Folland

Chap. 1~7. Core materials for measure and integration theory.
point set topology
functional analysis

Chap 8~11. topics

§0.1. 0.5 terminology rest: referred to as needed.

"Note and References."

2.1. The language of set theory

$\mathbb{N}$. positive $0 \notin \mathbb{N}$.

- iff.

- A. B. $\neg A$ $\neg B$. (mathematical assertions)

$A \Rightarrow B$ iff. $\neg B \Rightarrow \neg A$

not same as reductio ad absurdum. :

assume A, $\neg B$. derive a contradiction

实际一样 ...

Sets. $\varnothing$. $\mathcal{P}(X) = \{E : E \subset X\}$

includes $E = X$.

$\mathcal{E}$ is a family of sets. $\bigcup_{E \in \mathcal{E}} E = \{x : x \in E \text{ for some } E \in \mathcal{E}\}$

$\bigcap_{E \in \mathcal{E}} E = \{x : x \in E \text{ for all } E \in \mathcal{E}\}$

---

indexed families of sets. $\mathcal{E} = \{E_\alpha : \alpha \in A\} = \{E_\alpha\}_{\alpha \in A}$

$\bigcup_{\alpha \in A} E_\alpha$

$\bigcap_{\alpha \in A} E_\alpha$

---

disjoint.

---

indexed by $\mathbb{N}$. $\{E_n\}_{n=1}^{\infty}$ $\{E_n\}_1^{\infty}$

limit superior. $\limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n = \{x : x \in E_n \text{ for infinitely many } n\}$

limit inferior $\liminf E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n = \{x : x \in E_n \text{ for all but finitely many } n\}$

E, F are sets   difference : $E \setminus F$

$= \{x : x \in E \text{ and } x \notin F\}$.

symmetric difference : $E \triangle F = (E \setminus F) \cup (F \setminus E)$.

E^c (in X). $E^c = X \setminus E$

deMorgan's law. $\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c$ $\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$.

Cartesian product. $X \times Y$.

(x, y).

relation. from X to Y. a subset of $X \times Y$.

If $Y = X$. speak of relation. on X.

R is a relation from X to Y write. xRy mean that $(x,y) \in R$.

- Equivalence relations.

$$\begin{cases} xRx \text{ for all } x \in X \\ xRy \text{ iff } yRx \\ xRz \text{ whenever } xRy \text{ and } yRz \text{ for some } y \end{cases}$$

The equivalence class of x : $\{y \in X : xRy\}$.

X is disjoint union of those equi. classes

- Orderings.

- Mappings. $f: X \to Y$.

  every $x \in X$. $\exists!\ y \in Y$ that xRy

  $y = f(x)$

  - $f: X \to Y$. $g: Y \to Z$. are mappings. $g \circ f$: composition.

    $g \circ f: X \to Z$. $g \circ f(x) = g(f(x))$

  - if $D \subset X$ and $E \subset Y$.

    image of D : $f(D) = \{f(x) : x \in D\}$

    inverse image of E : $f^{-1}(E) = \{x : f(x) \in E\}$.

$f^{-1} : P(Y) \to P(X)$. Commutes with union, intersections and complements

$$f^{-1}\left(\bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(E_\alpha) \qquad f^{-1}\left(\bigcap_{\alpha \in A} E_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(E_\alpha).$$

$$f^{-1}(E^c) = \left(f^{-1}(E)\right)^c$$

$$f : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y).$$

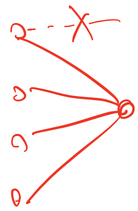
$$f\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f(E_\alpha). \quad \checkmark$$

$$f\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} f(E_\alpha) \quad \times$$

$$f(E^c) = \left(f(E)\right)^c \quad \times$$

?

Mapping 's def!



$f : X \to Y.$ $\quad$ X: domain of f

$f(X)$: range of f.

injective : $f(x_1) = f(x_2)$ only when $x_1 = x_2$

surjective : $f(X) = Y$

bijective .

$\Downarrow$

$\exists\, f^{-1} \; Y \to X.$ s.t. $\underset{\displaystyle id_X}{\underset{\shortparallel}{f^{-1}\circ f}} \quad \underset{\displaystyle id_Y.}{\underset{\shortparallel}{f\circ f^{-1}}}$

$A \subset X.$ $\quad (f|_A) : A \to Y$

$(f|_A)(x) = f(x)$ for $x \in A.$

Sequence. mapping from $\mathbb{N} \to X$.

finite. sequence. $\{1, \dots, n\} \to X$.

if $f: \mathbb{N} \to X$ is a sequence and $g: \mathbb{N} \to \mathbb{N}$ satisfies $g(n) < g(m)$. whenever $n<m$.

$\Rightarrow$ $f \circ g$: subseqence. of f.

$f(n) = x_n$. speak of. sequence $\{x_n\}_1^\infty$.

$\{X_\alpha\}_{\alpha \in A}$ is an indexed family of sets.

their Cartesian product $\prod_{\alpha \in A} X_\alpha$. is the set of all maps.

$$f: A \to \bigcup_{\alpha \in A} X_\alpha. \quad \text{s.t. } f(\alpha) \in X_\alpha. \quad \forall \alpha \in A$$

前面定义的 $X_1 \times X_2$

是这里 $f: \{1, 2\} \to X_1 \cup X_2$ 的 $(f(1), f(2))$.

if $X = \prod_{\alpha \in A} X_\alpha$, $\alpha \in A$.

we define the α th projection or coordinate map $\pi_\alpha: X \to X_\alpha$.

by $\pi_\alpha(f) = f_\alpha$

write x. or x_α instead of f and $f(\alpha)$.

Call x_α. the α th coordinate of x.

if $X_\alpha \equiv Y$. $\prod_{\alpha \in A} X_\alpha = Y^A$.

Y^A : set of all mappings from A to Y.

if $A = \{1, \dots, n\}$. Y^A is denoted by Y^n.

0.2 Orderings

A partial ordering on a nonempty set X is a Relation on X.

- if xRy and yRz $\Rightarrow$ xRz
- if xRy and yRx $\Rightarrow$ $x=y$
- if xRx for all x.

---

if R also satisfies

- if $x, y \in X$ then either xRy or yRx.

R: linear (total ordering)

e.g. X a set. $\mathcal{P}(E)$: partially ordered by inclusion.

$\mathbb{R}$. linearly ordered

R: denote by $\leq$

$<$: $\leq$ & $\neq$.

---

order isomorphic if.

$\exists$ bijection. $f: X \to Y$. s.t. $x_1 \leq x_2$ iff $f(x_1) \leq f(x_2)$

X : $\leq$. partially ordered.

a maximal (resp. minimal) element of X is an $x \in X$

s.t. only $y \in X$. satisfying $x \leq y$ (resp. $x \geq y$) is x itself.

if $E \subset X$. an upper bound for E is an element $x \in X$.

s.t. $y \leq x$ for all $y \in E$.

If. X is linearly ordered by $\leq$.

and every nonempty subsets of X. has a (necess unique). minimal element.

$\Rightarrow$ X is said to be well ordered by $\leq$

$\mathbb{N}$ for example ✓

0.1. (The Hausdorff Maximal Principle.)

Every partially ordered set has a maximal linearly ordered subset.

$\Updownarrow$

0.2 (Zorn's Lemma). If X is a partially ordered set and every linearly ordered subset of X has an upper bound. then

X has a maximal element

0.1 $\Rightarrow$ 0.2 X have a maximal linearly ordered subset

then its upper bound. is maximal.

0.2 $\Rightarrow$ 0.1 $\{$ all linearly ordered subset of X. $\} = S$

$\downarrow$

partially ordered. by inclusion.

every linearly ordered subsets of S

has an upper bound ?

$\hookrightarrow$ union of all elements

$\Rightarrow \exists$ maximal. element $\square$

0.3. (The well ordering Principle). Every nonempty set X can be well ordered.

pf by zorn lemma

0.4 (The Axiom of Choice) ? actually logically equivalent with 0.1 & 0.2

If $\{X_\alpha\}_{\alpha\in A}$ is a nonempty collection of nonempty sets

then $\prod_{\alpha\in A} X_\alpha$ is nonempty

pf. Let $X = \bigcup_{\alpha\in A} X_\alpha$. Pick a well ordering on X.

and for $\alpha \in A$. let $f(\alpha)$ be the minimal element of X_α Then $f \in \prod_{\alpha \in A} X_\alpha$.

0.5 Corollary

If $\{X_\alpha\}_{\alpha \in A}$ is a disjoint collection of nonempty sets, there is a set $Y \subset \bigcup_{\alpha \in A} X_\alpha$ s.t. $Y \cap X_\alpha$ contains precisely one element for each $\alpha \in A$

Pf. Take $Y = f(A)$. where $f \in \prod_{\alpha \in A} X_\alpha$