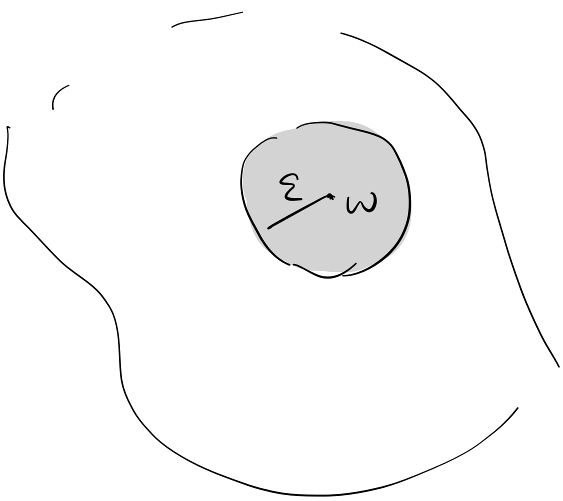
Maximal module principle.

Cor. Let $U \xrightarrow[\text{holo}]{f} \mathbb{C}$ If $\exists z_0 \in U$ s.t. $|f(z)| \leq |f(z_0)|$

U: path-connected. open

then. f is a constant function

Pf. For any $w \in U$



and any $\varepsilon > 0$. s.t. $\overline{B_\varepsilon(w)} \subseteq U$ we.

have $$f(w) = \frac{1}{2\pi i} \int_{\partial B_w(w)} \frac{f(z)}{z-w}\, dz$$

$$f(w) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(w+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\, i \varepsilon e^{i\theta}\, d\theta.$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(w+\varepsilon e^{i\theta})\, d\theta. \quad \left(\Rightarrow |f(w)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(w+\varepsilon e^{i\theta})|\, d\theta \right)$$

Let $M := |f(z_0)|$

$$\Rightarrow. \quad M - |f(w)| \geq \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\left(M - |f(w+\varepsilon e^{i\theta})| \right)}_{\geq 0} d\theta. \geq 0$$

Consider $V := \{ w \in U \mid |f(w)| = M \}$.

$\downarrow$ 单点

$\Rightarrow$ preimage 闭 即 V 闭

$$w \in V \Rightarrow 0 = M - f(w) \geqslant \frac{1}{2\pi} \int_0^{2\pi} \left(M - |f(w + \varepsilon e^{i\theta})| \right) d\theta \geqslant 0$$

$$\Rightarrow \quad \| \quad 0$$

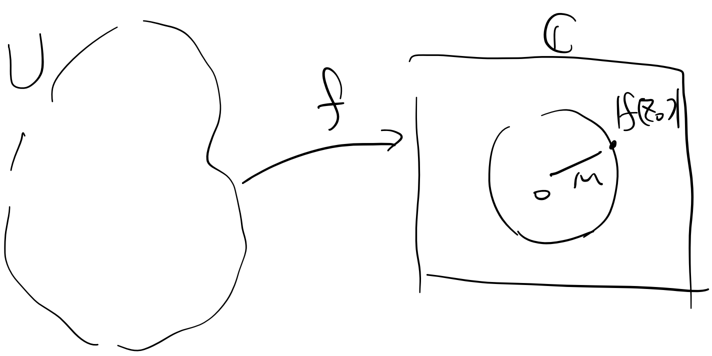
$$\Rightarrow M - |f(w + \varepsilon e^{i\theta})| = 0. \quad 0 \leq \theta \leq 2\pi.$$

$$\Rightarrow B_\varepsilon(w) \subseteq V \Rightarrow V \underset{\text{open}}{\subseteq} U.$$

由 $U \subseteq \mathbb{R}^n$ path-connected.

& $V \underset{\text{open}}{\subseteq} U$. $U \setminus V \underset{\text{open}}{\subseteq} U$. $\Rightarrow V = U$ 或 ϕ.

$\Rightarrow V = U$. $\Rightarrow \forall z \in U$. $|f(z)| = M$. $\square$



现在已经打到了圆周上

下证都打到 $f(z_0)$.

考虑 $M > 0$ 的情况 $U = \mathrm{Re} f$ $V = \mathrm{Im} f$.

$$\Rightarrow u^2 + v^2 = M^2 \xrightarrow{\frac{\partial}{\partial x}} 2\left(u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x}\right) = 0$$

$$\xrightarrow{\frac{\partial}{\partial y}} 2\left(u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y}\right) = 0$$

由 Cauchy-Rieman Equation

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ -\frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$|u|>0.$ $\Rightarrow$ u, v 不全为0. $\Rightarrow$ 有非零解 $\Rightarrow$ det

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0$$

$$\Rightarrow \underset{\frac{\partial v}{\partial y}}{\underset{\parallel}{\frac{\partial u}{\partial x}}} \equiv 0 \equiv \underset{-\frac{\partial u}{\partial y}}{\underset{\parallel}{\frac{\partial v}{\partial x}}}$$

$\Rightarrow$ u, v. const $\Rightarrow$ $f =$ const.

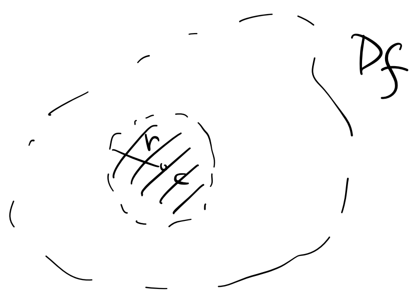
(C) Isolated. singularities and the Laurent expansion

<u>Convention</u> f. domain of f : D_f.

<u>Def</u>. Let f be a func with $D_f \subseteq \mathbb{C}$ and $c \in \mathbb{C}$

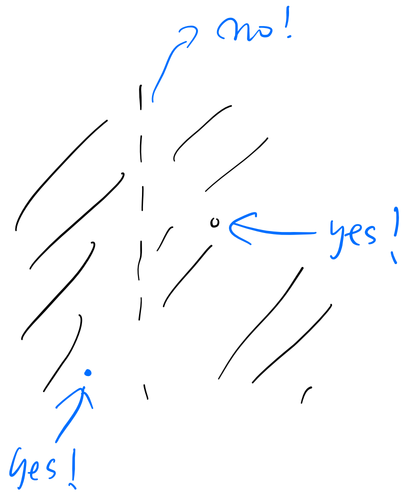
We say that "c is an isolated singularity." or "f has an isolated singularity at $z = c$."

$\exists\, r>0.$ $B_r(c) \setminus \{c\} \subseteq D_f$

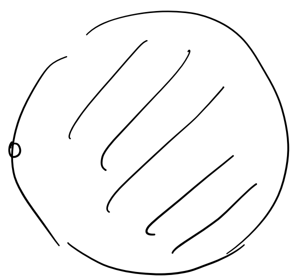


i.e. f is defined near c but not necessarily at c

Example. (1). $\dfrac{z}{(\operatorname{Re} z)(z-1)}$ has domain



(2) $\sum_{n=0}^{\infty}(-1)^n \dfrac{z^n}{n}$ has domain $\{z \in \mathbb{C} \mid |z| \leq 1 \text{ 且 } z \neq -1.\}$



$B_1(0)$ 中所有点为孤立奇点.

Def. (1) A (formal) Laurent series centered at c is a formal sum

$$\sum_{n=-\infty}^{\infty} \underset{\underset{\mathbb{C}}{\Uparrow}}{a_n} (z-\underset{\underset{\mathbb{C}}{\Uparrow}}{c})^n.$$

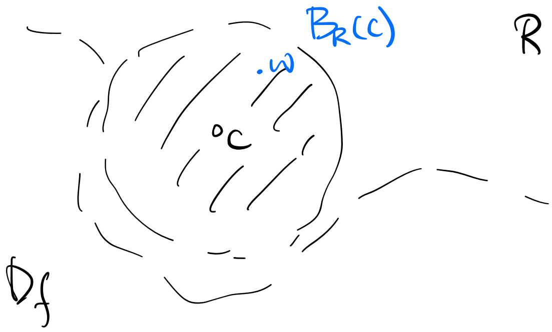
(2). We say that $\sum_{n=-\infty}^{\infty} a_n(z-c)^n$ converges (to S) if

both $\sum_{n=0}^{\infty} a_n(z-c)^n$ and $\sum_{m=1}^{\infty} a_{-m}(z-c)^{-m}$ converges

and sum $\to S$

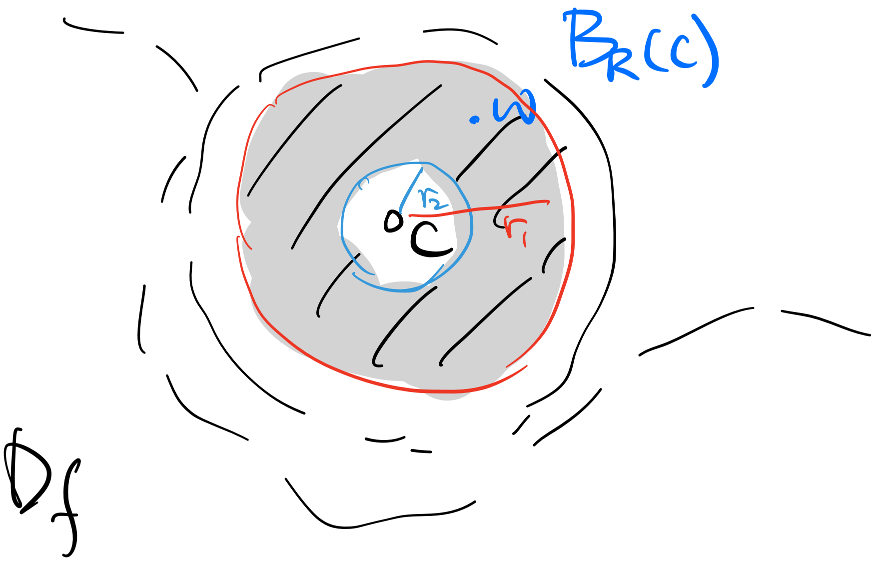
Absolute
Uniform $\Big\}$ defined similarly

We now show that a holo func f coincides, near an iso sing, with a (convergent) Laurent series centered at c



$R := \sup\{r>0 \mid B_r(c)\setminus\{c\} \subseteq D_f\}$

and concider $w \in B_R(c)\setminus\{c\}$



For any (fixed) $w \in B_R(c)\setminus\{c\}$. choose $0<r_2<|w-c|<r_1<R$

and. let $D := B_{r_1}(c)\setminus\overline{B_{r_2}(c)}$

↳ a open annulus

Cauchy integral formula $\Longrightarrow$ $$f(w) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z-w}\,dz$$

$$= \underbrace{\frac{1}{2\pi i}\int_{\partial B_{r_1}(c)}\frac{f(z)}{z-w}\,dz}_{(\mathrm{I})} - \underbrace{\frac{1}{2\pi i}\int_{\partial B_{r_2}(c)}\frac{f(z)}{z-w}\,dz}_{(\mathrm{II})}$$

(I). : If $z \in \partial B_{r_1}(c)$. then $|w-c|<r_1$. and hence

$$\frac{1}{z-w} = \frac{1}{(z-c)-(w-c)} = \frac{1}{z-c}\,\frac{1}{1-\underbrace{\frac{w-c}{z-c}}_{|\ \ |<1}} = \sum_{n=0}^{\infty}\frac{1}{z-c}\left(\frac{w-c}{z-c}\right)^n$$

$$(I) = \frac{1}{2\pi i}\int_{\partial B_{r_1}(c)}\left[\sum_{n=0}^{\infty}\frac{f(z)}{z-c}\left(\frac{w-c}{z-c}\right)^n\right]dz \ \underset{\text{uniformly converges}}{\text{"}=\text{"}}\ \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial B_r(c)}\frac{f(z)}{(z-c)^{n+1}}dz\right)(w-c)^n.$$

ex. show that $-(II) = \sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial B_{r_2}(c)} f(z)(z-c)^{m+1}\,dz\right)\frac{1}{(w-c)^{m+1}}$

$\Rightarrow$. for a (fixed.) $w \in B_R(c)\setminus\{c\}$

$$f(w) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial B_{r_1}(c)}\frac{f(z)}{(z-c)^{n+1}}dz\right)(w-c)^n$$

$$+\sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial B_{r_2}(c)} f(z)(z-c)^{m+1}\,dz\right)\frac{1}{(w-c)^{m+1}}$$

However, r_1. r_2 can be replaced by any $r\in(0,R)$ without altering the integrals.

$$f(w) = \sum_{n=-\infty}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{\partial B_r(c)}\frac{f(z)}{(z-c)^{n+1}}dz\right)}_{\stackrel{!!}{a_n(f;c)}}(w-c)^n \quad \begin{pmatrix} 0<r<R \\ w\in B_R(c)\setminus\{c\}\end{pmatrix}$$

Laurant series

Ex (1) Given $a_n \in \mathbb{C}$. $(n \in \mathbb{Z})$. if there are $z_1, z_2 \in \mathbb{C}$ with $|z_2| < |z_1|$

s.t. both $\sum_{n=-\infty}^{\infty} a_n (z_1)^n$ and $\sum_{n=-\infty}^{\infty} a_n (z_2)^n$ converge. then, $\sum_{n=-\infty}^{\infty} a_n z^n$

converges uniformly and absolutely on the closed annulus

$\overline{B_{R_1}(0)} \setminus B_{R_2}(0)$ for any R_1, R_2 s.t.

$$|z_2| < R_2 < R_1 < |z_1|$$

(2). Does there exist $a_n \in \mathbb{C}$ $(n \in \mathbb{Z})$. s.t.

$\sum_{n=-\infty}^{\infty} a_n z^n$. converges everywhere on $\{ z \in \mathbb{C} \mid |z| = 1 \}$

but nowhere else ?

Def. If f is a holo. func. which has an iso. sing. at c.

we call $a_{-1}(f; c)$ the residue of f at c,

denoted by $\operatorname{Res}_{z=c} f(z)$. $\operatorname{Res}_c f(z)$

Lemma. If f is holo. and

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z-c)^n \text{ for some } \begin{cases} r_0 > 0. \\ b_n \in \mathbb{C},\ n \in \mathbb{Z} \\ \text{and } c \in \mathbb{C} \end{cases} \text{ on } B_{r_0}(c) \setminus \{c\}$$

then $b_n = a_n(f; c) = \frac{1}{2\pi i} \int_{\partial B_r(c)} \frac{f(z)}{(z-c)^{n+1}} \, dz \quad (n \in \mathbb{Z})$

Pf.

$$2\pi i\, a_{-1}(f;c) = \int_{\partial B_r(c)} f(z)\,dz$$

$$= \int_{\partial B_r(c)} \underbrace{\left(\sum_{n=-\infty}^{\infty} b_n (z-c)^n\right)}_{\text{uni conv}} dz.$$

$$\partial B_r(c) = \gamma(\theta) = c + re^{i\theta}, \quad 0 \le \theta \le 2\pi.$$

$$= \sum_{n=-\infty}^{\infty} b_n \int_{\partial B_r(c)} (z-c)^n\,dz.$$

$$= \sum_{n=-\infty}^{\infty} b_n \int_0^{2\pi} r^n e^{in\theta}\, i r e^{i\theta}\, d\theta.$$

$$= \sum_{n=-\infty}^{\infty} b_n \int_0^{2\pi} i r^{n+1} e^{i(n+1)\theta}\, d\theta.$$

$$\begin{cases} n=-1 \text{ 时 } & \int_0^{2\pi} i\,d\theta = 2\pi i \\ n\ne -1 \text{ 时. } & \int_0^{2\pi} i\, r^{n+1} e^{i(n+1)\theta}\, d\theta = 0. \end{cases}$$

$$= 2\pi i\, b_{-1} \quad \Rightarrow \quad b_{-1} = a_{-1}(f;c)$$

$b_n = a_n(f,c)$ for all $n \in \mathbb{Z}$ 一样的操作

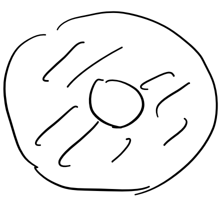
Ex. If $a_n \in \mathbb{C}$, $b_n \in \mathbb{C}$ $(n \in \mathbb{Z})$ are s.t.

$\sum_{n=-\infty}^{\infty} a_n z^n$ and $\sum_{n=-\infty}^{\infty} b_n z^n$ both converge on $B_{R_1}(0) \setminus \overline{B_{R_2}(0)}$

收敛到的一样

for some $0 < R_2 < R_1$ and coincide with each other.

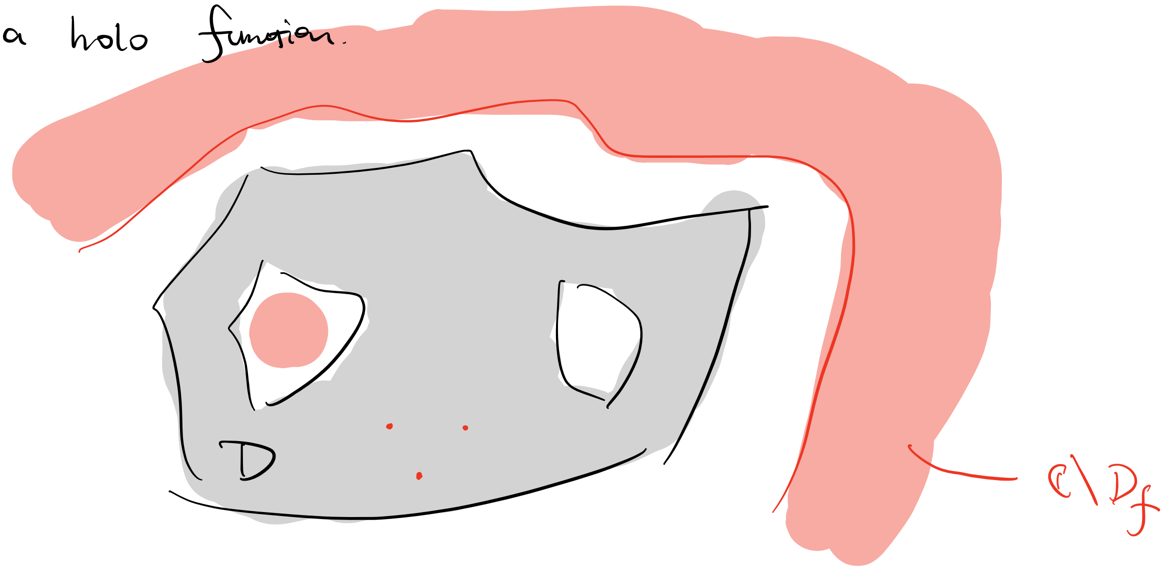
then $a_n = b_n$ $(n \in \mathbb{Z})$.



Theorem

Let $\underset{\text{open}}{D \subseteq \mathbb{C}}$ be a Green domain ($\Rightarrow \overline{D}$ cpt.),

and f a holo function.



If $\partial D \subseteq D_f$, i.e. f is defined along ∂D

Every point of D is an iso. sing of f.

then (1). $\overline{D} \setminus D_f$ is a finite set

(2). $$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{c \in D} \operatorname{Res}_{z=c} f(z)$$

由(1)、实际只需考虑有限和

Pf. (1) (use cpt. condition)

(2) 挖红点圆盘. use Cauchy integral theorem.