Algebra. Lec 8. Sylow theorem & application.

Thm. 64. $|G| = p^n \cdot m$. $(p, m) = 1$

①. H: p-subgroup of G $\Rightarrow$ H is contained a p-Sylow

② all p-Sylow are conjugate

③ $Syl_p(G) \triangleq$ the set of all p-Sylow subgroups of G

$$n_p \triangleq |Syl_p(G)| \equiv 1 \pmod p$$

Rmk. ①. if $n_p = |Syl_p(G)| = 1$.

$$|S| = 1 = \{P\}$$

i.e. $\forall g \in G$. $gPg^{-1} = P \Rightarrow P \trianglelefteq G$

Rmk. ② G finite p-group $\Rightarrow$ solvable

pf. class equation.

$$|G| = |Z(G)| + \underbrace{\sum_i (G : G_{x_i})}_{\text{divisible by } p.}$$

$\Rightarrow$ $Z(G)$ non-trivial.

$G \supset Z(G) \supset \{e\}$.

$G/Z(G) \quad \cdots \quad \{Z(G)\}$ $\Rightarrow$ ✓

by induction

Lem. 6.7. $|G|$. finite. p: smallest prime dividing $|G|$

$H \leq G$. $(G:H) = p \Rightarrow H \trianglelefteq G$

Pf. Let. $N_H = N \Rightarrow H \leq N \leq G$

则 由 $(G:H) = p \Rightarrow N = G$ ✓ Lag.

or $N = H$

consider $N = H$. $|\text{the orbit of } H \text{ under Conjugation}| = |\{gHg^{-1} \mid g \in G\}|$

G action

use G-action to understand. $= (G:N_H)$

$= p$

$G \xrightarrow{\text{iso / hom}} \mathrm{Perm}(\{H_1, \ldots, H_p\}) \cong S_p$

hom

$g \longmapsto \pi_g : H_i \longrightarrow gH_ig^{-1}$

$\ker(\varphi) = N_{H_1} \cap N_{H_2} \cap \cdots \cap N_{H_p}$

$$\begin{array}{c} \shortparallel \\ H \end{array}$$

$\Rightarrow \ker(\varphi) \leq H$

$\ker(\varphi) = H \quad \Rightarrow \quad H \trianglelefteq G$

$\ker(\varphi) \underset{\neq}{\leq} H \quad \underbrace{(G:\ker\varphi) \overset{\text{1st. iso}}{=} (G:H)\,(H:\ker(\varphi))}_{\text{Lag}} = |\operatorname{im}(\varphi)| \,\Big|\, |S_p|$

$\Rightarrow \quad p \cdot (H:\ker(\varphi)) \mid p!$

$(H:\ker(\varphi)) \mid (p-1)!$

$\Rightarrow$ get a prime factor in $(H:\ker(\varphi))$ $\square$

contradiction

Prop (application of Sylow)

p, q. distinct prime $\quad |G| = pq \quad \Rightarrow \quad G$ solvable

pf. Let $p<q$. $\Rightarrow \quad P \in \mathrm{Syl}_p(G) \quad n_p = 1, 1+p, 1+2p, \dots \quad n_p \mid q$

$Q \in \mathrm{Syl}_q(G) \quad n_q = 1, 1+q, 1+2q \dots \quad n_q \mid p$

$\Rightarrow n_q = 1$

$\Rightarrow \quad Q \trianglelefteq G$

$$G = G_0 \supset \underbrace{Q}_{} \supset \{e\} \quad \Rightarrow \text{solvable}$$

G_0/Q Q

p q

cyclic   cyclic

eg.: $|G| = 35 = \underset{p<q}{5 \times 7}$. $\Rightarrow$ G . cyclic.

$H_5 \in Syl_5(G)$ $n_5 = 1, 6, 11,$ $n_5 \mid 7$

$H_7 \in Syl_7(G)$ $n_7 = 1, 8, 15$ $n_7 \mid 5$

$\Rightarrow n_5 = 1, \ n_7 = 1$

Consider. $H_5 \xrightarrow{\phi} Aut(H_7)$

$x \longmapsto \phi_x : H_7 \to H_7$

$y \mapsto xyx^{-1}$ for $Syl_7(G)$ has only one element. so. $xH_7x^{-1} = H_7$

$\Rightarrow \phi_x \in Aut(H_7)$

$\Rightarrow im(\phi) \simeq H_5/Ker(\phi)$

$im(\phi) \leq Aut(H_7)$

$\Rightarrow |im(\phi)| \ \big| \ |Aut(H_7)|$ $Aut(H_7) \simeq (\mathbb{Z}/7\mathbb{Z})^{\times} \simeq (\mathbb{Z}/6\mathbb{Z})$

$\Rightarrow |im(\phi)| \ \big| \ 6_{=q-1}$

$\Rightarrow |im(\phi)| = 1, 2, 3, 6$ $\Rightarrow$ $\underset{p \nmid q-1}{|im(\phi)| = 1}$ $\Rightarrow \phi$: trivial

$\forall x \in H_5. \ y \in H_7 \ xy = yx$

Consider $\begin{pmatrix} H_5 = \langle x \rangle \\ H_7 = \langle y \rangle \end{pmatrix}$

$\langle x \rangle \times \langle y \rangle \longrightarrow G$ hom.

$(x^{n_1}, y^{n_2}) \longmapsto (x^{n_1} y^{n_2})$ surjective. and $|\langle x \rangle \times \langle y \rangle| = |G|$

$\Rightarrow \quad \langle x \rangle \times \langle y \rangle \simeq G$

$$\begin{array}{c} \langle x \rangle \times \langle y \rangle \simeq G \\ \text{\rotatebox{90}{$\simeq$}} \\ (\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/7\mathbb{Z}) \simeq \mathbb{Z}/35\mathbb{Z} \end{array}$$

$\downarrow$

$(5,7)=1$

G : cyclic

eg. $|G| = 2 \times 7$

$n_2 = 1, 3, 5, 7$, $\quad n_2 | 7 \quad \Rightarrow n_2 = 1, 7$

$n_7 = 1, 8, \ldots \quad n_7 | 2 \quad \Rightarrow n_7 = 1$

$\Rightarrow$ isomorphic classes of groups

① $\mathbb{Z}/14\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$

② D_{14}.

prime.

p^2

$p \times q$.

1 2 3 4 5 6 7 8 9 10 11

Δ

5 classes

finite abelian group.

[illegible]

12 13 14 15

Δ

Artin.
Conrad.

Solvable (ex 27)

order < 60 solvable. $|A_5| = 60$ not solvable for A_5 is simple

[1] • $\mathbb{Z}/p\mathbb{Z}$: cyclic $\overset{\text{(abelian)}}{\Longrightarrow}$ solvable.

[2] • $|G| = pq$. G : solvable.

[3] • G. p-group. G : solvable

[4] • $H \trianglelefteq G$ G : solvable $\Longleftrightarrow$ G/H, H solvable,

[5] • $H \leq G$. G : solvable $\Longrightarrow$ H : solvable

(hint : normal tower of G. $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\}$

$\Longrightarrow$ ~ of H. $H \cap G_0 \supset H \cap G_1 \supset \cdots \supset \{e\}$)

[1] [2] [3] Do

1 2 3 4 5 6 7 8 9 10 11 $\overset{2^2 \times 3}{12}$ ②

13 14 15 16 17 $\overset{2 \times 3^2}{18}$ ① 19 $\overset{2^2 \times 5}{20}$ ① 21 22 23 $\overset{2^3 \times 3}{24}$ ②

25 26 27 $\overset{2^2 \times 7}{28}$ ① 29 $\overset{2 \times 3 \times 5}{30}$ ③ 31 32 33 34 35 $\overset{2^2 \times 3^2}{36}$ ②

37 38 39 $\overset{2^3 \times 5}{40}$ ① 41 $\overset{2 \times 3 \times 7}{42}$ ① 43 $\overset{2^2 \times 11}{44}$ ① 45 46 47 $\overset{2^4 \times 3}{48}$ ②

49 $\overset{2 \times 5^2}{50}$ ① 51 $\overset{2^2 \times 13}{52}$ ① 53 $\overset{2 \times 3^3}{54}$ ① 55 $\overset{2^3 \times 7}{56}$ 57 58 59 60

⓪ $G = S_4 \supset A_4 \supset \{e, (12)(34), (13)(24), (14)(23)\} \supset \{e\}$ with indices 2, 3, 4

i.e. S_4 is solvable

① $|G| = 28 = 2^2 \times 7$

$n_2 = 1, 7 \qquad n_2 \mid 7$

$n_7 = 1, \qquad n_7 \mid 2$

$\Rightarrow \quad n_2 = 1, 2, \quad n_7 = 1 \quad \Rightarrow \quad \underset{\downarrow}{\overset{Syl_7(G)}{}} Q \trianglelefteq G \quad \Rightarrow \quad G \underbrace{\supset Q}_{4} \underbrace{\supset \{e\}}_{7}$

② $|G| = 12 = 2^2 \cdot 3 \quad \begin{cases} n_2 = 1, 3 \\ n_3 = 1, 4 \end{cases}$ if $n_2 = 1$ or $n_3 = 1$

$|G| = 24. = 2^3 \cdot 3 \quad \begin{cases} n_2 = 1, 3 \\ n_3 = 1, 4 \end{cases} \Rightarrow$ induction ✓

if $n_3 = 4$.

$G \xrightarrow{\phi} Perm(Syl_3(G)) \simeq S_4$

$g \longmapsto T_g : Syl_3(G) \to Syl_3(G)$ hom

$P \longmapsto gPg^{-1}$

$G/ker(\phi) \simeq im(\phi) \leq S_4$ solvable

[5]

$\Rightarrow im(\phi)$ Solvable

and $ker(\phi) \leq G$, $\neq$ $\Rightarrow$ $G/ker(\phi)$ solvable and $ker(\phi)$ solvable [4] $\Rightarrow$ G solvable

if $n_3 = 1$

$Syl_3(G) \ni Q \trianglelefteq G$

$G \underbrace{\supset Q}_{8} \underbrace{\supset \{e\}}_{3}$ ✓

$$|G| = 2^4 \cdot 3 \qquad \left\{ \begin{array}{l} n_2 = 1, 3 \\ n_3 = 1, 4, 16. \end{array} \right.$$

if $n_2 = 3$.

Consider $G \xrightarrow{\phi} \mathrm{Perm}(\mathrm{Syl}_3(G)) \simeq S_3$

$$g \longmapsto \pi_g : \mathrm{Syl}_3(G) \longrightarrow \mathrm{Syl}_3(G)$$

$$\underline{P} \longmapsto gPg^{-1}$$

$\mathrm{im}(\phi) \trianglelefteq S_3$ solvable.

$\ker(\phi) \neq G.$ $\Rightarrow$ $G/\ker(\phi) \simeq \mathrm{im}(\phi)$ solvable & $\ker(\phi)$ solvable. $\Rightarrow$ G solvable

① & ②

18. 2×3^2. $n_2 = 1, 3, \cancel{5}, \cancel{7}, 9$; $n_3 = 1, \cancel{4}$ by ①

20 $2^2 \times 5$. $n_2 = 1, \cancel{3}, 5$, ; $n_5 = 1, \cancel{6}$ by ①

40 $2^3 \times 5$ $n_2 = 1, 5$; $n_5 = 1$ by ①

42. $2 \times 3 \times 7$.

$n_2 = 1, 3, 7$

$n_3 = 1, 7$ by ①

$n_7 = 1$

44 $2^2 \times 11$

$n_2 = 1, 11$ by ①

$n_{11} = 1$

50 2×5^2

$n_2 = 1, 5, 25$ by ①

$n_5 = 1$

36 $2^2 \times 3^2$.

$n_2 = 1, 3, 9$ if $n_3 = 4$.

$n_3 = 1, 4$. S_4 ✓ by ②

54. 2×3^3

$n_2 = 1, 3, 9, 27$ by ①

$n_3 = 1$

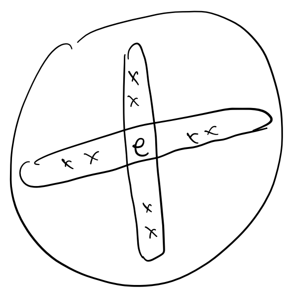
③ $|G| = 30 = 2 \times 3 \times 5$

$n_2 = 1, 3, 5, 15$

$n_3 = 1, 10$

$n_5 = 1, 6$

if $n_5 = 6$.



by prop of cyclic groups

Counting: $(5-1) \times 6 + 1 = 25$

if $n_3 = 10 \Rightarrow$ too much. elements $\Rightarrow n_3 = 1$ ✓

$|G| = 56 = 2^3 \times 7$

$n_2 = 1, 7$

$n_7 = 1, 8$

if $n_7 = 8$. $8 \times (7-1) + 1 = 49$

if $n_2 \neq 1$

$|Q| = 8$ $\Rightarrow$ too much elems. ✓

$\cap$

$Syl_2(G)$