Week 3.

Measure.

Let (X,d) metric space. $2^X = \{\text{all subsets of } X\}$

Def. (Outer Measure) A map $\mu : 2^X \to [0,+\infty]$. is an outer measure.

if. 1) $\mu(\phi)=0$

2) $\mu(A) \le \mu(B)$ $\forall A \subset B$

3). $\mu(\bigcup_{j=1}^{\infty} A_j) \le \sum_{j=1}^{\infty} \mu(A_j)$ $\forall A_j \subset X$.

We say a subset $A \subset X$. is μ-measurable if $\mu(S) = \mu(S \setminus A) + \mu(S \cap A)$ $\forall S \subset X$

$$\mu(S\setminus A) = \mu(S \cap A^c)$$

Rmk. ① If A is μ-measurable $\iff$ A^c is μ-measurable

② A is μ-measurable $\iff$ $\mu(S) \geq \mu(S\setminus A) + \mu(S \cap A)$ $\forall S \subset X$.

Def. A collection $\mathcal{S} \subset 2^X$. is a σ-algebra, if

1) $\phi, X \in \mathcal{S}$

2). If $A \in \mathcal{S}$. then $A^c \in \mathcal{S}$

3). If $A_1, A_2, \dots \in \mathcal{S}$. then $\bigcup_{n=1}^{\infty} A_i \in \mathcal{S}$

Rmk. If $A_1, A_2, \dots \in \mathcal{S}$. then $\bigcap_{n=1}^{\infty} A_i \in \mathcal{S}$.

Pf $\left(\bigcap_{j=1}^{\infty} A_j = X \setminus \bigcup_{j=1}^{\infty} A_j^c \right.$, $A_j^c \in \mathcal{S}$. $\left. \bigcup_{j=1}^{\infty} A_j^c \in \mathcal{S} \right)$

Lemma. The collection $\mathcal{M}$ of all μ-measurable subsets is a σ-algebra.

Furthermore. 1). $A_j \in \mathcal{M}$. $A_j \cap A_i = \phi$ then

$$\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$$

2) Increasing sequence: $A_1 \subset A_2 \subset \cdots \in \mathcal{M}$. then

$$\mu(\bigcup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mu(A_j)$$

3). Decreasing sequence: $A_1 \supset A_2 \supset \dots \in \mathcal{M}$ and $\mu(A_1) < +\infty$

then

$$\mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} \mu(A_j)$$

Pf σ-algebra is clear ...

1). $A_i \cap A_j = \phi$ set $S = A_i \cup A_j$

Since $A_i \in \mathcal{M} \Rightarrow \mu(A_i \cup A_j) = \mu(S) = \mu(S \cap A_i) + \mu(S \setminus A_i)$

$$= \mu(A_i) + \mu(A_j)$$

By induction

$$\mu\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} \mu(A_i)$$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{+\infty} A_i\right) \geq \lim_{N\to\infty} \mu\left(\bigcup_{i=1}^{N} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{N} \mu(A_i)$$

$$= \sum_{i=1}^{\infty} \mu(A_i)$$

By induction. of outer measure

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \bigcup_{n=1}^{\infty} \mu(A_i)$$ ▨

2). $\Rightarrow$ 互不相交.

set $A_0 = \phi$. by 1)

$$\mu\left(\bigcup_{n=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} (A_i \setminus A_{i-1})\right)$$

$$\overset{(1)}{=} \sum_{i=1}^{\infty} \mu(A_i \setminus A_{i-1})$$

$$= \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu(A_i \setminus A_{i-1})$$

$$= \lim_{n\to\infty} \mu(A_n)$$ ▨

3) . Set $E_j = A_1 \setminus A_j$ then

$$E_1 \subset E_2 \subset E_3 \subset \dots \in \mathcal{M}$$

and $\mu(A_1) = \mu(A_j) + \mu(E_j) \quad \forall j \geq 1$. by (1) and $\bigcup_{j=1}^{\infty} E_j = A_1 \setminus \left(\bigcap_{j=1}^{\infty} A_j\right)$ (*)

$$\overset{(*)}{\Rightarrow} \mu(A_1) = \mu\left(\bigcup_{j=1}^{\infty} E_j\right) + \mu\left(\bigcap_{j=1}^{\infty} A_j\right). \quad \underline{\text{by (1)}}$$

$$= \lim_{j\to\infty} \mu(E_j) + \mu\left(\bigcap_{j=1}^{\infty} A_j\right) \quad \text{by (2)}$$

$$= \lim_{j\to+\infty}\left(\mu(A_1) - \mu(A_j)\right) + \mu\left(\bigcap_{j=1}^{\infty} A_j\right)$$

$$= \mu(A_1) - \lim_{j\to+\infty} \mu(A_j) + \mu\left(\bigcap_{j=1}^{\infty} A_j\right)$$

$$\Rightarrow \mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j\to+\infty} \mu(A_j) \qquad \blacksquare$$

<u>Def</u> (Borel set) We say the smallest σ-algebra containing all <u>open</u> subsets are Borel sets.

<u>Thm</u>. (Caratheodory's criterion) <u>Let (X,d)</u> metric space, μ be an outer measure on X.

if $\mu(A\cup B) = \mu(A) + \mu(B)$

for all $A, B \subset X$. with $d(A,B) = \inf\{d(a,b) : a\in A, b\in B\} > 0$

then all <u>Borel sets</u> are μ-measurable

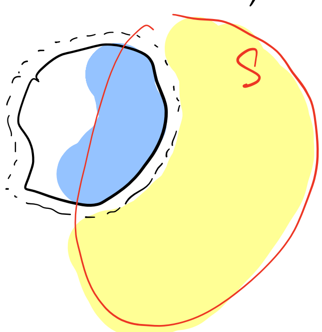
由于 Borel sets's definition
且 μ is a σ-algebra

<u>pf</u>. It suffices to show all closed set C is measurable

i.e. to prove

$$\mu(S) \geq \mu(S\setminus C) + \mu(S\cap C), \quad \forall S\subset X$$
$$(\mu(S) < +\infty)$$

Set $C_j = \{x\in X : d(x,C) \leq \frac{1}{j}\}$.



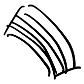
$d(S\setminus C_j, S\cap C) > 0$.

$$\Rightarrow \mu(S) \geq \mu\left((S\setminus C_j)\cup(S\cap C)\right) = \mu(S\setminus C_j) + \mu(S\cap C)$$

need to prove. $\lim\limits_{j\to+\infty} \mu(S\setminus C_j) = \mu(S\setminus C)$

Since C is closed.

$$S\setminus C = \{x\in S : d(x,C)>0\}$$

不相切 

$$\Rightarrow S\setminus C = (S\setminus C_j) \cup \left(\bigcup_{k=j}^{\infty} R_k\right), \text{ where } R_k = \left\{x\in S : \frac{1}{k+1} < d(x,C) \leq \frac{1}{k}\right\}$$

$$\Rightarrow \mu(S\setminus C_j) \leq \mu(S\setminus C) \leq \mu(S\setminus C_j) + \mu\left(\bigcup_{k=j}^{\infty} R_k\right)$$

only need to show. $\lim\limits_{j\to+\infty} \mu\left(\bigcup_{k=j}^{\infty} R_k\right) = 0$

Note that $\mu\left(\bigcup_{k=j}^{\infty} R_k\right) \leq \sum_{k=j}^{\infty} \mu(R_k)$

Only need to show $\sum_{k=1}^{\infty} \mu(R_k) < +\infty$

<u>Noting</u>. $d(R_k, R_{k+2}) > 0$. $\forall k$

$$\sum_{k=1}^{N} \mu(R_{2k}) = \mu\left(\bigcup_{k=1}^{N} R_{2k}\right) \leq \mu(S) < +\infty$$

$$\sum_{k=1}^{N} \mu(R_{2k-1}) = \mu\left(\bigcup_{k=1}^{N} R_{2k-1}\right) \leq \mu(S) < +\infty$$

$$\Rightarrow \sum_{k=1}^{\infty} \mu(R_k) < +\infty$$

<u>Def</u>. We say a outer measure μ is Borel, if any Borel set is μ-measurable

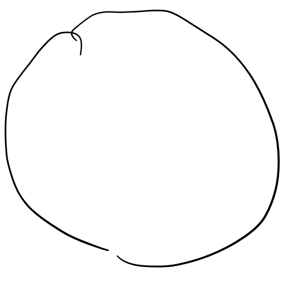
We say a outer measure μ is Borel-regular if μ is Borel, and

$\forall A\subset X$, $\exists$ Borel set $B\supset A$ s.t. $\mu(A)=\mu(B)$

<u>Def</u>. (Hausdoff measure)

Let (X,d) be a metric space, $A\subset X$. $t\geq 0$.

the t-Hausdoff measure of A:



$$\mathcal{H}^t(A) := \lim_{\delta\to 0} \mathcal{H}^t_\delta(A)$$

开球覆盖

and $\mathcal{H}^t_\delta(A) := \inf\left\{\sum \omega_t r_i^t : A\subset \bigcup B_{r_i}(x_i),\ r_i\leq \delta\right\}$

where $\omega_t = \frac{\pi^{t/2}}{\Gamma(\frac{t}{2}+1)} > 0$ $\left(\omega_n = Vol(B_1(0)),\ B_1(0) \subseteq \mathbb{R}^n \right)$

Rmk 1) $\mathcal{H}^t_\delta(A) \le \mathcal{H}^t_{\delta'}(A)$ $\forall \delta' \le \delta$

2) $\mathcal{H}^t_\delta$, $\mathcal{H}^t$ are outer measure.

Lemma The Hausdoff measure $\mathcal{H}^t$, is Borel regular, $\forall t \geq 0$

Pf. (use creterion)

① $\forall A, B \subset X$. if $d(A,B) > 2\delta$

then. $\mathcal{H}^t_\delta(A \cup B) = \mathcal{H}^t_\delta(A) + \mathcal{H}^t_\delta(B)$

$\Downarrow$

$\mathcal{H}^t(A \cup B) = \mathcal{H}^t(A) + \mathcal{H}^t(B)$ $\forall d(A,B) > 2\delta$

$\Rightarrow \forall A, B \subset X$. if $d(A,B) > 0$, then

$\mathcal{H}^t(A \cup B) = \mathcal{H}^t(A) + \mathcal{H}^t(B)$

$\Rightarrow$ by criterion. $\mathcal{H}^t$ is Borel.

② For $A \subset X$. $\forall \delta > 0$. $\exists$ covering $\mathcal{B} = \{B_{r_i}(x_i), r_i \le \delta\}$ of A

S.T. $\sum \omega_t r_i^t - \delta \le \mathcal{H}^t(A) \le \sum \omega_t r_i^t + \delta$

Set $U_\delta = \bigcup B_{r_i}(x_i)$ is open.

$\Rightarrow \mathcal{H}^t_\delta(A) \le \mathcal{H}^t_\delta(U_\delta) \le \sum_i \mathcal{H}^t_\delta(B_{r_i}(x_i)) \le \omega_t r_i^t$

$\Rightarrow \mathcal{H}^t(A) \geq \mathcal{H}^t_\delta(U_\delta) - \delta$ (*)

Let $U = \bigcap_{0 \le \delta < 1} U_\delta$. is Borel. and $A \subset U$ $\Rightarrow \mu(A) \le \mu(U)$

by (*). $\mathcal{H}^t(A) \geq \mathcal{H}^t_\delta(U) - \delta$

$\Rightarrow \mathcal{H}^t(A) \geq \lim_{\delta \to +\infty} \mathcal{H}^t_\delta(U) = \mathcal{H}^t(U)$ ∎

Def (Hausdoff dimension)

$A \subset X$.

$\dim_H A := \dim A := \inf\{ t \geq 0 : \mathcal{H}^t(A) = 0 \}$

$$= \sup\{t \geq 0 : \mathcal{H}^t(A) = +\infty\}$$

Ex. 1). Let $(\mathbb{Q}, d)$. rational points in $\mathbb{R}$, $d(x,y) = |x-y|$

then $\dim_H \mathbb{Q} = 0$

2) $\dim_H \mathbb{R}^n = n$

Pf. 1) $\forall t > 0$. show $\mathcal{H}^t(\mathbb{Q}) = 0$

since $\mathbb{Q}$ is countable, $a_1, a_2, a_3, \dots$

$\forall \delta \quad \mathbb{Q} \subseteq \bigcup_i B_{\delta \cdot 2^{-i}}(a_i)$

$$\Rightarrow \mathcal{H}^t_\delta(\mathbb{Q}) \leq \sum_i \omega_t (\delta \cdot 2^{-i})^t = \omega_t \delta^t \sum_i 2^{-it} \leq C_t \, \omega_t \, \delta^t$$

$$\Rightarrow \mathcal{H}^t(\mathbb{Q}) = \lim_{\delta \to 0} \mathcal{H}^t_\delta(\mathbb{Q}) \leq 0$$

$\Rightarrow \dim_H \mathbb{Q} = 0$

2) 几何测度论中 ⇒ $\mathbb{R}^n$ 中. H 测度 = L 测度.

$\forall B_R(0) \subseteq \mathbb{R}^n$.

$\forall B_{r_i}(x_i) \supset B_R(0) \qquad r_i < \delta$

$$\Rightarrow \sum \omega_t r_i^t = \sum \omega_n r_i^n \cdot \frac{\omega_t}{\omega_n} r_i^{t-n} \leq \delta^{t-n} \frac{\omega_t}{\omega_n} \sum \omega_n r_i^n \to 0$$

$\Rightarrow \dim B_R(0) \leq n$, $\mathcal{H}^n(B_R(0)) > 0 \quad \Rightarrow \dim_H B_R(0) = n$

Since $\mathbb{R}^n = \bigcup_{i=1}^{\infty} B_i(0)$

Since $\mathcal{H}^t(B_i(0)) = 0 \quad \forall t > n$

$$\Rightarrow \mathcal{H}^t(\mathbb{R}^n) = \lim_{i \to \infty} \mathcal{H}^t(B_i(0)) = 0$$

$\Rightarrow \dim_H \mathbb{R}^n \leq n$. since $\dim_H B_1(0) = n$ and $B_1(0) \subset \mathbb{R}^n$

$$\Rightarrow \dim_H \mathbb{R}^n = n$$

Def. (Minkowski dimension) $A \subset X$

$$\overline{\dim}_M A = \overline{\lim_{\varepsilon \to 0}} \frac{\log N(A,\varepsilon)}{\log(\frac{1}{\varepsilon})}$$

$$\underline{\dim}_M A = \underline{\lim_{\varepsilon \to 0}} \frac{\log N(A,\varepsilon)}{\log(\frac{1}{\varepsilon})}$$

where $N(A,\varepsilon) = \inf \# \{ k : \exists B_\varepsilon(x_1), \dots, B_\varepsilon(x_k) \text{ s.t. } A \subset \bigcup_{i=1}^{k} B_\varepsilon(x_i) \}$

Rmk. In $\mathbb{R}^n$. $N(B_1(0), \varepsilon) \simeq \dfrac{Vol(B_1(0))}{Vol(B_\varepsilon(0))} \simeq C(n)\,\varepsilon^{-n}$

$$\frac{\log N(B_1(0)), \varepsilon)}{\log(\frac{1}{\varepsilon})} \simeq \frac{\log C_n + n \log(\frac{1}{\varepsilon})}{\log(\frac{1}{\varepsilon})} \to n$$

If $\overline{\dim}_M A = \underline{\dim}_M A$, we call $\dim_M A = \overline{\dim}_M A = \underline{\dim}_M A$

Rmk. $\dim_M(\mathbb{Q}) = 1$ in $\mathbb{R}$

$\dim_M(\mathbb{Q}^n) = n$ in $\mathbb{R}^n$