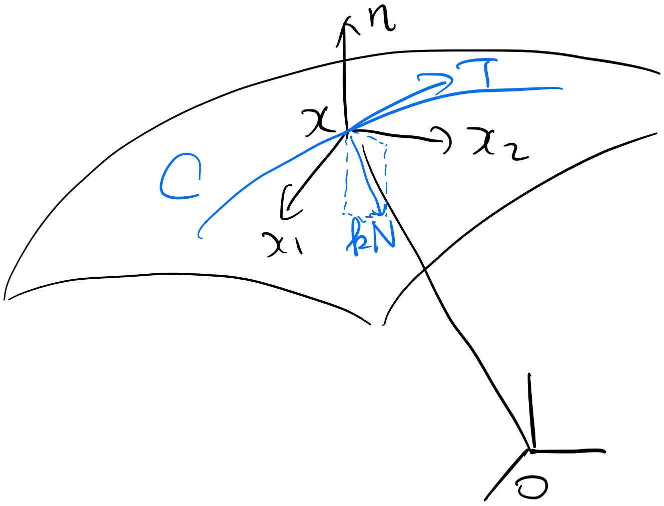


$$\{x;\ x_1, x_2, n\}$$

类似 Frenet 公式

$$\begin{cases} x_\alpha = \dfrac{\partial x}{\partial u^\alpha} \\ x_{\alpha\beta} = \Gamma_{\alpha\beta}^{\gamma} x_\gamma + h_{\alpha\beta} n & \text{Gauss 公式} \\ n_\alpha = -h_\alpha^\beta x_\beta & \text{Weingarten 公式} \end{cases}$$

$$\Gamma_{\alpha\beta}^{\tau} = \frac{1}{2} g^{\sigma\tau}\left(\frac{\partial g_{\alpha\beta}}{\partial u^\sigma} + \frac{\partial g_{\sigma\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\sigma}}{\partial u^\beta}\right)$$

$$C:\ x(s) = x(u^1(s), u^2(s)).$$

$$\dot{T} = kN = k_g Q + k_n n \qquad Q = n \times T.$$

k_g 测地曲率. $\qquad k_g Q = \left(\Gamma_{\alpha\beta}^{\sigma} \dfrac{du^\alpha}{ds}\dfrac{du^\beta}{ds} + \dfrac{d^2u^\sigma}{ds^2}\right) x_\sigma$

Prop. C 测地线 $\Leftrightarrow$ 沿着 C. $k_g = 0$

ODE. $\Gamma_{\alpha\beta}^{\sigma} \dfrac{du^\alpha}{ds}\dfrac{du^\beta}{ds} + \dfrac{d^2u^\sigma}{ds^2} = 0$

$u^\alpha(0) = u_0^\alpha$ —— start point

$\frac{du^\alpha}{ds}(0) = v_0^\alpha$ —— start vector $\quad V = V_0^\alpha x_\alpha$

ODE理论

$\Longrightarrow$ $\exists!$ 测地线

$\forall\, v \in T_pM. \quad \|v\| = s$

过P点 以 $\frac{v}{\|v\|}$ 为初始向量的测地线

$\gamma_p(s)$. 取 $\gamma_p(s)$ 上一点 Q, s.t. 沿测地线 Q

到P点的距离为 $\underline{s}$. ↪ 对应 v 的长度

$$\exp_p: \quad T_pM \longrightarrow M, \qquad v \longmapsto Q$$

$\exp_p$ 在 p 的邻域内是一个微分同胚.

$Q = \exp_p(v)$

若在P点处切平面 T_pM 处 选取 幺正标架 $\{\overset{P}{\underset{\shortparallel}{}}\hspace{-0.3em}0; e_1, e_2\}$.

$\frac{v}{\|v_0\|} = v = v_0^1 e_1 + v_0^2 e_2$

$v = \|v\| v_0 = s v_0 = s v_0^1 e_1 + s v_0^2 e_2 \qquad \{s v_0^1, s v_0^2\}$

将 $\{sv_0^1, sv_0^2\}$ 作为 对应 Q点的 坐标

$\Updownarrow$

Q

由此 得到Q的坐标 称为 法坐标系. (Normal)

$$\begin{cases} y^1 = sv_0^1 \\ y^2 = sv_0^2 \end{cases} \Rightarrow$$ 指数坐标系下法坐标系 测地线参数方程

Thm. 在曲面 M 上一点 p 的邻域的法坐标系下

$$g_{11}(p) = g_{22}(p) = 1. \quad g_{12}(p) = 0 \qquad \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}(0) = 0$$

证 $\{0, e_1, e_2\} \in T_pM$

e_α 为初始向量的测地线 y^α— 参数曲线

y^α. 弧长参数. $\dfrac{\partial x}{\partial y^\alpha} = e_\alpha \Rightarrow g_{11}(p) = g_{22}(p) = 1$

$g_{12}(p) = 0$

测地线：$\boxed{y^\alpha = y_0^\alpha s.}$ 代入

$$\begin{cases} \Gamma^\alpha_{\beta\gamma} \dfrac{du^\beta}{ds}\dfrac{du^\gamma}{ds} + \dfrac{d^2u^\alpha}{ds^2} = 0 \\ u^\alpha(0) = u_0^\alpha \\ \dfrac{du^\alpha}{ds}(0) = v_0^\alpha \end{cases}$$

$$\Gamma^\alpha_{\beta\gamma} \frac{dy^\beta}{ds}\frac{dy^\gamma}{ds} + \frac{d^2y^\alpha}{ds^2} = 0$$

$$y^\alpha(0) = 0$$

$$\frac{dy^\alpha}{ds}(0) = y_0^\alpha$$

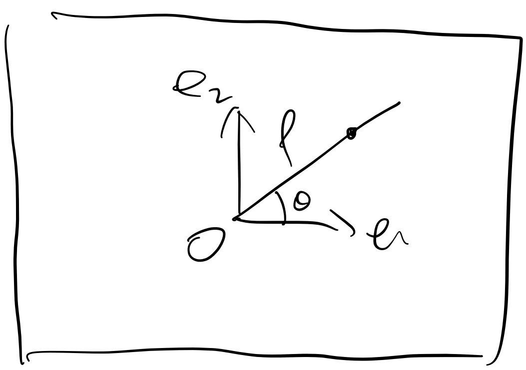
代入 $\Rightarrow \quad 0+\Gamma^{\alpha}_{\beta\gamma}(p)\, y_0^{\beta} y_0^{\gamma}=0 \qquad \forall\, (y_0^1, y_0^2)$

$$\Rightarrow \Gamma^{\alpha}_{\beta\gamma}=0$$

$$\left.\frac{\partial g_{\alpha\beta}}{\partial y^{\gamma}}\right|_p = (x_\alpha x_\beta)_\gamma = x_{\alpha\gamma} x_\beta + x_\alpha x_{\beta\gamma}$$

$$= \left(\Gamma^{\sigma}_{\alpha\gamma} x_\sigma + h_{\alpha\gamma} n\right) x_\beta + x_\alpha \left(\Gamma^{\delta}_{\beta\gamma} x_\delta + h_{\beta\gamma} n\right)$$

$$=0$$ #



$T_pM \quad \{O, e_1, e_2\}$

$\{y^1, y^2\}$ 直角坐标

$\{\rho, \theta\}$ 极坐标

$\exp_p V = Q$

$\{y_1, y_2\}$ 法坐标系

$\{\rho, \theta\}$ 测地极坐标系

$\theta=$ 常数 $\longleftrightarrow$ 测地射线

$\rho=$ 常数 $\longleftrightarrow$ 测地圆

$\bar{y}_1=\rho$

$\bar{y}_2=\theta$

$$\begin{cases} y^1 = \rho\cos\theta \\ y^2 = \rho\sin\theta \end{cases}$$

Thm 在关于 P 点为中心的测地极坐标系下.

$$\bar{g}_{11} = 1 \quad \bar{g}_{12} = 0. \quad \lim_{\rho\to 0}\sqrt{\bar{g}_{22}} = 0$$

$$\lim_{\rho\to 0}\left(\sqrt{\bar{g}_{22}}\right)_\rho = 1$$

极坐标下. 原点有奇性.

① Pf. $(\rho, \theta) = (\bar{y}^1, \bar{y}^2)$.

$$ds^2 = \bar{g}_{11}(d\rho)^2 + 2\bar{g}_{12}\,d\rho\,d\theta + \bar{g}_{22}(d\theta)^2$$

$\rho = \bar{y}^1$ 为弧长参数. $x_\rho = \frac{\partial x}{\partial \bar{y}^1}$ 单位向量.

$$\bar{g}_{11} = x_{\bar{y}^1} x_{\bar{y}^1} = 1$$

---

② θ = 常数 $= \bar{y}^2$ 为测地线

2代入.

$$\begin{cases} \Gamma^\sigma_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} + \frac{d^2u^\sigma}{ds^2} = 0 \\ u^\alpha(0) = u^\alpha_0 \\ \frac{du^\alpha}{ds}(0) = v^\alpha_0 \end{cases}$$

$$\Rightarrow 0 + \bar{\Gamma}^2_{11}\, 1\cdot 1 = 0$$

$$\bar{\Gamma}_{11}^{2} = \frac{1}{2}\bar{g}^{2\alpha}\left(\frac{\partial \bar{g}_{\alpha 1}}{\partial \bar{y}^{1}} + \frac{\partial \bar{g}_{1\alpha}}{\partial \bar{y}^{1}} - \frac{\partial \bar{g}_{11}}{\partial \bar{g}^{\alpha}}\right)$$

$$= \bar{g}^{22}\frac{\partial \bar{g}_{21}}{\partial \bar{y}^{1}} = 0$$

$$\|$$

$$\frac{\partial \bar{g}_{21}}{\partial \rho} = 0 \qquad \bar{g}_{21} \text{ 与}\rho\text{无关}$$

$$\Rightarrow \bar{g}_{21}(\alpha) = \lim_{\rho\to a} \bar{g}_{21} \qquad \bar{g}_{21} = x_\theta \cdot x_\rho.$$

$$\begin{cases} x_\theta = -x_{y^1}\rho\sin\theta + x_{y^2}\rho\cos\theta \\ x_\rho = x_{y^1}\cos\theta + x_{y^2}\sin\theta \end{cases}$$

$$\Rightarrow x_\theta \cdot x_\rho = -g_{11}\rho\sin\theta\cos\theta + g_{22}\rho\sin\theta\cos\theta$$

$$+ g_{12}\rho(\sin^2\theta - \cos^2\theta)$$

$$\rho \to 0. \quad x_\theta \cdot x_\rho = 0 \quad \Rightarrow \bar{g}_{12}(\theta) = 0$$

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③

$$\begin{pmatrix} \bar{g}_{11} & \bar{g}_{12} \\ \bar{g}_{21} & \bar{g}_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{g}_{22} \end{pmatrix} \quad \det(\bar{g}_{\alpha\beta}) = \bar{g}_{22}$$

$$\sqrt{\det(\bar{g}_{\alpha\beta})} = \sqrt{\bar{g}_{22}}$$

$$= \sqrt{\det g_{\alpha\beta}} \cdot \frac{\partial(y^1, y^2)}{\partial(\rho, \theta)}.$$

$$= \rho \sqrt{\det(g_{\alpha\beta})}$$

$$\lim_{\rho \to 0} \sqrt{\bar{g}_{22}} = \lim_{\rho \to 0} \rho \sqrt{\det g_{\alpha\beta}} = 0$$

$$\left(\sqrt{\bar{g}_{22}}\right)_\rho = \left(\rho \sqrt{\det g_{\alpha\beta}}\right)_\rho = \sqrt{\det g_{\alpha\beta}} + \rho \left(\sqrt{\det g_{\alpha\beta}}\right)_\rho$$

$$\left(\sqrt{\det g_{\alpha\beta}}\right)_\rho = \frac{1}{2\sqrt{\det g_{\alpha\beta}}} G^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial \rho}$$

代数余子式

$$\frac{\partial g_{\alpha\beta}}{\partial \rho} = \underbrace{\frac{\partial g_{\alpha\beta}}{\partial y^1}}_{=0} \cos\theta + \underbrace{\frac{\partial g_{\alpha\beta}}{\partial y^2}}_{=0} \sin\theta$$

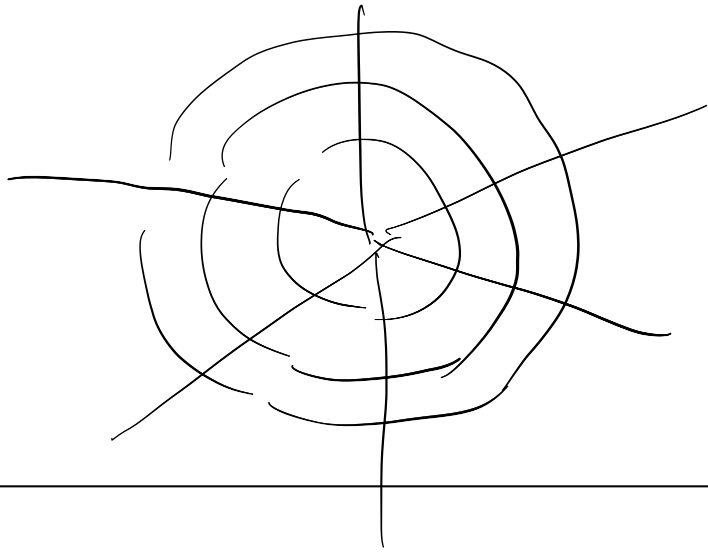
$\rho \to 0$

$$\Rightarrow \left(\sqrt{\bar{g}_{22}}\right)_\rho \to \sqrt{\det g_{\alpha\beta}} = 1 \qquad \#$$

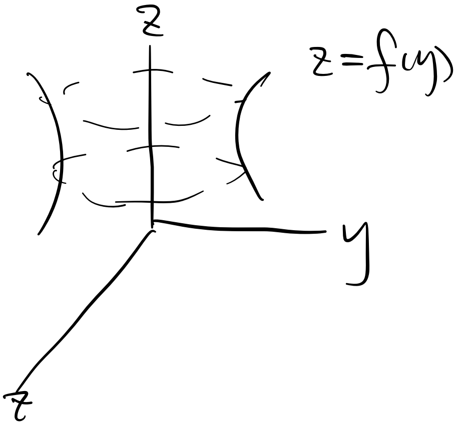
$$ds^2 = (d\rho)^2 + \bar{g}_{22}(d\theta)^2$$

$$\lim_{\rho\to 0} g_{22} = 0 \qquad \lim_{\rho\to 0}\left(\sqrt{g_{22}}\right)_\rho = 1$$

$$\left.\begin{array}{l}\theta\text{常数} \iff \text{测地线} \\ \rho\text{常数} \iff \text{测地圆}\end{array}\right\}\text{正交}$$



例 $K = -\frac{1}{a^2}$ 旋转面



$$x(u^1,u^2) = (u^2\cos u^1,\ u^2\sin u^1,\ f(u^2))$$

$$K = \frac{f'f''}{u^2(1+(f')^2)^2} = -\frac{1}{a^2}$$

$$\frac{\frac{1}{2}\,d((f')^2+1)}{(1+(f')^2)^2} = -\frac{1}{a^2}\,\frac{1}{2}\,d(u^2)^2$$

$$\frac{1}{1+(f')^2} = \frac{(u^2)^2}{a^2} + C \quad 取\ C=0$$

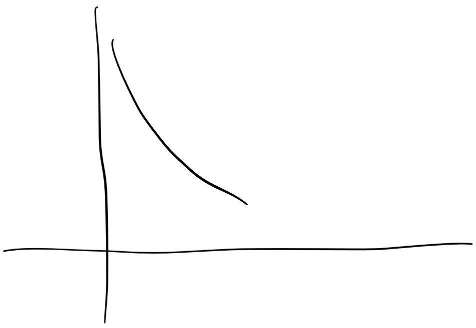
$$f' = \pm \frac{\sqrt{a^2-(u^2)^2}}{u^2}$$

取"−" $u^2 = a\cos\varphi$

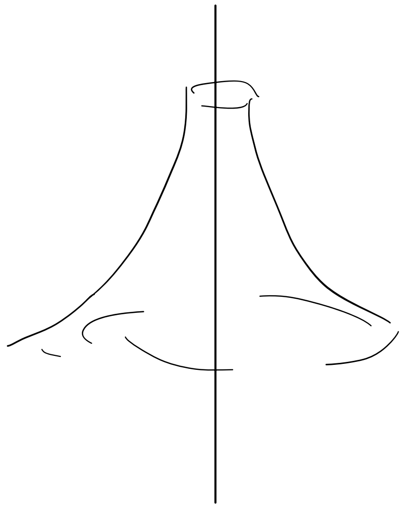
$$\Rightarrow f(\varphi) = \int a \frac{\sin^2\varphi}{\cos\varphi}\, d\varphi$$

$$\begin{cases} z = a\left[\ln(\sec\varphi+\tan\varphi) - \sin\varphi\right] + C \\ y = a\cos\varphi \end{cases}$$

曳物线



伪球面 $\Downarrow$



正常数 ⇒ 球面

测地挠率

P点处挠率 ⇒ 测地挠率 T_g

$$T_g = -\frac{dB}{ds}\cdot N = B\,\frac{dN}{ds} = \dot{N}\cdot(T\times N)$$

$$= (\dot{N}, T, N)$$

测地线 $kN = \underset{\overset{\|}{0}}{k_g}\, a + k_n\, n \Rightarrow N = \pm n$

$$\Rightarrow T_g = (\dot{n}, T, n) = \left(n_\alpha \frac{du^\alpha}{ds}, x_\beta \frac{du^\beta}{ds}, n\right)$$

$$= (n_\alpha \times x_\beta)\,\frac{(x_1\times x_2)}{\sqrt{\det g_{\alpha\beta}}}\,\frac{du^\alpha}{ds}\,\frac{du^\beta}{ds}$$

$$= \frac{1}{\sqrt{\det g}}\Big((n_1\times x_1)(x_1\times x_2)\left(\frac{du^1}{ds}\right)^2$$

$$+\left((n_1 \times x_2) + (n_2 \times x_1)\right)(x_1 \times x_2)\left(\frac{du^1}{ds}\frac{du^2}{ds}\right)$$

$$+ (n_2 \times x_2)(x_1 \times x_2)\left(\frac{du^2}{ds}\right)^2 \Bigg)$$

$$= (n_1 \, x_1)(x_1 x_2) - (n_1 x_2)(x_1 x_1)$$

$$= -h_{11} g_{12} + h_{12} g_{11}$$

$$T_g = \frac{1}{\sqrt{\det g}} \underbrace{\begin{vmatrix} \left(\frac{du^2}{ds}\right)^2 & -\frac{du^1}{ds}\frac{du^2}{ds} & \left(\frac{du^1}{ds}\right)^2 \\ g_{11} & g_{12} & g_{22} \\ h_{11} & h_{12} & h_{22} \end{vmatrix}}_{\text{曲线的ODE}}$$