§0. Categories, Functors, and Natrual Transformations.

(0.1) Categories.

A category $\mathcal{C}$ consists of

(a). (objects) $Ob(\mathcal{C})$. the class of objects in $\mathcal{C}$.

(denote as $X \xrightarrow{f} Y$, $f \in Hom_{\mathcal{C}}(X,Y)$)

(b) (morphisms) $\forall X, Y \in Ob(\mathcal{C})$, we have a set $Hom_{\mathcal{C}}(X,Y)$.

s.t. $Hom_{\mathcal{C}}(X,Y) = Hom_{\mathcal{C}}(X',Y') \Leftrightarrow X = X', Y = Y'$

(c). (Composition law) $\forall X, Y, Z \in Ob(\mathcal{C})$. we have a map.

$$\begin{aligned} Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) &\xrightarrow{\circ} Hom_{\mathcal{C}}(X,Y) \\ (f, g) &\longmapsto g \circ f \end{aligned}$$

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & \circlearrowright & Z \end{tikzcd}$$

which satisfy the following two axioms

(1). (Associativity)

$$X \xrightarrow{f} Y, \quad Y \xrightarrow{g} Z, \quad Z \xrightarrow{h} W$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr] & Z \arrow[d, "h"] \\ X \arrow[u, "f"] \arrow[ur] \arrow[r] & W \end{tikzcd}$$

(2) (identity) $\forall X \in Ob(\mathcal{C}) \quad \exists X \xrightarrow{I_X} X.$

s.t. $h \circ I_X = h. \quad I_X \circ k = k$

$\forall X \xrightarrow{h} H. \quad K \xrightarrow{k} X$

$$\begin{tikzcd}
X \arrow[dr, "h"] \arrow[d, "I_X"'] & \\
X \arrow[r, phantom, "\circlearrowright"] \arrow[r, "h"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[d, "I_X"] \\
K \arrow[ur, "k"] \arrow[r, "k"'] \arrow[ur, phantom, "\circlearrowright"'] & X
\end{tikzcd}$$

Example.

(1) $\mathcal{C} = (Set)$ $\quad (Top)$

(Ab) $\quad (Top\ Gp)$

(Mod_R)

ring

(2) $\mathcal{C}^{op}$ (the opposite of $\mathcal{C}$)

$ob(\mathcal{C}) := ob(\mathcal{C})$

$Hom_{\mathcal{C}^{op}}(X, Y) := Hom_{\mathcal{C}}(Y, X).$

$$Hom_{\mathcal{C}^{op}}(X, Y) \times Hom_{\mathcal{C}^{op}}(Y, Z) \xrightarrow{\circ_{\mathcal{C}^{op}}} Hom_{\mathcal{C}^{op}}(X, Z)$$

$$(f, g) \qquad\qquad g \circ_{\mathcal{C}^{op}} f$$

$$X \underset{\mathcal{C}}{\overset{f}{\longleftarrow}} Y,\ Y \underset{\mathcal{C}}{\overset{g}{\longleftarrow}} Z \qquad\qquad Z \underset{\mathcal{C}}{\xrightarrow{f \circ g}} X$$

Terminology

$X, X' \in Ob(\mathcal{C})$. $X \xrightarrow[\mathcal{C}]{f} X'$

f is an isomorphism. $\Longleftrightarrow \exists\, X' \xrightarrow{\tilde{f}} X$.

s.t. $\tilde{f} \circ f = I_X$

$f \circ \tilde{f} = I_{X'}$

(0.2) (Functors)

①. $\mathcal{C}$, $\mathcal{C}'$. categories

A covariant / contravariant functor $F: \mathcal{C} \to \mathcal{C}'$ (not a map just write like this)

or write as $\mathcal{C} \xrightarrow{F} \mathcal{C}'$

Consists of (a) a rule of associating to each $X \in Ob(\mathcal{C})$ an object

$$F(X) \in Ob(\mathcal{C}')$$

(b). a map $Hom_{\mathcal{C}}(X,Y) \xrightarrow{F} Hom_{\mathcal{C}'}(F(X), F(Y))$

for each pair $X, Y. \in Ob(\mathcal{C})$. $F(Y), F(X)$,

s.t. $F(I_X) = I_{F(X)}$ and

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "g\circ f"'] \arrow[rr, phantom, shift left=3, "\circlearrowleft"] & & Z \end{tikzcd} \longrightarrow \begin{tikzcd} & F(Y) \arrow[dr, "F(g)"] & \\ F(X) \arrow[ur, "F(f)"] \arrow[rr, "F(g\circ f)"'] \arrow[rr, phantom, shift left=3, "\circlearrowleft"] & & F(Z) \end{tikzcd}$$

i.e $F(g\circ f) = F(g) \circ F(f)$

$F(f) \circ F(g)$ 反过来

Examples

(1). $\mathcal{C} \xrightarrow{op} \mathcal{C}^{op}$

$$\begin{cases} X^{op} = X \\ \mathrm{Hom}_{\mathcal{C}}(X,Y) \ni X \xrightarrow{f} Y \xrightarrow{op} X \xrightarrow[\mathcal{C}]{f} Y \in \mathrm{Hom}_{\mathcal{C}^{op}}(Y,X) \end{cases}$$

$\Rightarrow$ Contravariant functor

(2). $\forall X \in Ob(\mathcal{C})$. $h_X : \mathcal{C} \to (Set)$

$h_X(Y) := \mathrm{Hom}_{\mathcal{C}}(Y,X)$. $\forall Y \in Ob(\mathcal{C})$.

$$\begin{tikzcd} h_X(f) := h_X(Y) \arrow[r, "\circ f"] \arrow[d, equal] & h_X(Y') \arrow[d, equal] & \forall Y' \xrightarrow[\mathcal{C}]{f} Y \\ \mathrm{Hom}_{\mathcal{C}}(Y,X) & \mathrm{Hom}_{\mathcal{C}}(Y',X) & \end{tikzcd}$$

is a map in (Set).

$\Rightarrow$ Contravariant functor

(a3). (Natural Transformations).

$\mathcal{C} \overset{F_1}{\underset{F_2}{\rightrightarrows}} \mathcal{C}'$: two functors of the same variance $+1$ / -1

(1) A <u>natural transformation</u> T. from F_1 to F_2. (denoted as $F_1 \xrightarrow{T} F_2$)

is a rule of associating to each $X \in Ob(\mathcal{C})$. a morphism.

$F_1(X) \xrightarrow[\mathscr{C}']{T(X)} F_2(X)$ s.t. for each $X \underset{\mathscr{C}\, f}{\overset{f}{\rightleftarrows}} Y$ we have

$$\begin{tikzcd} F_1(X) \arrow[r, "T(X)"] \arrow[d, "F_1(f)"'] \arrow[dr, phantom, "\circlearrowright"] & F_2(X) \arrow[d, "F_2(f)"] \\ F_1(Y) \arrow[r, "T(Y)"] & F_2(Y) \end{tikzcd}$$

- $\mathscr{C} \underset{F_2}{\overset{F_1}{\rightrightarrows}} \mathscr{C}'$ $\quad F_1 \xrightarrow{T} F_2$

  $\xrightarrow[F_3]{}$ $\quad F_2 \xrightarrow{S} F_3$.

$\rightsquigarrow F_1 \xrightarrow{S \circ T} F_3 \qquad F_1(X) \xrightarrow{T(X)} F_2(X) \xrightarrow{S(X)} F_3(X).$

(2). A natural transformation $F_1 \xrightarrow{T} F_2$ is called a natural equivalence if

$F_1(X) \xrightarrow[\mathscr{C}]{T(X)} F_2(X)$ is an isom. for each $X \in Ob(\mathscr{C})$