W3 L1

P39. 4.8. <u>Rmk.</u> $F(z_1, z_2)$ 双全纯函数

则 $F(z, \bar{z})$ 求复偏导.可以分开算. $\Rightarrow$ P75

例. "<u>复可微点集</u>"

$\frac{\partial f}{\partial \bar{z}} = f(z)$. → 零点即为复可微点集.

Chap 5.

$f = u + iv$

$f: \Omega \to \mathbb{R}^2$. $(x, y) \to (u(x,y), v(x,y))$

$$J_f(z_0) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}(z_0)$$

<u>Prop 5.1.</u> $\det(J_f(z_0)) = \left|\frac{\partial f}{\partial z}(z_0)\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}(z_0)\right|^2$

若 f 全纯. $\det(J_f(z_0)) = |f'(z)|^2$

△几何证明.

Thm. 5.1 ✓

$\Rightarrow$ 推广到复可微.

$\Downarrow$

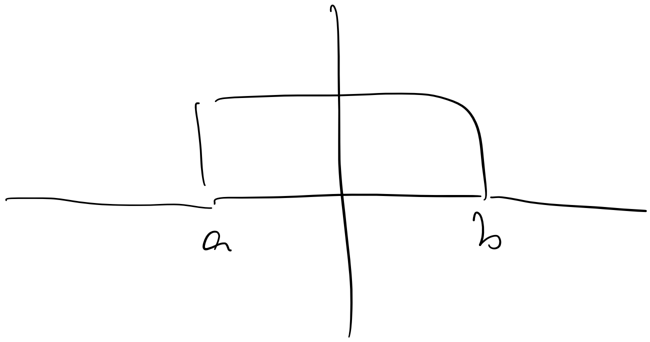
Thm 5.2 ✓

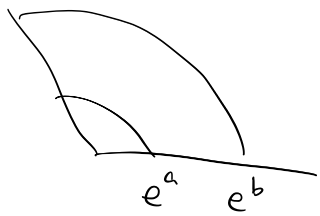
讨论复可微意义 $\overset{\text{Thm.5.3.}}{\Longleftrightarrow}$ 保角. $\boxed{\text{分析与几何}}$

两种看法. 1° 引入曲线严格定义. $f'(p) = \dfrac{\eta'(0)}{\gamma'(0)}$

2° 由CR方程,结合Jacobi矩阵. 旋转+伸缩矩阵.

指数函数 $e^x \cdot e^{iy}$. 处处共形.





实可微

$$\det(J_f(z_0)) = \left|\frac{\partial f}{\partial z}(z_0)\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}(z_0)\right|^2$$

$$\|Jx\|_{max} := \sqrt{\lambda_{max}(J^T J)}$$

$$T_f : h \longmapsto f(z_0) + \frac{\partial f}{\partial z}(z_0)\, h + \frac{\partial f}{\partial \bar{z}}(z_0)\, \bar{h}$$

$$\|Jx\|_{min} = \sqrt{\lambda_{min}(J^T J)}.$$

$$\frac{\partial f}{\partial z} = a.\quad \frac{\partial f}{\partial \bar{z}} = b$$

$$J = \begin{pmatrix} \mathrm{Re}(a+b) & -\mathrm{Im}(a-b) \\ \mathrm{Im}(a+b) & \mathrm{Re}(a-b) \end{pmatrix}$$

$$J^T J = \begin{pmatrix} \mathrm{Re}(a+b) & \mathrm{Im}(a+b) \\ -\mathrm{Im}(a-b) & \mathrm{Re}(a-b) \end{pmatrix} \begin{pmatrix} \mathrm{Re}(a+b) & -\mathrm{Im}(a-b) \\ \mathrm{Im}(a+b) & \mathrm{Re}(a-b) \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{Re}^2(a+b) + \mathrm{Im}^2(a+b) & -\mathrm{Re}(a+b)\,\mathrm{Im}(a-b) + \mathrm{Im}(a+b)\,\mathrm{Re}(a-b) \\ -\mathrm{Im}(a-b)\,\mathrm{Re}(a+b) + \mathrm{Re}(a-b)\,\mathrm{Im}(a+b) & \mathrm{Re}^2(a-b) + \mathrm{Im}^2(a-b) \end{pmatrix}$$

$$= \begin{matrix} |a+b|^2 & \\ & |a-b|^2 \end{matrix}$$

4.5

T8. $f = u + iv$ $\Omega \to \mathbb{C}$ 全纯

$\Rightarrow \frac{\partial f}{\partial \bar{z}} = 0$

现在看 $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$

$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$

$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)i\right)$

$= \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}i = \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x}i$

1) u 为常数 $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial \bar{z}} = 0$

2) $u = v^2 \Rightarrow \frac{\partial f}{\partial \bar{z}} = 2v\left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}i\right) = \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x}i$

$\Rightarrow 2v\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}$

$-2v\frac{\partial v}{\partial y} = \frac{\partial v}{\partial x}$

$2v\left(-2v\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y}$

$\Rightarrow v$ 为常数

5.5 习题

1. f在D上全纯 且$f'\neq 0 \Rightarrow f$在D上共形

(1). $u_0+iv_0 \in f(D)$

$Re(f(z)) = u_0$ 设曲线为 $\gamma_1(t)$ $\gamma_2(t)$ $\gamma_1(0)=\gamma_2(0)=u_0+iv_0$

$Im(f(z)) = v_0$ $f(z)=f(\gamma_1(t))$

有 $Re(f(z)) = Re(f(\gamma_1(t))) = u_0$

即 $Re(\lambda_1(t)) = u_0 \Rightarrow Re(\lambda_1'(t)) = 0$

$\Rightarrow \lambda_1'(t) \perp \lambda_2'(t)$

同理 $Im(\lambda_2(t)) = v_0 \Rightarrow Im(\lambda_2'(t)) = 0$

由 $\lambda_1(t)$ 与 $\lambda_2(t)$ 在 在 u_0+iv_0处 正交

且由 f 共形 $\Rightarrow$ $\gamma_1(t)$ 与 $\gamma_2(t)$ 在 $t=0$ 处正交

(2). $r_0 e^{i\theta_0} \in f(D) \setminus \{0\}$ $(r_0>0)$

$|f(z)| = r_0$

$\arg f(z) = \theta_0$

设曲线为 $\gamma_1(t)$ $\gamma_2(t)$. $\gamma_1(0)=\gamma_2(0)=r_0 e^{i\theta_0}$

$|f(\gamma_1(t))| = r_0$ $|\lambda_1(t)| = r_0$. $\Rightarrow \lambda_1(t)$ 与 $\lambda_2(t)$ 在 $r_0 e^{i\theta_0}$ 处正交

$\arg f(\gamma_2(t)) = \theta_0$ $\arg \lambda_2(t) = \theta_0$. $\Rightarrow \gamma_1(t)$ 与 $\gamma_2(t)$ 在 $t=0$ 处正交.