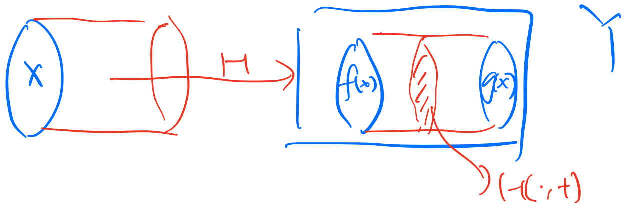
§1

映射的同伦



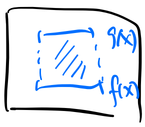
Def. $f, g: X \to Y$ 连续. $A \subset X$. 说 f, g 相对于 A 同伦

$\Longleftrightarrow$ $\exists H: X \times [0,1] \to Y$ s.t.
$$\begin{cases} H(x,0) = f & \\ H(x,1) = g & , x \in X \\ H(a,t) = a & , a \in A \end{cases}$$

(H 连续)

例 (1) $f, g: pt \to \mathbb{R}$. $f(x) = 0$ $g(x) = 1$

$$H(x,t) = t$$

(2). $\underset{0 \quad\quad\quad 1}{\text{———}} = I \longrightarrow$ 

$\mathbb{R}^2$

$f: I \to \mathbb{R}^2$. $f(x) = (x, 0)$

$g: I \to \mathbb{R}^2$ $g(x) = (x, 1)$

$$H(x,t) = (x,t)$$

(3) $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

- $\mathrm{Const}(x) = 0$
- $\mathrm{id}(x) = x$

$$H(x,t) = tx. \quad \begin{cases} H(x,0) = \mathrm{const} \\ H(x,1) = \mathrm{id} \end{cases}$$

$\mathrm{const}(x) = \vec{y}$?

$$\begin{cases} H(x,t) = x + t(y-x) \end{cases} \qquad H(x,0) = \mathrm{id}, \quad H(x,1) = y$$

相对同伦的例子

(1) $f\ S^1 \to \mathbb{R}$.

$\exists\ \underset{\in}{pt} \mapsto 0$.

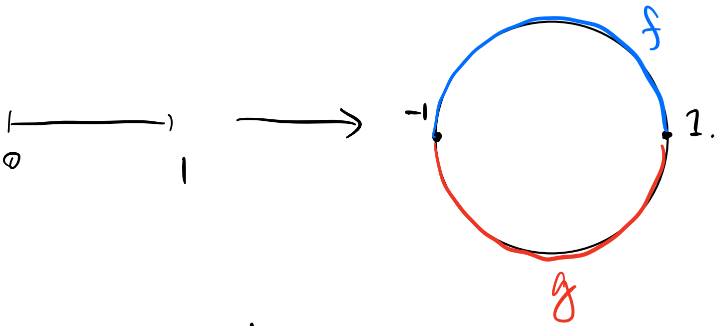
$\Rightarrow f \overset{H}{\simeq} (\mathrm{const}: S^1 \to 0)$ rel pt.

$$H(x,t) = f(x)\cdot t$$

then $H(pt,t) = f(pt)\cdot t = 0$.

(2) $f, g: I \to S^1$

$\{0,1\} \mapsto \{1,-1\}$.



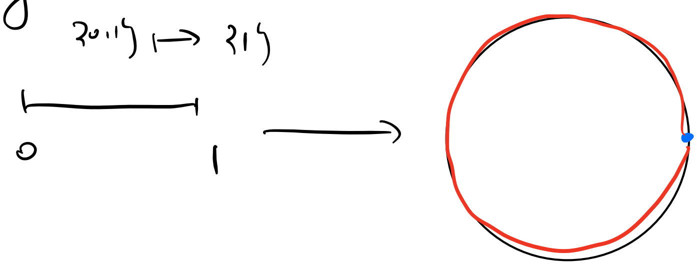
rel $\{0,1\}$.

不能相对同伦

(基本群).

$f, g\ \ I \to S^1$

$\{0,1\} \mapsto \{1\}$



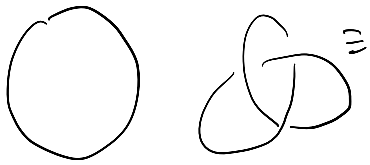
rel $\{0,1\}$

不能收缩到一个点.

not rel ? $f: [0,1] \to S^1$

$H(x,t) = f(tx) \Rightarrow f \overset{H}{\simeq} \mathrm{const}$.

(3). $S^1 \longrightarrow \mathbb{R}^3$



三叶结.

H

同伦

区别?

不仅. $H(x,t_0): S^1 \longrightarrow \mathbb{R}^3$ $\quad \forall t_0 \in [0,1]$

同时要满足 $H(x,t_0)$ $\quad S^1 \hookrightarrow \mathbb{R}^3$ $\quad \forall t_0 \in [0,1]$

同痕(isotopic) > 同伦 (homotopic).

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§2

同伦等价的拓扑空间

Def. X、Y 称为同伦等价的 $\Longleftrightarrow$ 存在 $X \underset{g}{\overset{f}{\rightleftarrows}} Y$. 且 $f\circ g \overset{H}{\simeq} id_Y$

f, g 连续 $\qquad g\circ f \overset{H}{\simeq} id_X$

f, g 称为同伦逆

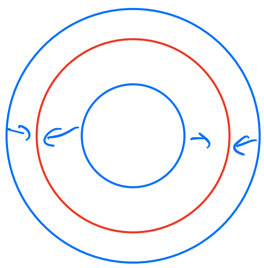
例. (1). $\mathbb{R}^n \overset{H}{\simeq} pt$. $\qquad f: \mathbb{R}^n \longrightarrow pt$

$g: pt \longrightarrow 0 \in \mathbb{R}^n$

$$\begin{cases} f\circ g = id_{pt} \\ g\circ f = const_0 \overset{H}{\simeq} id_{\mathbb{R}^n}. \end{cases}$$

(2) $\{1 \le |z| \le 2\} = X \subset \mathbb{R}^2$ $\qquad Y = S^1$

$X \overset{H}{\simeq} Y$.



Def.(强)形变收缩核 $A\subset X$ 是一个(强)形变收缩核 $\Leftrightarrow$

- $\exists H: X\times[0,1]\to X$ $\begin{cases} H(x,0)=x \\ H(x,1)\in A & \forall x\in X, a\in A \\ H(a,1)=a \end{cases}$ 形变收缩
- 强. $H(a,t)=a.$ $\forall t\in[0,1]$

Lem. 若 $A\subset X$ 为一个形变收缩核 $\Rightarrow A\simeq X$.

Pf. $f: A \xhookrightarrow{inc} X$

$g: X\longrightarrow A.$ let $g(x)=H(x,1)$

$g\circ f|_A = id_A.$

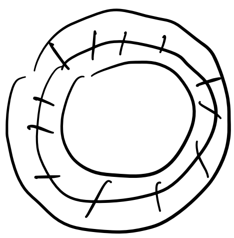
$f\circ g: X\to X.$ $\Rightarrow H(x,t)$ 给出 $H(x,0)=id_X$ 到 $H(x,1)=f\circ g$ 的同伦.

$\| $

$H(x,1)$

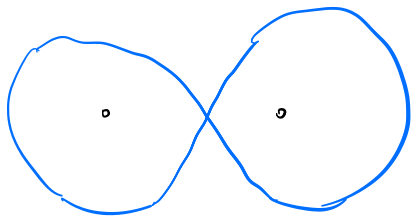
eg. (1) $\mathbb{R}^n\simeq pt$

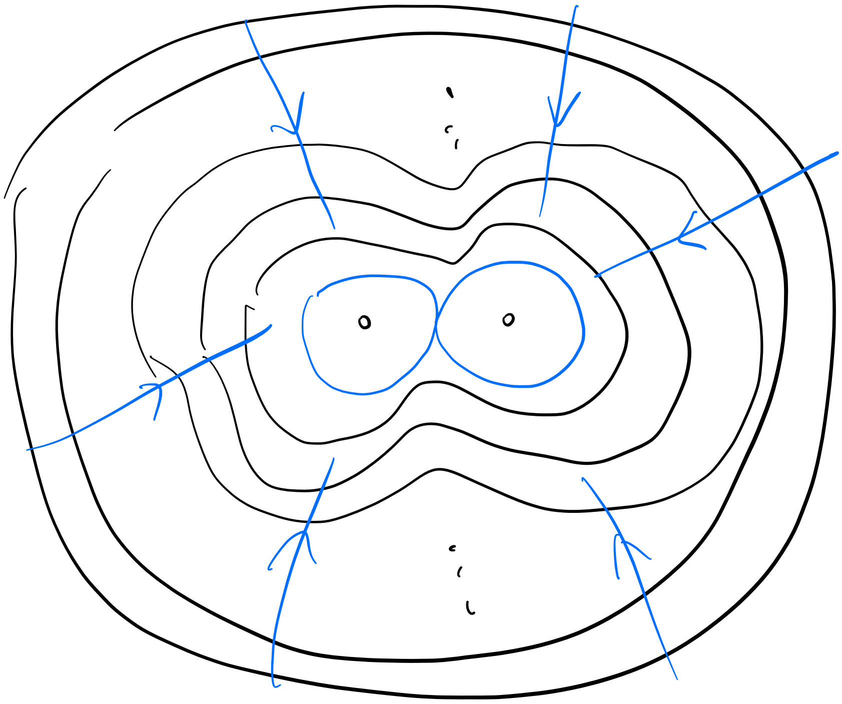
$H(\vec{x},t)=t\vec{x}$. $\Rightarrow$ 给出从 $\mathbb{R}^n$ 到原点的形变收缩

(2).  $=\{z\in\mathbb{C} \mid \frac{1}{2}\leqslant|z|\leqslant 2\}\simeq S^1$

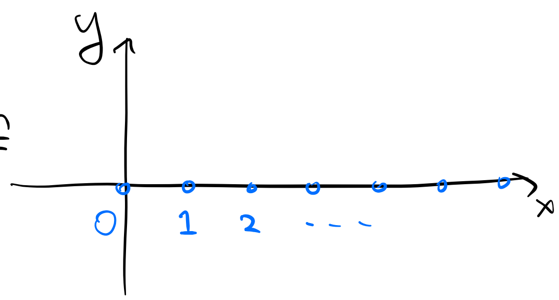
$H(\vec{x},t)=\dfrac{\vec{x}}{(t+(1-t)|\vec{x}|)}$. 形变收缩到 $\{|z|=1\}$

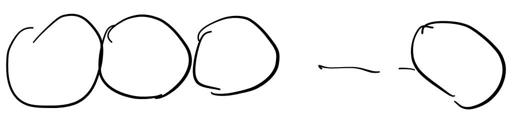
(3). $\mathbb{R}^2 \setminus \{pt \amalg pt\}$

直观: $\overset{H}{\simeq}$  $S^1 \wedge S^1$



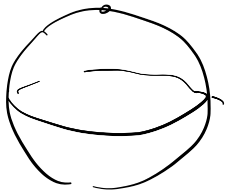
$\mathbb{R}^2 \setminus \{n \text{ pts}\}$. $\cong$



$\overset{H}{\simeq}$ 

$S^1 \wedge S^1 \wedge \cdots \wedge S^1$

(4) $S^2 \setminus \{n \text{ pts}\} = \bigwedge_{i=1}^{n-1} S^1$



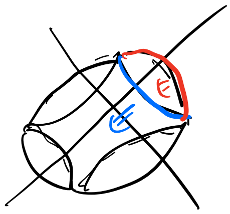
因为去掉一个点 ≅ 平面

(5) $\mathbb{R}^3 \setminus \{2 \text{ pts}\}$. $\quad S^2 \wedge S^2$

(6) $\mathbb{R}^3 \setminus \{a \text{ line}\} \overset{H}{\simeq} S^1 \times \mathbb{R}$ 柱面 $\overset{H}{\simeq} S^1 \times \{0\}$

$\mathbb{R}^3 \setminus \{\text{two lines}\}$?

① 相交

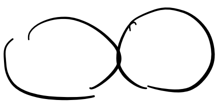


1.1

⇓

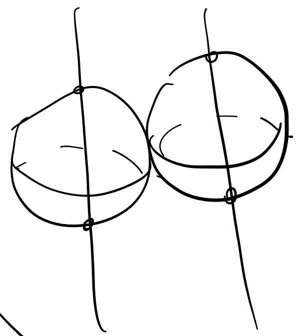
② 不相交.

2.1

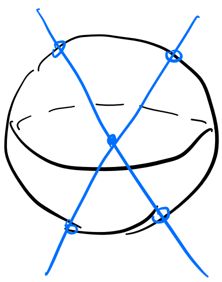


$S^1 \wedge S^1$

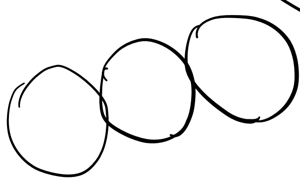
2.2



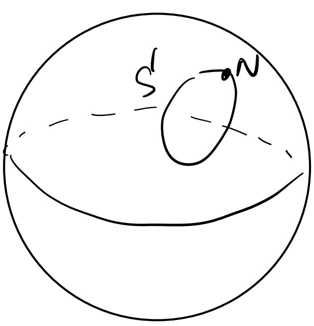
1.2



$\Rightarrow$

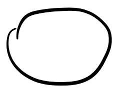


(7). $S^3 \setminus S^1$

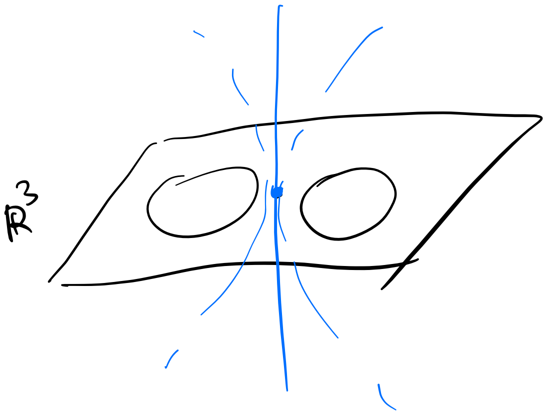


$\Rightarrow$ $\mathbb{R}^3$中去一条直线

$\simeq \mathbb{R}^3 \setminus$ 直线



(8). $\underbrace{\mathbb{R}^3 \setminus 两个圆}_{在xy平面上不相交的圆}$



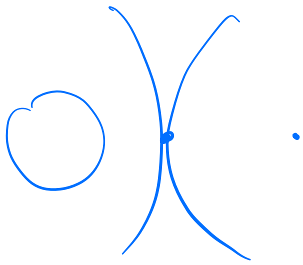
$\Rightarrow (\mathbb{R}^3\setminus 圆)\wedge(\mathbb{R}^3\setminus 圆)$

$\simeq S^1\wedge S^1\wedge S^2\wedge S^2$

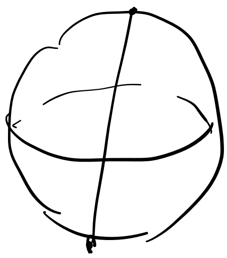
怎么看? $\mathbb{R}^3\setminus 圆 = S^3\setminus pt\setminus 圆$

$= S^3\setminus 点\setminus 直\setminus 点$

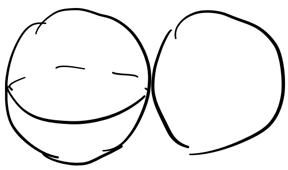
$= \mathbb{R}^3\setminus 直\setminus 点.$



$= S^1\wedge S^2$

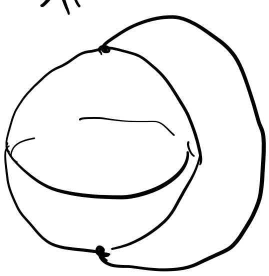
(9) 

S^2 + 线段.

$\overset{H}{\simeq}$ 

S||

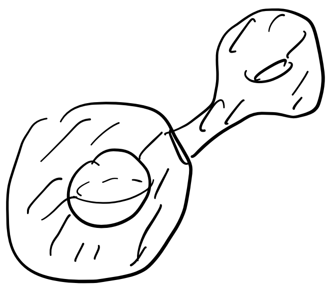
$S^2\wedge S^1$

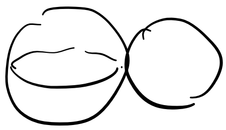


法一

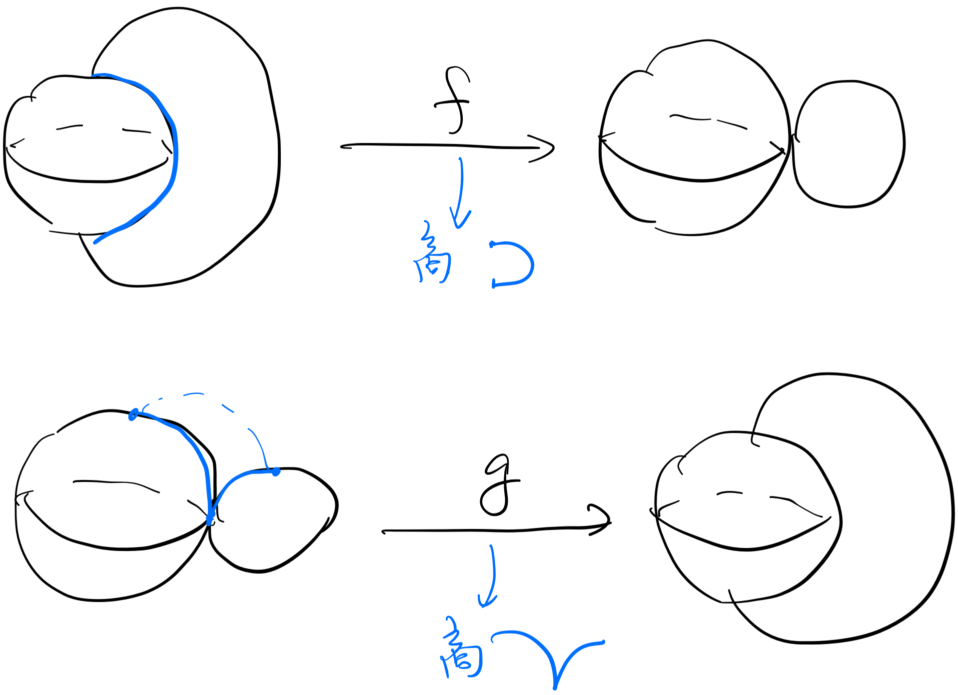
$\overset{H}{\simeq}$

$\overset{H}{\simeq}$



$H \simeq$ 

法二.



$f\circ g$ $g\circ f$ 需要 $\sim id_X$, id_Y.

基本群.

- $SO(n)$ 特殊正交群
  正交矩阵 & $\det=1$
  (旋转)

  $O(n)$
  正交矩阵.
  (旋转或反射)

(复) $SU(n)$

(复) $U(n)$

- 自由生成群 $\langle a, b\rangle$

  取 a, b，考虑所有以 a，b，a^{-1}，b^{-1} 为字符的

  字符串. 比如

  $$a\,a\,b\,b\,b^{-1}a^{-1}b^{-1}a^{-1}a^{-1}$$

  (若有 aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b$ 则忽略.)

  $$\sim aaba^{-1}b^{-1}a^{-1}a^{-1}$$

则字符串集形成群.

乘法即串连起来

$\langle a_1, \cdots, a_n\rangle$ n个元素的自由生成群

指无更多关系限制

$Y \xrightarrow{f} X$. f 连续 $\underbrace{[Y, X]}_{\text{同伦类}} = \mathrm{Map}(Y, X)/\sim$ 同伦

- X : 球面 Y : 球面. $\Rightarrow$ 球面同伦群 difficult.

last important.

- Y. 球面. $X = $ CW-Complex, $X' = $ CW-complex. $\Rightarrow$ $\left(\begin{matrix} \pi_i(X) \\ \cong \\ \pi_i(X') \end{matrix} \Rightarrow X \sim X' \right)$ $\pi_i(X) \to [S^i, X]$

first step? Foundamental group

Def. 取基点 $* \in X$, 考虑. $\mathrm{Map}\left(([0,1], \{0,1\}), (X, *)\right)$

means rel $\{0,1\}$

means $\{0,1\} \to \{*\}$

$$= \{ \gamma : [0,1] \to X.\ \gamma(0) = \gamma(1) = * \}$$

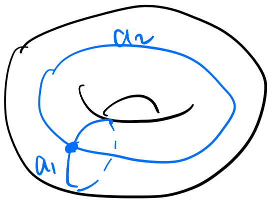
$$\pi_1(X, *) = \mathrm{Map}\left(([0,1], \{0,1\}), (X, *)\right)/\sim$$

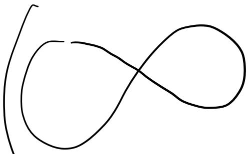
$\gamma \sim \gamma' \Leftrightarrow \gamma$ 与 γ' rel $\{0,1\}$ 同伦. (定端同伦)

① why is $\sim$

② why is group

- $\pi_1(T^2) \overset{?}{=} \langle a_1, a_2 \mid a_1 a_2 a_1^{-1} a_2^{-1} \rangle$



- $\pi_1\Big($  $\Big) = \langle a_1, a_2 \rangle$

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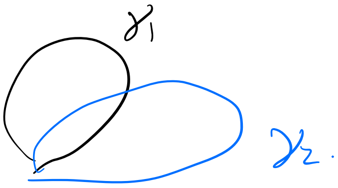
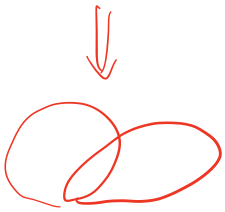
(1)



$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t), & 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$



well-defined ?

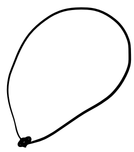
$\gamma_1 \sim \gamma_1'$ $H_1(s,t)$

$\gamma_2 \sim \gamma_2'$ $H_2(s,t)$

$\Longrightarrow \gamma_1 * \gamma_2 \sim \gamma_1' * \gamma_2'$

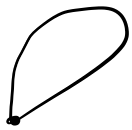
$$H(s,t) = \begin{cases} H(s,2t) & t \in [0,\frac{1}{2}) \\ H(s,2t-1) & t \in [\frac{1}{2},1] \end{cases}$$

(2)



引理. $\gamma'(t)$ 是 $\gamma(t)$ 的 reparametrization.(再参数化)

那么 $\gamma'(t) \sim \gamma(t)$ 同伦.

① 单位元. $e:[0,1]\longrightarrow *$　　$e*\gamma(t)$  $\gamma(2t-1)\sim\gamma(t)$.

② 逆元. $\gamma(t)$.　　$\gamma^{-1}(t)=\gamma(1-t)$

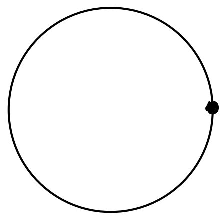
③ 结合律.　　<u>分段</u>

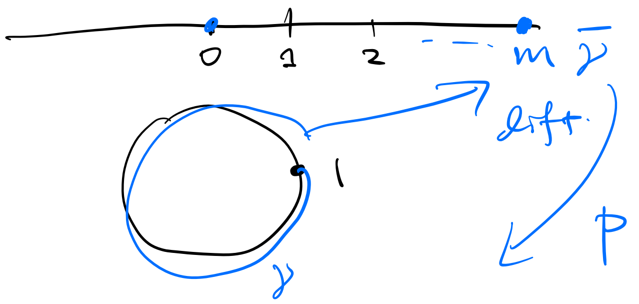
例　$\pi_1(S^1)$　　考虑. $p:\mathbb{R}\longrightarrow S^1$

$$t\longmapsto e^{2\pi i t}$$

$p(0)=1$　　$p(\mathbb{Z})=1$

设　$\gamma:[0,1]\longrightarrow S^1$.　$\gamma(0)=\gamma(1)=1=*$





$$\begin{tikzcd}
{[0,1]} \arrow[r, "\bar\gamma"] \arrow[dr, "\gamma"'] & \mathbb{R} \arrow[d, "p"] \\
& S^1
\end{tikzcd}$$

(1). 0到m的定端同伦投影到S^1上，相对$*$定端同伦

(2) 任何0到m的道路都定端同伦 use $H(s,x)=\gamma s+(1-s)\gamma'$

(3). 同伦类只跟提升的端点有关.　　$\Rightarrow\ \pi_1(S^1)=\mathbb{Z}$

How to computes.

Van-Kampen. $U, V \subset X$ 开集 $+$ ($U\cap V$ 连通 $+$ 路)

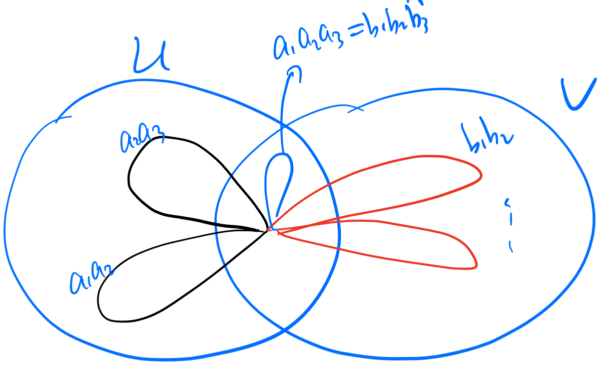
$U \cup V = X$, $* \in U\cap V$

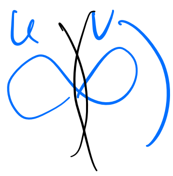
$$\pi_1(X,*) = \pi_1(U,*) * \pi_1(V,*) \big/ i_1\alpha \sim i_2\alpha$$

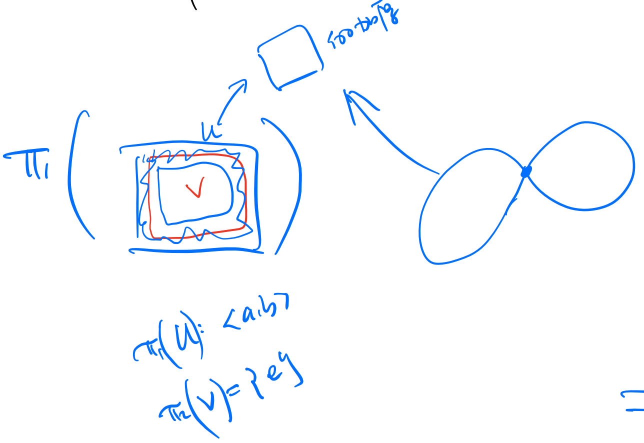
加入一些关系

$\alpha \in \pi_1(U\cap V)$. $i_1 : U\cap V \to U$

$i_2 : U\cap V \to V$.



例. $\pi_1\Big($  $\Big) = \langle a, b\rangle$



$\Rightarrow$ $i_1(1) = aba^{-1}b^{-1}$ in U

$i_2(1) = 1$ in V

$$\Rightarrow \langle a, b\rangle \big/ aba^{-1}b^{-1}$$