

W1L1

0, 1
局部理论

2
曲线
整体

3
曲面
整体

作业 12/14 100 } 50%
小测 3次 300 }

期末 50%

E^3 中曲线

$\{0; E_1, E_2, E_3\}$ 右手系

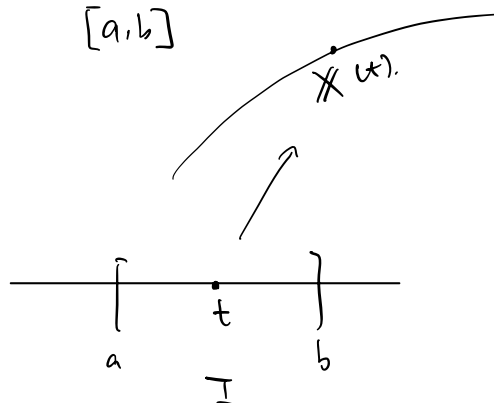
$$\langle E_i, E_j \rangle = \delta_{ij}$$

$$\forall \vec{x} \in E^3 \quad x = x^1 E_1 + x^2 E_2 + x^3 E_3 = \sum_{i=1}^3 x^i E_i = x^i E_i$$

曲线 C 是 $\gamma: I \rightarrow E^3$ 同胚.

\parallel
 (a, b)
 $[a, b]$
 (a, b)
 $[a, b]$

γ : 连续 γ^{-1} : 连续
 γ : 1-1 映射



局部 γ . $\gamma: I \rightarrow E^3$

$$t \mapsto \gamma(t) \in E^3$$

$$\gamma(t) = \gamma^i(t) E_i = (\gamma^1(t), \gamma^2(t), \gamma^3(t))$$

$X(t)$ 可微 $\Leftrightarrow x^i(t)$ 可微 $i=1,2,3$

$$\frac{dX}{dt} = \frac{dx^i}{dt} E_i$$

正则曲线 如果曲线 $C: X=X(t)$, 是可微的 且 $\|X'(t)\| \neq 0, \forall t \in I$.
 > 0 .

称 C 是正则曲线.

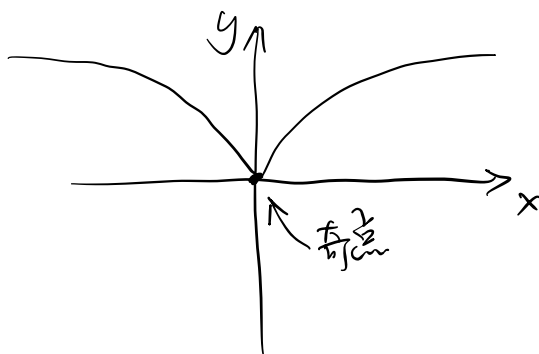
$$\left\| \frac{dX}{dt} \right\|^2 = \sum_{i=1}^3 \left(\frac{dx^i}{dt} \right)^2 > 0 \quad (\text{i.e., } \exists i, \frac{dx^i}{dt} \neq 0, \forall t \in I)$$

若 $\exists t_0 \in I, \left\| \frac{dX}{dt}(t_0) \right\| = 0$, 则称 C 在 $X(t_0)$ 点是奇异的.

↓
 奇点 \rightarrow 叉点.

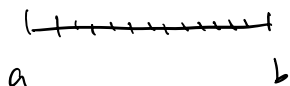
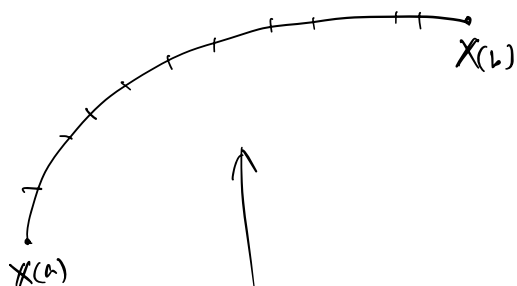
e.g. $X(t) = (t^3, t^2, 0)$

$$y = x^{\frac{2}{3}}$$



$X'(t) = \frac{dX}{dt}$ 称为曲线 C 在 $X(t)$ 处 切向量

弧长 $C: X: I=(a,b) \rightarrow X(I) \subset E^3$



$$L(X, A) = \sum_{i=1}^n |X(t_i) - X(t_{i-1})|$$

A 分割 $a=t_0 < \dots < t_n=b$

A 范数.

$$|A| = \max_i |t_i - t_{i-1}|$$

$$S = \lim_{|A| \rightarrow 0} l(s, A)$$

下面 $x(t)$ 与 $x(s)$ 同义!

$$= \lim_{|A| \rightarrow 0} \sum_{i=1}^n |x(t_i) - x(t_{i-1})|$$

$$= \lim_{|A| \rightarrow 0} \sum_{i=1}^n \frac{|x(t_i) - x(t_{i-1})|}{t_i - t_{i-1}} (t_i - t_{i-1})$$

$$= \int_a^b \left| \frac{dx}{dt} \right| dt$$

中值定理 \Rightarrow 积分定义

$$S(t) = \int_a^t \left| \frac{dx}{dt} \right| dt \quad . \text{ 上限: } t \text{ 的可微函数.}$$

Prop. S 与参数 t 的选取无关

证.

$$t = t(\tau) \Rightarrow S(t) = S(t(\tau))$$

$0 = t(0)$ (assume) (同起点)

$$S(\tau) = \int_0^\tau \left| \frac{dx}{d\tau} \right| d\tau$$

$\frac{dt}{d\tau} > 0$ (正比)

$$= \int_0^t \left| \frac{dx}{dt} \frac{dt}{d\tau} \right| \frac{d\tau}{dt} dt$$

$$= \int_0^t \left| \frac{dx}{dt} \right| dt \quad \square$$

拿出 $\frac{dt}{d\tau}$

另取. 弧长为参数 $x = x(s)$

$$\frac{ds}{dt} = \left| \frac{dx}{dt} \right| = \frac{ds}{ds} = 1 \Rightarrow \left| \frac{dx}{ds} \right| = 1. \text{ 单位向量}$$

反之 \Leftarrow . 若 $\left| \frac{dx}{ds} \right| = 1$. $S(t) = \int_0^t \left| \frac{dx}{ds} \right| ds = t \Rightarrow$ 弧长为参数.

Prop. 曲线的参考量为弧长的充要条件为 $|\frac{dx}{dt}|=1, \forall t \in I$

Prop. 弧长与坐标选取无关

$\{E_i\}, \{e_i\}$ 正标架.

$E_i = a_i^j e_j$ $A = \{a_i^j\}$ 正交矩阵.

review

$$A^T A = I$$

$$(E_1 \ E_2 \ E_3) = (e_1 \ e_2 \ e_3) A$$

$$= (e_1 \ e_2 \ e_3) \begin{pmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{pmatrix}$$

$$x(t) = x^i(t) E_i = y^j(t) e_j$$

$$\Rightarrow y^j(t) = x^i(t) a_i^j$$

$$= x^i(t) a_i^j e_j$$

$$S(t) = \int_0^t \left(\sum_{i=1}^3 \left(\frac{dy^i}{dt} \right)^2 \right)^{\frac{1}{2}} dt \neq \int_0^t \left(\sum_{i=1}^3 \left(\frac{dx^i}{dt} \right)^2 \right)^{\frac{1}{2}} dt$$

||

$$\int_0^t \left(\left(\frac{dy^1}{dt} \right)^2 + \left(\frac{dy^2}{dt} \right)^2 + \left(\frac{dy^3}{dt} \right)^2 \right)^{\frac{1}{2}} dt$$

$$= \int_0^t \dots \text{全部展开} = \int_0^t \left(\sum_{i=1}^3 \left(\frac{dx^i}{dt} \right)^2 \right)^{\frac{1}{2}} dt \quad \square$$

以弧长为参数

$$\frac{dx}{ds} = \dot{x} \quad \frac{d^2x}{ds^2} = \ddot{x} \quad \frac{d^3x}{ds^3} = \dddot{x} \quad \frac{d^4x}{ds^4} = x^{(4)}$$

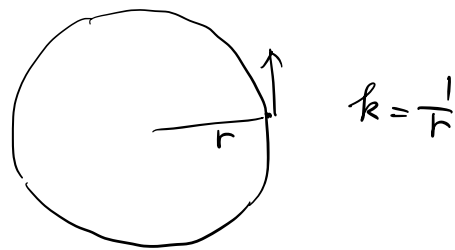
正则曲线. $\frac{dx}{ds} = \dot{x} = T(s)$ (单位切向量)

$T(s)$ 的变化率刻画曲线弯曲程度.

$$|\dot{x}| = |T(s)| = k(s) \quad \text{曲率}$$

$$\geq 0$$

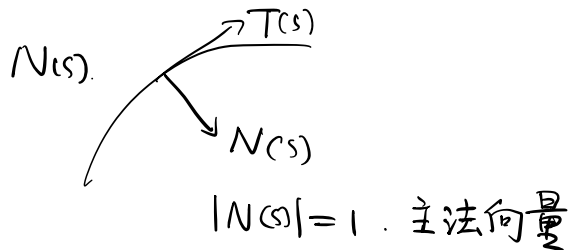
若 $k(s) \neq 0$ 称 $\frac{1}{k(s)}$ 为曲率半径



$$|T(s)| = 1 \Leftrightarrow T \cdot T = 1$$

$$\Rightarrow \dot{T} \cdot T = 0 \text{ 正交}$$

$$\dot{T}(s) = k(s) N(s)$$



$$\rho(s) = \frac{1}{k(s)} \quad \rho(s) N(s) = P(s) : \text{曲率中心}$$

$B = T \times N$ $\{T, N, B\}$ 右手系标架.
 \downarrow
 从法向量.

$\text{span}(T, N)$ 密切平面

$\text{span}(N, B)$ 法平面

$\text{span}(B, T)$ 从切平面

Frenet 标架.. 活动标架.

$$\dot{T} = k(s) N(s)$$

$$B \cdot B = 1 \Rightarrow \dot{B} \cdot B = 0$$

$$B \cdot T = 0 \Rightarrow \dot{B} \cdot T + B \cdot \dot{T} = 0$$

$$\dot{B} \cdot T = -B \cdot \dot{T} = -B \cdot k N = 0$$

$$\Rightarrow \dot{B} \parallel N \quad \text{令 } \dot{B} = -\tau(s) N$$

$\tau(s)$ 挠率.

若 $k(s) \neq 0 \Rightarrow N(s)$ 唯一. $\Rightarrow \dot{B}$ 唯一.
 \downarrow
 若 $k=0$, $N(s)$ 的选取任意. ...

$$-\dot{B} \cdot N = \tau(s)$$

$$B \cdot N = 0 \Rightarrow \dot{B} \cdot N + B \cdot \dot{N} = 0 \Rightarrow \tau(s) = B \cdot \dot{N}$$

$$\begin{aligned}
&= B \left(\frac{1}{k} \dot{T} \right)' \\
&= (T \times N) \cdot \left[\left(\frac{1}{k} \right)' \dot{T} + \frac{1}{k} \ddot{T} \right] \\
&= \frac{1}{k} (T \times N) \ddot{T} \\
&= \frac{1}{k} \left(T \times \frac{1}{k} \dot{T} \right) \ddot{T} \\
&= \frac{1}{k^2} (T, \dot{T}, \ddot{T}) \\
&= \frac{1}{|\ddot{x}|^2} (\dot{x}, \ddot{x}, \ddot{x})
\end{aligned}$$

"几何量" 与参数、坐标系选取无关. (空间运动不变量).

$$\{O_1, E_1, E_2, E_3\}$$

$$\{O_2, e_1, e_2, e_3\}$$

$$C: x_1(s), x_2(s)$$

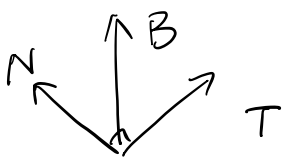
$$\Rightarrow x_2(s) = A x_1(s) + B \quad \left. \begin{array}{l} A: \text{正交} \\ B: \text{常向量} \end{array} \right\} \text{在求导下, } A, B \text{ 不变.}$$

$$k(s): \ddot{x}_2(s) = A \ddot{x}_1(s)$$

$$|\ddot{x}_2(s)|^2 = |A \ddot{x}_1(s)|^2 = \ddot{x}_1(s)^T A^T A \ddot{x}_1(s) = \ddot{x}_1(s)^T \ddot{x}_1(s)$$

=

$$\tau(s): \frac{(\dot{x}_2, \ddot{x}_2, \ddot{x}_2)}{|\ddot{x}_2|^2} = \frac{(A\dot{x}_1, A\ddot{x}_1, A\ddot{x}_1)}{|\ddot{x}_1|^2} \rightarrow \text{想几何意义不变.}$$



$$\parallel \frac{(\dot{x}_1, \ddot{x}_1, \ddot{x}_1)}{|\ddot{x}_1|^2}$$

$$N \cdot N = 1 \Rightarrow \dot{N} \cdot N = 0$$

另一种证法. 设 $\dot{N} = aT + bB$

$$N(s) = B(s) \times T(s)$$

$$\dot{N}(s) = \dot{B} \times T + B \times \dot{T} = -\tau N \times T + B \times kN = \tau B - kT$$

$$a = \dot{N} \cdot T = -N \cdot \dot{T} = -k(s)$$

$$b = \dot{N} \cdot B = -N \cdot \dot{B} = \tau(s)$$

Conclusion

1. 曲线表示 Frenet 标架.

2. "几何量". 弧长. 曲率. 挠率. 与坐标选取. 与参数选取无关.

△ 正交矩阵复习. ✓?

$$\Delta (a \times b) \cdot c = (a, b, c) = \det(a \ b \ c)$$