W3L2

Prop. 直纹面是可展曲面，则必定是柱面或锥面或某一曲线的切线曲面

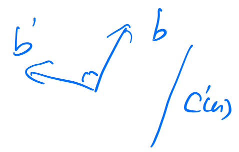
设 $x(u,v)=a(u)+v\,b(u)$ 为可展曲面. 有 $(a',b,b')=0$

由 $|b|=1 \Rightarrow b'\cdot b=0$. $b'\perp b$

① if $b\times b'=0 \Longleftrightarrow b'\parallel b \Leftrightarrow b'=0 \Rightarrow$ 柱面

② if $b\times b'\neq 0$ $b\neq$ 常向量

将原方程写成. $x(u,v)=c(u)+\tilde{v}\,b(u)$



s.t. $c'(u)\parallel b(u) \Rightarrow c'(u)\cdot b'(u)=0$

令 $c(u)=a(u)+f(u)\,b(u)$ f 待定.

why $c'\parallel b$?

$c'(u)=a'(u)+f'(u)\,b(u)+f(u)\,b'(u)$

$\Longrightarrow (c',b,b')=0$

乘 b' $0=a'\cdot b'+f\cdot|b'|^2 \to f(u)=\dfrac{a'\cdot b'}{|b'|^2}$

then why not $c'\parallel b'$

$$x(u,v)=\underbrace{a(u)+f(u)\,b(u)}_{c(u)}\underbrace{-f(u)\,b(u)+v\,b(u)}_{(\underbrace{v-f(u)}_{\tilde{v}})\,b(u)}$$

then if $c'\neq 0$

we have $c'\parallel b \Rightarrow$ 切线面

$$x(\tilde{u},\tilde{v})=c(\tilde{u})+\tilde{v}\,b(\tilde{u}) \qquad \left.\begin{cases}\tilde{u}=u\\ \tilde{v}=v-f(u)\end{cases}\right\}\text{参数变换}$$

验证: $(c',b,b')=(a'+f'b+fb',\,b,\,b')=0$

此时. 若 1) $c'(u)\neq 0$

$c'\cdot b'=0$ $\quad b'\cdot b=0$

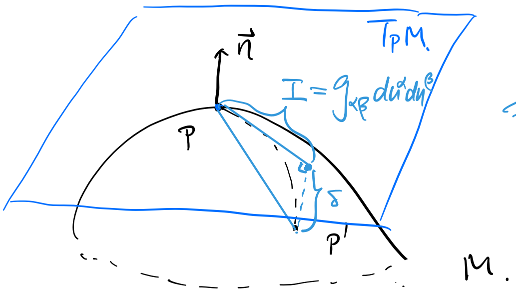
又由 $(c',b,b')=0 \Rightarrow c'\parallel b \Rightarrow$ 切线面

2) $c'(u)=0$

$\Rightarrow$ 锥面

下周课堂练习. 周四

曲面的第二基本形式



$\Longrightarrow$ 几乎可以用 $\frac{\mathrm{II}}{\mathrm{I}}$ 来表现

$p(u^1, u^2)$

$p'(u^1+\Delta u^1, u^2+\Delta u^2)$

$$\overrightarrow{PP'} = \Delta x = x(u^1+\Delta u^1, u^2+\Delta u^2) - x(u^1, u^2)$$

$$= x_1(u^1,u^2)\Delta u^1 + x_2(u^1,u^2)\Delta u^2 + \frac{1}{2}\left(x_{11}(u^1,u^2)(\Delta u^1)^2 + 2x_{12}(u^1,u^2)(\Delta u^1)(\Delta u^2) + x_{22}(u^1,u^2)(\Delta u^2)^2\right) + \cdots$$

$$\delta = \overrightarrow{PP'}\cdot\vec{n} = \frac{\overrightarrow{PP'}\cdot(x_1\times x_2)}{|x_1\times x_2|} = \frac{1}{2}\left(x_{11}\cdot n(\Delta u^1)^2 + 2x_{12}\cdot n\,\Delta u^1\Delta u^2 + x_{22}\,n(\Delta u^2)^2\right) + \cdots$$

$$= \frac{1}{2}k_n(ds)^2 \Longrightarrow k_n = \frac{\mathrm{II}}{(ds)^2} = \frac{\mathrm{II}}{\mathrm{I}}$$

$$2\delta \simeq x_{11}\cdot n(\Delta u^1)^2 + 2x_{12}\cdot n\,\Delta u^1\Delta u^2 + x_{22}\,n(\Delta u^2)^2$$

$$\begin{cases} x_{11}\cdot n = h_{11} = L \\ x_{12}\cdot n = h_{12} = M \\ x_{22}\,n = h_{22} = N \end{cases} \qquad x_{\alpha\beta}\cdot n = h_{\alpha\beta}$$

$$= h_{11}(\Delta u^1)^2 + 2h_{12}\Delta u^1\Delta u^2 + h_{22}(\Delta u^2)^2$$

记 2δ 的主要部分 $\Rightarrow$ 二次微分形式

$$\mathrm{II} = h_{11}(du^1)^2 + 2h_{12}du^1du^2 + h_{22}(du^2)^2$$

$$= h_{\alpha\beta}du^\alpha du^\beta$$

$$x_\alpha\cdot n = 0 \Longrightarrow \overbrace{x_{\alpha\beta}\cdot n}^{h_{\alpha\beta}} + x_\alpha\cdot n_\beta = 0$$

$$h_{\alpha\beta} = x_{\alpha\beta}\cdot n = -x_\alpha n_\beta = -x_\beta n_\alpha$$

$$\mathrm{II} = -(x_\alpha\cdot n_\beta)du^\alpha du^\beta = -(dx, dn)$$

$$h_{\alpha\beta} = x_{\alpha\beta}\cdot\frac{x_1\times x_2}{|x_1\times x_2|} = \frac{(x_1, x_2, x_{\alpha\beta})}{|x_1\times x_2|}$$

$\mathrm{I} = (dx, dx)$

衡量的切空间的内在度量.

即 测地线距离 (不管法方向)

$\mathrm{II} = -(dx, dn)$

衡量曲面如何沿着法方向弯曲

即法向变化的量度

Ex 旋转面

$$x(u^1,u^2) = \left(f(u^2)\cos u^1,\ f(u^2)\sin u^1,\ g(u^2) \right)$$

$$x_1 = (-f\sin u^1,\ f\cos u^1,\ 0)$$

$$x_2 = (f'\cos u^1,\ f'\sin u^1,\ g')$$

$$g_{11} = f^2 \cdot\quad g_{12} = 0 \quad g_{22} = (f')^2 + (g')^2$$

$$n = \frac{x_1 \times x_2}{|x_1 \times x_2|}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -f\sin u^1 & f\cos u^1 & 0 \\ f'\cos u^1 & f'\sin u^1 & g' \end{vmatrix}$$

$$= \frac{(g'\cos u^1,\ g'\sin u^1,\ -f')}{\sqrt{(f')^2+(g')^2}}$$

$$x_{11} = (-f\cos u^1,\ -f\sin u^1,\ 0)$$

$$x_{12} = (-f'\sin u^1,\ f'\cos u^1,\ 0)$$

$$x_{22} = (f''\cos u^1,\ f''\sin u^1,\ g'')$$

$$h_{11} = x_{11}\cdot n = \frac{-g'f}{\sqrt{(f')^2+(g')^2}}$$

$$h_{12} = 0$$

$$h_{22} = \frac{f''g' - f'g''}{\sqrt{(f')^2+(g')^2}}$$

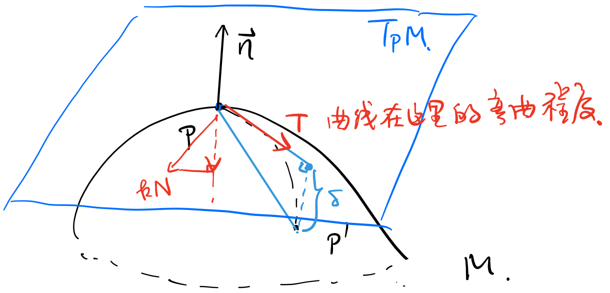
取 $\begin{cases} f = r\cos u^2 \\ g = r\sin u^2 \end{cases} \Rightarrow$ 球面

$$\begin{cases} g_{22} = r^2 \\ g_{12} = 0 \\ g_{11} = r^2\cos^2 u^2 \end{cases} \qquad \left.\begin{matrix} h_{22} = -r \\ h_{12} = 0 \\ h_{11} = -r\cos^2 u^2 \end{matrix}\right)$$

$$h_{\alpha\beta} = -\frac{1}{r} g_{\alpha\beta}$$

$$\mathrm{II} = h_{\alpha\beta}\, du^\alpha du^\beta = -(dx, dn)$$

另法 ☆ 用嵌入曲线去研究.



$$-\frac{(dx, dn)}{(dx, dx)}.$$

$C \subset M.$ $x(s) = x(u^1(s), u^2(s))$

决定T方向

$$T = x_1 \frac{du^1}{ds} + x_2 \frac{du^2}{ds} = x_\alpha \frac{du^\alpha}{ds}$$

$$kN = \dot{T} = \frac{d}{ds}\left(x_\alpha \frac{du^\alpha}{ds}\right) = x_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} + x_\alpha \frac{d^2u^\alpha}{ds^2}$$

$$k_n(T) := (kN \cdot) n = (x_{\alpha\beta} n) \frac{du^\alpha}{ds} \frac{du^\beta}{ds}$$

$$= h_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds}$$

$$= \frac{h_{\alpha\beta}\, du^\alpha du^\beta}{ds^2} = \frac{\mathrm{II}}{\mathrm{I}} = \frac{\mathrm{II}(du^1, du^2)}{\mathrm{I}(du^1, du^2)} = \mathrm{II}\left(\frac{du^1}{ds}, \frac{du^2}{ds}\right)$$

沿着T方向的法曲率

如果 $v = x_1\lambda + x_2\mu$ 单位向量

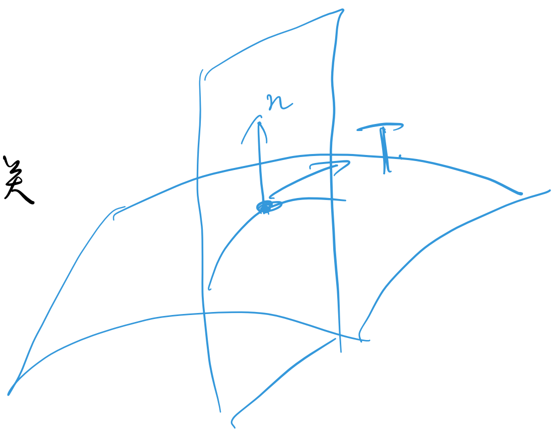
$$k_n(v) = \frac{\mathrm{II}(\lambda,\mu)}{\mathrm{I}(\lambda,\mu)} = \frac{h_{11}\lambda^2 + 2h_{12}\lambda\mu + h_{22}\mu^2}{g_{11}\lambda^2 + 2g_{12}\lambda\mu + g_{22}\mu^2} = h_{11}\lambda^2 + 2h_{12}\lambda\mu + h_{22}\mu^2$$

$$\|v\|^2 = 1 = (x_1\lambda + x_2\mu)(x_1\lambda + x_2\mu) = \mathrm{I}(\lambda,\mu)$$

若 $v = \lambda x_1 + \mu x_2$ 不为单位向量

$$k_n(v) = \frac{\mathrm{II}(\lambda,\mu)}{\mathrm{I}(\lambda,\mu)} = \frac{\mathrm{II}(\lambda,\mu)}{\|v\|^2}$$

Meusnier定理 $\quad k_n$：仅与T有关



△ 第二基本形式与参数选择无关

$M: \ x = x(u^1,u^2)$

$= \bar{x}(\bar{u}^1,\bar{u}^2) \qquad \begin{cases} \bar{u}^1 = \bar{u}^1(u^1,u^2) \\ \bar{u}^2 = \bar{u}^2(u^1,u^2) \end{cases}$

$\mathrm{II} = h_{\alpha\beta}\, du^\alpha du^\beta$ 　　已有 $n(u^1,u^2) = \bar{n}(\bar{u}^1,\bar{u}^2)$

$= \bar{h}_{\alpha\beta}\, d\bar{u}^\alpha d\bar{u}^\beta$

$$\bar{h}_{\alpha\beta} = \bar{x}_{\alpha\beta}\, n = \frac{\partial}{\partial \bar{u}^\beta}\left(\frac{\partial \bar{x}}{\partial \bar{u}^\alpha}\right) n = \frac{\partial}{\partial \bar{u}^\beta}\left(\frac{\partial \bar{x}}{\partial u^\gamma}\frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\right) n$$

$$= \frac{\partial}{\partial u^\sigma}\left(\frac{\partial \bar{x}}{\partial u^\gamma}\frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\right)\frac{\partial u^\sigma}{\partial \bar{u}^\beta}\cdot n$$

$$= \cdots = x_{\sigma\gamma}\frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\frac{\partial u^\sigma}{\partial \bar{u}^\beta} \quad ✓$$

$\mathrm{I} = g_{\alpha\beta}\, du^\alpha du^\beta$ 正定

$(g_{\alpha\beta})^{-1} =: (g^{\alpha\beta})$ ← 矩阵的逆

注 $g_{\alpha\gamma}\, g^{\gamma\beta} = \delta_\alpha^\beta = \begin{cases} 1, & \alpha=\beta \\ 0, & \alpha\neq\beta \end{cases}$

$\Leftrightarrow G G^{-1} = I$

$h_\alpha^\beta \Rightarrow \begin{pmatrix} h_1^1 & h_1^2 \\ h_2^1 & h_2^2 \end{pmatrix} = \mathrm{II}\,\mathrm{I}^{-1}$

$\Rightarrow \begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix} = \mathrm{I}^{-1}\mathrm{II}$ 　actually.

Self-disjointed. 实内积空间 $\mathbb{R}^n$ & $\langle\,,\rangle$

$\langle T(x), y\rangle = \langle x, T(y)\rangle$

$\Leftrightarrow A = A^T$ （实对称矩阵）

$\Rightarrow$ 特征值都为实数，

且 可正交对角化

Weingarten 变换. $T_pM \to T_pM$

$\{x_\alpha\}$

$W(x_\alpha) = h_\alpha^\beta x_\beta$ 　 $h_\alpha^\beta = h_{\alpha\gamma}\, g^{\gamma\beta} = h_{\alpha\gamma}(g_{\gamma\beta})^{-1}$

$= \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1}$

↓ 自共轭

$W(x_1,x_2) = (x_1,x_2)\begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix}$ 　 $= \mathrm{II}\,\mathrm{I}^{-1}$

$\Rightarrow (W(x_\alpha))\, x_\beta = h_\alpha^\gamma x_\gamma x_\beta = h_\alpha^\gamma g_{\gamma\beta} = \underline{h_{\alpha\beta}} = \underline{h_{\beta\alpha}} = x_\alpha(W(x_\beta))$

自共轭线性变换

$\Rightarrow W(dx) = W(x_\alpha du^\alpha) = h_\alpha^\beta x_\beta du^\alpha \quad \Rightarrow W(dx)\cdot dx = \mathrm{II}$

$h_\alpha^\gamma = h_{\alpha\sigma}\, g^{\sigma\gamma}$ 　用度量矩阵（逆）把指标拉下来.

$$k_n(dx) = k_n(du^1/du^2) = \frac{h_{\alpha\beta}\, du^\alpha du^\beta}{g_{\alpha\beta}\, du^\alpha du^\beta} = \frac{(W(dx))\cdot dx}{|dx|^2}$$

$W(dx) = W(x_\alpha du^\alpha) = du^\alpha\, h_\alpha^\beta x_\beta.$

$W(dx)\cdot dx = du^\alpha\, W(x_\alpha)\cdot x_\beta\, du^\beta = h_{\alpha\beta}\, du^\alpha du^\beta$

$dN = -W(dx)$ 　（可以这么理解. 实际直接定义 W 变换也没什么问题

Weingarten 变换有两个实的特征值. k_1, k_2. $\Rightarrow$ 对应的特征向量 e_1, e_2. s.t. $e_\alpha e_\beta = \delta_\beta^\alpha$ 且

$e_\alpha e_\beta = \delta_\beta^\alpha$. 且. $W(e_\alpha) = k_\alpha e_\alpha$

$\Rightarrow$ 任一单位向量 T 可表为 $T = \cos\theta\, e_1 + \sin\theta\, e_2$

Prop. (Euler公式)

$$k_n(\theta) = \frac{(W(T))\cdot T}{|T|^2} = \left(W(\cos\theta e_1 + \sin\theta e_2)\right)(\cos\theta e_1 + \sin\theta e_2)$$

$$= k_1 \cos^2\theta + k_2 \sin^2\theta$$

$$\frac{dk_n}{d\theta} = (k_2 - k_1)\sin 2\theta.$$

若 $k_1 \neq k_2$. 则 $\theta = 0, \frac{\pi}{2}$ 时. 法曲率达到极值.

故称 e_α 为主方向. k_α 为主曲率.

若某一点有 $k_1 = k_2$ 则称为脐点、则各方向主曲率均相等. 为 λ

$\lambda = 0$: 平点. $\lambda \neq 0$. 圆点.

$(h_{\alpha\beta} - \lambda g_{\alpha\beta})\, du^\alpha du^\beta = 0$ 这是由于. $k_n(dx) = \dfrac{h_{\alpha\beta}\, du^\alpha du^\beta}{g_{\alpha\beta}\, du^\alpha du^\beta}$

$\Rightarrow \boxed{h_{\alpha\beta} = \lambda g_{\alpha\beta}}$