

Thm Let  $\mathcal{X} = \{ \text{all bdd closed subset } Z \subset X \}$  Then

1) If  $(X, d)$  is complete, then  $(\mathcal{X}, d_H)$  is complete metric space  
( $\Leftarrow$  也有)

2)  $(X, d)$  is compact  $\Leftrightarrow (\mathcal{X}, d_H)$  is compact

where  $d_H$  is "Hausdorff distance"

$$d_H(Z_1, Z_2) \triangleq \inf \{ \varepsilon : Z_1 \subset B_\varepsilon(Z_2), Z_2 \subset B_\varepsilon(Z_1) \}$$

Pf. 1) Check  $(\mathcal{X}, d_H)$  is a metric space.

a)  $d_H(Z_1, Z_2) = d_H(Z_2, Z_1) \geq 0$

b)  $d_H(Z_1, Z_2) = 0 \Leftrightarrow Z_1 = Z_2$

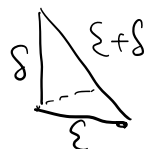
pf. if  $d_H(Z_1, Z_2) = 0$ , assume  $Z_1 \neq Z_2 \exists x \in Z_1 \setminus Z_2$

$r_0 := d(x, Z_2) = \inf \{ d(x, y) : y \in Z_2 \} > 0$ . since  $Z_2$  is closed,

$$\Rightarrow B_{\frac{r_0}{2}}(Z_2) \not\ni x \quad \text{so } d_H(Z_1, Z_2) = 0 \text{ is false.}$$

c).

$$Z_1 \subset B_\varepsilon(Z_2), Z_2 \subset B_\delta(Z_3)$$



$$\Rightarrow Z_1 \subset B_{\varepsilon+\delta}(Z_3) \quad \text{why?}$$

$$\Rightarrow d_H(Z_1, Z_1) \leq d_H(Z_1, Z_2) + d_H(Z_2, Z_3)$$

- If  $(X, d)$  is complete

Given  $\{Z_1, Z_2, \dots, Z_n, \dots\} \subset \mathcal{X}$ . Cauchy seq.

$$d_H(Z_i, Z_j) \leq \varepsilon_i \rightarrow 0 \quad j \geq i \quad \Rightarrow Z_j \subset B_{\varepsilon_i}(Z_i) \quad j \geq i$$

Construction Define  $Z_i = \bigcup_{j \geq i} Z_j$  之后的所有并.

let  $\hat{Z}_i = \text{closure of } \tilde{Z}_i$  由于无穷并,  $\tilde{Z}_i$  不一定 closed  $\Rightarrow \hat{Z}_i$

$\hat{Z}_i \supset \hat{Z}_{i+1} \supset \hat{Z}_{i+2} \supset \dots$  递减 (后面集合少, 故并起来少)

let  $Z_\infty \triangleq \bigcap_{i \geq 1} \hat{Z}_i$  实际就是  $\bigcap_{i \geq 1} \overline{\bigcup_{j \geq i} \tilde{Z}_j}$ . 闭包是为了保持闭集  
直接得到

claim.  $d_H(Z_i, Z_\infty) \rightarrow 0$ .  $Z_\infty$  closed bounded  $\in \mathcal{X}$

we have  
pf.  $d_H(Z_i, \hat{Z}_i) \leq \varepsilon_i$   
 $d_H(Z_i, \hat{Z}_j) \leq \varepsilon_i \quad j \geq i$   
 $d_H(\hat{Z}_i, \hat{Z}_j) \leq \varepsilon_i \quad j \geq i$

$$\Rightarrow Z_\infty \subseteq B_{\varepsilon_i}(Z_i)$$

另证 want to prove  $Z_i \subseteq B_{10\varepsilon_i}(Z_\infty)$

随便选的. 我们希望能证出  $Z_i \in B_{10\varepsilon_i}(Z_\infty) \Rightarrow \text{Contradiction}$

Argue by construction. Assume  $\exists z_i \in Z_i \setminus B_{10\varepsilon_i}(Z_\infty)$

for  $z_i \in Z_i$   
 $\forall j \geq i, \exists z_j \in \hat{Z}_j$  s.t.  
 $d(z_i, z_j) \leq 2\varepsilon_i$   
注意  $\varepsilon_i$   
 $Z_j \supset \hat{Z}_j$   $j \geq i$

let  $i_0 = i, i_1 > i_0$  s.t.  $\varepsilon_{i_1} \leq \frac{\varepsilon_{i_0}}{10}$

$\Rightarrow$  找  $i_0 < i_1 < i_2 < \dots$  s.t.  $\varepsilon_{i_k} \leq \frac{1}{10} \varepsilon_{i_{k-1}}$

$\downarrow \quad \downarrow \quad \downarrow$   
 $z_{i_0} \quad z_{i_1} \quad z_{i_2}$   
 $\parallel$   
 $z_i$

s.t.  $d(z_{i_k}, z_{i_{k-1}}) \leq 2\varepsilon_{i_{k-1}} \leq 2(\frac{1}{10})^{k-1} \varepsilon_{i_0}$

$\Rightarrow \{z_{i_0}, z_{i_1}, z_{i_2}, \dots\}$  is Cauchy seq

$$\text{let } z_\infty = \lim_{i_k \rightarrow \infty} z_{i_k} \Rightarrow z_\infty \in \bigcap_k \bigcap_{i \geq k} Z_i \Rightarrow z_\infty \in Z_\infty$$

$$\Rightarrow d(z_i, z_\infty) \leq \sum_{k=i}^\infty d(z_{i_k}, z_{i_{k+1}}) < 6 \varepsilon_i \quad \times$$

2). ① Assume  $(X, d)$  compact.

want to prove  $(X, d_H)$  is compact

It suffices to show  $(X, d_H)$  is totally bounded.  $\star$

Since  $(X, d)$  is totally bounded.  $\forall \varepsilon > 0, \exists$  finite  $\varepsilon$ -net.

$$T = \{x_1, x_2, \dots, x_k\} \text{ s.t. } X \subseteq \bigcup_{i=1}^k B_\varepsilon(x_i)$$

$$\forall z \in X, \text{ define } T_z = \{x_i \in T : d(x_i, z) < \varepsilon\} \neq \emptyset \subseteq T.$$

and  $\underline{T_z} \in X$ .

$$\text{and } d_H(z, T_z) < \varepsilon$$

let  $\{T_1, T_2, \dots, T_\ell\} \subset X$  be all subset of  $T$  (finite)

$$\Rightarrow X \subseteq \bigcup_{i=1}^\ell B_\varepsilon(T_i) \Rightarrow X \text{ is compact}$$

② Assume  $(X, d_H)$  compact.

want to prove  $(X, d)$  is compact

It suffices to show  $(X, d_H)$  is totally bounded.  $\star$

choose  $\{x_1, x_2, x_3, \dots\} \subseteq X$  s.t.  $d(x_i, x_j) > \varepsilon, i \neq j$

$$\Rightarrow \{x_1, x_2, x_3, \dots\} \subseteq X \quad d(x_i, x_j) > \varepsilon$$

Since  $(X, d_X)$  is compact  $\Rightarrow$  finite  $\Rightarrow X$  is compact.  $\square$

Ex.  $Z_1 = (0, 1)$   $Z_2 = [0, 1] \subseteq \mathbb{R}$

$$d_X(Z_1, Z_2) = 0 \quad \text{so closed is needed!!}$$


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Def. let  $(X, d_X)$ ,  $(Y, d_Y)$  metric space. we say map  $f: X \rightarrow Y$  is conti. iff

$$d_X(x, x_i) \rightarrow 0 \Rightarrow d_Y(f(x), f(x_i)) \rightarrow 0$$

Let  $\gamma: [0, 1] \rightarrow X$  be a continue map, call it a curve

Def.  $L[\gamma] = \sup \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1}))$  over all division of  $[0, 1]$   
 $0 = t_0 < t_1 < \dots < t_n = 1$

Ex.  $L[\gamma] \geq d(\gamma(0), \gamma(1))$   $L[\gamma] \geq d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(1))$

Def. (length space) We say a metric space  $(X, d)$  is a length space or geodesic space if.

1.  $X$  is path connected. i.e.  $\forall x, y \in X \exists$  curve  $\gamma: [0, 1] \rightarrow X$  s.t.

$$\gamma(0) = x, \gamma(1) = y$$

$$2) \forall x, y \in X. \exists \gamma: [0,1] \rightarrow X, \text{ connecting } x, y. \text{ s.t. } L[\gamma] = d(x, y)$$

Ex. 1).  $(\mathbb{R}^n, d)$  length space

2)  $(S^n, d)$  length space, ~~is it?~~  $\frac{d}{2}$ ? surface with induced by metric

Thm. Let  $(Y, d)$  be complete, then the following are equivalent.

1).  $Y$  is a length space

2).  $\forall y_1, y_2 \in Y. \exists$  midpoint  $y_3 \in Y$  of  $y_1, y_2$  i.e.

$$d(y_1, y_3) = d(y_2, y_3) = \frac{1}{2} d(y_1, y_2)$$

Pf

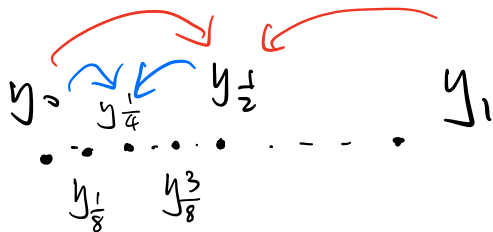
$$(1) \Rightarrow (2) \quad \forall y_1, y_2 \in Y. \exists \gamma. L[\gamma] = d(y_1, y_2)$$

$$\text{chose } y_3 \in \gamma \text{ s.t. } d(y_1, y_3) = \frac{1}{2} d(y_1, y_2).$$

$$d(y_1, y_2) \leq d(y_1, y_3) + d(y_2, y_3) \leq L[\gamma] = d(y_1, y_2)$$

$$\Rightarrow y_3 \text{ is midpoint}$$

$$(2) \Rightarrow (1) \quad \forall y_0, y_1 \in Y, \text{ find a curve } \gamma. \text{ s.t. } L[\gamma] = d(y_0, y_1)$$



Define  $T = \left\{ \frac{i}{2^k}, 0 \leq i \leq 2^k, k \geq 1 \right\}$  dense in  $[0,1]$

one can check  $\forall t, s \in T$  we have  $y_t, y_s$ . (use 三角不等式)  
 s.t.  $d(y_t, y_s) = |t-s| d(y_0, y_1)$

In particular  $\forall t_0 < t_1 < \dots < t_N, t_i \in T$

$$d(y_{t_0}, y_{t_N}) = \sum_{i=0}^{N-1} d(y_{t_i}, y_{t_{i+1}})$$

(完备性) ↓

Def  $\gamma: T \rightarrow X, \gamma(t) = y_t$  then  $\gamma$  is continue △

since  $X$  is complete. we can extend  $\gamma$  to  $[0,1]$

$\Rightarrow L[\gamma] = d(y_0, y_1) \Rightarrow Y$  is length space  $\square$

Def (boundedly compact)  $(X, d)$  is boundedly compact if any bold closed subset of  $Y$  is compact

Ex.  $(C([0,1]), d_{\max})$  not locally compact  
 not boundedly compact

Thm (??). If  $(X, d)$  is locally compact, complete and.

$$\overline{B_R(x)} = \overline{B_R(x)} \quad \forall R > 0, x \in X$$

then  $(X, d)$  is totally compact.

Thm. Let  $(X, d)$  be a locally compact, complete, length space. then  $(X, d)$  is totally compact

$$\Downarrow \\ \overline{B_R(x)} = \overline{B_R(x)}$$

pf. let  $x \in X$ . since locally compact

$$\Rightarrow \exists r_0 > 0 \text{ s.t. } \overline{B_{r_0}(x)} \text{ is compact}$$

all  $\overline{B_r(x)}$  is compact  $\Rightarrow \forall 0 < r < r_0$ .  $\overline{B_r(x)}$  is compact. (紧闭集)

then if  $E$  is closed and bounded Define.  $R = \sup \{r : \overline{B_r(x)} \text{ is compact}\}$

$\Downarrow$   $E \subset \overline{B_r(x)}$  for some  $x, r$  then  $E$  is compact.  $\Uparrow$  It suffices to prove  $\boxed{\text{Need to prove } R = +\infty}$  now we only have  $R \geq r_0$ .

By contradiction, assume  $R < +\infty$

Claim 1  $\overline{B_R(x)}$  is compact

pf.  $\forall \varepsilon > 0$ , find finite  $\varepsilon$ -net

since  $\overline{B_{R-\frac{\varepsilon}{4}}(x)}$  is compact.

$$\Rightarrow \exists \text{ finite } \frac{\varepsilon}{3}\text{-net of } \overline{B_{R-\frac{\varepsilon}{4}}(x)} \\ \{x_1, x_2, \dots, x_k\}$$

$\Rightarrow$

$$\overline{B_{R-\frac{\varepsilon}{4}}(x)} \subseteq \bigcup_{i=1}^k \overline{B_{\frac{\varepsilon}{3}}(x_i)}$$

$$\overline{B}_R(x) \subseteq B_{\frac{\epsilon}{3}}(\overline{B}_{R-\frac{\epsilon}{4}}(x)) \subseteq \bigcup_{i=1}^k B_{\epsilon}(x_i) \Rightarrow \overline{B}_R(x) \text{ is compact} \quad \square$$

$\forall y \in \overline{B}_R(x) \Rightarrow \exists \overline{B}_y(y) \text{ is compact. (locally compact)}$

$$\Rightarrow \overline{B}_R(x) \subseteq \bigcup_{y \in \overline{B}_R(x)} \overline{B}_y(y). \quad (\text{open covering})$$

$$\Rightarrow \exists \text{ finite } \overline{B}_R(x) \subseteq \bigcup_{j=1}^k \overline{B}_{y_j}(y_j) =: U$$

$\Rightarrow U \text{ is compact. (由 } U \text{ 为有限个紧集的并)}$

$\Rightarrow \text{any closed subset of } U \text{ is compact}$

Claim 2

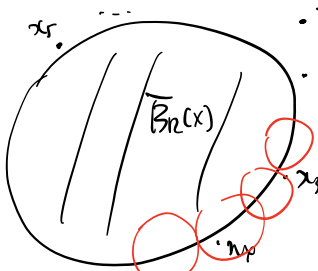
$$\exists \delta > 0, \text{ s.t. } B_{\delta}(\overline{B}_R(x)) \subset U$$

pf. by contradiction.

$$\exists \{x_i \notin U\} \text{ s.t. } d(x_i, \overline{B}_R(x)) \leq 2^{-i}$$

Since  $\overline{B}_R(x)$  is compact  $\Rightarrow \exists \text{ subseq } \{x_{i'}\} \subset \{x_i\}$   
 s.t.  $x_{i'} \rightarrow x_{\infty}$ .  $\exists r > 0$   
 $x_{\infty} \in \overline{B}_R(x)$  and  $B_r(x_{\infty}) \subset U$ .  $\xrightarrow{x}$

pf.



$\exists y_{\alpha}, 1 \leq \alpha \leq k_i$   
 $\forall i, \bigcup_{\alpha=1}^{k_i} B_{2^{-i}}(y_{\alpha}) \supset \overline{B}_R(x)$   
 $\Rightarrow \bigcup_{\alpha=1}^{k_i} B_{2^{-i}}(y_{\alpha}) \supset B_{2^{-i}}(\overline{B}_R(x))$



$$\Rightarrow \exists \uparrow B_{2r^i}(y_n) \text{ 中无 } \uparrow x_i$$

by claim 2.

$$\overline{B_{R+\frac{1}{2}}}(x) \subseteq B_2(\overline{B_R}(y)) \subset U$$

$\downarrow$   
compact.

$$\text{---} \times \Rightarrow R = +\infty. \quad \square$$

本节得.

$$1^\circ (X, d) \text{ complete} \Leftrightarrow (X, d_H) \text{ complete.}$$

$$(X, d) \text{ compact} \Leftrightarrow (X, d_H) \text{ compact.}$$

$$2^\circ \text{ continue function} \quad \text{length space} \quad \underline{\text{equivalent definition.}}$$

$$3^\circ \text{ length space} + \text{locally compact} + \text{complete} \Rightarrow \text{boundedly compact}$$