W1L1

$\underbrace{0,\ 1}$	2	3
局部理论	曲线	曲面
	整体	整体

作业 12/14 100
小测 3次 300 } 50%

期末 50%

E^3 中曲线 → E^3中曲线

$\{O; E_1, E_2, E_3\}$. 右手系

$\langle E_i, E_j \rangle = \delta_{ij}$

$\forall \vec{x} \in E^3 \quad x = x^1 E_1 + x^2 E_2 + x^3 E_3 = \sum_{i=1}^{3} x^i E_i = x^i E_i$

曲线C是 $X: I \longrightarrow E^3$ 同胚.

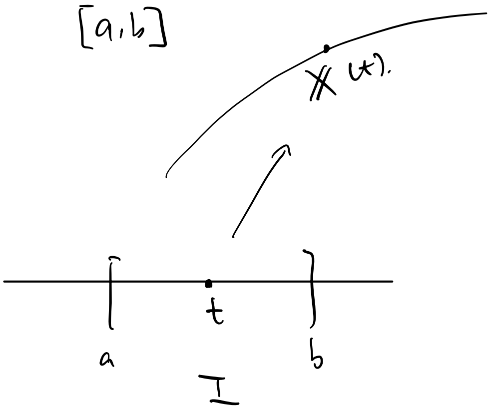
$I = (a,b)$

$[a,b]$

$[a,b)$

$[a,b]$

$\boxed{X: \text{连续} \quad X^{-1}\ \text{连续} \qquad X: 1\text{-}1\ \text{到上}}$



局部定义. $X: I \longrightarrow E^3$

$t \longmapsto X(t) \in E^3$

$X(t) = x^i(t) E_i = (x^1(t), x^2(t), x^3(t))$.

$X(t)$ 可微 $\Longleftrightarrow$ $x^i(t)$ 可微 $i=1,2,3$ $\qquad \dfrac{dX}{dt} = \dfrac{dx^i}{dt} E_i$

正则曲线：如果曲线 C. $X = X(t)$，是可微的 且 $\|X'(t)\| \neq 0$, $\forall t \in I$.
>0.

称 C 是正则曲线.

$$\left\|\frac{dX}{dt}\right\|^2 = \sum_{i=1}^{3} \left(\frac{dx^i}{dt}\right)^2 > 0 \quad \left(\text{i.e. } \exists i.\ \frac{dx^i}{dt} \neq 0.\ \forall t \in I\right)$$

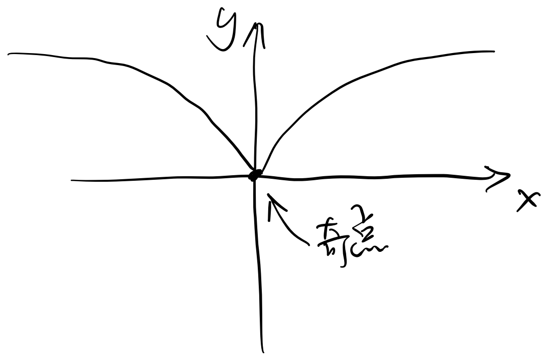
若 $\exists t_0 \in I$. $\left\|\dfrac{dX}{dt}(t_0)\right\| = 0$，则称 C 在 $X(t_0)$ 点是奇异的.

↓

奇点 → 难.

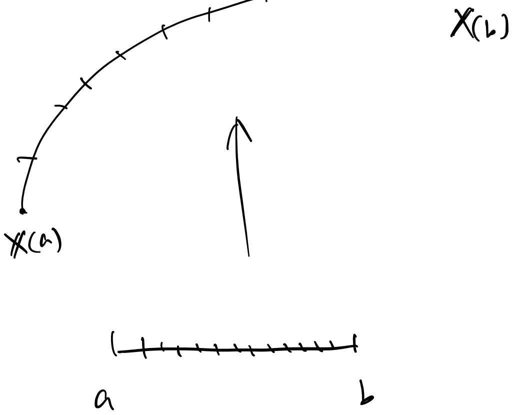
e.g. $X(t) = (t^3, t^2, 0)$

$y = x^{\frac{2}{3}}$



$X'(t) = \dfrac{dX}{dt}$ 称为曲线 C 在 $X(t)$ 处 切向量

弧长 $\quad C$: $X: I = (a,b) \longrightarrow X(I) \subset E^3$



$l(X, \Delta)\ \sum_{i=1}^{n} |X(t_i) - X(t_{i-1})|$

Δ 分割 $\quad a = t_0 < \cdots < t_n = b$

Δ 范数.

$|\Delta| = \max_i |t_i - t_{i-1}|$.

$$S=\lim_{|\mathcal{A}|\to 0} l(s,\mathcal{A})$$

下面 $x(t)$ 与 $\gamma(t)$ 同义！

$$=\lim_{|\mathcal{A}|\to 0}\sum_{i=1}^{n}|x(t_i)-x(t_{i-1})|$$

$$=\lim_{|\mathcal{A}|\to 0}\sum_{i=1}^{n}\frac{|x(t_i)-x(t_{i-1})|}{t_i-t_{i-1}}(t_i-t_{i-1})$$

中值定理 $\Rightarrow$ 积分定义

$$=\int_a^b\left|\frac{dx}{dt}\right|dt$$

$$S(t)=\int_a^t\left|\frac{dx}{dt}\right|dt$$

. 变上限：t 的可微分函数.

Prop. s 与参数 t 的选取无关

Pf.

$t=t(\tau)\Rightarrow S(t)=S(t(\tau))$

$0=t(0)$ (assume) (同起点)

$\frac{dt}{d\tau}>0$ (正比)

$$S(\tau)=\int_0^{\tau}\left|\frac{dx}{d\tau}\right|d\tau$$

$$=\int_0^t\left|\frac{dx}{dt}\frac{dt}{d\tau}\right|\frac{d\tau}{dt}\,dt$$

拿出来

$$=\int_0^t\left|\frac{dx}{dt}\right|dt \qquad \square$$

若取. 弧长为参数 $\quad x=x(s)$

$$\frac{ds}{dt}=\left|\frac{dx}{dt}\right|=\frac{ds}{ds}=1 \quad\Rightarrow\quad \left|\frac{dx}{dt}\right|=1.\ \text{单位向量}$$

反之 $\Leftarrow$. 若 $\left|\frac{dx}{dt}\right|=1$. $S(t)=\int_0^t\left|\frac{dx}{dt}\right|dt=t \Rightarrow$ 以弧长为参数.

Prop. 曲线的参变量为弧长的充要条件为 $\left|\frac{dx}{dt}\right|=1,\ \forall t\in I$

Prop. 弧长与坐标系选取无关

$\{E_i\}$ $\{e_i\}$ 幺正标架.

$E_i = a_i^j e_j$ $A=\{a_i^j\}$ 正交矩阵. review

$A^T A = I$

$$(E_1\ E_2\ E_3) = (e_1\ e_2\ e_3)A = (e_1\ e_2\ e_3)\begin{pmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{pmatrix}$$

$$x(t) = x^i(t)E_i = y^j(t)e_j \quad \Rightarrow \quad y^j(t) = x^i(t)a_i^j$$

$$= x^i(t)\,a_i^j e_j$$

$$S(t) = \int_0^t \left(\sum_{j=1}^3 \left(\frac{dy^j}{dt}\right)^2\right)^{\frac12} dt \overset{?}{=} \int_0^t \left(\sum_{i=1}^3 \left(\frac{dx^i}{dt}\right)^2\right)^{\frac12} dt$$

$$\| $$

$$\int_0^t \left(\left(\frac{dy^1}{dt}\right)^2 + \left(\frac{dy^2}{dt}\right)^2 + \left(\frac{dy^3}{dt}\right)^2\right)^{\frac12} dt$$

$$= \int_0^t \cdots \underset{\text{全部展开}}{} = \int_0^t \left(\sum_{i=1}^3 \left(\frac{dx^i}{dt}\right)^2\right)^{\frac12} dt. \quad \square$$

以弧长为参数

$$\frac{dx}{ds} = \dot{x} \quad \frac{d^2x}{ds^2} = \ddot{x} \quad \frac{d^3x}{ds^3} = \dddot{x} \quad \frac{d^4x}{ds^4} = x^{(4)}$$

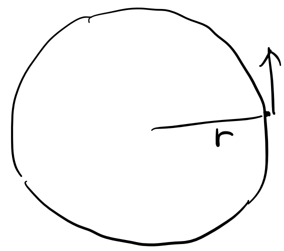
正则曲线. $\frac{dx}{ds} = \dot{x} = T(s)$ (单位切向量)

$T(s)$ 的变化率刻画曲线弯曲程度.

$$|\dot{T}(s)| = k(s) \geq 0 \quad \text{曲率.}$$

$$|\ddot{x}| =$$

若 $k(s)\neq 0$, >0. 称 $\frac{1}{k(s)}$ 为曲率半径

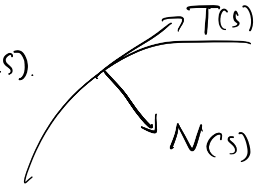


$k=\frac{1}{r}$

$|T(s)|=1 \Leftrightarrow T\cdot T=1$

$\Rightarrow \dot{T}\cdot T=0$ 正交

$N(s)$.



$\dot{T}(s)=k(s)N(s)$

$|N(s)|=1$. 主法向量

$x(s)+\frac{1}{k(s)}N(s)=\rho(s)$: 曲率中心

$B=T\times N$ $\{x(s);T,N,B\}$ 右手系标架.

从法向量.

span(T,N) 密切平面

span(N,B). 法平面

span(B,T). 从切平面

Frenet 标架.. 活动标架.

$\dot{T}=k(s)N(s)$

$B\cdot B=1 \Rightarrow \dot{B}\cdot B=0$

$B\cdot T=0 \Rightarrow \dot{B}\cdot T+B\cdot\dot{T}=0$

$\dot{B}T=-B\cdot\dot{T}=-BkN=0$

$\Rightarrow \dot{B}\parallel N$ 令 $\dot{B}=-\tau(s)N$ $\tau(s)$ 挠率.

若 $k(s)\neq 0 \Rightarrow N(s)$ 唯一. $\Rightarrow \dot{B}$ 唯一.

若 $=0$. $N(s)$ 的选取任意了 ...

$-\dot{B}N=\tau(s)$

$B\cdot N=0 \Rightarrow \dot{B}N+B\dot{N}=0 \Rightarrow \tau(s)=B\dot{N}$

$$= B\left(\frac{1}{k}\dot{T}\right)'$$

$$= (T\times N)\cdot\left[\left(\frac{1}{k}\right)'\dot{T} + \frac{1}{k}\ddot{T}\right]$$

$$= \frac{1}{k}(T\times N)\ddot{T}$$

$$= \frac{1}{k}\left(T\times \frac{1}{k}\dot{T}\right)\ddot{T}$$

$$= \frac{1}{k^2}(T, \dot{T}, \ddot{T})$$

$$= \frac{1}{|\ddot{x}|^2}(\dot{x}, \ddot{x}, \dddot{x})$$

"几何量"与参数、坐标系选取无关。（空间运动不变量）.

$\{O_1, E_1, E_2, E_3\}$

$\{O_2, e_1, e_2, e_3\}$

C: $x_1(s)$. $x_2(s)$

$$\Rightarrow x_2(s) = A x_1(s) + B \quad \begin{cases} A: \text{正交} \\ B: \text{常向量} \end{cases}$$

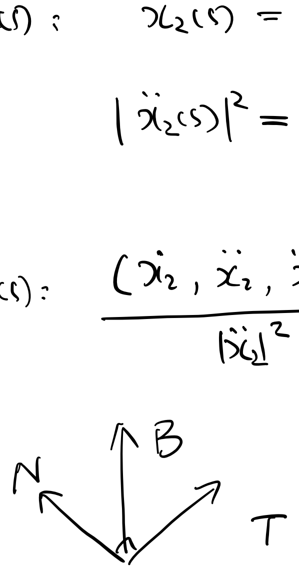
在求导下，A, B 不变.

$k(s)$: $\ddot{x}_2(s) = A\ddot{x}_1(s)$.

$$|\ddot{x}_2(s)|^2 = |A\ddot{x}_1(s)|^2 = \ddot{x}_1(s)^T \underbrace{A^T A}_{=I} \ddot{x}_1(s) = \ddot{x}_1(s)^T \ddot{x}_1(s)$$

$$\tau(s): \frac{(\dot{x}_2, \ddot{x}_2, \dddot{x}_2)}{|\ddot{x}_2|^2} = \frac{(A\dot{x}_1, A\ddot{x}_1, A\dddot{x}_1)}{|\ddot{x}_1|^2} = \frac{(\dot{x}_1, \ddot{x}_1, \dddot{x}_1)}{|\ddot{x}_1|^2}$$

→ 想几何意义 不变.



$N\cdot N = 1 \Rightarrow \dot{N} N = 0$

另一种方法. 设 $\dot{N} = aT + bB$

$a = \dot{N}T \overset{*}{=} -N\dot{T} = -k(s)$

$b = \dot{N}B = -N\dot{B} = \tau(s)$.

$$N(s) = B(s)\times T(s)$$

$$\dot{N}(s) = \dot{B}\times T + B\times\dot{T} = -\tau N\times T + B\times kN = \tau B - kT$$

Conclusion

1. 曲线表示 Frenet 标架.

2. "几何量". 弧长. 曲率、挠率. 与坐标选取.与参数选取无关.

△. 正交矩阵复习. ✓ ?

△ $(a\times b)\cdot c = (a,b,c) = \det(a\ b\ c)$