

# ASP Final

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## 1 Details of Beamformers

- assume uniform linear array(ULA) with N isotropic antennas with inter-element spacing  $d = \frac{\lambda}{2}$
- the steering vector for ULAs

$$\mathbf{a}(\theta_i) = [1 \ \exp(j\pi \sin \theta) \ \exp(j2\pi \sin \theta) \ \dots \ \exp(j(N-1)\pi \sin \theta)]^T$$

- array data model

$$\mathbf{x}(t) = \mathbf{a}(\theta)s_1(t) + \mathbf{n}(t) = \mathbf{1}(Ae^{j2\pi ft}) + \mathbf{n}(t)$$

$$\begin{aligned} & \text{where } \mathbb{E}[A] = \mathbf{0} \quad \text{and} \quad \mathbb{E}[|A|^2] = \sigma_1^2 \\ & \text{where } \mathbb{E}[\mathbf{n}(t)] = \mathbf{0} \quad \text{and} \quad \mathbb{E}[\mathbf{n}(t)\mathbf{n}^H(t)] = \sigma_n^2 \mathbf{I} \end{aligned}$$

### 1.1 The beamformer with uniform weights

#### 1.1.1 weight vector

$$\mathbf{w} = \mathbf{1}/N$$

#### 1.1.2 beamformer output

$$\begin{aligned} y(t) &= \mathbf{w}^H \mathbf{x}(t) \\ &= (Ae^{j2\pi ft}) + \left(\frac{1}{N}\mathbf{1}^H \mathbf{n}(t)\right) \end{aligned} \tag{1}$$

#### 1.1.3 beam pattern

$$\begin{aligned} B_\theta(\theta) &= \left(\frac{1}{N}\mathbf{1}^H\right)\mathbf{a}(\theta) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} e^{j\pi \sin \theta \cdot n} \\ &= e^{j\frac{N-1}{2}\pi \sin(\theta)} \times \frac{1}{N} \times \frac{\sin(\frac{N}{2}\pi \sin \theta)}{\sin(\frac{1}{2}\pi \sin \theta)} \end{aligned} \tag{2}$$

- DOA  $\theta_1 = 0$  :N-times SNR enhancement
- DOA  $\theta_1 = \theta$  :SNR depending on  $B_\theta(\theta)$
- The maximum of  $B_\theta(\theta)$  occurs at  $\theta=0$

## 1.2 The beamformer with array steering

### 1.2.1 weight vector

$$\mathbf{w} = \frac{1}{N} \mathbf{a}(\theta_s) = \frac{1}{N} [1 \quad e^{j(\pi \sin \theta_s)} \quad e^{j(\pi \sin \theta_s) \times 2} \quad \dots \quad e^{j(\pi \sin \theta_s) \times (N-1)}]^T$$

### 1.2.2 beam pattern

$$\begin{aligned} B_\theta(\theta) &= \mathbf{w}^H \mathbf{a}(\theta) \\ &= e^{j\frac{N-1}{2}(\pi \sin \theta - \pi \sin \theta_s)} \times \frac{1}{N} \times \frac{\sin[\frac{N}{2}(\pi \sin \theta - \pi \sin \theta_s)]}{\sin[\frac{1}{2}(\pi \sin \theta - \pi \sin \theta_s)]} \end{aligned} \quad (3)$$

- The maximum of  $|B_\theta(\theta)|$  occurs at  $\theta = \theta_s$

## 1.3 The MVDR beamformer

### 1.3.1 Interference rejection

- $\mathbf{x}(t) = \underbrace{\mathbf{a}(\theta_s)\mathbf{s}(t)}_{\text{Signal}} + \underbrace{\mathbf{a}(\theta_i)\mathbf{i}(t)}_{\text{Interference}} + \underbrace{\mathbf{n}(t)}_{\text{Noise}}$
- $\mathbf{R} = \underbrace{\sigma_s^2 \mathbf{a}(\theta_s)\mathbf{a}^H(\theta_s)}_{\text{Signal}} + \underbrace{\sigma_i^2 \mathbf{a}(\theta_i)\mathbf{a}^H(\theta_i)}_{\text{Interference}} + \underbrace{\sigma_n^2 \mathbf{I}}_{\text{Noise}} \quad \text{where } \mathbf{R} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^H(t)]$
- $\mathbf{y}(t) = \mathbf{w}^H \mathbf{x}(t)$   
 $= \underbrace{\mathbf{w}^H \mathbf{a}(\theta_s)\mathbf{s}(t)}_{\text{Signal}} + \underbrace{\mathbf{w}^H \mathbf{a}(\theta_i)\mathbf{i}(t)}_{\text{Interference}} + \underbrace{\mathbf{w}^H \mathbf{n}(t)}_{\text{Noise}}$

### 1.3.2 Optimization problem (with the distortionless constraint)

$$\begin{aligned} \arg \min_{\mathbf{w}} \quad & \mathbb{E}[|y(t)|^2] \\ \text{subject to} \quad & \mathbf{w}^H \mathbf{a}(\theta_s) = 1, \\ & y(t) = \mathbf{w}^H \mathbf{x}(t) \end{aligned} \quad (4)$$

### 1.3.3 weight vector

$$\mathbf{w}_{MVDR} = \frac{\mathbf{R}^{-1} \mathbf{a}(\theta_s)}{\mathbf{a}^H(\theta_s) \mathbf{R}^{-1} \mathbf{a}(\theta_s)}$$

## 1.4 The LCMV beamformer

### 1.4.1 Interference rejection

- $\mathbf{x}(t) = \underbrace{\mathbf{a}(\theta_s)\mathbf{s}(t)}_{\text{Signal}} + \underbrace{\mathbf{a}(\theta_i)\mathbf{i}(t)}_{\text{Interference}} + \underbrace{\mathbf{n}(t)}_{\text{Noise}}$
- $\mathbf{R} = \underbrace{\sigma_s^2 \mathbf{a}(\theta_s)\mathbf{a}^H(\theta_s)}_{\text{Signal}} + \underbrace{\sigma_i^2 \mathbf{a}(\theta_i)\mathbf{a}^H(\theta_i)}_{\text{Interference}} + \underbrace{\sigma_n^2 \mathbf{I}}_{\text{Noise}} \quad \text{where } \mathbf{R} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^H(t)]$

- $\mathbf{y}(t) = \mathbf{w}^H \mathbf{x}(t)$

$$= \underbrace{\mathbf{w}^H \mathbf{a}(\theta_s) \mathbf{s}(t)}_{\text{Signal}} + \underbrace{\mathbf{w}^H \mathbf{a}(\theta_i) \mathbf{i}(t)}_{\text{Interference}} + \underbrace{\mathbf{w}^H \mathbf{n}(t)}_{\text{Noise}}$$

#### 1.4.2 Optimization problem (with the more constraints)

$$\begin{aligned} & \arg \min_{\mathbf{w}} \quad \mathbb{E}[|y(t)|^2] \\ & \text{subject to} \quad \mathbf{C}^H \mathbf{w} = \mathbf{g}, \\ & \quad y(t) = \mathbf{w}^H \mathbf{x}(t) \end{aligned} \tag{5}$$

#### 1.4.3 weight vector

$$\mathbf{w}_{LCMV} = \mathbf{R}^{-1} \mathbf{C} (\mathbf{C}^H \mathbf{R}^{-1} \mathbf{C})^{-1} \mathbf{g}$$

## 2 Denoise $\tilde{\theta}_s(t)$ and $\tilde{\theta}_i(t)$

- I use EMD(Empirical Mode decomposition) to denoise  $\tilde{\theta}_s(t)$  and  $\tilde{\theta}_i(t)$ .

### 2.1 Advatages of EMD algorithm

- When we use EMD, we could easily denoise  $\tilde{\theta}_s(t)$  and  $\tilde{\theta}_i(t)$  in time domain, rather than transforming the signals into frequency domain then progressing.
- When we use EMD, we could denoise  $\tilde{\theta}_s(t)$  and  $\tilde{\theta}_i(t)$  well.

### 2.2 EMD(take denoising $\tilde{\theta}_s(t)$ for example)

#### 2.2.1 EMD details

**STEP 1:**let  $y(n)=x(n)$  and  $n=1$  and  $k=1$ ,where  $x(n)$  is the sample

**STEP 2:**Find the local peaks

**STEP 3:**Connect local peaks and get  $e_{max}(n)$

**STEP 4:**Find the local dips

**STEP 5:**Connect local dips and get  $e_{min}(n)$

**STEP 6:**Copute the mean  $m(n)=\frac{e_{max}(n)+e_{min}(n)}{2}$

**STEP 7:**Compute the residue  $d(n)=y(n)-m(n)$

**STEP 8:**Find  $d_{max}(n)$ :the local peaks of  $d(n)$

**STEP 9:**Find  $d_{min}(n)$ :the local dips of  $d(n)$

**STEP 10:**Check whether the local peaks of  $d(n)$  are all greater than 0

**STEP 11:**Check whether the local dips of  $d(n)$  are all smaller than 0

**STEP 12:**Check whether  $|\frac{u_1(n)+u_0(n)}{2}|$  is smaller than  $thr$ ,where  $u_1(n)$  and  $u_0(n)$  are the upper and lower envelope of  $d(n)$  respectively

**STEP 13:**If step 10 to step 12 are satisfied,set  $c_n(n)=d(n)$  and continue to step 14,otherwise,set  $y(n)=d(n)$  and  $k=k+1$  and repeat step 2 to step 6(if k is greater than 1000,stop the iteration)

**STEP 14:**Calculate  $x_o(n)=x(n)-\sum_{s=1}^n c_s(n)$

**STEP 15:**Check whether  $x_o(n)$  is a function with no more than one extreme point(let  $n_{max}$ :the total number of the local maxima of  $x_o(n)$  and  $n_{min}$ :the total number of the local minima of  $x_o(n)$ )

**STEP 16:**If step 15 is satisfied,the empirical mode decomposition is completed,otherwise,set  $y(n)=x_o(n)$  and  $n=n+1$  and repeat step 2 to step 7

#### 2.2.2 EMD algorithm

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**Algorithm 1** An algorithm with EMD

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**Require:** the samples  $\tilde{\theta}_s(n)$  and the threshold  $thr$

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     $y(n) \leftarrow \tilde{\theta}_s(n)$ 
     $x_o(n) \leftarrow \tilde{\theta}_s(n)$ 
     $thr \leftarrow 0.2$ 
     $k \leftarrow 1$ 
    while 1 do
        while 1 do
             $m(n) = \frac{e_{max}(n)+e_{min}(n)}{2}$ 
             $k \leftarrow k + 1$ 
             $d(n) = y(n) - m(n)$ 
            if  $d_{max}(n) > 0 \wedge d_{min}(n) < 0 \wedge |\frac{u_1(n)+u_0(n)}{2}| < thr \vee k > 1000$  then
                 $c_n(n) \leftarrow d(n)$ 
                break
            else
                 $y(n) \leftarrow d(n)$ 
            end if
        end while
         $x_o(n) = \tilde{\theta}_s(n) - \sum_{s=1}^n c_s(n)$ 
        if  $n_{max} + n_{max} \leq 3$  then
            break
        else
             $y(n) \leftarrow x_o(n)$ 
             $n \leftarrow n + 1$ 
        end if
    end while
     $\hat{\theta}_s(n) \leftarrow x_o(n)$ 
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### 3 Plot the estimated DOAs $\hat{\theta}_s(n)$ and $\hat{\theta}_i(n)$

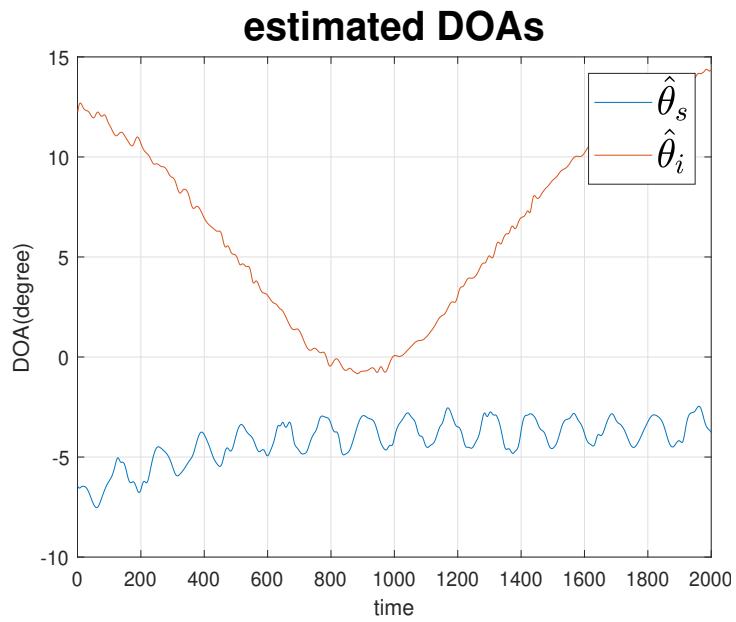


Figure 1: estimated DOAs

#### 3.1 The noisy DOAs $\tilde{\theta}_s(n)$ and $\tilde{\theta}_i(n)$

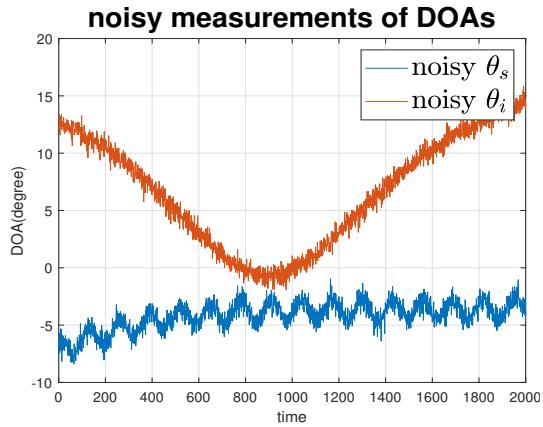


Figure 2: noisy DOAs

## 4 Design my beamformer

### 4.1 Advantages of my beamformer

Compared to the LCMV beamformer, my beamformer could beamform the signal in the time interval, in which DOAs of the source signal and the interference signal are similar. Thus, the estimated signal will be more similar to the source signal.

### 4.2 Details

**STEP 1:** Beamforming the signal with array steering beamformer (the most similar to the real signal)

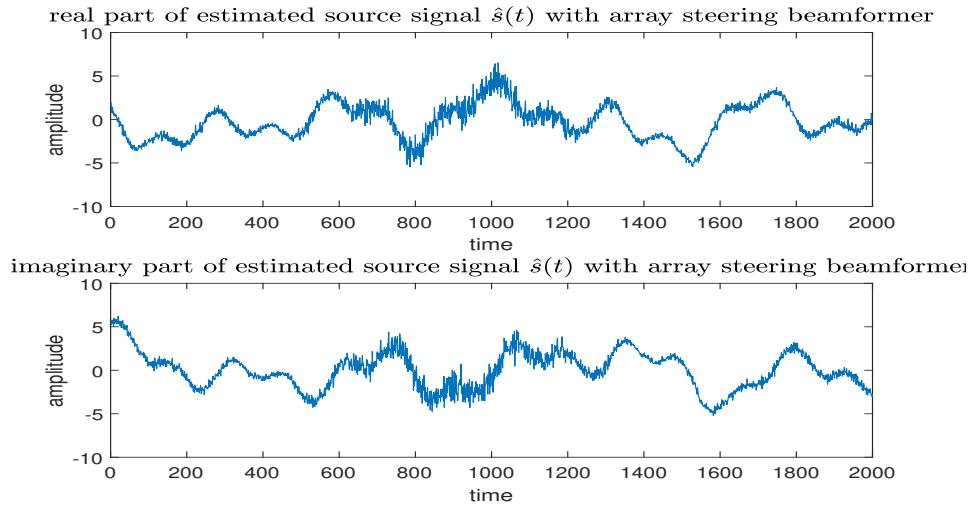


Figure 3: signal with array steering beamformer

**STEP 2:** Beamforming the signal with uniform weighting beamformer

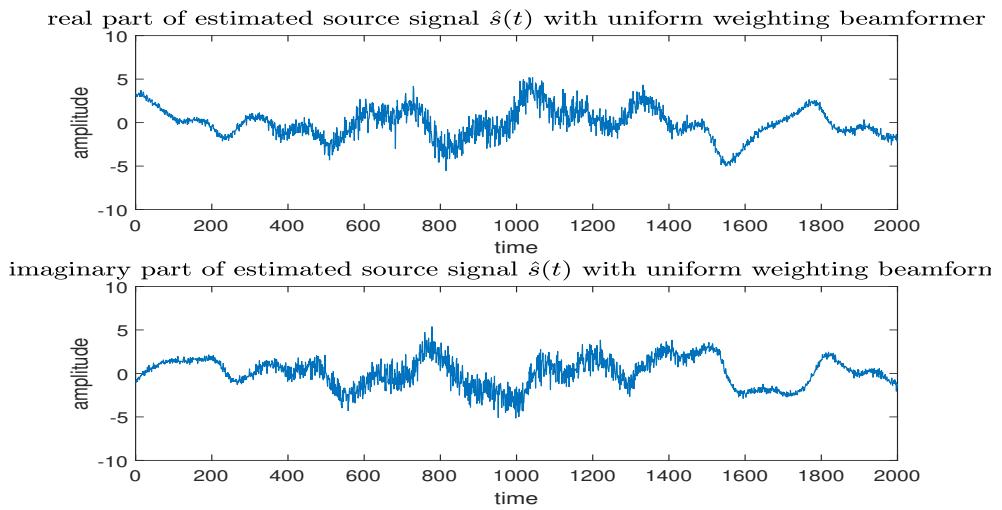


Figure 4: signal with uniform weighting beamformer

**STEP 3:** Calculate  $\hat{\mathbf{R}}$  with SMI(Sample Matrix Inversion)

**STEP 4:** Using  $\hat{\mathbf{R}}$  to beamform the signal with MVDR beamformer

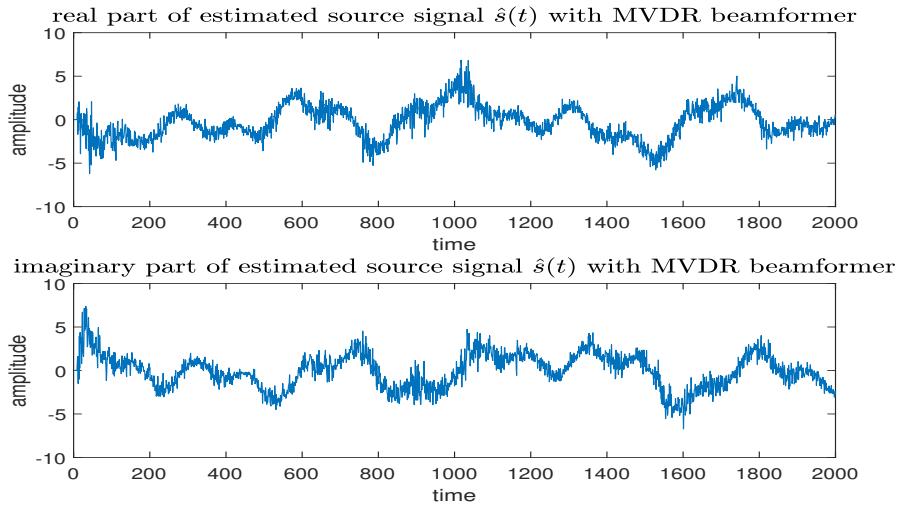


Figure 5: signal with MVDR beamformer

**STEP 5:** Using  $\hat{\mathbf{R}}$  to beamform the signal with LCMV beamformer

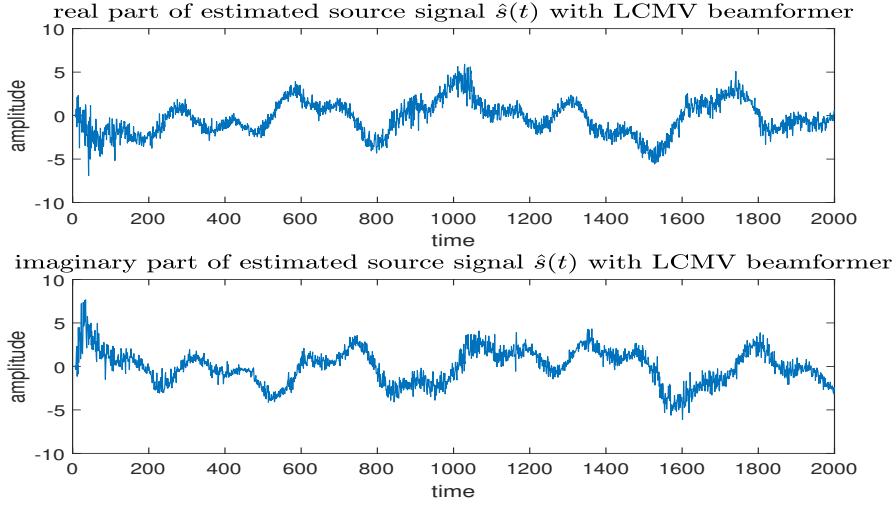


Figure 6: signal with LCMV beamformer

**STEP 6:** Based on LCMV beamformer, add more constraints. First, let  $\mathbf{C} = [\mathbf{a}(\theta_s) \mathbf{a}(\theta_i - 1) \mathbf{a}(\theta_i) \mathbf{a}(\theta_i + 1)]$   
**STEP 7:** By observing the estimated DOAs, we could find that, the interference is very serious from time 670 to 1172. So I change the vector  $\mathbf{g} = [1 \ 10^{-4} \ 10^{-4} \ 10^{-4}]^T$  from time 670 to 1172.

### 4.3 Algorithm

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#### Algorithm 2 An algorithm with SMI

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**Require:**  $\sigma_o^2$  and  $\mu$  and steering vector  $\mathbf{a}(\theta_s)$  and the sample  $x(n)$

- 1:  $\mathbf{P}(0) = \frac{1}{\sigma_o^2} \mathbf{I}$
  - 2:  $\hat{\mathbf{w}}(0) = \frac{\mathbf{a}(\theta_s)}{N}$
  - 3: **for**  $K=1,2,\dots,2000$  **do**
  - 4:      $\mathbf{g}(K) = \frac{\mu^{-1} \mathbf{P}(K-1) \mathbf{x}(K)}{1 + \mu^{-1} \mathbf{x}^H(K) \mathbf{P}(K-1) \mathbf{x}(K)}$
  - 5:      $\mathbf{P}(K) = \mu^{-1} \mathbf{P}(K-1) - \mu^{-1} \mathbf{g}(K) \mathbf{x}^H(K) \mathbf{P}(K-1)$
  - 6: **end for**
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**Algorithm 3** An algorithm with my beamformer

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**Require:**  $\sigma_o^2$  and  $\mu$  and steering vector  $\mathbf{a}(\theta_s)$  and interference steering vector  $\mathbf{a}(\theta_i)$  and the sample  $x(n)$

```

1:  $\mathbf{P}(0) = \frac{1}{\sigma_o^2} \mathbf{I}$ 
2:  $\hat{\mathbf{w}}(0) = \frac{\mathbf{a}(\theta_s)}{N}$ 
3: for  $K=1,2,\dots,2000$  do
4:    $\mathbf{C}(K) = [\mathbf{a}(\theta_s)(K) \mathbf{a}(\theta_i - 1)(K) \mathbf{a}(\theta_i)(K) \mathbf{a}(\theta_i + 1)(K)]$ 
5:    $\mathbf{g}(K) = \frac{\mu^{-1} \mathbf{P}(K-1) \mathbf{x}(K)}{1 + \mu^{-1} \mathbf{x}^H(K) \mathbf{P}(K-1) \mathbf{x}(K)}$ 
6:    $\mathbf{P}(K) = \mu^{-1} \mathbf{P}(K-1) - \mu^{-1} \mathbf{g}(K) \mathbf{x}^H(K) \mathbf{P}(K-1)$ 
7:   if  $K \geq 672 \wedge K \leq 1172$  then
8:      $\mathbf{g} \leftarrow [1 \ 10^{-4} \ 10^{-4} \ 10^{-4}]^T$ 
9:   else
10:     $\mathbf{g} \leftarrow [1 \ 0 \ 0 \ 0]^T$ 
11:   end if
12:    $\mathbf{w}_{LCMV}(K) = \mathbf{P} \mathbf{C} (\mathbf{C}^H \mathbf{P} \mathbf{C})^{-1} \mathbf{g}$ 
13:    $\mathbf{y}(K) = \mathbf{w}_{LCMV}(K) \mathbf{x}(K)$ 
14: end for

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## 5 Plot the real and imaginary parts of estimated source signal $\hat{s}(t)$

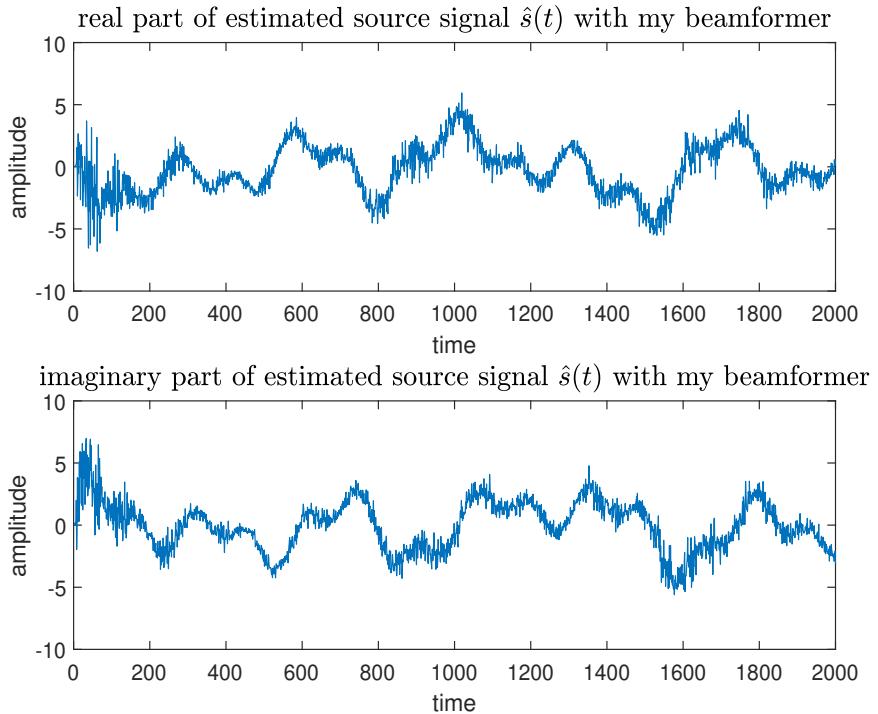


Figure 7: signal with my beamformer