(A)

Cauchy integral theorem $\quad \mathbb{R} \nearrow \gamma \hookrightarrow \mathbb{C} \xrightarrow{f} \mathbb{C}$

$\int_\gamma f(z)\,dz \qquad \gamma(t) : z = x(t) + y(t)i \longrightarrow f(z).$

$dz = dx + i dy$

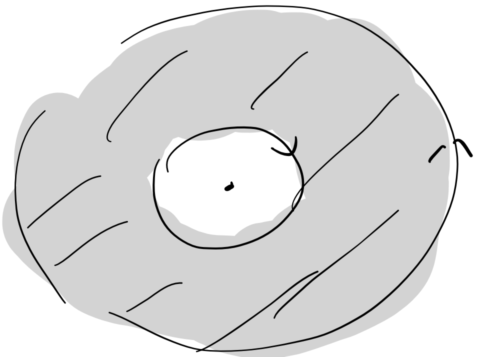
$f(z) = u(x,y) + i v(x,y)$

$\int_\gamma f(z)\,dz$

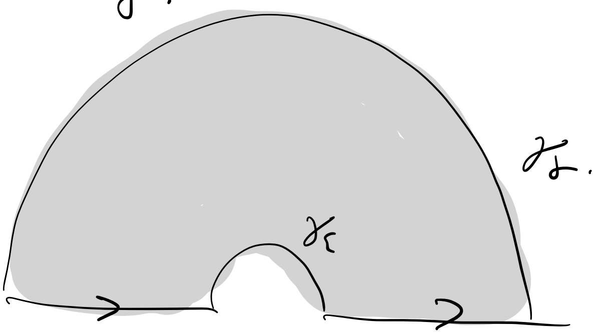
$\int_\gamma (u(x,y) + i v(x,y))(dx + i dy)$

$= \int_\gamma (u dx - v dy) + i \int_\gamma (v dx + u dy)$

$$\begin{cases} \dfrac{\partial(-v)}{\partial x} - \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}. \end{cases}$$



$\frac{1}{z}$



$\int_0^{+\infty} \frac{\sin x}{x}\,dx.$

变量替换 $\quad \mathbb{R} \to \underset{z}{\mathbb{C}} \to \underset{u}{\mathbb{C}} \to \underset{f(u)}{\mathbb{C}}$

Ex.

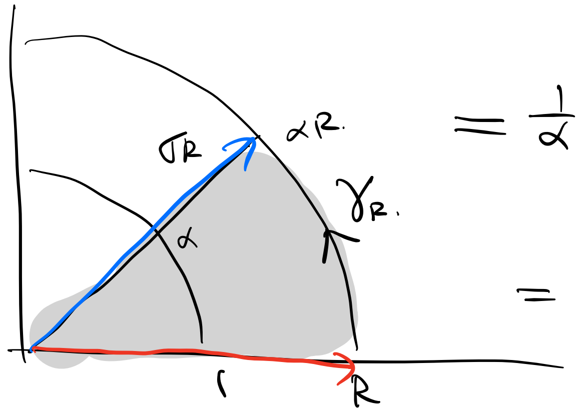
$\int_0^{\infty} \cos(x^2)\,dx \qquad \int_0^{\infty} \sin(x^2)\,dx \qquad$ Fresnel's integrals

$\int_0^R \cos(x^2)\,dx - i \int_0^R \sin(x^2)\,dx = \int_0^R e^{-ix^2}\,dx.$

Let. $\alpha = e^{\frac{\pi i}{4}} = \frac{1+i}{\sqrt{2}} \Rightarrow \alpha^2 = i$

$$= \frac{1}{\alpha}\int_0^R e^{-(\alpha x)^2}\alpha\,dx = \frac{1}{\alpha}\underbrace{\int_0^{\alpha R} e^{-z^2}\,dz}\Rightarrow \sigma_R$$

而 $\int e^{-x^2}dx$ 已知.



$$= \frac{1}{\alpha}\left(\underbrace{\int_0^R e^{-x^2}dx}_{\downarrow\ \frac{\sqrt{\pi}}{2}} + \int_{\gamma_R} e^{-z^2}dz\right)$$

$$= \frac{1}{\alpha}\cdot\frac{\sqrt{\pi}}{2}$$

$\gamma_R(\theta) = Re^{i\theta}$

$$\int_0^{\frac{\pi}{4}} e^{-R^2(\cos(2\theta)+i\sin(2\theta))}Re^{i\theta}i\,d\theta$$

$\downarrow$

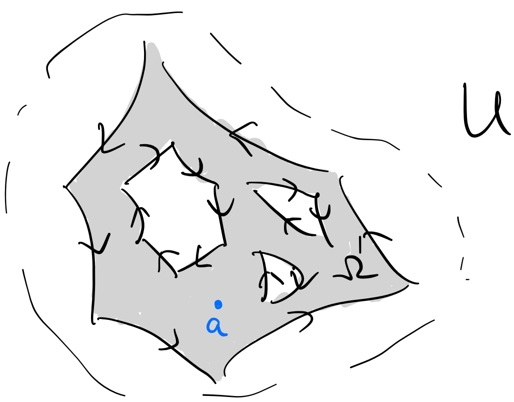
0

$$\Rightarrow \int_0^\infty \cos(x^2)\,dx - i\int_0^\infty \sin x^2\,dx$$

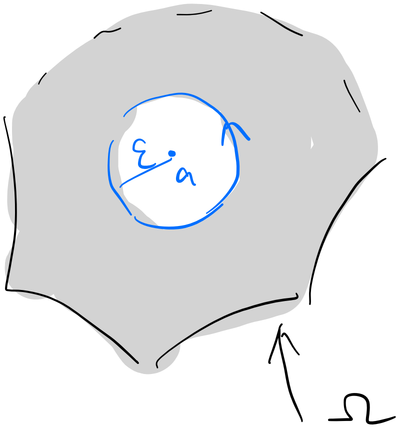
$$= \frac{1}{\alpha}\frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\frac{\pi}{2}} - i\,\frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

(B) If $\mathbb{C} \supset U$ open $\xrightarrow{F} \mathbb{C}$ is holo. Cauchy integral Formula.

$U \supset \bar{\Omega}$ with Ω a Green domain.



then $F(a) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{F(z)}{z-a}\,dz$ for every $a\in\Omega$



$\frac{F(z)}{z-a}$ is holo. on $\Omega \setminus \overline{B_\varepsilon(a)}$ ($= D$) for a sufficiently small $\varepsilon > 0$

Cauchy int. thm $\Rightarrow \int_{\partial D} \frac{F(z)}{z-a}\,dz = 0$

$\partial D = \partial\Omega - \partial B_\varepsilon a$

反向

$$\Rightarrow \frac{1}{2\pi i}\int_{\partial\Omega}\frac{F(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_{\partial B_\varepsilon(a)}\frac{F(z)}{z-a}\,dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\frac{F(a+\varepsilon e^{i\theta})\, i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}}\,d\theta$$

$a+\varepsilon e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$.

$$= \frac{1}{2\pi}\int_0^{2\pi} F(a+\varepsilon e^{i\theta})\,d\theta.$$

$$\left| \frac{1}{2\pi}\int_0^{2\pi} F(a+\varepsilon e^{i\theta})\,d\theta - F(a) \right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}\left|F(a+\varepsilon e^{i\theta})-F(a)\right|d\theta \longrightarrow 0 \text{ as } \varepsilon \to 0^+$$

since F is conti at a. $\square$