

外代数

V 是 \mathbb{R} 上的 m 维线性空间

$\{\sigma_1, \sigma_2, \dots, \sigma_m\}$ 基

$$\xi \in V, \quad \xi = \xi^\alpha \sigma_\alpha$$

在 V 的元素之间定义一个运算, 外乘 (外积 " \wedge ") (形式定义)

$$\textcircled{1} \quad 1^\circ \sigma_1, \dots, \sigma_m$$

$$2^\circ \sigma_\alpha \wedge \sigma_\beta = -\sigma_\beta \wedge \sigma_\alpha \Rightarrow \sigma_\alpha \wedge \sigma_\alpha = 0$$

$$\alpha < \beta \quad \sigma_1 \wedge \sigma_2 \quad \sigma_1 \wedge \sigma_3 \quad \dots \quad \sigma_{m-1} \wedge \sigma_m \quad \text{新元素} \quad \binom{m}{2}$$

⋮

$$k^\circ \quad \sigma_{\alpha_1} \wedge \sigma_{\alpha_2} \wedge \dots \wedge \sigma_{\alpha_k} \quad 1 \leq \alpha_1 < \dots < \alpha_k \leq m \quad \binom{m}{k}$$

$$m^\circ \quad \sigma_1 \wedge \dots \wedge \sigma_m \quad \binom{m}{m} = 1.$$

② 对于 固定的 k ($k \geq 2$).

有如下性质

$$\begin{aligned} & \sigma_{\alpha_1} \wedge \dots \wedge \sigma_{\alpha_p} \wedge \dots \wedge \sigma_{\alpha_q} \wedge \dots \wedge \sigma_{\alpha_k} \\ &= - \sigma_{\alpha_1} \wedge \dots \wedge \sigma_{\alpha_q} \wedge \dots \wedge \sigma_{\alpha_p} \wedge \dots \wedge \sigma_{\alpha_k}. \end{aligned} \quad \text{反对称性}$$

$$\text{if } \sigma_{\alpha_p} = \sigma_{\alpha_q} \text{ 则 } = 0$$

③ 利用线性扩张, 外乘可以推广到整个 V 上

$$\begin{aligned} \xi &= \xi^\alpha \sigma_\alpha, \quad \eta = \eta^\beta \sigma_\beta & \xi \wedge \eta &= (\xi^\alpha \sigma_\alpha) \wedge (\eta^\beta \sigma_\beta) \\ & & &= \sum_{\alpha, \beta} \xi^\alpha \eta^\beta \sigma_\alpha \wedge \sigma_\beta \end{aligned}$$

$$= \sum_{\alpha < \beta} \xi^\alpha \eta^\beta \sigma_\alpha \wedge \sigma_\beta$$

$$+ \sum_{\alpha < \beta} \xi^\alpha \eta^\beta \sigma_\alpha \wedge \sigma_\beta$$

$$= \sum_{\alpha < \beta} \xi^\alpha \eta^\beta \sigma_\alpha \wedge \sigma_\beta$$

$$\downarrow + \sum_{\alpha < \beta} \xi^\beta \eta^\alpha \sigma_\beta \wedge \sigma_\alpha$$

交换 α 与 β

$$= \sum_{\alpha < \beta} (\xi^\alpha \eta^\beta - \xi^\beta \eta^\alpha) \sigma_\alpha \wedge \sigma_\beta$$

$$= \sum_{\alpha < \beta} \begin{vmatrix} \xi^\alpha & \xi^\beta \\ \eta^\alpha & \eta^\beta \end{vmatrix} \sigma_\alpha \wedge \sigma_\beta$$

$$\zeta = \zeta^\gamma \sigma_\gamma \quad \xi \wedge \eta \wedge \zeta = \sum_{\alpha, \beta, \gamma} \xi^\alpha \eta^\beta \zeta^\gamma \sigma_\alpha \wedge \sigma_\beta \wedge \sigma_\gamma$$

$$= \sum_{\alpha < \beta < \gamma} \begin{vmatrix} \xi^\alpha & \xi^\beta & \xi^\gamma \\ \eta^\alpha & \eta^\beta & \eta^\gamma \\ \zeta^\alpha & \zeta^\beta & \zeta^\gamma \end{vmatrix} \sigma_\alpha \wedge \sigma_\beta \wedge \sigma_\gamma.$$

V 中任意 k 个元素作外积 得到的称为 k -元素 V_k

$$V_k = \text{span} \{ \sigma_{\alpha_1} \wedge \dots \wedge \sigma_{\alpha_k} ; 1 \leq \alpha_1 < \dots < \alpha_k \leq m \} \quad k\text{-空间}$$

$$\dim V_k = \binom{m}{k}$$

$$V_1 = V. \quad V_0 = \mathbb{R} \text{ 称为 } 1\text{-元}$$

$$V_m = \{ \sigma_1 \wedge \dots \wedge \sigma_m \} \quad \dim V_m = 1$$

$$\Lambda(V) = \bigoplus_{k=0}^m V_k \quad \leftarrow \text{外代数}$$

$$G(V)$$

$$\dim \Lambda(V) = \binom{m}{0} + \binom{m}{1} + \dots + \binom{m}{m} = 2^m$$

$$\text{set } \xi = \sigma_{\alpha_1} \wedge \dots \wedge \sigma_{\alpha_k} \quad k\text{-元集}$$

$$\eta = \sigma_{\beta_1} \wedge \dots \wedge \sigma_{\beta_s} \quad s\text{-元集}$$

$$\xi \wedge \eta = (\sigma_{\alpha_1} \wedge \dots \wedge \sigma_{\alpha_k}) \wedge (\sigma_{\beta_1} \wedge \dots \wedge \sigma_{\beta_s})$$

$$= \sigma_{\alpha_1} \wedge \dots \wedge \sigma_{\alpha_k} \wedge \sigma_{\beta_1} \wedge \dots \wedge \sigma_{\beta_s} \quad (k+s)\text{-元集}$$

$$\text{若 } k+s > m, \quad \xi \wedge \eta = 0$$

" \wedge " 有如下运算律 $\wedge(V)$ ^{分属} \wedge -结合代数 \rightarrow 张量的元素

1° 结合律 ξ - k 元集, η - s 元集, ζ - r 元集

$$(\xi \wedge \eta) \wedge \zeta = \xi \wedge (\eta \wedge \zeta) = \xi \wedge \eta \wedge \zeta$$

$$w_0 + w_1 + \dots + w_m$$

$$w_k \in V_k$$

2° 分配律 $a, b, c \in \mathbb{R}$

$$(a\xi + b\eta) \wedge \zeta = a\xi \wedge \zeta + b\eta \wedge \zeta$$

$$\xi \wedge (a\eta + b\zeta) = \xi \wedge a\eta + \xi \wedge b\zeta$$

命题 1) $\varphi \in V_p, \psi \in V_q$

$$\varphi \wedge \psi = (-1)^{pq} \psi \wedge \varphi$$

$$\text{设 } \varphi = \sigma_{\alpha_1} \wedge \dots \wedge \sigma_{\alpha_p}$$

$$\psi = \sigma_{\beta_1} \wedge \dots \wedge \sigma_{\beta_q}$$

$$\varphi \wedge \psi = (\sigma_{\alpha_1} \wedge \dots \wedge \sigma_{\alpha_p}) \wedge (\sigma_{\beta_1} \wedge \dots \wedge \sigma_{\beta_q})$$

$$= (-1)^{pq} \psi \wedge \varphi$$

$$2) \varphi_r = a_r^\alpha \sigma_\alpha \quad 1 \leq r \leq k \leq m$$

$$\varphi_r \in V_1, k \uparrow \quad 1 \leq \alpha \leq m.$$

$$\varphi_1 \wedge \varphi_2 \wedge \dots \wedge \varphi_k = \sum_{\alpha_1 < \dots < \alpha_k} \begin{vmatrix} a_1^{\alpha_1} & \dots & a_k^{\alpha_1} \\ \vdots & \ddots & \vdots \\ a_1^{\alpha_k} & \dots & a_k^{\alpha_k} \end{vmatrix} \sigma_{\alpha_1} \wedge \dots \wedge \sigma_{\alpha_k}$$

外微分形式 $\mathbb{R}^m = \{ (u^1, \dots, u^m) \mid u^\alpha \in \mathbb{R}, 1 \leq \alpha \leq m \}$

$$u = (u^1, \dots, u^m)$$

以 du^1, \dots, du^m 为基, 生成 \mathbb{R} 上的线性空间 $V(u)$

$$V(u) = L[u]$$

$$\triangleq \text{span} \{ du^1, \dots, du^m \}$$

$$\wedge(L[u]) \quad V_k = \wedge^k(V) = \wedge^k(L[u]), \quad k\text{-元} \frac{\pm}{\pm} \quad k\text{-空间}$$

$$\forall \varphi \in \wedge^k(L[u]).$$

$$\varphi = \sum_{\alpha_1 < \dots < \alpha_k} a_{\alpha_1 \dots \alpha_k} du^{\alpha_1} \wedge \dots \wedge du^{\alpha_k}$$

$$\text{当 } u \text{ 变化时, } u \in U \subset \mathbb{R}^m \quad \varphi(u) = \sum_{\alpha_1 < \dots < \alpha_k} a_{\alpha_1 \dots \alpha_k}(u) du^{\alpha_1} \wedge \dots \wedge du^{\alpha_k}$$

称为 U 上的 k 次微分形式

k -形式

$$\varphi(u) = \sum_{1 \leq \alpha_1 < \dots < \alpha_k \leq m} a_{\alpha_1 \dots \alpha_k}(u) du^{\alpha_1} \wedge \dots \wedge du^{\alpha_k}$$

$$= \sum_{1 \leq \alpha_1 < \dots < \alpha_k \leq m} \tilde{a}_{\alpha_1 \dots \alpha_k}(u) du^{\alpha_1} \wedge \dots \wedge du^{\alpha_k}$$

其中 $\tilde{a}_{\alpha_1 \dots \alpha_k}(u)$ 关于 $\alpha_1, \dots, \alpha_k$ 是反对称的.

$$a_{ij} = \underbrace{\frac{1}{2}(a_{ij} + a_{ji})}_{\text{对称}} + \underbrace{\frac{1}{2}(a_{ij} - a_{ji})}_{\text{反对称}}$$

一对一对去看只有反对称的部分
留了下来

$$= k! \sum_{1 \leq \alpha_1 \leq \dots \leq \alpha_k} a_{\alpha_1 \dots \alpha_k}(u) du^{\alpha_1} \wedge \dots \wedge du^{\alpha_k}$$