

W2.L1

曲线论基本定理.

$K(s)$  可微,  $\tau(s)$  连续

ODE.

$$\begin{cases} x = x(s) \\ \dot{x}(s) = T(s) \\ \dot{T}(s) = K(s)N(s) \\ \dot{N}(s) = -K(s)T(s) + \tau(s)B(s) \\ \dot{B}(s) = -\tau(s)N(s) \end{cases}$$

$$\{x(s); T(s), N(s), B(s)\}$$

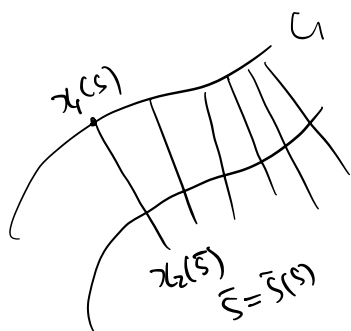
初值条件.

$$\begin{cases} x(0) = x_0 \\ T(0) = T_0 \\ N(0) = N_0 \\ B(0) = B_0 \end{cases}$$

$\Rightarrow$  有解.



Bertrand. 曲线



$x_1(s)$  与  $x_2(s)$  处相同  $N$

(沿线) 证. 对应点间距离相同

对应点切向呈定角

Pl.  $C_1: x = x_1(s)$ ,  $s$  为弧长

$N_1(s)$

$C_2: x = x_2(\bar{s})$ ,  $\bar{s}$  为弧长

$N_2(\bar{s}(s))$

对应关系  $\bar{s} = \bar{s}(s)$  有  $N_2(\bar{s}(s)) = \lambda N_1(s)$

$$x_2(\bar{s}(s)) = x_1(s) + \lambda(s) N_1(s)$$

对  $s$  求导.

$$T_2(\bar{s}) \frac{d\bar{s}}{ds} = T_1(s) + \dot{\lambda} N_1 + \lambda(-K_1(s)T_1(s) + \tau_1(s)B_1(s))$$

$$\text{点乘 } N_1(s) \Rightarrow 0 = 0 + \dot{\lambda} + 0 \Rightarrow \dot{\lambda} = 0 \Rightarrow \lambda(s) \text{ 为常数.}$$

$$\frac{d}{ds} (T_1(s) \cdot T_2(\bar{s}(s))) = k(s) \underbrace{N_1(s) \cdot T_2(\bar{s}(s))}_{=0} + \underbrace{T_1(s) \cdot N_2(\bar{s})}_{=0} \frac{d\bar{s}}{ds} = 0$$

$$\Rightarrow T_1(s) \cdot T_2(\bar{s}(s)) = \cos \theta \text{ 为常数}$$

## 曲面论

$D \subset E^2$ .  $D$  为 连通区域 坐标为  $u^1, u^2$

$x: D \rightarrow E^3$  同胚.

$$x(u, v) = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2))$$

$C^k$  曲面  $M$ . ( $k \geq 2$ ): 满足 ①  $x^i(u^1, u^2)$  具有  $k$  阶连续偏导.

② Jacobian 矩阵.

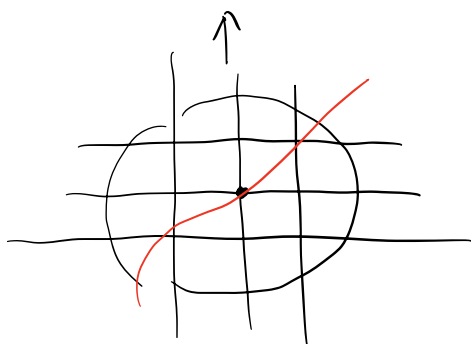
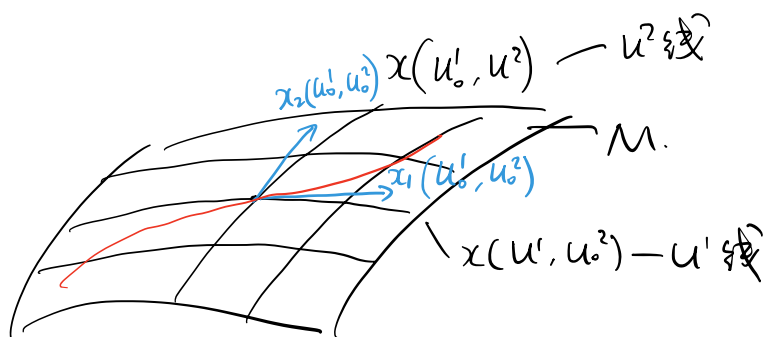
$$\begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^2}{\partial u^1} & \frac{\partial x^3}{\partial u^1} \\ \frac{\partial x^1}{\partial u^2} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^3}{\partial u^2} \end{pmatrix}_{2 \times 3} \triangleq \left( \frac{\partial x^i}{\partial u^a} \right)_{\substack{1 \leq i \leq 3 \\ 1 \leq a \leq 2}}$$

$x_1 = \frac{\partial x}{\partial u^1}$   
 $x_2 = \frac{\partial x}{\partial u^2}$

的秩处处为 2

$\Leftrightarrow x_1$  与  $x_2$  线恒无关

$$\Leftrightarrow x_1 \times x_2 \neq 0$$



构成参数曲线网

$\text{span} \{x_1, x_2\}$   
切平面  $T_{x(u_0^1, u_0^2)} M$

单位法向量.  $n(u^1, u^2) = \frac{x_1 \times x_2}{|x_1 \times x_2|} (u^1, u^2) \Rightarrow$  构成向量场 (光滑)

参数表示不唯一.

参数变换.  $\bar{u}^\alpha = \bar{u}^\alpha(u^1, u^2)$

$\frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(u^1, u^2)} \neq 0$ . 则切平面不变. 单位法向量最多差一个符号.

若  $\frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(u^1, u^2)} > 0 \Rightarrow$  单位法向量不变

$$x = x(u^1, u^2) = \bar{x}(\bar{u}^1, \bar{u}^2) = \bar{x}(\bar{u}^1(u^1, u^2), \bar{u}^2(u^1, u^2))$$

$$\left\{ \begin{array}{l} \frac{\partial x}{\partial u^1} = \frac{\partial \bar{x}}{\partial \bar{u}^1} \frac{\partial \bar{u}^1}{\partial u^1} + \frac{\partial \bar{x}}{\partial \bar{u}^2} \frac{\partial \bar{u}^2}{\partial u^1} \\ \frac{\partial x}{\partial u^2} = \frac{\partial \bar{x}}{\partial \bar{u}^1} \frac{\partial \bar{u}^1}{\partial u^2} + \frac{\partial \bar{x}}{\partial \bar{u}^2} \frac{\partial \bar{u}^2}{\partial u^2} \end{array} \right. \quad \begin{array}{l} (x_1, x_2) \\ \parallel \\ (\frac{\partial x}{\partial u^1}, \frac{\partial x}{\partial u^2}) \end{array} \quad \begin{array}{l} (\bar{x}_1, \bar{x}_2) \\ \parallel \\ (\frac{\partial \bar{x}}{\partial \bar{u}^1}, \frac{\partial \bar{x}}{\partial \bar{u}^2}) \end{array}$$

相互线性表示  $\Rightarrow \text{span}\{x_1, x_2\} = \text{span}\{\bar{x}_1, \bar{x}_2\}$

$$x_1 \times x_2 = (\bar{x}_1 \times \bar{x}_2) \frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(u^1, u^2)}$$

曲面  $M$  上的一条曲线  $C$ .

$$I \longrightarrow x(I) \subset M.$$

$$t \longmapsto (u^1(t), u^2(t)) \xrightarrow{x} x(u^1(t), u^2(t))$$

$C$  的弧长.  $s(t) = \int_0^t \left| \frac{dx}{dt} \right| dt$

$$\Rightarrow \frac{ds}{dt} = \left| \frac{dx}{dt} \right|$$

即. 在  $u^1, u^2$  平面中.  $du^1, du^2 \Rightarrow M$  中的  $ds$  大小

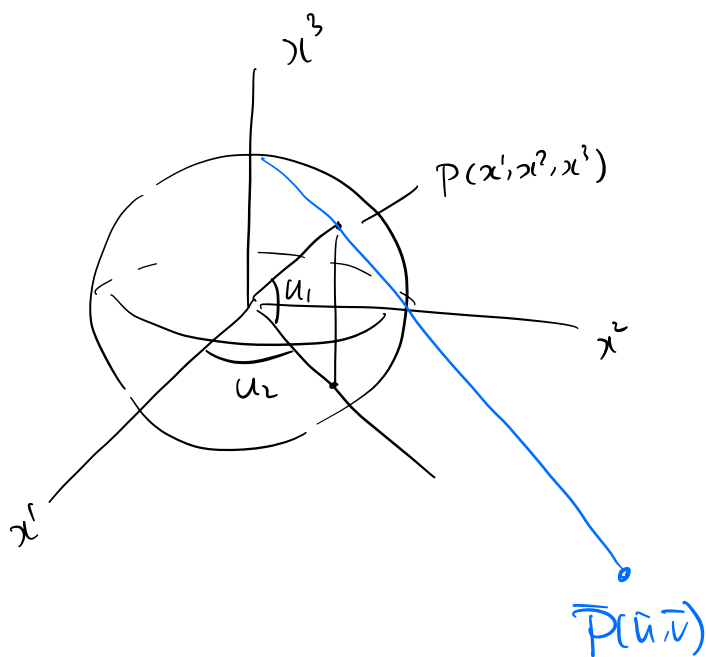
$$I = ds^2 = |dx|^2 = |x_1 du^1 + x_2 du^2|^2 = x_1 \cdot x_1 du^1 + 2x_1 x_2 du^1 du^2 + x_2 \cdot x_2 du^2$$

$$\sum x_\alpha x_\beta = g_{\alpha\beta} = g_{11} du^1 + 2g_{12} du^1 du^2 + g_{22} du^2$$

$$\left( \text{第一基本形式} \right) = g_{\alpha\beta} du^\alpha du^\beta \quad \text{对称张量} \quad \triangle$$

$$g_{11} = E, \quad g_{12} = F, \quad g_{21} = G$$

球面



$$-\frac{\pi}{2} \leq u^1 \leq \frac{\pi}{2}$$

$$0 \leq u^2 \leq 2\pi$$

$$x^3 = r \cdot \sin u^1$$

$$x^1 = r \cos u^1 \cos u^2$$

$$x^2 = r \cos u^1 \sin u^2$$

$$x_1 = (-r \sin u^1 \cos u^2, -r \sin u^1 \sin u^2, r \cos u^1)$$

$$x_2 = (-r \cos u^1 \sin u^2, r \cos u^1 \cos u^2, 0)$$

$$E = r^2, \quad F = 0, \quad G = r^2 \cos^2 u^1$$

$$I = r^2 (du^1)^2 + r^2 \cos^2 u^1 (du^2)^2 = I((du^1, du^2), (du^1, du^2)) \\ = I(du^1, du^2)$$

球极投影. 
$$\begin{cases} x^1 = \frac{2r^2 \bar{u}_1}{r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2} \\ x^2 = \frac{2r^2 \bar{u}_2}{r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2} \\ x^3 = \frac{r(\bar{u}_1^2 + (\bar{u}_2)^2 - r^2)}{r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2} \end{cases}$$

$$x_1 = \left( \frac{2r^2(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2) - 2\bar{u}_1(2r^2\bar{u}_1)}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2}, \frac{-2\bar{u}_1(2r^2\bar{u}_2)}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2}, \frac{2r\bar{u}_1(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2) - 2\bar{u}_1 r((\bar{u}_1)^2 + (\bar{u}_2)^2 - r^2)}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2} \right)$$

$$= \left( \frac{2r^2(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2}, \frac{-4r^2(\bar{u}_1\bar{u}_2)}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2}, \frac{2r\bar{u}_1(2r^2)}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2} \right)$$

$$x_2 = \left( \frac{-4r^2(\bar{u}_1\bar{u}_2)}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2}, \frac{2r^2(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2}, \frac{2r\bar{u}_2(2r^2)}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2} \right)$$

$$g_{11} = \frac{4r^4}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2} \left( (r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2 + 4(\bar{u}_1\bar{u}_2)^2 + 4r^2(\bar{u}_1)^2 \right)$$

$$= \frac{4r^4}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2} (r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2 = \frac{4r^4}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2}$$

同理  $g_{22} = \frac{4r^4}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2}$

$$g_{12} = x_1 \cdot x_2 = \frac{4r^4}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2} \left( (r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)(-2\bar{u}_1\bar{u}_2) + (r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)(-2\bar{u}_1\bar{u}_2) + 4r^2\bar{u}_1\bar{u}_2 \right)$$

$$= \frac{4r^4}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2} \times 0 = 0$$

$$\Rightarrow I = \frac{4r^4}{(r^2 + (\bar{u}_1)^2 + (\bar{u}_2)^2)^2} \cdot ((d\bar{u}_1)^2 + (d\bar{u}_2)^2) \Rightarrow \text{等温系数.}$$

第一基本形式在参数变换下变化?

$$\bar{x}(\bar{u}^1, \bar{u}^2) = \bar{x}(\bar{u}^1(u^1, u^2), \bar{u}^2(u^1, u^2)) = x(u^1(\bar{u}^1, \bar{u}^2), u^2(\bar{u}^1, \bar{u}^2)).$$

$$\bar{u}^\alpha = \bar{u}^\alpha(u^1, u^2)$$

$$I = g_{\alpha\beta} du^\alpha du^\beta \stackrel{?}{=} \bar{g}_{\alpha\beta} d\bar{u}^\alpha d\bar{u}^\beta$$

第一基本形式系数

张量

$$\bar{g}_{\alpha\beta} = \bar{x}_\alpha \bar{x}_\beta = x_\gamma \frac{\partial u^\gamma}{\partial \bar{u}^\alpha} x_\delta \frac{\partial u^\delta}{\partial \bar{u}^\beta}$$

$$= x_\gamma x_\delta \left( \frac{\partial u^\gamma}{\partial \bar{u}^\alpha} \frac{\partial u^\delta}{\partial \bar{u}^\beta} \right) = g_{\gamma\delta} \frac{\partial u^\gamma}{\partial \bar{u}^\alpha} \frac{\partial u^\delta}{\partial \bar{u}^\beta}$$

$$\frac{\partial \bar{x}}{\partial \bar{u}^\alpha} = \frac{\partial x}{\partial u^1} \frac{\partial u^1}{\partial \bar{u}^\alpha} + \frac{\partial x}{\partial u^2} \frac{\partial u^2}{\partial \bar{u}^\alpha}$$

$$\Rightarrow \bar{g}_{\alpha\beta} d\bar{u}^\alpha d\bar{u}^\beta = g_{\gamma\delta} \frac{\partial u^\gamma}{\partial \bar{u}^\alpha} \frac{\partial u^\delta}{\partial \bar{u}^\beta} d\bar{u}^\alpha d\bar{u}^\beta = g_{\gamma\delta} du^\gamma du^\delta$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \text{ 正定.}$$

$$g_{11}, g_{22} > 0$$

Lagrang 恒等式

$$(x_1 \times x_2)(x_1 \times x_2) = (x_1 \cdot x_1)(x_2 \cdot x_2) - (x_1 \cdot x_2)(x_1 \cdot x_2)$$

$$\det \begin{pmatrix} E & F \\ F & G \end{pmatrix} = (x_1 \cdot x_1)(x_2 \cdot x_2) - (x_1 \cdot x_2)(x_1 \cdot x_2) = |x_1 \times x_2|^2 > 0$$

活动标架.

$$\{x(u^1, u^2); x_1(u^1, u^2), x_2(u^1, u^2), n(u^1, u^2)\}.$$

自然标架.

Schmidt 正交化

$$\begin{cases} e_1 = \frac{x_1}{|x_1|} = \frac{x_1}{|g_{11}|} \\ e_2 = \frac{x_2 - (x_2 \cdot e_1) e_1}{|x_2 - (x_2 \cdot e_1) e_1|} \end{cases}$$

$$x_1 = \sqrt{g_{11}} e_1$$

$$x_2 = \frac{g_{12}}{\sqrt{g_{11}}} e_1 + \frac{1}{\sqrt{g_{11}}} \sqrt{g_{11}g_{22} - g_{12}^2} e_2$$

实际随意正交标系:

$$\begin{cases} x_1 = a_1^1 e_1 + a_1^2 e_2 \\ x_2 = a_2^1 e_1 + a_2^2 e_2 \end{cases}$$

基底用

下标. 坐标用上标.

矩阵. 上标对应基底

下标对应结果

$$x_\alpha = a_\alpha^\beta e_\beta \quad (a_\alpha^\beta) \text{ 非退化的系数矩阵.}$$

$\{e_1, e_2\}$   
正交

$$dx = x_\alpha du^\alpha = (a_\alpha^\beta e_\beta) du^\alpha$$

$$= \underbrace{a_\alpha^\beta du^\alpha}_{\omega^\beta} e_\beta = \omega^\beta e_\beta.$$

$\omega^\beta \rightarrow$  一次微分形式

$$ds^2 = |dx|^2 = \omega^2 + \omega^2$$