全纯函数

holomorphic

Ω 平面区域 → 连通开集

Ex: $f(z)=z^2$ $\quad g(z)=\bar{z}$ $\quad h(z)=|z|^2$

$$z_0\in\Omega,\quad \frac{f(z_0+h)-f(z_0)}{h}=\frac{(z_0+h)^2-z_0^2}{h}=\frac{2hz_0+h^2}{h}=2z_0+h\to 2z_0$$

$$\frac{g(z_0+h)-g(z_0)}{h}=\frac{\bar{h}}{h}$$ 不存在 limit

$h(z)=z\bar{z}$ 只有 $z=0$ 有导数.

C-R d.e. 用了 看 $\mathbb{R}^2\to\mathbb{R}^2$ 的导 → 这个一下子就完蛋了!

{ 也写了两个方向趋近

$g\circ f(z)$.

$$\frac{g\circ f(z+h)-g\circ f(z)}{h}=\frac{g\circ f(z+h)-g\circ f(z)}{f(z+h)-f(z)}\;\frac{f(z+h)-f(z)}{h}$$

$$=g'(f(z))f'(z)$$

formal derivative

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

$$f(z) = u(x,y) + i\,v(x,y)$$

$$f(\bar z) = u(x,-y) + i\,v(x,-y)$$

$$\overline{f(\bar z)} = u(x,-y) - i\,v(x,-y)$$

$$\overline{f(z)} = u(x,y) - i\,v(x,y).$$

X

we have $\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar f}{\partial \bar z}$

check

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\overline{\frac{\partial f}{\partial z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial \bar f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - i\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$$

复偏导. 定义式. 1 形式定义 (直接给公式) 即用算子理解.

2. 增量定义. (增量方程).

记忆式 (不严格). $f(z) \to f(x,y)$

$$\begin{cases} x = \frac{1}{2}(z+\bar z) \\ y = \frac{-i}{2}(z-\bar z) \end{cases}$$

$\Rightarrow$ use 链式法则.

且 $z, \bar z$ 形式上独立

满足"偏导"的各种性质

4.4 例子

Ex 4.2 $f:\Omega\to\mathbb{C}$ holo.

$g:\Omega\to\mathbb{C}$ 反holo

$$\Rightarrow \begin{cases} f(z_0+\Delta z)-f(z_0)=f'(z_0)\Delta z+o(\Delta z) \\ \bar g(z_0+\Delta z)-\bar g(z_0)=(\bar g)'(z_0)\Delta z+o(\Delta z) \end{cases}$$

$$\hookrightarrow \quad g(z_0+\Delta z)-g(z_0)=\overline{(\bar g)'(z_0)}\ \overline{\Delta z}+o(\Delta z)$$

$$(fg)(z_0+\Delta z)=\left(f(z_0)+f'(z_0)\Delta z+o(\Delta z)\right)\left(g(z_0)+\overline{(\bar g)'(z_0)}\ \overline{\Delta z}+o(\Delta z)\right)$$

$$=f(z_0)g(z_0)+f'(z_0)g(z_0)\Delta z+f(z_0)\overline{(\bar g)'(z_0)}\,\overline{\Delta z}+o(\Delta z)$$

$$\frac{\partial(fg)}{\partial z}(z_0)=f'(z_0)g(z_0) \qquad \frac{\partial(fg)}{\partial \bar z}(z_0)=f(z_0)\overline{(\bar g)'(z_0)} \qquad (4.7).$$

$\square$

Rmk. g 为 $\bar z$ 的多项式. $g(z)=p(\bar z)$

$\hookrightarrow$ $p(\bar z)$ 为 $\bar z$ 的复多项式

$$\overline{\bar g'(z_0)}=\overline{\left(\bar p(\bar z)\right)'}=\overline{\left(\frac{\partial \bar p}{\partial z}(z_0)\right)}=\frac{\partial p}{\partial \bar z}(z_0)=p'(\bar z_0)$$

$$\Rightarrow \frac{\partial(fg)}{\partial \bar z}(z_0)=f(z_0)\,p'(\bar z_0) \qquad (4.8).$$

Notes. 如果复值函数对 $z, \bar{z}$ 可分离分量

$\Rightarrow$ 求复偏导时 按独立变量即可

将 $z, \bar{z}$ 视作独立的变量，只是技巧 并非代表 $z, \bar{z}$ 无关。

In particular. f 为 $z, \bar{z}$ 的多项式

$$f = \sum_{k=0}^{m}\sum_{j=0}^{n} a_{k,j} z^{k} \bar{z}^{j}$$

利用 (4-7) (4-8)

$$\begin{cases} \dfrac{\partial f}{\partial z} = \sum_{k=1}^{m}\sum_{j=0}^{n} k a_{k,j} z^{k-1} \bar{z}^{j} \\ \dfrac{\partial f}{\partial \bar{z}} = \sum_{k=0}^{m}\sum_{j=1}^{n} j a_{k,j} z^{k} \bar{z}^{j-1} \end{cases}$$

Ex 4.3

求 $\mathbb{C}$ 上实可微函数，复可微点集恰好为圆周 ∂D

Sol. 圆周方程：$|z|^2-1=0$ 的零点集

$\Rightarrow$ D.E. $\dfrac{\partial f}{\partial \bar{z}} = |z|^2-1 = z\bar{z}-1$

只要是多项式

以上关于 $\bar{z}$ 积分 ($z, \bar{z}$ 独立变量)

$$f(z) = \frac{1}{2} z\bar{z}^2 - \bar{z} + C. = \frac{1}{2}|z|^2\bar{z} - \bar{z} + C \qquad \square$$

Rmk. 若 $g(z)$, holo. $g+f$ also satisfy.

Ex 4.4.

$f:\Omega\to\mathbb{C}$ 全纯，且 $|f|$ 为常数，则 f 为常数.

Pf. for. f holo $\Rightarrow$ $\dfrac{\partial f}{\partial \bar z}=0$

$f\bar f=$ const. $\dfrac{\partial \bar f}{\partial z}=\overline{\left(\dfrac{\partial f}{\partial \bar z}\right)}=0$

$g(w)=w\bar w$. $|f|=g\circ f$

$$\frac{\partial |f|}{\partial z}=\frac{\partial g}{\partial w}\frac{\partial f}{\partial z}+\frac{\partial g}{\partial \bar w}\frac{\partial \bar f}{\partial z}$$

$$=\bar w f'+0.=\bar w f'=\bar f f'=0\Rightarrow f'=0.$$

$\Rightarrow f$ 为常值函数. $\square$

HW.

1. $f(z)=\dfrac{xy^2(x+iy)}{x^2+y^4}$ $z=x+iy\neq 0$ $f(0)=0.$

①. $x=t\cos\theta$, $y=t\sin\theta$. $\theta\in[0,2\pi)$

$$\lim_{l\ni z\to 0}\frac{f(z)-f(0)}{z}=\lim_{t\to 0.}\frac{f(z)}{z}=\lim_{t\to 0.}\frac{t^3\cos\theta\sin^2\theta}{t^2\cos^2\theta+t^4\sin^4\theta}.$$

$$= \lim_{t\to 0} \frac{t\cos\theta \sin^2\theta}{\cos^2\theta + t^2\sin^4\theta}.$$

若 $\theta = \frac{\pi}{2} \Rightarrow \lim\limits_{t\to 0}\frac{t\cos\theta\sin^2\theta}{\cos^2\theta+t^2\sin^4\theta} = \lim\limits_{t\to 0}\frac{t\cdot 0}{t^2} = 0$

$x = y^2.$ $\quad y\to 0$

$$\lim_{y\to 0}\frac{f(z)-f(0)}{z} = \lim_{y\to 0}\frac{y^4}{y^4+y^4} = \frac{1}{2} \neq 0$$

2. f 在区域 Ω 上全纯.

$g(z) = \overline{f(\bar z)}$, 区域 $\Omega^* = \{\zeta ; \zeta^* \in \Omega\}$ 上全纯.

$$\frac{\partial g(z)}{\partial \bar z} = \frac{\partial\, \bar f \circ \overset{=\bar z}{h(z)}}{\partial \bar z} = \frac{\partial \bar f}{\partial z}\frac{\partial h}{\partial \bar z} + \frac{\partial \bar f}{\partial \bar z}\frac{\partial \bar h}{\partial \bar z} = \overline{\left(\frac{\partial f}{\partial \bar z}\right)}\overset{1}{\overset{\shortparallel}{\frac{\partial h}{\partial \bar z}}} + \overline{\left(\frac{\partial f}{\partial z}\right)}\overset{0}{\overset{\shortparallel}{\frac{\partial \bar h}{\partial \bar z}}}$$

$$= \overline{\left(\frac{\partial f}{\partial \bar z}\right)(\bar z)} = 0$$

$\Rightarrow \bar z \in \Omega^*$

$$\frac{\partial g(z)}{\partial z} = \frac{\partial\, \bar f \circ \overset{=\bar z}{h(z)}}{\partial z} = \frac{\partial \bar f}{\partial z}\overset{0}{\overset{\shortparallel}{\frac{\partial h}{\partial z}}} + \frac{\partial \bar f}{\partial \bar z}\overset{1}{\overset{\shortparallel}{\frac{\partial \bar h}{\partial z}}}$$

$$= \overline{\left(\frac{\partial f}{\partial z}\right)}(\bar z) = \overline{f'(\bar z)}.$$

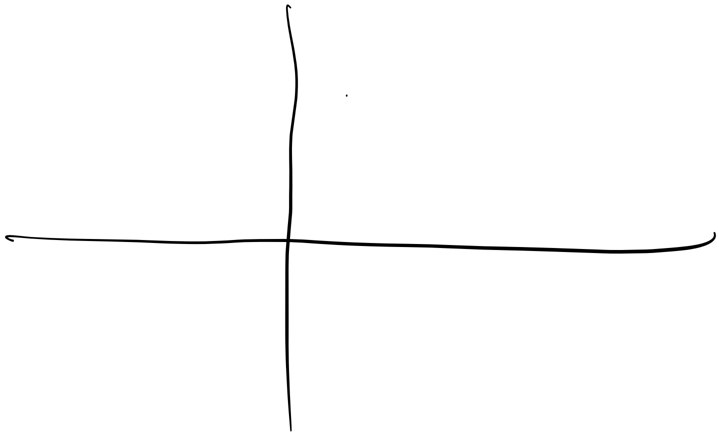
3. $f(z) = x^3y^2 + i x^2 y^3$

$$\frac{\partial f(z)}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

$$= \frac{1}{2}\left(3x^2y^2 + 2xy^3 i + i\left(2x^3y + 3x^2y^2 i\right)\right)$$

$$= \frac{1}{2}\left(\cancel{3x^2y^2} - \cancel{3x^2y^2} + (2xy^3 + 2x^3y)i\right).$$

$$= (xy)(x^2+y^2)\, i = 0$$



x 轴

y 轴

5. (1). $f(z) = \mathrm{Re}(z)$

$$= \frac{1}{2}(z + \bar{z})$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \neq 0.$$

(2). $f(z) = |z|^3 = (z\bar{z})^{\frac{3}{2}} = z^{\frac{3}{2}}\bar{z}^{\frac{3}{2}}$

$$\frac{\partial f}{\partial \bar{z}} = z^{\frac{3}{2}}\,\frac{3}{2}\cdot\bar{z}^{\frac{1}{2}} = \frac{3}{2}z|z| = 0$$

$z = 0$

或 $|z| = 0$ （$z=0$处 可微）

(3). $f(z) = z(z-1)\dfrac{z+\bar{z}}{2}$

$$= \frac{1}{2} z^2 (z-1) + \frac{\bar{z}}{2} z(z-1)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} z(z-1) = 0$$

$$z = 0, \quad z-1=0$$

$$z = 1.$$

6. $f(z) = az^2 + z\bar{z}$

$$\frac{\partial f}{\partial z} = 2az + \bar{z}$$

$$\frac{\partial f}{\partial \bar{z}} = z$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} = (a+1)z + \bar{z}$$