

W3 L1

$T_{x(u,v)} M$
 $\text{Span}(x_1, x_2)$

当成某条曲线的切向量.

$\gamma(t) = x(u^1(t), u^2(t))$
 $\frac{d\gamma}{dt} = \vec{v} = \frac{du^1}{dt} x_1 + \frac{du^2}{dt} x_2$
 $\frac{du^1}{dt} = a, \frac{du^2}{dt} = b$

$\sigma(u^1, u^2) = (\bar{u}^1, \bar{u}^2)$
 $\begin{cases} \bar{u}^1 = \bar{u}^1(u^1, u^2) \\ \bar{u}^2 = \bar{u}^2(u^1, u^2) \end{cases}$

$\bar{\gamma}(t) = \sigma(\gamma(t)) = \bar{x}(\bar{u}^1(u^1(t), u^2(t)), \bar{u}^2(u^1(t), u^2(t)))$

$\frac{d\bar{\gamma}(t)}{dt} \Big|_{t=0} \triangleq \sigma^*(\vec{v})$
 $\begin{aligned} \bar{\pi} &= \bar{x}_{\bar{u}^1} \frac{d\bar{u}^1}{dt}(0) + \bar{x}_{\bar{u}^2} \frac{d\bar{u}^2}{dt}(0) \\ &= \bar{x}_1 \left(\frac{\partial \bar{u}^1}{\partial u^1} \frac{du^1}{dt}(0) + \frac{\partial \bar{u}^1}{\partial u^2} \frac{du^2}{dt}(0) \right) \\ &\quad + \bar{x}_2 \left(\frac{\partial \bar{u}^2}{\partial u^1} \frac{du^1}{dt}(0) + \frac{\partial \bar{u}^2}{\partial u^2} \frac{du^2}{dt}(0) \right) = a \bar{x}_1 + b \bar{x}_2 \end{aligned}$

$\left| \frac{\partial(\bar{u}^1, \bar{u}^2)}{\partial(u^1, u^2)} \right| \neq 0$

$I = ds^2 = g_{\alpha\beta} du^\alpha du^\beta$

$\bar{I} = g_{\bar{\alpha}\bar{\beta}} d\bar{u}^\alpha d\bar{u}^\beta$

若满足 $\bar{I} = \varphi^2(u^1, u^2) I$. $\Rightarrow \sigma$ -共形的 Conformal (保角的)

$$\sigma^*: T_x M \rightarrow T_{\sigma(x)} M \quad \text{切映射}$$

$$\sigma_*(a x_1 + b x_2) = a \bar{x}_1 + b \bar{x}_2 = a \sigma_*(x_1) + b \sigma_*(x_2) \quad .$$

下面讨论保角性

$$\left. \begin{aligned} \vec{a} &= a^\alpha \chi_\alpha \\ \vec{b} &= b^\alpha \chi_\alpha \end{aligned} \right\} \in T_x M \quad \Rightarrow \quad \begin{aligned} \sigma^*(\vec{a}) &= a^\alpha \sigma^*(\chi_\alpha) = a^\alpha \bar{\chi}_\alpha \\ \sigma^*(\vec{b}) &= b^\alpha \sigma^*(\chi_\alpha) = b^\alpha \bar{\chi}_\alpha \end{aligned}$$

$$\begin{aligned}
\cos \angle(\sigma^*(\vec{a}), \sigma^*(\vec{b})) &= \frac{(\sigma^*(\vec{a}), \sigma^*(\vec{b}))}{\|\sigma^*(\vec{a})\| \|\sigma^*(\vec{b})\|} \\
&= \frac{a^\alpha b^\beta \bar{x}_\alpha \bar{x}_\beta}{(a^\alpha a^\tau \bar{x}_\alpha \bar{x}_\tau)^{\frac{1}{2}} (b^\gamma b^\delta \bar{x}_\gamma \bar{x}_\delta)^{\frac{1}{2}}} \\
&= \frac{a^\alpha b^\beta \bar{g}_{\alpha\beta}}{\dots} = \frac{\varphi^2 a^\alpha b^\beta g_{\alpha\beta}}{\dots} \\
&= \cos \angle(\vec{a}, \vec{b})
\end{aligned}$$

Thm. 任何曲面. 必在 局部与平面共形对应

即. 任何曲面的第一基本形式. 都可以写成

$$I = \varphi^2(u, v) (du^2 + dv^2)$$

参数网 (u, v) 称为 等温参数网.

S.-S. Chern. 论文, 现代几何学之局部.

↓

$$ds^2 = E dp^2 + 2F dp dq + G dq^2$$

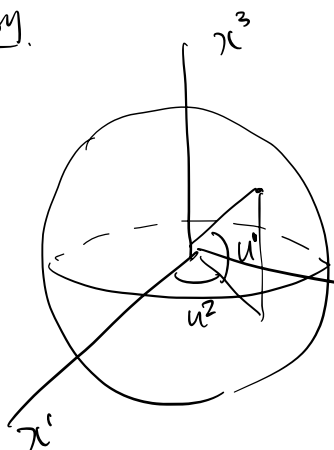
$$= \left(\sqrt{E} dp + \frac{F + i\sqrt{G}}{\sqrt{E}} dq \right) \overset{\text{det}}{\left(\sqrt{E} dp + \frac{F - i\sqrt{G}}{\sqrt{E}} dq \right)}$$

$$\lambda \left(\right) = du + i dv$$

$$\bar{\lambda} \left(\right) = du - i dv$$

$$\Rightarrow \lambda \bar{\lambda} ds^2 = du^2 + dv^2 \quad \dots \quad \underline{\text{PDE}}$$

134.



$$x^2 = r^2 \quad \text{i.e.} \quad \sum_{i=1}^3 (x^i)^2 = r^2$$

$$\begin{cases} x^1 = r \cos u^1 \cos u^2 \\ x^2 = r \cos u^1 \sin u^2 \\ x^3 = r \sin u^1 \end{cases}$$

$$\begin{cases} -\frac{\pi}{2} < u^1 < \frac{\pi}{2} \\ 0 < u^2 < 2\pi \end{cases}$$

$$I = r^2 (du^1)^2 + r^2 \cos^2 u^1 (du^2)^2$$

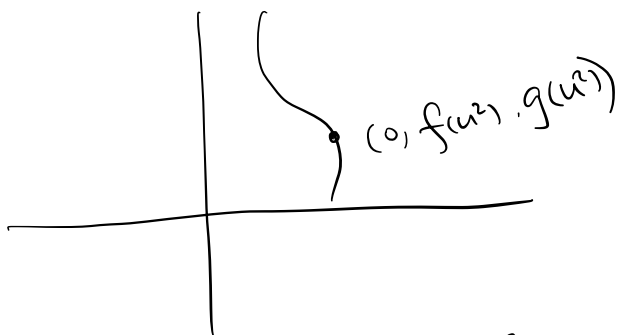
球极投影, $x^1 = \frac{2r^2 u^1}{r^2 + (u^1)^2 + (u^2)^2}$ $x^2 = \frac{2r^2 u^2}{r^2 + (u^1)^2 + (u^2)^2}$ $x^3 = r \frac{(u^1)^2 + (u^2)^2 - r^2}{(u^1)^2 + (u^2)^2 + r^2}$

$(u^1, u^2, 0)$ 在平面上

$$\Rightarrow I = \frac{4}{[1 + \alpha((u^1)^2 + (u^2)^2)]^2} [(du^1)^2 + (du^2)^2], \quad \alpha = \frac{1}{r^2}$$

$\Rightarrow u^1, u^2$ 为等度参数

135.



$$\Rightarrow x(u^1, u^2) = (f(u^2) \cos u^1, f(u^2) \sin u^1, g(u^2))$$

$$x_1 = (-f \sin u^1, f \cos u^1, g)$$

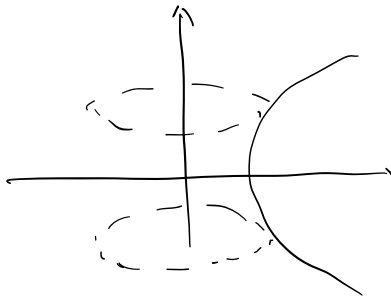
$$x_2 = (f' \cos u^1, f' \sin u^1, g')$$

$$\Rightarrow I = f^2 (du^1)^2 + [f'^2 + g'^2] (du^2)^2$$

悬链线



\Rightarrow



\Rightarrow

$$\begin{cases} f(t) = f(u^2) = a \cosh \frac{t}{a} \\ g(u) = g(u^2) = u^2 \end{cases}$$

$$\Rightarrow \begin{cases} f' = \sinh \frac{u^2}{a} \\ g' = 1 \end{cases}$$

$$\Rightarrow I' = \cosh^2 \frac{u^2}{a} [a(du^1)^2 + (du^2)^2]$$

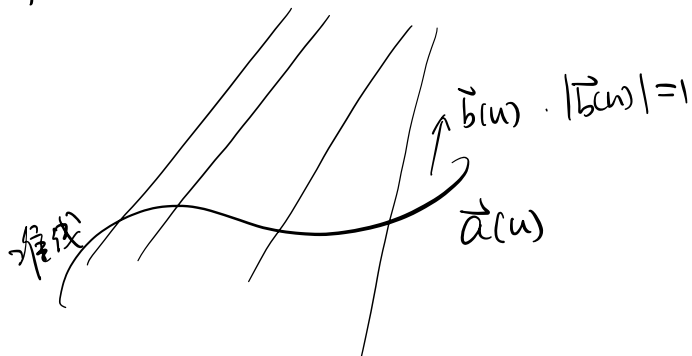
螺旋面 $x(u^1, u^2) = (f(u^2) \cos u^1, f(u^2) \sin u^1, g(u^2) + a u^1)$

$$I = (f'^2 + a^2) (du^1)^2 + 2ag' du^1 du^2 + (f'^2 + g'^2) (du^2)^2$$

令 $f(u^2) = u^2$
 $g(u^2) = 0 \Rightarrow$ 正螺面 $I_1 = (a^2 + (u^2)^2) (du^1)^2 + (du^2)^2$

若令 $u^1 = u^1, u^2 = a \sinh \frac{u^2}{a} \Rightarrow I_1 = I$. 故正螺面与悬链面
 成等距对应.

例. 直纹面



$$x(u, v) = a(u) + v b(u)$$

$$\begin{aligned} b \cdot b &= 1 \\ \Rightarrow \underline{b \cdot b'} &= 0 \end{aligned}$$

$$x_u = x_1 = a' + v b'$$

$$x_v = x_2 = b$$

$$\begin{cases} g_{11} = |a' + v b'|^2 \\ g_{12} = a' \cdot b \\ g_{22} = 1 \end{cases}$$

$$\Rightarrow I = |a' + vb'|^2 du^2 + 2a'b' du dv + (b')^2 dv^2$$

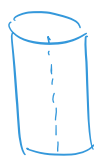
In particular, 1° $b(u)$ is constant \Rightarrow 柱面.

2° $a(u)$ is constant \Rightarrow 锥面.

3° $\vec{b} \parallel \vec{a}'$ 切线面.



4° define. **可展曲面**. 沿着任意一条直母线每一点处的



"剪开"

曲面的切平面重合

Thm. 直纹面 $\mathbf{r}(u, v) = \mathbf{a}(u) + v\mathbf{b}(u)$, $|\mathbf{b}|=1$ 是可展曲面 $\Leftrightarrow (a', b, b') = 0$

证. 任取直母线 (v -线) $u = u^0$

$$P(u^0, v^0), Q(u^0, v^0 + \Delta v) \quad \Delta v \neq 0$$

P, Q 两点处的法向量

$$(a' + vb') \times b \quad (a' + (v + \Delta v)b') \times b$$

$$\text{故. 可展} \Leftrightarrow 0 = [(a' + vb') \times b] \times [(a' + (v + \Delta v)b') \times b]$$

$$\Delta = \Delta v^2 (a' \times b) (b' \times b)$$

$$= -\Delta v^2 (a', b, b') b.$$