W3L2

Prop. 直纹面是可展曲面，则必定是柱面或锥面或某一曲线的切线曲面

设 $x(u,v)=a(u)+v\,b(u)$ 为可展曲面. 有 $(a',b,b')=0$

由 $|b|=1 \Rightarrow b'\cdot b=0$. $b'\perp b$

① if $b\times b'=0 \Leftrightarrow b'\parallel b \Leftrightarrow b'=0 \Rightarrow$ 柱面

② if $b\times b'\neq 0$ $b\neq$ 常向量

将原方程写成 $x(u,v)=C(u)+\tilde{v}\,b(u)$

s.t. $C'(u)\parallel b(u) \Rightarrow C'(u)\cdot b'(u)=0$

令 $C(u)=a(u)+f(u)\,b(u)$ f 待定.

$C'(u)=a'(u)+f'(u)\,b(u)+f(u)\,b'(u)$

乘 b' $0=a'\cdot b'+f\cdot|b'|^2 \rightarrow f(u)=\dfrac{a'\cdot b'}{|b'|^2}$

$$x(u,v)=\underbrace{a(u)+f(u)b(u)}_{C(u)}-\underbrace{f(u)b(u)+v\,b(u)}_{\underbrace{(v-f(u))}_{\tilde{v}}b(u)}$$

$x(\tilde{u},\tilde{v})=C(\tilde{u})+\tilde{v}\,b(\tilde{u})$ $\begin{cases}\tilde{u}=u\\ \tilde{v}=v-f(u)\end{cases}$ 参数变换

验证: $(c',b,b')=(a'+f'b+fb',b,b')=0$

此时.若 1) $C'(u)\neq 0$

$c'\cdot b'=0$ $b'\cdot b=0$

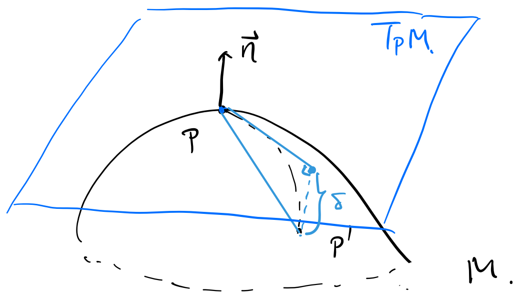
又由 $(c',b,b')=0 \Rightarrow c'\parallel b \Rightarrow$ 切线面

2) $C'(u)=0$

$\Rightarrow$ 锥面

下周课堂练习. 周四

曲面的第二基本形式



$P(u^1, u^2)$

$P'(u^1+\Delta u^1, u^2+\Delta u^2)$

$$\overrightarrow{PP'} = \Delta x = x(u^1+\Delta u^1, u^2+\Delta u^2) - x(u^1, u^2)$$

$$= x_1(u^1,u^2)\Delta u^1 + x_2(u^1,u^2)\Delta u^2 + \frac{1}{2}\left(x_{11}(u^1,u^2)(\Delta u^1)^2 + 2x_{12}(u^1,u^2)(\Delta u^1)(\Delta u^2) + x_{22}(u^1,u^2)(\Delta u^2)^2\right) + \cdots$$

$$\delta = \overrightarrow{PP'}\cdot\vec{n} = \frac{\overrightarrow{PP'}\cdot(x_1\times x_2)}{|x_1\times x_2|} = \frac{1}{2}\left(x_{11}\cdot n(\Delta u^1)^2 + 2x_{12}\cdot n\,\Delta u^1\Delta u^2 + x_{22}n(\Delta u^2)^2\right) + \cdots$$

$$2\delta \simeq x_{11}\cdot n(\Delta u^1)^2 + 2x_{12}\cdot n\,\Delta u^1\Delta u^2 + x_{22}n(\Delta u^2)^2$$

$$\begin{cases} x_{11}\cdot n = h_{11} = L \\ x_{12}\cdot n = h_{12} = M \\ x_{22}n = h_{22} = N \end{cases} \qquad x_{\alpha\beta}\cdot n = h_{\alpha\beta}$$

$$= h_{11}(\Delta u^1)^2 + 2h_{12}\Delta u^1\Delta u^2 + h_{22}(\Delta u^2)^2$$

记 2δ 的主要部分 ⇒ 二次微分形式

$$\mathrm{II} = h_{11}(du^1)^2 + 2h_{12}du^1du^2 + h_{22}(du^2)^2$$

$$= h_{\alpha\beta}du^\alpha du^\beta$$

$$x_\alpha\cdot n = 0 \Rightarrow x_{\alpha\beta}\cdot n + x_\alpha\cdot n_\beta = 0$$

$$h_{\alpha\beta} = x_{\alpha\beta}\cdot n = -x_\alpha n_\beta = -x_\beta n_\alpha$$

$$\mathrm{II} = -(x_\alpha\cdot n_\beta)du^\alpha du^\beta = -(dx, dn)$$

$$h_{\alpha\beta} = x_{\alpha\beta}\cdot\frac{x_1\times x_2}{|x_1\times x_2|} = \frac{(x_1, x_2, x_{\alpha\beta})}{|x_1\times x_2|}$$

Ex 旋转面

$$x(u^1,u^2) = (f(u^2)\cos u^1, f(u^2)\sin u^1, g(u^2))$$

$$x_1 = (-f\sin u^1, f\cos u^1, 0)$$

$$x_2 = (f'\cos u^1, f'\sin u^1, g')$$

$$g_{11} = f^2 \cdot g_{12} = 0 \quad g_{22} = (f')^2 + (g')^2$$

$$n = \frac{x_1 \times x_2}{|x_1 \times x_2|}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -f\sin u^1 & f\cos u^1 & 0 \\ f'\cos u^1 & f'\sin u^1 & g' \end{vmatrix}$$

$$= \frac{(g'\cos u^1, g'\sin u^1, -f')}{\sqrt{(f')^2 + (g')^2}}$$

$$x_{11} = (-f\cos u^1 . -f\sin u^1, 0)$$

$$x_{12} = (-f'\sin u^1, f'\cos u^1, 0)$$

$$x_{22} = (f''\cos u^1, f''\sin u^1, g'')$$

$$h_{11} = x_{11} \cdot n = \frac{-g'f}{\sqrt{(f')^2 + (g')^2}}$$

$$h_{12} = 0$$

$$h_{22} = \frac{f''g' - f'g''}{\sqrt{(f')^2 + (g')^2}}$$

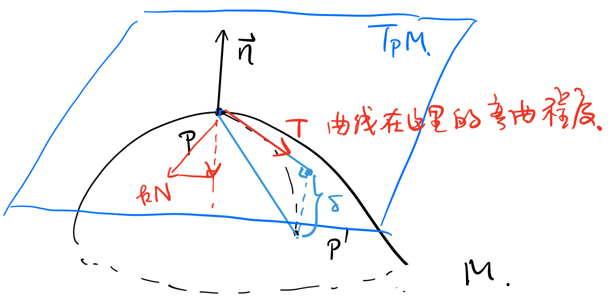
取 $\begin{cases} f = r\cos u^2 \\ g = r\sin u^2 \end{cases} \Rightarrow$ 球面

$$\begin{cases} g_{22} = r^2 \\ g_{12} = 0 \\ g_{11} = r^2\cos^2 u^2 \end{cases} \quad \begin{cases} h_{22} = -r \\ h_{12} = 0 \\ h_{11} = -r\cos^2 u^2 \end{cases}$$

$$h_{\alpha\beta} = -\frac{1}{r} g_{\alpha\beta}$$

$$\mathrm{II} = h_{\alpha\beta}\, du^\alpha du^\beta = -(dx, dn)$$

另法 ☆ 用嵌入曲线去研究.



$C \subset M.$ $\quad x(s) = x(u^1(s), u^2(s))$

决定T方向

$$T = x_1 \frac{du^1}{ds} + x_2 \frac{du^2}{ds} = x_\alpha \frac{du^\alpha}{ds}$$

$$kN = \dot{T} = \frac{d}{ds}\left(x_\alpha \frac{du^\alpha}{ds}\right) = x_{\alpha\beta} \frac{du^\alpha}{ds}\frac{du^\beta}{ds} + x_\alpha \frac{d^2u^\alpha}{ds^2}$$

$$k_n(T) := (kN\cdot)n = (x_{\alpha\beta} n)\frac{du^\alpha}{ds}\frac{du^\beta}{ds}$$

$$= h_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}$$

$$= \frac{h_{\alpha\beta}\, du^\alpha du^\beta}{ds^2} = \frac{\mathrm{II}}{\mathrm{I}} = \frac{\mathrm{II}(du^1, du^2)}{\mathrm{I}(du^1, du^2)} = \mathrm{II}\left(\frac{du^1}{ds}, \frac{du^2}{ds}\right)$$

沿着T方向的法曲率

如果 $v = x_1\lambda + x_2\mu$ 单位向量

$$k_n(v) = \frac{\mathrm{II}(\lambda,\mu)}{\mathrm{I}(\lambda,\mu)} = \frac{h_{11}\lambda^2 + 2h_{12}\lambda\mu + h_{22}\mu^2}{g_{11}\lambda^2 + 2g_{12}\lambda\mu + g_{22}\mu^2} = h_{11}\lambda^2 + 2h_{12}\lambda\mu + h_{22}\mu^2$$

$$\|v\|^2 = 1 = (x_1\lambda + x_2\mu)(x_1\lambda + x_2\mu) = \mathrm{I}(\lambda,\mu)$$

若 $v = \lambda x_1 + \mu x_2$ 不为单位向量

$$k_n(v) = \frac{\mathrm{II}(\lambda,\mu)}{\mathrm{I}(\lambda,\mu)} = \frac{\mathrm{II}(\lambda,\mu)}{\|v\|^2}$$

Meusnier定理 $\quad k_n$：仅与T有关

△ 第二基本形式与参数选择无关

$M: \ x = x(u^1, u^2)$

$= \bar{x}(\bar{u}^1, \bar{u}^2)$ $\quad \begin{cases} \bar{u}^1 = \bar{u}^1(u^1, u^2) \\ \bar{u}^2 = \bar{u}^2(u^1, u^2) \end{cases}$

$\mathrm{II} = h_{\alpha\beta} du^\alpha du^\beta$ 　　已有 $n(u^1, u^2) = \bar{n}(\bar{u}^1, \bar{u}^2)$

$= \bar{h}_{\bar{\alpha}\bar{\beta}} d\bar{u}^\alpha d\bar{u}^\beta$

$\bar{h}_{\bar{\alpha}\bar{\beta}} = \bar{x}_{\bar{\alpha}\bar{\beta}} n = \frac{\partial}{\partial \bar{u}^\beta}\left(\frac{\partial \bar{x}}{\partial \bar{u}^\alpha}\right) n = \frac{\partial}{\partial \bar{u}^\beta}\left(\frac{\partial \bar{x}}{\partial u^\gamma}\frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\right) n$

$= \frac{\partial}{\partial u^\sigma}\left(\frac{\partial \bar{x}}{\partial u^\gamma}\frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\right)\frac{\partial u^\sigma}{\partial \bar{u}^\beta} \cdot n$

$= \cdots = \bar{x}_{\sigma\gamma} \frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\frac{\partial u^\sigma}{\partial \bar{u}^\beta}.$ ✓

$\mathrm{I} = g_{\alpha\beta} du^\alpha du^\beta$ 正定

$(g_{\alpha\beta})^{-1} =: (g^{\alpha\beta})$ ← 矩阵的逆

记 $g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma$

$\Leftrightarrow G G^{-1} = I$

Weingarten 变换. $T_pM \to T_pM$

$\{x_\alpha\}$

$W(x_\alpha) = h_\alpha^\beta x_\beta \qquad h_\alpha^\beta = h_{\alpha\nu} g^{\nu\beta}$

↓

自共轭