WIL. 0, 1 局部理论. 2 曲线 整体 3 曲面 整体

作业 12/14 100
小测. 3次 300
} 50%

期末 50%

E^3 中曲线

$\{O; E_1, E_2, E_3\}$. 右手系

$\langle E_i, E_j \rangle = \delta_{ij}$

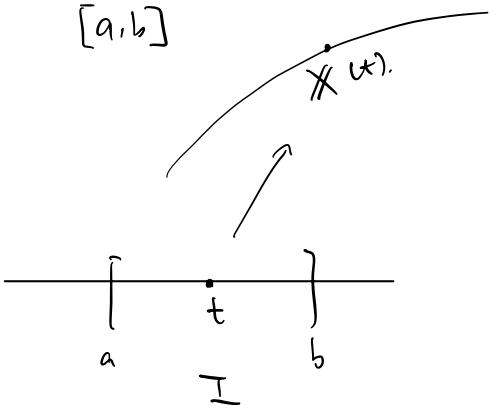
$\forall \vec{x} \in E^3 \quad x = x^1 E_1 + x^2 E_2 + x^3 E_3 = \sum_{i=1}^{3} x^i E_i = x^i E_i$

曲线C是 $X: I \longrightarrow E^3$ 同胚.

I = (a,b) / $[a,b]$ / $[a,b)$ / $[a,b]$

X: 连续 X^{-1} 连续.

x: 1-1 到上



局部定义. $X: I \longrightarrow E^3$

$t \longmapsto X(t) \in E^3$

$X(t) = x^i(t) E_i = (x^1(t), x^2(t), x^3(t))$.

$X(t)$ 可微 $\Leftrightarrow$ $x^i(t)$ 可微 $i=1,2,3$ $\quad \frac{dX}{dt} = \frac{dx^i}{dt} E_i$

正则曲线：如果曲线 C $X = X(t)$，是可微的，且 $\|X'(t)\| \neq 0$，$\forall t \in I$.

称 C 是正则曲线.

$$\left\|\frac{dX}{dt}\right\|^2 = \sum_{i=1}^{3} \left(\frac{dx^i}{dt}\right)^2 > 0 \quad (\text{i.e. } \exists i.\ \frac{dx^i}{dt} \neq 0.\ \forall t \in I)$$

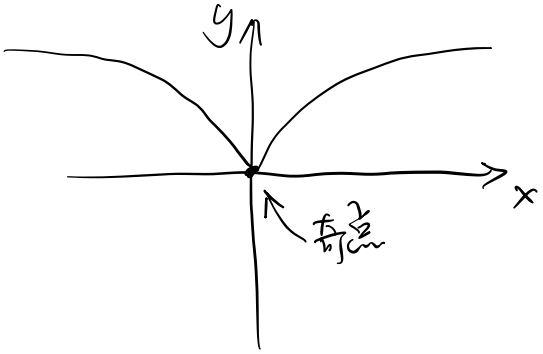
若 $\exists t_0 \in I$. $\left\|\frac{dX}{dt}(t_0)\right\| = 0$，则称 C 在 $X(t_0)$ 点是奇异的.

↓

奇点 → 难.

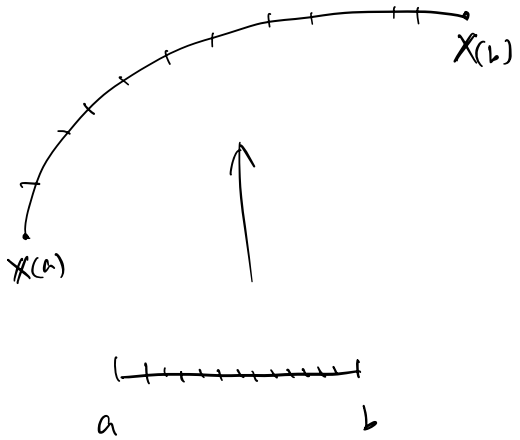
e.g. $X(t) = (t^3, t^2, 0)$

$y = x^{\frac{2}{3}}$



$X'(t) = \frac{dX}{dt}$ 称为曲线 C 在 $X(t)$ 处 切向量

弧长 $\quad C: X: I=(a,b) \longrightarrow X(I) \subset E^3$



$l(X, \Delta) \sum_{i=1}^{n} |X(t_i) - X(t_{i-1})|$

Δ 分割 $\quad a = t_0 < \cdots < t_n = b$

Δ 范数.

$$|\Delta| = \max_i |t_i - t_{i-1}|.$$

$$S = \lim_{|\Delta|\to 0} l(s, \Delta)$$

下面 $x(t)$ 与 $x(s)$ 同义！

$$= \lim_{|\Delta|\to 0} \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|$$

$$= \lim_{|\Delta|\to 0} \sum_{i=1}^{n} \frac{|x(t_i) - x(t_{i-1})|}{t_i - t_{i-1}} (t_i - t_{i-1})$$

↳ 中值定理 $\Rightarrow$ 积分定义

$$= \int_a^b \left|\frac{dx}{dt}\right| dt$$

$$S(t) = \int_a^t \left|\frac{dx}{dt}\right| dt$$. 变上限：t 的可微分函数 .

Prop. s 与参数 t 的选取无关

即.

$t = t(\tau) \Rightarrow S(t) = S(t(\tau))$

$0 = t(0)$ (assume) (同起点)

$\frac{dt}{d\tau} > 0$ (正比)

$$S(\tau) = \int_0^\tau \left|\frac{dx}{d\tau}\right| d\tau$$

$$= \int_0^t \left|\frac{dx}{dt}\frac{dt}{d\tau}\right| \frac{d\tau}{dt} dt$$

↳ 拿出来

$$= \int_0^t \left|\frac{dx}{dt}\right| dt \quad \square$$

若取 . 弧长为参数 $x = x(s)$

$$\frac{ds}{dt} = \left|\frac{dx}{dt}\right| = \frac{ds}{ds} = 1 \Rightarrow \left|\frac{dx}{dt}\right| = 1$$ 单位向量

反之 $\Leftarrow$ 若 $\left|\frac{dx}{dt}\right| = 1$. $S(t) = \int_0^t \left|\frac{dx}{dt}\right| dt = t \Rightarrow$ 以弧长为参数 .

Prop. 曲线的参变量为弧长的充要条件为 $\left|\frac{dx}{dt}\right|=1 \quad \forall t\in I$

Prop. 弧长与坐标系选取无关

$\{E_i\}$ $\{e_i\}$ 为正标架.

$E_i = a_i^j e_j$ $A=(a_i^j)$ 正交矩阵. ☆

review

$A^TA = I$

$(E_1\ E_2\ E_3) = (e_1\ e_2\ e_3)A$

$= (e_1\ e_2\ e_3)\begin{pmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{pmatrix}$

$x(t) = x^i(t)E_i = y^j(t)e_j$

$= x^i(t)a_i^j e_j$

$\Rightarrow y^j(t) = x^i(t)a_i^j$

$$s(t) = \int_0^t \left(\sum_{j=1}^3 \left(\frac{dy^j}{dt}\right)^2\right)^{\frac{1}{2}} dt \neq \int_0^t \left(\sum_{i=1}^3 \left(\frac{dx^i}{dt}\right)^2\right)^{\frac{1}{2}} dt$$

$$\| $$

$$\int_0^t \left(\left(\frac{dy^1}{dt}\right)^2 + \left(\frac{dy^2}{dt}\right)^2 + \left(\frac{dy^3}{dt}\right)^2\right)^{\frac{1}{2}} dt$$

$$= \int_0^t \cdots \text{全部展开} = \int_0^t \left(\sum_{i=1}^3 \left(\frac{dx^i}{dt}\right)^2\right)^{\frac{1}{2}} dt \qquad \square$$

以弧长为参数

$\frac{dx}{ds} = \dot{x}$ $\frac{d^2x}{ds^2} = \ddot{x}$ $\frac{d^3x}{ds^3} = \dddot{x}$ $\frac{d^4x}{ds^4} = x^{(4)}$

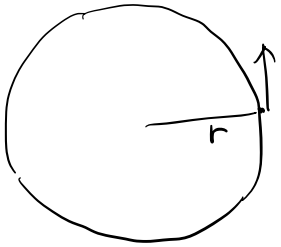
正则曲线. $\frac{dx}{ds} = \dot{x} = T(s)$ (单位切向量)

$T(s)$ 的变化率刻画曲线弯曲程度.

$|\dot{T}(s)| = k(s)$ 曲率.

$|\dot{x}| = |\ddot{x}| \geq 0$

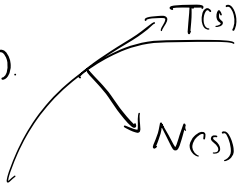
若 $k(s)\neq 0$ >0. 称 $\frac{1}{k(s)}$ 为曲率半径



$k=\frac{1}{r}$

$|T(s)|=1 \Leftrightarrow T\cdot T=1$

$\Rightarrow \dot{T}\cdot T=0$ 正交



$\dot{T}(s)=k(s)N(s)$

$|N(s)|=1$. 主法向量

$x(s)+\frac{1}{k(s)}N(s)=\rho(s)$: 曲率中心

$B=T\times N$ $\{x(s);T,N,B\}$ 右手系标架.

↓从法向量.

Span (T,N) 密切平面

Span (N,B) 法平面

Span (B,T) 从切平面

Frenet 标架.. 活动标架.

$\dot{T}=k(s)N(s)$

$B\cdot B=1 \Rightarrow \dot{B}\cdot B=0$

$B\cdot T=0 \Rightarrow \dot{B}T+B\dot{T}=0$

$\dot{B}T=-B\dot{T}=-Bk N=0$

$\Rightarrow \dot{B}\parallel N$ 令 $\dot{B}=-\tau(s)N$ $\tau(s)$ 挠率.

若 $k(s)\neq 0 \Rightarrow N(s)$ 唯一. $\Rightarrow \dot{B}$ 唯一.

↓若 $=0$. $N(s)$ 的选取任意了 ...

$-\dot{B}N=\tau(s)$

$B\cdot N=0 \Rightarrow \dot{B}N+B\dot{N}=0 \Rightarrow$ $\tau(s)=B\dot{N}$

$$= B\left(\frac{1}{k}\dot{T}\right)'$$

$$= (T\times N)\cdot\left[\left(\frac{1}{k}\right)'\dot{T}+\frac{1}{k}\ddot{T}\right]$$

$$= \frac{1}{k}(T\times N)\ddot{T}$$

$$= \frac{1}{k}\left(T\times\frac{1}{k}\dot{T}\right)\ddot{T}$$

$$= \frac{1}{k^2}(T,\dot{T},\ddot{T})$$

$$= \frac{1}{|\dot{x}|^2}(\dot{x},\ddot{x},\dddot{x})$$

"几何量"与参数、坐标系选取无关. (空间运动不变量).

$\{O_1, E_1, E_2, E_3\}$

$\{O_2, e_1, e_2, e_3\}$

C: $x_1(s)$ $x_2(s)$

$\Rightarrow x_2(s) = Ax_1(s) + B$ $\begin{cases} A: \text{正交} \\ B: \text{常矢量} \end{cases}$ 在求导下, A, B不变.

$k(s)$: $\ddot{x}_2(s) = A\ddot{x}_1(s)$

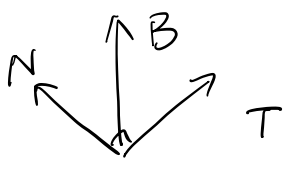
$$|\ddot{x}_2(s)|^2 = |A\ddot{x}_1(s)|^2 = \ddot{x}_1(s)^T A^T A\,\ddot{x}_1(s) = \ddot{x}_1(s)^T\ddot{x}_1(s)$$

$$A^TA = I$$

$\tau(s)$: $\dfrac{(\dot{x}_2,\ddot{x}_2,\dddot{x}_2)}{|\ddot{x}_2|^2} = \dfrac{(A\dot{x}_1, A\ddot{x}_1, A\dddot{x}_1)}{|\ddot{x}_1|^2}$ → 想几何意义不变.

$$\| $$

$$\frac{(\dot{x}_1,\ddot{x}_1,\dddot{x}_1)}{|\ddot{x}_1|^2}$$



$N(s) = B(s)\times T(s)$

$\dot{N}(s) = \dot{B}\times T + B\times\dot{T} = -\tau N\times T + B\times kN = \tau B - kT$

Conclusion

1. 曲线表示 Frenet 标架.

2. "几何量". 弧长. 曲率、挠率. 与坐标选取. 与参数选取 无关.

△ 正交矩阵复习. ✓ ?

△ $(a\times b)\cdot c = (a,b,c) = \det(a\ b\ c)$