极小曲面 $H=0$.

• 设 $y=f(z)$ 绕 z 轴旋转一周形成的旋转面 M. 则 M 为极小曲面 $\Longleftrightarrow$ M 为悬链面. ($y=a\,\mathrm{ch}\frac{z}{a}$)

$$M:\ x(u^1,u^2)=\left(f(u^2)\cos u^1,\ f(u^2)\sin u^1,\ u^2\right)$$

$$g_{11}=f^2 \quad g_{12}=0 \quad g_{22}=(f')^2+1$$

$$h_{11}=\frac{-f}{\sqrt{1+(f')^2}} \quad h_{12}=0 \quad h_{22}=\frac{f''}{\sqrt{1+(f')^2}}$$

$$0=H=\frac{1}{2}\frac{h_{11}g_{22}-2h_{12}g_{12}+h_{22}g_{11}}{f^2(1+(f')^2)}=\frac{1}{2}\frac{1}{\sim}\left(-f\sqrt{1+(f')^2}+\frac{f^2f''}{\sqrt{1+(f')^2}}\right)$$

$$\Longleftrightarrow \quad f\sqrt{1+(f')^2}=\frac{f^2f''}{\sqrt{1+(f')^2}}$$

$$\overset{f\neq 0}{\Longleftrightarrow} \quad \sqrt{1+(f')^2}=\frac{ff''}{\sqrt{1+(f')^2}}$$

$$1+(f')^2=ff''$$

$$f'(1+(f')^2)=(ff'')f'$$

$$\frac{f'}{f}=\frac{f''f'}{(1+(f')^2)}=\frac{\frac{1}{2}d\,((f')^2+1)}{1+(f')^2}$$

$$\ln f = \frac{1}{2}\ln(1+(f')^2) + \ln c.$$

$$\Rightarrow f = c\sqrt{1+(f')^2}$$

$$\left(\frac{f}{c}\right)^2 = 1+(f')^2$$

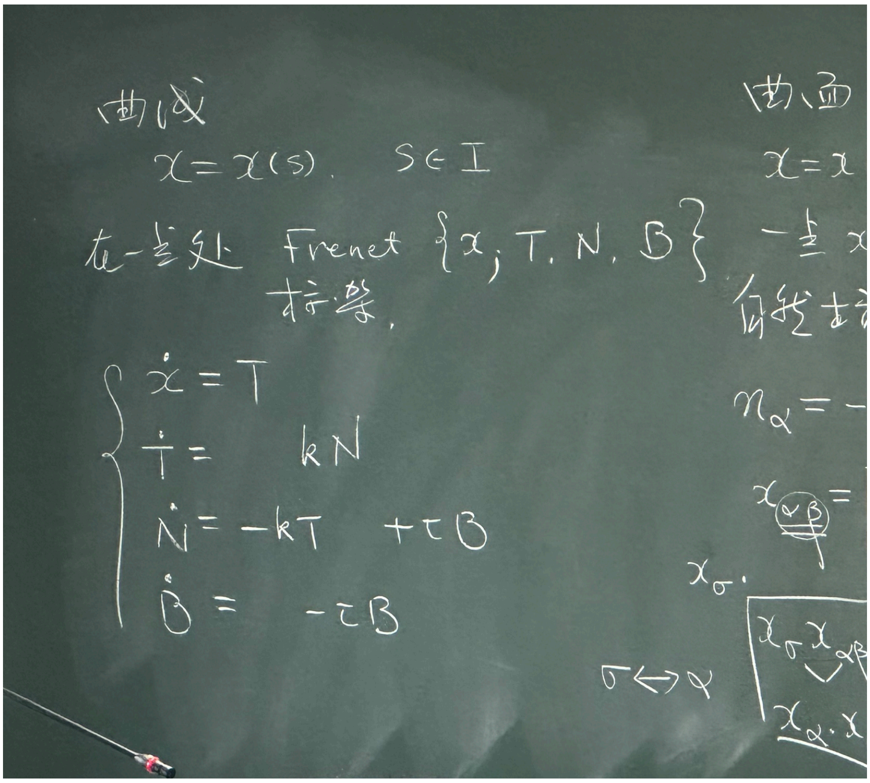
$$f' = \pm\sqrt{\left(\frac{f}{c}\right)^2 - 1}$$

$$\frac{d\left(\frac{f}{c}\right)}{\sqrt{\left(\frac{f}{c}\right)^2-1}} = \pm dz\,\frac{1}{c}$$

$$\pm\left(\frac{z}{c}+b\right) = \mathrm{ch}^{-1}\left(\frac{f}{c}\right)$$

$$f = c\,\mathrm{ch}\left(\frac{t}{c}+b\right).$$

极小曲面 $\rightsquigarrow$ 几何测度论?



曲面. $x=x(u^1,u^2)$ $(u^1,u^2)\in D\subset E^2$

一点 $x(u_0^1,u_0^2)=x_0$. $\{x_0, x_1, x_2, n\}$. 自然标架.

$n_\alpha=-h_\alpha^\beta x_\beta$ Weingarten公式

$x_{\alpha\beta}=\Gamma_{\alpha\beta}^\gamma x_\gamma+h_{\alpha\beta}n$ $\Gamma_{\alpha\beta}^\gamma=\Gamma_{\beta\alpha}^\gamma$ Guass公式.

$$x_\sigma\cdot x_{\alpha\beta}=\Gamma_{\alpha\beta}^\gamma x_\gamma x_\sigma=g_{\gamma\sigma}\Gamma_{\alpha\beta}^\gamma$$

$\sigma\leftrightarrow\alpha$ $$x_\alpha\cdot x_{\sigma\beta}=g_{\gamma\alpha}\Gamma_{\sigma\beta}^\gamma$$

$$(x_\alpha x_\sigma)_\beta=x_\sigma\cdot x_{\alpha\beta}+x_\alpha x_{\sigma\beta}$$

$$\frac{\partial g_{\alpha\sigma}}{\partial u^\beta}=g_{\gamma\sigma}\Gamma_{\alpha\beta}^\gamma+\cancel{g_{\gamma\alpha}\Gamma_{\sigma\beta}^\gamma} \quad ①$$

同理有 $\alpha\leftrightarrow\beta$ $$\frac{\partial g_{\beta\sigma}}{\partial u^\alpha}=g_{\gamma\sigma}\Gamma_{\beta\alpha}^\gamma+\cancel{g_{\gamma\beta}\Gamma_{\sigma\alpha}^\gamma} \quad ②$$

$\sigma\leftrightarrow\beta$. $$\frac{\partial g_{\alpha\beta}}{\partial u^\sigma}=\cancel{g_{\gamma\beta}\Gamma_{\alpha\sigma}^\gamma}+\cancel{g_{\gamma\alpha}\Gamma_{\beta\sigma}^\gamma} \quad ③$$

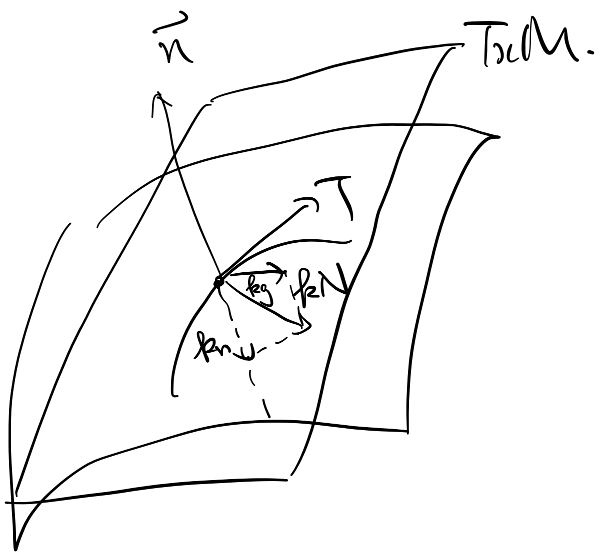
①+②−③

$$\underbrace{g^{\sigma\tau}2g_{\gamma\sigma}}_{2\delta_\gamma^\tau}\Gamma_{\alpha\beta}^\gamma=\left(\frac{\partial g_{\alpha\beta}}{\partial u^\sigma}+\frac{\partial g_{\sigma\beta}}{\partial u^\alpha}-\frac{\partial g_{\alpha\sigma}}{\partial u^\beta}\right)g^{\sigma\tau}$$

$$\Gamma_{\alpha\beta}^\tau=\frac{1}{2}g^{\sigma\tau}\left(\frac{\partial g_{\alpha\beta}}{\partial u^\sigma}+\frac{\partial g_{\sigma\beta}}{\partial u^\alpha}-\frac{\partial g_{\alpha\sigma}}{\partial u^\beta}\right).$$

Christoffel 符号 (克氏符号)

内蕴几何量.



$\dot{T} = kN$

$k_n = kN \cdot n$

kN 在 T_xM 上投影 ?

$kN = f^\sigma x_\sigma + k_n n$

$$g^{\alpha\beta}(x_\alpha \cdot kN) = f^\sigma (x_\sigma x_\alpha) = g_{\sigma\alpha} f^\sigma g^{\alpha\beta}$$

$$\| $$

$$\delta_\sigma^\beta f^\sigma = f^\beta$$

$$\Rightarrow f^\beta = g^{\alpha\beta}(x_\alpha k N) = g^{\alpha\beta}(x_\alpha \cdot \dot{T}) = g^{\alpha\beta}\left(x_\alpha \cdot \frac{d}{ds}\left(x_\gamma \frac{du^\gamma}{ds}\right)\right)$$

$$= g^{\alpha\beta} x_\alpha \left(x_{\gamma\sigma} \frac{du^\sigma}{ds} \frac{du^\gamma}{ds} + x_\gamma \frac{d^2 u^\gamma}{ds^2} \right)$$

$$= g^{\alpha\beta} x_\alpha \left(\Gamma^\tau_{\gamma\sigma} x_\tau \frac{du^\sigma}{ds} \frac{du^\gamma}{ds} + h_{\gamma\sigma} n \frac{du^\sigma}{ds} \frac{du^\gamma}{ds} + x_\gamma \frac{d^2 u^\gamma}{ds^2} \right)$$

$$= g^{\alpha\beta} g_{\alpha\tau} \Gamma^\tau_{\gamma\sigma} \frac{du^\sigma}{ds} \frac{du^\gamma}{ds} + g^{\alpha\beta} g_{\alpha\gamma} \frac{d^2 u^\gamma}{ds^2}$$

$$= \Gamma^\beta_{\gamma\sigma} \frac{du^\sigma}{ds} \frac{du^\gamma}{ds} + \frac{d^2 u^\beta}{ds^2}$$

$\Rightarrow$

$$kN = \left(\Gamma^\sigma_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} + \frac{d^2 u^\sigma}{ds^2} \right) x_\sigma + k_n \cdot n$$

$$= k_g \cdot Q + k_n \cdot n \Rightarrow k^2 = k_n^2 + k_g^2$$

Q 单位向量.

$$k_g Q = \left(\Gamma^\sigma_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} + \frac{d^2 u^\sigma}{ds^2} \right) x_\sigma$$

k_g 测地曲率

$Q \cdot T = 0$

$Q n = 0$. $Q = n \times T$

$$k_g = kN\cdot Q = kN\cdot(n\times T) = (\dot{T}, n, T)$$

Liouville公式…

Def. $k_g \equiv 0$ 的曲线，称为测地线

测地线的微分方程.

$$\frac{d^2u^\gamma}{ds^2} + \underbrace{\Gamma^\gamma_{\alpha\beta}}_{\text{内蕴}}\, du^\alpha du^\beta = 0$$

测地线是内蕴概念.

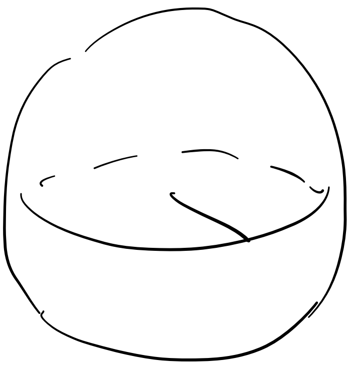
$$\begin{cases} \dfrac{d^2u^\gamma}{ds^2} + \Gamma^\gamma_{\alpha\beta}\, du^\alpha du^\beta = 0 \\[2ex] \dfrac{du^\gamma}{ds}(0) = v_0^\gamma \\[2ex] u^\gamma(0) = u_0^\gamma \end{cases}$$

存在唯一解

(方程观点)

- 曲面上直线一定为测地线
- 非直曲线为测地线.

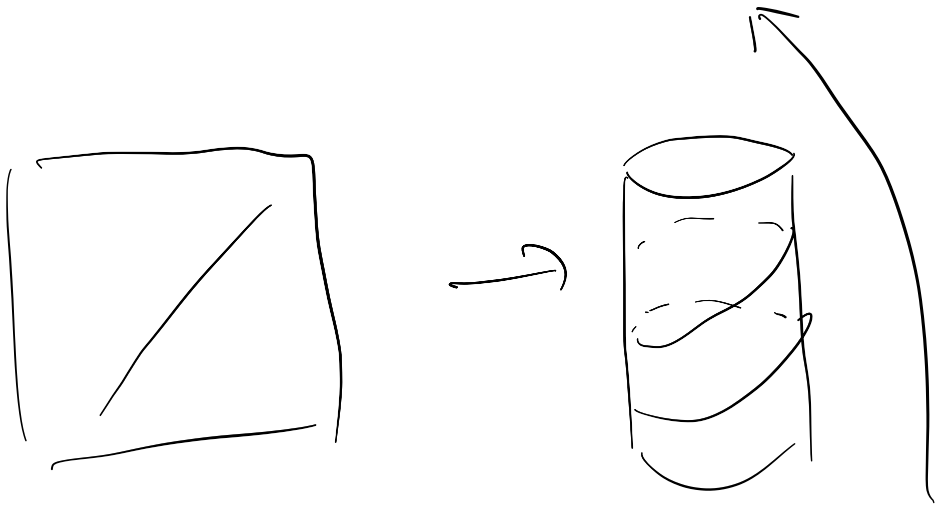
$N \parallel n$ （几何观点）.



因为测地线是内蕴的!

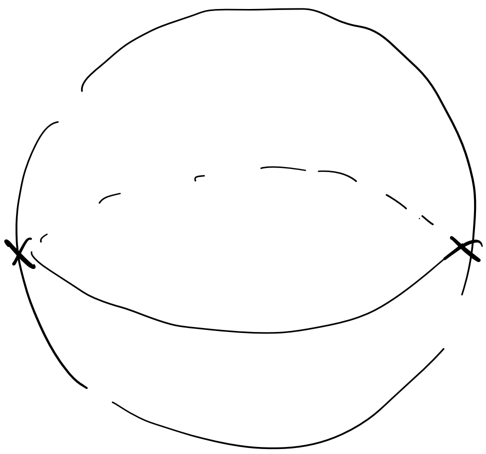
等距 $\varphi: M_1 \to M_1$ 仍保持测地线

$\boxed{ds_1^2 = ds_2^2}$

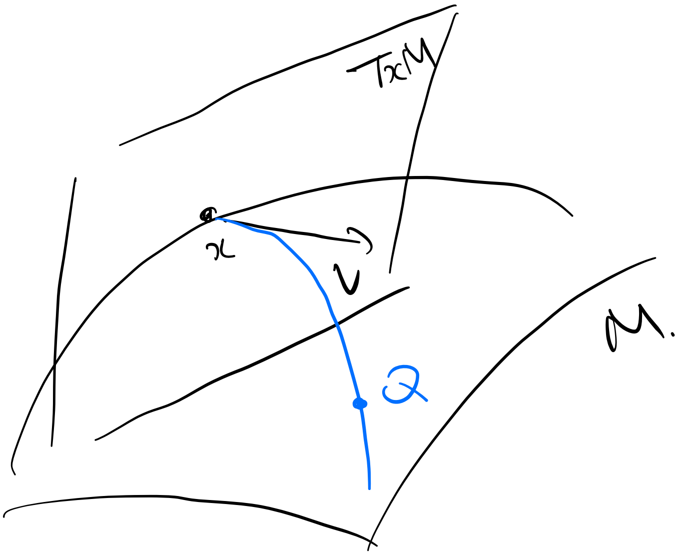


• 测地线为两点间最短曲线（小范围内）

球面上两点



割点.



$\forall x\in M.\ \ v\in T_xM.$ 过x点与 $\frac{v}{\|v\|}$ 为初始向量的测地线唯一(局部).

测地线上取Q点. s.t.

Q沿测地线到x点的距离$=\|v\|$

$T_xM \longrightarrow M$ 指数映射.

$\exp_x : v \longmapsto Q$

$\exp_x(v)=Q$

$T_xM:\{x;\ e_1, e_2\}$ 幺正.

$v = y^1 e_1 + y^2 e_2$ $\{y^1, y^2\}$ 为v的坐标.

令$\{y^1, y^2\}$为Q点的坐标.

$$Q \in M \longleftrightarrow \{y^1, y^2\} \in \mathbb{R}^2.$$

$$V = y^1 e_1 + y^2 e_2.$$

法坐标

$\{x; e_1, e_2\}$ 幺正标架.

Prop. 指数映射在小邻域内是微分同胚.

x点处的法坐标系

$$\frac{V}{\|V\|} = V_0. \qquad V = \underset{\|V\|}{\underset{\parallel}{s}} \cdot V_0 = s(y_0^1 e_1 + y_0^2 e_1).$$

$$\exp_x V = Q\,(s y_0^1,\ s y_0^2).$$

“测地线方程”

$$\begin{cases} y^1 = s y_0^1 \\ y^2 = s y_0^2. \end{cases}$$