W2L.

高斯消元法 可行性条件.

$a_{ii}^{(i-1)} \neq 0$. 或(i). 看记号.

Q. ① $\det A \neq 0 \Rightarrow$ 顺序主子式 $\neq 0$?

② 高斯消元法 改变 A 的行列式?

主元太小

Ex. $\begin{pmatrix} 0.001 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\Rightarrow L = \begin{pmatrix} 1 & 0 \\ 1000 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 0.001 & 1 \\ 0 & -998 \end{pmatrix}$

$Ly=b \quad \begin{pmatrix} 1 & 0 \\ 1000 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -997 \end{pmatrix}$

$Ux=y \quad \begin{pmatrix} 0.001 & 1 \\ 0 & -998 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -997 \end{pmatrix} \Rightarrow$ 精度太低就直接 = 0.

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{997}{998}\right) / 0.001 \\ \frac{997}{998} \end{pmatrix} \simeq \begin{pmatrix} 1.002 \\ 0.998 \end{pmatrix}$.

选主元消元法

1° 全主元 Gauss 消元法.

$$(A, b) \xrightarrow{k-1 \text{ 步消元}} (A^{(k)}, b^{(k)}) = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\ & & \ddots & & \\ & & a_{kk}^{(k)} \cdots & a_{kn}^{(k)} & b_k^{(k)} \\ & & \vdots & \vdots & \vdots \\ & & a_{nk}^{(k)} \cdots & a_{nn}^{(k)} & b_n^{(k)} \end{pmatrix}$$

选主元. $|a_{pq}^{(k)}| = \max\limits_{k\le i,j\le n} |a_{ij}^{(k)}|$. 交换到 $a_{kk}^{(k)}$ 位置上

$\Longrightarrow (\tilde{A}^{(k)}, \tilde{b}^{(k)})$

<u>初等置换矩阵</u>. $\overset{\text{行}}{P_k} = I_{kp}$, $\overset{\text{列}}{Q_k} = I_{kq}$ $\quad P_k^2 = Q_k^2 = I$

$$\tilde{A}^{(k)} = P_k A^{(k)} Q_k \qquad A^{(k+1)} = L_k \tilde{A}^{(k)} = L_k P_k A^{(k)} Q_k$$

$$U = A^{(n)} = L_{n-1} P_{n-1} A^{(n-1)} Q_{n-1}$$

$$= L_{n-1} P_{n-1} L_{n-2} P_{n-2} \cdots L_1 P_1 A \underbrace{Q_1 Q_2 \cdots Q_{n-1}}_{Q}.$$

$$P = P_{n-1} P_{n-2} \cdots P_1. \qquad L = P (L_{n-1} P_{n-1} L_{n-2} P_{n-2} \cdots L_1 P_1)^{-1}$$

$$PAQ = LU$$

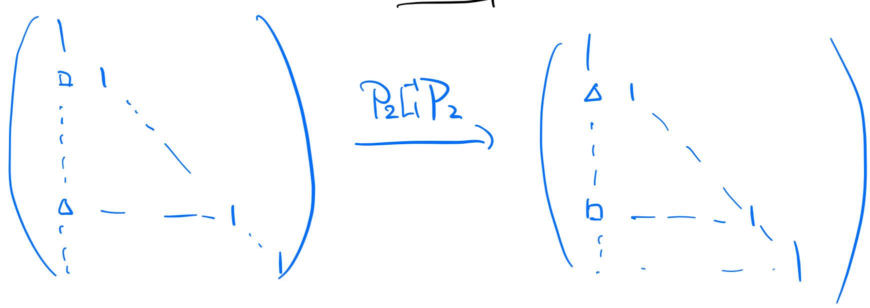
Q. L是什么?

$$L = P_{n-1} P_{n-2} \cdots P_2 P_1 P_1 L_1^{-1} P_2 L_2^{-1} P_3 L_3^{-1} \cdots P_{n-1} L_{n-1}^{-1}$$

$$= P_{n-1} P_{n-2} \cdots \underline{P_2 L_1^{-1} P_2} L_2^{-1} \cdots P_{n-1} L_{n-1}^{-1}$$

$$= P_{n-1} P_{n-2} \cdots P_3 \underline{\tilde{L}_1^{-1} L_2^{-1} P_3} \cdots P_{n-1} L_{n-1}^{-1}$$

$= \cdots\cdots = $ <u>仍为下三角</u>



$= \tilde{L}_1^{-1}$

Thm. $A\in\mathbb{R}^{n\times n}$. 则存在排列矩阵 $P.Q.\in\mathbb{R}^{n\times n}$. 以及单位下三角阵 L 和上三角阵 U.

s.t. $PAQ=LU$，且 L 的所有元素满足 $|L_{ij}|\le 1$.

U 的非零对角元的个数 $=\operatorname{rank}(A)$

进行 $\sum_{k=1}^{n-1}(n-k+1)^2 \sim \frac{1}{3}n^3$ 次比较. 算法复杂度. $O(n^3)$

① $PAQ=LU$ $\qquad \underline{PAQ}\,\underline{Q^Tx}=\underline{Pb} \Rightarrow L\underbrace{UQ^Tx}_{=y}=Pb$

② $Ly=Pb\Rightarrow y$

③ $Uz=y$ $\qquad \Rightarrow U\underbrace{Q^Tx}_{=z}=y$

④ $x=Qz$ $\qquad \Rightarrow Q^Tx=z.$

2°. 列主元消元法. (减少比较次数).

$$(A,b)\xrightarrow{k-1\text{步消元}}(A^{(k)},b^{(k)})=\begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ & a_{22}^{(2)} & \cdots & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\ & & \ddots & & & \\ & & & a_{kk}^{(k)} & \cdots\ a_{kn}^{(k)} & b_k^{(k)} \\ & & & \vdots & \vdots & \vdots \\ & & & a_{nk}^{(k)} & \cdots\ a_{nn}^{(k)} & b_n^{(k)} \end{pmatrix}$$

选主元 $\qquad |a_{pk}^{(k)}|=\max_{k\le i\le n}|a_{ik}^{(k)}|$ $\qquad$ 交换 第 p 行与第 k 行

$$\longrightarrow(\widetilde{A}^{(k)},\widetilde{b}^{(k)})\xrightarrow{\text{消元}}(A^{(k+1)},b^{(k+1)}).$$

设初等置换矩阵. $P_k=I_{kp}$. $\qquad \widetilde{A}^{(k)}=P_kA^{(k)}$

$$A^{(k+1)}=L_k\widetilde{A}^{(k)}=L_kP_kA^{(k)}$$

$$\Rightarrow U = A^{(n)} = L_{n-1}P_{n-1}A^{(n-1)} = L_{n-1}P_{n-1}\cdots L_1P_1A$$

记排列矩阵 $P = P_{n-1}P_{n-2}\cdots P_1$. $L = P(L_{n-1}P_{n-1}L_{n-2}P_{n-2}\cdots L_1P_1)^{-1}$ 为单位下三角

$PA = LU$. $PAx = Pb$

$\Rightarrow \underbrace{Ux}_{=y}$: $LUx = Pb$

① $PA = LU$

② $Ly = Pb \Rightarrow y$ $\Rightarrow Ux = y$

③ $Ux = y \Rightarrow x$

比较次数. $\sum_{k=1}^{n-1}(n-k+1) \sim O(n^2)$

Guass 消元

<u>Conclusion</u>. 列主元. 全主元. 数值稳定性方面差不多

但计算量减少一点

应用. 中小型. 稠密方程组