**Thm** Let $\mathcal{X} = \{\text{all bdd closed subset } Z \subset X\}$ Then

1) If $(X, d)$ is complete, then $(\mathcal{X}, d_H)$ is complete metric space ($\Leftarrow$ 也有)

2). $(X, d)$ is compact $\Leftrightarrow$ $(\mathcal{X}, d_H)$ is compact

where $d_H$ is "Hausdoff distance"

$$d_H(Z_1, Z_2) \triangleq \inf\{\varepsilon : Z_1 \subset B_\varepsilon(Z_2),\ Z_2 \subset B_\varepsilon(Z_1)\}$$

Pf. 1). Check $(\mathcal{X}, d_H)$ is a metric space.

a) $d_H(Z_1, Z_2) = d_H(Z_2, Z_1) \geq 0$

b) $d_H(Z_1, Z_2) = 0 \Leftrightarrow Z_1 = Z_2$

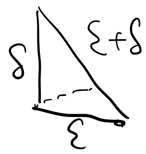
Pf. If $d_H(Z_1, Z_2) = 0$, assume $Z_1 \neq Z_2$ $\exists x \in Z_1 \setminus Z_2$

$r_0 := d(x, Z_2) = \inf\{d(x, y) : y \in Z_2\} > 0$. since $Z_2$ is closed.

$\Rightarrow B_{\frac{r_0}{2}}(Z_2) \not\ni x$. 与 $d_H(Z_1, Z_2) = 0$ ※.

c).

$Z_1 \subset B_\varepsilon(Z_2)$. $Z_2 \subset B_\delta(Z_3)$



$\Rightarrow \underline{Z_1 \subset B_{\varepsilon+\delta}(Z_3)}$ ☆ why?

$\Rightarrow d_H(Z_1, Z_3) \leq d_H(Z_1, Z_2) + d_H(Z_2, Z_3)$ ✓

· If $(X, d)$ is complete

Given $\{Z_1, Z_2, \dots, Z_n, \dots\} \subseteq \mathcal{X}$. Cauchy seq.

$d_H(Z_i, Z_j) \leq \varepsilon_i \to 0 \quad j \geq i \quad \Rightarrow Z_j \subset B_{\varepsilon_i}(Z_i) \quad j \geq i$

Construction: Define $\tilde{Z}_i = \bigcup_{j \geq i} Z_j$ 之后的所有的并.

let $\hat{Z}_i$ = closure of $\tilde{Z}_i$  由于无穷并 $\tilde{Z}_i$ 不一定 closed $\Rightarrow \hat{Z}_i$

$\hat{Z}_i \supset \hat{Z}_{i+1} \supset \hat{Z}_{i+2} \supset \cdots$  递减（后面集合少，故并起来少了）.

Let $Z_\infty \triangleq \bigcap_{i \geq 1} \hat{Z}_i$  实际就是 $\bigcap_{i\geq 1} \overline{\bigcup_{j \geq i} Z_j}$. 闭包是为了保持闭集

直接得到

claim. $d_H(Z_i, Z_\infty) \to 0$.  $Z_\infty$ closed bounded $\in \mathcal{X}$

pf. we have

$d_H(Z_i, \tilde{Z}_i) \leq \varepsilon_i$

$\cup$

$d_H(Z_i, \tilde{Z}_j) \leq \varepsilon_i \quad j \geq i$

$d_H(\hat{Z}_i, \hat{Z}_j) \leq \varepsilon_i, \quad j \geq i$

$\Rightarrow Z_\infty \subseteq B_{\varepsilon_i}(Z_i)$

另一方面 want to prove $Z_i \subseteq B_{10\varepsilon_i}(Z_\infty)$

随便选的. 我们希望证出 $z_i \in B_{10\varepsilon_i}(Z_\infty)$ $\Rightarrow$ contradiction

Argue by construction  Assume $\exists\, z_i \in Z_i \setminus B_{10\varepsilon_i}(Z_\infty)$

for. $z_i \in Z_i$  $\forall j \geq i$. $\exists\, z_j \in \tilde{Z}_j$ s.t. $d(z_i, z_j) \leq 2\varepsilon_i$

接近 $\varepsilon_i$

$Z_j \quad \tilde{Z}_j \quad j \geq i$

let $i_0 = i$, $i_1 > i_0$ . s.t. $\varepsilon_{i_1} \leq \frac{\varepsilon_{i_0}}{10}$

$\Rightarrow$ 找 $i_0 < i_1 < i_2 < \cdots$ s.t. $\varepsilon_{i_k} \leq \frac{1}{10} \varepsilon_{i_{k-1}}$

$z_{i_0} \quad z_{i_1} \quad z_{i_2}$  s.t. $d(z_{i_k}, z_{i_{k-1}}) \leq 2\varepsilon_{i_{k-1}} \leq 2\left(\frac{1}{10}\right)^{k-1} \varepsilon_{i_0}$

$\parallel$

$z_i$

可推.

$\Rightarrow \{z_{i_0}, z_{i_1}, z_{i_2}, \cdots\}$ is Cauchy seq let $z_\infty = \lim_{i_k \to \infty} z_{i_k} \Rightarrow z_\infty \in \hat{Z}_k \ \forall k. \Rightarrow z_\infty \in Z_\infty$

$\Rightarrow d(z_i, z_\infty) \leq \sum_k d(z_{i_k}, z_{i_{k-1}}) < 6\,\varepsilon_{i_0}$ $\times$

2). ① Assume $(X, d)$ compact.

want to prove $(X, d_H)$ is compact

It suffices to show $(X, d_H)$ is totally bounded ☆

Since $(X, d)$ is totally bounded. $\forall \varepsilon > 0$, $\exists$ finite $\varepsilon$-net.

$T = \{x_1, x_2, \dots, x_k\}$ s.t. $X \subseteq \bigcup_{i=1}^{k} B_\varepsilon(x_i)$

$\forall Z \in X$. define $T_Z = \{x_i \in T : d(x_i, Z) < \varepsilon\} \neq \phi$. $\subseteq T$.

and $T_Z \in X$.

and $d_H(Z, T_Z) < \varepsilon$

let $\{T_1, T_2, \dots, T_\ell\} \subset X$ be all subset of $T$ (finite)

$\Rightarrow X \subseteq \bigcup_{i=1}^{\ell} B_\varepsilon(T_i)$. $\Rightarrow X$ is compact

② Assume $(X, d_H)$ compact.

want to prove $(X, d)$ is compact

It suffices to show $(X, d_H)$ is totally bounded ☆

choose $\{x_1, x_2, x_3, \dots\} \subseteq X$. s.t. $d(x_i, x_j) > \varepsilon$, $i \neq j$

$\Rightarrow \{ \{x_1\} \{x_2\} . \{x_3\} , \dots \} \subseteq \mathcal{X} \quad d_H(\{x_i\}, \{x_j\}) > \varepsilon$

Since $(\mathcal{X}, d_H)$ is compact $\Rightarrow$ finite $\Rightarrow$ $X$ is compact. $\square$

**Rmk.** $Z_1 = (0,1) . Z_2 = [0,1] \subseteq \mathbb{R}$

$d_H(Z_1, Z_2) = 0$ so closed is needed !!

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**Def.** let $(X, d_X)$ , $(Y, d_Y)$ metric space . we say map $f: X \to Y$. is conti. iff

$$d_X(x, x_i) \to 0 \Rightarrow d_Y(f(x_i), f(x)) \to 0$$

Let $\gamma : [0,1] \to X$ be a continue map , call it. a curve

**Def.** $L[\gamma] = \sup \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1}))$ over all division of $[0,1]$

$0 = t_0 < t_1 < \dots < t_n = 1$

**Rmk.** $L[\gamma] \geq d(\gamma(0), \gamma(1))$ . $L[\gamma] \geq d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(1))$.

**Def** (length space) We say a metric space $(X, d)$ is a length space or geodesic space if.

1). $X$ is path connected . i.e. $\forall x, y \in X$ $\exists$ curve $\gamma : [0,1] \to X$. s.t.

$\gamma(0) = x,\ \gamma(1) = y$

2) $\forall x, y \in X.\ \exists \gamma : [0,1] \to X$, connecting $x, y$. s.t. $L[\gamma] = d(x,y)$

Ex. 1). $(\mathbb{R}^n, d)$. length. space

黎曼度量?

2) $(S^n, d)$ length space. , surface with induced by metric

**Thm.** Let $(Y, d)$ be complete, then the following are equivalent.

1). $Y$ is a length space

2). $\forall y_1, y_2 \in Y$. $\exists$ midpoint. $y_3 \in Y$. of $y_1, y_2$ i.e.

$$d(y_1, y_3) = d(y_2, y_3) = \frac{1}{2} d(y_1, y_2)$$

Pf

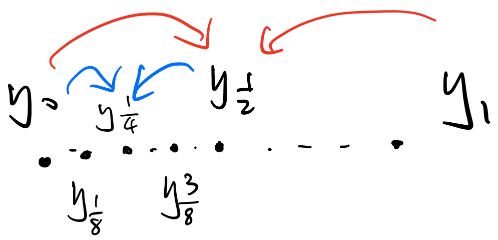
(1) $\Rightarrow$ (2) $\quad \forall y_1, y_2 \in Y.\ \exists \gamma,\ L[\gamma] = d(y_1, y_2)$

Choose $y_3 \in \gamma$ s.t. $d(y_1, y_3) = \frac{1}{2} d(y_1, y_3)$.

$$d(y_1, y_2) \leq d(y_1, y_3) + d(y_2, y_3) \leq L[\gamma] = d(y_1, y_2)$$

$\Rightarrow y_3$ is midpoint

(2) $\Rightarrow$ (1) $\quad \forall y_0, y_1 \in Y$, find a curve $\gamma$. s.t. $L[\gamma] = d(y_0, y_1)$




**Define** $T = \left\{ \frac{i}{2^k},\ 0 \leq i \leq 2^k,\ k \geq 1 \right\}$ dense in $[0,1]$

one can check $\forall t, s \in T$. we have $y_t, y_s$. (use 三角不等式)

s.t. $d(y_t, y_s) = |t-s|\, d(y_0, y_1)$

In particular. $\forall t_0 < t_1 < \cdots < t_N$. $t_i \in T$

$$d(y_{t_0}, y_{t_N}) = \sum_{i=0}^{N-1} d(y_{t_i}, y_{t_{i+1}})$$

(完备性)

**Def** $\gamma : T \to X$. $\gamma(t) = y_t$ . then $\gamma$ is continue

since $X$ is complete. we can extend $\gamma$ to $[0,1]$

$\Rightarrow L[\gamma] = d(y_0, y_1) \Rightarrow Y$ is length space $\square$

**Def** (boundedly compact) $(X, d)$ is boundly compact if any bold closed subset of $Y$ is compact

Ex. $(C([0,1]), d_{max})$ not locally compact
not boldly compact

**Thm** (??) If $(X, d)$ is locally compact, complete and

$$\overline{B_R}(x) = \overline{B_R(x)} \quad \forall R>0, \ x \in X$$

then $(X, d)$ is bdally compact.

**Thm** Let $(X, d)$ be a locally compact, complete, length space. then $(X, d)$ is bdally compact

$\Downarrow$ $\overline{B_R}(x) = \overline{B_R(x)}$

Pf. let $x \in X$. since locally compact

$$\Rightarrow \exists \ r_0 > 0. \text{ s.t. } \overline{B_{r_0}}(x) \text{ is compact}$$

all $\overline{B_r}(x)$ is compact

$$\Rightarrow \forall \ 0 < \rho < r_0 . \ \overline{B_\rho}(x) \text{ is compact. } (紧闭子集)$$

then if E is closed and bounded

Define. $R = \sup \{ r : \overline{B_r}(x) \text{ is compact} \}$

$\Downarrow$ $\Uparrow$ It suffices to prove

$E \subset \overline{B_r}(x)$ for some $x, r$ then E is compact.

$\boxed{\text{Need to prove } R = +\infty}$ now we only have $R \geq r_0$.

By contradiction, assume $R < +\infty$

**Claim 1** $\overline{B_R}(x)$ is compact

Pf. $\forall \varepsilon > 0$. find finite $\varepsilon$-net

Since $B_{R-\frac{\varepsilon}{4}}(x)$ is compact.

$$\Rightarrow \exists \text{ finite } \underbrace{\tfrac{\varepsilon}{3}\text{-net}}_{\{x_1, x_2, \dots, x_k\}} \text{ of } \overline{B_{R-\frac{\varepsilon}{4}}}(x)$$

$\Rightarrow$

$$\overline{B_{R-\frac{\varepsilon}{4}}}(x) \subseteq \bigcup_{i=1}^{k} B_{\frac{\varepsilon}{3}}(x_i).$$

$$\overline{B_R}(x) \subseteq B_{\frac{\varepsilon}{3}}\left(\overline{B_{R-\frac{\varepsilon}{4}}}(x)\right) \subseteq \bigcup_{i=1}^{k} B_\varepsilon(x_i) \Rightarrow \overline{B_R}(x) \text{ is compact}$$

$\square$

$\forall y \in \overline{B_R}(x) \Rightarrow \exists\ \overline{B_{r_y}}(y)$ is compact. (locally compact)

$\Rightarrow \overline{B_R}(x) \subseteq \bigcup_{y \in \overline{B_R}(x)} B_{r_y}(y).$ (open covering)

$\Rightarrow \exists$ finite $\overline{B_R}(x) \subseteq \bigcup_{j=1}^{k} B_{r_{y_j}}(y_j) =: U$

$\Rightarrow \overline{U}$ is compact. (由 $\overline{U}$ 为有限个紧集的并)

$\Rightarrow$ any closed subset of $U$ is compact

**Claim 2** $\exists\ \varepsilon > 0$, s.t. $B_\varepsilon(\overline{B_R}(x)) \subset U$

Pf. by contradiction.

$\exists\ \{x_i \notin U\}$ s.t. $d(x_i, \overline{B_R}(x)) \leq 2^{-i}$

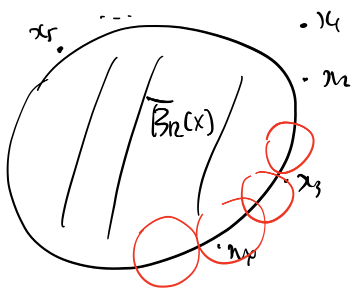
Since $\overline{B_R}(x)$ is compact $\Rightarrow \exists$ subseq $\{x_{i'}\} \subset \{x_i\}$ Cauchy seq

s.t. $\cancel{x_{i'}}\ U$ $x_{i'} \longrightarrow x_\infty$.

$\exists\ r > 0$

$x_\infty \in \overline{B_R}(x)$ and $B_r(x_\infty) \subset U$. $\longrightarrow$ ×

Pf.



$\exists\ y_\alpha, 1 \leq \alpha \leq k_i$

$\forall i$, $\bigcup_{\alpha=1}^{k_i} B_{2^{-i}}(y_\alpha) \supset \overline{B_R}(x)$

$\Rightarrow \bigcup_{\alpha=1}^{k_i} B_{2\cdot 2^{-i}}(y_\alpha) \supset B_{2^{-i}}(\overline{B_R}(x))$

$\Rightarrow \exists$ 一个 $B_{2\cdot 2^{-i}}(y_\alpha)$ 中无限个 $x_i$

$\uparrow$

$n$

<u>by claim 2.</u> $\overline{B_{R+\frac{\varepsilon}{2}}}(x) \subseteq B_\varepsilon(\overline{B_R(x)}) \subset U$

$\downarrow$

compact.

$\longrightarrow\!\!\!\!\!\!\times \Rightarrow R = +\infty.$ $\square$

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本节课.

1° $(X,d)$ complete $\Longleftrightarrow$ $(\mathcal{X}, d_H)$ complete.

$(X,d)$ compact $\Longleftrightarrow$ $(\mathcal{X}, d_H)$ compact.

2° Continue function. length space. <u>equivalent definition.</u>

3° length space + locally compact + complete $\Rightarrow$ boundedly compact