

Algebra. Lec 18 Separable extension.

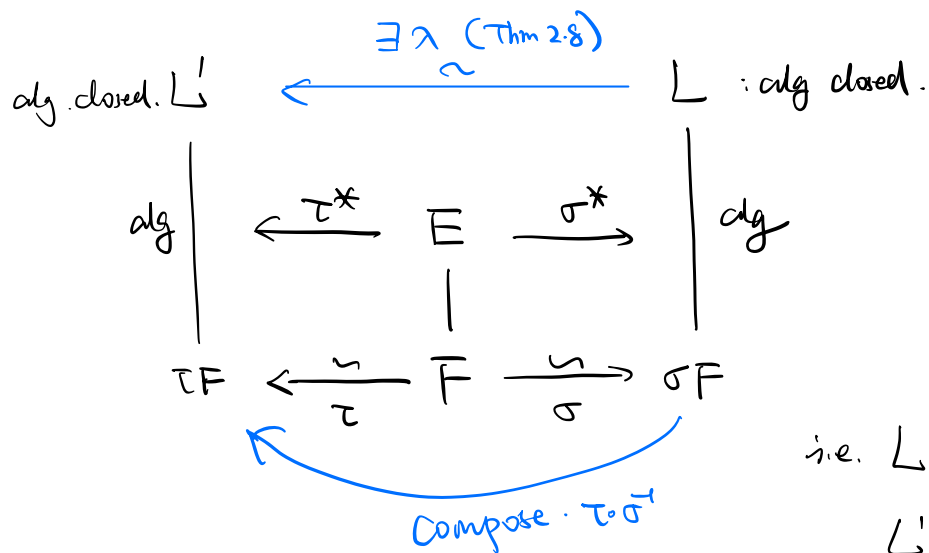
E. alg ext of F

$\sigma: F \xrightarrow{\text{emb.}} L: \text{alg closed.}$

$S_\sigma: \left. \begin{array}{l} \text{the set of all extension of } \sigma \\ \text{to } E \hookrightarrow L \end{array} \right\}$

Study the extension of σ to \bar{E}

$S_\tau: \left. \begin{array}{l} \text{the set of all extension of } \tau \\ \text{to } E \hookrightarrow L' \end{array} \right\}$



i.e. L . alg closure of σF
 L' alg closure of τF

if $\sigma^* \in S_\sigma$

then $\lambda \circ \sigma^* \in S_\tau. \Rightarrow S_\sigma, S_\tau$ bijective as set.

$$|S_\sigma| = |S_\tau| \quad \left(\begin{array}{l} \text{indep of the choice of} \\ \sigma, \tau, \\ L, L' \end{array} \right)$$

When $|S_\sigma| = |S_\tau| < \infty$. $|S_\sigma| = |S_\tau| \triangleq [E:F]_S$. separable degree of E over F

Thm. 4.1 $E \supset F \supset k$

$$\Rightarrow [E:k]_s = [E:F]_s [F:k]_s.$$

Moreover if E is finite over k (i.e. $[E:k] < \infty$)

$$\text{then } [E:k]_s < \infty. \quad [E:k]_s \leq [E:k]$$

Pf. ① $\sigma: k \hookrightarrow L$: alg. closed

$\{\sigma_i\}_{i \in I}$: the family of ext of σ to $F \hookrightarrow L$

$$(\sigma_i: F \hookrightarrow L)$$

for each $i \in I$. $\{\tau_{ij}\}_{j \in J_i}$: the family of ext of σ_i to E

$$\begin{array}{c} \sigma_i: F \hookrightarrow L \\ \downarrow \\ \tau_{ij}: E \hookrightarrow L \end{array}$$

claim $\{\tau_{ij}\}$: all embeddings $E \hookrightarrow L$ over k .



$$\text{then } |\{\tau_{ij}\}| = [E:F]_s [F:k]_s$$

any embedding $E \hookrightarrow L$ must be one of τ_{ij} □

② $[E:k] < \infty$.

\therefore we can obtain E by

$$k \subset k(\alpha_1) \subset k(\alpha_1, \alpha_2) \subset \dots \subset k(\alpha_1, \dots, \alpha_n) = E$$

$$\begin{array}{c} \parallel \\ F_0 \end{array} \quad \begin{array}{c} \parallel \\ F_1 \end{array} \quad \dots \quad \begin{array}{c} \parallel \\ F_r \end{array}$$

$$F_{v+1} = F_v(\alpha_{v+1}) : \alpha_{v+1} \text{ alg}/F_v \quad \text{By prop 2.7.}$$

$$[F_v(\alpha_{v+1}) : F_v]_s \leq [F_v(\alpha_{v+1}) : F_v]$$

$$\therefore [E:k]_s \leq [E:k]$$

Rmk. (Cor 4.2). E . field over k .

$$E \supset F \supset k.$$

$$[E:k]_s = [E:k] \Leftrightarrow [E:F]_s = [E:F]$$

$$[F:k]_s = [F:k]$$

Rmk. Later

$$[E:k]_s \mid [E:k]$$

$$\therefore \text{can define } [E:k] = [E:k]_s [E:k]_i \quad \curvearrowright \text{ inseparable degree}$$

def. E . finite / k .

• We say E is a separable extension over k .

$$\text{if } [E:k] = [E:k]_s$$

• α . alg / k . is said to be separable over k .

if $k(\alpha)$ is separable extension over k . (i.e. $\text{Irr}(\alpha, k, x)$ has no

multiple root

- $f(x) \in k[x]$ is called **separable** if it has no multiple root.

in k^a

$\text{Irr}(\alpha, k, x)$
is sep.

Rmk. if α is a root of a sep polynomial $g(x) \in k[x]$

$$\Rightarrow \text{Irr}(\alpha, k, x) \mid g(x) \quad \text{and } \text{Irr}(\alpha, k, x) \text{ separable.}$$

\Downarrow
 α is sep

- if $k \subset F \subset K$ $\alpha \in K$, α : separable $/k$.

$$\therefore k(\alpha) : \text{sep} / k. \quad \text{Irr}(\alpha, k, x) \text{ sep polynomial}$$

$$[k(\alpha) : k]_s = [k(\alpha) : k] \xrightarrow{\text{prop 2.7}} \text{no multiple roots}$$

$$\text{Irr}(\alpha, F, x) \mid \text{Irr}(\alpha, k, x) \Rightarrow \text{Irr}(\alpha, F, x) \text{ separable.}$$

$$\therefore [k(\alpha) : F]_s = [k(\alpha) : F]$$

$$\therefore \alpha : \text{separable} / F.$$

Thm. 4.3

E . finite ext $/k$.

$$E : \text{separable} / k. \quad (\text{i.e. } [E : k]_s = [E : k])$$

$$\Leftrightarrow \text{each elem of } E \text{ is sep} / k.$$

Pf. \Rightarrow . Assume. $E : \text{sep} / k$.

Let $\alpha \in E$

Consider the tower $k \subset k(\alpha) \subset E$.

$$\therefore \text{we have } [k(\alpha) : k]_s = [k(\alpha) : k]$$

$$[E:k(\alpha)]_s = [E:k(\alpha)]$$

$\therefore \alpha$ is sep.

\Leftarrow Assume $\forall \alpha \in E$. α : separable / k

Write: $E = k(\alpha_1, \dots, \alpha_n)$. α_i : alg / k . α_i sep / k

Consider

$$k \subset k(\alpha_1) \subset k(\alpha_1, \alpha_2) \subset \dots \subset k(\alpha_1, \dots, \alpha_n) = E$$

$\underbrace{\quad\quad\quad}_{\alpha_1 \text{ sep}/k} \quad \underbrace{\quad\quad\quad}_{\alpha_2 \text{ sep}/k} \quad \dots \quad \underbrace{\quad\quad\quad}_{\alpha_n \text{ sep}/k(\alpha_1, \dots, \alpha_{n-1})}$
 \Downarrow
 $\alpha_i \text{ sep}/k(\alpha_1, \dots, \alpha_{i-1})$ by Remark.

$$\begin{aligned}
 [E:k]_s &= [k(\alpha_1, \dots, \alpha_n) : k(\alpha_1, \dots, \alpha_{n-1})]_s \cdots [k(\alpha_1) : k]_s \\
 &= [k(\alpha_1, \dots, \alpha_n) : k(\alpha_1, \dots, \alpha_{n-1})] \cdots [k(\alpha_1) : k] \\
 &= [E:k]
 \end{aligned}$$

□

Rmk. E . arbitrary alg ext of k .

We define E to be separable / k .

if every ext $k(\alpha_1, \dots, \alpha_n)$: separable / k

$\alpha_i \in E$ \downarrow why finite? prop 1.6.

Rmk $E: \text{alg ext} / k$.

(Thm 4.4). generated by $\{\alpha_i\}_{i \in I}$

if each α_i is sep/k

$$\Rightarrow E \text{ sep}/k$$

pf. by definition. is clear

Thm 4.5 Separable extensions

form a distinguished class of exts

$$\text{pf } \textcircled{1} E \supset F \supset k$$

$$\text{Assume } E: \text{sep}/k \Rightarrow \left\{ \begin{array}{l} \text{every elem of } E. \text{ sep}/F \\ \text{every elem of } F. \text{ sep}/k \end{array} \right.$$

Conversely $E/F \text{ sep}$ $F/k. \text{ sep}$

$$\bullet \text{ if } [E:k] < \infty$$

$$[E:k]_s = [E:F]_s [F:k]_s$$

$$= [E:F][F:k] = [E:k] \Rightarrow E \text{ sep}/k$$

$$\bullet \text{ if } [E:k] \text{ not finite } \alpha \in E. \alpha \text{ is a root of a sep polynomial}$$
$$f(x) \in F[x] \quad (E/F \text{ sep})$$

$$f(x) = a_0 + a_1x + \dots + a_nx^n, a_i \in F$$

$$a_i \text{ sep}/k \cdot \text{alg}/k$$

$$F_0 \triangleq k(a_0, a_1, \dots, a_n)$$

Consider

$$k \subset F_0 \subset F_0(\alpha)$$

sep
finite

prop. 1.6.

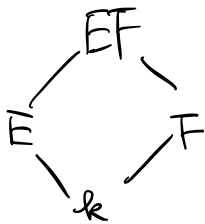
also F_0 coeff.

$$\Rightarrow F_0(\alpha) : \text{sep}/k$$

$$\alpha \in F_0(\alpha) : \text{sep}/k$$

Why!!!

②



$$E/k \text{ sep.}$$

$$E, F \subset L$$

$$\forall \alpha \in E. \alpha : \text{sep}/k \Rightarrow \alpha : \text{sep}/F$$

$$\uparrow$$

$$E \subset L$$

$\therefore EF$: generated by all elements of E over F

Thm 4.4

$$\Rightarrow EF \text{ sep}/F$$

$$\xrightarrow{\text{alg}} \text{ and } [E:k]_s \leq [E:k] < \infty$$

Rmk.

E finite ext of k .

$$k \subset E \subset E^a$$

$$K \triangleq \bigcap_{\substack{k \subset E \subset \tilde{K} \subset E^a \\ \tilde{K} \text{ normal}/k}} \tilde{K} = \left(\begin{array}{l} \text{Smallest} \\ \text{normal extension of } k, \text{ in } E^a \\ \text{containing } E \end{array} \right)$$

How to prove?

Lem. $\{K_i\}_{i \in I}$ $\Rightarrow \bigcap_{i \in I} K_i : \text{normal}/k$
 $K_i \text{ normal}/k$

let $f(x) \in k[x]$ irred $\alpha \in \bigcap_{i \in I} K_i$ $f(\alpha) = 0$

i.e. $f(x)$ has a root in K_i , $i \in I$

$\Rightarrow f(x)$ has all roots in K_i $\forall i \in I$

\Rightarrow all roots $\in \bigcap_{i \in I} K_i$

diag.



$\{\sigma_1, \dots, \sigma_n\}$ all distinct embeddings of E in E^a (finite)

$\sigma_i : E \hookrightarrow E^a$

$$\underbrace{(\sigma_1 E)(\sigma_2 E) \dots (\sigma_n E)} \triangleq k' = k$$

① K' is normal ext / k

pf. $\tau: K' \hookrightarrow E^a \implies (\tau\sigma_1, \dots, \tau\sigma_n) = (\sigma_1, \dots, \sigma_n)$

$$\begin{array}{c} E^a \\ | \\ E \\ | \\ k \end{array}$$

$\xleftrightarrow{\text{permutation}}$
 permutation.

• $\tau\sigma_i: E \rightarrow \sigma_i E \rightarrow \tau\sigma_i E$. : an embedding $E \hookrightarrow E^a$.

$\therefore \tau\sigma_i = \sigma_j$ for some j

• if $\tau\sigma_i = \tau\sigma_j$ i.e. $\tau\sigma_i(x) = \tau\sigma_j(x) \quad \forall x \in E$

$$\therefore \sigma_i(x) = \sigma_j(x) \quad \forall x \in E$$

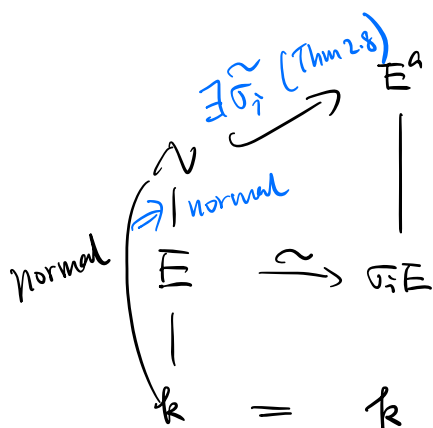
$$\therefore \sigma_i = \sigma_j$$

$$\therefore \tau K' = \tau((\sigma_1 E)(\sigma_2 E) \dots (\sigma_n E)) = K' \quad \therefore K' \text{ normal}$$

② Any normal extension N , containing E , must contain $\sigma_i E$. $\forall i$

$$\bigcap_{i=1}^n E^a$$

$\implies N \supseteq K'$: the K' is the smallest



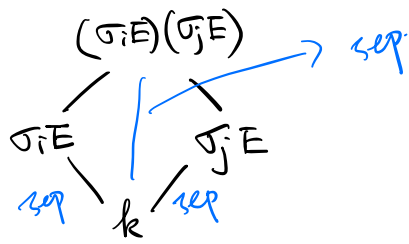
$$\text{im}(\tilde{\sigma}_i) = N \supseteq \sigma_i E$$

□

Rmk. E . sep ext / k

$\{\sigma_1, \dots, \sigma_n\}$ all distinct embedding of $E \hookrightarrow E^a$

$\therefore \sigma_i E$. sep / k



$\Rightarrow K = (\sigma_1 E) \cdots (\sigma_n E)$. smallest normal
ext / k
containing E
and sep.