

Algebra. Lec 17. Normal extensions.

$k$ . field.  $f \in k[x]$ .  $\deg f \geq 1$ .

$K$  is called splitting field of  $f \in k[x]$

if it's an ext of  $k$ . s.t.  $f$  splits into linear factors in  $K$   
(smallest)  $\Rightarrow \text{alg}/k$ .

**Thm. 3.1**  $K$ . a splitting field of  $f \in k[x]$

$E$ : another splitting field of  $f$

$$\Rightarrow \exists \sigma: E \xrightarrow{\sim} K. \quad \sigma|_k = \text{id}_k$$

pf.  $k^a$ . alg. closure of  $k$   
(alg closed.  $\text{alg}/k$ )

$K: \text{alg}/k$ .

$\Rightarrow k^a: \text{alg}/k \quad \therefore k^a$  is also an alg closure of  $k$   
(i.e.)  $k^a \simeq k^a$

$$\exists \sigma: E \hookrightarrow k^a \quad (\text{By Thm 2.8}).$$

(including identity on  $k$ )

in  $E$   $f(x) = c(x-\beta_1)(x-\beta_2)\cdots(x-\beta_n) \quad \beta_i \in E, c \in k$

then  $E = k(\beta_1, \dots, \beta_n)$

$$f(x) = f^\sigma(x) \quad (\text{for } f(x) \in k[x])$$

$$f(x) = c(x - \sigma\beta_1) \cdots (x - \sigma\beta_n).$$

$$\quad \quad \quad \bigcap_{k^a} \quad \quad \quad \bigcap_{k^a}$$

in  $k^a[x]$  ( $k \subseteq k^a$ )

$f(x) = c(x - \alpha_1) \cdots (x - \alpha_n)$   $\alpha_i \in k$ . for  $k$  is a splitting field.

we have  $k^a[x]$  is a UFD

then  $(\sigma\beta_1, \dots, \sigma\beta_n) = (\alpha_1, \dots, \alpha_n)$   
up to permutation.

$\therefore \sigma\beta_i \in k$

$\Rightarrow \sigma E \subseteq k$

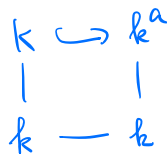
$\Rightarrow \sigma E = k$

$\Rightarrow \sigma: E \xrightarrow{\sim} k \subseteq k^a$

□

Thm 33.  $k$  alg ext/ $k$

$k \subseteq k \subseteq k^a$ .



TFAE. NOR1. every embedding  $k \hookrightarrow k^a$  induce an automorphism of  $k$

NOR2  $k$  is the splitting field of a family  $\{f_i(x)\}_{i \in I}$   
 $f_i(x) \in k[x]$

NOR3.  $f(x) \in k[x]$   $f(x)$  irred.  $f(x)$  has a root in  $k$ .

$\Rightarrow f(x)$  splits into linear factors in  $k$

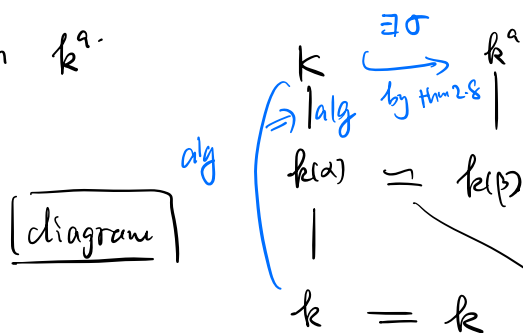
pf. **NOR1  $\Rightarrow$  NOR2.**  $\nearrow \text{Irr}(\alpha, k, x)$

Let  $\alpha \in k$ .  $P_\alpha(x) \in k[x]$  irred poly of  $\alpha$ .

let  $\beta$  be a root of  $P_\alpha(x)$  in  $k^a$ .

$$\therefore k(\alpha) \xrightarrow{\sim} k(\beta)$$

$$\alpha \mapsto \beta$$



by NOR1. the extension  $\sigma$  induces an automorphism of  $k$

$\therefore \sigma(\alpha) = \beta \in k$  ( $\sigma$  is the extension of  $\sigma|_k$ )

$\therefore$  every root of  $P_\alpha(x)$  is in  $k$

$\therefore k$  is the splitting field of  $\{P_\alpha(x)\}_{\alpha \in k}$ .

**NOR2  $\Rightarrow$  NOR1.**

$\{f_i\}_{i \in I}$  a family of polynomials in  $k[x]$ .

$K$ : splitting field.

let  $\alpha$  be a root of some  $f_i$

$$\bigcap_{i \in I} K$$

$$\begin{array}{ccc} k & \xrightarrow[\text{Thm 2.8}]{\exists \sigma} & k^a \\ \uparrow \text{alg} & & \uparrow \text{alg} \\ k & \xrightarrow{\sim} & k \\ | & & | \\ k & = & k \end{array}$$

$0 = f_i(\alpha) \xrightarrow{\sigma} f_i^\sigma(\sigma\alpha) = f_i(\sigma\alpha) \Rightarrow \sigma\alpha$  is also a root of  $f_i$

i.e.  $\sigma: \{\text{roots of } f_i\} \rightarrow \{\text{roots of } f_i\}$

$$\therefore \sigma: \begin{array}{ccc} & \xrightarrow{\text{emb.}} & \\ \text{alg} \left( \begin{array}{c} K \\ | \\ K \end{array} \right) & & \text{alg} \left( \begin{array}{c} K \\ | \\ K \end{array} \right) \\ & \xrightarrow{\quad} & \end{array} \Rightarrow \text{by Lem 2.1. } \sigma \text{ is an iso.}$$

**NOR1  $\Rightarrow$  NOR3**  $f(x) \in K[x]$  irred.

if  $\exists \alpha \in K. \tau.t. f(\alpha) = 0$  let  $\beta$  be another root of  $f(x)$ .

$$\begin{array}{ccc} & \xrightarrow{\exists \text{ emb. by thm 2.8}} & \\ \text{alg} \left( \begin{array}{c} K \\ | \\ K(\alpha) \end{array} \right) & & \text{alg} \left( \begin{array}{c} K^a \\ | \\ K(\beta) \end{array} \right) \\ \alpha \in K(\alpha) \xrightarrow{\sim} K(\beta) \ni \beta & \Rightarrow \exists \sigma: K \hookrightarrow K^a. & (\text{Thm 2.8}) \\ \downarrow & & \\ K & = & K \end{array} \therefore \sigma: K \hookrightarrow K \text{ by NOR1}$$

we have  $\beta = \sigma(\alpha) \in K$ .

i.e. all roots of  $f$  are in  $K$

**NOR3  $\Rightarrow$  NOR1**

$$\text{alg} \left( \begin{array}{cc} K & K^a \\ | & | \\ K & = & K \end{array} \right) \quad \sigma: K \hookrightarrow K^a$$

let  $\alpha \in K. P(x) = \text{Irr}(\alpha, K, x) \quad P(\alpha) = 0.$

$0 = \sigma(P(\alpha)) = P(\sigma\alpha) \quad \therefore \sigma\alpha$  is another root

by NOR3.  $\sigma\alpha \in K$ .

$$\sigma: \begin{array}{ccc} K & \hookrightarrow & K \\ & \searrow & \swarrow \\ & K & \end{array} \text{ by Lem 2.1 } \Rightarrow \sigma: K \hookrightarrow K. \quad \square$$

Rmk. NOR 1., NOR 2., NOR 3. normal extension.

Rmk.  $\deg = 2$  extension is normal.

(i.e.  $[k(\alpha):k] = 2$ )

$$p(x) = \text{Irr}(\alpha, k, x) \quad \cdot \quad \deg p(x) = 2.$$

roots  $\alpha, \beta \Rightarrow \alpha + \beta, \alpha\beta \in k \Rightarrow \beta \in k(\alpha)$   
(coeff of  $p(x)$ ).

$\therefore k(\alpha)$ : splitting field of  $p(x)$ .

eg.  $E = \mathbb{Q}(\sqrt[4]{2}) \supset F = \mathbb{Q}(\sqrt{2}) \supset \mathbb{Q}$

degree 4 (over  $\mathbb{Q}$ )  
degree 2 (over  $F$ )      degree 2 (over  $\mathbb{Q}$ )

$$\text{Irr}(\sqrt[4]{2}, \mathbb{Q}, x) = x^4 - 2 \quad (\text{Eisenstein}).$$

$\left\{ \begin{array}{l} E: \text{normal ext of } F \\ F: \text{normal ext of } \mathbb{Q} \end{array} \right.$

But  $E$  is not normal ext of  $\mathbb{Q}$ .

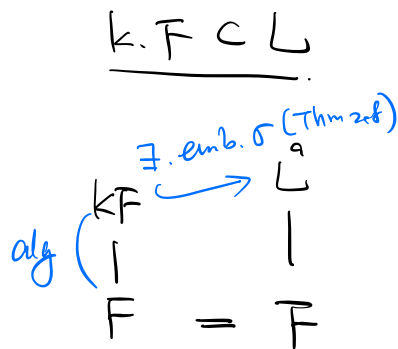
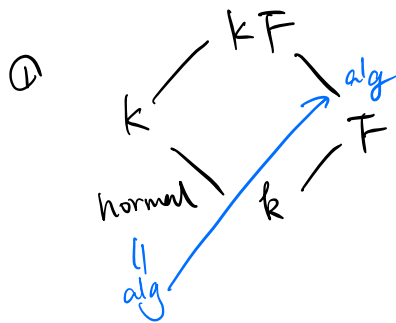
$\therefore$  normal ext is not distinguished.

Thm. 3.4.

- normal extension remains normal under lifting
- if  $K \supset E \supset k$ .  $K$  normal /  $k \Rightarrow K$  normal /  $E$
- $K_1, K_2$  normal /  $k$ . contained in  $L$

$$\Rightarrow K_1 K_2 \text{ normal } / k.$$

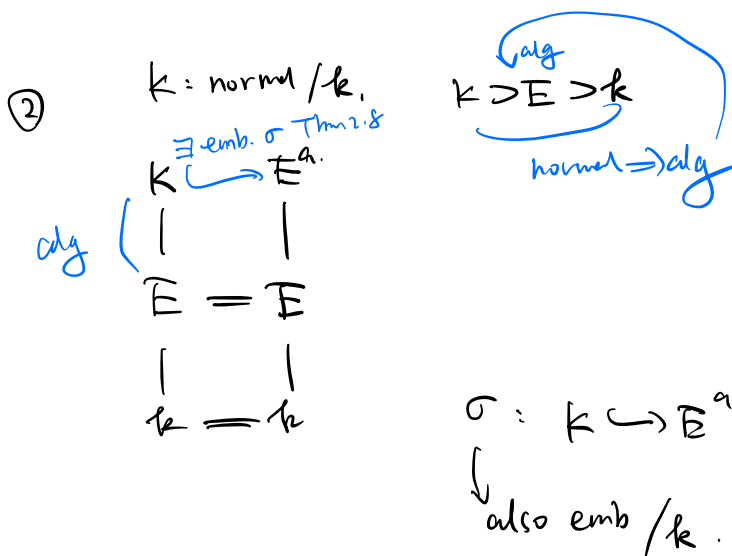
pf.



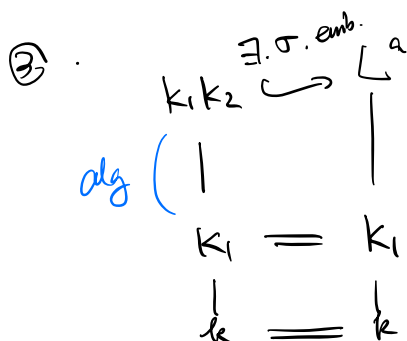
$$\sigma: KF \hookrightarrow L^a$$

$$\sigma|_F = \text{id}_F \quad \sigma|_K: K \xrightarrow{\sim} K \text{ (NORI)}.$$

$$(KF)^\sigma = K^\sigma F^\sigma = KF \Rightarrow \text{NORI} \Rightarrow KF: \text{normal over } F.$$



$$\Rightarrow \sigma|_K: K \xrightarrow{\sim} K \Rightarrow K \text{ normal } / E$$



$$\sigma(K_1 K_2) = \sigma(K_1) \sigma(K_2) = K_1 K_2$$

$$\sigma(K_1 \cap K_2) = K_1^\sigma \cap K_2^\sigma = K_1 \cap K_2$$

□

HW. Lang. Chap V.

ex. 7. 8. 10. 21