Algebra. Lec 3. Isomorphism theorem. 2 & 3.

Tower of subgroups.

review 1st. iso thm.

$f: G \to G'$ $\quad H = \ker(f)$

$$\begin{tikzcd} G \arrow[r, two heads, "\varphi"] \arrow[rrr, bend right, "f"'] & G/H \arrow[r, "\sim"] & \operatorname{im}(f) \arrow[r, hook, "j"] & G' \end{tikzcd}$$

j inclusion

φ. can. proj

(*).

$$\begin{tikzcd} x \in G \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[r, bend left] & G' \ni f(x) \\ xN \quad G/N \arrow[ur, "\exists! f_*"'] & \end{tikzcd}$$

$N \trianglelefteq G$. $N \leq \ker(f)$

$\Rightarrow \exists! f_*$, s.t. ?

2nd. 3rd. iso thm.

- $H \trianglelefteq G$. $K \trianglelefteq G$. $\underbrace{K \leq H}_{\Rightarrow K \trianglelefteq H}$. (3rd).

$\Rightarrow (G/K)/(H/K) \cong G/H.$

Pf. $G/K \xrightarrow{\varphi} G/H$ | well-defined? $x_1K = x_2K$.

$xK \longmapsto xH$ | $x_2^{-1}x_1 \in K. \leq H$

$\Rightarrow x_1H = x_2H$.

hom $\varphi(x_1K x_2K) = \varphi(x_1x_2K) = x_1x_2H$

$= x_1H \cdot x_2H = \varphi(x_1K)\varphi(x_2K)$

surjective clearly

$\ker(\varphi) = \{xK \mid x\in G,\ xH = H\} = H/K$

by 1st iso. $\Rightarrow$ $(G/K)/(H/K) \cong G/H$

- 2nd iso thm.

$H, K \leq G$. $\underline{H \leq N_K}$

$H\cap K \leq H$ is clear

$\Rightarrow$ $\left\{\begin{array}{l} \cdot\ H\cap K \trianglelefteq H \\ \cdot\ HK = KH \leq G \\ \cdot\ K \trianglelefteq HK. \\ \cdot\ \boxed{H/H\cap K \cong HK/K} \end{array}\right.$

$\left(\forall h\in H\quad h(H\cap K)h^{-1} \subseteq H\cap(hKh^{-1}) \subseteq H\cap K\right)$

$($ $hkh^{-1} \subseteq K \Rightarrow hk \subseteq Kh \Rightarrow HK \subseteq KH.$

$h^{-1}kh \subseteq K \Rightarrow kh \subseteq hK \Rightarrow KH \subseteq HK.$

subgroup criterion. $(h_1k_1)(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1}$

$= h_1 h_2^{-1}h_2 k_1k_2^{-1}h_2^{-1}$

$= \underset{\in H}{h_1h_2^{-1}}\cdot\underset{\in K}{(h_2k_1k_2^{-1}h_2^{-1})}$

$\in HK.$ $)$

$(\forall k\in K,\ h_1k_1 \in HK$

$(h_1k_1)k(h_1k_1)^{-1} = h_1(k_1kk_1^{-1})h_1^{-1} \in K.)$

- $H \longrightarrow HK/K = \{hkK \mid h\in H, k\in K\} = \{hK \mid h\in H\} = $ "H/K"

  $x \longmapsto xK$ surjective

$\varphi(x_1x_2) = x_1x_2K = (x_1K)(x_2K) = \varphi(x_1)\varphi(x_2)$. hom.

$\ker(\varphi) = \{x\in H \mid xK = K\} = H\cap K \Rightarrow H/H\cap K \cong HK/K$.

Rmk. $\cdot$ $f: G \to G'$ hom.

$$\begin{array}{c} G' \\ \triangledown\!| \\ H' \end{array}$$

let $H \triangleq f^{-1}(H')$ pre-image

inverse-image of H'.

$= f^{-1}(H' \cap \mathrm{Im}(f))$.

$\Rightarrow H \trianglelefteq G$ $\Big(\forall x \in G.\ f(xHx^{-1})$

$= f(x)f(H)f(x^{-1}) \in H'$

$\Rightarrow xHx^{-1} \subseteq H. \Big)$

- $G \xrightarrow{f} G' \xrightarrow[\text{(can. proj).}]{\varphi} G'/H'$

$\ker(\varphi\circ f) = \{x\in G \mid f(x)\in H'\} = H.$

$$G/H \simeq \mathrm{Im}(\varphi \circ f) \hookrightarrow G'/H'$$

we obtain an injection from

$$\bar{f} \quad G/H \hookrightarrow G'/H'$$

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, hook] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & G \arrow[r, two heads] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & G/H \arrow[r] \arrow[d, "\bar{f}"] & 0 & \text{exact seq} \\
0 \arrow[r] & H' \arrow[r, hook] & G' \arrow[r, two heads, "\varphi"'] & G'/H' \arrow[r] & 0 & \text{exact seq}
\end{tikzcd}$$

universal property.
(1st. iso theorem ②).

clearly commutative

$\ker(\varphi \circ f) = H$

if f surjective

$\Rightarrow \bar{f}$ isomorphism.

Hw. 03.

① [DF]. Sec 3.2. ex. 9. (Cauchy thm)

② [L]. chap I. ex. 14

③ $G_1 \trianglelefteq G_2$, $G_2 \trianglelefteq G_3 \overset{?}{\Longrightarrow} G_1 \trianglelefteq G_3$

④ Read [DF] chap 3. Thm 20. (4th or lattice iso thm).

- $G = G_0 \supset G_1 \supset \cdots \supset G_n$. tower of subgroups.

The tower is said to be normal if $G_{i+1} \trianglelefteq G_i$ $i = 0, \ldots, m-1$.

The tower is abelion if $G_{i+1} \trianglelefteq G_i$ and G_i / G_{i+1} is ablian.

The tower is cyclic if $G_{i+1} \trianglelefteq G_i$ and G_i / G_{i+1} is cyclic

(G is cyclic if $\exists\, x \in G$. s.t. $G = \langle x \rangle$.).

- $f \quad G \to G'$

$G' = G_0' \supset G_1' \supset G_2' \supset \cdots \supset G_m'$: a normal tower

let $G_i = f^{-1}(G_i')$.

$\Rightarrow G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_m$ form a normal tower

(check : $G_{i+1} \trianglelefteq G_i$. $\forall x \in G_i$ $\cdot$ $f(x G_{i+1} x^{-1}) = \underset{\substack{\cap \\ G_i'}}{f(x)}\ \underset{\substack{\cap \\ G_{i+1}'}}{f(G_{i+1})}\ \underset{\substack{\cap \\ G_i'}}{f(x^{-1})} \subseteq G_{i+1}'$

$\Rightarrow x G_{i+1} x^{-1} \subseteq G_{i+1}$)

$$\begin{array}{ccc} G_i & \xrightarrow{f} & G_i' \\ \triangledown\!/\!/ & & \triangledown\!\mid \\ G_{i+1} & & G_{i+1}' \end{array}$$

$$G_i / G_{i+1} \hookrightarrow G_i' / G_{i+1}'$$

G_i' abelian tower $\Rightarrow$ G_i abelian tower. Ex.

G_i' cylic tower $\Rightarrow$ G_i cylic tower.

Def A refinement of a tower $G = G_0 \supset G_1 \supset \cdots \supset G_m$.

is a tower, which can be can be obtained by inserting

a finite # of subgroups in the given tower

Def. G is said to be solvable if it has an abelian tower

ending in $G_m = \{e\}$.

i.e. $\exists$ a normal tower $G = G_0 \supset G_1 \supset \cdots \supset G_m = \{e\}$.

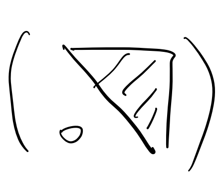
G_i / G_{i+1} abelian group.

Rmk. sovable group

$\rightsquigarrow$ Galois Theory.

e.g. k field.

$G = GL(n, k) = \left(\begin{array}{l}\text{the } n\times n \text{ invertible}\\ \text{matices } / k.\end{array}\right)$

$T = T(n,k)$: $\left(\begin{array}{c}\text{upper triangular group}.\\ \det \neq 0.\end{array}\right)$ 

verify.

$D =$ diagonal matices. $\det \neq 0$.

$N =$ matices. with 0 on and below diagonal

$U = I + N$. Check $U \leq G$ (unipotent).

- $A \in N \Rightarrow A^m = 0$ for some m. (A nilpotent).

- $(I - A)^{-1} = I + A + \cdots + A^{m-1}$.

- $$\begin{array}{ccc} T & \longrightarrow & D \\ \rotatebox{90}{$\in$} & & \\ A & \longmapsto & \begin{pmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{nn} \end{pmatrix} = \operatorname{diag}(A) \end{array}$$

$\ker(\phi) = U$.

observe. $N^{r-1} = \left\{ \begin{pmatrix} 0 & 0 & \cdots & 0 & a_{1r} & & \\ 0 & 0 & \cdots & 0 & & a_{2,r+1} & \\ & & & & & & \ddots \\ & & O. & & & & a_{n-r+1,n} \end{pmatrix} \right\}$

(product of ideals).

let $U_r = I + N^r$

$T \supset U = U_1 = I + N \supset U_1 \supset U_2 \supset \cdots U_n = I$

Claim. • U_r: subgroup of G or T.

• $U_{r+1} \trianglelefteq U_r$

• $U_r / U_{r+1} \cong (k^{n-r}, +) \longrightarrow$ abelian group.

i.e. is an abelian. tower of T.

eg. sepecialize to $n=5$

$U_1 = \left\{ \begin{pmatrix} 1 & & & & \\ & 1 & & * & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \right\} \supset U_2 = \left\{ \begin{pmatrix} 1 & 0 & & & \\ & 1 & 0 & * & \\ & & 1 & 0 & \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \right\} \supset U_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 & * & \\ & 1 & 0 & 0 & \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \right\}$

$\Rightarrow$ check U_r. subgroup

U_3 is normal in U_2 ? $\quad y = I - B \in U_3 \quad B^2 = 0$

$x = I - A \in U_2. \quad A^2 \neq 0 \quad A^3 = 0.$

$$\Rightarrow \quad x^{-1} = I + A + A^2$$

$$xyx^{-1} = (I-A)(I+B)(I+A+A^2)$$

$$= I - (I-A)\,B\,(I+A+A^2) \in U_3.$$

- $g : U_2 \longrightarrow (k^3, +)$.

$$\begin{pmatrix} 1 & 0 & a_{13} & & \\ & 1 & 0 & a_{24} & \\ & & 1 & 0 & a_{35} \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \longmapsto (a_{13}, a_{24}, a_{35}) \qquad \text{hom.}$$

$$\ker(g) = U_3.$$

$$\Rightarrow \quad U_2 / U_3 \simeq (k^3, +)$$