From zero to General Relativity in 10 pages

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1 What is this document?

Maybe you have a scientific education, but chose a path other than physics. Or maybe you are not a scientist at all, but you still remember pretty well those basic concepts introduced in high school. Either way, when you hear about theories of modern fundamental physics you are fascinated: after all, that's the closest we humans can get to understand our Universe. But you also think that learning those theories requires many years of specialized education, or at least a good amount of dedication and time, which you currently might not have. While that is true, I think that you can grasp part of the beauty of modern physics with relatively little effort.

There are four known fundamental forces in nature. Our best description of three of them is given by a framework called Quantum Field Theory. The fourth one, the force of gravity, is a little bit special and has its own theory: it's General Relativity (GR), developed by Einstein between 1905 and 1915, and still valid today. The immense beauty, elegance and experimental success of GR contributed to consacrating Einstein as one of the best-known geniuses in human history. You can't go through your life without learning GR. You owe it to yourself. If you still aren't excited, read this article of the New York Times from 1919 [1].

2 Newton's gravity

Newton was a genius too. His *second law of dynamics* reads

$$\mathbf{F} = m\mathbf{a}\,, \tag{2.1}$$

meaning that the acceleration $\mathbf{a}$ of a body is proportional to the force $\mathbf{F}$ acting on it; the proportionality coefficient defines the mass m of the body.[1] Equation (2.1) is useful because it allows one to calculate how things move. Recall that if the position of a body as function of time is $\mathbf{r}(t) = (x(t), y(t), z(t))$, its acceleration is defined as the second derivative in time: $\mathbf{a} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$. So (2.1) is really saying: if you know the force acting on a body, you can solve a differential equation and determine $\mathbf{r}(t)$, that is, its position at any time.

What exactly is the value of $\mathbf{F}$ depends on the situation. For example, a positive electrical charge will experience a force directed towards negative charges, and away from other positive charges. Or, when a body slides on the floor, it experiences a friction force directed opposite to its direction of motion, which slowes it down. Newton realized, however, that there is a certain force that *everything* is subject to: it's the force of gravity. On the surface of the Earth, it is responsible for making apples fall down from trees. In outer space, it makes the planets orbit the Sun. According to Newton, the force of gravity between two objects of masses M and m is

$$\mathbf{F} = \frac{GMm}{r^2}\hat{\mathbf{r}}\,, \tag{2.2}$$

[1] $\mathbf{F}$ and $\mathbf{a}$ are boldface to denote that they are vectors; instead, m is just a real number and so is in regular font.

where G is a proportionality constant, r is the distance between the two objects and $\hat{\mathbf{r}}$ is a unit vector signaling that the force attracts each body to the other.

There is something weird about (2.2): why would a force depend on the mass of a body? Recall that we defined m as the proportionality coefficient between $\mathbf{F}$ and $\mathbf{a}$. If (2.2) is plugged in (2.1), then m cancels out. *Any* body will experience the same acceleration, and thus move exactly in the same way, when attracted by the gravity of another body (as in the famous Galileo experiment on the leaning tower of Pisa). This seems to make the mass useless, in some sense. Nevertheless, (2.2) and (2.1) together give an extremely accurate description of how gravity works.

3 Special Relativity

In 1905, Einstein revolutionized Newton's mechanics by introducing Special Relativity. Instead of imagining particles in 3-dimensional space, changing their position as time flows, Einstein pictured a 4-dimensional *spacetime*, where the fourth dimension is time itself. Any point (or *event* in time and space) is described by four coordinates: $(ct, \mathbf{r}) = (ct, x, y, z)$, where c is the speed of light.[2] To remind ourselves that we are dealing with 4-dimensional objects, we denote the points (and the 4-dimensional vectors) with an explicit index, such as x^μ. The index μ can take the values $0, 1, 2, 3$ and we have $x^0 = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$.

Time and space, however, are not quite treated the same way. For example, consider two points $x_1^\mu = (ct_1, x_1, y_1, z_1)$ and $x_2^\mu = (ct_2, x_2, y_2, z_2)$. Their distance in space is given by Pithagora's theorem, $\Delta r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. To find the (squared) distance in spacetime, instead, we add a negative term,

$$\Delta s^2 = -(ct_2 - ct_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \,. \tag{3.1}$$

Sometimes, the same formula is written as

$$\Delta s^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \,, \tag{3.2}$$

where $\Delta x^\mu = x_2^\mu - x_1^\mu$ and we defined the matrix $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.[3] To reduce clutter, the sums are generally omitted, so that we simply write $\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$.

The concept of spacetime is useful because it allows to describe physical phenomena from the point of view of different observers, that move with respect to each other. One of the principles of Special Relativity, for example, is that every observer sees the light travelling at the same speed c. A definition like (3.1) makes this natural: if a light ray moves from (x_1, y_1, z_1) to (x_2, y_2, z_2) with speed c, then $\Delta s = 0$. So, both observers will compute the same vanishing distance. In other words, the relative distances in spacetime are the same regardless of how you move within it.

Time and space are not the only two quantities that are put on similar footing in Special Relativity. Another example are the energy E and momentum p of a body. Recall that, in

[2] It is customary to multiply the time t by the speed of light c, so that the four coordinates all have the dimension of a length.

[3] In other words, $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$ and $\eta_{ij} = 0$ if $i \neq j$.

Newton's mechanics, a body with mass m that moves with velocity $\mathbf{v}$ has $E = \frac{1}{2}mv^2$ and $\mathbf{p} = m\mathbf{v}$. In Special Relativity, they become part of a single object, the 4-momentum $p^\mu = (E/c, \mathbf{p})$.[4] Differently from Newton's mechanics, however, the energy of a body at rest (i.e., with $v = 0$) is not zero, but equals $E = mc^2$. The mass, defined by Newton as the proportionality constant between $\mathbf{F}$ and $\mathbf{a}$, is now interpreted as one part of an object's energy. While p^μ is useful to tell the energy and momentum of a single body, we can describe the content a whole region of spacetime by specifying the *density* of energy and momentum at each location within it. For example, a region with many massive objects will have a high energy density, while an empty region will have a vanishing energy density. As E and $\mathbf{p}$ are incorporated in p^μ, their densities are incorporated in an object called the *energy-momentum tensor*, or $T^{\mu\nu}$. Like $\eta_{\mu\nu}$ above, $T^{\mu\nu}$ is a 4×4 matrix. For example, its 00 ("top-left") entry is $T^{00} = \rho c^2$, where ρ is the energy density.

There is one last crucial concept we need from Special Relativity. While light always travels at speed c, all bodies with $m > 0$ can only move at speeds smaller than the speed of light, $v < c$. Among the consequences of this, instant communication within the spacetime is not possible. For any physical entity (including information, which must be transmitted physically) to travel from one place to another, a minimum *travel time* is required, equal to the time it takes for the light do the same journey.

4 The equivalence principle

After learning Special Relativity, we find another disturbing property of Newton's formula for the gravitational force, (2.2). The force depends on the distance between the two bodies. So if one of the two is moved, the other one will *immediately* feel a change in the force acting on it. We could use this to communicate information faster than light! This is clearly not compatible with Special Relativity, so we need a new theory of gravity.

In the effort to find it, Einstein started from the other suspicious property of (2.2), i.e., its dependence on the mass m, or the fact that all bodies fall exactly at the same rate. Could this curious fact hide a deeper truth? Imagine being inside an elevator. Suddently, the cables break and you and the elevator start falling down. You and the elevator have the same acceleration, and thus do not move with respect to each other. All the other objects in the elevator also do not move with respect to you. All particles in your body have exactly the same acceleration: no two of them are pushed or pulled against each other, they all fall perfectly synchronized, so there is no way for your nerves and brain to feel that you are actually being pulled down by gravity. So... are you actually being pulled down by gravity at all? Is there any way to distinguish this situation from being in a sealed elevator, that is not moving and sits instead in outer space, far from all stars, planets and any force? Einstein postulated that the answer must be no, a hypothesis known as the equivalence principle.

5 Curved spacetime

Taking the equivalence principle seriously requires acknowledging that the motion of an object under the force of gravity has nothing to do with its mass. But then, we cannot describe that

[4] The dependence of E and $\mathbf{p}$ on $\mathbf{v}$ are different than in Newton's mechanics, but we will not need them explicitly.

motion by means of Newton's second law. We must instead use something that does not depend on the object itself, something that is somehow equal for all objects. It must be the spacetime, Einstein thought.

However, the spacetime of Special Relativity is boring, or *flat*. We only need to be given $\eta_{\mu\nu}$, and then we are able to compute distances between any two points. Because $\eta_{\mu\nu}$ is a constant, the spacetime looks exactly the same at any point. To hope to obtain any interesting effect, we should find a way to deform it. So imagine that instead of $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ we measured distances with a *metric*, $g_{\mu\nu}$, whose entries depend on the point x^μ of the spacetime. If the difference of, say, the x coordinates of two points is 1, then we may say they are actually a bit closer or a bit farther away than 1 metre, depending on where exactly they are in the spacetime. An example of curved space is the surface of the Earth: two points that have same latitude and differ by 1 degree in longitude are closer together if they are near the poles, and farther away if they are at the equator.

Clearly, such a metric would only be useful to measure distances between points very close to each other (so $g_{\mu\nu}$ is approximately the same at both points), otherwhise it would not be clear which of the two values of $g_{\mu\nu}$ one should use. Given two points whose infinitesimally small difference is $x_2^\mu - x_1^\mu = \mathrm{d}x^\mu$, let their (squared) distance be

$$ds^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \tag{5.1}$$

where again we omitted the sums over μ and ν. Whenever formula (5.1) differs from the special relativistic result, $ds^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, we say that the spacetime is curved. How curved exactly, it depends on the metric $g_{\mu\nu}$. Just as an example, distances in a curved spacetime might look something like this:

$$ds^2 = -c^2\,\mathrm{d}t^2 + a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,, \tag{5.2}$$

where $a(t)$ is some unspecified fuction of the time. In the spacetime (5.2), time flows unperturbed everywhere, but spatial distances grow (or shrink) over time depending on $a(t)$. In fact, (5.2) is how we describe mathematically the expansion of our Universe.

6 Geodesics

Let us now try to use curved spacetime, rather than Newton's forces, to describe the motion of a body. In flat spacetime, we know that if no force acts on an body, then it will move in a straight line, as implied by (2.1) with $\mathbf{F} = 0$. It is also well known that a straight line between two points is the shortest possible path connecting them. For how boring the uniform motion of a body might be, it has the interesting property of minimizing the distance between the points it visits. It is very common in physics that the equations describing the motion of things come from similar principles of minimum.

So, let us try to generalize this idea. Imagine a particle that moves along a path $x^\mu(\lambda)$, where λ is just a parameter used to label the points along the path. The total distance between two points on the path of a body will be given by summing (that is, taking the integral of) all the infinitesimally small distances ds along the path,

$$S = \int ds = \int \sqrt{g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}\,. \tag{6.1}$$

The requirement that S is minimum can be translated into a condition on $g_{\mu\nu}$ and the path $x^\mu(\lambda)$ through standard techniques of variational calculus. The condition turns out to be

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0\,, \tag{6.2}$$

where we omitted sums over the indices α and β and defined the Christoffel symbols

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\beta\nu}}{\partial x^\alpha} + \frac{\partial g_{\nu\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right), \tag{6.3}$$

where $g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$ and we omitted a sum over ν.[5] The formula might look a bit complicated, but that is not important here. All it matters is where it comes from. Whatever the solution of (6.2) is, it must describe the shortest path between two given points of the spacetime. These paths are called *geodesics*: for example, on the surface of a sphere, they correspond to circles of maximal radius (like the meridians on Earth).

The straight line path in flat space generalizes to a geodesic in curved spacetime. This new intuition makes the equivalence principle much less mysterious: all bodies fall in the same way because they all just follow geodesics. Their mass is irrelevant, and that's why it cancels out when Newton's formula is plugged in $\mathbf{F} = m\mathbf{a}$. In General Relativity, there is no force of gravity. It is the curvature of spacetime that tells matter how to move. The Earth does not orbit the Sun because it is attracted to it; instead, the Earth travels *straight* in the *curved* spacetime around the Sun, resulting in it moving in an apparent cirlce.

Geodesics also remove the disturbing "action at a distance" of Newton's gravity: formula (6.2) shows that the motion of a body only depends on the spacetime metric calculated at the same point where the particle is. There is no instant communication between different points of the spacetime: everything happens "locally" and compatibility with Special Relativity is restored.

7 Intermezzo: indices, vectors, tensors and invariants

You might have wondered by this point what is going on with all these indices. Sometimes we write them as a superscript, like an exponent (as in x^μ or $g^{\mu\nu}$), other times as a subscript (as in $g_{\mu\nu}$), and now with $\Gamma^\mu_{\alpha\beta}$ things seem to be going out of control. The reason for this apparent inconsistency has to do with the wish of being able to describe phenomena from the point of view of any observer, regardless of their position, motion, or choice of coordinates. Quantities without indices, like ds, are just real numbers. They only have to do with spacetime itself, and every observer must agree on their value (as we already discussed in Special Relativity). Quantities with indices, instead, are more subjective: for example, I might choose to orient my cartesian axes in a different way than another observer, and so the coordinates of my vectors, like $\mathrm{d}x^\mu$,

[5] From here on, all sums over repeated indices (that is, which appear more than once) will be silently omitted.

will look different from theirs. But the two of us will still agree on the length of the vector: after all, it's just the spacetime distance between its tip and its tail. As we saw in (5.1), the formula is $g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$. More generally, given a vector V^μ, its squared length $g_{\mu\nu}V^\mu V^\nu$ will be the same for all observers.

So we define objects with "lower indices" precisely to help us build quantities that are *invariant*, i.e., independent of the observer. For example, we define $V_\nu = g_{\mu\nu}V^\mu$, so that the squared length is $V_\nu V^\nu$. A similar operation can be done with any other object with indices, such as the energy-momentum tensor $T^{\mu\nu}$: we can define $T_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}$ so that $T_{\mu\nu}T^{\mu\nu}$ is invariant. We can even mix things up, and build invariant quantities like $T_{\mu\nu}V^\mu V^\nu$. This notation is particularly useful because all invariant quantities are ultimately real numbers: so all the indices they contain must be summed over. Because we only sum over repeated indices, which appear once as "upper" and once as "lower", you can imagine that those pairs of indices cancel out.

It is important that non-repeated, or *free* indices, are the same on the left and right-hand sides of an equation. For example, $T_{\mu\nu}V^\mu = A_\nu B^\mu{}_\mu$ is a sensible equation (ν is free and lower on both sides), while $T^{\mu\nu}V_\mu A_\nu = g^{\mu\alpha}B_\mu$ is not (the left-hand side is an invariant, while the right-hand has a free upper α).

8 Curvature tensors

Let us go back to physics. The idea of curved spacetime and geodesics worked very well with the equivalence principle. But then, if we want to compute how things move, we need someone to tell us *how exactly* is spacetime curved. All information about gravity must somehow be contained in $g_{\mu\nu}$ and its curvature.

From the equivalence principle, we also know that a uniform gravitational field is actually indistinguishable from no field at all, because all geodesisc run parallel to each other. In a non-uniform field, instead, two objects in our elevator would follow slightly different geodesics, so we could realize we actually live in a gravitational field. This gives us the idea to quantify the curvature of spacetime by looking at how two closely spaced geodesics deviate from each other. Consider two nearby particles. By using the geodesic equation (6.2) for both of them, we can compute how their distance B^μ changes as they travel each along their own geodesic. After some calculations, we find

$$\frac{\mathrm{d}^2B^\mu}{\mathrm{d}\lambda^2} = -R^\mu{}_{\nu\rho\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}B^\rho\,, \tag{8.1}$$

where

$$R^\mu{}_{\nu\rho\sigma} = \frac{\partial\Gamma^\mu_{\nu\sigma}}{\partial x^\rho} - \frac{\partial\Gamma^\mu_{\nu\rho}}{\partial x^\sigma} + \Gamma^\mu_{\rho\alpha}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\sigma\alpha}\Gamma^\alpha_{\nu\rho}\,. \tag{8.2}$$

This complicated object is known in differential geometry as the *Riemann tensor*, and is indeed used by mathematicians to quantify the curvature of mathematical spaces. The fact that we seemingly accidentally stumbled upon it, by computing how two geodesics deviate from each other, is a very good sign that we are on the right track towards a geometric description of gravity. Physics here is telling us very clearly what is the right kind of mathematics needed for the problem.

Other "curvature tensors" mathematicians often use are the Ricci tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ (basically summing up some compontents of the Riemann tensor) and the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ (i.e., the trace of the Ricci tensor). Given a metric $g_{\mu\nu}$, we can first compute the Christoffel symbols through (6.3), then plug them into (8.2), and from there compute the Ricci tensor and scalar. It's a long computation, but conceptually straightforward.

9 The Einstein equations

Now that we have the appropriate machinery to quantify the curvature of the spacetime, we are ready to write down an equation that links it to where the mass is. At that point, the circle will close: spacetime tells matter how to move (through (6.2)), matter tells spacetime how to curve. The equation we are looking for cannot be derived from pure reasoning. It's a new physical law, a bit like $\mathbf{F} = m\mathbf{a}$ but for the curvature of spacetime. To be able to write it down, some physicist's intuition must be used. One aspect that will help us is that the fundamental laws of Nature are often jaw-droppingly elegant and simple.

From Special Relativity, we already have a tool to describe the matter content of a region of spacetime: it is the energy-momentum tensor $T_{\mu\nu}$. On the other hand, the various objects we have to describe the curvature of spacetime are $R^{\mu}{}_{\nu\rho\sigma}$, $R_{\mu\nu}$ and R. Einstein's first guess was thus the simplest possible equation, with correctly placed indices, that one can write with these quantities:

$$R_{\mu\nu} \stackrel{?}{=} \kappa T_{\mu\nu} \,, \tag{9.1}$$

where κ is some proportionality constant. But this is not the correct guess. The reason is that there is a physical property satisfied by the right-hand side but not by the left-hand side: it is the conservation of energy and momentum. In flat spacetime, it would read as $\frac{\partial T^{\mu\nu}}{\partial x^{\mu}} = 0$, meaning that energy and momentum are indeed constant over the spacetime. In curved spacetime, this becomes $\nabla^{\mu} T_{\mu\nu} = 0$, where ∇^{μ} is a derivative that incorporates correcting terms that cancel out the changes to $T_{\mu\nu}$ due to variations in spacetime curvature.[6] The left-hand side of (9.1) instead satisfies

$$\nabla^{\mu} R_{\mu\nu} = \nabla^{\mu} \left(\frac{1}{2} g_{\mu\nu} R \right) \neq 0 \,. \tag{9.2}$$

So (9.1) could never be compatible with energy conservation. But the problem is not insurmountable: we can just use (9.2) to subtract off the left-hand side the spurious term. The second guess is then:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \,,} \tag{9.3}$$

and this time it is truly the correct one. We also fixed $\kappa = 8\pi G/c^4$, as with this value (9.3) can be shown to reproduce Newton's predictions at small velocities and weak gravity. Please, stare at the Einstein equations a little bit more. It will never be enough anyway.

On the left-hand side, we have the curvature tensors: extremely intricate functions of the spacetime metric and its derivatives, they contain information on how spacetime is curved and

[6]Explicitly, $\nabla^{\mu} T_{\mu\nu} = g^{\mu\alpha} (\partial T_{\mu\nu}/\partial x^{\alpha} - \Gamma^{\beta}_{\alpha\mu} T_{\beta\nu} - \Gamma^{\beta}_{\alpha\nu} T_{\mu\beta})$.

warped. On the right-hand side, we have the enegy and momentum of the matter in the spacetime. The constant of proportionality involves together the gravitational constant G as well as the speed of light c, telling us that this is truly a relativistic theory of gravity. Both sides are 4×4 symmetric matrices, giving in total $4^2 - \frac{4\times(4-1)}{2} = 10$ independent partial nonlinear differential equations for the metric components. Such a short and and elegant equation, yet so complicate to solve. The special relativistic boring spacetime was a theatre where events took place. The general relativistic spacetime becomes instead the protagonist of the show: it is a dynamical entity and it changes in reponse to what it hosts.

As we did for geodesics, the Einstein equations can also be derived from a principle of minimum. In this case too, minimality, elegance and simplicity seem to be the favorite attributes of Nature. The minimized functional is

$$S = \int R\,\mathrm{dV} + S_\mathrm{m}\,, \tag{9.4}$$

where R is the Ricci scalar, $\mathrm{d}V$ is the spacetime volume element and S_m is a term that depends on the matter content of the spacetime. S_m gives rise to $T_{\mu\nu}$, while $\int R\,\mathrm{dV}$ gives rise to the left-hand side of (9.3). Of all measures of curvature, only R is indices-free and thus invariant. Instead, $R^{\mu}{}_{\nu\rho\sigma}$ and $R_{\mu\nu}$ could not have appeared in an integral over the spacetime, because they depend on the observer. The simplest and only sensible possibility is thus the one realized in Nature.

Some people would say that GR ends here. Together, (9.3) and (6.2) (or (9.4) alone) fully specify the interplay of matter and spacetime, and, after all, these equations are all we need. But I disagree: finding the equations of Nature is only one part of a physicist's job; the other part is to solve them. And believe me, the solutions of Einstein equations are mind-blowing to say the least.

10 Black holes

Let us look for the simplest possible solutions to the Einstein equations. First of all, consider a vacuum spacetime with no matter in it, i.e., $T_{\mu\nu} = 0$. Then, let us impose spherical symmetry and, accordingly, label points with spherical coordinates (ct, r, θ, ϕ).[7] Then, the Einstein equations admit the following solution, known as the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2\,\mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{10.1}$$

Apart from the fundamental constants G and c, there is only one other parameter: it is M, the mass of the object at the centre of space.

But wait—didn't we just say this spacetime was vacuum? How can there be something with nonzero mass at the centre? Let us proceed with caution. If $M = 0$, then (10.1) becomes $ds^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$, which is the spherical coordinate equivalent of the familiar flat space $ds^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ from Special Relativity. If $M \neq 0$, then we

[7]You can imagine r as the distance to the centre, $\pi/2 - \theta$ as the latitude and ϕ as the longitude.

can compute the Christoffel symbols and thus the geodesic motion of particles, which indeed at $r \gg 2GM/c^2$ become nearly identical to the motion of particles orbiting a body of mass M in Newton's gravity. So, it really looks like the Schwarzschild metric is describing the gravity of a spherical body. There are however small but significant corrections, such as those producing the precession of the perihelion of Mercury and the bending of light passing by the Sun [1]—the two earliest experimental confirmation of Einstein's theory.

But at smaller values of the radial distance r, things start to get very weird. When $r < 2GM/c^2$, we see that g_{00} (the coefficient of $c^2\,\mathrm{d}t^2$) turns positive, while g_{rr} (the coefficient of $\mathrm{d}r^2$) turns negative. Remember that time contributed to distances with a sign different from the three spatial coordinates, cf. (3.1). It's as if space and time exchanged roles at radial distances smaller than $2GM/c^2$: instead of being forced to move forward in time, a body is forced to move inwards in space! This means that whatever enters the region $r < 2GM/c^2$ will never be able to escape. Not even light: we have found a black hole. The surface $r = 2GM/c^2$ is called the *event horizon*, and is the point of no return.

We looked for the simplest possible solution to the Einstein equations and found a seemingly absurd object, composed of no matter at all, but able to trap forever whatever comes too close to it. What is at the very centre of it? At $r = 0$, we seem to get divergences in (10.1), both in g_{00} and g_{rr}. This is called a gravitational *singularity*, meaning that gravity becomes infinitely strong. Nobody knows what this means—it looks like General Relativity is predicting its own failure at the centre of a black hole. Most physicists believe that a theory of Quantum Gravity will be needed to fully understand what is going on.

The Schwarzschild metric hides a few other surprises, namely a white hole, a wormhole and a parallel universe. Yes, it is all contained in that seemingly innocent metric written in (10.1). If you think it is crazy it's because it is. But I am not going to explain those things here: if I made you curious to learn more, then the goal of this document has been achieved and you don't need it anymore.

11 Cosmology

Not just black holes: the Einstein equations also describe our entire universe. We have already written the metric

$$ds^2 = -c^2\,\mathrm{d}t^2 + a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,. \tag{11.1}$$

We see that if $a(t)$ grows with time, then spatial distances increase: an expanding universe! Conversely, a decreasing function $a(t)$ would describe a contracting universe.

The Einstein equations allow us to calculate the function $a(t)$: if we know the energy-momentum tensor of the universe (in particular, its average density ρ and pressure p), then we get two differential equations known as Friedmann equations:

$$\frac{1}{a^2}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{8\pi G}{3}\rho\,, \tag{11.2}$$

$$\frac{1}{a}\frac{\mathrm{d}^2a}{\mathrm{d}t^2} = -\frac{4\pi G}{3}(\rho + 3p)\,. \tag{11.3}$$

A universe containing only light would expand as $a(t) \sim t^{1/2}$. A universe made of many particles at rest would instead have $a(t) \sim t^{2/3}$. In both cases, the Einstein equations are able to predict the fate of our universe, which seems to be an eternal expansion, whose rate slows down over time. Surely our universe will fall somewhere in between $t^{1/2}$ and $t^{2/3}$, given that it contains both light and particles, right?

No. Astronomical observations suggest instead that $a(t) \sim e^{Ht}$, where H is a constant. An exponential growth that can only be achieved if the universe contains a very strange form of energy, called dark energy. Nobody really knows what that is, but it is here around us.

12 Gravitational waves

Let us go back to the case of a vacuum universe. Instead of looking for a spherical object like a black hole, one might wonder what if we had an *almost* flat spacetime. So, consider the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{12.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is our old friend, and $h_{\mu\nu}$ is a very small perturbation. Then, the Einstein equations predict that $h_{\mu\nu}$ obeys the same equation known to describe a vibrating string, the propagation of sound in air, and electromagnetic waves (i.e., light). We conclude that gravity propagates waves, which travel at the speed of light. These tiny ripples of spacetime travel through our universe, carrying the secrets of the events that generated them, such as violent collisions of black holes, or quantum mechanical fluctuations that took place when our Universe was a fraction of a second old.

13 Epilogue

The power of mathematics and logical reasoning to describe our universe is unbelievable, and honestly mysterious. The beauty of the fundamental laws of Nature is even more astonishing, as if it was a prize given to the scientist for the ability to discover them. Admittedly, it is hard to find a theory more intellectually satisfying than General Relativity, but chances are you are still underestimating its beauty, because so much could not be said in these few pages. I hope I trasmitted all my enthusiasm and fascination to you.

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References

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