Holomorphy and conformality

For any $\Omega \subseteq \mathbb{C}$ we let $\Omega_{\mathbb{R}} := \{ (x,y) \in \mathbb{R}^2 \mid x+iy \in \mathbb{C} \}$

- $\{\text{maps from } \Omega \text{ to } \mathbb{C}\} \longrightarrow \{\text{maps from } \Omega_{\mathbb{R}} \text{ to } \mathbb{R}^2\}$

$$\Omega \xrightarrow{f} \mathbb{C} \longmapsto \Omega_{\mathbb{R}} \xrightarrow{f} \mathbb{R}^2 \quad \text{is a bijection}$$

If $\Omega \overset{\text{open}}{\subseteq} \mathbb{C}$, then under this corespondence,

holo maps $\Omega \xrightarrow{f} \mathbb{C}$ correspond to C^1 maps $\Omega_{\mathbb{R}} \to \mathbb{R}^2$

$$(x,y) \mapsto (u(x,y), v(x,y))$$

which satisfy the Cauchy-Riemann equations.

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases} \quad \text{i.e. } f_{\mathbb{R}}'(x,y) \text{ is of the form } \begin{pmatrix} A & -B \\ B & -A \end{pmatrix}.$$

$$\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$$

则γ在$\mathbb{R}^2$中的切向量由Jacobi矩阵作用后得到$(f\circ\gamma)$在$\mathbb{R}^2$中的切向量

Therefore, a holo map f transforms tangent vectors of paths by

rotation after a scaling

↳ multiplication by $\sqrt{A^2+B^2}$

and hence, preserves oriented angles between tangent vectors when $A^2+B^2 \neq 0$.

for $A^2+B^2 > 0$.

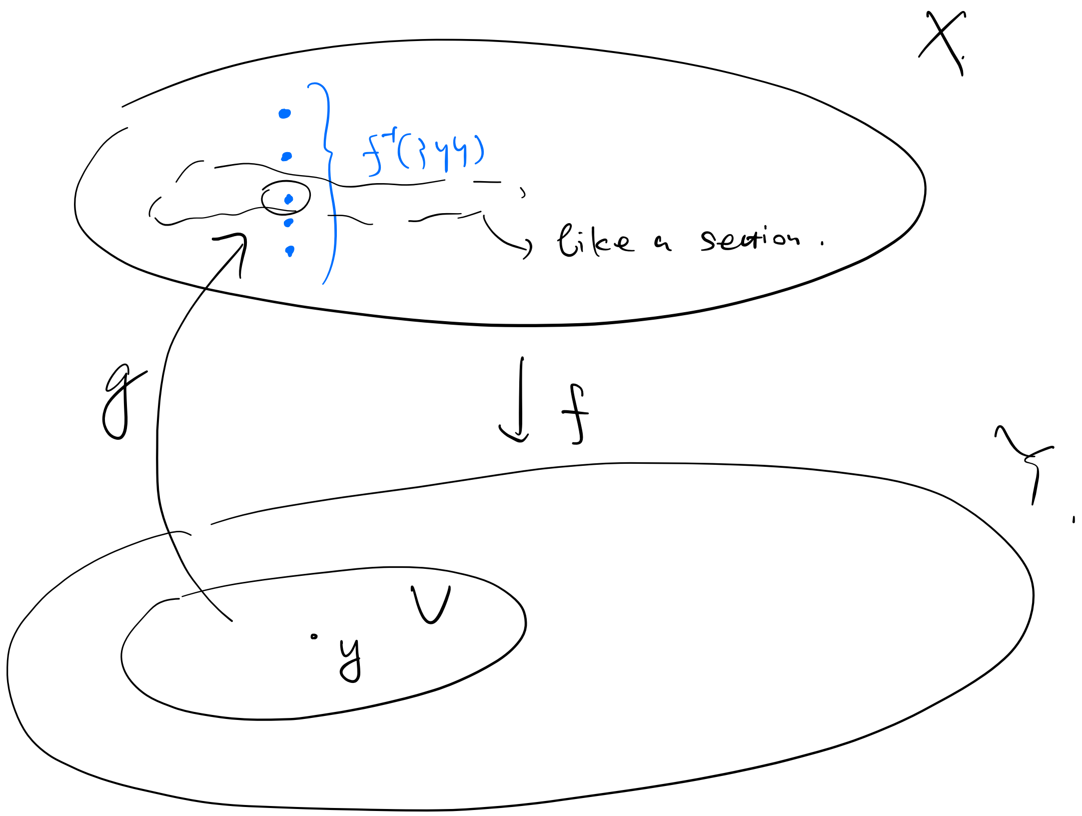
局部的保角，
也就是在每个点处
进行了一种扭转.

例. $f(z) = z^2$

Terminology. (只限于这里讨论)

Given a map $X \xrightarrow{f} Y$, any map $X \xleftarrow{g} V$ with $V \subseteq Y$ s.t. $(f \circ g)(y) = y$ for all $y \in V$ is called a (partial) inverse of f (on/over V)

- In many situations in mathematics g is called "a section of f on/over V"
- Such a g (if it exists) must be an injection.
- For any map $\begin{matrix} Y \\ \cup| \\ V \end{matrix} \xrightarrow{g} X$, the condition $(f \circ g)(y) = y$ for all $y \in V$ is equivalent to $\forall y \in V,\ g(y) \in f^{-1}(\{y\})$.



Q. For a holo map $\underset{\Omega}{\overset{\mathbb{C}}{\cup\text{open}}} \xrightarrow{f} \mathbb{C}$, has a partial inverse $\Omega \xleftarrow{g} V$ on a open set $V \subseteq \mathbb{C}$, also holomorphic?

It is direct to see that if g is C^1 then g is holo.

<u>Prop.</u> Let $\Omega \xrightarrow{f} \mathbb{C}$ be a holo map and $z_0 = x_0 + iy_0 \in \Omega$.

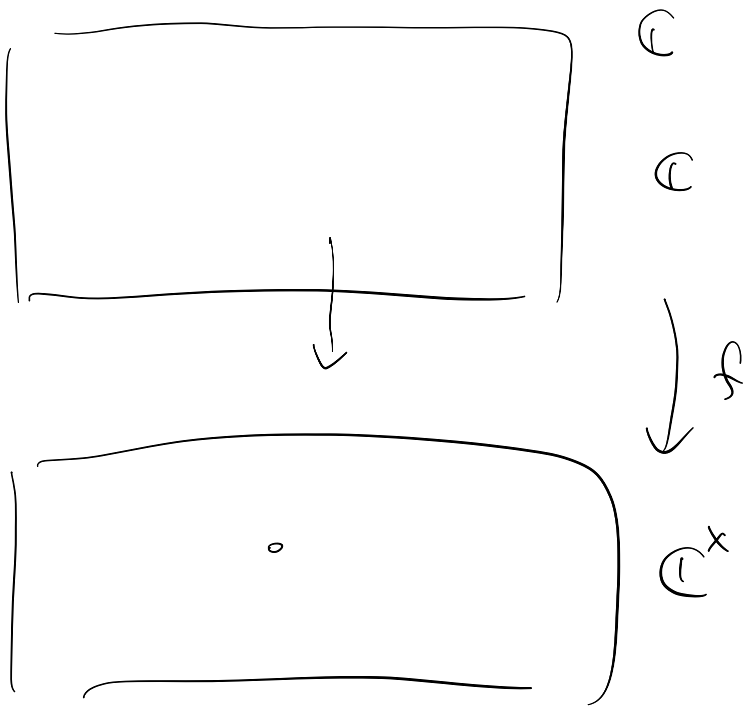
If $f'(z_0) \neq 0$, then $\exists\ \underset{z_0}{\overset{}{U \ni}} \underset{\text{open}}{\subseteq} \Omega$ ($\Rightarrow U \underset{\text{open}}{\subseteq} \mathbb{C}$) s.t.

① $f(U) \underset{\text{open}}{\subseteq} \mathbb{C}$ ② $U \xrightarrow{f} f(U)$ is an bijection, and

③ $U \xleftarrow{f^{-1}} f(U)$ is holo

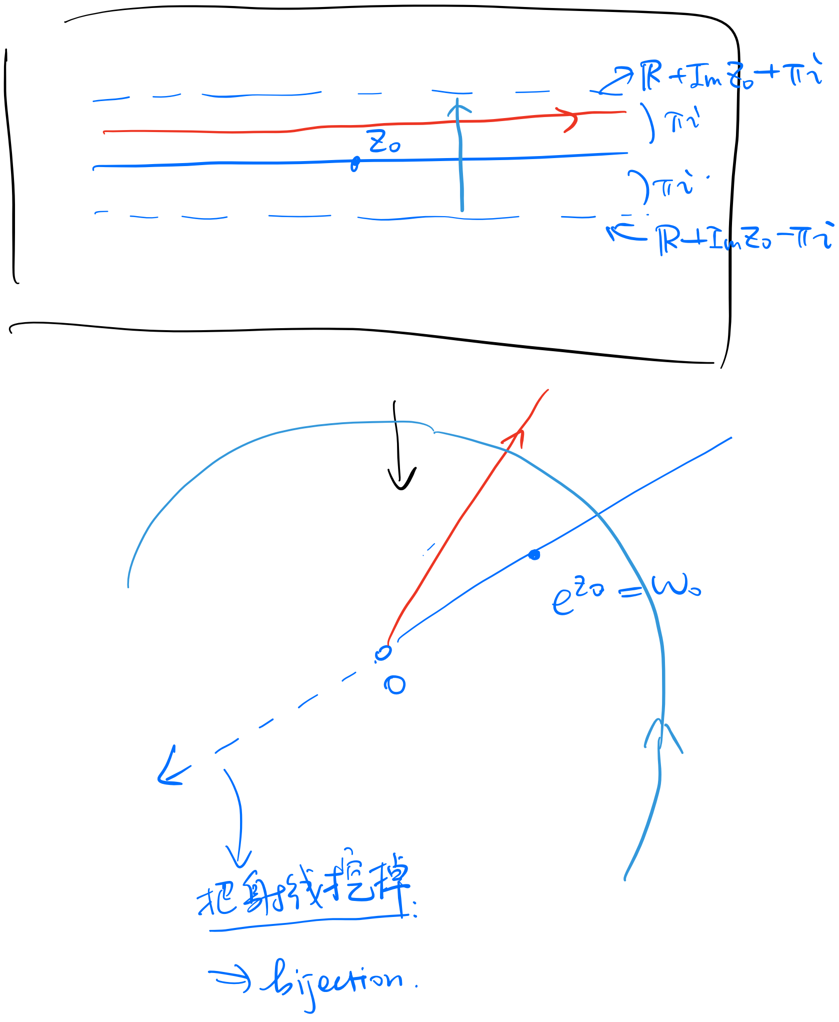
(用反函数定理)

<u>Example</u>. $f(z) = e^z \left(= \exp(z)\right) = e^x(\cos y + i \sin y)$.



$f'(z) = e^z \neq 0$

Def. A continuous partial inverse g of $f(z) = e^z$ on an open set V is called a logrithm func on V.
$V \subset \mathbb{C}^\times$



Prop. For any $w_0 \in \mathbb{C}^\times$ and $z_0 \in \mathbb{C}$ s.t. $e^{z_0} = w_0$ and $w_0 \in V \in \mathbb{C}^\times$ (a path-connected open set), there exist at most one logrithm function g on V s.t. $g(w_0) = z_0$

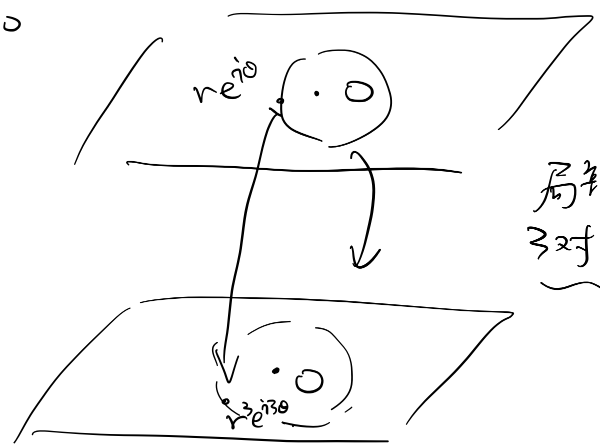
Q. Given a holo. map $\Omega \xrightarrow{f} \mathbb{C}$ and $z_0 \in \Omega$.

if $f'(z_0)=0$, does there exist any $V \underset{\text{open}}{\subseteq} \mathbb{C}$ on which a
$f(z_0)$ $\in$

Conti. partial inverse exists?

$\mathbb{R} \to \mathbb{R}$? $f(x)=x^3$.

$$\begin{array}{ccc} z & \mathbb{C} & \\ \downarrow & \downarrow f & \\ z^3 & \mathbb{C} & \end{array} \qquad f'(0)=0$$



局部
3对1的映射

不行

⋮

<u>Cor</u>. If Ω is a path-connected open set in $\mathbb{C}$ and

$\Omega \xrightarrow{f} \mathbb{C}$ is a nonconstant holo map. then, $\forall A \subseteq \Omega$.

$$\left(A \underset{\text{open}}{\subseteq} \Omega \Rightarrow f(A) \underset{\text{open}}{\subseteq} \mathbb{C} \right)$$

(open mapping theorem)

简介 7~10 刻?