(D) Classification of isolated singularities

Let f be a holo func, which has an iso. sing at $z=c$. We have seen that if $r_0>0$. is st. $B_{r_0}(c)\setminus\{c\}\subseteq D_f$. we have

$$f(z)=\sum_{-\infty}^{\infty} a_n(z-c)^n \quad \left(z\in B_{r_0}(c)\setminus\{c\}\right). \quad \text{where} \quad a_n=\frac{1}{2\pi i}\int_{\partial B_r(c)}\frac{f(w)}{(w-c)^{n+1}}\,dw. \quad \begin{cases} n\in\mathbb{Z}. \\ 0<r<r_0 \end{cases}$$

Def. $\mathrm{ord}_{z=c} f(z)$ (or $\mathrm{ord}_c f$) $:= \inf\{n\in\mathbb{Z} \mid a_n\neq 0\}$

(0) (Convention. $\inf\phi=-\infty$). $f=0$ near c. $\Leftrightarrow$ $\mathrm{ord}_c f=\infty$

(1). f has a zero of order n_0 at c. if $\mathrm{ord}_c f=n_0\in\mathbb{N}\cup\{0\}$

(2) f has a pole of order n_0 at c if $-\mathrm{ord}_c f=n_0\in\mathbb{N}$

(3). f has an essential singularity at c if $\mathrm{ord}_c f=-\infty$

- In case (1)., we may extend f holomorphically across $z=c$. simply by setting $f(c):=a_0$ (在圆盘上直接=幂级数. $\Rightarrow$ holo)

  Therefore c is called a removable singularity of f.

  example: $\frac{\sin z}{z}$. at $z=0$

- In case (1) and (2), there exists a unique $m\in\mathbb{Z}$. st.

  $\lim_{z\to c}(z-c)^m f(z)$. exists and is nonzero Actually. $m=-\mathrm{ord}_c f$

$a_n = \frac{1}{2\pi i}\int_{\partial B_r(c)} \frac{f(z)}{(z-c)^{n+1}}\,dz = 0$ if $n<0$ and $f(z)$ have limit

In particular (1) $\Longleftrightarrow \lim_{z\to c} f(z)$ exists in $\mathbb{C}$

(2) $\Longleftrightarrow \lim_{z\to c} \frac{1}{|f(z)|} = 0$ $\left(\begin{array}{l}\text{written as } \lim_{z\to c}|f(z)| = \infty \\ \lim_{z\to c} f(z) = \infty\end{array}\right)$

(1)

$z = c + re^{i\theta}$. $\theta \in (0, 2\pi)$

$n<0$

$f(z)(z-c)^m \to A \neq 0.$

$\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(c+re^{i\theta})}{r^{n+1} e^{i(n+1)\theta}}\, r i e^{i\theta}\, d\theta$

$\frac{1}{|f(z)|} \to \lim_{z\to c}\left|\frac{(z-c)^m}{A}\right| = 0$

$= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(c+re^{i\theta})}{r^n e^{in\theta}}\, d\theta$

Besides. $\mathrm{Res}_c f = \lim_{z\to c} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-c)^m f(z)\right)$

$= \frac{1}{2\pi}\int_0^{2\pi} f(c+re^{i\theta})\, r^m e^{im\theta}\, d\theta \quad m=-n>0$

Sometimes we use this expression to compute $\mathrm{Res}_c f$ if c is a pole of f

$= \frac{1}{2\pi} r^m \int_0^{2\pi} f(c+re^{i\theta})\, e^{im\theta}\, d\theta$

$\overset{?}{=} 0$

(2). Consider $\frac{1}{f(z)}$

Therefore (3) $\Longleftrightarrow$ neither does $\lim_{z\to c} f(z)$ exists in $\mathbb{C}$ nor $\lim_{z\to c}|f(z)| = \infty$

都有可能 $\left\{\begin{array}{l}① .\ \exists\, c_n \in D_f\ (n\in\mathbb{N}).\ \text{s.t. } c_n \to c.\ f(c_n) \to \text{a given } L \in \mathbb{C} \\ ② .\ \exists\, c_n \in D_f\ (n\in\mathbb{N}).\ \text{s.t. } c_n \to c\ \ \frac{1}{|f(c_n)|} \to 0 \text{ as } n\to\infty \\ ③ .\ \exists\, c_n \in D_f\ (n\in\mathbb{N})\ \text{s.t. } c_n \to c\ \ f(c_n) \text{ does not have a limit in } \mathbb{C}\cup\{\infty\}\end{array}\right.$

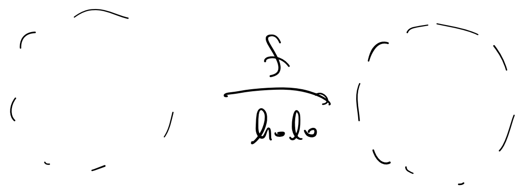
Picard's great theorem

Maximal Module principle → 代数基本定理.

Schwarz's lemma.

Let $D = B_1(0) \subseteq \mathbb{C}$.

If $D \xrightarrow{f} D$ is a holo func s.t. $f(0) = 0$.



then. $\begin{cases} \forall z \in D. \ |f(z)| \le |z| \\ |f'(0)| \le 1 \end{cases}$

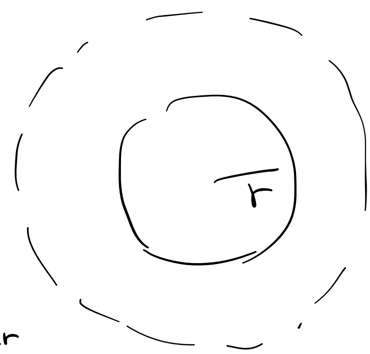
旋转

Moreover, if $\exists z_0 \in D$ $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$, then $\exists c \in \mathbb{C}$ s.t. $|c| = 1$ and $\forall z \in D$ $f(z) = cz$.

$f(z)$. $f(0)=0$ 泰勒展开/z 没有负项 "可能去奇异点"

Pf. Let $g(z) := \begin{cases} \frac{f(z)}{z} & z \in D \setminus \{0\} \\ f'(0). & z = 0. \end{cases}$

$\Rightarrow g(z)$. holo on D.



let. $0 < r < 1$.

$$\max_{z \in \overline{B_r(0)}} |g(z)| = |g(z_r)| \text{ for some } z_r \in \partial B_r(0)$$

→ 如果在里面 则由极大模原理 → Constant = 0. ✓

$$|g(z_r)| = \frac{|f(z_r)|}{|z_r|} \le \frac{1}{r}$$

令 $r \to 1^-$. $|g(z)| \leq 1$ for all $z \in D$

In other word, $\dfrac{|f(z)|}{|z|} \leq 1$. if $z \in D \setminus \{0\}$

$\}$ $|f'(0)| \leq 1$

if. $\exists z_0 \in D$ $|f(z_0)| = |z_0|$ $\Rightarrow$ $|g(z)|$ attains its maximum in D

or $|f'(0)| = 1$

$\xrightarrow{\text{maximal module principle}}$ $g = \text{const. } C$ $\Rightarrow$ $f(z) = cz$ for all $z \in D$

and $|c| = 1$

Ex. If $D \xrightarrow[\text{holo}]{f} D$ is a bijection. and. $f(0) = 0$, then $\exists$ $c \in \mathbb{C}$

$\Rightarrow$ f^{-1} is holo (why?)

$\Rightarrow$ use $g(z) \leq 1$ and $g^{-1}(z) \leq 1$

$\Rightarrow$ $g(z) = 1$

s.t. $|c| = 1$ and. $f(z) = cz$ for all $z \in \mathbb{C}$