W2L1.

$\mathbb{C} \in$ 度量空间 & 拓扑空间 开、闭、紧 连通 等概念.

$\rho(z_1, z_2) = |z_1 - z_2| \Rightarrow$ 完备性.

闭区间套

$\sim \mathbb{R}^2$ ($\mathbb{R}^n$) 欧氏空间 有界闭 $\Leftrightarrow$ 紧.

有了空间的性质

$\Downarrow$

建立函数. 积分（测度）...

复值函数. 性质

$\mathbb{C}$ 紧化 $\Rightarrow \hat{\mathbb{C}} \simeq S^2$

在 S^2 上新的度量.

平面几何

球面几何.

1. ∞ $\quad \mathbb{C} \ni a \pm \infty = \infty$

$\frac{a}{\infty} = 0$

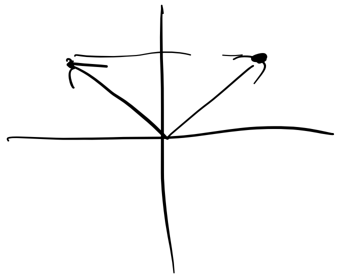
$0 \neq b \cdot \infty = \infty$

$\frac{b}{0} = \infty$

极限. $|z| \to +\infty$

$\Longrightarrow$

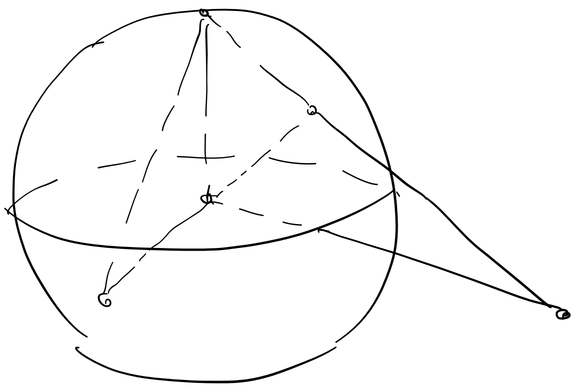
2. $$\frac{2x_1}{|z|^2+1} = -\frac{2x_2}{|w|^2+1}$$



$$\frac{2y_1}{|z|^2+1} = -\frac{2y_2}{|w|^2+1}$$

$$\frac{|z|^2-1}{|z|^2+1} = -\frac{|w|^2-1}{|w|^2+1}$$

$$|z|^2|w|^2 - \cancel{|w|^2} + \cancel{|z|^2} - 1 = (-1)\left(|z|^2|w|^2 + \cancel{|w|^2} - \cancel{|z|^2} - 1\right).$$



$$|z|^2|w|^2 = 1$$

$$|z||w| = 1$$

$$|z| = \frac{1}{|w|}$$

$$\Longrightarrow \frac{x_1}{\frac{1}{|w|^2}+1} = \frac{-x_2}{|w|^2+1} \qquad \frac{|w|^2 x_1}{1+|w|^2} = \frac{-x_2}{|w|^2+1}$$

$$|w|^2 x_1 = -x_2$$

$$\frac{y_1}{\frac{1}{|w|^2}+1} = \frac{-y_2}{|w|^2+1}$$

$$|w|^2 y_1 = -y_2$$

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}$$

$$\Longrightarrow z\cdot\bar{w} = (x_1+iy_1)(x_2-iy_2)$$

$$= x_1x_2 + (x_2y_1 - x_1y_2)i + y_1y_2$$

$$= x_1x_2 + y_1y_2$$

$$= |w|^2(-x_1^2) + |w|^2(-y_1^2)$$

$$= -|w|^2|z|^2 = -1$$

$\Longleftarrow$ 1. 有 $\begin{cases} x_1x_2+y_1y_2=-1 \\ x_2y_1-x_1y_2=0 \end{cases} \Rightarrow \dfrac{y_1}{y_2}=\dfrac{x_1}{x_2}=:t$

$|w||z|=1$

$$t x_2^2+t y_2^2=-1$$

$$\left(\frac{2x_1}{|z|^2+1},\ \frac{2y_1}{|z|^2+1},\ \frac{|z|^2-1}{|z|^2+1}\right)$$

考虑. $t=\dfrac{-1}{|w|^2}=-|z|^2$

$$\left(\frac{2x_2}{|w|^2+1},\ \frac{2y_2}{|w|^2+1},\ \frac{|w|^2-1}{|w|^2+1}\right)$$

$\Rightarrow x_1=-|z|^2x_2$

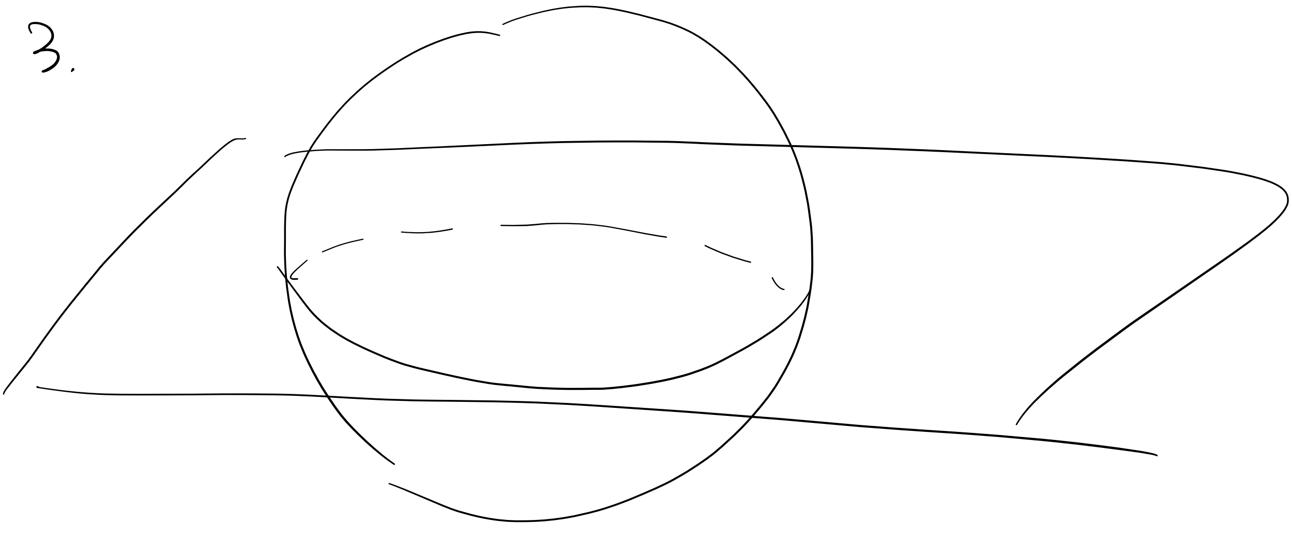
$y_1=-|z|^2y_2$

$$\Rightarrow \frac{|z|^2-1}{|z|^2+1}=-\frac{|w|^2-1}{|w|^2+1}$$

$$\frac{2x_1}{|z|^2+1}+\frac{2x_2}{|w|^2+1}=\frac{2(|w|^2x_1+x_2)}{|w|^2+1}$$

$\square$

3.



$$\varphi(z)=\varphi(x+iy)=\left(\frac{2x}{|z|^2+1},\ \frac{2y}{|z|^2+1},\ \frac{|z|^2-1}{|z|^2+1}\right)$$

$$\varphi\left(\frac{1}{\bar z}\right)=\varphi\left(\frac{1}{x-iy}\right)=\varphi\left(\frac{x+iy}{x^2+y^2}\right).$$

$$=\left(\frac{\frac{2x}{|z|^2}}{\frac{1}{|z|^2}+1},\ \frac{\frac{2y}{|z|^2}}{\frac{1}{|z|^2}+1},\ \frac{\frac{1}{|z|^2}-1}{\frac{1}{|z|^2}+1}.\right)$$

$$= \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{1-|z|^2}{1+|z|^2} \right)$$

4. $\phi(z) = f(z) + g(z) + h(z)$

$$\begin{cases} f(wz) = f(z) \\ g(wz) = wg(z) \\ h(wz) = w^2 h(z) \end{cases}$$

Construction

$$f(z) = \frac{\phi(z) + \phi(wz) + \phi(w^2 z)}{3}$$

$$h(z) = \frac{\phi(z) + w\phi(wz) + w^2\phi(w^2 z)}{3}$$

$\phi(wx) + w\phi(w^2 x) + w^2\phi(x)$

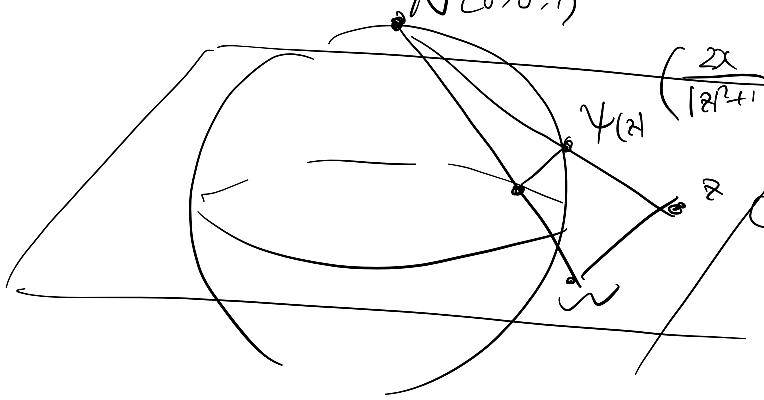
$$g(z) = \frac{\phi(z) + w^2\phi(wz) + w\phi(w^2 z)}{3}$$

唯一性？ 设. $\phi = f + g + h = f' + g' + h'$

$$f - f' = -(g - g') - (h - h')$$

由 f, f', g, g', h, h' 性质 $\Rightarrow f - f' = g - g' = h - h' = 0$

5. Nz. $N\Phi(z)$



$$\left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$

-2

$$(x^2+y^2+1)\cdot\left(\frac{4x^2+4y^2+4}{(|z|^2+1)^2}\right)$$

$$= (x^2+y^2+1)\,\frac{4(x^2+y^2+1)}{(x^2+y^2+1)^2} = 4$$