*) \Rightarrow \frac{df}{dz}(z_0) = \frac{1}{2}\left(\frac{\partial u}{\partial x}(z_0) + \frac{\partial v}{\partial y}(z_0) + i\left(\frac{\partial v}{\partial x}(z_0) - \frac{\partial u}{\partial y}(z_0)\right)\right)$

$$= \frac{1}{2}\left(\frac{\partial u + iv}{\partial x}(z_0) + \frac{\partial(-ui+v)}{\partial y}(z_0)\right)$$

$$= \frac{1}{2}\left(\frac{\partial u + iv}{\partial x}(z_0) - i\frac{\partial u + vi}{\partial y}(z_0)\right)$$

$$\Rightarrow \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

<u>Example</u>. (1) $f(z) = \bar{z}$ $(z \in \mathbb{C})$ is not cpx. diff. at $z_0 \in \mathbb{C}$

(way1) for every $z_0 \in \mathbb{C}$. $\lim\limits_{h\to 0} \frac{\overline{z_0+h} - \bar{z}_0}{h} = \lim\limits_{h\to 0} \frac{\bar{h}}{h}$ (沿不同轴不一样)

(way2). $f_{\mathbb{R}}(x,y) = \begin{pmatrix} x \\ -y \end{pmatrix}$. $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

(2) $f, g : U \to \mathbb{C}$ are both cpx. diff, then

$f \pm g$, $f \cdot g$ and f/g (if $g(z) \neq 0$) are all cpx. diff. at z_0 and

$(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0)$, $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$, and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g(z_0)}$$